1).

To show that the estimator $\hat{\beta} = (X'X)^{-1} X'y$ is unbiased under the Gauss-Markov assumptions, we need to show that the expected value of the estimator equals the true parameter value, i.e., $E[\hat{\beta}] = \beta$. This requires some of the Gauss-Markov assumptions.

The Gauss-Markov assumptions are:

1. Linearity in parameters: The model can be written as $\boldsymbol{Y = X\beta + u}$

2. Random Sampling: The observations (yi, Xi) are drawn randomly from the population.

3. No perfect multicollinearity: The matrix of predictors X has full column rank, meaning that no independent variable is a perfect linear function of other explanatory variables.

4. Zero conditional mean: The error u has an expected value of zero given any value of the explanatory variables. In other words, $\boldsymbol{E(u|X) = 0}$

5. Homoscedasticity: The error u has the same variance given any value of the explanatory variables. In other words, $Var(u|X) = \sigma^2$

6. No serial correlation: The errors are uncorrelated across observations.

To show that the estimator βH is unbiased, we rely primarily on assumptions **1, 3, and 4**.

Here's the proof:

$$\begin{aligned} E[\beta] &= E[(X'X)^{-1} X'y] \\ &= E[(X'X)^{-1} X'(X\beta + u)] \\ &= E[(X'X)^{-1} X'X\beta + (X'X)^{-1} X'u] \\ &= \beta + E[(X'X)^{-1} X'u] \end{aligned}$$

From assumption 4 $(E[u|X] = 0)$, we have $E[X'u] = X'E[u] = 0$, because the expectation of u given any X is 0, thus the expectation of the product is also 0.

This gives us $E[\hat{\beta}] = \beta + 0 = \beta$.

So, the estimator $\hat{\beta}$ is unbiased under the Gauss-Markov assumptions.

2). Log file seen Appendix

a)

```
% a) Generate X and Y

X = (1:50)';

e = normrnd(0,1,50,1);

Y = 1 + 2*X + e;



% Add a column of ones to X for the intercept term
```

```
X = [ones(50,1), X];


% Estimate the coefficients

beta = (X'*X)\(X'*Y); % estimate the coefficients


disp(['Estimates of coefficients: ', num2str(beta')]);
```

Estimates of coefficients: 1.0736 1.9953

b) Yes, the estimates of coefficients are close to 1,2.

c)

```
% c) Initialize the matrix B to store coefficients

B = zeros(100,2);

B(1,:) = beta';
```

d)

```
% d) Run the simulation 100 times

for i = 2:100

    % Generate X and Y

    X = (1:50)';

    e = normrnd(0,1,50,1);

    Y = 1 + 2*X + e;


    % Add a column of ones to X for the intercept term

    X = [ones(50,1), X];
```

```
    % Estimate the coefficients

    beta = (X'*X)\(X'*Y);



    % Store the coefficients in the matrix B

    B(i,:) = beta';

end
```

e)

```
%e) Compute the mean and standard deviation of the coefficients across the rows of B

B_mean = mean(B);

B_std = std(B);



disp(['Mean of coefficients: ', num2str(B_mean)]);

disp(['Standard deviation of coefficients: ', num2str(B_std)]);
```

a. **Mean of coefficients: 0.94188 2.0023**

b. **Standard deviation of coefficients: 0.26734 0.0089618**

c. The true coefficients (β) for your model are 1 and 2.

1. Mean of Estimated Coefficients: From the simulation results, the means of the estimated coefficients are 0.94188 and 2.0023, which are close to the true values of 1 and 2 respectively. This implies that the OLS estimator is unbiased, as the expected value of the estimator is approximately equal to the true parameter value.

2. Standard Deviation of Estimated Coefficients: The standard deviations of the estimated coefficients are 0.26734 and 0.0089618. These values represent the variability or dispersion of the estimated coefficients from the mean. A smaller standard deviation implies a more precise estimate, as the estimated values are more concentrated around the mean. Therefore, for the second coefficient (with the true value of 2), the estimated beta seems to have a very low variance, which is good.

So in conclusion, the OLS estimator performs well under these circumstances in terms of unbiasedness and precision (low variance), as reflected by the first two moments of the estimator.

