Poisson process

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1 Preface

Poisson process plays an important role in stochastic process, just as Bernulli experiment in 1d random variables. Though simple, it is widely used and worthy spending times on. We first start from one version of definition of poisson process, deduce the memoryless and independent/stationary increment properties. We will describe possion process from viewpoints of interarrival times X_i, sum process $S_n = \sum_i^n X_i$ and counting process $N(t)$ and play with their relations and distributions. We also show the equivalence of different versions of definitions of poisson process. Finally we introduce the combination and devision of poisson process.

2 Review of stochastic process

We will give a brief review of stochastic process. Specially, we focus on discrete stochastic process. Continuous stochastic process is defined in a similar way as discrete random process.

Definition 2.1. *A collection of random variables $\{X_i | i \in \mathbb{N}\}$ with order is called a stochastic process.*

If we see a stochastic process $\{X_i | i \in \mathbb{N}\}$ as a infinite-dimensional random vector $\vec{X}$, then it maps from its infinite sample space Ω to a sequence $\{x_n\}$. When we determine a sample point $s \in \Omega$, we then get a real squence $X_n(s) = x_n$. This squence $\{x_n\}$ is called a **sample function** or **sample trajectory**. If we fix a index i, then X_i is simply a random variable.

For examples, $E(t) = E\cos(\omega t + \Phi)$ with $E \sim \text{Unif}(-1, 1)$ and $\Phi \sim \text{Unif}(0, 2\pi)$ is a continuous stochastic process, for which the sample function is simply a cosine function $e(t) = e\cos(\omega t + \phi)$. $\{Y_i | n \in \mathbb{N}\}$ with $Y_i \sim \text{Bern}(p)$ is a discrete stochastic process, for which each trial is independent.

3 Definitions and properties of Poisson process

There are three ways to define poisson process, and they are, with no doubts, equivalent. We are going to start with specific one of them.

3.1 Poisson process by exponential distribution

Definition 3.1. *We call $\{N(t)|t \in [0, \infty)\}$ an **arrival process** (counting process), if it counts the number of arrivals (occurance of events) until time t.*

For example, if we count how many of customers coming into the shop , then the counting process $N(t) = n$ for specific t represents there are totally n customers coming within time interval $[0, t]$.

Actually there are different equivalent descriptions of an arrival process.

- Counting process $\{N(t)|t \in [0, \infty)\}$. $N(t) = n$ represents totally n arrivals up to time t.
- Arrival epochs $\{S_n | n \in \mathbb{N}\}$. $S_n = t$ represents the n^{th} arrival comes at time t.
- Interarrival times $\{X_i | i \in \mathbb{N}\}$. $X_i = t$ represents the waiting time from $i^{th} - 1$ arrival to i^{th} arrival is t.

These three stochastic process actually characterize the same process. For example, we note that $S_n = \sum_i X_i$, and event "$S_n < t$" is equivalent to $N(t) > n$.

Definition 3.2. *We call $\{N(t)|t \in [0, \infty)\}$ a **renewal process**, if it is an arrival process with **i.i.d.** interarrival times $\{X_i | i \in \mathbb{N}\}$.*

Renewal process is a broad topic and its assumption of $i.i.d.$ interarrival times is both easier to handle and stemming from realistic meanings. Poisson process is a special renewal process.

Definition 3.3 (First Definition of Poisson process)**.** *We call $\{N(t)|t \in (0, \infty]\}$ a **poisson process**, if it is an renewal process with $X_i \sim$ Expo(λ).*

Exponential distribution is just like the continuous version of Bernulli distribution, in the sense that they are both memoryless (and they are the only two). We will see in a few setions later that poisson process is exactly a continuous Bernulli process. The memoryless property gives us great amount of convenience. Suppose there is a poisson process going on, and for certain time t we restart counting, and we expect the memoryless property will forget the past and begin a new poisson process. The claim is stated as the following theorem.

