Introduction and Basic Implementation for Finite Element Methods

Chapter 6: Finite elements for 2D steady Stokes equation

Xiaoming He
Department of Mathematics & Statistics
Missouri University of Science & Technology

Outline

Outline

Target problem

- Consider the 2D Stokes equation:

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\ \mathbf{u} = \mathbf{g} \qquad on \quad \partial\Omega. \end{cases}$$

  where

$$\mathbf{u}(x, y) = (u_1,\ u_2)^t,\ \ \mathbf{g}(x, y) = (g_1,\ g_2)^t,\ \ \mathbf{f}(x, y) = (f_1,\ f_2)^t.$$

- The stress tensor $\mathbb{T}(\mathbf{u}, p)$ is defined as

$$\mathbb{T}(\mathbf{u}, p) = 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}$$

  where ν is the viscosity and the deformation tensor

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$$

Weak formulation

- Since p appears in the equation without any derivative, then, if $(\mathbf{u}, p)$ is a solution, then $(\mathbf{u}, p+c)$ is also a solution where c is a constant. Hence we need to impose additional condition for p. Here are three regular choices:
- (1) Fix p at one point in the domain Ω.
- (2) Apply a stress or Robin boundary condition (at least in the normal direction) on part of the boundary $\partial\Omega$.
- (3) Apply $\int_\Omega p dxdy = 0$. (A good reference: On the finite element solution of the pure Neumann problem, Pavel Bochev, R. B. Lehoucq, SIAM Review, 47(1): 50-66, 2005.)

Target problem

- In more details, the deformation tensor can be written as

$$\mathbb{D}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

- Hence the stress tensor can be written as

$$\mathbb{T}(\mathbf{u}, p) = \begin{pmatrix} 2\nu\frac{\partial u_1}{\partial x} - p & \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & 2\nu\frac{\partial u_2}{\partial y} - p \end{pmatrix}$$

Weak formulation

- First, take the inner product with a vector function $\mathbf{v}(x,y) = (v_1,\, v_2)^t$ on both sides of the Stokes equation:

$$
\begin{aligned}
& -\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \\
\Rightarrow \quad & -(\nabla \cdot \mathbb{T}(\mathbf{u}, p)) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v} \quad \text{in } \Omega \\
\Rightarrow \quad & -\int_{\Omega} (\nabla \cdot \mathbb{T}(\mathbf{u}, p)) \cdot \mathbf{v} \; dxdy = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \; dxdy.
\end{aligned}
$$

- Second, multiply the divergence free equation by a function $q(x,y)$:

$$
\begin{aligned}
\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad & (\nabla \cdot \mathbf{u}) q = 0 \\
\Rightarrow \quad & \int_{\Omega} (\nabla \cdot \mathbf{u}) q \; dxdy = 0.
\end{aligned}
$$

- $\mathbf{u}(x,y)$ and $p(x,y)$ are called trail functions and $\mathbf{v}(x,y)$ and $q(x,y)$ are called test functions.

Weak formulation

- Using integration by parts in multi-dimension:

$$\int_{\Omega} (\nabla \cdot \mathbb{T}) \cdot \mathbf{v} \; dxdy = \int_{\partial\Omega} (\mathbb{T}\mathbf{n}) \cdot \mathbf{v} \; ds - \int_{\Omega} \mathbb{T} : \nabla \mathbf{v} \; dxdy,$$

where $\mathbf{n} = (n_1, \, n_2)^t$ is the unit outer normal vector of $\partial\Omega$, we obtain

$$\int_{\Omega} \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} \; dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \; dxdy.$$

Here,

$$\begin{aligned} A : B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}. \end{aligned}$$

Weak formulation

- Using the above definition for $A : B$, it is not difficult to verify (an independent study project topic) that

$$\begin{aligned}\mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} &= (2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}) : \nabla \mathbf{v} \\ &= 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - p(\nabla \cdot \mathbf{v}).\end{aligned}$$

- Hence we obtain

$$\begin{aligned}&\int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy \\ &- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy, \\ &- \int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0.\end{aligned}$$

Here we multiply the second equation by -1 in order to keep the matrix formulation symmetric later.

Weak formulation

- Since the solution on the domain boundary $\partial\Omega$ are given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega$.
- Hence

$$\int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy,$$

$$-\int_\Omega (\nabla \cdot \mathbf{u})q\ dxdy = 0.$$

Weak formulation

- Weak formulation in the vector format: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \, dxdy &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dxdy, \\ -\int_\Omega (\nabla \cdot \mathbf{u}) q \, dxdy &= 0, \end{aligned}$$

  for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $q \in L^2(\Omega)$.

- Let $a(\mathbf{u}, \mathbf{v}) = \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dxdy$, $b(\mathbf{u}, q) = -\int_\Omega (\nabla \cdot \mathbf{u}) q \, dxdy$, and $(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dxdy$.

- Weak formulation: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ s. t.

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}), \\ b(\mathbf{u}, q) &= 0, \end{aligned}$$

  for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $q \in L^2(\Omega)$.

