Macroeconomics Homework 1

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Problem 1: Malthusian Model Simulation

(a): Creating Table for the first 15 period

Note that one important assumption here is that all output Y is consumed by the labor force (i.e. $Y_t = C_t$)

Table 1: Malthusian Model Simulation Results

period t	L	N_t	Y_t	C_t	$\frac{C_t}{N_t}$	Birth	Death	N_{t+1}	$\frac{L}{N_{t+1}}$
0	100	50.00	10.10	10.10	0.2020	5.051	2.980	52.07	1.92
1	100	52.07	10.31	10.31	0.1980	5.154	3.145	54.08	1.85
2	100	54.08	10.51	10.51	0.1943	5.253	3.307	56.03	1.78
3	100	56.03	10.69	10.69	0.1909	5.346	3.464	57.91	1.73
4	100	57.91	10.87	10.87	0.1877	5.436	3.617	59.73	1.67
5	100	59.73	11.04	11.04	0.1848	5.520	3.765	61.48	1.63
6	100	61.48	11.20	11.20	0.1822	5.601	3.908	63.18	1.58
7	100	63.18	11.35	11.35	0.1797	5.677	4.047	64.81	1.54
8	100	64.81	11.50	11.50	0.1775	5.750	4.181	66.38	1.51
9	100	66.38	11.64	11.64	0.1753	5.819	4.310	67.89	1.47
10	100	67.89	11.77	11.77	0.1734	5.885	4.434	69.34	1.44
11	100	69.34	11.90	11.90	0.1716	5.948	4.555	70.73	1.41
12	100	70.73	12.01	12.01	0.1699	6.007	4.670	72.07	1.39
13	100	72.07	12.13	12.13	0.1683	6.064	4.781	73.35	1.36
14	100	73.35	12.23	12.23	0.1668	6.117	4.888	74.58	1.34
15	100	74.58	12.34	12.34	0.1654	6.168	4.990	75.76	1.32

(b) Phase Diagram for Population (N_t vs. N_{t+1})

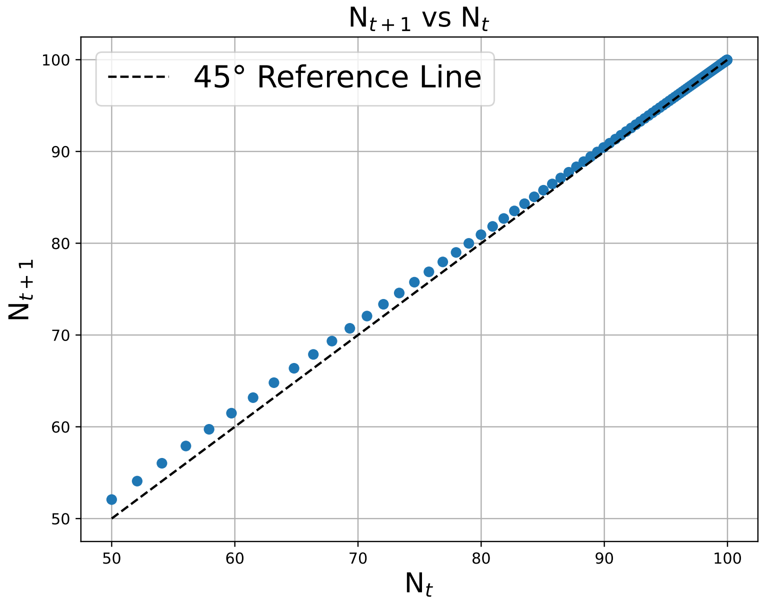

Figure 1: Phase diagram plotting N_{t+1} against N_t with a 45° reference line, illustrating how the population evolves over time under the Malthusian model. Where the curve intersects the 45° line indicates a steady-state population.

(c) Time Paths of N_t and $\frac{C_t}{N_t}$

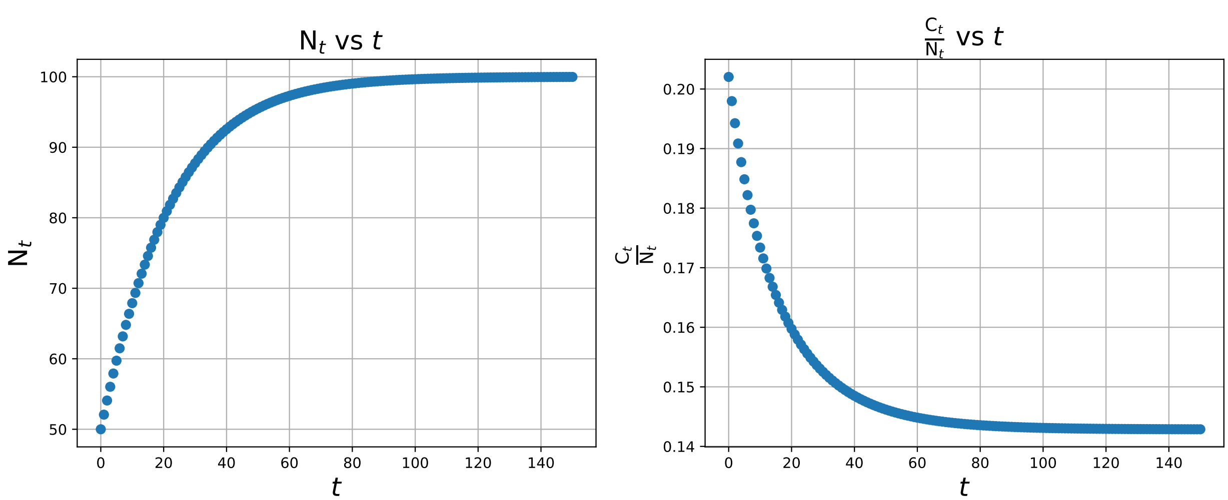

Figure 2: Left panel: The evolution of population N_t over time. Right panel: Consumption per worker $\left(\frac{C_t}{N_t}\right)$ over time. Both demonstrate how population growth and per-worker consumption adjust in the Malthusian setting.

