Metric space

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1 Preface

This is a note of metric space at an introductary level. Since this is not a note discussing about pure math, it will not focus too much on the details of proofs. The note mainly covers topics like metric, topology in metric space, continuity, completeness and compactness. Metric space gives us a chance to see function, sequence, convergence, continuity, etc. in a higher and more abstract perpestive.

2 Metric space and topology

Metric space comes naturally when we want to tell about how to measure ”distance”. For a set of elements X, we define so-called ”metric” on this space as a function $d : X \to X$ which meets the following:

Definition 2.1. *Let X be some set. Fucntion $d : X \to X$ is said to be a* ***metric****, if*

- *Non-negative: $d(x, y) \geq 0$ for any $x, y \in X$ and equality holds only for $x = y$.*
- *Symmetric: $d(x, y) = d(x, y)$ for any $x, y \in X$.*
- *Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.*

Then set X along with metric d is called a metric space (X, d).

The structure of a metric space is simple: it only tells about distance between elements, and that’s the whole story. But it turns out many normal and frequently used concepts stem from metric space.

2.1 Sequences and convergence

A key idea that metric space can provide and allow us to discuss about is "convergence". Intuitively, convergence means we can get close to some element arbitrarily. This is formalized in a metric space.

Definition 2.2. *A sequence $\{x_n\} \subset X$ converges to some $x \in X$, if for any $\epsilon > 0$, we can find some integer $N > 0$, for all index $n > N$,*

$$d(x_n, x) \leq \epsilon. \tag{1}$$

We also denote it as $\lim_n x_n = x$ for short.

A few points is worth being noted. First, when we say convergence, the convergent point x must be in X, because we cannot define metric for elements outside our space X. So when we say $\{x_n\}$ converges, we actually mean $\{x_n\}$ converges in X. Second, convergent point x is unique, which can be easily proved by considering distance between two convergent point x, y

$$d(x, y) \leq d(x, x_n) + d(y, x_n) \leq 2\epsilon. \tag{2}$$

An arbitrary small ϵ implies $d(x, y) = 0$. So convergent point (or limit) is unique.

2.2 Topology in metric space

Some topology terminology are quite useful for the discussion afterwards.

Definition 2.3. ***Open ball*** *$B_d(a, \delta)$of with center $a \in X$ and radius δ is defined as a set in X such that*

$$B_d(a, \delta) = \{x \in X | d(x, a) < \delta\}. \tag{3}$$

Definition 2.4. *A subset A in metric space (X, d) is said to be* ***open****, if for any $a \in A$, we can find a open ball $B_d(a, \delta)$ such that*

$$B_d(a, \delta) \subset A. \tag{4}$$

Be careful that whether a subset is open strongly depends on the metric space X.

We can show that open ball is open, and that's why it is called "open" ball. To prove this, consider open ball $B_d(a, \delta)$ and an element $b \in B_d(a, \delta)$. Make a open ball $B_d(b, \epsilon)$ with $\epsilon = \delta - d(a, b)$, then we find for any element $x \in B_d(b, \epsilon)$, we have

$$d(x, a) \leq d(x, b) + d(a, b) \leq \epsilon + d(a, b) = \delta. \tag{5}$$

It implies $x \in B_d(a, \delta)$. So $B_d(b, \epsilon) \subset B_d(a, \delta)$, and by definition we prove that $B_d(a, \delta)$ is open.

Theorem 2.1. *(1) Metric space X itself is open. Empty set $\emptyset$ is open.*

(2) Any union of open sets is open.

(3) Ant finite intersection of open sets is open.

Proof 2.1. *(1) Since any open ball must in X, so by definition X is open. Empty set is open from vacuous truth, that is since no element can be found in $\emptyset$, so the false assumption leads to a true conclusion all the time.*

(2) Suppose A_i is open and consider $\cup_i A_i$. For any $a \in \cup_i A_i$, it must be in certain A_j. Because $a \in A_j$ and A_j is open, then we can always find a open ball $B_d(a, \delta) \subset A_j \subset \cup_i A_i$. Thus $\cup_i A_i$ is open.

(3) Suppose A_i is open and consider $\cap_i A_i$. For any $a \in \cap_i A_i$, it lies in all A_j. For each A_j, we can find a open ball $B_d(a, \delta_j) \subset A_j$. Let us define $\delta = \inf_j\{\delta_j\}$, and for open ball $B_d(a, \delta)$, it is obvious that $B_d(a, \delta) \subset B_d(a, \delta_j) \subset A_j$ for each j. So $B_d(a, \delta) \subset \cap_i A_i$. Thus $\cap_i A_i$ is open. But notice that $\inf_j\{\delta_j\}$ may be 0 if there are infite number of A_j. So only finite intersection of A_j makes sure it is open.

