VARIATION ON A VARIATIONAL PRINCIPLE

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Abstract. We prove that alternating links in the thickened torus remain hyperbolic after certain augmentations.

1. Introduction

Alice TODO

Definitions, Notations, and Conventions

Always assume surface Σ (our case $\Sigma = \mathbb{T}^2$) is oriented.

define cellular decomp of surface

V, E, F are set of vertices, edges, faces.

$\vec{E}$ is the set of oriented edges. We may identify an oriented edge $\vec{e}$ with the pair $(f_{\vec{e}}, e)$, where $f_{\vec{e}}$ is the face to the left of $\vec{e}$.

Recall circle pattern.

A circle pattern is determined by the radius of the circle C_f associated to each face, r_f, and the angle that each edge subtends in adjacent faces, $\varphi_{\vec{e}}$. (see figure TODO) Thus determines, and is determined by a point in $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, where

- $\underline{\mathfrak{R}} := \mathbb{R}_+^F = \{(r_f)_{f \in F} | r_f \in \mathbb{R}_+\}$
- $\underline{\mathfrak{Q}} := \mathbb{R}^{\vec{E}} = \{(\varphi_{\vec{e}})_{\vec{e} \in \vec{E}} | \varphi_{\vec{e}} \in \mathbb{R}\}$

but clearly not every point $c \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ determines a circle pattern.

On $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, there are several functions to consider:

- $\Phi_f = 2 \sum_{\vec{e} \in \partial f} \varphi_{\vec{e}}$, measuring the cone angle at the center of C_f
- $\theta_e = \pi - \varphi_{\vec{e}} - \varphi_{\overleftarrow{e}}$
- $l_{\vec{e}} = r_{f_{\vec{e}}} \sin \varphi_{\vec{e}}$

These fit together to give maps to the following spaces:

- $\underline{\boldsymbol{\Phi}} := \mathbb{R}^F = \{(\Phi_f)_{f \in F} | \Phi_f \in \mathbb{R}\}$
- $\underline{\boldsymbol{\Theta}} := \mathbb{R}^E = \{(\theta_e)_{e \in E} | \theta_e \in \mathbb{R}\}$
- $\underline{\mathfrak{L}} := \mathbb{R}^{\vec{E}} = \{(l_{\vec{e}})_{\vec{e} \in E} | l_{\vec{e}} \in \mathbb{R}\}$

Our main argument is to deform "degenerate" circle patterns, where adjacent circles may be identical or tangent, into ones that don't look so degenerate, hence it is conveneint to extend the notion of circle pattern:

Definition 1.1. An *extended circle pattern* is $c = ((r_f), (\varphi_{\vec{e}})) \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ such that $l_{\vec{e}} = l_{\overleftarrow{e}}$ for all edges $e \in E$. We denote by $\underline{\mathfrak{C}}$ the set of extended circle patterns.

$$\underline{\mathfrak{C}} = \{l_{\vec{e}} = l_{\overleftarrow{e}}\} \quad \subseteq \quad \begin{tikzcd} \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}} \arrow[r, "\Theta"] \arrow[d, "\Phi"] & \underline{\boldsymbol{\Theta}} \\ \underline{\boldsymbol{\Phi}} & \end{tikzcd}$$

[Check this] The usual notion of circle pattern would be restricted to those $c \in \underline{\mathfrak{C}}$ with $\varphi_{\vec{e}} \in (0, \pi)$ and $\theta_e \in (0, \pi)$. One may consider deforming a circle pattern so as to have some θ_e approach 0 or π. The limit $\theta_e \to \pi$ is easy to picture, one simply gets that the two circles $C_f, C_{f'}$ of the adjacent faces become tangent. The limit $\theta_e \to 0$ is a bit more complex, as the final shape of Q_e (the quadrangle associated to e) depends also on $\varphi_{\vec{e}}$ (which equals $\pi - \varphi_{\overleftarrow{e}}$ in the limit. If we parametrize circle patterns by (r_f) and (θ_e) as in TODO [?BS], the limiting shape would depend on how r_f approaches $r_{f'}$, some sort of blow up stuff TODO.

We will mostly be working with extended circle patterns that mostly "look normal", with all $\varphi_{\vec{e}}$ in some range $(-\varepsilon, \pi + \varepsilon)$.