(d) Effect of a Permanent TFP Shock ($z \to z' = 5$ at $t = 80$)

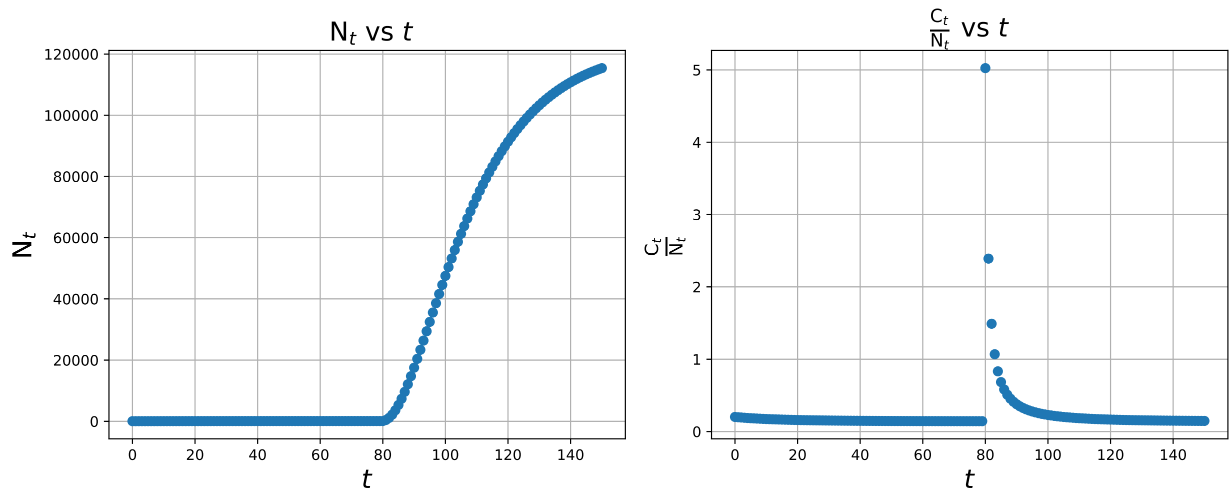

Figure 3: After $t = 80$, total factor productivity (z) permanently increases to 5. The left panel shows how population N_t grows more rapidly over time under higher productivity, while the right panel depicts the resulting path of consumption per worker $\frac{C_t}{N_t}$. The new shock changes the long-run equilibrium in line with Malthusian dynamics (more resources trigger faster population growth).

Problem 2: Solow Model Simulation

(a): Table for the first 15 period

Table 2: Solow Model Simulation Results

period t	K_t	N_t	$\frac{\mathrm{K}_t}{\mathrm{N}_t} = k_t$	Y_t	C_t	S_t	$\frac{\mathrm{C}_t}{\mathrm{N}_t} = c_t$	K_{t+1}	N_{t+1}	N_{t+1}	$\frac{\mathrm{K}_{t+1}}{\mathrm{N}_{t+1}}$
0	1000.00	100.00	10.00	2529.82	1897.37	632.46	18.97	1332.46	120.00	120.00	11.10
1	1332.46	120.00	11.10	3198.95	2399.21	799.74	20.00	1732.46	144.00	144.00	12.03
2	1732.46	144.00	12.03	3995.79	2996.84	998.95	20.80	2211.67	172.80	172.80	12.80
3	2211.67	172.80	12.80	4945.63	3709.22	1236.41	21.45	2784.57	207.36	207.36	13.43
4	2784.57	207.36	13.43	6079.00	4559.25	1519.75	21.99	3468.95	248.83	248.83	13.95
5	3468.95	248.83	13.95	7432.62	5574.46	1858.16	22.40	4286.42	298.60	298.60	14.36
6	4286.42	298.60	14.36	9050.68	6788.01	2262.67	22.76	5263.16	358.32	358.32	14.69
7	5263.16	358.32	14.69	10986.21	8239.66	2746.55	23.00	6430.77	429.98	429.98	14.96
8	6430.77	429.98	14.96	13302.90	9977.17	3325.73	23.21	7827.26	515.98	515.98	15.18
9	7827.26	515.98	15.18	16077.20	12057.90	4019.30	23.38	9498.38	619.17	619.17	15.34
10	9498.38	619.17	15.34	19400.86	14550.64	4850.22	23.51	11499.09	743.01	743.01	15.48
11	11499.09	743.01	15.48	23383.98	17537.98	5846.00	23.61	13895.36	891.61	891.61	15.58
12	13895.36	891.61	15.58	28158.68	21119.01	7039.67	23.69	16766.42	1069.93	1069.93	15.67
13	16766.42	1069.93	15.67	33883.50	25412.63	8470.87	23.74	20207.37	1283.92	1283.92	15.74
14	20207.37	1283.92	15.74	40748.68	30561.51	10187.17	23.79	24332.33	1540.70	1540.70	15.79
15	24332.33	1540.70	15.79	48982.52	36736.89	12245.63	23.86	29278.26	1848.84	1848.84	15.83

