VARIATION ON A VARIATIONAL PRINCIPLE

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ABSTRACT. We prove that alternating links in the thickened torus remain hyperbolic after certain augmentations.

1. Introduction

Alice TODO

Definitions, Notations, and Conventions

Always assume surface Σ (our case $\Sigma = \mathbb{T}^2$) is oriented.

define cellular decomp of surface

V, E, F are set of vertices, edges, faces. No bigon faces.

$\vec{E}$ is the set of oriented edges. We may identify an oriented edge $\vec{e}$ with the pair $(f_{\vec{e}}, e)$, where $f_{\vec{e}}$ is the face to the left of $\vec{e}$.

When we use $\overleftarrow{e}$ to refer to an oriented edge, it refers to the oppositely oriented edge to $\vec{e}$. If we construct an expression with both $\vec{e}$ and $\overleftarrow{e}$, it will always be (anti-)symmetrical in the two orientations, and we assume that an arbitrary choice has been made.

Recall circle pattern.

A circle pattern is determined by the radius of the circle C_f associated to each face, r_f, and the angle that each edge subtends in adjacent faces, $\varphi_{\vec{e}}$. (see figure TODO) Thus determines, and is determined by a point in $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, where

- $\underline{\mathfrak{R}} := \mathbb{R}_+^F = \{(r_f)_{f\in F} | r_f \in \mathbb{R}_+\}$
- $\underline{\mathfrak{Q}} := \mathbb{R}^{\vec{E}} = \{(\varphi_{\vec{e}})_{\vec{e}\in\vec{E}} | \varphi_{\vec{e}} \in \mathbb{R}\}$

but clearly not every point $c \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ determines a circle pattern.

On $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, there are several functions to consider:

- $\Phi_f = 2\sum_{\vec{e}\in\partial f} \varphi_{\vec{e}}$, measuring the cone angle at the center of C_f
- $\theta_e = \pi - \varphi_{\vec{e}} - \varphi_{\overleftarrow{e}}$
- $l_{\vec{e}} = r_{f_{\vec{e}}} \sin \varphi_{\vec{e}}$

These fit together to give maps to the following spaces:

- $\underline{\mathbf{\Phi}} := \mathbb{R}^F = \{(\Phi_f)_{f\in F} | \Phi_f \in \mathbb{R}\}$
- $\underline{\mathbf{\Theta}} := \mathbb{R}^E = \{(\theta_e)_{e\in E} | \theta_e \in \mathbb{R}\}$
- $\underline{\mathfrak{L}} := \mathbb{R}^{\vec{E}} = \{(l_{\vec{e}})_{\vec{e}\in E} | l_{\vec{e}} \in \mathbb{R}\}$

Our main argument is to deform "degenerate" circle patterns, where adjacent circles may be identical or tangent, into ones that don't look so degenerate, hence it is conveneint to extend the notion of circle pattern:

Definition 1.1. Let Γ be a graph on the torus. An *extended circle pattern* on Γ is $c = ((r_f), (\varphi_{\vec{e}})) \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ such that $l_{\vec{e}} = l_{\overleftarrow{e}}$ for all edges $e \in E$. We denote by $\underline{\mathfrak{C}}$ the set of extended circle patterns.

$$\begin{array}{ccccc} \underline{\mathfrak{C}} = \{l_{\vec{e}} = l_{\overleftarrow{e}}\} & \subseteq & \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}} & \xrightarrow{\Theta} & \underline{\mathbf{\Theta}} \\ & & \Big\downarrow{\scriptstyle\Phi} & & \\ & & \underline{\mathbf{\Phi}} & & \end{array}$$

[TODO Check this] The usual notion of circle pattern would be restricted to those $c \in \underline{\mathfrak{C}}$ with $\varphi_{\vec{e}} \in (0, \pi)$ and $\theta_e \in (0, \pi)$. One may consider deforming a circle pattern so as to have some θ_e approach 0 or π. The limit $\theta_e \to \pi$ is easy to picture, one simply gets that the two circles $C_f, C_{f'}$ of the adjacent faces become tangent. The limit $\theta_e \to 0$ is a bit more complex, as the final shape of Q_e (the quadrangle associated to e) may depend on $\varphi_{\vec{e}}$ or $\varphi_{\overleftarrow{e}}$. If we parametrize circle patterns by (r_f) and (θ_e) as in TODO [?BS], the limiting shape would depend on the relationship between r_f and $r_{f'}$. As our main result depends on extending [?BS] result to such limits and beyond, we find it more convenient to describe extended circle patterns by (r_f) and $(\varphi_{\vec{e}})$.

