VARIATION ON A VARIATIONAL PRINCIPLE

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Abstract. We prove that alternating links in the thickened torus remain hyperbolic after certain augmentations.

1. Introduction

Alice TODO

Definitions, Notations, and Conventions

Always assume surface Σ (our case $\Sigma = \mathbb{T}^2$) is oriented.

define cellular decomp of surface

V, E, F are set of vertices, edges, faces. No bigon faces.

$\vec{E}$ is the set of oriented edges. We may identify an oriented edge $\vec{e}$ with the pair $(f_{\vec{e}}, e)$, where $f_{\vec{e}}$ is the face to the left of $\vec{e}$.

When we use $\overleftarrow{e}$ to refer to an oriented edge, it refers to the oppositely oriented edge to $\vec{e}$. If we construct an expression with both $\vec{e}$ and $\overleftarrow{e}$, it will always be (anti-)symmetrical in the two orientations, and we assume that an arbitrary choice has been made.

Recall circle pattern.

A circle pattern is determined by the radius of the circle C_f associated to each face, r_f, and the angle that each edge subtends in adjacent faces, $\varphi_{\vec{e}}$. (See Figure 1) Thus determines, and is determined by a point in $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, where

- $\underline{\mathfrak{R}} := \mathbb{R}_+^F = \{(r_f)_{f \in F} | r_f \in \mathbb{R}_+\}$
- $\underline{\mathfrak{Q}} := \mathbb{R}^{\vec{E}} = \{(\varphi_{\vec{e}})_{\vec{e} \in \vec{E}} | \varphi_{\vec{e}} \in \mathbb{R}\}$

but clearly not every point $c \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ determines a circle pattern.

On $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, there are several functions to consider:

- $\Phi_f = 2\sum_{\vec{e} \in \partial f} \varphi_{\vec{e}}$, measuring the cone angle at the center of C_f
- $\theta_e = \pi - \varphi_{\vec{e}} - \varphi_{\overleftarrow{e}}$
- $l_{\vec{e}} = 2 r_{f_{\vec{e}}} \sin \varphi_{\vec{e}}$

These fit together to give maps to the following spaces:

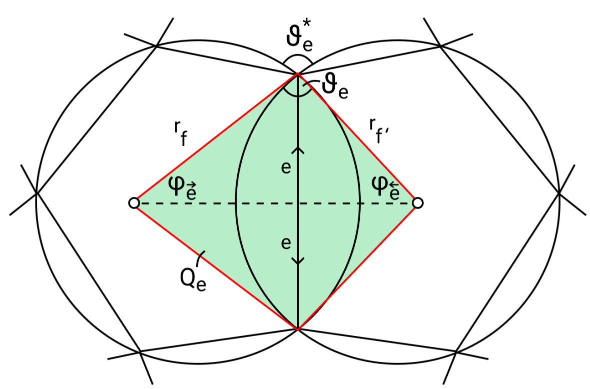

Figure 1. Circle pattern in terms of radii r_f and angles $\varphi_{\vec{e}}$

- $\underline{\boldsymbol{\Phi}} := \mathbb{R}^F = \{(\Phi_f)_{f\in F} | \Phi_f \in \mathbb{R}\}$
- $\underline{\boldsymbol{\Theta}} := \mathbb{R}^E = \{(\theta_e)_{e\in E} | \theta_e \in \mathbb{R}\}$
- $\underline{\mathfrak{L}} := \mathbb{R}^{\vec{E}} = \{(l_{\vec{e}})_{\vec{e}\in E} | l_{\vec{e}} \in \mathbb{R}\}$

Our main argument is to deform "degenerate" circle patterns, where adjacent circles may be identical or tangent, into ones that don't look so degenerate, hence it is conveneint to extend the notion of circle pattern:

Definition 1.1. Let Γ be a graph on the torus. An *extended circle pattern* on Γ is $c = ((r_f), (\varphi_{\vec{e}})) \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ such that $l_{\vec{e}} = l_{\overleftarrow{e}}$ for all edges $e \in E$. We denote by $\underline{\mathfrak{C}}$ the set of extended circle patterns.

$$\begin{tikzcd} \underline{\mathfrak{C}} = \{l_{\vec{e}} = l_{\overleftarrow{e}}\} & \subseteq & \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}} \arrow[r, "\Theta"] \arrow[d, "\Phi"] & \underline{\boldsymbol{\Theta}} \\ & & \underline{\boldsymbol{\Phi}} & \end{tikzcd}$$

