UCI Paul Merage School of Business

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MFIN 290: Financial Econometrics

Lecture 5-1

Last Time

- Cointegration
- Error Correction Models

- Yule Walker Equations, Moment Restrictions
- AR(2), ARMA(1,1) examples

Today

- ARCH

ARCH

- Thus far in the course, when we have looked at time series effects and dependencies, we have focused on dependencies in the first moment.

- $y = \beta y_{t-1} + u_t + \gamma u_{t-1}$

- How does the expected value of y depend on previous values of y or previous values of u?

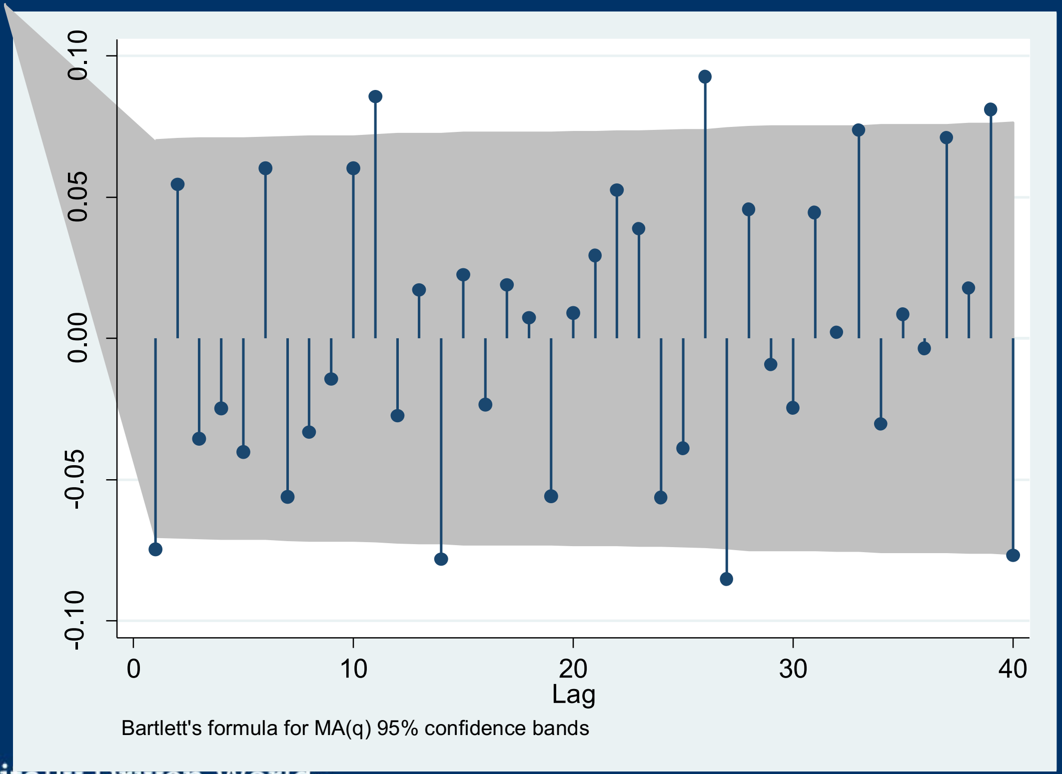
ARCH

- There is usually no autocorrelation in financial prices (why does this make sense)?