We will mostly be working with extended circle patterns that "looks normal", with all $\varphi_{\vec{e}}$ in the range $[0, \pi)$, so we do not dwell on the meaning of negative $\varphi_{\vec{e}}$'s. [1]

Definition 1.2. An extended circle pattern is said to be *face non-singular* if $\Phi_f = 2\pi$ for all faces f; it is said to be *vertex non-singular* if $\sum_{e \ni v} \theta_e = 2\pi$ for all vertices v. Finally it is said to be *non-singular* if it is both.

TODO non-degenerate circle pattern?

Definition 1.3. Given some extended circle pattern $c \in \underline{\mathfrak{C}}$, we say a face f is *convex* if for all edges $\vec{e} \in \partial f$, we have $\varphi_{\vec{e}} \in [0, \pi/2)$. We say f is *thin* if exactly two edges $\vec{e}, \vec{e'} \in \partial f$ have nonzero $\varphi_\bullet$, and furtheremore, $0 < \varphi_{\vec{e}} = \pi - \varphi_{\vec{e'}} < \pi/2$.

TODO figures for convex, thin faces

Definition 1.4. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, an edge e is *short* if it has length 0, $l_{\vec{e}} = l_{\overleftarrow{e}} = 0$; it is *long* otherwise.

Definition 1.5. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, a *thick path* is a sequence of faces $f_0, f_1, \ldots, f_n$ such that f_i, f_{i+1} share a long edge. A *thick cycle* is a thick path with $f_0 = f_n$.

Lemma 1.6. *Let c be an extended circle pattern whose faces are either convex or thin. Then $\underline{\mathfrak{C}}$ is a manifold in a neighborhood of c.*

Proof. $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = \sin\varphi_{\vec{e}} dr_{f_{\vec{e}}} + r_{f_{\vec{e}}} \cos\varphi_{\vec{e}} d\varphi_{\vec{e}} - \sin\varphi_{\overleftarrow{e}} dr_{f_{\overleftarrow{e}}} - r_{f_{\overleftarrow{e}}} \cos\varphi_{\overleftarrow{e}} d\varphi_{\overleftarrow{e}}$, and since having only convex of thin faces implies $\cos\varphi_{\vec{e}} \neq 0$ for all $\vec{e}$, we see that the differentials $d(l_{\vec{e}} - l_{\overleftarrow{e}})$ are linearly independent at c, hence are linearly independent in a neighborhood of c. □

Lemma 1.7. *Let c be a face non-singular extended circular pattern such that $\varphi_{\vec{e}} \geq 0$ for all $\vec{e} \in \vec{E}$. TODO no cycle of thin faces? It's a bit weaker than that... Then $\underline{\mathfrak{C}}$ is a manifold in a neighborhood of c.*

Proof. We need to show that the differentials $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = \sin\varphi_{\vec{e}} dr_{f_{\vec{e}}} + r_{f_{\vec{e}}} \cos\varphi_{\vec{e}} d\varphi_{\vec{e}} - \sin\varphi_{\overleftarrow{e}} dr_{f_{\overleftarrow{e}}} - r_{f_{\overleftarrow{e}}} \cos\varphi_{\overleftarrow{e}} d\varphi_{\overleftarrow{e}}$ are linearly independent at c. If $\varphi_{\vec{e}} \neq \pi/2$, then the $d\varphi_{\vec{e}}$ term makes $d(l_{\vec{e}} - l_{\overleftarrow{e}})$ linearly independent from the rest; likewise for $\varphi_{\overleftarrow{e}} \neq \pi/2$. If both equal $\pi/2$, then $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = dr_{\vec{e}} - dr_{\overleftarrow{e}}$. consider f with $\vec{e} \in \partial f$.

By face non-singularity, $\sum_{\vec{e} \in f} \varphi_{\vec{e}} = \pi$, and since there are no bigon faces,

□

1.1. **Modifications to Extended Circle Patterns.** Let Γ be a graph on the torus, and let c be a non-singular extended circle pattern on Γ. There are two simple operations to modify Γ: cutting a face along a new edge, and splitting a vertex in two then joining them by a new edge. We describe how to associate an extended circle pattern c' to the new graph.