The notion of circle pattern that is considered in [?BandS] would be restricted to those $c \in \underline{\mathfrak{C}}$ with $\varphi_{\vec{e}} \in (0, \pi)$ and $\theta_e \in (0, \pi)$. They parametrize circle patterns by (r_f) and (θ_e) (or (θ_e^*)), and call $(\varphi_{\vec{e}})$ a coherent angle system. One may consider deforming a circle pattern so as to have some θ_e approach 0 or π. The limit $\theta_e \to \pi$ is easy to picture, one simply gets that the two circles $C_f, C_{f'}$ of the adjacent faces become tangent. The limit $\theta_e \to 0$ is a bit more complex, as the final shape of Q_e (the quadrangle associated to e) may depend on $\varphi_{\vec{e}}$ or $\varphi_{\overleftarrow{e}}$. If we parametrize circle patterns by (r_f) and (θ_e) as in [?BandS], the limiting shape would depend on the relationship between r_f and $r_{f'}$. As our main result depends on extending [?BS] result to such limits and beyond, we find it more convenient to describe extended circle patterns by (r_f) and $(\varphi_{\vec{e}})$.

We will mostly be working with extended circle patterns that "look normal", with all $\varphi_{\vec{e}}$ in the range $[0, \pi)$, so we do not dwell on the meaning of negative $\varphi_{\vec{e}}$'s. [1]

Definition 1.2. An extended circle pattern is said to be *face non-singular* if $\Phi_f = 2\pi$ for all faces f; it is said to be *vertex non-singular* if $\sum_{e\ni v} \theta_e = 2\pi$ for all vertices v. Finally it is said to be *non-singular* if it is both.

Definition 1.3. Let f be a non-singular face (i.e. $\Phi_f = 2\pi$), such that for all edges $\vec{e} \in \partial f$, $\varphi_{\vec{e}} \in [0, \pi)$. We say f is *thin* if exactly two edges of f have nonzero $\varphi_\bullet$; we say it is *thick* otherwise. We say it is *convex* if for all edges $\vec{e} \in \partial f$, we have $\varphi_{\vec{e}} \in [0, \pi/2)$.

Definition 1.4. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, an edge e is *short* if it has length 0, $l_e = 0$; it is *long* otherwise.

Definition 1.5. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, a *wide path* is a sequence of faces $f_0, f_1, \ldots, f_n$ such that f_i, f_{i+1} share a long edge. A *wide cycle* is a wide path with $f_0 = f_n$.

Lemma 1.6. *Let c be a face non-singular extended circular pattern such that $\varphi_{\vec{e}} \geq 0$ for all $\vec{e} \in \vec{E}$. Furthermore, suppose that there is no cycle of edges $e_0, e_1, \ldots, e_n = e_0$ such that e_i, e_{i+1} belong to the same face, and $\varphi_{\vec{e_i}} = \varphi_{\overleftarrow{e_i}} = \pi/2$. Then $\underline{\mathfrak{C}}$ is a manifold in a neighborhood of c.*

Proof. We need to show that the differentials $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = 2\sin\varphi_{\vec{e}} dr_{f_{\vec{e}}} + 2r_{f_{\vec{e}}} \cos\varphi_{\vec{e}} d\varphi_{\vec{e}} - 2\sin\varphi_{\overleftarrow{e}} dr_{f_{\overleftarrow{e}}} - 2r_{f_{\overleftarrow{e}}} \cos\varphi_{\overleftarrow{e}} d\varphi_{\overleftarrow{e}}$ are linearly independent at c. If $\varphi_{\vec{e}} \neq \pi/2$, then the $d\varphi_{\vec{e}}$ term makes $d(l_{\vec{e}} - l_{\overleftarrow{e}})$ linearly independent from the rest (no other differential $d(l_{\vec{e'}} - l_{\overleftarrow{e'}})$ contains such component); likewise for $\varphi_{\overleftarrow{e}} \neq \pi/2$.

