Problem Set 4 (Part 1)

Q1 – Lack of Overlap?

Recall from lecture that certain causal assumptions, including ignorability (also called unconfoundedness) and common support (also called overlap or positivity), need to be met in order to identify the average treatment effect (ATE). Under these assumptions, we saw that we could use two different approaches to do this identification: covariate adjustment using the so-called "adjustment formula" or inverse probability weighting (IPW). In this problem, you will explore and gain intuition for why one of these causal assumptions, particularly overlap/positivity, is important for estimating the ATE. Secondly, in the context of a concrete example, you will think through the interpretation of the ATE in a setting where there is a positivity violation.

To begin, we will define some notation. Let Y denote an outcome of interest and $T \in \{0, 1\}$ denote a binary treatment. Furthermore, let $Y(t)$ denote the potential outcome of an individual under treatment, $T = t$. Finally, let X denote all other covariates. Recall the adjustment formula used to estimate the ATE,

$$\mathbb{E}_{x\sim p(x)}[Y(1) - Y(0)] = \mathbb{E}_{x\sim p(x)}[\mathbb{E}[Y|T = 1, X = x] - \mathbb{E}[Y|T = 0, X = x]]$$

Recall as well the formula for IPW,

$$\begin{aligned}\mathbb{E}_{x\sim p(x)}[Y(1) - Y(0)] &= \mathbb{E}_{x\sim p(x|T=1)}\left[\frac{p(T = 1)}{p(T = 1|X)}Y\right] - \mathbb{E}_{x\sim p(x|T=0)}\left[\frac{p(T = 0)}{p(T = 0|X)}Y\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{1}(t_i = 1)}{\hat{p}(t_i = 1|x_i)}y_i - \frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{1}(t_i = 0)}{\hat{p}(t_i = 0|x_i)}y_i,\end{aligned}$$

where n is the total number of individuals in the cohort.

[3 points] 2.1 Consider an alternate form of the IPW estimator, called IPW-T, where the only difference is that the true propensity scores are given to you (i.e. you do not need to estimate them). Further assume that the true ATE is finite, i.e. $\mathbb{E}_{x\sim p(x)}[Y(1) - Y(0)] < \infty$. Show that if there is a positivity violation, i.e. $p(t_i = t|x_i) = 0$ for some unit x_i, then IPW-T is a biased estimator, i.e. the estimate you get using IPW-T is not finite.

For the remaining two parts, we will work with a toy example. Suppose that $X \in \{-1, 0, 1\}$ and ignorability holds. The observed outcomes under binary treatment for the entire cohort (of four units) are shown in Table 1. Note that there is lack of overlap when $X = -1$ and $X = 0$.

	$T = 0$	$T = 1$
$x_1 = -1$	?	0.1
$x_2 = 0$	0.6	?
$x_3 = 1$	?	0.4
$x_4 = 1$	0.5	?

Table 1: Observed Outcomes for Toy Example; ? represents the Unobserved Outcomes

[4 points] 2.2 Suppose that the true potential outcomes are linear as a function of X, as below,

$$\mathbb{E}[Y(1)|X] = \alpha_1 + \beta_1 \cdot X$$
$$\mathbb{E}[Y(0)|X] = \alpha_0 + \beta_0 \cdot X$$

Under this stuctural assumption, fill in Table 1 with the potential outcomes. What is the ATE?

[4 points] 2.3 Now suppose that the true potential outcomes differ up to a constant effect, δ,

$$\mathbb{E}[Y(1)|X] = f(X) + \delta$$
$$\mathbb{E}[Y(0)|X] = f(X),$$

where f is a potentially non-linear function. As in the previous part, fill in Table 1 with the potential outcomes under this assumption. What is the ATE?

[4 points] 2.4 Now suppose that we make no parametric assumptions on the potential outcome functions (as in the previous two parts). Show that the ATE is nonidentifiable, i.e. you can get multiple potential outcomes (and thus ATEs) that are consistent with the data. *(Hint: Show a graph of the outcomes as a function of X for both interventions under the two parametric assumptions from the previous two parts.)* In general, what does this mean about identifiability of the ATE when there is lack of overlap?