Cutting a face: Consider a face f, and suppose we added a new edge e' so that f is split into two faces f' and f", so that $\partial f' = \{\vec{e_1}, \ldots, \vec{e_k}, \vec{e'}\}$, and $\partial f" = \{\overleftarrow{e'}, \vec{e_{k+1}}, \ldots \vec{e_n}\}$. Then for the new extended circle pattern c', we set $r_{f'}(c') = r_{f"}(c') = r_f(c)$, $\varphi_{\vec{e'}} = \pi - \varphi_{\vec{e_1}} - \ldots - \varphi_{\vec{e_k}}$, $\varphi_{\overleftarrow{e'}} = \pi - \varphi_{\vec{e_{k+1}}} - \ldots \varphi_{\vec{e_n}}$, and $r_\bullet(c') = r_\bullet(c)$, $\varphi_\bullet(c') = \varphi_\bullet(c)$ for all other faces, edges. It is straightforward to check that c' is non-singular.

Consider a vertex v of the

[1] Intuitively, one can visualize decreasing $\varphi_{\vec{e}}$ from ε to $-\varepsilon$ as moving the black vertices of the quadrangle Q_e past each other, thus flipping Q_e along its long axis and making it have negative area.

TODO: make a list of images of vectors, e.g. $\Theta_*(\frac{\partial}{\partial\varphi_{\vec{e}}}) = -\frac{\partial}{\partial\theta_e}$.

Our general strategy for obtaining a (non-singular) circle pattern is to start with an assignment $\underline{\theta} = (\theta_\bullet)$ for which we know an extended circle pattern exists (say from results of [?BS]), and a path γ in $\underline{\mathbf{\Theta}}$ starting at $\underline{\theta}$. We then attempt to lift γ to a path $\tilde{\gamma}$ in $\underline{\mathfrak{C}}$, so that $\tilde{\gamma}$ remains (face) non-singular (vertex non-singularity is already determined by γ).

Note that since $\sum_{e\in E}\theta_e = 2\pi|E| - \sum_{\vec{e}\in\vec{E}}\varphi_{\vec{e}} = 2\pi|E| - \sum_{f\in F}\Phi_f$, we see that maintaining face non-singularity of $\tilde{\gamma}$ forces $\sum\theta_e$ to be constant. We will show that this is the only obstruction on γ to the lifting to such $\tilde{\gamma}$.

To that end, let L be the $(|E|-1)$-plane distribution on $\underline{\mathbf{\Theta}}$ tangent to the level sets of $\sum_{e\in E}\theta_e$. The following proposition proves it (avoid using 'it'?) up to first order at a point:

Proposition 2.1. *Let $c \in \underline{\mathfrak{C}}$ be an extended circle pattern such that*

- *$\Phi_f = 2\pi$ for all $f \in F$;*
- *all faces are eithe convex or thin, with at least one convex face;*
- *every pair of faces is connected by a thick path.*

Let $K_c = \ker(\Phi_*|_c : T_c\underline{\mathfrak{C}} \to T_{\Phi(c)}\underline{\mathbf{\Phi}})$. Then $\Theta_*(K_c) = L_{\Theta(c)}$.

In other words, for any vector $a \in T_{\Theta(c)}\underline{\mathbf{\Theta}}$ with sum of components 0, one can vary c so that its first order change in $\theta_\bullet$ is a, and also remains face non-singular up to first order.

Before we prove this, it is convenient to first prove the following, which will establish some useful notation:

Lemma 2.2. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Then Φ is a submersion in a neighbourhood of c.*

Proof. We construct vectors $\beta_f \in T_c\underline{\mathfrak{C}}$ so that $\Phi_*(\beta_f) = \frac{\partial}{\partial\Phi_f}$. The vector

$$\alpha_f := \frac{\partial}{\partial r_f} - \sum_{\vec{e}\in\partial f}\frac{\tan\varphi_{\vec{e}}}{r_f}\frac{\partial}{\partial\varphi_{\vec{e}}} \in T(\underline{\mathfrak{R}}\times\underline{\mathfrak{Q}}) \tag{2.1}$$

doesn't change $l_{\vec{e}}$, so is in $T_c\underline{\mathfrak{C}}$. Intuitively, α_f is like pulling the center of C_f up off the plane, increasing r_f and decreasing all φ's. Its pushforward under Φ is simply $\Phi_*(\alpha_f) = (-\frac{2}{r_f}\sum_{\vec{e}\in\partial f}\tan\varphi_{\vec{e}})\frac{\partial}{\partial\Phi_f}$.