If we have open set, we can also define closeness.

Definition 2.5. *A subset A in X is said to be closed, if complement A^c is open.*

Use this definition, we will have the following theorem.

Theorem 2.2. *(1) Metric space X itself is closed. Empty space $\emptyset$ is closed.*
(2) Any intersection of closed sets is closed.
(3) Any finite union of closed sets is closed.

Proof 2.2. *(1) Since $X^c = \emptyset$ and $\emptyset$ is open, so X is closed. Since $\emptyset^c = X$ and X is open, so $\emptyset$ is closed.*
(2,3) Simply by doing complement of (2,3) in Theorem 2.1 can we prove these.

Though at the first glance it is strange to say a set is both open and closed. But remember "open" and "closed" is not opposite here: A set that is not open does not necessirily means it is closed. Some sets can be neither open or closed, so it will not be weired to say X and $\emptyset$ is both open and closed.

Definition 2.6. *Let A be a subset of X. $a \in A$ is said to be* ***interior*** *of A, if there exists a radius δ such that $B_d(a, \delta) \subset A$. Interior of A, usually denoted as A°, refers to all interior points in A.*

Definition 2.7. *Let A be a subset of X. $x \in X$ is said to be a* ***limit point*** *of A, if there is a sequence $\{a_n\}$ which converges to x.*

Definition 2.8. *Let A be a subset of X. $x \in X$ is said to be in the* ***closure*** *of A, if for all radius $\delta > 0$, there are always some elements $a \in B_d(x, \delta)$. Closure of A is usually denoted as $\bar{A}$.*

Limit point should not be confused with limit. Limit point x of $\{a_n\}$ is not necessarily in A. Only when $x \in A$, we say x is the limit of $\{a_n\}$.

Theorem 2.3. *Interior A° of A is a open set.*

Proof 2.3. *The definition of A° shows that for any $a \in A^\circ$, we can find a open ball $B_d(a, \delta) \subset A$. What we want to prove is there exists soem ϵ such that $B_d(a, \epsilon) \subset A^\circ$. Let $\epsilon = \delta/2$, and consider open ball $B_d(a, \epsilon)$. We would like to shou that this open ball $B_d(a, \epsilon)$ is in A°. So let us first consider $x \in B_d(a, \epsilon)$. We need to prove x is also in A°. By definition, it means there exists a radius γ such that open ball $B_d(x, \gamma) \subset A$. let $\gamma = \delta/2$ and $y \in B_d(x, \gamma)$. We find that*

$$d(a, y) \leq d(a, x) + d(x, y) \leq \epsilon + \gamma = \delta. \tag{6}$$

So it implies $y \in B_d(a, \delta)$. Note that since $a \in A^\circ$, so $B_d(a, \delta) \subset A$. So $y \in A$. Because y is an element in $B_d(x, \gamma)$, so $B_d(x, \gamma) \subset A$ and x is a interior point by definition. So $B_d(a, \epsilon)$ consists of interior points and $B_d(a, \epsilon) \subset A^\circ$, which proves that A° is open.

Theorem 2.4. *Closure $\bar{A}$ of A is a closed set.*

Proof 2.4. *The proof is constructed in a similar way of proving interior A° is open. To prove $\bar{A}$ is closed, we need to prove $\bar{A}^c$ is open. Let $x \in \bar{A}^c$ be an arbitrary element, then by definition of closure, we know that there is some $\delta > 0$ such that for all $a \in A$, $a \notin B_d(x, \delta)$. Now we are going to prove $\bar{A}^c$ is open, which requires us to prove for $x \in \bar{A}^c$, there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subset \bar{A}^c$. Let $\epsilon = \delta/2$. To prove $B_d(x, \epsilon) \subset \bar{A}^c$, we need to prove for any $y \in B_d(x, \epsilon)$, we have $y \in \bar{A}^c$. $y \in \bar{A}^c$ implies there exists some $\gamma > 0$ such that for all $a \in A$, $a \notin B_d(y, \gamma)$. Consider triangle inequality:*

$$d(a, y) + d(x, y) \geq d(x, a) \geq \delta \Rightarrow d(a, y) \geq \frac{\delta}{2}. \tag{7}$$

So just let $\gamma = \delta/2$, then open ball $B_d(y, \gamma)$ does not contain any $a \in A$. So $y \in \bar{A}^c$. Since y is an arbitrary element in $B_d(x, \epsilon)$, so $B_d(x, \epsilon) \subset \bar{A}^c$. Because x is an arbitrary element in $\bar{A}^c$, so $\bar{A}^c$ is open, and thus $\bar{A}$ is closed.