Theorem 3.1. *Suppose $\{N(t)|t \in [0, \infty)\}$ is a poisson process and $\{Z_i | i \in \mathbb{N}\}$ represents a new counting process defined by beginning counting from time t. Then Z_i are i.i.d Expo(λ) distributed and independent of the past $\{N(\tau)|\tau \in [0, t]\}$.*

Proof 3.1. *Let us first consider Z_1, namely the waiting time for the next arrival from time t. Conditioning on $N(t) = n$ and $S_n = \tau$, we find*

$$P(Z_1 > z | N(t) = n, S_n = \tau) = P(X_{n+1} > t - \tau + z | N(t) = n, S_n = \tau). \tag{1}$$

Note that $N(t) = n, S_n = \tau$ is equivalent to say $S_n = \tau, X_{n+1} > t - \tau$. So

$$P(X_{n+1} > t - \tau + z | N(t) = n, S_n = \tau) = P(X_{n+1} > t - \tau + z | S_n = \tau, X_{n+1} > t - \tau). \quad (2)$$

Since $S_n = \sum_i^n X_i$ and X_i are i.i.d., so $S_n = \tau$ here has no impact on X_{n+1}. So we then have

$$P(X_{n+1} > t-\tau+z | S_n = \tau, X_{n+1} > t-\tau) = P(X_{n+1} > t-\tau+z | X_{n+1} > t-\tau) = P(X_{n+1} > z) = e^{-\lambda z}. \quad (3)$$

Overall we conclude that

$$P(Z_1 > z | N(t) = n, S_n = \tau) = e^{-\lambda z}. \quad (4)$$

Note that this conditional probability does not really depend on $N(t)$ and S_n. We can drop the condition and we get the cdf of Z_1

$$P(Z_1 \leq z) = 1 - e^{-\lambda z}. \quad (5)$$

This is what we expect to see: We cut in at time t, and we ignore the history and begin a new exponential waiting time. Also note that $Z_m = X_{N(t)+m}$, so the joint distribution of $\{Z_i\}$ conditioning on $N(t) = n, S_n = \tau$ is

$$\begin{aligned} &P(Z_1 > z_1, Z_2 > z_2, ..., Z_k > z_k | N(t) = n, S_n = \tau) \\ &= P(X_{n+1} > t - \tau + z_1, X_{n+2} > z_2, ..., X_{n+k} > z_k | X_{n+1} > t - \tau) \\ &P(X_{n+1} > z_1) P(X_{n+2} > z_2) ... P(X_{n+k} > z_k) = e^{-\lambda \sum_{i=1}^k z_i}. \end{aligned} \quad (6)$$

So indeed Z_i are i.i.d. exponentially distributed and are independent of the history. This implies if we begin a new counting in a poisson process from any time t, we get a fresh new poisson process again.

3.2 Independent increment and stationary increment process

Definition 3.4. *An arrival process $\{N(t) | t \in [0, \infty)\}$ is called a* ***independent increment process****, if for any disjoint time intervals $[0, t_1], [t_1, t_2], ...[t_{k-1}, t_k]$, random variables $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), ..., N(t_k) - N(t_{k-1})$ are independent.*

Independent increment implies that for any disjoint time intervals, the number of arrivals are independent.

Definition 3.5. *An arrival process $\{N(t) | t \in [0, \infty)\}$ is called a* ***stationary increment process****, if $N(t + \tau) - N(t)$ has the same distribution with $N(\tau)$.*

Stationary increment process suggests that number of arrivals only depends on the length of waiting time, not on the beginning time.

Theorem 3.2. *Poisson process is both* ***independent increment process*** *and* ***stationary increment process****.*

Proof 3.2. *Note that $N(t + \tau) - N(t)$ can be interpreted as a counting process starting from time t. We have seen that starting counting from t, we obtain a new poisson process which is independent of the history. To be specific,*

$$P(N(t + \tau) - N(t) > n) = P(\sum_i^n Z_i < \tau) = P(\sum_i^n X_i < \tau) = P(N(\tau) > n), \quad (7)$$

and

$$\begin{aligned}P(N(t)>n, N(t+\tau)-N(t)>m) &= P(\sum_i^n X_i<t, \sum_i^m Z_i<\tau)\\ &= P(\sum_i^n X_i<t)P(\sum_i^m Z_i<\tau) = P(N(t)>n)P(N(t+\tau)-N(t)>m).\end{aligned} \tag{8}$$

Therefore, poisson process is both independent increment process and stationary increment process.