Weak formulation

- In more details,

$$
\begin{aligned}
&\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= &\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix} \\
&: \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) & \frac{\partial v_2}{\partial y} \end{pmatrix} \\
= &\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\
&+\frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}.
\end{aligned}
$$

Weak formulation

- Hence

$$
\begin{aligned}
&\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= \ & \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
&+ \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}.
\end{aligned}
$$

- Then

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \ dxdy \\
= \ & \int_\Omega \nu \Big(2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + 2\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
&+ \frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x} \Big) \ dxdy.
\end{aligned}
$$

Weak formulation

- We also have

$$\int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy = \int_\Omega \left(p\frac{\partial v_1}{\partial x} + p\frac{\partial v_2}{\partial y} \right)\ dxdy,$$
$$\int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy = \int_\Omega (f_1 v_1 + f_2 v_2)\ dxdy,$$
$$\int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = \int_\Omega \left(\frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right)\ dxdy.$$

Weak formulation

- Weak formulation in the scalar format: find $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$, and $p \in L^2(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \nu\Big(2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + 2\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
&+\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}\Big)\ dxdy \\
&-\int_\Omega \left(p\frac{\partial v_1}{\partial x} + p\frac{\partial v_2}{\partial y}\right)\ dxdy \\
&= \int_\Omega (f_1 v_1 + f_2 v_2)\ dxdy. \\
&-\int_\Omega \left(\frac{\partial u_1}{\partial x}q + \frac{\partial u_2}{\partial y}q\right)\ dxdy = 0.
\end{aligned}$$

for any $v_1 \in H_0^1(\Omega)$, $v_2 \in H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

Galerkin formulation

- Consider a finite element space $U_h \subset H^1(\Omega)$ for the velocity and a finite element space $W_h \subset L^2(\Omega)$ for the pressure. Define U_{h0} to be the space which consists of the functions of U_h with value 0 on the Dirichlet boundary.

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Galerkin formulation

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Galerkin formulation

- In more details of the vector format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy,$$
$$-\int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Galerkin formulation

- In our numerical example, $U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$ are chosen to be the finite element spaces with the quadratic global basis functions $\{\phi_j\}_{j=1}^{N_b}$ and linear global basis functions $\{\psi_j\}_{j=1}^{N_{bp}}$, which are defined in Chapter 2. They are called Taylor-Hood finite elements.
- Why do we choose the pairs of finite elements in this way?
- Stability of mixed finite elements: inf-sup condition.

$$\inf_{0 \neq q_h \in W_h} \sup_{0 \neq \mathbf{u}_h \in U_h \times U_h} \frac{b(\mathbf{u}_h, q_h)}{\|\nabla \mathbf{u}_h\|_0 \, \|q_h\|_0} > \beta,$$

where $\beta > 0$ is a constant independent of mesh size h.
- See other course materials and references for the theory and more examples of stable mixed finite elements for Stokes equation.

Galerkin formulation

- In the scalar format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $u_{1h} \in U_h$, $u_{2h} \in U_h$, and $p_h \in W_h$ such that

$$\int_\Omega \nu \Big(2\frac{\partial u_{1h}}{\partial x}\frac{\partial v_{1h}}{\partial x} + 2\frac{\partial u_{2h}}{\partial y}\frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{1h}}{\partial y}$$
$$+ \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{2h}}{\partial x} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{2h}}{\partial x}\Big)\ dxdy$$
$$- \int_\Omega \left(p_h \frac{\partial v_{1h}}{\partial x} + p_h \frac{\partial v_{2h}}{\partial y} \right)\ dxdy$$
$$= \int_\Omega (f_1 v_{1h} + f_2 v_{2h})\ dxdy.$$
$$- \int_\Omega \left(\frac{\partial u_{1h}}{\partial x} q_h + \frac{\partial u_{2h}}{\partial y} q_h \right)\ dxdy = 0.$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

Outline

Discretization formulation

Recall the following definitions from Chapter 2:

- N: number of mesh elements.
- N_m: number of mesh nodes.
- E_n $(n = 1, \cdots, N)$: mesh elements.
- Z_k $(k = 1, \cdots, N_m)$: mesh nodes.
- N_l: number of local mesh nodes in a mesh element.
- P:information matrix consisting of the coordinates of all mesh nodes.
- T: information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.

Discretization formulation

- We only consider the nodal basis functions (Lagrange type) in this course.
- N_{lb}: number of local finite element nodes (=number of local finite element basis functions) in a mesh element.
- N_b: number of the finite element nodes ($=$ the number of unknowns $=$ the total number of the finite element basis functions).
- X_j $(j = 1, \cdots, N_b)$: finite element nodes.
- P_b: information matrix consisting of the coordinates of all finite element nodes.
- T_b: information matrix consisting of the global node indices of the finite element nodes of all the mesh elements.

Discretization formulation

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j, \quad p_h = \sum_{j=1}^{N_{bp}} p_j\psi_j$$

  for some coefficients $u_{1j},\ u_{2j} \quad (j = 1, \cdots, N_b)$, and $p_j \quad (j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for u_{1j}, $u_{2j} \quad (j = 1, \cdots, N_b)$, and $p_j \quad (j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h = (u_{1h}, u_{2h})^t$ and p_h.

Discretization formulation

- For the first equation in the Galerkin formulation, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Discretization formulation

- Set $\mathbf{v}_h = (\phi_i, 0)^t$, i.e., $v_{1h} = \phi_i$ and $v_{2h} = 0$ $(i = 1, \cdots, N_b)$, in the first equation of the Galerkin formulation. Then

$$
\begin{aligned}
&2\int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{1j}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{1j}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial y}\ dxdy \\
&+\int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{2j}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial y}\ dxdy - \int_\Omega \left(\sum_{j=1}^{N_{bp}} p_j\psi_j\right)\frac{\partial \phi_i}{\partial x}\ dxdy \\
&= \int_\Omega f_1\phi_i dxdy.
\end{aligned}
$$