Now consider the set E' of edges e that have $\varphi_{\vec{e}} = \varphi_{\overleftarrow{e}} = \pi/2$. Consider a maximal sequence of faces $f_0, \ldots, f_n$ with no repetitions and f_i, f_{i+1} share an edge $e_i \in E'$. By the condition on no cycles, $f_0 \neq f_n$. By face non-singularity, any face can meet at most two edges from E', thus if $e \in E'$, it is contained in a unique such string of faces/edges. Thus the differentials $d(l_{\vec{e_i}} - l_{\overleftarrow{e_i}}) = 2dr_{f_i} - 2dr_{f_{i+1}}$ are linearly independent. $\square$

[1] Intuitively, one can visualize decreasing $\varphi_{\vec{e}}$ from ε to $-\varepsilon$ as moving the black vertices of the quadrangle Q_e past each other, thus flipping Q_e along its long axis and making it have negative area.

2. Main ?? results

Our general strategy for obtaining a (non-singular) circle pattern is to start with an assignment $\underline{\theta} = (\theta_\bullet)$ for which we know an extended circle pattern exists (say from results of [?BandS]), and a path γ in $\underline{\mathbf{\Theta}}$ starting at $\underline{\theta}$. We then attempt to lift γ to a path $\tilde{\gamma}$ in $\underline{\mathfrak{C}}$, so that $\tilde{\gamma}$ remains (face) non-singular (vertex non-singularity is already determined by γ).

Note that since $\sum_{e \in E} \theta_e = 2\pi|E| - \sum_{\vec{e} \in \vec{E}} \varphi_{\vec{e}} = 2\pi|E| - \sum_{f \in F} \Phi_f$, we see that maintaining face non-singularity of $\tilde{\gamma}$ forces $\sum \theta_e$ to be constant. We will show that this is the only obstruction on γ to the lifting to such $\tilde{\gamma}$.

To that end, let L be the $(|E|-1)$-plane distribution on $\underline{\mathbf{\Theta}}$ tangent to the level sets of $\sum_{e \in E} \theta_e$. The following proposition proves that we can deform some circle patterns up to first order at a point:

Proposition 2.1. *Let $c \in \underline{\mathfrak{C}}$ be a non-singular extended circle pattern on a graph with no bigons or self-loops such that there is no wide cycle of thin faces, and each edge meets two distinct faces. Let $K_c = \ker(\Phi_*|_c : T_c\underline{\mathfrak{C}} \to T_{\Phi(c)}\underline{\mathbf{\Phi}})$. Then $\Theta_*(K_c) = L_{\Theta(c)}$.*

In other words, for any vector $a \in T_{\Theta(c)}\underline{\mathbf{\Theta}}$ with sum of components 0, one can vary c so that its first order change in $\theta_\bullet$ is a, and also remains face non-singular up to first order.

Before we prove this, it is convenient to first prove the following, which will establish some useful notation:

Lemma 2.2. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Then Φ is a submersion in a neighbourhood of c.*

Proof. We construct vectors $\beta_f \in T_c\underline{\mathfrak{C}}$ so that $\Phi_*(\beta_f) = \frac{\partial}{\partial \Phi_f}$. The vector

$$\alpha_f := \frac{\partial}{\partial r_f} - \sum_{\vec{e} \in \partial f} \frac{\tan \varphi_{\vec{e}}}{r_f} \frac{\partial}{\partial \varphi_{\vec{e}}} \in T(\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}) \tag{2.1}$$

doesn't change $l_{\vec{e}}$, so is in $T_c\underline{\mathfrak{C}}$. Intuitively, α_f is like pulling the center of C_f up off the plane, increasing r_f and decreasing all φ's. Its pushforward under Φ is simply $\Phi_*(\alpha_f) = (-\frac{2}{r_f}\sum_{\vec{e} \in \partial f} \tan \varphi_{\vec{e}}) \frac{\partial}{\partial \Phi_f}$.