- But that doesn't mean there isn't anything we can say here…

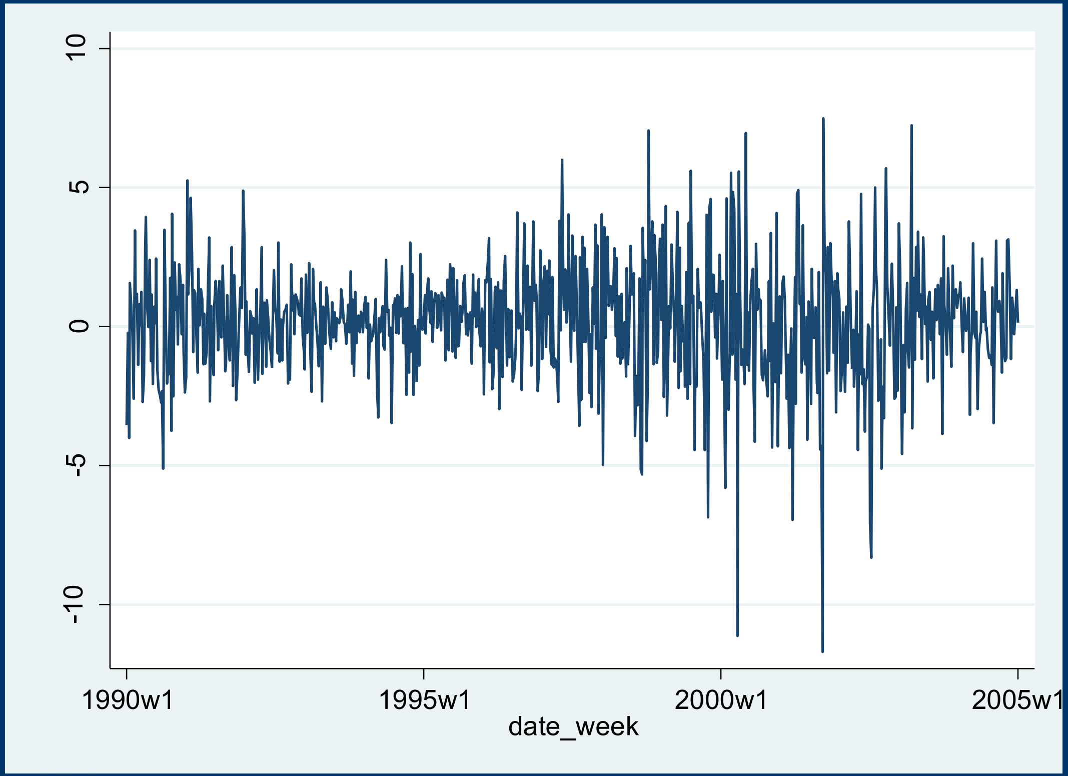
ARCH

- Below: AC plot of weekly S&P 500 returns from Jan 1990 – Dec 2004. Nothing is a perfect textbook example, but this is pretty close…



ARCH

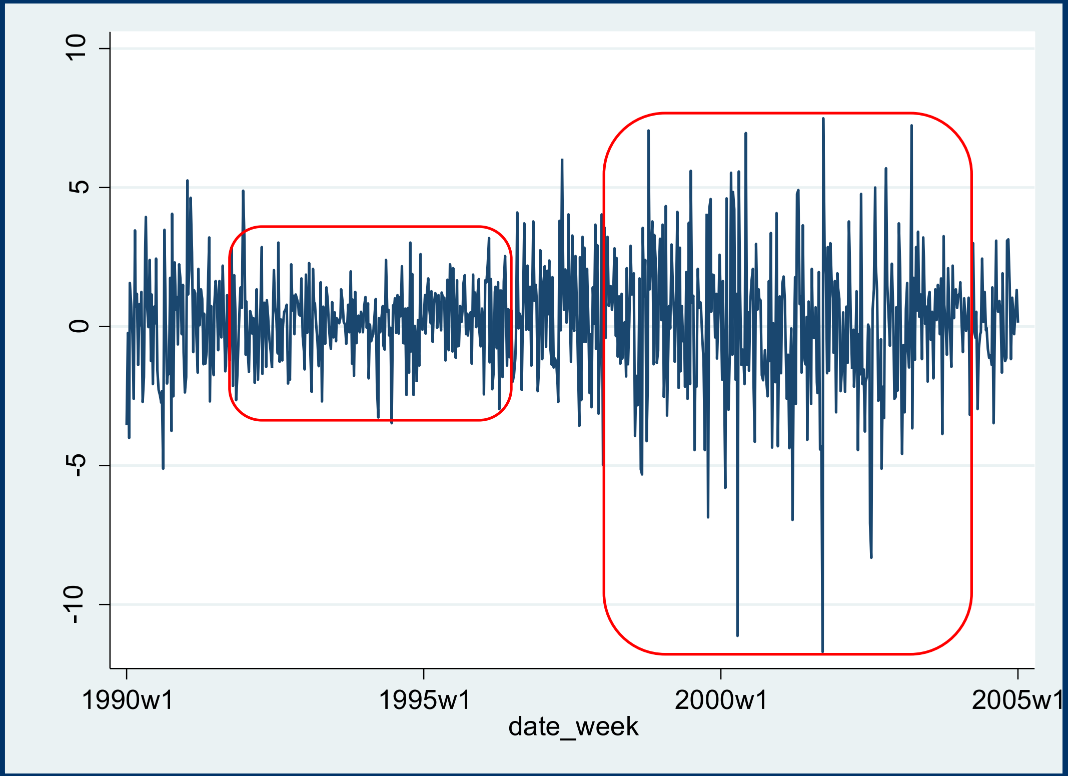
- Below: Weekly S&P 500 returns from Jan 1990 – Dec 2004



What do we see?

