Intro to system function

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This article introduce concepts like zero-input response, zero-state response, system function in system analysis. And we show how these concepts work when the system is represented by a differential/difference equation.

1 Continuous time system

A continuous time system is defined as an operator: $S : F \to F$, here F is a signal (function) space. Usually systems can be represented by a differential equation and certain initial conditions. We use operator $\hat{p} = \frac{d}{dt}$ to write a differential equation as

$$A(\hat{p})y(t) = B(\hat{p})x(t), \quad y^{(n)}(0) = y_0^{(n)}. \tag{1}$$

By defining $\hat{p}^{-1}x = \int_{-\infty}^{t} x(\tau)d\tau$, we can rewrite the equation as

$$y(t) = \frac{A(\hat{p})}{B(\hat{p})}x(t) = S(\hat{p})x(t). \tag{2}$$

Operator $S(\hat{p})$ is called the system operator. The function $S(\cdot)$ is also called the system function.

1.1 Homogenous solution and particular solution

To solve a equation like

$$A(\hat{p})y(t) = B(\hat{p})x(t), \quad y^{(n)}(0) = y_0^{(n)}, \tag{3}$$

our typical method is to split the solution into two parts: homogenous solution and particular solution.

Def: A homogeous solution is a function $y_h(t)$ that meets the equation with $x(t) = 0$ and the a undecided initial conditions:

$$A(\hat{p})y_h(t) = 0. \tag{4}$$

Def: A particular solution is a function $y_p(t)$ that meets the equation with random initial conditions:

$$A(\hat{p})y_p(t) = B(\hat{p})x(t). \tag{5}$$

"Undecided" means uncertain constants, and "random" means you can pick up any initial conditions you like. Then the solution will be $y(t) = y_h(t)+y_p(t)$, and we can use initial conditions $y^{(n)}(0)$ to determine undecide constant in $y_h(t)$.

Example: Consider equation $\frac{d}{dt}y + 2y = \cos t$ and initial condition $y(0) = 1$. The homogenous solution is $y_h(t) = c_0 e^{-2t}$, and the particular solution is $y_p(t) = \frac{1}{5}\sin t + \frac{2}{5}\cos t$. So the solution of the equation is

$$y(t) = c_0 e^{-2t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t = \frac{3}{5}e^{-2t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t. \tag{6}$$

1.2 System response: zero-input response and zero-state response

A homogenous solution and a particular solution are not giving too much physical meanings. In system analysis we attempt to find a particular solution with clear physical meanings.

Let us reconsider the following equation,

$$A(\hat{p})y(t) = B(\hat{p})x(t), \quad y^{(n)}(0) = y_0^{(n)}. \tag{7}$$

Intuition tells us that the response $y(t)$ should be able to be splited into two parts: One part induced from the initial condition $y^{(n)}(0) = y_0^{(n)}$, and another part induced from the input $x(t)$. We call the former zero-input response, the latter zero-state input response,

$$y(t) = S(x(t)) = S_{zs}(x(t)) + S_{zi}(x(t)) = y_{zs}(t) + y_{zi}(t). \tag{8}$$

The formal definition is as follows.

Def: Zero-state response $y_{zs}(t)$ is a function that meets the equation with properties (including initial conditions) determined from the physical system.

When we talk about "system", we are actually refering to the zero-state response (we use index "xs" to highlight). it is important to figure out which "system" we are refering to (total system response = zero-input response + zero-state response). In practice, our system is linear, time-invariant, and causal, then these properties determine a specific initial conditions for the zero-state response. Linearity implies

$$S_{zs}(\alpha x_1 + \beta x_2) = \alpha S_{zs}(x_1) + \beta S_{zs}(x_2). \tag{9}$$

Time-invariance implies, if $y_{zs}(t) = S_{zs}(x(t))$, then

$$y_{zs}(t') = S_{zs}(x(t + t')). \tag{10}$$

Causality leads to relaxed initial condition: If $x(t) = 0$ when $t < t_0$, then $y_{zs}(t) = 0$ when $t < t_0$. Causality assures the linearity and time invariance. A system with linearity and time invariance nearly covers all cases we concern, and we call such system a LTI system.