Definition 1.2. An extended circle pattern is said to be *face non-singular* if $\Phi_f = 2\pi$ for all faces f; it is said to be *vertex non-singular* if $\sum_{e \ni v} \theta_e = 2\pi$ for all vertices v. Finally it is said to be *non-singular* if it is both.

non-degenerate circle pattern?

Definition 1.3. Given some extended circle pattern $c \in \underline{\mathfrak{C}}$, we say a face f is *convex* if for all edges $\vec{e} \in \partial f$, we have $\varphi_{\vec{e}} \in [0, \pi/2)$. We say f is *thin* if exactly two edges $\vec{e}, \vec{e'} \in \partial f$ have nonzero $\varphi_\bullet$, and furtheremore, $0 < \varphi_{\vec{e}} = \pi - \varphi_{\vec{e'}} < \pi/2$.

TODO figures for convex, thin faces

Definition 1.4. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, an edge e is *short* if it has length 0, $l_{\vec{e}} = l_{\overleftarrow{e}} = 0$; it is *long* otherwise.

Definition 1.5. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, a *thick path* is a sequence of faces $f_0, f_1, \ldots, f_n$ such that f_i, f_{i+1} share a long edge. A *thick cycle* is a thick path with $f_0 = f_n$.

Definition 1.6. A *slit-convex circle pattern* is an extended cirlce pattern $c \in \underline{\mathfrak{C}}$ such that all its faces are either convex or thin.

Lemma 1.7. *$\underline{\mathfrak{C}}$ is a manifold near slit-convex circle patterns.*

Proof. $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = \sin\varphi_{\vec{e}} dr_{f_{\vec{e}}} + r_{f_{\vec{e}}} \cos\varphi_{\vec{e}} d\varphi_{\vec{e}} - \sin\varphi_{\overleftarrow{e}} dr_{f_{\overleftarrow{e}}} - r_{f_{\overleftarrow{e}}} \cos\varphi_{\overleftarrow{e}} d\varphi_{\overleftarrow{e}}$, and since having only convex of thin faces implies $\cos\varphi_{\vec{e}} \neq 0$, we see the differentials are linearly independent at a slit-convex circle pattern c. ☐

2. Main ?? results

Our general strategy for obtaining a (non-singular) circle pattern is to start with an assignment $\underline{\theta} = (\theta_\bullet)$ for which we know an extended circle pattern exists (say from results of [?BS]), and a path γ in $\underline{\boldsymbol{\Theta}}$ starting at $\underline{\theta}$. We then attempt to lift γ to a path $\tilde{\gamma}$ in $\underline{\mathfrak{C}}$, so that $\tilde{\gamma}$ remains (face) non-singular (vertex non-singularity is already determined by γ).

Note that since $\sum_{e \in E} \theta_e = 2\pi|E| - \sum_{\vec{e} \in \vec{E}} \varphi_{\vec{e}} = 2\pi|E| - \sum_{f \in F} \Phi_f$, we see that maintaining face non-singularity of $\tilde{\gamma}$ forces $\sum \theta_e$ to be constant. We will show that this is the only obstruction on γ to the lifting to such $\tilde{\gamma}$.

To that end, let L be the $(|E|-1)$-plane distribution on $\underline{\boldsymbol{\Theta}}$ tangent to the level sets of $\sum \theta_v$. The following proposition proves it (avoid using 'it'?) up to first order at a point:

Proposition 2.1. *Let $c \in \underline{\mathfrak{C}}$ be an extended circle pattern such that*

- *$\Phi_f = 2\pi$ for all $f \in F$;*
- *all faces are eithe convex or thin, with at least one convex face;*
- *every pair of faces is connected by a thick path.*

Let $K_c = \ker(\Phi_*|_c : T_c\underline{\mathfrak{C}} \to T_{\Phi(c)}\underline{\boldsymbol{\Phi}})$. Then $\Theta_*(K_c) = L_{\Theta(c)}$.

In other words, for any vector $b = b^e \partial_{\theta_e}$ with sum of coefficients 0, one can vary c so that its first order change in $\theta_\bullet$ is b, and also remains face non-singular up to first order.