If f is a convex face, then the $\tan \varphi_{\vec{e}}$ are all non-negative, being 0 if and only if e is short, and because $\Phi_f = 2\pi$, at least three edges are long. So $\Phi_*(\alpha_f)$ is a negative multiple of $\frac{\partial}{\partial \Phi_f}$; we choose β so that

$$\beta_f := \beta \cdot \alpha_f; \quad \Phi_*(\beta_f) = \frac{\partial}{\partial \Phi_f} \tag{2.2}$$

If f is thick but not convex, let $\vec{e}$ be an edge with largest $\varphi_{\vec{e}}$, so that $\varphi_{\vec{e}} \geq \pi/2$. If $\varphi_{\vec{e}} > \pi/2$, then since tan is convex, and f has at least two other long edges, we see that $\Phi_*(\alpha_f) = -\sum \frac{\tan \varphi_{\vec{e'}}}{r_f} > 0$, so we can define β_f as in (2.2). If $\varphi_{\vec{e}} = \pi/2$, we can take

$$\beta_f = \frac{1}{2} \frac{\partial}{\partial \varphi_{\vec{e}}}$$

If f is thin, we actually have that $\alpha_f \in K_c$, since the components sum to 0, so we need a different approach. First suppose f shares an edge e with a thick face f', with $\vec{e} \in \partial f$. If $\varphi_{\vec{e}} = \pi/2$, we can take $\beta_f = \frac{1}{2}\frac{\partial}{\partial \varphi_{\vec{e}}}$ as before. Otherwise, we can increase $l_{\vec{e}}$ by varying $\varphi_{\vec{e}}$ while holding r_f constant, and increase $l_{\overleftarrow{e}}$ by increasing $r_{f'}$. This will affect both $\Phi_f, \Phi_{f'}$, so we use $\beta_{f'}$ to make $\Phi_{f'}$ constant. We can repeat this procedure with f' set to this thin face, and f set to another thin face adjacent to it, etc. □

Proof. We construct a vector $u_e \in K_c \subseteq T_c\underline{\mathfrak{C}}$ for each edge e and show that $\{\Theta_*(u_e)\}_{e \in E}$ spans an $|E|-1$ dimensional space, thus must be equal L_{Θ_c}. The vectors u_e will have the following property: if $\Theta_*(u_e) = \sum_{e' \in E} a^{e'} \frac{\partial}{\partial \theta_{e'}}$, then

- $a^e = 1$;
- $a^{e'} \leq 0$ for all $e' \neq e$;
- $\sum_{e' \in E} a^{e'} = 0$ (follows directly from $u_e \in K_c$)

Furthermore, these u_e's collectively satisfy the following connectivity property: consider the graph G whose vertex set is the set of edges E, and we connect two long edges e, e' by an edge if there is some e'' such that $a^e, a^{e'}$ are both nonzero in $\Theta_*(u_{e''})$; then G is connected.

Let us first suppose we have constructed such u_e, and show that these properties ensure that $\{\Theta_*(u_e)\}_{e\in E}$ spans an $|E|-1$ dimensional space. This is a simple exercise in linear algebra, but we show it for completeness.

Put the $\Theta_*(u_e)$'s into a $E\times E$ matrix, denoted M, so that the e-th row corresponds to $\Theta_*(u_e)$. By virtue of $u_e\in K_c$, we have that $(1\ 1\ \cdots\ 1)^T$ is in the null space of M; our goal is to show that it spans the null space.

Suppose $b=(b_e)^T$ is in the null space of M. Let $|b_e|$ be the largest among components of b; rescale b so that $b_e=1$. The e-th component of Mb is $1-\sum_{e'\neq e}a^{e'}b_{e'}$, where $a^{e'}$ are the components of $\Theta_*(u_e)$. Since $\sum_{e'\neq e}|a^{e'}|=1$, this can be 0 if and only if for all e' with $a^{e'}<0$, we have exactly $b_{e'}=1$. We then repeat this argument on the other e' that appear in $\Theta_*(u_e)$. By connectedness of G, this implies $b=(1\ \cdots\ 1)^T$, thus M has rank $|E|-1$.

Before constructing the u_e's, let us first construct several other auxiliary vectors that are not necessarily in $T_c\underline{\mathfrak{C}}$. Generally they keep Φ_f's constant, but have an effect on the lengths of oriented edges $l_{\vec{e}}$. The u_e's will be built from a combination of these. All discussion of changes will refer to changes in the first order.

Let f be a convex face, and $\vec{e}\in\partial f$. Define

$$\eta_{\vec{e}}=\frac{1}{2r_f\cos\varphi_{\vec{e}}}\frac{\partial}{\partial\varphi_{\vec{e}}}-\frac{1}{r_f\cos\varphi_{\vec{e}}}\beta_f \tag{2.3}$$

$\eta_{\vec{e}}$ keeps Φ_f constant, but increases $l_{\vec{e}}$ at unit speed, i.e. $dl_{\vec{e}}(\eta_{\vec{e}})=1$.