(b) Phase Diagram (k_t vs. k_{t+1})

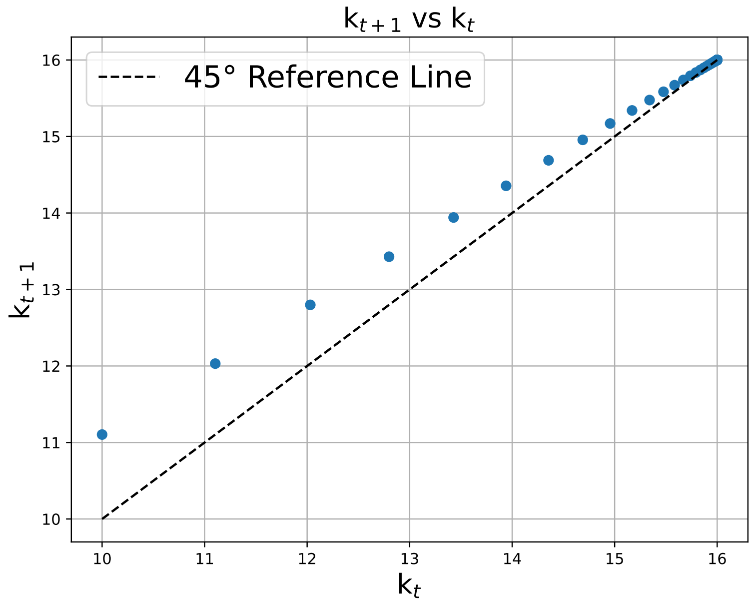

Figure 4: Phase diagram plotting k_{t+1} against k_t with a 45° reference line, illustrating how the economy converges to a steady-state capital per worker.

(c) Time Paths of k_t and c_t

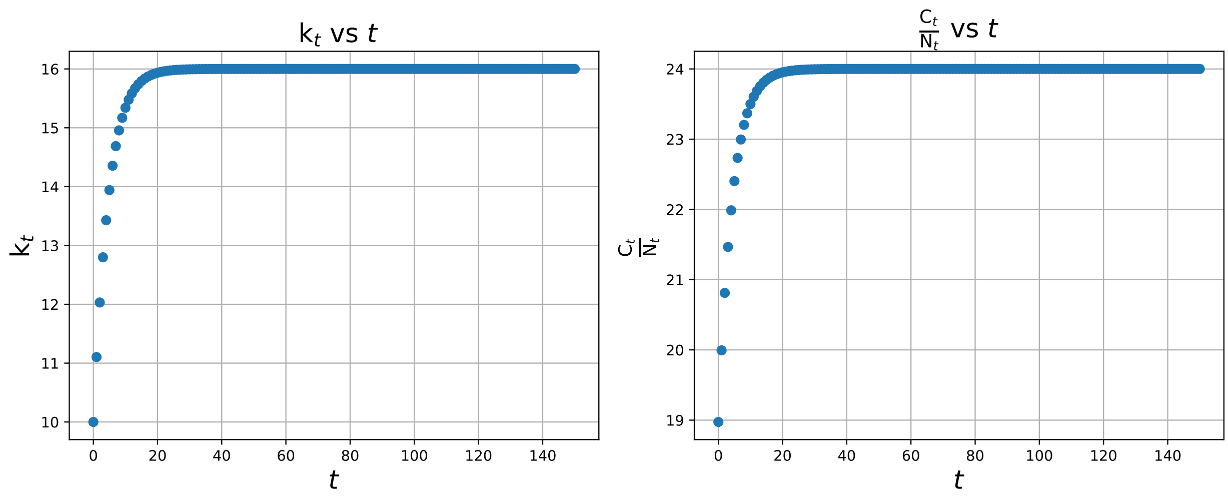

Figure 5: The left panel shows the evolution of k_t (capital per worker) over time, while the right panel shows the evolution of c_t (consumption per worker) over time. Both series converge to their respective steady-state values.

(d) Effect of a Permanent TFP Shock ($z \to z' = 5$ at $t = 80$)