Before we prove this, it is convenient to first prove the following:

Lemma 2.2. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Then Φ is a submersion in a neighbourhood of c.*

Proof. We construct vectors $\beta_f \in T_c\underline{\mathfrak{C}}$ so that $\Phi_*(\beta_f) = \frac{\partial}{\partial \Phi_f}$. The vector

$$\alpha_f := \frac{\partial}{\partial r_f} - \sum_{\vec{e} \in \partial f} \frac{\tan\varphi_{\vec{e}}}{r_f} \frac{\partial}{\partial \varphi_{\vec{e}}} \in T(\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}) \tag{2.1}$$

doesn't change $l_{\vec{e}}$, so is in $T_c\underline{\mathfrak{C}}$. Intuitively, α_f is like pulling the center of C_f up off the plane, increasing r_f and decreasing all φ's. Its pushforward under Φ is simply $\Phi_*(\beta_f) = \frac{1}{r_f} \sum_{\vec{e} \in \partial f} \tan\varphi_{\vec{e}} \frac{\partial}{\partial \Phi_f}$.

If f is a convex face, then the $\tan\varphi_{\vec{e}}$ are all non-negative, being 0 if and only if e is short. so $\Phi_*(\alpha_f)$ is a negative multiple of $\frac{\partial}{\partial\Phi_f}$; we choose β so that

$$\beta_f := \beta \cdot \alpha_f; \quad \Phi_*(\beta_f) = \frac{\partial}{\partial\Phi_f} \tag{2.2}$$

If f is thin, the coefficients sum to 0 so that $\Phi_*(\beta_f) = 0$, so we need a different approach. First suppose f shares an edge e with a convex face f'. We can increase l_e by increasing $\varphi_{\vec{e}}, \varphi_{\overleftarrow{e}}$ while holding $r_f, r_{f'}$ constant. This will affect both $\Phi_f, \Phi_{f'}$, so we use $\beta_{f'}$ to make $\Phi_{f'}$ constant. More explicitly, consider

$$\alpha_e = \frac{1}{2r_f\cos\varphi_{\vec{e}}}\frac{\partial}{\partial\varphi_{\vec{e}}} + \frac{1}{2r_{f'}\cos\varphi_{\overleftarrow{e}}}\frac{\partial}{\partial\varphi_{\overleftarrow{e}}} \tag{2.3}$$

Then we can take

$$\beta_f = r_f\cos\varphi_{\vec{e}}(\alpha_e - \frac{1}{r_{f'}\cos\varphi_{\overleftarrow{e}}}\beta_{f'}) \tag{2.4}$$

which increases $l_{\vec{e}}, l_{\overleftarrow{e}}$ at unit speed, $dl_{\vec{e}}(\alpha_e) = dl_{\overleftarrow{e}}(\alpha_e) = 1$. We can repeat this procedure with f' set to this thin face, and f set to another thin face adjacent to it, etc. □

Proof of Proposition 2.1. We construct a vector $u_e \in K_c \subseteq T_c\underline{\mathfrak{C}}$ for each long edge e such that if $\Theta_*(u_e) = \sum_{e'\in E} a^{e'}\frac{\partial}{\partial\theta_{e'}}$,

- $a^e = 1$;
- $a^{e'} \leq 0$ for all $e' \neq e$;
- $a^{e'} = 0$ for short edges e'.

First suppose that e is between two convex faces f, f' (so $\vec{e} \in \partial f, \overleftarrow{e} \in \partial f'$). Observe that α_e from (2.3) increases Φ_f and $\Phi_{f'}$, and we can compensate using β_f from (2.2), so we consider

$$w_e = \alpha_e - \frac{1}{\cos\varphi_{\vec{e}}}\beta_f - \frac{1}{\cos\varphi_{\overleftarrow{e}}}\beta_{f'} \in K_c$$

For $\vec{e'} \in \partial f \cup \partial f' \backslash \{\vec{e}, \overleftarrow{e}\}$, the $\frac{\partial}{\partial\varphi_{\vec{e'}}}$ component only appears in β_f or $\beta_{f'}$, which is positive by construciton (this requires f, f' to be convex faces); thus, $d\theta_{e'}(w_e) \geq 0$ for such e', and is equal 0 if and only if e' is short. Finally, since $w_e \in K_c$, it leaves the sum $\sum_{e\in E}\theta_e$ constant, so $d\theta_e(w_e) =: \mu < 0$, so we take

$$u_e = \mu^{-1} \cdot w_e \tag{2.5}$$

Intuitively,

$$u_e := \mu \cdot [\alpha_e - \frac{1}{\cos\varphi_{\vec{e}}}\beta_f - \frac{1}{\cos\varphi_{\overleftarrow{e}}}\beta_{f'}]$$

□

Proposition 2.3. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. TODO full lifting statement.*

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