3.3 Dig into distribution of S_n and $N(t)$

Our poisson process is defined by its interarrival times $X_i \sim \text{Expo}(\lambda)$. Now we are going to deduce the distribution of $N(t)$ and S_n, both of which are equivalent description for poisson process.

Theorem 3.3. *The followings are true for S_n in poisson process:*

- *Joint pdf $f_{\boldsymbol{S}}(\boldsymbol{s}) = \lambda^n e^{-\lambda s_n}$ for $0 < s_1 < s_2 < ... < s_n$.*
- *Marginal distribution $S_n \sim$ Gamma(n, λ).*
- *Conditional joint pdf $f_{\boldsymbol{S}|S_{n+1}}(\boldsymbol{s}|t) = \frac{n!}{t^n}$ for $0 < s_1 < s_2 < ... < s_n < t$.*
- *Conditional marginal distribution $S_i|S_{n+1} = t \sim t \cdot$ Beta$(i, n+1-i)$.*

Proof 3.3. *To deduce the joint pdf of **S**, we first note that $S_n = \sum_i^n X_i$ and we have already known the joint pdf of **X**, which is $f_{\boldsymbol{X}}(\boldsymbol{x}) = \lambda^n \exp(-\lambda \sum_i^n x_i)$. By a variables transformation, we have*

$$f_{\boldsymbol{S}}(\boldsymbol{s}) = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{s}}\right| f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{X}}(\boldsymbol{x}) = \lambda^n e^{-\lambda s_n}, \quad 0 < s_1 < s_2 < ... < s_n, \tag{9}$$

where the determinant of jacobian $\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{s}}\right| = 1$.

To find marginal distribution, we can simpliy integrate joint pdf $f_{\boldsymbol{S}}(\boldsymbol{s})$. Here we give an alternative method, that is to use convolution and induction. We claim that $S_n \sim$ Gamma(n, λ), for which the pdf is

$$f_{S_n}(s) = \frac{1}{\Gamma(n)} \lambda^n s^{n-1} e^{-\lambda s}. \tag{10}$$

we are going to prove this by induction. $f_{S_1}(s) = f_{X_1}(s) = \lambda \exp(-\lambda s)$ is obviously true. Consider conditioning on $S_n = s'$

$$\begin{aligned}f_{S_{n+1}}(s) &= \int_0^s f_{S_{n+1}|S_n}(s|s') f_{S_n}(s') ds' = \int_0^s f_{X_{n+1}}(s-s') f_{S_n}(s') ds'\\ &= \int_0^s \lambda e^{-\lambda(s-s')} \frac{\lambda^n (s')^{n-1} e^{-\lambda s'}}{\Gamma(n)} = \frac{\lambda^{n+1} s^n e^{-\lambda s}}{\Gamma(n+1)}.\end{aligned} \tag{11}$$

Thus by induction, we prove that $S_n \sim$ Gamma(n, λ). Furthermore, the conditional probability is easy to calculate

$$f_{\boldsymbol{S}|S_{n+1}}(\boldsymbol{s}|t) = \frac{f_{\boldsymbol{S},S_{n+1}}(\boldsymbol{s}, t)}{f_{S_{n+1}}(t)} = \frac{\lambda^{n+1} e^{-\lambda t}}{\lambda^{n+1} t^n e^{-\lambda t}/\Gamma(n+1)} = \frac{n!}{t^n}. \tag{12}$$

*Finally, we turn to the conditional marginal distribution of $S_i|S_{n+1} = t$. We note that $f_{\boldsymbol{S}|S_{n+1}}(\boldsymbol{s}|t)$ does not depend on **s**, so it is uniform for **s** expect for that fact that $s_1 < s_2 < \ldots < s_n$. We realize that this is the order statistics for uniform distribution. So basically given $S_{n+1} = t$, S_i is just the i^{th} arrival time of n uniformly points in time interval $[0, t]$.*

From the analysis above, it is worthy noting that $\sum_i^n \text{Expo}(\lambda) = \text{Gamma}(n, \lambda)$. Basically, it says a Gamma distribution $\text{Gamma}(n, \lambda)$ can be interpreted as the waiting time for n arrivals in a possion process with rate λ. Also, given the $n^{th} + 1$ arrival is at time t, the n arrivals before is the order random variables for n uniformly distributed random variables with interval $[0, t]$.

