VARIATION ON A VARIATIONAL PRINCIPLE

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Abstract. We prove that alternating links in the thickened torus remain hyperbolic after certain augmentations.

1. Introduction

Alice TODO

Definitions, Notations, and Conventions

Always assume surface Σ (our case $\Sigma = \mathbb{T}^2$) is oriented.

define cellular decomp of surface

V, E, F are set of vertices, edges, faces.

$\vec{E}$ is the set of oriented edges. We may identify an oriented edge $\vec{e}$ with the pair $(f_{\vec{e}}, e)$, where $f_{\vec{e}}$ is the face to the left of $\vec{e}$.

Recall circle pattern.

A circle pattern is determined by the radius of the circle C_f associated to each face, r_f, and the angle that each edge subtends in adjacent faces, $\varphi_{\vec{e}}$. (see figure TODO) Thus determines, and is determined by a point in $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, where

- $\underline{\mathfrak{R}} := \mathbb{R}_+^F = \{(r_f)_{f\in F} | r_f \in \mathbb{R}_+\}$
- $\underline{\mathfrak{Q}} := \mathbb{R}^{\vec{E}} = \{(\varphi_{\vec{e}})_{\vec{e}\in\vec{E}} | \varphi_{\vec{e}} \in \mathbb{R}\}$

but clearly not every point $c \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ determines a circle pattern.

On $\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$, there are several functions to consider:

- $\Phi_f = 2\sum_{\vec{e}\in\partial f} \varphi_{\vec{e}}$, measuring the cone angle at the center of C_f
- $\theta_e = \pi - \varphi_{\vec{e}} - \varphi_{\overleftarrow{e}}$
- $l_{\vec{e}} = r_{f_{\vec{e}}} \sin\varphi_{\vec{e}}$

These fit together to give maps to the following spaces:

- $\underline{\boldsymbol{\Phi}} := \mathbb{R}^F = \{(\Phi_f)_{f\in F} | \Phi_f \in \mathbb{R}\}$
- $\underline{\boldsymbol{\Theta}} := \mathbb{R}^E = \{(\theta_e)_{e\in E} | \theta_e \in \mathbb{R}\}$
- $\underline{\mathfrak{L}} := \mathbb{R}^{\vec{E}} = \{(l_{\vec{e}})_{\vec{e}\in E} | l_{\vec{e}} \in \mathbb{R}\}$

Our main argument is to deform "degenerate" circle patterns, where adjacent circles may be identical or tangent, into ones that don't look so degenerate, hence it is conveneint to extend the notion of circle pattern:

Definition 1.1. An *extended circle pattern* is $c = ((r_f), (\varphi_{\vec{e}})) \in \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}$ such that $l_{\vec{e}} = l_{\overleftarrow{e}}$ for all edges $e \in E$. We denote by $\underline{\mathfrak{C}}$ the set of extended circle patterns.

$$\underline{\mathfrak{C}} = \{l_{\vec{e}} = l_{\overleftarrow{e}}\} \quad \subseteq \quad \begin{tikzcd} \underline{\mathfrak{R}} \times \underline{\mathfrak{Q}} \arrow[r, "\Theta"] \arrow[d, "\Phi"] & \underline{\boldsymbol{\Theta}} \\ \underline{\boldsymbol{\Phi}} & \end{tikzcd}$$

[Check this] The usual notion of circle pattern would be restricted to those $c \in \underline{\mathfrak{C}}$ with $\varphi_{\vec{e}} \in (0, \pi)$ and $\theta_e \in (0, \pi)$. One may consider deforming a circle pattern so as to have some θ_e approach 0 or π. The limit $\theta_e \to \pi$ is easy to picture, one simply gets that the two circles $C_f, C_{f'}$ of the adjacent faces become tangent. The limit $\theta_e \to 0$ is a bit more complex, as the final shape of Q_e (the quadrangle associated to e) depends also on $\varphi_{\vec{e}}$ (which equals $\pi - \varphi_{\overleftarrow{e}}$ in the limit. If we parametrize circle patterns by (r_f) and (θ_e) as in TODO [?BS], the limiting shape would depend on how r_f approaches $r_{f'}$, some sort of blow up stuff TODO.