ARCH

- Below: Weekly S&P 500 returns from Jan 1990 – Dec 2004



Clear periods of lower and higher volatility.

AND it looks like the periods of low and high volatility tend to follow one another.

ARCH

- Consider a model given by
- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- Where u_t is white noise, unit variance, uncorrelated, standard normal as usual

ARCH

- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- $E(u_t^2) = 1$
- $E(e_t|x_t, e_{t-1}) = 0$
- $E(e_t|x_t) = 0$
- $E(y_t|x_t) = x_t\beta$

- These are the classical regression assumptions. We can get $\hat{\beta}$ unbiased through OLS

ARCH

- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- $var(e_t|e_{t-1}) = E(e_t^2|e_{t-1})$
- $= E(\left[u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}\right]^2 = \alpha_0 + \alpha_1 e_{t-1}^2$

- Here, the **variance** of e_t is autoregressive. Called "Conditionally Heteroskedastic"

ARCH

- “Conditionally Heteroskedastic”
- Not with respect to the x values, but with earlier values of e_t ...
- “Auto Regressive Conditionally Heteroskedastic”

ARCH

- This has implications for pricing securities (such as options) whose value depends on future volatilities.

- If we model volatility as flat, when it is really conditionally heteroskedastic, our option prices will be systematically too high when times are quiet (assuming the option becomes more valuable when volatility increases) and too low when times are turbulent.

- You can see this in traded option prices relative to vanilla Black Scholes! Using a GARCH or ARCH model helps mitigate the implied vol smile for out of the money options.

ARCH

- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
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ARCH

- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- $var(e_t) = \alpha_0 + \alpha_1 var(e_{t-1})$
- If weakly stationary => $var(e_t) = var(e_{t-1}) \Rightarrow ?$

ARCH

- $y_t = x_t\beta + e_t$
- $e_t = u_t\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}$
- $var(e_t) = \frac{\alpha_0}{1-\alpha_1}$
- Note: $|\alpha_1| < 1$ here for this to be finite and positive (like a variance has to be)

ARCH

- But then… e_t is distributed normal with mean 0 and variance $\frac{\alpha_0}{1-\alpha_1}$

- => we have our Gauss Markov Assumptions => OLS is the minimum variance linear unbiased estimator for β

- So can't we just run OLS here? What are we missing?

ARCH

- There is a more efficient NON-LINEAR estimator!
- We estimate these models through maximum likelihood (later in this course).
- Engle (1982) derived the function needed here (and won a Nobel prize for it!). The likelihood function is in his paper for those interested.

Extensions to ARCH

- Higher Order Processes

- $\sigma_t^2 = E(e_t^2 | e_{t-s}\ for\ all\ S > 0)$
- Given all of the relevant information known on period t… usually written compactly, say ψ_t:

- $\sigma_t^2 = E(e_t^2 | \psi_t)$
-
- $\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + \ \ldots + \alpha_n e_{t-n}^2$

Extensions to ARCH

- GARCH

- $y_t = x_t\beta + e_t$
- $e_t|\psi_t \sim N(0, \sigma_t^2)$

- $\sigma_t^2 = \alpha_0 + \delta_1\sigma_{t-1}^2 + \cdots + \delta_n\sigma_{t-n}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_n e_{t-n}^2$

Extensions to ARCH

- GARCH

- $y_t = x_t\beta + e_t$
- $e_t|\psi_t \sim N(0, \sigma_t^2)$

- $\sigma_t^2 = \alpha_0 + \delta_1\sigma_{t-1}^2 + \cdots + \delta_n\sigma_{t-n}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_n e_{t-n}^2$

- The conditional variance is defined as an ARMA process in e_t^2
- GARCH(p,q) where p = order or AR component, q = order of MA component

Extensions to ARCH

- GARCH
- Some extra conditions are required to ensure that the specification is covariance stationary…
- Ensuring that σ_t^2 is stable doesn't guarantee that all of the higher moments are.

- Bollerslev (1986) ("Generalized Autoregressive Conditional Heteroskedasticity" *Journal of Econometrics*) contains some of these conditions. Worth keeping this caveat in mind with these models and looking them up/checking as needed.

- Often the case that relatively simple GARCH models (such as (1,1)) perform as well characterizing real data as a longer (say order 8) ARCH.

Extensions to ARCH

- ARCH in Mean
- What if the conditional mean of y depends in part on the variance?

Extensions to ARCH

- ARCH in Mean
- What if the conditional mean of y depends in part on the variance?
- Intuition:
- The returns on a portfolio should be increasing with its risk, which may vary over time.