Discretization formulation

- Set $\mathbf{v}_h = (0, \phi_i)^t$, i.e., $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$, in the first equation of the Galerkin formulation. Then

$$
\begin{aligned}
&2\int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{2j}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{1j}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial x}\ dxdy \\
&+ \int_\Omega \nu \left(\sum_{j=1}^{N_b} u_{2j}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial x}\ dxdy - \int_\Omega \left(\sum_{j=1}^{N_{bp}} p_j\psi_j\right)\frac{\partial \phi_i}{\partial y}\ dxdy \\
&= \int_\Omega f_2\phi_i dxdy.
\end{aligned}
$$

- Set $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$ in the second equation of the Galerkin formulation. Then

$$
-\int_\Omega \left(\sum_{j=1}^{N_b} u_{1j}\frac{\partial \phi_j}{\partial x}\right)\psi_i\ dxdy - \int_\Omega \left(\sum_{j=1}^{N_b} u_{2j}\frac{\partial \phi_j}{\partial y}\right)\psi_i\ dxdy = 0.
$$

Discretization formulation

- Simplify the above three sets of equations, we obtain

$$\sum_{j=1}^{N_b} u_{1j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right)$$

$$+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right) + \sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right) = \int_\Omega f_1\phi_i dxdy,$$

$$\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\ dxdy\right)$$

$$+\sum_{j=1}^{N_b} u_{2j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right)$$

$$+\sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right) = \int_\Omega f_2\phi_i dxdy,$$

$$\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i\ dxdy\right) + \sum_{j=1}^{N_{bp}} p_j * 0 = 0.$$

Matrix formulation

- Define

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\,dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\,dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\,dxdy\right]_{i,j=1}^{N_b}, \quad A_4 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\,dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\,dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\,dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}$$

$$A_7 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial x}\psi_i\,dxdy\right]_{i=1,j=1}^{N_{bp},N_b}, \quad A_8 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial y}\psi_i\,dxdy\right]_{i=1,j=1}^{N_{bp},N_b}$$

- Define a zero matrix $\mathbb{O}_1 = [0]_{i=1,j=1}^{N_{bp},N_{bp}}$ whose size is $N_{bp} \times N_{bp}$. Then

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_4 & 2A_2 + A_1 & A_6 \\ A_7 & A_8 & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Each matrix above can be obtained by Algorithm I-3 in Chapter 3.
- It is not difficult to verify (an independent study project topic) that
$$A_4 = A_3^t, \quad A_7 = A_5^t, \quad A_8 = A_6^t.$$
- Hence the matrix A is actually symmetric:

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Define the load vector

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy \right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Matrix formulation

- Define the unknown vector

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vec{X}_3 \end{pmatrix}$$

where

$$\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}, \quad \vec{X}_3 = [p_j]_{j=1}^{N_{bp}}.$$

- Then we obtain the linear algebraic system

$$A\vec{X} = \vec{b}.$$

Outline

Dirichlet boundary condition

- Basically, the Dirichlet boundary condition $\mathbf{u} = \mathbf{g}$ (i.e., $u_1 = g_1$ and $u_2 = g_2$) provides the solutions at all boundary finite element nodes.
- Since the coefficient u_{1j} and u_{2j} in the finite element solutions $u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j$ and $u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$ are actually the numerical solutions at the finite element node X_j $(j = 1, \cdots, N_b)$ when nodal basis functions are used, we actually know those u_{1j} and u_{2j} which are corresponding to the boundary finite element nodes.
- Recall that boundarynodes(2,:) store the global node indices of all boundary finite element nodes.
- If $m \in \; boundarynodes(2,:)$, then the m^{th} equation is called a boundary node equation for u_1 and the $(N_b + m)^{th}$ equation is called a boundary node equation for u_2.
- Set nbn to be the number of boundary nodes;

Dirichlet boundary condition

- One way to impose the Dirichlet boundary condition is to replace the boundary node equations in the linear system by the following equations

$$
\begin{aligned}
u_{1m} &= g_1(X_m) \\
u_{2m} &= g_2(X_m).
\end{aligned}
$$

  for all $m \in \ boundarynodes(2,:)$. This is similar to $u_m = g(X_m)$ in Chapter 3. We already discussed about this in Chapter 5.
- Since the Dirichlet boundary condition only involves u_1 and u_2, not p, only the first two rows of the 3×3 block matrix A need to be modified for the Dirichlet boundary condition. This is similar to how we handle Dirichlet boundary condition in Chapter 5. Hence we can still use Algorithm III-3 in Chapter 5.

Dirichlet boundary condition

Recall Algorithm III-3 from Chapter 5:

- Deal with the Dirichlet boundary conditions:

$FOR\ k = 1, \cdots, nbn$:

If $boundarynodes(1, k)$ shows Dirichlet condition, then

$i = boundarynodes(2, k)$;

$A(i, :) = 0$;

$A(i, i) = 1$;

$b(i) = g_1(P_b(:, i))$;

$A(N_b + i, :) = 0$;

$A(N_b + i, N_b + i) = 1$;

$b(N_b + i) = g_2(P_b(:, i))$;

$ENDIF$

END

Additional treatment for the solution uniqueness

Recall:

- Since p appears in the equation without any derivative, then, if $(\mathbf{u}, p)$ is a solution, then $(\mathbf{u}, p + c)$ is also a solution where c is a constant. Hence we need to impose additional condition for p. Here are three regular choices:
- (1) Fix p at one point in the domain Ω.
- (2) Apply a stress or Robin boundary condition (at least in the normal direction) on part of the boundary $\partial\Omega$.
- (3) Apply $\int_\Omega p dxdy = 0$.