Let f be a thick but not convex face. There is exactly one edge $\vec{e}\in\partial f$ with $\varphi_{\vec{e}}\geq\pi/2$. If $\varphi_{\vec{e}}>\pi/2$, we will define $\eta_{\vec{e}}$ as in (2.3). If $\varphi_{\vec{e}}=\pi/2$, we will define

$$\eta_{\vec{e}}=\frac{1}{2}\frac{\partial}{\partial r_f}+\Big(\sum_{\vec{e'}\neq\vec{e}}\frac{\sin\varphi_{\vec{e'}}}{2r_f\cos\varphi_{\vec{e'}}}\Big)\frac{\partial}{\partial\varphi_{\vec{e}}}-\sum_{\vec{e'}\neq\vec{e}}\frac{\sin\varphi_{\vec{e'}}}{2r_f\cos\varphi_{\vec{e'}}}\frac{\partial}{\partial\varphi_{\vec{e'}}} \tag{2.4}$$

Then it is easy to see that $dl_{\vec{e}}(\eta_{\vec{e}})=1$ while $dl_{\vec{e'}}(\eta_{\vec{e}})=0$ for all other $\vec{e'}\neq\vec{e}$, and $\eta_{\vec{e}}$ keeps Φ_f constant.

Now let $\vec{e'}\in\partial f$ be another edge. We define

$$\gamma_{\vec{e'}}=\frac{1}{2r_f\cos\varphi_{\vec{e'}}}\Big(\frac{\partial}{\partial\varphi_{\vec{e'}}}-\frac{\partial}{\partial\varphi_{\vec{e}}}\Big) \tag{2.5}$$

Then $dl_{\vec{e'}}=1$, $dl_{\vec{e}}=-\frac{\cos\varphi_{\vec{e}}}{\cos\varphi_{\vec{e'}}}\geq 0$, and keeps Φ_f constant.

Finally, let f be a thin face. If $\vec{e}\in\partial f$ is a short edge, i.e. $\varphi_{\vec{e}}=0$, then we define $\eta_{\vec{e}}$ as in (2.3). If $\vec{e}$ is long, and $\varphi_{\vec{e}}\neq\pi/2$, we define $\gamma_{\vec{e}}$ as in (2.5); if $\varphi_{\vec{e}}=\pi/2$, we define

$$\xi_{\vec{e}}=\frac{1}{2}\frac{\partial}{\partial r_f} \tag{2.6}$$

Clearly $dl_{\vec{e}}(\xi_{\vec{e}})=dl_{\vec{e'}}(\xi_{\vec{e}})=1$ for the other edge e' with $\varphi_{\vec{e'}}=\pi/2$.

In summary, $\eta_{\vec{e}},\gamma_{\vec{e}},\xi_{\vec{e}}$ always increases $l_{\vec{e}}$ at unit speed, and keeps $\Phi_{f_{\vec{e}}}$ constant. $\eta_{\vec{e}}$ does not affect the lenghts of other edges in the face $f_{\vec{e}}$, but the other two might affect another edge - this other edge $\vec{e'}$ is unique, and $l_{\vec{e}},l_{\vec{e'}}$ either both increase or decrease. Furthermore, $l_{\vec{e'}}\geq l_{\vec{e'}}$.

For any oriented edge $\vec{e}$, there is exactly one such vector defined for it; call that vector $\zeta_{\vec{e}}$.

Now we can put these vectors together to construct the desired u_e's. To illustrate the main idea of the construction, first suppose that e is an edge between two convex faces f,f'. Then we can consider

$$w_e=\eta_{\vec{e}}+\eta_{\overleftarrow{e}}$$

Since $dl_{\vec{e}}(w_e)=dl_{\overleftarrow{e}}(w_e)=1$, and $dl_{\vec{e'}}(w_e)=0$ for all other edges, $w_e\in T_c\underline{\mathfrak{C}}$; furthermore, η_- does not affect Φ_-, so $w_e\in K_c$. For $\vec{e'}\in\partial f\cup\partial f'\backslash\{\vec{e},\overleftarrow{e}\}$, the $\frac{\partial}{\partial\varphi_{\vec{e'}}}$ component only appears in β_f or $\beta_{f'}$, which is positive by construciton (because f,f' are convex faces); thus, $d\theta_{e'}(w_e)\geq 0$ for such e', and is equal to 0 if and only if e' is short. Finally, since $w_e\in K_c$, it leaves the sum $\sum_{e\in E}\theta_e$ constant, so $d\theta_e(w_e)<0$, so we can take