Extensions to ARCH

- ARCH in Mean
- $y_t = \beta x_t + \delta\sigma_t^2 + e_t$
- $var(e_t|\psi_t) = ARCH(q)$
- This makes a lot of sense, but is pretty rare.... Why don't we see this more?

Extensions to ARCH

- ARCH in Mean
- $y_t = \beta x_t + \delta \sigma_t^2 + e_t$
- $var(e_t | \psi_t) = ARCH(q)$
- Mis-specification of the variance function will give you inconsistent estimates of σ_t^2, which can give you inconsistent estimates of β!
- Unlike in weighted least squares/GLS from earlier in the course here, the "right" weights are determined by (and thus correlated with) the disturbances themselves.

$\Rightarrow$ If there is mis-specification in our ARCH model, the coefficients we care about will be inconsistent.

Extensions to ARCH

- ARCH in Mean
- $y_t = \beta x_t + \delta \sigma_t^2 + e_t$
- $var(e_t|\psi_t) = ARCH(q)$
- Can have an efficient estimate in one case, but it might be inconsistent?
- Sounds like a Hausman test!

Extensions to ARCH

- T-ARCH
- Idea is that the effect of the residual will be asymmetric. “Good” news may have less of an impact on volatility than “bad” news.

- Conditional Variance function is asymmetric:
- $h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \gamma D_{t-1} e_{t-1}^2 + \beta h_{t-1} + u_t + others \ldots$

- $D_t = \begin{cases} 1 \; e_t < 0 \\ 0 \; e_t \geq 0 \end{cases}$

Extensions to ARCH

Title

[TS] **arch** — Autoregressive conditional heteroskedasticity (ARCH) family of estimators

Syntax

arch *depvar* [*indepvars*] [*if*] [*in*] [*weight*] [, *options*]

options	Description
Model	
noconstant	suppress constant term
arch(*numlist***)**	ARCH terms
garch(*numlist***)**	GARCH terms
saarch(*numlist***)**	simple asymmetric ARCH terms
tarch(*numlist***)**	threshold ARCH terms
aarch(*numlist***)**	asymmetric ARCH terms
narch(*numlist***)**	nonlinear ARCH terms
narchk(*numlist***)**	nonlinear ARCH terms with single shift
abarch(*numlist***)**	absolute value ARCH terms
atarch(*numlist***)**	absolute threshold ARCH terms
sdgarch(*numlist***)**	lags of s_t
earch(*numlist***)**	new terms in Nelson's EGARCH model
egarch(*numlist***)**	lags of ln(s_t^2)
parch(*numlist***)**	power ARCH terms
tparch(*numlist***)**	threshold power ARCH terms
aparch(*numlist***)**	asymmetric power ARCH terms
nparch(*numlist***)**	nonlinear power ARCH terms
nparchk(*numlist***)**	nonlinear power ARCH terms with single shift
pgarch(*numlist***)**	power GARCH terms
constraints(*constraints***)**	apply specified linear constraints
collinear	keep collinear variables

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Extensions to ARCH

Model 2	
archm	include ARCH-in-mean term in the mean-equation specification
archmlags(*numlist*)	include specified lags of conditional variance in mean equation
archmexp(*exp*)	apply transformation in *exp* to any ARCH-in-mean terms
arima(*#p*, *#d*, *#q*)	specify ARIMA(*p*,*d*,*q*) model for dependent variable
ar(*numlist*)	autoregressive terms of the structural model disturbance
ma(*numlist*)	moving-average terms of the structural model disturbances
Model 3	
distribution(*dist* [#])	use *dist* distribution for errors (may be **gaussian**, **normal**, **t**, or **ged**; default is **gaussian**)
het(*varlist*)	include *varlist* in the specification of the conditional variance
savespace	conserve memory during estimation
Priming	
arch0(xb)	compute priming values on the basis of the expected unconditional variance; the default
arch0(xb0)	compute priming values on the basis of the estimated variance of the residuals from OLS
arch0(xbwt)	compute priming values on the basis of the weighted sum of squares from OLS residuals
arch0(xb0wt)	compute priming values on the basis of the weighted sum of squares from OLS residuals, with more weight at earlier times
arch0(zero)	set priming values of ARCH terms to zero
arch0(#)	set priming values of ARCH terms to #
arma0(zero)	set all priming values of ARMA terms to zero; the default
arma0(p)	begin estimation after observation p, where p is the maximum AR lag in model
arma0(q)	begin estimation after observation q, where q is the maximum MA lag in model
arma0(pq)	begin estimation after observation (p + q)
arma0(#)	set priming values of ARMA terms to #
condobs(#)	set conditioning observations at the start of the sample to #