If f is a convex face, then the $\tan\varphi_{\vec{e}}$ are all non-negative, being 0 if and only if e is short, and because $\Phi_f = 2\pi$, atleast three edges are long. So $\Phi_*(\alpha_f)$ is a negative multiple of $\frac{\partial}{\partial\Phi_f}$; we choose β so that

$$\beta_f := \beta\cdot\alpha_f; \quad \Phi_*(\beta_f) = \frac{\partial}{\partial\Phi_f} \tag{2.2}$$

If f is thin, we actually have that $\alpha_f \in K_c$, since the components sum to 0, so we need a different approach. First suppose f shares an edge e with a convex face f', with $\vec{e} \in \partial f$. We can increase l_e by varying $\varphi_{\vec{e}}, \varphi_{\overleftarrow{e}}$ while holding $r_f, r_{f'}$ constant. This will affect both $\Phi_f, \Phi_{f'}$, so we use $\beta_{f'}$ to make $\Phi_{f'}$ constant. More explicitly, consider

$$\alpha_e = \frac{1}{2r_f\cos\varphi_{\vec{e}}}\frac{\partial}{\partial\varphi_{\vec{e}}} + \frac{1}{2r_{f'}\cos\varphi_{\overleftarrow{e}}}\frac{\partial}{\partial\varphi_{\overleftarrow{e}}} \tag{2.3}$$

which increases $l_{\vec{e}}, l_{\overleftarrow{e}}$ at half unit speed, $dl_{\vec{e}}(\alpha_e) = dl_{\overleftarrow{e}}(\alpha_e) = 1/2$. Then we can take

$$\beta_f = r_f\cos\varphi_{\vec{e}}(\alpha_e - \frac{1}{r_{f'}\cos\varphi_{\overleftarrow{e}}}\beta_{f'}) \tag{2.4}$$

We can repeat this procedure with f' set to this thin face, and f set to another thin face adjacent to it, etc. □

Proof of Proposition 2.1. We construct a vector $u_e \in K_c \subseteq T_c\underline{\mathfrak{C}}$ for each edge e and show that $\{\Theta_*(u_e)\}_{e\in E}$ spans an $|E|-1$ dimensional space, thus must be equal L_{Θ_c}.

For long edges e, u_e will have the following property: if $\Theta_*(u_e) = \sum_{e'\in E} a^{e'}\frac{\partial}{\partial\theta_{e'}}$, then

- $a^e = 1$;
- $a^{e'} \leq 0$ for all $e' \neq e$;

- $a^{e'} = 0$ for short edges e'.
- $\sum_{e' \in E} a^{e'} = 0$ (follows directly from $u_e \in K_c$)

Furthermore, these u_e's collectively satisfy the following connectivity property: consider the graph G_{long} whose vertex set is the set of long edges, and we connect two long edges e, e' by an edge if there is some e" such that $a^e, a^{e'}$ are both nonzero in $\Theta_*(u_{e"})$; then G_{long} is connected.

In addition, for short edges e, u_e have the following property: again writing $\Theta_*(u_e) = \sum_{e' \in E} a^{e'} \frac{\partial}{\partial \theta_{e'}}$, one has

- $a^e = 1$;
- $a^{e'} = 0$ for other short edges e'.
- $\sum_{e' \in E} a^{e'} = 0$ (follows directly from $u_e \in K_c$)

Let us first suppose we have constructed such u_e, and show that these properties ensure that $\{\Theta_*(u_e)\}_{e \in E}$ spans an $|E|-1$ dimensional space. This is a simple exercise in linear algebra, but we show it for completeness.

Put the $\Theta_*(u_e)$'s into a $E \times E$ matrix, denoted M, so that the e-th row corresponds to u_e. By virtue of $u_e \in K_c$, we have that $(1\ 1\ \cdots\ 1)^T$ is in the null space of M; our goal is to show that it spans the null space.

Suppose $b = (b_e)^T$ is in the null space of M. Reorder E, if necessary, so that long edges come before short edges. Then by the last property of u_e's for long edges, the top right block of long-by-short entries are 0. Thus the vector $b_{\text{long}} := (b_e)^T_{e \in \text{long}}$ is in the null space of the submatrix M_{long} of long-by-long entries.

Let $|b_e|$ be the largest among components of b_{long}; rescale b so that $b_e = 1$. The e-th component of $M_{\text{long}} b_{\text{long}}$ is $1 - \sum_{e' \neq e} a^{e'} b_{e'}$, where $a^{e'}$ are the components of $\Theta_*(u_e)$. Since $\sum_{e' \neq e} |a^{e'}| = 1$, this can be 0 if and only if for all e' with $a^{e'} < 0$, we have exactly $b_{e'} = 1$. By connectedness of G_{long}, this implies $b_{\text{long}} = (1\ \cdots\ 1)^T$.