We will mostly be working with extended circle patterns that mostly "look normal", with all $\varphi_{\vec{e}}$ in some range $(-\varepsilon, \pi+\varepsilon)$.

Definition 1.2. An extended circle pattern is said to be *face non-singular* if $\Phi_f = 2\pi$ for all faces f; it is said to be *vertex non-singular* if $\sum_{e\ni v}\theta_e = 2\pi$ for all vertices v. Finally it is said to be *non-singular* if it is both.

non-degenerate circle pattern?

Definition 1.3. Given some extended circle pattern $c \in \underline{\mathfrak{C}}$, we say a face f is *convex* if for all edges $\vec{e} \in \partial f$, we have $\varphi_{\vec{e}} \in [0, \pi/2)$. We say f is *thin* if exactly two edges $\vec{e}, \vec{e'} \in \partial f$ have nonzero $\varphi_\bullet$, and furtheremore, $0 < \varphi_{\vec{e}} = \pi - \varphi_{\vec{e'}} < \pi/2$.

TODO figures for convex, thin faces

Definition 1.4. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, an edge e is *short* if it has length 0, $l_{\vec{e}} = l_{\overleftarrow{e}} = 0$; it is *long* otherwise.

Definition 1.5. Given an extended circle pattern $c \in \underline{\mathfrak{C}}$, a *thick path* is a sequence of faces $f_0, f_1, \ldots, f_n$ such that f_i, f_{i+1} share a long edge. A *thick cycle* is a thick path with $f_0 = f_n$.

Definition 1.6. A *slit-convex circle pattern* is an extended cirlce pattern $c \in \underline{\mathfrak{C}}$ such that all its faces are either convex or thin.

Lemma 1.7. *$\underline{\mathfrak{C}}$ is a manifold near slit-convex circle patterns.*

Proof. $d(l_{\vec{e}} - l_{\overleftarrow{e}}) = \sin\varphi_{\vec{e}} dr_{f_{\vec{e}}} + r_{f_{\vec{e}}}\cos\varphi_{\vec{e}} d\varphi_{\vec{e}} - \sin\varphi_{\overleftarrow{e}} dr_{f_{\overleftarrow{e}}} - r_{f_{\overleftarrow{e}}}\cos\varphi_{\overleftarrow{e}} d\varphi_{\overleftarrow{e}}$, and since having only convex of thin faces implies $\cos\varphi_{\vec{e}} \neq 0$, we see the differentials are linearly independent at a slit-convex circle pattern c. ☐

2. Main ?? results

TODO: make a list of images of vectors, e.g. $\Theta_*(\frac{\partial}{\partial\varphi_{\vec{e}}}) = -\frac{\partial}{\partial\theta_e}$.

Our general strategry for obtaining a (non-singular) circle pattern is to start with an assignment $\underline{\theta} = (\theta_\bullet)$ for which we know an extended circle pattern exists (say from results of [?BS]), and a path γ in $\underline{\boldsymbol{\Theta}}$ starting at $\underline{\theta}$. We then attempt to lift γ to a path $\tilde{\gamma}$ in $\underline{\mathfrak{C}}$, so that $\tilde{\gamma}$ remains (face) non-singular (vertex non-singularity is already determined by γ).

Note that since $\sum_{e\in E}\theta_e = 2\pi|E| - \sum_{\vec{e}\in\vec{E}}\varphi_{\vec{e}} = 2\pi|E| - \sum_{f\in F}\Phi_f$, we see that maintaining face non-singularity of $\tilde{\gamma}$ forces $\sum\theta_e$ to be constant. We will show that this is the only obstruction on γ to the lifting to such $\tilde{\gamma}$.