ARCH

- Usually we aren't concerned with volatility for its own sake, we want it for something else (option pricing is one example).
- ARCH style models are usually estimated, and the easiest way (IMO) to demonstrate the ARCH impacts is through simulation.
- A good exercise would be to see the effect an ARCH or GARCH model would have on option prices for the S&P 500 and compare that to actual observed prices…

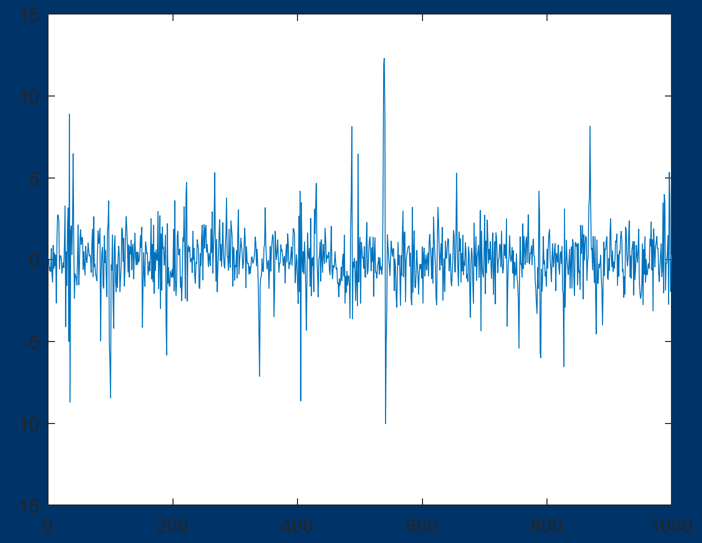
ARCH Simulation

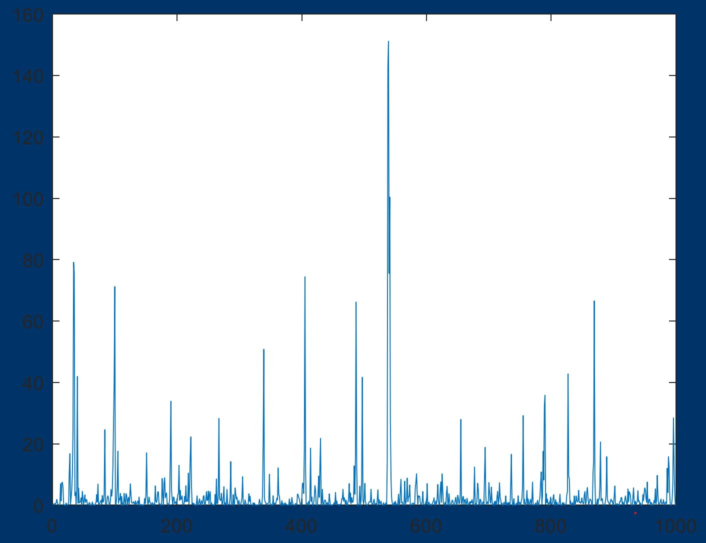
- MATLAB!

```
clear
alpha = [1 0.8];
eps = zeros(1000,1);
u = mvnrnd(0,1,1000);
T = [1:1:1000]';

for i =2:1000
    eps(i) = u(i)*(alpha(1) + alpha(2)*(eps(i-1))^2)^(0.5);
End

plot(T,eps)

eps_sq = eps.^2;
plot(T,eps_sq)

autocorr(eps)
autocorr(eps_sq)
parcorr(eps_sq)
```

