Some Notes on Many-body Physics

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Throughout the notes, we use:

$$k = (\mathbf{k}, i\omega_n) \tag{1}$$

$$q = (\mathbf{q}, i\Omega_m) \tag{2}$$

where the Matsubara frequencies are:

$$\omega_n = \frac{(2n+1)\pi}{\beta} = (2n+1)\pi T \tag{3}$$

$$\Omega_m = \frac{2m\pi}{\beta} = 2m\pi T \tag{4}$$

where T is the temperature and β is the inverse temperature.

1 Single-orbital case

1.1 Non-interacting Green function in the normal state

In the normal state, the electron-phonon self-energy

$$G_0(k) = G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}} \tag{5}$$

Doing analytical continuation leads to:

$$G(\mathbf{k}, \omega + i\delta) = \frac{1}{\omega + i\delta - \epsilon_{\mathbf{k}}} \tag{6}$$

where δ is an infinitesimal number. The spectral function is then:

$$A(\mathbf{k},\omega) = -\frac{1}{\pi}\mathrm{Im}G(\mathbf{k},\omega+i\delta) \tag{7}$$

For a particular example, we use:

$$\epsilon_{\mathbf{k}} = -2t_1\left[\cos(k_x)+\cos(k_y)\right] + 4t_2\cos(k_x)\cos(k_y) - 2t_3\left[\cos(2k_x)+\cos(2k_y)\right] \tag{8}$$

where $t_1 = 0.38$ eV, $t_2 = 0.32t_1$ and $t_3 = 0.5t_2$.

Please use a contour plot to show the $A(\mathbf{k},\omega)$.

1.2 Non-interacting Green function in the superconducting state

First we introduce Pauli matrices:

$$\hat{\tau}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\tau}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

We use $\hat{A}$ on the symbol A to refer to a matrix.

The non-interacting Green function in the superconducting state is:

$$\hat{G}_{\mathrm{sc}}^{-1}(k) = \hat{G}_0^{-1}(k) - \hat{\Sigma}_{\mathrm{sc}}(k) \tag{10}$$

where $\hat{G}_0^{-1}(k)$ is

$$\hat{G}_0^{-1}(k) = i\omega_n\hat{\tau}_0 - \epsilon_{\mathbf{k}}\hat{\tau}_3 \tag{11}$$

The self-energy $\hat{\Sigma}(k)$ is:

$$\hat{\Sigma}(k) = i\omega_n\left(1 - Z(k)\right)\hat{\tau}_0 + \chi(k)\hat{\tau}_3 + \phi(k)\hat{\tau}_1 + \overline{\phi}(k)\hat{\tau}_2 \tag{12}$$

We can choose a gauge so that $\overline{\phi}(k) = 0$.

In the superconducting state with a weak coupling, we may set:

$$Z(k) \to 1 \quad \chi(k) \to 0 \quad \Delta(k) = \frac{\phi(k)}{Z(k)} \to \phi(k) = \Delta(k) \tag{13}$$

Therefore we have:

$$\hat{\Sigma}_{\rm sc}(k) = \Delta(k)\hat{\tau}_1 \tag{14}$$

Here we consider a special case: d-wave superconductivity in two dimensions. We have:

$$\Delta(k) = \Delta_{\mathbf{k}} = \Delta_0 \left[\cos(k_x) - \cos(k_y)\right]/2 \tag{15}$$

where $\Delta_0 = 35$ meV. We note that the frequency dependence in $\Delta(k)$ disappears.

Combining everything, we have:

$$\hat{G}^{-1}_{\rm sc}(k) = i\omega_n\hat{\tau}_0 - \epsilon_{\mathbf{k}}\hat{\tau}_3 - \Delta_{\mathbf{k}}\hat{\tau}_1 \tag{16}$$

Taking the inverse leads to:

$$\hat{G}_{\rm sc}(k) = \frac{i\omega_n\hat{\tau}_0 + \epsilon_{\mathbf{k}}\hat{\tau}_3 + \Delta_{\mathbf{k}}\hat{\tau}_1}{(i\omega_n)^2 - E^2_{\mathbf{k}}} \tag{17}$$

where $E_{\mathbf{k}}$ is:

$$E_{\mathbf{k}} = \sqrt{\epsilon^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}}} \tag{18}$$

Doing analytical continuation leads to:

$$\hat{G}_{\rm sc}(\mathbf{k}, \omega + i\delta) = \frac{(\omega + i\delta)\hat{\tau}_0 + \epsilon_{\mathbf{k}}\hat{\tau}_3 + \Delta_{\mathbf{k}}\hat{\tau}_1}{(\omega + i\delta)^2 - E^2_{\mathbf{k}}} \tag{19}$$

Next we calculate the spectral function:

$$A(\mathbf{k}, \omega) = -\frac{1}{\pi}\mathrm{Im}\left[\hat{G}_{\rm sc}\right]_{11}(\mathbf{k}, \omega + i\delta) \tag{20}$$

Use a contour plot for $A(\mathbf{k}, \omega)$ along $\mathbf{k}$ path from $(0, 0)$ to $(\pi, 0)$. Use a contour plot for $A(\mathbf{k}, \omega = 0)$ for a two-dimensional $\mathbf{k}$ plane $0 \leq k_x \leq \pi$ and $0 \leq k_y \leq \pi$.