Theorem 3.4. *In Poisson process $\{N(t)|t \in [0, \infty)\}$, counting $N(t) \sim$ Poisson(λt).*

Proof 3.4. *Consider event that $t < S_{n+1} < t + \delta$, where δ is a small interval. We can condition it on*

$$P(t < S_{n+1} < t + \delta) = P(t < S_{n+1} < t + \delta, S_n < t) + P(t < S_{n+1} < t + \delta, S_{n-1} < t, t < S_n < S_{n+1}) \\ + \ldots \tag{13}$$

Note that the first term $P(t < S_{n+1} < t + \delta, S_n < t) = P(N(t) = n, N(t + \delta) - N(t) = 1)$ is the leading term. The second term $P(t < S_{n+1} < t + \delta, S_{n-1} < t, t < S_n < S_{n+1}) = P(N(t) = n - 1, N(t + \delta) - N(t) = 2)$ is the higher order of δ and thus can be neglected (and so as the third term, etc.). So when $\delta \to 0$,

$$\begin{aligned} P(t < S_{n+1} < t + \delta) &= P(N(t) = n, N(t + \delta) - N(t) = 1) = P(N(t) = n)P(N(\delta) = 1) \\ &= P(N(t) = n)P(S_1 < \delta, S_2 > \delta) = P(N(t) = n) \int_0^\delta ds_1 \int_\delta^\infty ds_2 \lambda^2 e^{-\lambda s_2} \\ &\approx P(N(t) = n)\lambda\delta. \end{aligned} \tag{14}$$

And we also know that

$$P(t < S_{n+1} < t + \delta) \approx f_{S_{n+1}}(t)\delta = \frac{\delta}{n!}\lambda^{n+1} t^n e^{-\lambda t} \tag{15}$$

Combine these two equations, we obtain the pmf of $N(t)$:

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \tag{16}$$

which is exactly the poisson pmf.

3.4 Alternative defintions of poisson process

The second version of poisson process definition is as follows.

Definition 3.6 (Second Definition of Poisson process)**.** *An arrival process $\{N(t)|t \in [0, \infty)\}$ is called a poisson process, if*

- *It is an independent increment process and a stationary increment process.*
- *$N(t) \sim$ Poisson(λt).*

Definition 3.7 (Third Definition of Poisson process)**.** *An arrival process $\{N(t)|t \in [0, \infty)\}$ is called a poisson process, if for any small δ*

- *It is an independent increment process and a stationary increment process.*
- $P(N(t+\delta) - N(t) = 0) = 1 - \lambda\delta + O(\delta)$.
- $P(N(t+\delta) - N(t) = 1) = \lambda\delta + O(\delta)$.
- $P(N(t+\delta) - N(t) = 2) = O(\delta)$.

These two definitions can be proved as equivalent to the first defintion.

4 Poisson process from Bernulli process with shrinking time

In this section, we are going to give a tuitive way to understand poisson process, that is, a continuous version of Bernulli process.

Definition 4.1. *We call a discrete counting process $N_i = \sum_i Y_i$ an Bernulli process, if $Y_i \sim$ Bern(p) and are i.i.d..*

Consider a continuous time interval $[0, t]$ and we split it into M subintervals with equal length $\delta = t/M$. For each small time interval $[n\delta, (n+1)\delta]$ we do an Bernulli trial with probability $p = \lambda\delta$. Then $N(t) = \sum_i^M Y_i$ is just an Bernulli process with pmf

$$P(N(t) = n) = P(\sum_i^M Y_i = n) = \binom{M}{n}(\lambda\frac{t}{M})^n(1 - \lambda\frac{t}{M})^{M-n} \xlongequal{M\to\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \tag{17}$$

We can see that if the Bernulli process probability $p \propto \delta$, namely the length of the subinterval, and we let its length goes to 0, then we get poisson distribution. Since Bernulli process is independent/stationary increment process (which is quite obvious), we conclude that when $\delta \to 0$, we find it is poisson process.