Furthermore, we can show a closure $\bar{A}$ is just the union of A and all limit points from A. So a closed set also contains all limit points.

3 Continuity

The continuity of a function is defined a metric space.

Definition 3.1. *Let X, Y are metric space and let $f : X \to Y$ be a function. f is said to be* ***continuous*** *at $x_0 \in X$, if for any $\epsilon > 0$, we can find a $\delta > 0$, for all $d_X(x, x_0) < \delta$, we have*

$$d_Y(f(x), f(x_0)) < \epsilon. \tag{8}$$

A useful property of continuous function is its relation with sequence.

Theorem 3.1. *Suppose f is continuous at x_0. For every sequence $\lim_n x_n \to x_0$, we have $\lim_n f(x_n) = f(x_0)$.*

Proof 3.1. *Since $\lim_n x_n \to x$, so for any $\delta > 0$, we can find a integr $N > 0$ and when $n > N$, $d_X(x_n, x_0) < \delta$. From the defintion of a continuous function, if we choose $n > N$ such that $d_X(x_n, x_)) < \delta$, then $d_Y(f(x_n), f(x_0)) < \epsilon$. So $\lim_n f(x_n) = f(x_0)$.*

The theorem tells us if a function is continuous, then we can switch the limitation and the function.

Definition 3.2. *f is said to be continuous on X, if*

$$\forall x_0 \in X, \forall \epsilon > 0, \exists \delta > 0, \forall x \in B_d(x_0, \delta), d_Y(f(x), f(x_0)) < \epsilon. \tag{9}$$

Here δ can depend on x. There are other stronger types of continuity, which are uniformly continuous and Lipschitz continuous.

Definition 3.3. *f is said to be* ***uniformly continuous*** *on X, if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_0 \in X, \forall x \in B_d(x_0, \delta), d_Y(f(x), f(x_0)) < \epsilon. \tag{10}$$

Definition 3.4. *f is said to be* ***Lipschitz continuous*** *on X, if there exists a constant L such that*

$$\forall x, y \in X, d_Y(f(x), f(y)) < L d_X(x, y). \tag{11}$$

Uniformly continuity is a stronger restriction than continuity. Lipschtiz continuity is obviously a stronger version even than uniformly continuity.

4 Completeness

From previous discussion, we knew that in metric space (X, d), a convergent squence must be a cauchy sequence, but conversely it is not true: some cauchy sequences do not converge in X. This leads to the concept of completeness.

Definition 4.1. *A metric space (X, d) is said to be **complete**, if every cauchy sequence converges in X.*

For a complete metric space, we can show the convergence by pointing out the sequence is a cauchy sequence, which is easier to prove than proving its convergence by definition. Also it is worth noting that completeness of a metric space depends on the metric itself, because whether a sequence is a cauchy sequence is related with metric definition.

Theorem 4.1. *A complete metric space A must be a closed set. A closed set A is not necessarily complete.*

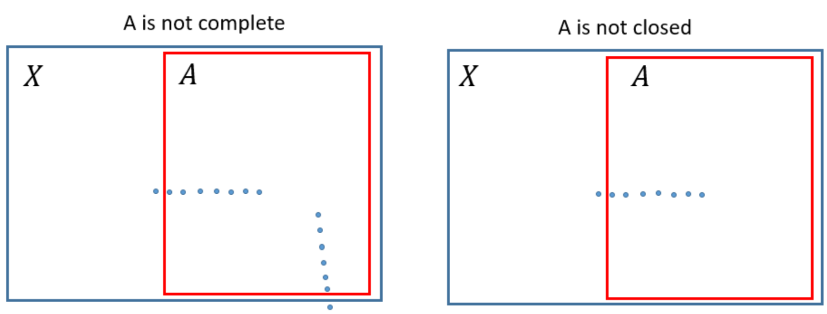

Figure 1: Not complete v.s. not closed

We just give a non-rigorous argument. Suppose a metric space is not closed, then there are some sequence $\{x_n\}$ converges to x_0 outside A but in X. Here it is worth noting that the concept of closed set depends on the largest space X we consider. x_0 must in X to be considered as a limiting point outside A, then we can say A is not closed. But if we consider completeness, as long as the $\{x_n\}$ converges to somewhere outside A (even outside X), the space A is not complete. So we can see that actually "A is not closed" implies "A is not complete". Thus "A is complete" implies "A is closed".