$$u_e:=(d\theta_e(w_e))^{-1}\cdot w_e \tag{2.7}$$

Now consider an arbitrary edge e between faces f,f'. We build u_e inductively, beginning with $w_e=\zeta_{\vec{e}}+\zeta_{\overleftarrow{e}}$, $F'=\{f,f'\}$, and $E'=\{e\}$. If w_e also affects $l_{\vec{e'}}$ for some edge $\vec{e'}\in\partial F'\backslash\{\vec{e''},\overleftarrow{e''}|e''\in E'\}$, we add an appriopriate positive multiple of $\zeta_{\overleftarrow{e'}}$ to w_e, add $f_{\overleftarrow{e'}}$ to F', and add e' to E'. Recall that when $\zeta_{\vec{e'}}$ affects the length of another edge e'', we always have $l_{\vec{e''}} \geq l_{\vec{e'}}$. Thus by the condition of non-existence of wide cycles of thin faces, this process must terminate. It is not hard to see that when an η_- term appears, the process terminates (for that direction); a γ_- term begets either another γ_- or η_- term; a ξ_- term begets either a ξ_-, γ_-, or η_- term. In other words, in the end we get some sum (ignoring possible positive coefficients)

$$w_e = (\xi_- + \xi_- + \cdots + \gamma_- + \gamma_- + \cdots + \eta_-) + (\xi_- + \xi_- + \cdots + \gamma_- + \gamma_- + \cdots + \eta_-)$$

where in each parenthesis, the first term affects the next term and so on.

Assume that there are no thin faces. We check that w_e has the desired properties. By construction, $w_e \in T_c\underline{\mathfrak{C}}$. Observe that the coefficient of $\frac{\partial}{\partial \varphi_{\vec{e}}}$ is positive in either $\eta_{\vec{e}}$ or $\gamma_{\vec{e}}$, so $d\theta_e(\Theta_*(w_e)) < 0$. There is a maximal sequence of thick faces $f_0 = f, f_1, \ldots, f_n$ and a sequence of edges $e_0 = e, e_1, \ldots, e_n$, where for $i \neq 0$, e_i is between f_{i-1} and f_i, and l_{e_i} increases under w_e; we label oriented edges so $\vec{e_i} \in \partial f_i$, $\overleftarrow{e_i} \in \partial f_{i+1}$. (Similar discussion for $f_0 = f'$). Consider one of the edges e_i, $i \neq 0$. Since l_{e_i} increases, the $\zeta_{\vec{e}_{i-1}}$ term from the previous edge e_{i-1} must have been $\gamma_{\vec{e}_{i-1}}$; in particular, $\varphi_{\overleftarrow{e_i}} > \pi/2$, w_e decreases $\varphi_{\overleftarrow{e_i}}$, and keeps $r_{f_{i-1}}$ constant. The edge e_i also contributes some $\zeta_{\vec{e_i}}$ term to w_e; it increases $\varphi_{\vec{e_i}}$ and either increases or keeps constant r_{f_i}. Then θ_{e_i} increases, i.e. $d\theta_{e_i}(\Theta_*(w_e)) > 0$, by the following simple exercise in elementary Euclidean geometry. Suppose an obtuse triangle $\triangle ABC$, $\angle B > \pi/2$, and we decrease $\angle B$, increase $\angle C$, keep length of $\overline{AB}$ constant, and either increase or keeps constant the length of $\overline{AC}$; then $\angle A$ increases. This can be proved by observing that the changes we are forcing on the triangle can be broken in two: the first is increase $\angle B$ and keeping lengths $\overline{AB}, \overline{AC}$ constant, and the second is keeping $\angle B$ and length $\overline{AB}$ constant, and increasing length $\overline{AC}$; each of these increases $\angle A$. We apply this to our situation with B, C being the centers of circles $C_{f_{i-1}}, C_{f_i}$ of the faces f_{i-1}, f_i. As before, we simply rescale w_e appropriately to obtain u_e.