Compared to poisson process, Bernulli process is easier to understand and get insight. For example, memoryless property is inherited naturally from Bernulli process. And it turns out understanding poisson process by Bernulli process is helpful for the following discussions.

5 Combing and spliting Poisson process

Consider two poisson process $\{N_1(t, \lambda_1)|t \in [0, \infty)\}$ and $\{N_2(t, \lambda_2)|t \in [0, \infty)\}$. What can we say about $N(t) = N_1(t) + N_2(t)$?

Theorem 5.1 (Combining poisson process)**.** *For two independent possion processes $\{N_1(t, \lambda_1)|t \in [0, \infty)\}$ and $\{N_2(t, \lambda_2)|t \in [0, \infty)\}$, their sum $N(t) = N_1(t) + N_2(t)$ is also a poisson process with $\lambda = \lambda_1 + \lambda_2$.*

Proof 5.1. *We can rigorously prove it by doing the convolution. But actually we can use the Bernulli process to understand it. For each subinterval, the arrival is either from N_1 or N_2, but not both because the probability is too small. So $p = p_1 + p_2$, thus $\lambda = \lambda_1 + \lambda_2$. We can strictly prove this by using the third defintion of poisson process.*

$$P(N(t+\delta)-N(t) = 0) = P(N_1(\delta) = 0)P(N_2(\delta) = 0) = (1-\lambda_1\delta)(1-\lambda_\delta) \approx 1-(\lambda_1+\lambda_2)\delta+O(\delta), \tag{18}$$

$$\begin{aligned} P(N(t+\delta) - N(t) = 1) &= P(N_1(\delta) = 1)P(N_2(\delta) = 0) + P(N_1(\delta) = 0)P(N_2(\delta) = 1) \\ &= (\lambda_1 + \lambda_2)\delta + O(\delta), \end{aligned} \tag{19}$$

$$P(N(t+\delta)-N(t)=2)=O(\delta). \tag{20}$$

So the equations above shows $N(t) \sim$ Poisson$((\lambda_1+\lambda_2)t)$. The properties of independent increment and stationary increment are naturally inherited from $N_1(t)$ and $N_2(t)$. To sum up, we show that $N(t)=N_1(t)+N_2(t)$ is indeed a poisson process with $\lambda=\lambda_1+\lambda_2$.

We consider another stochastic process. We first do a poisson process $N(t)$, and for each arrival, we do an Bernulli trial and add to $N_1(t)$ if it is a success, otherwise add to $N_2(t)$.

Theorem 5.2 (Splitting poisson process)**.** *Let $\{N(t,\lambda)| \in [0,\infty)\}$ be a poisson process. For each arrival, we do an Bernulli trial with success probability p to determine which this arrival belongs to: if success, add it to $N_1(t)$; otherwise add it to $N_2(t)$. Then $N_1(t) \sim$ Poisson(λpt) and $N_2(t) \sim$ Poisson$(\lambda(1-p)t)$.*

Proof 5.2. *Consider the joint pdf of $N_1(t)$ and $N_2(t)$:*

$$\begin{aligned}&P(N_1(t)=n,N_2(t)=m)=P(N_1(t)=n,N_2(t)=m|N(t)=n+m)P(N(t)=n+m)\\&=\binom{n+m}{n}p^n(1-p)^m\frac{\lambda^{n+m}e^{-\lambda t}}{(n+m)!}=\frac{(\lambda pt)^n e^{-\lambda pt}}{n!}\frac{(\lambda(1-p)t)^m e^{-\lambda(1-p)t}}{m!}.\end{aligned} \tag{21}$$

Thus from the equation above, we conclude that $N_1(t) \sim$ Poisson(λpt) and $N_2(t) \sim$ Poisson$(\lambda(1-p)t)$.

This result is easy to understand from the viewpoint of Bernulli process. For each subinterval, we first do a Bern$(\lambda\delta)$ to dertermine if there is an arrival, and next do another Bernulli process Bern(p) to determine which catergory it belongs to. This process is equivalent to say for each subinterval $N_1(t)$ does a Bern$(\lambda\delta p)$ and $N_2(t)$ does another Bern$(\lambda(1-p)\delta)$. These two descriptions are equivalent because δ is small and thus we can exclude the case that both of them are successful, for which the probability is neglected.