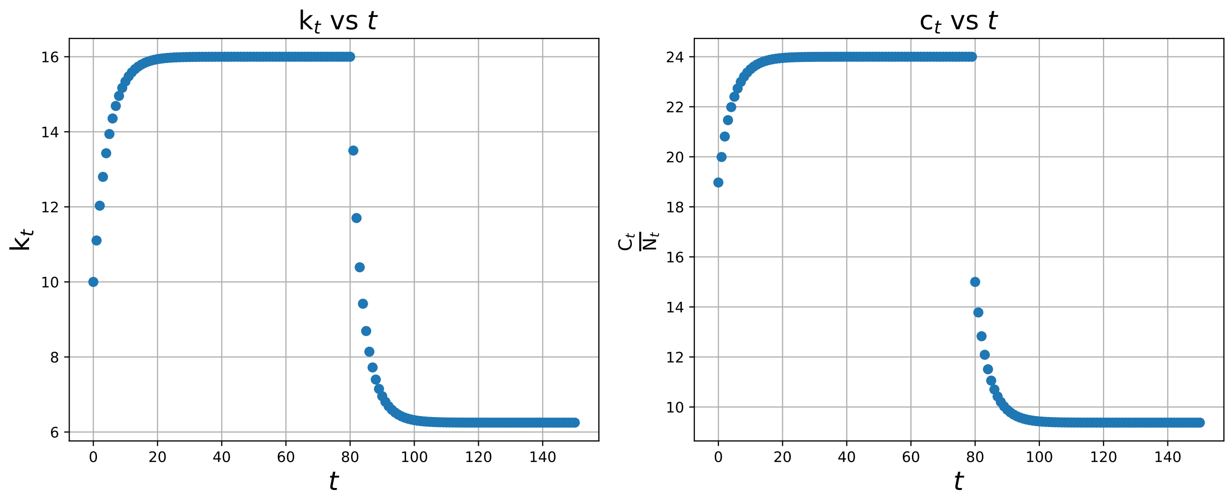

Figure 6: Impact of a permanent increase in TFP at $t = 80$. The figure re-plots k_t and c_t to show how the economy transitions to a new, lower steady state.

(e) Effect of a Permanent Increase in Savings Rate ($s \to s' = 0.5$ at $t = 80$)

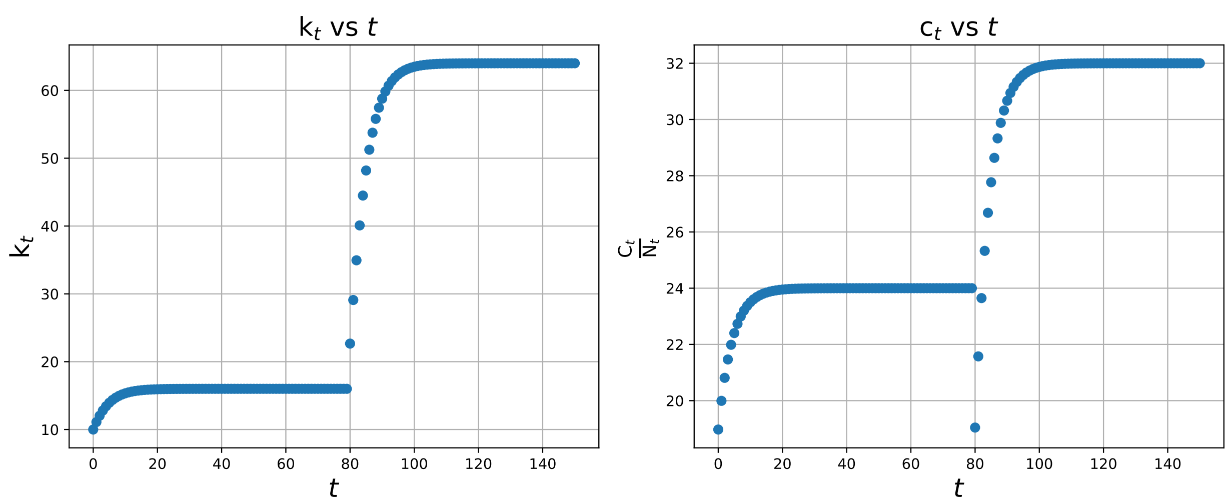

Figure 7: Illustration of a sudden increase in the savings rate s from 0.25 to 0.5 at $t = 80$. As savings rises, the capital stock k_t grows faster, causing c_t also to increase.

Problem 3: Hybrid Model

(a): Table for the first 15 period

Table 3: Hybird Model Simulation Results

period t	K_t	N_t	$\frac{K_t}{N_t} = k_t$	Y_t	C_t	S_t	$\frac{C_t}{N_t} = c_t$	$\frac{S_t}{N_t}$	Net Population growth	K_{t+1}	N_{t+1}	$\frac{K_{t+1}}{N_{t+1}}$
0	1000.00	100.00	10.00	2529.82	1897.37	632.46	18.97	6.32	0.00	1332.46	1038.68	1.28
1	1332.46	1038.68	1.28	9411.48	7058.61	2352.87	6.79	2.27	938.68	3285.59	4464.12	0.74
2	3285.59	4464.12	0.74	30638.29	22978.72	7659.57	5.15	1.72	4364.12	9959.48	15507.07	0.64
3	9959.48	15507.07	0.64	99419.88	74564.91	24854.97	4.81	1.60	15407.07	31826.61	51238.82	0.62
4	31826.61	51238.82	0.62	323061.14	242295.86	80765.29	4.73	1.58	51138.82	103043.91	167262.86	0.62
5	103043.91	167262.86	0.62	1050269.91	787702.43	262567.48	4.71	1.57	167162.86	334698.22	544387.79	0.61
6	334698.22	544387.79	0.61	3414844.04	2561133.03	853711.01	4.71	1.57	544287.79	1087999.76	1770515.53	0.61
7	1087999.76	1770515.53	0.61	1.110e+07	8.328e+06	2.776e+06	4.70	1.57	1770415.53	3537438.79	5757222.40	0.61
8	3537438.79	5757222.40	0.61	3.610e+07	2.707e+07	9.026e+06	4.71	1.57	5757122.40	1.150e+07	1.8720e+07	0.61
9	1.150e+07	1.8720e+07	0.61	1.1739e+08	8.8042e+07	2.9347e+07	4.70	1.57	1.8720e+07	3.7399e+07	6.0869e+07	0.61
10	3.7399e+07	6.0869e+07	0.61	3.8169e+08	2.8627e+08	9.5424e+07	4.70	1.57	6.0869e+07	1.2160e+08	1.9792e+08	0.61
11	1.2160e+08	1.9792e+08	0.61	1.2411e+09	9.3082e+08	3.1027e+08	4.70	1.57	1.9792e+08	3.9539e+08	6.4354e+08	0.61
12	3.9539e+08	6.4354e+08	0.61	4.0354e+09	3.0266e+09	1.0089e+09	4.70	1.57	6.4354e+08	1.2856e+09	2.0925e+09	0.61
13	1.2856e+09	2.0925e+09	0.61	1.3121e+10	9.8410e+09	3.2803e+09	4.70	1.57	2.0925e+09	4.1803e+09	6.8037e+09	0.61
14	4.1803e+09	6.8037e+09	0.61	4.2665e+10	3.1998e+10	1.0666e+10	4.70	1.57	6.8036e+09	1.3592e+10	2.2123e+10	0.61
15	1.3592e+10	2.2123e+10	0.61	1.3873e+11	1.0404e+11	3.4681e+10	4.70	1.57	2.2123e+10	4.4196e+10	7.1932e+10	0.61