Outline

Universal framework of the finite element method

Recall from Chapter 3:

- Generate the mesh information: matrices P and T;
- Assemble the matrices and vectors: local assembly based on P and T only;
- Deal with the boundary conditions: boundary information matrix and local assembly;
- Solve linear systems: numerical linear algebra.

Algorithm

- Generate the mesh information matrices P and T.
- Assemble the stiffness matrix A by using Algorithm I. (We will choose Algorithm I-3 in class)
- Assemble the load vector $\vec{b}$ by using Algorithm II. (We will choose Algorithm II-3 in class)
- Deal with the Dirichlet boundary condition by using Algorithm III-3.
- Fix the pressure at one point in the domain Ω.
- Solve $A\vec{X} = \vec{b}$ for $\vec{X}$ by using a direct or iterative method.

Algorithm

Recall Algorithm I-3 from Chapter 3:

- Initialize the matrix: $A = sparse(N_b^{test}, N_b^{trial})$;
- Compute the integrals and assemble them into A:

$FOR\ n = 1, \cdots, N$

$\quad FOR\ \alpha = 1, \cdots, N_{lb}^{trial}$

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}^{test}$

$\quad\quad\quad$ Compute $r = \int_{E_n} c \frac{\partial^{r+s}\varphi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad\quad$ Add r to $A(T_b^{test}(\beta, n), T_b^{trial}(\alpha, n))$.

$\quad\quad END$

$\quad END$

END

Algorithm

- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 1,\ q = 0,\ c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_1.
- Call Algorithm I-3 with $r = 0,\ s = 1,\ p = 0,\ q = 1,\ c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_2.
- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 0,\ q = 1,\ c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_3.
- Call Algorithm I-3 with $r = 0,\ s = 0,\ p = 1,\ q = 0,\ c = -1$, basis type of p for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_5.
- Call Algorithm I-3 with $r = 0,\ s = 0,\ p = 0,\ q = 1,\ c = -1$, basis type of p for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_6.
- Generate a zero matrix $\mathbb{O}$ whose size is $N_{bp} \times N_{bp}$.
- Then the stiffness matrix
  $A = [2A_1 + A_2 \quad A_3 \quad A_5; A_3^t \quad 2A_2 + A_1 \quad A6; A_5^t \quad A_6^t \quad \mathbb{O}]$.

Algorithm

Recall Algorithm II-3 from Chapter 3:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Algorithm

- Call Algorithm II-3 with $p = q = 0$ and $f = f_1$ to obtain b_1.
- Call Algorithm II-3 with $p = q = 0$ and $f = f_2$ to obtain b_2.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$.
- Then the load vector $\vec{b} = [b_1; b_2; \vec{0}]$.

Algorithm

Recall Algorithm III-3 from Chapter 5:

- Deal with the Dirichlet boundary conditions:

$FOR\ k = 1, \cdots, nbn$:

If $boundarynodes(1,k)$ shows Dirichlet condition, then

$i = boundarynodes(2,k)$;

$A(i,:) = 0$;

$A(i,i) = 1$;

$b(i) = g_1(P_b(:,i))$;

$A(N_b + i,:) = 0$;

$A(N_b + i, N_b + i) = 1$;

$b(N_b + i) = g_2(P_b(:,i))$;

$ENDIF$

END

Measurements for errors

- L^∞ norm error:

$$\|\mathbf{u}-\mathbf{u}_h\|_\infty = \max\left(\|u_1-u_{1h}\|_\infty, \|u_2-u_{2h}\|_\infty\right),$$
$$\|u_1-u_{1h}\|_\infty = \sup_\Omega |u_1-u_{1h}|,$$
$$\|u_2-u_{2h}\|_\infty = \sup_\Omega |u_2-u_{2h}|,$$
$$\|p-p_h\|_\infty = \sup_\Omega |p-p_h|.$$

Measurements for errors

- L^2 norm error:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 = \sqrt{\|u_1-u_{1h}\|_0^2 + \|u_2-u_{2h}\|_0^2},$$

$$\|u_1-u_{1h}\|_0 = \sqrt{\int_\Omega (u_1-u_{1h})^2 dxdy},$$

$$\|u_2-u_{2h}\|_0 = \sqrt{\int_\Omega (u_2-u_{2h})^2 dxdy},$$

$$\|p-p_h\|_0 = \sqrt{\int_\Omega (p-p_h)^2 dxdy}.$$

Measurements for errors

- H^1 semi-norm error:

$$|\mathbf{u}-\mathbf{u}_h|_1 = \sqrt{|u_1-u_{1h}|_1^2 + |u_2-u_{2h}|_1^2},$$

$$|u_1-u_{1h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_1-u_{1h})}{\partial x}\right)^2 + \left(\frac{\partial(u_1-u_{1h})}{\partial y}\right)^2 dxdy},$$

$$|u_2-u_{2h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_2-u_{2h})}{\partial x}\right)^2 + \left(\frac{\partial(u_2-u_{2h})}{\partial y}\right)^2 dxdy},$$

$$|p-p_h|_1 = \sqrt{\int_\Omega \left(\frac{\partial(p-p_h)}{\partial x}\right)^2 + \left(\frac{\partial(p-p_h)}{\partial y}\right)^2 dxdy}.$$

- Basic idea: call Algorithm IV and Algorithm V in Chapter 3 for each of u_1, u_2, and p; then plug the results into the above formulas for the errors of $\mathbf{u}$ and p.