Returning to the full M, it is now easy to see that for a short edge e, by looking at the e-th row of M, we have $b_e = \sum a^{e'} = 1$, so we are done.

Now we construct the u_e's. First suppose that e is a long edge between two convex faces f, f' (so $\vec{e} \in \partial f$, $\overleftarrow{e} \in \partial f'$). Observe that α_e from (2.3) increases Φ_f and $\Phi_{f'}$, and we can compensate using β_f from (2.2), so we consider

$$w_e = \alpha_e - \frac{1}{\cos \varphi_{\vec{e}}} \beta_f - \frac{1}{\cos \varphi_{\overleftarrow{e}}} \beta_{f'} \in K_c$$

For $\vec{e'} \in \partial f \cup \partial f' \backslash \{\vec{e}, \overleftarrow{e}\}$, the $\frac{\partial}{\partial \varphi_{\vec{e'}}}$ component only appears in β_f or $\beta_{f'}$, which is positive by construciton (this requires f, f' to be convex faces); thus, $d\theta_{e'}(w_e) \geq 0$ for such e', and is equal 0 if and only if e' is short. Finally, since $w_e \in K_c$, it leaves the sum $\sum_{e \in E} \theta_e$ constant, so $d\theta_e(w_e) < 0$, so we can take

$$u_e := (d\theta_e(w_e))^{-1} \cdot w_e \tag{2.5}$$

In general, we consider a thick path of faces $f_0, \ldots, f_n$, such that all faces except f_0 and f_n are thin, and f_i are distinct except possibly $f_0 = f_n$. [2] Let e_i be the long edge between f_i and f_{i+1}, and orient so that $\vec{e_i} \in \partial f_i$, $\overleftarrow{e_i} \in \partial f_{i+1}$.

We will construct u_{e_i} for $i = 1, \ldots, n$.

We want to increase the lengths of e_i's all together. Let $E' = \{\vec{e_i}, \overleftarrow{e_i} | i = 0, 1, \ldots, n-1\}$. Consider

$$x := \sum_{i=0,\ldots,n-1} \alpha_{e_i} = \sum_{\vec{e} \in E'} \frac{1}{2 r_{f_{\vec{e}}} \cos \varphi_{\vec{e}}} \frac{\partial}{\partial \varphi_{\vec{e}}}$$

with α_e from (2.3). It is clear that x increases all oriented e_i lengths $l_{\vec{e}}$ equally fast, i.e. $dl_{\vec{e_i}}(x) = dl_{\overleftarrow{e_i}}(x) = 1/2$ for all i, so $x \in T_c \underline{\mathfrak{C}}$. It is also clear that for the thin faces f_i, x makes one of the angles $\varphi_{\overleftarrow{e_{i-1}}}, \varphi_{\vec{e_i}}$ increase and the other decrease, both at the same rate, thus $d\Phi_{f_i}(x) = 0$. However, the vector x increase Φ_{f_0} and Φ_{f_n}, so it is compensated using β_f's, so that

$$y := x - \frac{1}{r_{f_0} \cos \varphi_{\vec{e_0}}} \beta_{f_0} - \frac{1}{r_{f_n} \cos \varphi_{\overleftarrow{e_{n-1}}}} \beta_{f_n} \in K_c$$

[2] Picture a closed accordian.

Before we modify y to give the desired u_{e_i}, we note that for $\vec{e} \in (\cup \partial f_i) \backslash E'$, the $\varphi_{\vec{e}}$-component in y is non-positive, and is 0 if and only if e is short; so $\Theta_*(-y)$ already satisfies three of the four properties desired in $\Theta_*(u_e)$.

Recall that for thin face f, α_f from (2.1) is in K_c, and we have

$$\Theta_*(\frac{r_{f_i}}{\tan \varphi_{\vec{e_i}}} \alpha_{f_i}) = \frac{\partial}{\partial \theta_{e_i}} - \frac{\partial}{\partial \theta_{e_{i-1}}}$$