ARCH Simulation

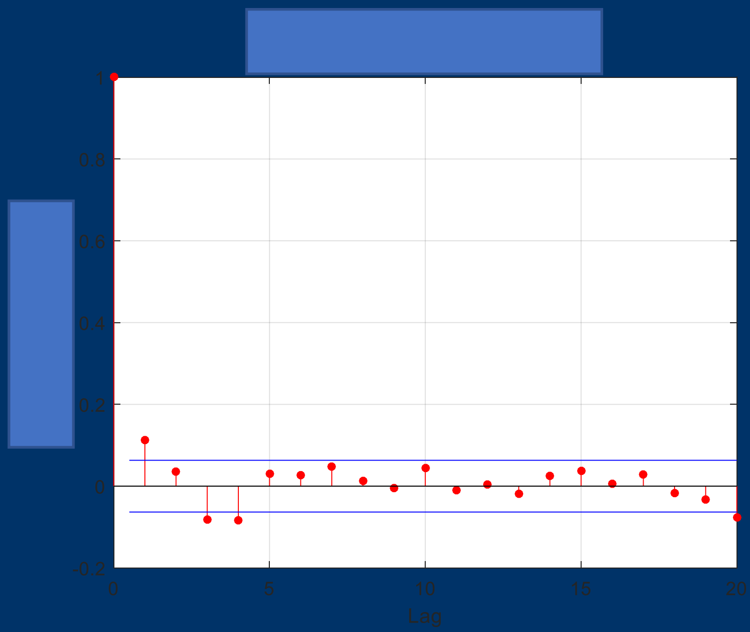
- `plot(T,eps)`
- `plot(T,eps_sq)`

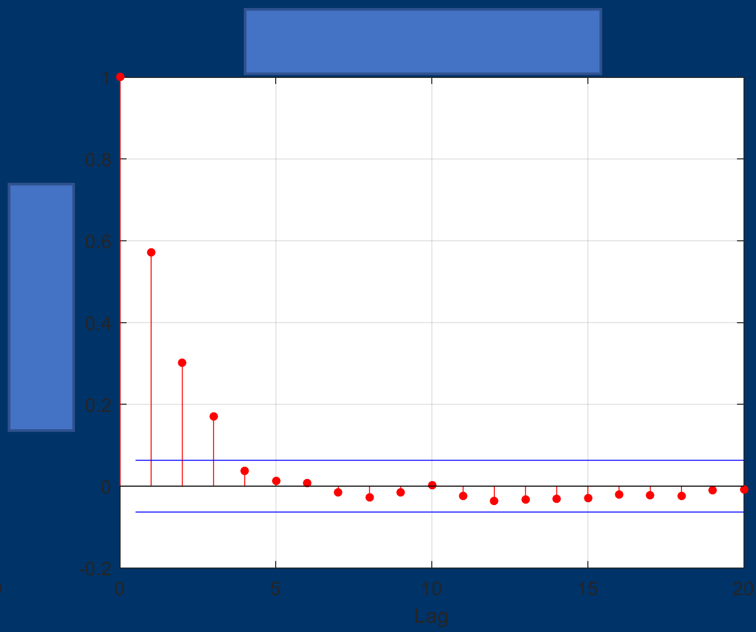


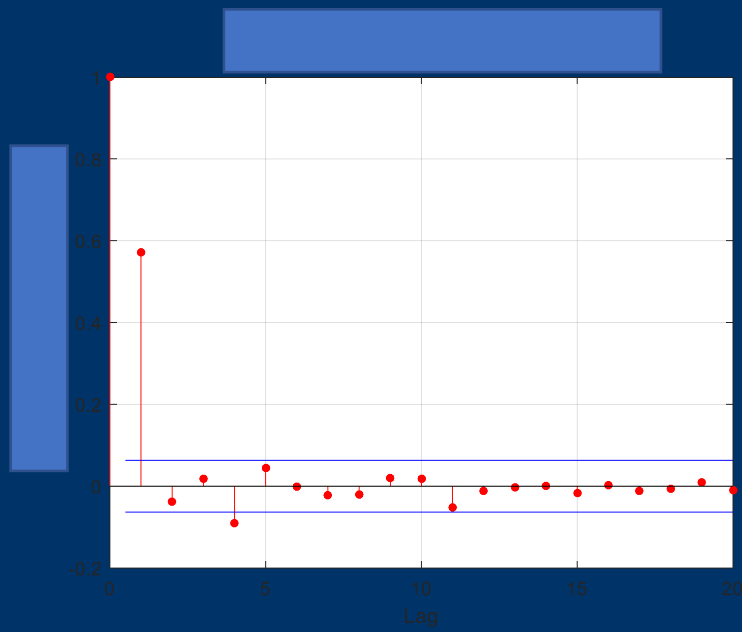


Which Graph is Which?