(b) Time Paths of k_t, c_t, and N_t

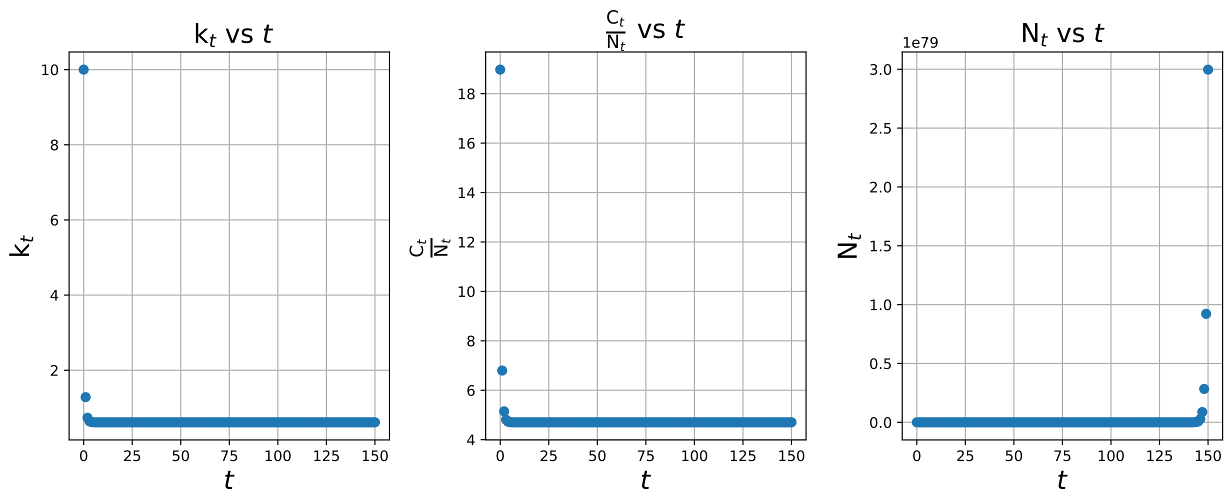

Figure 8: In the hybrid model, population growth is endogenous. Plots show how k_t, c_t, and N_t evolve over time and tend toward a particular growth path.

(c) Effect of a Permanent TFP Shock ($z \to z' = 5$ at $t = 80$)

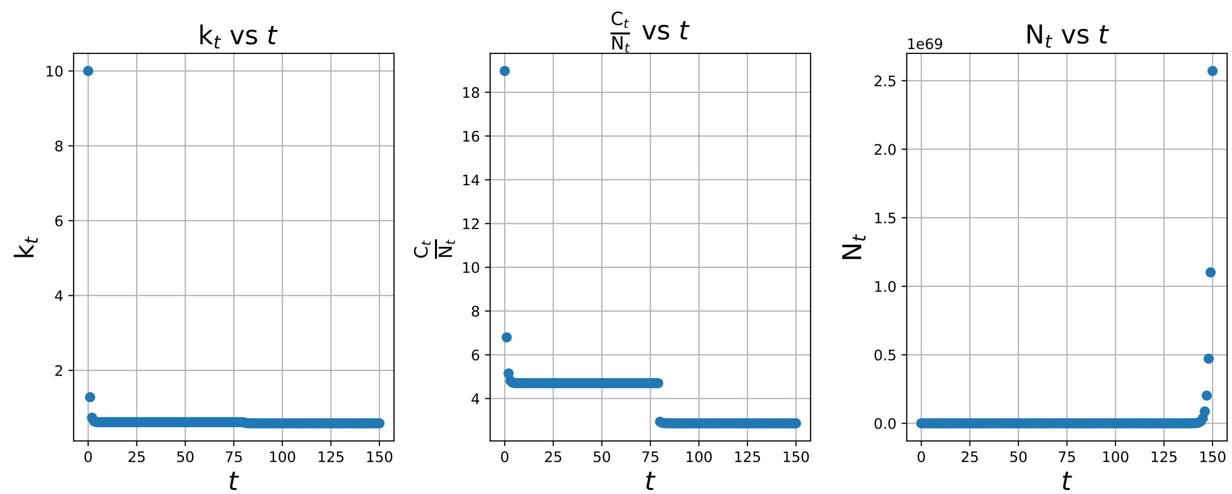

Figure 9: Impact of a permanent increase in TFP at $t = 80$ on k_t, c_t, and N_t in the hybrid model. Lower productivity decreases output, alters the saving/population dynamics, and shifts the paths of all three variables.