A zero-state response of a LTI system can be easily solved by method of impulse response (green function in math) or laplace transform . We are going introduce them in a second.

Theorem (Convolution theorem): For a LTI system, if we define impulse response $h(t) = S(\delta(t))$, then for input $x(t)$, the output is

$$y_{zs}(t) = S_{zs}(x(t)) = (x * h)(t), \tag{11}$$

where $*$ refers to convolution.

Proof: The key idea to choose a basis and use linearity and time-invarince. Since we already known $S(\delta(t)) = h(t)$, then according to linearity and time-invarince, we can expand input $x(t)$ upon the basis $\delta(t)$:

$$S_{zs}(x(t)) = S_{zs}\left(\int x(\tau)\delta(t-\tau)d\tau\right) = \int x(\tau)S_{zs}(\delta(t-\tau))d\tau = \int x(\tau)h(t-\tau)d\tau = (x*h)(t). \tag{12}$$

Thus we prove the convolution theorem. This is one way to obtain zero-state input $y_{zs}(t)$: first calculate impulse response $h(t)$, and do the convolution $y = x * h$.

Another method to find $y_{zs}(t)$ is to choose a different basis, the eigenstates of our LTI system. It is easy to show e^{st} is an eigenstate of LTI system by convolution theorem, with eigene quation

$$S_{zs}(e^{st}) = H(s)e^{st}, \tag{13}$$

where $H(s) = \int h(t)e^{-st}dt$ is the expansion coefficient of $h(t)$ on basis e^{st}. We call this transform Laplace transform. By expanding $x(t)$ on basis e^{st} we get

$$S_{zs}(x(t)) = S_{zs}\left(\frac{1}{2\pi i}\int X(s)e^{st}ds\right) = \frac{1}{2\pi i}\int X(s)S_{zs}(e^{st})ds = \frac{1}{2\pi i}\int H(s)X(s)e^{st}ds. \tag{14}$$

It is clearer for LTI system represented by differential equation $y_{sz}(t) = S(\hat{p})x_{sz}(t)$. System opertors have eigenstates like e^{st} with the following eigen equation

$$S(\hat{p})e^{st} = S(s)e^{st}. \tag{15}$$

So it is obvious that system function $S(s)$ is equal to $H(s)$. Thus we also call $H(s)$ system function. For differential equation problem, it is pretty easy to find the system function H by simple algebra.

Example: Consider equation $\frac{d}{dt}y + 2y = x(t) = \cos t$ and initial condition $y(0) = 1$. Please find the zero-state response.

We first seak for system function $H(s)$, eigenvalue of e^{st}. Let $x(t) = e^{st}$, and $y(t) = H(s)e^{st}$

$$(s+2)H(s) = 1. \tag{16}$$

Therefore $H(s) = 1/(2+s)$. Suppose the system is causal, then ROC is $Re(s) > -2$. Input $x(t) = \cos t = \frac{1}{2}(e^{it} + e^{-it})$. Thus the zero-state response is

$$y_{zs}(t) = \frac{1}{2}\left(H(i)e^{it} + H(-i)e^{-it}\right) = \frac{1}{5}\sin t + \frac{2}{5}\cos t, \tag{17}$$

Which is the same as particular solution in the previous discussion. Then let us talk about zero-input response.

Def: zero-input response is a function $y_{zi}(t)$ that meets the equation with $x(t) = 0$ and the initial conditions $y_{zi}^{(n)}(0) = y^{(n)}(0) - y_{zs}^{(n)}(0)$.

Zero-input response is of less interest, because it reflects the history of the system, and does not depend on the input $x(t)$. Let us take the same example to clarify.