- `autocorr(eps)`
- `autocorr(eps_sq)`
- `parcorr(eps_sq)`

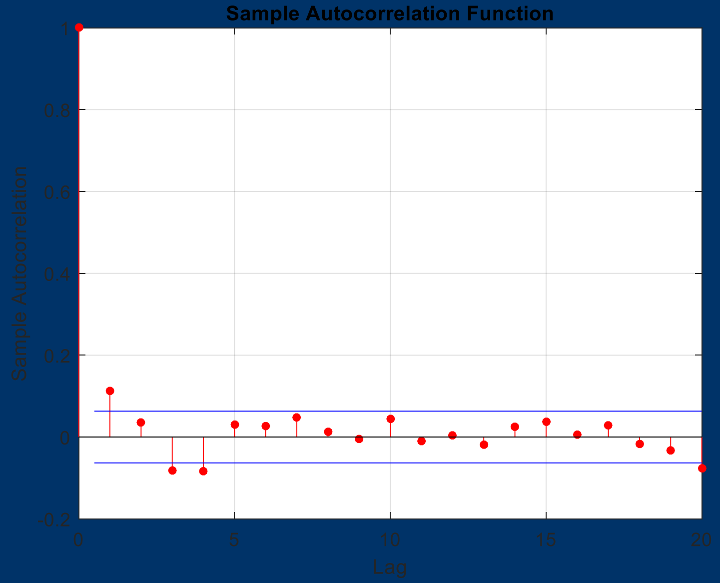


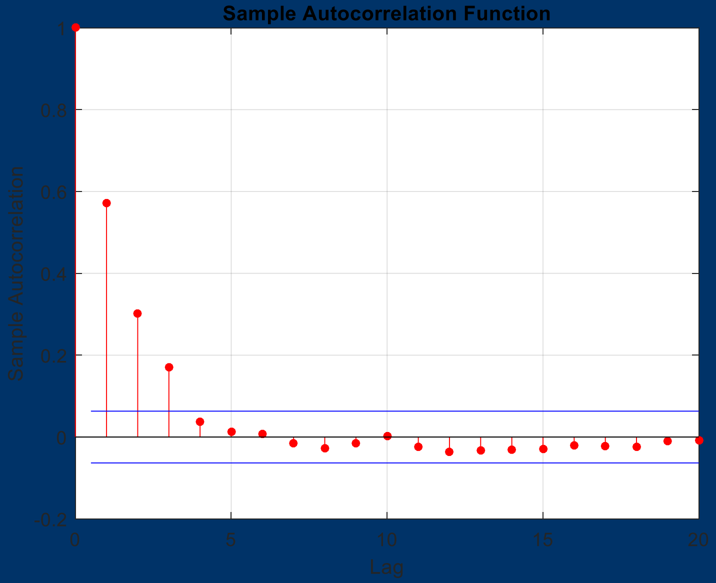


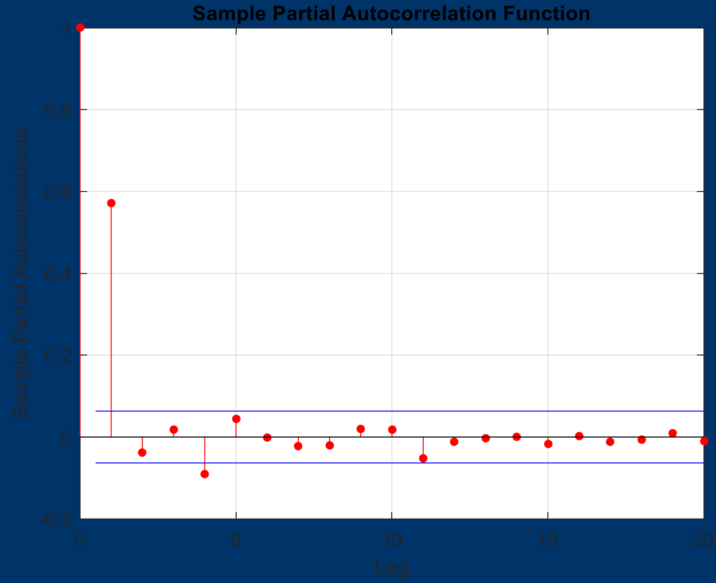


Which Graph is Which?