3)

f(x,y)	Y = 0	Y = 1	f(x)
X=-10	0.18	0	0.18
X=0	0	0.3	0.3
X=10	0.07	0.45	0.52
f(y)	0.25	0.75	1

4)

a.

$$
\begin{aligned}
E(Z) &= E[\frac{X+Y}{2}] \\
&= \frac{E(X+Y)}{2} \\
&= \frac{[E(X)+E(Y)]}{2} \\
&= (\mu_x + \mu_y)/2 \\
&= \mu
\end{aligned}
$$

b.

$$
\begin{aligned}
var(Z) &= var\left[\frac{(X+Y)}{2}\right] \\
&= \frac{var(X+Y)}{4} \\
&= \frac{E((X+Y)^2) - \left(E(X+Y)\right)^2}{4} \\
&= \frac{E(X^2+Y^2+2XY) - \left(E(X)^2 + E(Y)^2 + 2E(X)E(Y)\right)}{4} \\
&= \frac{E(X^2) + E(Y^2) + 2E(XY) - E(X)^2 - E(Y)^2 - 2E(X)E(Y)}{4} \\
&= \frac{E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2E(XY) - 2E(X)E(Y)}{4} \\
&= \frac{var(X) + var(Y) + 2(E(XY) - E(X)E(Y))}{4}
\end{aligned}
$$

If the X and Y are independent, $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

$$
\begin{aligned}
var(Z) &= \frac{var(X) + var(Y) + 2\left(E(XY) - E(X)E(Y)\right)}{4} \\
&= \frac{var(X) + var(Y)}{4} \\
&= \frac{\sigma^2}{2}
\end{aligned}
$$

i.

if $Cov(X,Y) = E(XY) - E(X)E(Y) = 0.5\sigma^2$:

$$var(Z) = \frac{var(X) + var(Y) + 2\big(E(XY) - E(X)E(Y)\big)}{4}$$
$$= \frac{\sigma^2 + \sigma^2 + \sigma^2}{4}$$
$$= \frac{3\sigma^2}{4}$$

5)

$$E(chi) = 264$$
$$var(Chi) = 2 * 264$$

$$\Pr(X \leq 297) = \Pr\left(\frac{X - 264}{\sqrt{2 * 264}} \leq \frac{297 - 264}{\sqrt{2 * 264}}\right)$$
$$= \Pr\,(\phi \leq \frac{297 - 264}{\sqrt{2 * 264}})$$
$$= \Pr\,(\phi \leq 1.44)$$
$$= 0.920589$$

6)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 * 2 + 3 * 1 + 3 * 6 & 1 * 4 + 3 * 5 + 3 * 2 \\ 2 * 2 + 4 * 1 + 1 * 6 & 2 * 4 + 4 * 5 + 1 * 2 \end{bmatrix} = \begin{bmatrix} 23 & 25 \\ 14 & 30 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 3 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 5 & 2 \end{bmatrix}$$

$$A'B' = \begin{bmatrix} 1 * 2 + 2 * 4 & 1 * 1 + 2 * 5 & 1 * 6 + 2 * 2 \\ 3 * 2 + 4 * 4 & 3 * 1 + 4 * 5 & 3 * 6 + 4 * 2 \\ 3 * 2 + 1 * 4 & 3 * 1 + 1 * 5 & 3 * 6 + 1 * 2 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 10 \\ 22 & 23 & 26 \\ 10 & 8 & 20 \end{bmatrix}$$

$$BA = (A'B')' = \begin{bmatrix} 10 & 22 & 10 \\ 11 & 23 & 8 \\ 10 & 26 & 20 \end{bmatrix}$$

Appendix: Log file

```
% a) Generate X and Y
X = (1:50)';
e = normrnd(0,1,50,1);
Y = 1 + 2*X + e;

% Add a column of ones to X for the intercept term
X = [ones(50,1), X];

% Estimate the coefficients
beta = (X'*X)\(X'*Y); % estimate the coefficients

disp(['Estimates of coefficients: ', num2str(beta')]);

% c) Initialize the matrix B to store coefficients
B = zeros(100,2);
B(1,:) = beta';

% d) Run the simulation 100 times
for i = 2:100
    % Generate X and Y
    X = (1:50)';
    e = normrnd(0,1,50,1);
    Y = 1 + 2*X + e;

    % Add a column of ones to X for the intercept term
    X = [ones(50,1), X];

    % Estimate the coefficients
    beta = (X'*X)\(X'*Y);

    % Store the coefficients in the matrix B
    B(i,:) = beta';
end

% e) Compute the mean and standard deviation of the coefficients across the rows of B
B_mean = mean(B);
B_std = std(B);

disp(['Mean of coefficients: ', num2str(B_mean)]);
disp(['Standard deviation of coefficients: ', num2str(B_std)]);
```