To that end, let L be the $(|E|-1)$-plane distribution on $\underline{\boldsymbol{\Theta}}$ tangent to the level sets of $\sum\theta_v$. The following proposition proves it (avoid using 'it'?) up to first order at a point:

Proposition 2.1. *Let $c \in \underline{\mathfrak{C}}$ be an extended circle pattern such that*

- *$\Phi_f = 2\pi$ for all $f \in F$;*
- *all faces are eithe convex or thin, with at least one convex face;*
- *every pair of faces is connected by a thick path.*

Let $K_c = \ker(\Phi_*|_c : T_c\underline{\mathfrak{C}} \to T_{\Phi(c)}\underline{\boldsymbol{\Phi}})$. Then $\Theta_*(K_c) = L_{\Theta(c)}$.

In other words, for any vector $a = a^e\partial_{\theta_e}$ with sum of coefficients 0, one can vary c so that its first order change in $\theta_\bullet$ is a, and also remains face non-singular up to first order.

Before we prove this, it is convenient to first prove the following:

Lemma 2.2. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Then Φ is a submersion in a neighbourhood of c.*

Proof. We construct vectors $\beta_f \in T_c\underline{\mathfrak{C}}$ so that $\Phi_*(\beta_f) = \frac{\partial}{\partial\Phi_f}$. The vector

$$\alpha_f := \frac{\partial}{\partial r_f} - \sum_{\vec{e}\in\partial f}\frac{\tan\varphi_{\vec{e}}}{r_f}\frac{\partial}{\partial\varphi_{\vec{e}}} \in T(\underline{\mathfrak{R}} \times \underline{\mathfrak{Q}}) \tag{2.1}$$

doesn't change $l_{\vec{e}}$, so is in $T_c\mathfrak{C}$. Intuitively, α_f is like pulling the center of C_f up off the plane, increasing r_f and decreasing all φ's. Its pushforward under Φ is simply $\Phi_*(\beta_f) = \frac{1}{r_f}\sum_{\vec{e}\in\partial f}\tan\varphi_{\vec{e}}\frac{\partial}{\partial\Phi_f}$.

If f is a convex face, then the $\tan\varphi_{\vec{e}}$ are all non-negative, being 0 if and only if e is short. so $\Phi_*(\alpha_f)$ is a negative multiple of $\frac{\partial}{\partial\Phi_f}$; we choose β so that

$$\beta_f := \beta\cdot\alpha_f;\quad \Phi_*(\beta_f) = \frac{\partial}{\partial\Phi_f} \tag{2.2}$$

If f is thin, we actually have that $\alpha_f \in K_c$, since the coefficients sum to 0 so that $\Phi_*(\alpha_f) = 0$, so we need a different approach. First suppose f shares an edge e with a convex face f'. We can increase l_e by increasing $\varphi_{\vec{e}}, \varphi_{\overleftarrow{e}}$ while holding $r_f, r_{f'}$ constant. This will affect both $\Phi_f, \Phi_{f'}$, so we use $\beta_{f'}$ to make $\Phi_{f'}$ constant. More explicitly, consider

$$\alpha_e = \frac{1}{2r_f\cos\varphi_{\vec{e}}}\frac{\partial}{\partial\varphi_{\vec{e}}} + \frac{1}{2r_{f'}\cos\varphi_{\overleftarrow{e}}}\frac{\partial}{\partial\varphi_{\overleftarrow{e}}} \tag{2.3}$$

Then we can take

$$\beta_f = r_f\cos\varphi_{\vec{e}}(\alpha_e - \frac{1}{r_{f'}\cos\varphi_{\overleftarrow{e}}}\beta_{f'}) \tag{2.4}$$

which increases $l_{\vec{e}}, l_{\overleftarrow{e}}$ at unit speed, $dl_{\vec{e}}(\alpha_e) = dl_{\overleftarrow{e}}(\alpha_e) = 1$. We can repeat this procedure with f' set to this thin face, and f set to another thin face adjacent to it, etc. $\square$

Proof of Proposition 2.1. We construct a vector $u_e \in K_c \subseteq T_c\mathfrak{C}$ for each edge e and show that $\{\Theta_*(u_e)\}_{e\in E}$ spans an $|E|-1$ dimensional space, thus must be equal L_{Θ_c}.