Numerical example

- Example 1: Use the finite element method to solve the following equation on the domain $\Omega = [0,1] \times [-0.25, 0]$:

$$
\begin{aligned}
-\nabla \cdot \mathbb{T}(\mathbf{u}, p) &= \mathbf{f} \quad \text{on } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \qquad in \quad \Omega, \\
u_1 &= e^{-y} \quad \text{on } x = 0, \\
u_1 &= y^2 + e^{-y} \quad \text{on } x = 1, \\
u_1 &= \frac{1}{16} x^2 + e^{0.25} \quad \text{on } y = -0.25, \\
u_1 &= 1 \quad \text{on } y = 0, \\
u_2 &= 2 \quad \text{on } x = 0, \\
u_2 &= -\frac{2}{3} y^3 + 2 \quad \text{on } x = 1, \\
u_2 &= \frac{1}{96} x + 2 - \pi \sin(\pi x) \quad \text{on } y = -0.25, \\
u_2 &= 2 - \pi \sin(\pi x) \quad \text{on } y = 0.
\end{aligned}
$$

Numerical example

- Here

$$\begin{aligned} f_1 &= -2\nu x^2 - 2\nu y^2 - \nu e^{-y} + \pi^2 \cos(\pi x)\cos(2\pi y), \\ f_2 &= 4\nu xy - \nu\pi^3 \sin(\pi x) + 2\pi(2 - \pi\sin(\pi x))\sin(2\pi y). \end{aligned}$$

- The analytic solution of this problem is

$$u_1 = x^2 y^2 + e^{-y}, \quad u_2 = -\frac{2}{3}xy^3 + 2 - \pi\sin(\pi x),$$
$$p = -(2 - \pi\sin(\pi x))\cos(2\pi y),$$

which can be used to compute the errors between the numerical solution and the analytic solution. We can also verify f_1 and f_2 above by plugging the analytic solutions into the Stokes equation.

Numerical example

- Let's code for the Taylor-Hood finite elements for the 2D Stokes equation together!
- Taylor-Hood finite elements: linear finite elements for the pressure and quadratic finite elements for the velocity.
- Open your Matlab!

Numerical example

h	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\mathbf{u}-\mathbf{u}_h\|_1$
$1/8$	1.6765×10^{-3}	3.5687×10^{-4}	2.0424×10^{-2}
$1/16$	2.0256×10^{-4}	4.4059×10^{-5}	5.0674×10^{-3}
$1/32$	2.5182×10^{-5}	5.4832×10^{-6}	1.2623×10^{-3}
$1/64$	3.1057×10^{-6}	6.8444×10^{-7}	3.1522×10^{-4}

Table: The numerical errors for quadratic finite elements of the velocity.

- Any Observation?
- Third order convergence $O(h^3)$ in L^2/L^∞ norm and second order convergence $O(h^2)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise quadratic functions.

Numerical example

h	$\|p - p_h\|_\infty$	$\|p - p_h\|_0$	$\|p - p_h\|_1$
$1/8$	1.3124×10^{-1}	2.1810×10^{-2}	1.2651×10^{0}
$1/16$	4.5401×10^{-2}	8.4643×10^{-3}	6.3072×10^{-1}
$1/32$	1.2473×10^{-2}	2.4475×10^{-3}	3.1369×10^{-1}
$1/64$	3.2434×10^{-3}	6.5205×10^{-4}	1.5658×10^{-1}

Table: The numerical errors for linear finite elements of the pressure.

- Any Observation?
- Second order convergence $O(h^2)$ in L^2/L^∞ norm and first order convergence $O(h)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise linear functions.

Outline

Stress boundary condition

- Consider

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\ \mathbb{T}(\mathbf{u}, p)\mathbf{n} = \mathbf{p} \qquad on \quad \partial\Omega. \end{cases}$$

where $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and

$$\mathbf{p}(x, y) = (p_1, p_2)^t, \ \mathbf{f}(x, y) = (f_1, f_2)^t.$$

- Recall

$$\int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \ dxdy$$
$$- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u})q \ dxdy = 0.$$

Stress boundary condition

- Hence

$$\int_{\Omega} 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_{\Omega} p(\nabla \cdot \mathbf{v})\ dxdy - \int_{\partial\Omega} \mathbf{p} \cdot \mathbf{v}\ ds$$
$$= \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\ dxdy,$$
$$- \int_{\Omega} (\nabla \cdot \mathbf{u}) q\ dxdy = 0.$$

- Is there anything wrong? The solution is not unique!
- Recall that

$$\mathbb{D}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- If $\mathbf{u} = (u_1, u_2)^t$ is a solution, then $\mathbf{u} + \mathbf{c}$ is also a solution where $\mathbf{c}$ is a constant vector.

Stress boundary condition

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbb{T}(\mathbf{u}, p)\mathbf{n} = \mathbf{p} \quad \text{on } \Gamma_S \subset \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D = \partial\Omega / \Gamma_S.
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy \\
&- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \;=\; \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0.
\end{aligned}
$$

Stress boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \quad & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\partial\Omega/\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \quad & \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \; ds.
\end{aligned}
$$

Stress boundary condition

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}\ ds,$$
$$-\int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here

$$\int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}\ ds = \int_{\Gamma_S} p_1 v_1\ ds + \int_{\Gamma_S} p_2 v_2\ ds,$$
$$H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Stress boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h\ ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Stress boundary condition

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j, \quad p_h = \sum_{j=1}^{N_{bp}} p_j\psi_j$$

for some coefficients $u_{1j},\ u_{2j} \quad (j = 1, \cdots, N_b)$, and $p_j \quad (j = 1, \cdots, N_{bp})$.