(d) Effect of a Permanent Increase in Savings Rate ($s \to s' = 0.5$ at $t = 80$)

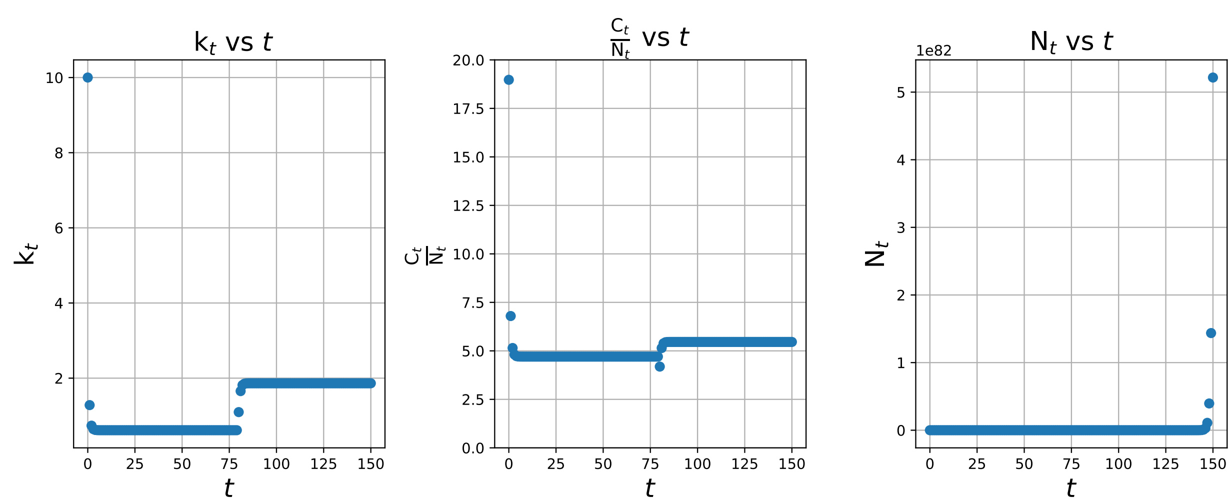

Figure 10: The saving rate s increases from 0.25 to 0.5 at $t = 80$ in the hybrid model. More saving raises investment, causing k_t to grow more quickly and eventually shift upward, which also changes population growth and consumption dynamics over time.

Problem 4: Hand Calculation of TFP Solow Model

(a): Law of motion of capital per effective worker

Starting from the law of motion of captial:

$$\mathrm{K}_{t+1} = (1-\delta)\mathrm{K}_t + s\mathrm{Y}_t \tag{1}$$

Then, we divided by the effective worker which is,

$$\mathrm{z}_{t+1}\mathrm{N}_{t+1} \tag{2}$$

We have,

$$\tilde{\mathrm{k}}_{t+1} = (1-\delta)\frac{\mathrm{K}_t}{\mathrm{z}_{t+1}\mathrm{N}_{t+1}} + s\frac{\mathrm{Y}_t}{\mathrm{z}_{t+1}\mathrm{N}_{t+1}} \tag{3}$$

Looking at eq.3, we can simplify with the fact that production function is $\mathrm{Y}_t = \mathrm{K}_t^{\alpha}(\mathrm{z}_t\mathrm{N}_t^{1-\alpha})$. And $\frac{\mathrm{Y}_t}{\mathrm{z}_{t+1}\mathrm{N}_{t+1}} = \frac{\tilde{\mathrm{k}}_t^{\alpha}}{(1+g)(1+n)}$. Now, we have law of motion of capital per effective worker :

$$\tilde{\mathrm{k}}_{t+1} = \frac{(1-\delta)\tilde{\mathrm{k}}_t + s\tilde{\mathrm{k}}_t^{\alpha}}{(1+g)(1+n)} \Rightarrow \tilde{\mathrm{k}}_{t+1}(1+g)(1+n) = (1-\delta)\tilde{\mathrm{k}}_t + s\tilde{\mathrm{k}}_t^{\alpha} \tag{4}$$

(b): Golden-rule saving rate to maximize $\tilde{\mathrm{c}}_t$

Recall that the golden-rule saving rate suggest the saving rate that maximize steady-state consumption per effective worker.
Now we write the law of motion for consumption per effective worker for steady-state as follow,

$$\tilde{\mathrm{k}}_{t+1}^{\star}(1+g)(1+n) = (1-\delta)\tilde{\mathrm{k}}_t^{\star} + s\tilde{\mathrm{k}}_t^{\star\alpha} \tag{5}$$

Note that the steady-state for capital per effective worker is as follow:

$$\tilde{\mathrm{k}}_{t+1}^{\star} = \tilde{\mathrm{k}}_t^{\star} = \tilde{\mathrm{k}}^{\star} \tag{6}$$

So we can rewrite the equation as follow:

$$\tilde{\mathrm{k}}^{\star}(1+g)(1+n) = (1-\delta)\tilde{\mathrm{k}}^{\star} + s\tilde{\mathrm{k}}^{\star\alpha} \Rightarrow (g+n+ng+\delta)\tilde{\mathrm{k}}^{\star} = s\tilde{\mathrm{k}}^{\star\alpha} \tag{7}$$

$$\tilde{\mathrm{c}}^{\star} = \tilde{\mathrm{y}}^{\star} - savings = \tilde{\mathrm{y}}^{\star} - s\tilde{\mathrm{k}}_t^{\star\alpha} = \tilde{\mathrm{y}}^{\star} - (g+n+ng+\delta)\tilde{\mathrm{k}}^{\star} \tag{8}$$

