Section 10.4 Matrix Inverses

Learning Objective: compute the inverse of a matrix, be able to solve a system of linear equations using the inverse of a matrix

The Identity Property for Matrix Multiplication
For any matrix A,

$$AI = A \text{ and } IA = A$$

where, in each case, I is the identity matrix of appropriate dimensions.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a\cdot 1 + b\cdot 0 & a\cdot 0 + b\cdot 1 \\ c\cdot 1 + d\cdot 0 & c\cdot 0 + d\cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1\cdot a + 0\cdot c & 1\cdot b + 0\cdot d \\ 0\cdot a + 1\cdot c & 0\cdot b + 1\cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

Thus, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix of any 2×2 matrix A. We denote it by $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Similarly, the $n \times n$ identity matrix for any positive integer n is

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Non-Singular Matrices and Matrix Inversion

Definition: (invertible matrix)
Let A be a square matrix. A is **invertible** (or **non-singular**) if there is a matrix B such that

$$AB = I \text{ and } BA = I$$

where I is the identity matrix. B is called the inverse (matrix) of A. (Notation: $B = A^{-1}$) If no inverse exists, A is non-invertible (or singular).

Note: A^{-1} doesn't mean $\frac{1}{A}$.

Example 1: Find the inverse of the matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution:

Finding a Multiplicative Inverse Matrix

To obtain A^{-1} for any $n \times n$ matrix A for which A^{-1} exists, follow these steps.

1. Form the augmented matrix $[A|I]$, where I is the $n \times n$ identity matrix.
2. Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$, if this is possible.
3. Matrix B is A^{-1}.

Example 2: Determine whether the given matrix is invertible, and if it is, find its inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

Solution:

A quick way to find the inverse of a 2×2 matrix:

When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example: The following matrix does not have an inverse.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Why do we need a matrix inverse?

Solving a System $AX = B$ Using Matrix Inverses
To solve a system of equations $AX = B$, where A is the square matrix of coefficients and A^{-1} exists, X is the matrix of variables, and B is the matrix of constants, first find A^{-1}. Then $X = A^{-1}B$.

Example 3: Use the inverse of the coefficient matrix to solve the linear system.

$$\begin{aligned} 5x + 4y &= 23 \\ 4x - 3y &= 6 \end{aligned}$$

Solution:

Remark: If a matrix has no inverse, then a corresponding system of equations will have either no solution or an infinite number of solutions.

For example, we found that this matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible.

Case 1:

$$\begin{gathered} x + 2y = 4 \\ 2x + 4y = 8 \end{gathered}$$

This is a dependent system and has infinitely many solutions.

Case 2:

$$\begin{gathered} x + 2y = 5 \\ 2x + 4y = 8 \end{gathered}$$

This is an inconsistent system and has no solution.