- For the first equation in the Galerkin formulation, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Stress boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$
\sum_{j=1}^{N_b} u_{1j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right)
+ \sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right) = \int_\Omega f_1\phi_i dxdy + \int_{\Gamma_S} p_1\phi_i\ ds
$$

$$
\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\ dxdy\right)
+ \sum_{j=1}^{N_b} u_{2j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right)
+ \sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right) = \int_\Omega f_2\phi_i dxdy + \int_{\Gamma_S} p_2\phi_i\ ds,
$$

$$
\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i\ dxdy\right) + \sum_{j=1}^{N_{bp}} p_j * 0 = 0.
$$

Stress boundary condition

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dxdy \right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dxdy \right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial y} \, dxdy \right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x} \, dxdy \right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y} \, dxdy \right]_{i=1,j=1}^{N_b,N_{bp}}$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

Stress boundary condition

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Recall the unknown vector

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vec{X}_3 \end{pmatrix}$$

where $\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \ \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}, \ \vec{X}_3 = [p_j]_{j=1}^{N_{bp}}$.

Stress boundary condition

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_1 \phi_i \ ds\right]_{i=1}^{N_b}, \quad \vec{v}_2 = \left[\int_{\Gamma_S} p_2 \phi_i \ ds\right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{v}$.
- Then we obtain the linear algebraic system

$$A\vec{X} = \widetilde{\vec{b}}.$$

- Similar to Chapter 5, we essentially only need repeat the code of Neumman condition in Chapter 3 for $\vec{v}_1$ and $\vec{v}_2$.

Stress boundary condition

Based on Algorithm VI-2 in Chapter 5, we obtain Algorithm VI-4:

- Initialize the vector: $v = sparse(2N_b + N_{bp}, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ k = 1, \cdots, nbe$:

$\quad IF\ boundaryedges(1,k)$ shows stress boundary, $THEN$

$\quad\quad n_k = boundaryedges(2,k)$;

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad\quad$ Compute $r = \int_{e_k} p_1 \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b}\, ds$;

$\quad\quad\quad v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r$;

$\quad\quad\quad$ Compute $r = \int_{e_k} p_2 \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b}\, ds$;

$\quad\quad\quad v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r$;

$\quad\quad END$

$\quad ENDIF$

END

Robin boundary conditions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{u} = \mathbf{q} \ \text{ on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega / \Gamma_R.
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \ dxdy \\
&- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \ = \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q \ dxdy = 0.
\end{aligned}
$$

Robin boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_R$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\partial\Omega/\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \ ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \ ds.
\end{aligned}
$$

Robin boundary condition

- The weak formulation is find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy + \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here

$$\int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} q_1 v_1 \; ds + \int_{\Gamma_R} q_2 v_2 \; ds,$$
$$\int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} r u_1 v_1 \; ds + \int_{\Gamma_R} r u_2 v_2 \; ds,$$
$$H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Robin boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$

$$+ \int_{\Gamma_R} r\mathbf{u}_h \cdot \mathbf{v}_h\ ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Robin boundary condition

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy - \int_\Omega p_h (\nabla \cdot \mathbf{v}_h) \; dxdy$$

$$+ \int_{\Gamma_R} r\mathbf{u}_h \cdot \mathbf{v}_h \; ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h \; ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Robin boundary condition

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then
$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j, \quad p_h = \sum_{j=1}^{N_{bp}} p_j\psi_j$$
for some coefficients u_{1j}, u_{2j} $(j = 1, \cdots, N_b)$, and p_j $(j = 1, \cdots, N_{bp})$.
- For the first equation in the Galerkin formulation, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.
- For the second equation in the Galerkin formulation, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Robin boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\,dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\,dxdy + \int_{\Gamma_R} r\phi_j\phi_i\,ds\right)\\
&+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\,dxdy\right) + \sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\,dxdy\right)\\
&= \int_\Omega f_1\phi_i dxdy + \int_{\Gamma_S} q_1\phi_i\,ds,
\end{aligned}
$$

Robin boundary condition

- and

$$\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\ dxdy\right)$$

$$+\sum_{j=1}^{N_b} u_{2j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}dxdy + \int_{\Gamma_R} r\phi_j\phi_i ds\right)$$

$$+\sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right)$$

$$= \int_\Omega f_2\phi_i dxdy + \int_{\Gamma_S} q_2\phi_i\ ds,$$

$$\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i\ dxdy\right)$$

$$+\sum_{j=1}^{N_{bp}} p_j * 0 = 0.$$

Robin boundary condition

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

Robin boundary condition

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Recall the unknown vector

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vec{X}_3 \end{pmatrix}$$

where $\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}, \quad \vec{X}_3 = [p_j]_{j=1}^{N_{bp}}.$

Robin boundary condition

- Define the additional vector from the Robin boundary condition:

$$\vec{w} = \begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{w}_1 = \left[\int_{\Gamma_S} q_1 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{w}_2 = \left[\int_{\Gamma_S} q_2 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{w}$.
- Since each of $\vec{w}_1$ and $\vec{w}_2$ is similar to the $\vec{w}$ for the Robin condition in Chapter 3, we essentially only need repeat the code of $\vec{w}$ in Chapter 3 for $\vec{w}_1$ and $\vec{w}_2$.

Robin boundary condition

- Define the additional matrix from the Robin boundary condition

$$R = [r_{ij}]_{i,j=1}^{N_b} = \left[\int_{\Gamma_R} r \phi_j \phi_i \, ds \right]_{i,j=1}^{N_b}.$$

- Since R is the same as the R in Chapter 3, the code for R is the same. But R needs to be added to the matrix A twice as showed above to obtain $\widetilde{A}$.