For long edges e, u_e will have the following property: if $\Theta_*(u_e) = \sum_{e'\in E} a^{e'}\frac{\partial}{\partial\theta_{e'}}$, then

- $a^e = 1$;
- $a^{e'} \leq 0$ for all $e' \neq e$;
- $a^{e'} = 0$ for short edges e'.

Furthermore, these u_e's collectively satisfy the following connectivity property: consider the graph G_{long} whose vertex set is the set of long edges, and we connect two long edges e, e' by an edge if there is some e" such that $a^e, a^{e'}$ are both nonzero in $\Theta_*(u_{e"})$; then G_{long} is connected.

In addition, for short edges e, u_e have the following property: again writing $\Theta_*(u_e) = \sum_{e'\in E} a^{e'}\frac{\partial}{\partial\theta_{e'}}$, one has

- $a^e = 1$;
- $a^{e'} = 0$ for other short edges e'.

Let us first suppose we have constructed such u_e, and show that these properties ensure that $\{\Theta_*(u_e)\}_{e\in E}$ spans an $|E|-1$ dimensional space. This is a simple exercise in linear algebra, but we show it for completeness.

Put the $\Theta_*(u_e)$'s into a $E\times E$ matrix, denoted M, so that the e-th row corresponds to u_e. By virtue of $u_e \in K_c$, we have that $(1\ 1\ \cdots\ 1)^T$ is in the null space of M; our goal is to show that it spans the null space.

Suppose $b = (b_e)^T$ is in the null space of M. Reorder E, if necessary, so that long edges come before short edges. Then by the last property of u_e's for long edges, the top right block of long-by-short entries are 0. Thus the vector $b_{\text{long}} := (b_e)^T_{e\in\text{long}}$ is in the null space of the submatrix M_{long} of long-by-long entries.

Let $|b_e|$ be the largest among components of b_{long}; rescale b so that $b_e = 1$. The e-th component of $M_{\text{long}}b_{\text{long}}$ is $1 - \sum a^{e'}b_{e'}$. Since $\sum a^{e'} = 1$, this can 0 if and only if for all e' with $a^{e'} < 0$, we have exactly $b_{e'} = 1$. By connectedness of G_{long}, this implies $b_{\text{long}} = (1\ \cdots\ 1)^T$.

Returning to the full M, it is now easy to see that for a short edge e, by looking at the e-th row of M, we have $b_e = \sum a^{e'} = 1$, so we are done.

Now we construct the u_e's. First suppose that e is a long edge between two convex faces f, f' (so $\vec{e}\in\partial f, \overleftarrow{e}\in\partial f'$). Observe that α_e from (2.3) increases Φ_f and $\Phi_{f'}$, and we can compensate using β_f from (2.2), so we consider

$$w_e = \alpha_e - \frac{1}{\cos\varphi_{\vec{e}}}\beta_f - \frac{1}{\cos\varphi_{\overleftarrow{e}}}\beta_{f'} \in K_c$$

For $\vec{e'} \in \partial f \cup \partial f' \backslash \{\vec{e}, \overleftarrow{e}\}$, the $\frac{\partial}{\partial \varphi_{\vec{e'}}}$ component only appears in β_f or $\beta_{f'}$, which is positive by construciton (this requires f, f' to be convex faces); thus, $d\theta_{e'}(w_e) \geq 0$ for such e', and is equal 0 if and only if e' is short. Finally, since $w_e \in K_c$, it leaves the sum $\sum_{e \in E} \theta_e$ constant, so $d\theta_e(w_e) < 0$, so we can take

$$u_e := (d\theta_e(w_e))^{-1} \cdot w_e \tag{2.5}$$

In general, we consider a thick path of faces $f_0, \dots, f_n$, such that all faces except f_0 and f_n are thin, and f_i are distinct except possibly $f_0 = f_n$. [1] Let e_i be the long edge between f_i and f_{i+1}, and orient so that $\vec{e_i} \in \partial f_i$, $\overleftarrow{e_i} \in \partial f_{i+1}$.

We will construct u_{e_i} for $i = 1, \dots, n$.