Write $\bar{\alpha}_{f_i} = \frac{r_{f_i}}{\tan \varphi_{\vec{e_i}}} \alpha_{f_i}$. Then it is easy to see that using $\bar{\alpha}_{f_i}$'s, we can "push around" the θ_{e_i}-components in $\Theta_*(y)$ so only one is nonzero. More precisely, if $\Theta_*(y) = \sum_e y^e \frac{\partial}{\partial \theta_e}$, the image under Θ_* of the vector

$$\bar{u}_{e_0} := y - y^{e_{n-1}} \bar{\alpha}_{f_{n-1}} - (y^{e_{n-2}} + y^{e_{n-1}}) \bar{\alpha}_{f_{n-2}} - \ldots - (y^{e_1} + \ldots + y^{e_{n-1}}) \bar{\alpha}_{f_1}$$

would have θ_{e_i}-components 0 for all i except 0; for $i = 0$, the θ_{e_0}-component is now $y^{e_0} + \ldots + y^{e_{n-1}}$, which is nonzero because $\sum_{e' \in E} y^{e'} = 0$ and $\sum_{e' \in E \backslash E'} y^{e'} > 0$. Therefore, we can take

$$u_{e_0} := (d\theta_{e_0}(\bar{u}_{e_0}))^{-1} \cdot \bar{u}_{e_0}; \quad u_{e_i} = u_{e_{i-1}} + \bar{\alpha}_{f_i}$$

We still need to show the collective connectivity property of the u_e's. It is easy to see that the long edges of a convex face are connected in G_{long} (for example, u_{e_0} defined above clearly does it for f_0). Furthermore, all the $\Theta_*(u_{e_i})$ have most components equal, so e_i are in the same connected component of G_{long}. Thus connectivity of G_{long} follows from the thick-path-connectivity of faces.

Finally, for short edges e, we simply define u_e as in (2.5) (here we do not worry about whether faces are thin or convex, as the non-positivity condition of $a^{e'}$ is absent). $\square$

Proposition 2.3. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Given a smooth path $\gamma : [0, \infty) \to \underline{\mathbf{\Theta}}$ starting at $\gamma(0) = \Theta(c)$ and has constant value $(\sum \theta_e) \circ \gamma$, there exists $\varepsilon > 0$ and a lift $\tilde{\gamma} : [0, \varepsilon] \to \underline{\mathfrak{C}}$ of $\gamma|_{[0,\varepsilon]}$ along Θ that starts at $\tilde{\gamma}(0) = c$ and has constant value $\Phi \circ \tilde{\gamma}$.*

Proof. By Lemma 2.2, Φ is a submersion at c, hence is a submersion in a neighborhood $U \in \underline{\mathfrak{C}}$ of c. Thus, the kernels $K_{c'} = \ker(\Phi_*|c' : T_{c'}\underline{\mathfrak{C}} \to T_{\Phi(c')}\underline{\mathbf{\Phi}})$ form an $|E|$-plane distribution $K \subset T\underline{\mathfrak{C}}|_U$ over U. By Proposition 2.1, the bundle map $\Theta_*|_K : K \to L$ is full rank at c, hence it is full in a possibly smaller neighborhood, which we redefine U to be.

Since $\sum_{e \in E} \theta_e = 2\pi|E| - \sum_{f \in F} \Phi_f$, the fullness of Φ_* also shows that $\sum \theta$ has no critical points in U; for s in some small interval around $(\sum \theta)(c)$, denote, by overloading notation, $L_s = (\sum \theta)^{-1}(s)$, $U_s = U \cap \Theta^{-1}(L_s)$. We may consider K as a family of $|E|$-plane distributions K_s over U_s.

The vector $\rho := \sum_{f \in F} r_f \frac{\partial}{\partial r_f}$ corresponds to scaling all radii by the same factor, and is clearly in $T\underline{\mathfrak{C}}$. It is obviously mapped to zero under Φ_* and Θ_*, so $\text{span}\{\rho\} = \ker(\Theta_*|_K) \subset K$. Thus, over U, we may split $K = \text{span}\{\rho\} \oplus K'$ (say as orthogonal decomposition w.r.t. some smooth metric on $\underline{\mathfrak{C}}$), so that $\Theta_*|c' : K'|_{c'} \simeq L_{\Theta(c')}$.

In other words, $K'|_{U_s}$ is a horizontal $(|E|-1)$-plane distribution that determines an Ehresmann connection of the fibre bundle $U_s \to L_s$ (after being appropriately restricted). Therefore, a short path γ starting at $\Theta(c)$ with constant $\sum \theta$, i.e. a path in L_s, $s = (\sum \theta)(c)$, can be lifted to a path $\tilde{\gamma}$ in U_s, so that $\tilde{\gamma}$ is always tangent to K', hence $\Phi(\tilde{\gamma})$ is constant. $\square$

TODO Note: no face can have its edges glued together; prevented by fact of alternating links

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