Now we consider the presence of thin faces. If all the ζ_- terms added to w_e are η_- or γ_-, then all arguments in the previous paragraph applies. The only situation in which ξ_- is used is when either $\zeta_{\vec{e}}$ or $\zeta_{\overleftarrow{e}}$ is ξ_-. Without loss of generality, $\zeta_{\vec{e}} = \xi_{\vec{e}}$, so that $\vec{e} \in \partial f$, f is a thin face with nonzero angles equal to $\pi/2$; let $\vec{e'}$ be the other edge with φ angle $\pi/2$. Suppose the next face, the one adjacent to e', is not thin. We simply add $\alpha \cdot (\frac{\partial}{\partial \vec{e}} - \frac{\partial}{\partial \vec{e'}})$ to w_e, where $\alpha > 0$ is chosen small enough so that, in combination with the next w_e term $\zeta_{\overleftarrow{e'}}$, $\theta_{e'}$ increases; for example, if the next term is $\zeta_{\overleftarrow{e'}} = \gamma_{\overleftarrow{e'}}$, then we choose $\alpha < \frac{1}{2r\cos\varphi_{\overleftarrow{e'}}}$, the coefficient of $\frac{\partial}{\partial \varphi_{\overleftarrow{e'}}}$ in $\gamma_{\overleftarrow{e'}}$. If the next face is also thin, we add the same term, and so on, until we reach a non-thin face. It is straighforward to check that θ_e decreases, and all other affected θ_-'s increase. $\square$

Proposition 2.3. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Given a smooth path $\gamma : [0, \infty) \to \underline{\boldsymbol{\Theta}}$ starting at $\gamma(0) = \Theta(c)$ and has constant value $(\sum \theta_e) \circ \gamma$, there exists $\varepsilon > 0$ and a lift $\tilde{\gamma} : [0, \varepsilon] \to \underline{\mathfrak{C}}$ of $\gamma|_{[0,\varepsilon]}$ along Θ that starts at $\tilde{\gamma}(0) = c$ and has constant value $\Phi \circ \tilde{\gamma}$.*

Proof. By Lemma 2.2, Φ is a submersion at c, hence is a submersion in a neighborhood $U \in \underline{\mathfrak{C}}$ of c. Thus, the kernels $K_{c'} = \ker(\Phi_*|c' : T_{c'}\underline{\mathfrak{C}} \to T_{\Phi(c')}\underline{\boldsymbol{\Phi}})$ form an $|E|$-plane distribution $K \subset T\underline{\mathfrak{C}}|_U$ over U. By Proposition 2.1, the bundle map $\Theta_*|_K : K \to L$ is full rank at c, hence it is full in a possibly smaller neighborhood, which we redefine U to be.

Since $\sum_{e\in E} \theta_e = 2\pi|E| - \sum_{f\in F} \Phi_f$, the fullness of Φ_* also shows that $\sum \theta$ has no critical points in U; for s in some small interval around $(\sum \theta)(c)$, denote, by overloading notation, $L_s = (\sum \theta)^{-1}(s)$, $U_s = U \cap \Theta^{-1}(L_s)$. We may consider K as a family of $|E|$-plane distributions K_s over U_s.

The vector $\rho := \sum_{f\in F} r_f \frac{\partial}{\partial r_f}$ corresponds to scaling all radii by the same factor, and is clearly in $T\underline{\mathfrak{C}}$. It is obviously mapped to zero under Φ_* and Θ_*, so $\mathrm{span}\{\rho\} = \ker(\Theta_*|_K) \subset K$. Thus, over U, we may split $K = \mathrm{span}\{\rho\} \oplus K'$ (say as orthogonal decomposition w.r.t. some smooth metric on $\underline{\mathfrak{C}}$), so that $\Theta_*|c' : K'|_{c'} \simeq L_{\Theta(c')}$.

In other words, $K'|_{U_s}$ is a horizontal $(|E|-1)$-plane distribution that determines an Ehresmann connection of the fibre bundle $U_s \to L_s$ (after being appropriately restricted). Therefore, a short path γ starting at $\Theta(c)$ with constant $\sum \theta$, i.e. a path in L_s, $s = (\sum \theta)(c)$, can be lifted to a path $\tilde{\gamma}$ in U_s, so that $\tilde{\gamma}$ is always tangent to K', hence $\Phi(\tilde{\gamma})$ is constant. $\square$

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