Laurent expansion, Z transform

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The purpose of this note is to review basic concepts in complex function (Cauchy intergal theorem, laurent expansion), and introduce Z transform, a useful tool in signal processing. Since Z transform is simply laurent expansion, by studying concept of laurent expansion, we can have a better understanding of Z transform.

1 Analytic function and Laurent expansion

1.1 Review of analytic function

Def: We say a complex function $f(z) : Z \to Z$ is **analytic** in region D, if $f(z)$ is differential in D.

For example, $f(z) = z$ is analytic in the whole complex plane except $z = \infty$, and $f(z) = 1/z$ is analytic in the whole complex plane except $z = 0$.

Theorem (Cauchy intergal theorem): If $f(z)$ is an analytic function in D, then

$$\oint_{\partial D} f(z)dz = 0, \tag{1}$$

where ∂D is the boundary of D in a counter-clock direction. We are not going to prove this, but a simple argument is, if there is no singularity in D, then because we integrate $f(z)$ with the same starting point and the ending point, the integral must be 0.

Theorem (Cauchy intergal formula): If $f(z)$ is analytic in D, then for all points $z_0 \in D$, we can express $f(z_0)$ as

$$f(z_0) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(z)}{z - z_0}dz. \tag{2}$$

Proof: We can see that $f(z)/(z-z_0)$ is analytic in $D/\{z_0\}$. According to Cauchy intergral theorem, the integral path can be done at the boundary of $z = z_0$:

$$\oint_{\partial D} \frac{f(z)}{z - z_0}dz = \lim_{\delta\to 0}\oint_{|z|=z_0+\delta} \frac{f(z)}{z - z_0}dz = f(z_0)\lim_{\delta\to 0}\oint_{|z|=z_0+\delta} \frac{1}{z - z_0}dz = 2\pi i f(z_0), \tag{3}$$

which proves the theorem.In the last step we have $\oint_{|z|=z_0+\delta} \frac{1}{z-z_0}dz = 2\pi i$ through simple calculation.

With Cauchy intergral formula, we can finally turn to laurent expansion, a generalized and more powerful version of taylor expansion. When we studied taylor expansion, which expands $f(x)$ near x_0, we know that the ROC (range of convergence) is limited to the nearest singularity around x_0. But laurent expansion is so powerful that we can expand $f(z)$ in form of z_0, even given that $f(z)$ is not well-defined at z_0.

Theorem (Laurent expansion): If $f(z)$ is analytic in a ring area $D = \{z : r < |z - z_0| < R\}$, then it can be expanded as

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n = \sum_{n=-\infty}^{+\infty}\left(\frac{1}{2\pi i}\oint_L \frac{f(z')}{(z' - z)^{n+1}}dz'\right)(z - z_0)^n, \tag{4}$$

where L is a closed curve within D.

Proof : From Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_{|z'-z_0|=R} \frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\oint_{|z'-z_0|=r} \frac{f(z')}{z'-z}dz'. \tag{5}$$

For the first term, we try to express it in terms of z_0:

$$\oint_{|z'-z_0|=R} \frac{f(z')}{z'-z}dz' = \oint_{|z'-z_0|=R} \frac{f(z')}{z'-z_0-(z-z_0)}dz' = \oint_{|z'-z_0|=R} \frac{1}{z'-z_0}\frac{f(z')}{1-\frac{z-z_0}{z'-z_0}}dz'. \tag{6}$$

Note that since $|z'-z_0| = R$ and $r < |z-z_0| < R$, so we can use taylor expansion to expand $\frac{1}{1-(z-z_0)/(z'-z_0)}$ near $(z-z_0)/(z'-z_0) = 0$, obtaining

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n. \tag{7}$$

Therefore the first integral term is now in form of the polynomials of $z-z_0$, which is

$$\oint_{|z'-z_0|=R} \frac{f(z')}{z'-z}dz' = \sum_{n=0}^{\infty}\left(\oint_{|z'-z_0|=R} \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n = \sum_{n=0}^{\infty}\left(\oint_{L} \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n. \tag{8}$$

For sake of convenience, in the last step we choose an intergal path L within D instead of $|z'-z_0| = R$, and they have the same integral value because $\frac{f(z')}{(z'-z_0)^{n+1}}$ is analytic in D. We then do the same to the second integral term:

$$\oint_{|z'-z_0|=r} \frac{f(z')}{z'-z}dz' = \oint_{|z'-z_0|=r} \frac{f(z')}{z'-z_0-(z-z_0)}dz' = -\oint_{|z'-z_0|=r} \frac{1}{z-z_0}\frac{f(z')}{1-\frac{z'-z_0}{z-z_0}}dz'. \tag{9}$$