And from maximized condition mentioned in class(equation 5, solow ppt page 23/36) the consumption is maximized when,

$$\tilde{\mathrm{y}}'^{\star} = (g+n+ng+\delta) \tag{9}$$

And because $\mathrm{y}_t^{\star} = \frac{\mathrm{Y}_t}{\mathrm{z}_t\mathrm{N}_t} = \tilde{\mathrm{k}}_t^{\alpha}$, we can rewrite the equation as follow:

$$\tilde{\mathrm{y}}'^{\star} = \alpha\tilde{\mathrm{k}}^{\star(\alpha-1)} = g+n+ng+\delta \tag{10}$$

Then $\tilde{k}^{\star} = (\frac{\alpha}{n+g+ng+\delta})^{\frac{1}{1-\alpha}}$ and by eq.10 we have,

$$s_{GR} = \alpha$$

(c): Growth rate of key variables in the steady state

(i): Growth rate of $\tilde{y}_t$ and $\tilde{k}_t$

In steady-state output per effective worker is constant becasue in steady-state captial per effective worker is constant(eq.7), so we have:

$$\frac{Y_{t+1}}{z_{t+1}N_{t+1}} = \frac{Y_t}{z_t N_t} = \tilde{k}^{\star\alpha} = constant \tag{11}$$

So growth rate of $\tilde{y}_t$ and $\tilde{k}_t = 0$

(ii): Growth rate of y_t and k_t

Growth rate of y_t and k_t is define as follow(note that $k_{t+1} = k_t = constant$ in steady-state):

$$\frac{y_{t+1}}{y_t} - 1 = \frac{z_{t+1}k_{t+1}^{\alpha}}{z_t k_t^{\alpha}} - 1 = 1 + g - 1 = g \tag{12}$$

(ii): Growth rate of K_t and Y_t

Growth rate of Y_t and K_t is define as follow:

$$\frac{Y_{t+1}}{Y_t} - 1 = \frac{y_{t+1}N_{t+1}}{y_t N_t} - 1 = (1+g)(1+n) - 1 = n + g + ng \tag{13}$$

(d): Market clearing condition with government

Recall before the assumption of a government present the market clearing conditions is:

$$Y_t = C_t + I_t \tag{14}$$

Now with the present of a government, we have:

$$Y_t = C_t + I_t + G_t \tag{15}$$

Where G_t is the government spending. And we can rewrite the equation as follow:

$$Y_t = C_t + I_t + G_t = C_t + I_t + \tau Y_t \tag{16}$$

(e): Law of motion of capital per effective worker with government

The law of motion of capital per effective worker with government is similar to the one without government, but we need to minus the government spending in the term of investments. So that's law of motion of captial is:

$$K_{t+1} = (1-\delta)K_t + s(1-\tau)Y_t \tag{17}$$

Then we divided by the effective worker which is,

$$z_{t+1}N_{t+1} \tag{18}$$

We have,

$$\tilde{\mathrm{k}}_{t+1} = (1-\delta)\frac{\mathrm{K}_t}{\mathrm{z}_{t+1}\mathrm{N}_{t+1}} + s(1-\tau)\frac{\mathrm{Y}_t}{\mathrm{z}_{t+1}\mathrm{N}_{t+1}} \tag{19}$$

Can from the result we got in eq.4, we can rewrite the equation as follow:

$$\tilde{\mathrm{k}}_{t+1} = \frac{(1-\delta)\tilde{\mathrm{k}}_t + s(1-\tau)\tilde{\mathrm{k}}_t^{\alpha}}{(1+g)(1+n)} \Rightarrow \tilde{\mathrm{k}}_{t+1}(1+g)(1+n) = (1-\delta)\tilde{\mathrm{k}}_t + s(1-\tau)\tilde{\mathrm{k}}_t^{\alpha} \tag{20}$$

Then we are done with the law of motion of capital per effective worker with government.

$\star$Note that if ng is small, we can ignore the ng term in all the equations above.

(f): Simulation of the model with government

(i): Approxmate steady state

we solve the approxmate steady state by using the following equation:

$$\tilde{\mathrm{k}}^{\star} = \frac{s(1-\tau)}{n+g+\delta}\tilde{\mathrm{k}}^{\star\alpha} \Rightarrow \tilde{\mathrm{k}}^{\star} = (\frac{s(1-\tau)}{n+g+\delta})^{\frac{1}{1-\alpha}} \tag{21}$$