Robin boundary condition

- Define the new matrix:

$$\widetilde{A} = \begin{pmatrix} 2A_1 + A_2 + R & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 + R & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

- Then we obtain the linear algebraic system

$$\widetilde{A}\vec{X} = \widetilde{\vec{b}}.$$

- Pesudo code? (Part of a project for you)

Dirichlet/stress/Robin mixed boundary condition

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbb{T}(\mathbf{u}, p)\mathbf{n} = \mathbf{p} \ \text{ on } \Gamma_S \subset \partial\Omega, \\
&\mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{u} = \mathbf{q} \ \text{ on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R).
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \ dxdy \\
&- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q \ dxdy = 0.
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/(\Gamma_S \cup \Gamma_R)$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \; ds + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds.
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \, dxdy + \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \, ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \, ds + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \, ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \, dxdy = 0,$$

  for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here $H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$.

- Code? Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.

Stress boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_n, \ \ \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_\tau \ \ \text{on } \Gamma_S \subset \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \ \ \text{on } \Gamma_D = \partial\Omega / \Gamma_S.
\end{aligned}
$$

where $\mathbf{n} = (n_1, \, n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Stress boundary condition in normal/tangential directions

- Recall

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy \\
&- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0.
\end{aligned}
$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.

Stress boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\partial\Omega/\Gamma_S} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_S} \left[(\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n})\mathbf{n} + (\tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n})\tau\right] \cdot \left[(\mathbf{n}^t\mathbf{v})\mathbf{n} + (\tau^t\mathbf{v})\tau\right] \; ds \\
= & \int_{\Gamma_S} (\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n})(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_S} (\tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n})(\tau^t\mathbf{v}) \; ds \\
= & \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \; ds.
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy + \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v})\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.

Stress boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}_h)\ ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}_h)\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Stress boundary condition in normal/tangential directions

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}_h)\ ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}_h)\ ds,$$

$$-\int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition in normal/tangential directions

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j, \quad p_h = \sum_{j=1}^{N_{bp}} p_j\psi_j$$

  for some coefficients u_{1j}, $u_{2j}\ \ (j = 1, \cdots, N_b)$, and $p_j\ \ (j = 1, \cdots, N_{bp})$.

- For the first equation in the Galerkin formulation, we choose $\mathbf{v}_h = (\phi_i, 0)^t\ (i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t\ (i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i\ (i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i\ (i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation, we choose $q_h = \psi_i\ (i = 1, \cdots, N_{bp})$.

Stress boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$\sum_{j=1}^{N_b} u_{1j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right)$$
$$+\sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right) = \int_\Omega f_1\phi_i dxdy + \int_{\Gamma_S} p_n\phi_i n_1\ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_1\ ds$$

$$\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\ dxdy\right)$$
$$+\sum_{j=1}^{N_b} u_{2j}\left(2\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right)$$
$$+\sum_{j=1}^{N_{bp}} p_j\left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right) = \int_\Omega f_2\phi_i dxdy + \int_{\Gamma_S} p_n\phi_i n_2\ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_2\ ds$$

$$\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i\ dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i\ dxdy\right) + \sum_{j=1}^{N_{bp}} p_j * 0 = 0.$$

Stress boundary condition in normal/tangential directions

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

Stress boundary condition in normal/tangential directions

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}, \ \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Recall the unknown vector

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vec{X}_3 \end{pmatrix}$$

where $\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}, \quad \vec{X}_3 = [p_j]_{j=1}^{N_{bp}}.$

Stress boundary condition in normal/tangential directions

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 + \vec{v}_2 \\ \vec{v}_3 + \vec{v}_4 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_n \phi_i n_1 \; ds \right]_{i=1}^{N_b}, \; \vec{v}_2 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_1 \; ds \right]_{i=1}^{N_b},$$
$$\vec{v}_3 = \left[\int_{\Gamma_S} p_n \phi_i n_2 \; ds \right]_{i=1}^{N_b}, \; \vec{v}_4 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_2 \; ds \right]_{i=1}^{N_b}$$
$$\vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{v}$.

Stress boundary condition in normal/tangential directions

- Then we obtain the linear algebraic system

$$A\vec{X} = \widetilde{\vec{b}}.$$

- Since each of $\vec{v}_i$ $(i = 1, 2, 3, 4)$ is similar to the $\vec{v}$ for the Neumann condition in Chapter 3, we can borrow the code of Neumman condition in Chapter 3 for $\vec{v}_i$ $(i = 1, 2, 3, 4)$.

- The major difference between $\vec{v}_i$ $(i = 1, 2, 3, 4)$ here and the $\vec{v}$ for the Neumann condition in Chapter 3 is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix $boundaryedges$.

Stress boundary condition in normal/tangential directions

Based on Algorithm VI-3 in Chapter 5, we obtain Algorithm VI-5:

- Initialize the vector: $v = sparse(2N_b + N_{bp}, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ k = 1, \cdots, nbe$:

$IF\ boundaryedges(1,k)$ shows stress boundary in normal/tangential directions, $THEN$

$n_k = boundaryedges(2,k)$;

$FOR\ \beta = 1, \cdots, N_{lb}$:

Compute

$$r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} n_1\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} \tau_1\ ds;$$

$v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r$;

Compute

$$r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} n_2\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} \tau_2\ ds;$$

$v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r$;

END

$ENDIF$

END

Robin boundary conditions in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n, \;\; \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \;\; \text{on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \;\; \text{on } \Gamma_D = \partial\Omega / \Gamma_R.
\end{aligned}
$$

where $\mathbf{n} = (n_1,\, n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Robin boundary conditions in normal/tangential directions

- Recall

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy \\
&- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0.
\end{aligned}
$$

- Since the solution on $\Gamma_D = \partial\Omega / \Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega / \Gamma_R$.