Since $|z'-z_0| = r$, we expand $\frac{1}{1-(z'-z_0)/(z-z_0)}$ near $(z'-z_0)/(z-z_0) = 0$, then we have

$$\frac{1}{1-\frac{z'-z_0}{z-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n = \sum_{n=-\infty}^{-1}\left(\frac{z'-z_0}{z-z_0}\right)^{-n-1}. \tag{10}$$

Thus

$$\oint_{|z'-z_0|=r} \frac{f(z')}{z'-z}dz' = -\sum_{n=-\infty}^{-1}\left(\oint_{|z'-z_0|=r} \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n = -\sum_{n=-\infty}^{-1}\left(\oint_{L} \frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n. \tag{11}$$

We then finish the proof by adding these two terms up.

1.2 ROC of laurent expansion

The laurent expansion of region D is unique, but it can be different for different regions (ROCs). For instance, $f(z) = \frac{1}{1-z}$ is analytic in $Z/\{1\}$, but the singularity $z = 1$ (we call it poles) implies that we can not have a uniform expansion on $Z/\{1\}$, so we have to do laurent expansion on $|z| < 1$ and $|z| > 1$. Let $z_0 = 0$ be our expansion point. We have

$$\frac{1}{1-z} = z + z^2 + z^3 + ... \tag{12}$$

when $|z| < 1$, and

$$\frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) \tag{13}$$

when $|z| > 1$.

2 Z transform

We are going to first have a review of discrete time system, and show the relation between Z transform and laurent transform.

A LTI (Linear time invariant) system is defined by its impulse response:

$$S(\delta[n]) = h[n], \quad S(x[n]) = x[n] * h[n]. \tag{14}$$

We find that $x[n] = z^n$ are the eigenstates of the system, whose eigenvalues are $H(z)$

$$S(z^n) = z^n * h[n] = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \equiv H(z) z^n. \tag{15}$$

We find that the system function $H(z)$ is a complex function and impulse response $h[n]$ are simply the coefficients of laurent expansion of $H(z)$ (almostly, there is a sign different). Thus we get such an inverse point of view, that we first have $H(z)$, and then we expand $H(z)$ in laurent series to study the signal $h[n]$ in time domain. Then some properties of $H(z)$ are pretty clear.

Property: Two different signals can have the same expression of $H(z)$.

It is obvious, because $H(z)$ with splited ROC have different expansion coefficients. generally, to specify a system, we need both $H(z)$ and ROC. $H(z) = 1/(1-z)$ is a typical example. And we should see that if $h[n]$ has finite length, then $H(z)$ does not have singularities in $Z/\{0, \infty\}$, so $H(z)$ is unique for this $h[n]$.

Property: If a signal $h[n]$ is right-side signal, then its ROC expands to $z = \infty$; If a signal $h[n]$ is left-side signal, then its ROC contracts to $z = 0$.

A right-side signal means $h[n] = 0$ when $n < n_0$. In the point of view laurent expansion, it means $H(z)$ has finite polynomials with positive exponent. Therefore $H(z)$ must converge when $z \to \infty$ (except $z = \infty$). In a word, a larger z make it easy for $H(z)$ to converge. The similar analysis can be applied to left-side signal. A special case for this is a causal signal. Since it does only hava terms like z^{-k}, $k > 0$, so its ROC expands to $z = \infty$ and contains $z = \infty$. If ROC of $H(z)$ does not contains $z = \infty$, it cannot be causal.

Property: Z transform at $|z| = 1$ is fourier transform.

According to the expression of laurent expansion, we have the inverse Z transform

$$h[n] = \oint_L H(z) z^{n-1} dz, \tag{16}$$

which is difficult to calculate. But if the ROC contains $|z| = 1$, we can let $L = \{z : |z| = 1\}$. This give us a easy way to switch from $H(z)$ to $h[n]$, and backward:

$$h[n] = \frac{1}{2\pi} \int_0^{2\pi} H(e^{i\omega}) e^{i\omega n} d\omega, \quad H(e^{i\omega}) = \sum_n h[n] e^{-i\omega n}. \tag{17}$$

This integral is easier to calculate. So if ROC contains $|z| = 1$, using fourier transform is a great idea. But sometimes $|z| = 1$ is not in the ROC, then we need Z transform to make analysis of the system.

In conclusion,

- Z transform is a laurent expansion at $z_0 = 0$, with $H(z)$ being the complex function, and $h[n]$ being the expansion coefficients.
- From this point of view, it is understandable that a system is specified by its system function $H(z)$ and ROC (because of the uniqueness of laurent expansion).

- Z transform at $|z| = 1$ is discrete time fourier transform (DTFT). If ROC does not contain $|z| = 1$, DTFT will not converge.