With the value given. We can get:

$$\tilde{k}_t^{\star} \approx 0.13 \tag{22}$$

(ii) Time Path of $\widetilde{k}_t$

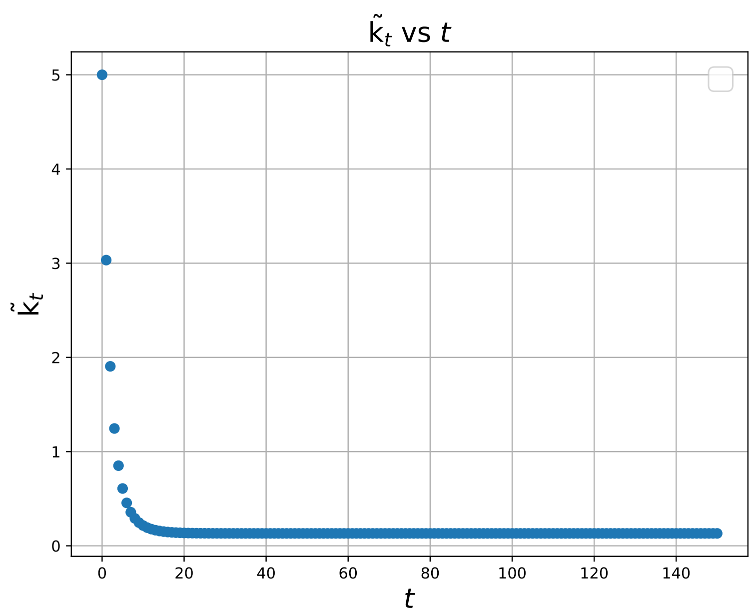

Figure 11: The evolution of capital per effective worker, $\widetilde{k}_t$, over time. As t increases, $\widetilde{k}_t$ converges to a steady-state level determined by the chosen parameters $(s, \alpha, \delta, n, g, \tau)$.

(iii) Time Path of $\widetilde{c}_t$

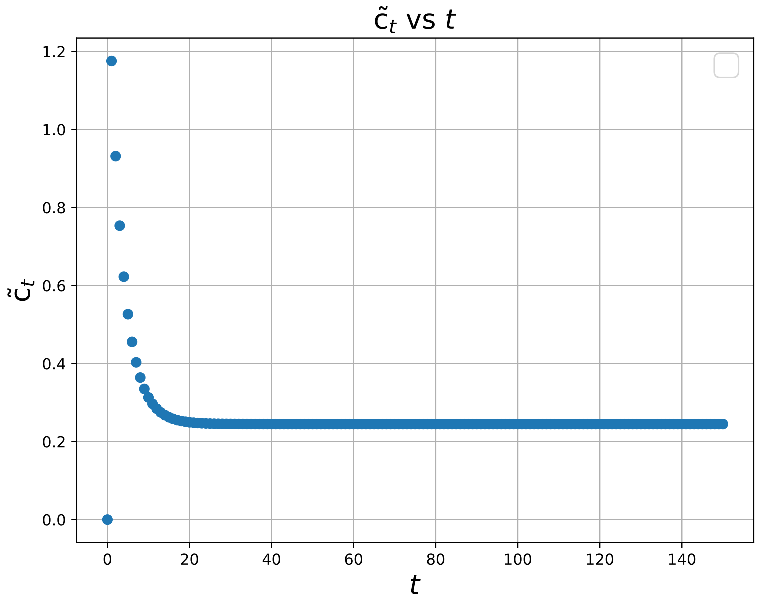

Figure 12: The evolution of consumption per effective worker, $\widetilde{c}_t$, over time. As the economy accumulates capital and productivity grows, $\widetilde{c}_t$ converges to its long-run path in tandem with $\widetilde{k}_t$.

(g) Effect of Varying τ

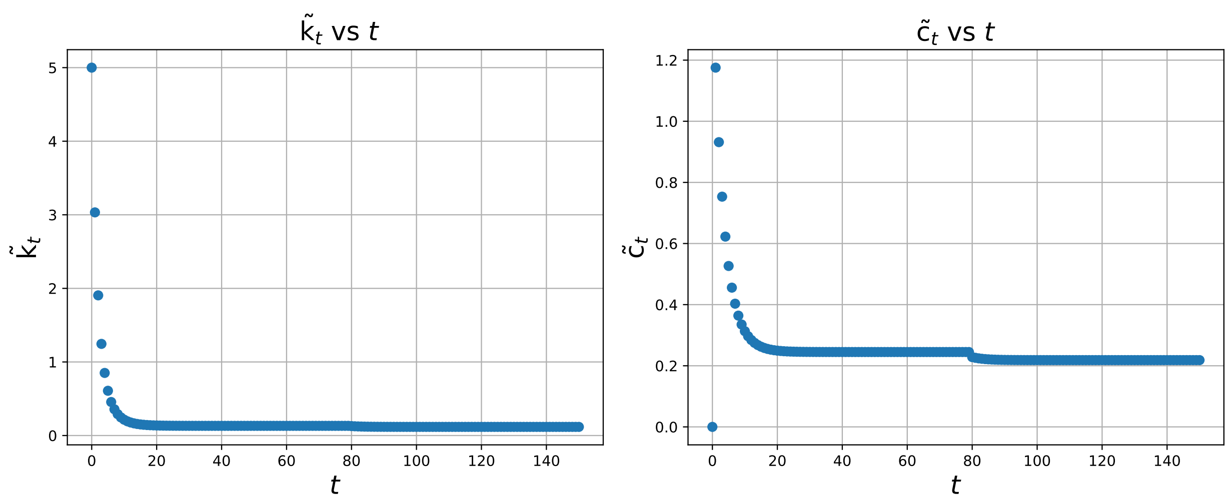

Figure 13: Comparison of $\widetilde{k}_t$ (left) and $\widetilde{c}_t$ (right) under different values of the tax rate τ. Higher taxes reduce the resources available for private saving, shifting the steady-state levels of both capital per effective worker and consumption per effective worker.