Theorem 4.2. *X is a complete metric space. A closed set $A \subset X$ is also complete.*

From the previous argument, it is easy to show that if X is complete, then $\{x_n\}$ cannot go to somewhere outside X, because $\{x_n\}$ with limiting point x_0 is cauchy and X is complete. Thus "A is closed" is equivalent to "A is complete".

4.1 Contraction mapping theorem

Then we turn to a useful application in complete metric space, that is the contraction mapping theorem. We first introduce what is a contraction.

Definition 4.2. *X is a metric space, and A is a subset of X. Function $f : X \to X$ is said to be a **contraction** on A, if*

- *$f(A) \subset A$.*

- *We have a contraction factor $\gamma < 1$ such that for any $x, y \in A$, $d(f(x), f(y)) \leq \gamma d(x, y)$.*

Intuitively, everytime we apply function f, it shrinks the space to a smaller scale, where elements are closer. Then we may imagine that after many times of shrinking, the set $f^N(A)$ will be compressed to a single point x^*. This unique point is called "fixed point".

Theorem 4.3 (Contraction mapping theorem)**.** *Let X be a complete metric space and f be a contraction on $A \subset X$. Then, there is a unique fixed point $x^* \in A$ such that $f(x^*) = x^*$. The sequence $x_{n+1} = f(x_n)$ converges to x^* for any $x_1 \in A$. Moreover, we have error bound $d(x^*, x_n) \leq \gamma^{n-1} d(x^*, x_1)$ and $d(x^*, x_{n+1}) \leq d(x_n, x_{n+1})\gamma/(1-\gamma)$.*

We are going to prove this whole theorem.

Proof 4.1. *We begin with fixed point. Let us first prove its uniqueness. Suppose there are two fixed points, x and y. Thus $f(x) = x$, $f(y) = y$. Consider distance between x and y, we get*

$$d(x, y) = d(f(x), f(y)) \leq \gamma d(x, y). \tag{12}$$

Note that only when $d(x, y) = 0$, we will have $0 \leq 0$. So $x = y$ by definition of metric.

Then let us prove the existence of fixed point. We must first prove that sequence $x_{n+1} = f(x_n)$ converges. In complete space X, it is enough to show that x_n is cauchy. Suppose $n > m$ and

$$d(x_n, x_m) \leq \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=m}^{n-1} \gamma^{i-1} d(x_2, x_1) \leq \frac{\gamma^m}{1-\gamma} d(x_2, x_1). \tag{13}$$

Then $\{x_n\}$ is a cauchy sequence. So limit $\lim x_n = x^*$ exists. Note that f is a contraction implies f is continuous, so $x^* = \lim x^n = \lim f(x_{n+1}) = f(\lim x_{n+1}) = f(x^*)$. So indeed fixed point x^* exists indeed. It is also worth noting that x_1 must be in A in order to make sure $\{x_n\} \subset A$.

Lastly we prove the following two error bound estimation.

$$d(x_n, x^*) = d(f(x_{n-1}), f(x^*)) \leq \gamma d(x_{n-1}, x^*) \leq ... \leq \gamma^{n-1} d(x_1, x^*). \tag{14}$$

And

$$d(x_{n+1}, x^*) = d(f(x_n), f(x^*)) \leq \gamma d(x_n, x^*) \leq \gamma d(x_n, x_{n+1}) + \gamma d(x_{n+1}, x^*). \tag{15}$$

By substracting $\gamma d(x_{n+1}, x^*)$ on both side we get

$$d(x^*, x_{n+1}) \leq \frac{\gamma}{1-\gamma} d(x_n, x_{n+1}). \tag{16}$$

5 Compactness

Compactness, intuitively, means a metric space is bounded.

Definition 5.1. *Metric space X is said to be* ***totally-bounded****, if for any $\epsilon > 0$, we can use a finite union of open balls $\cup_i B_d(x_i, \epsilon)$ to cover X, that is*

$$\cup_i B_d(x_i, \epsilon) = X, \quad x_i \in X. \tag{17}$$

Definition 5.2. *Metric space X is said to be* ***compact****, if and only if X is complete and totally-bounded.*

Theorem 5.1. *A closed subset A of a compact metric space X is also compact.*

An important conclusion of compact space is the following theorem.

Theorem 5.2. *Let X be a compact metric space. Then for sequence $\{x_n\} \subset X$, there exists a subsequence $\{x_{z_i}\}$ such that $\{x_{z_i}\}$ converges.*