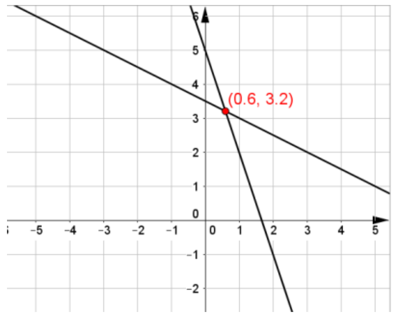
Chapter 10 Matrices

Many mathematical models require finding the solutions of several equations and the solutions must satisfy all of the equations in the model. A set of equations related in this way is called a **system of equations**. The learning objective in this chapter is to know how to solve systems of equations.

Section 10.1 Solution of Linear Systems

Learning Objective: Solve systems of linear equations in matrix form using row operations, determine if a system has a unique solution, no solution or infinitely many solutions



Example 1: Solve the system

$$\begin{aligned} 3x + y &= 5 \\ x + 2y &= 7 \end{aligned}$$

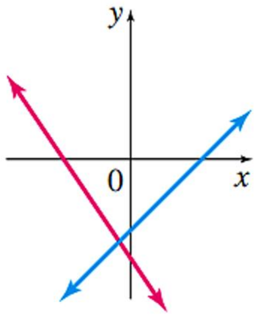
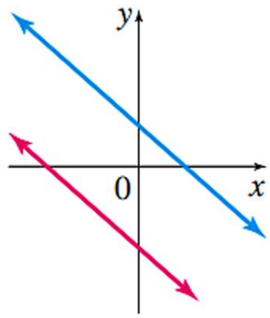
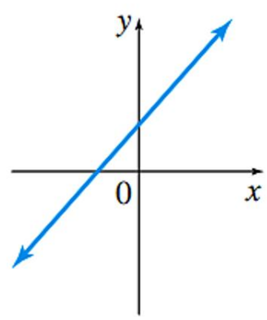
Solution:

Method 1:

$$y = 5 - 3x$$
$$x + 2y = 7 \Rightarrow x + 2(5 - 3x) = 7$$
$$\Rightarrow x + 10 - 6x = 7$$
$$\Rightarrow\ -5x = -3 \Rightarrow x = \frac{3}{5}$$
$$\Rightarrow y = 5 - 3 \times \frac{3}{5} = 5 - \frac{9}{5} = \frac{25 - 9}{5} = \frac{16}{5}$$

Method 2:

$$\begin{matrix} 3x + y = 5 \\ x + 2y = 7 \end{matrix} \Rightarrow \begin{matrix} 3x + y = 5 \\ -3x - 6y = -21 \end{matrix} \Rightarrow -5y = -16 \Rightarrow y = 16/5$$
$$3x + 16/5 = 5 \Rightarrow 3x = 9/5 \Rightarrow x = 3/5$$

Types of Solutions for Two Equation in Two Unknowns		
Case 1: The two graphs are lines intersecting at a single point. The system has a **unique solution**, and it is given by the coordinates of this point. 	**Case 2:** The graphs are distinct parallel lines. When this is the case, the system is **inconsistent**; that is, there is no solution common to both equations. 	**Case 3:** The graphs are the same line. In this case, the equations are said to be **dependent**, since any solution of one equation is also a solution of the other. There are infinitely many solutions. 

More than two variables?

$$\begin{cases} x + 2y + 3z = 15 \\ 2x + y - z = 7 \\ x - y + z = 10 \end{cases}$$

What is a matrix?

- In mathematics, a matrix is a rectangular array of numbers enclosed by brackets.
- The plural is matrices.
- Each number in the array is an element or entry.
- Matrices are described by the number of rows and columns. This is a 2x3 matrix:

$$\begin{bmatrix} 2 & 0 & 7 \\ 1 & 2 & 5 \end{bmatrix}$$

Matrices can be used to represent systems of linear equations. For example,

$3x + y = 5$
$x + 2y = 7$

This matrix is called an **augmented matrix**. We use a vertical bar to separate the constants in the last column from the coefficients of the variables.

Since any algebraic operation done to the equation of a linear system is equivalent to performing the same operation on the rows of the augmented matrix, we summarize the row operations could be used below.

Elementary Row Operators

For any augmented matrix A of a system of equations, the following operations produce the augmented matrix of an equivalent system:

1. Exchange two rows of A. (Denoted by $R_i \leftrightarrow R_j$)
2. Multiply a row of A by a constant $c \neq 0$. (Denoted by $cR_i \to R_i$)
3. Add a multiple of one row of A to another row. (Denoted by $aR_i + R_j \to R_j$)

Examples of Row Operations

$$\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right]$$

Interchanging two rows

Multiplying the elements of a row by any nonzero real number.

Adding a nonzero multiple of the elements of one row to the corresponding elements of a nonzero multiple of another row

The Gauss-Jordan Method

Goal: Use the elementary row operations to transform a system of equations into one from which the solutions can be immediately read.

Step 1: Put the equations in the proper form with the variables on the left and the constants on the right of the equals sign.

Step 2: Write the system as an augmented matrix.

Step 3: Use the elementary row operations on the augmented matrix to transform the matrix so that it has zeros above and below the diagonal and 1's on the diagonal to the left of the vertical bar.

Step 4: Once this is done, the solution can be read from the matrix.

Example 2: Use the Gauss-Jordan Method to solve the linear system in Example 1.

$$3x + y = 5$$
$$x + 2y = 7$$

Solution:

Example 3: Use the Gauss-Jordan Method to solve the linear system:

$$4x + 5y = 10$$
$$7x + 8y = 19$$

Solution:

Example 4: Use the Gauss-Jordan Method to solve the linear system:

$$x + 2y + 3z = 2$$
$$2x + 2y - 3z = 27$$
$$3x + 2y + 5z = 10$$

Solution:

How about no unique solution?

Example 5: Use the Gauss-Jordan Method to solve the linear system:

$$2x - 2y + 3z - 4w = 6$$
$$3x + 2y + 5z - 3w = 7$$
$$4x + y + 2z - 2w = 8$$

Solution:

$$\left[\begin{array}{cccc|c} 2 & -2 & 3 & -4 & 6 \\ 3 & 2 & 5 & -3 & 7 \\ 4 & 1 & 2 & -2 & 8 \end{array}\right] \Rightarrow \cdots \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & \frac{17}{9} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{4}{9} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{9} \end{array}\right]$$