- `autocorr(eps)`   `autocorr(eps_sq)`   `parcorr(eps_sq)`

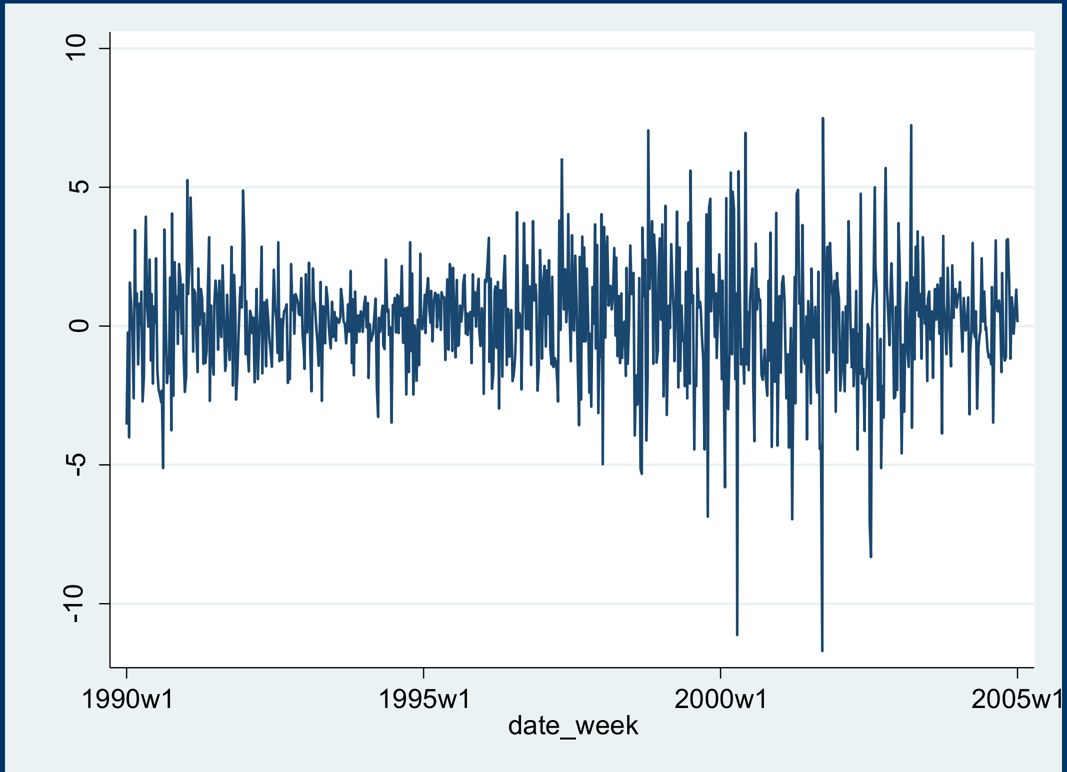






Testing for ARCH effects

- You should always inspect the data first. Sometimes the eyeball test can be the most obvious!



Testing for ARCH effects

- Can we be more quantitative?
- Idea: We can get consistent estimates (exc. ARCH – in – mean) with plain vanilla OLS.
- Use the estimated error terms to see if there is any autocorrelation (or MA terms) in a regression of $\tilde{e}_t^2$ (the fitted values) on a constant and the q lagged values.
- If there are no ARCH effects, a joint test of coefficient significance will show it.

Testing for ARCH effects

- $\widehat{e_t^2} = \alpha_0 + \alpha_1 \widehat{e_{t-1}^2} + v_t$
- $H_0\colon \alpha_1 = 0$ (no ARCH effects)

- Test statistic $= (T - q)R^2$
- T = number of obs
- Q = number of lags in the regression
- $R^2 = R^2$ from the regression above. Distributed χ_q^2.
- This form is generic for longer lags – with one lag, a t-test gets you to the same place.

Testing for ARCH effects

- Just like all heteroskedasticity, keep an eye on relevance.
- Depending on the application, OLS may yield an unbiased estimate, just not an efficient one.
- How much we care about characterizing the volatility series is usually a function of:
- 1) The question being asked
- 2) The consequences to mis-specifying the volatility relationship

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Testing for ARCH effects

- Example using S&P data

```
. reg r

      Source |       SS           df       MS      Number of obs   =       781
-------------+----------------------------------   F(0, 780)       =      0.00
       Model |           0         0           .   Prob > F        =         .
    Residual |  3598.50032       780  4.61346195   R-squared       =    0.0000
-------------+----------------------------------   Adj R-squared   =    0.0000
       Total |  3598.50032       780  4.61346195   Root MSE        =    2.1479

------------------------------------------------------------------------------
           r |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   .1582282   .0768578     2.06   0.040     .0073555    .3091008
------------------------------------------------------------------------------

. estat archlm, lags(1/5)
LM test for autoregressive conditional heteroskedasticity (ARCH)
---------------------------------------------------------------------------
    lags(p)  |          chi2               df                 Prob > chi2
-------------+-------------------------------------------------------------
       1     |         50.741               1                   0.0000
       2     |         50.956               2                   0.0000
       3     |         58.378               3                   0.0000
       4     |         59.240               4                   0.0000
       5     |         59.422               5                   0.0000
---------------------------------------------------------------------------
         H0: no ARCH effects          vs.  H1: ARCH(p) disturbance
```

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Testing for ARCH effects

- Example using S&P data

- Two points to think about.
- Second seems MA-style, less
- plausible…