Robin boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_R} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_R} \left[(\mathbf{n}^t \mathbb{T}(\mathbf{u},p)\mathbf{n})\mathbf{n} + (\tau^t \mathbb{T}(\mathbf{u},p)\mathbf{n})\tau\right] \cdot \left[(\mathbf{n}^t\mathbf{v})\mathbf{n} + (\tau^t\mathbf{v})\tau\right] \; ds \\
= & \int_{\Gamma_S} (\mathbf{n}^t \mathbb{T}(\mathbf{u},p)\mathbf{n})(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_S} (\tau^t \mathbb{T}(\mathbf{u},p)\mathbf{n})(\tau^t\mathbf{v}) \; ds \\
= & \left[\int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \; ds\right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \; ds\right],
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy \\
&+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v})\ ds \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v})\ ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u})q\ dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.

Robin boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$
$$+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u}_h)(\mathbf{n}^t\mathbf{v}_h)\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u}_h)(\tau^t\mathbf{v}_h)\ ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}_h)\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}_h)\ ds,$$
$$-\int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Robin boundary condition in normal/tangential directions

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h)\ dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h)\ dxdy$$

$$+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u}_h)(\mathbf{n}^t\mathbf{v}_h)\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u}_h)(\tau^t\mathbf{v}_h)\ ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\ dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}_h)\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}_h)\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h\ dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Robin boundary condition in normal/tangential directions

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then
$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j, \quad p_h = \sum_{j=1}^{N_{bp}} p_j\psi_j$$
for some coefficients $u_{1j},\ u_{2j} \quad (j = 1, \cdots, N_b)$, and $p_j \quad (j = 1, \cdots, N_{bp})$.

- For the first equation in the Galerkin formulation, we choose $\mathbf{v}_h = (\phi_i, 0)^t \ (i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t \ (i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i \ (i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i \ (i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation, we choose $q_h = \psi_i \ (i = 1, \cdots, N_{bp})$.

Robin boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$\sum_{j=1}^{N_b} u_{1j}\Big(2\int_{\Omega} \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_{\Omega} \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy$$

$$+\int_{\Gamma_R} (r n_1 \phi_j)(\phi_i n_1)\ ds + \int_{\Gamma_R} (r \tau_1 \phi_j)(\phi_i \tau_1)\ ds\Big)$$

$$+\sum_{j=1}^{N_b} u_{2j}\Big(\int_{\Omega} \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_{\Gamma_R} (r n_2 \phi_j)(\phi_i n_1)\ ds$$

$$+\int_{\Gamma_R} (r \tau_2 \phi_j)(\phi_i \tau_1)\ ds\Big) + \sum_{j=1}^{N_{bp}} p_j \left(-\int_{\Omega} \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right)$$

$$= \int_{\Omega} f_1 \phi_i dxdy + \int_{\Gamma_R} q_n \phi_i n_1\ ds + \int_{\Gamma_R} q_\tau \phi_i \tau_1\ ds,$$

Robin boundary condition in normal/tangential directions

- and

$$\sum_{j=1}^{N_b} u_{1j}\Big(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x}\ dxdy + \int_{\Gamma_R} (r n_1 \phi_j)(\phi_i n_2)\ ds$$

$$+ \int_{\Gamma_R} (r \tau_1 \phi_j)(\phi_i \tau_2)\ ds\Big) + \sum_{j=1}^{N_b} u_{2j}\Big(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y}\ dxdy$$

$$+ \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x}\ dxdy + \int_{\Gamma_R} (r n_2 \phi_j)(\phi_i n_2)\ ds$$

$$+ \int_{\Gamma_R} (r \tau_2 \phi_j)(\phi_i \tau_2)\ ds\Big) + \sum_{j=1}^{N_{bp}} p_j \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\ dxdy \right)$$

$$= \int_\Omega f_2 \phi_i dxdy + \int_{\Gamma_R} q_n \phi_i n_2\ ds + \int_{\Gamma_R} q_\tau \phi_i \tau_2\ ds,$$

Robin boundary condition in normal/tangential directions

- and

$$\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i \ dxdy\right)+\sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i \ dxdy\right)$$

$$+\sum_{j=1}^{N_{bp}} p_j * 0 = 0.$$

Robin boundary condition in normal/tangential directions

- Matrix formulation? Pesudo code? (Part of a project for you)
- Similar to the previous ones for Robin condition, we need to add eight sub-matrices and four sub-vectors into the block linear system.
- The major difference is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix $boundaryedges$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \qquad in \quad \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \qquad in \quad \Omega, \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_n, \;\; \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_\tau \;\; \text{on } \Gamma_S \subset \partial\Omega, \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n, \;\; \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \;\; \text{on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \;\; \text{on } \Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R).
\end{aligned}
$$

where $\mathbf{n} = (n_1, \, n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Recall

$$\int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy$$
$$-\int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$
$$-\int_\Omega(\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S\cup\Gamma_R)$ is given by $\mathbf{u}=\mathbf{g}$, then we can choose the test function $\mathbf{v}(x,y)$ such that $\mathbf{v}=0$ on $\partial\Omega/(\Gamma_S\cup\Gamma_R)$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Combining the above derivation for stress and Robin boundary conditions in normal/tangential directions, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \left[\int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& + \left[\int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \ ds \right],
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Weak formulation: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ s.t.

$$\int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy$$

$$+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v})\ ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v})\ ds$$

$$+ \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v})\ ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u})q\ dxdy = 0,$$

  for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.
- Code? Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.