Example: Consider equation $\frac{d}{dt}y + 2y = x(t) = \cos t$ and initial condition $y(0) = 1$. Please find the zero-input response.

The equation to solved is $A(\hat{p})y_{zi}(t) = 0$. Assume $y_{zi}(t) = ce^{st}$, then the equation becomes $cA(s)e^{st} = 0$. We need to find a s such that $A(s) = 0$. In this example, $A(s) = s + 2$, so $s = -2$. Thus $y_{zi}(t) = ce^{-2t}$. In order to determine c, since we have already known that $y_{zs}(t) = \frac{1}{5}\sin t + \frac{2}{5}\cos t$, so $y_{zs}(0) = \frac{2}{5}$, so $y_{zi}(0) = c = y(0) - y_{zs}(0) = \frac{3}{5}$. Finally we find zero-input response as

$$y_{zi}(t) = \frac{3}{5}e^{-2t}. \tag{18}$$

By adding up y_{zi} and y_{zs}, we get exactly what we solved before

$$y(t) = y_{zi}(t) + y_{zs}(t) = \frac{3}{5}e^{-2t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t. \tag{19}$$

2 Discrete time system

What we will do for discrete time system is similar to what we did in continuous time system. The total system response can be splited into zero-input response and zero-state response. The key idea to get zero-state response is mainly including two approaches:

- Use basis $\delta[n]$ to find impulse response $S_{zs}(\delta[n]) = h[n]$, and use convolution $y_{zs}[n] = (h * x)[n]$.
- Use eigenstates z^n as basis to find system function $H(z)$ by eigen equation $S_{zs}(z^n) = H(z)z^n$, and use spectrum decomposition $y_{zs}[n] = \frac{1}{2\pi i} \oint X(z)H(z)z^n dz$.

Discrete time system can be represented by a difference equation,

$$\sum_k a_k y[n-k] = \sum_k b_k x[n-k]. \tag{20}$$

By plugging in eigenstate $x[n] = z^n$, $y[n] = H(z)z^n$, we obtain the system function

$$H(z) = \frac{\sum_k b_k z^{-k}}{\sum_k a_k z^{-k}} \equiv \frac{B(z)}{A(z)}. \tag{21}$$

This result is reasonable, since if we see coefficients a_k and b_k as signals, then the difference equation is

$$(a * y)[n] = (b * x)[n]. \tag{22}$$

In Z domain, we thus have

$$A(z)Y(z) = B(z)X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)}. \tag{23}$$

3 Summary

A system is an operator S that map $x(t)$ to $y(t)$,

$$S(x(t)) = y(t). \tag{24}$$

In system analysis, we decompose S into two parts, namely zero-input response and zero-state response

$$y(t) = S(x(t)) = S_{zi}(x(t)) + S_{zs}(x(t)) = y_{zi}(t) + y_{zs}(t). \tag{25}$$

To solve $y_{zs}(t)$, we are encouraged to choose a good basis. Basis $\delta(t)$ (or $\delta[n]$) gives us convolution theorem

$$y_{zs}(t) = (x * h)(t), \tag{26}$$

where $h(t)$ is the impulse response. Basis e^{st} (or z^n) gives us spectrum decomposition

$$y_{zs}(t) = \frac{1}{2\pi i} \int X(s)H(s)e^{st} ds, \tag{27}$$

where $H(s)$ and $X(s)$ are representation of $h(t)$ and $x(t)$ in laplace domain. $H(s)$ is so important in the sense that it contains all information about the system, and thus it is given a name called system function. In differential equations $A(\hat{p})y(t) = B(\hat{p})x(t)$, the system function is $H(s) = \frac{B(s)}{A(s)}$. In difference equations $\sum_k a_k y[n-k] = \sum_k b_k x[n-k]$, the system function is $H(z) = \frac{\sum_k b_k z^{-k}}{\sum_k a_k z^{-k}} = \frac{B(z)}{A(z)}$.