We want to increase the lengths of e_i's all together. Let $E' = \{\vec{e_i}, \overleftarrow{e_i} | i = 1, \dots, n\}$. Consider

$$x := \sum_{i=0,\dots,n-1} \alpha_{e_i} = \sum_{\vec{e} \in E'} \frac{1}{2 r_{f_{\vec{e}}} \cos \varphi_{\vec{e}}} \frac{\partial}{\partial \varphi_{\vec{e}}}$$

with α_e from (2.3). It is clear that x increases all oriented e_i lengths $l_{\vec{e}}$ equally fast, i.e. $dl_{\vec{e_i}}(x) = dl_{\overleftarrow{e_i}}(x) = 1$ for all i, so $x \in T_c\underline{\mathfrak{C}}$. It is also clear that for the thin faces f_i, x makes one of the angles $\varphi_{\overleftarrow{e_{i-1}}}, \varphi_{\vec{e_i}}$ increase and makes the other decrease, both at the same rate, thus $d\Phi_{f_i}(x) = 0$.

The vector x increase Φ_{f_0} and Φ_{f_n}, so it is compensated using β_f's, so that

$$y := x - \frac{1}{r_{f_0} \cos \varphi_{\vec{e_0}}} \beta_{f_0} - \frac{1}{r_{f_n} \cos \varphi_{\overleftarrow{e_{n-1}}}} \beta_{f_n} \in K_c$$

Before we modify y to give the desired u_{e_i}, we note that for $\vec{e} \in (\cup \partial f_i) \backslash E'$, the coefficient of $\frac{\partial}{\partial \varphi_{\vec{e}}}$ in y is non-positive, and is 0 if and only if e is short; so $\Theta_*(y)$ satisfies two of the three properties desired in u_e.

Recall that for thin face f, α_f from (2.1) is in K_c, and we have

$$\Theta_*\left(\frac{r_{f_i}}{\tan \varphi_{\vec{e_i}}} \alpha_{f_i}\right) = \frac{\partial}{\partial \theta_{e_i}} - \frac{\partial}{\partial \theta_{e_{i-1}}}$$

Write $\bar{\alpha}_{f_i} = \frac{r_{f_i}}{\tan \varphi_{\vec{e_i}}} \alpha_{f_i}$. Then it is easy to see that using $\bar{\alpha}_{f_i}$'s, we can push the coefficients of $\frac{\partial}{\partial \theta_{e_i}}$'s in $\Theta_*(y)$ around so only one is nonzero. More precisely, if $\Theta_*(y) = \sum_e y^e \frac{\partial}{\partial \theta_e}$, the image under Θ_* of the vector

$$\bar{u}_{e_0} := y - y^{e_{n-1}} \bar{\alpha}_{f_{n-1}} - (y^{e_{n-2}} + y^{e_{n-1}}) \bar{\alpha}_{f_{n-2}} - \dots$$

would have $\frac{\partial}{\partial \theta_{e_i}}$ coefficients 0 for all i except 0. Finally, we can take

$$u_{e_0} := (d\theta_{e_0}(\bar{u}_{e_0}))^{-1} \cdot \bar{u}_{e_0}; \quad u_{e_i} = u_{e_{i-1}} + \bar{\alpha}_{f_i}$$

We still need to show the collective connectivity property of the u_e's. It is easy to see that the long edges of a convex face are connected in G_{long} (for example, u_{e_0} defined above clearly does it for f_0). Furthermore, all the $\Theta_*(u_{e_i})$ share most of the same coefficients, so e_i are in the same connected component of G_{long}. Thus connectivity of G_{long} follows from the thick-path-connectivity of faces.

Finally, for short edges e, we simply define u_e as in (2.5) (here we do not worry about whether faces are thin or convex, as there is no non-positivity condition of $a^{e'}$ is absent). $\square$

Proposition 2.3. *Let $c \in \underline{\mathfrak{C}}$ be as in Proposition 2.1. Given a path $\gamma : [0, \varepsilon] \to \underline{\mathbf{\Theta}}$ starting at $\gamma(0) = \Theta(c)$ and has constant value $(\sum \theta_e) \circ \gamma$, there exists a lift $\tilde{\gamma} : [0, \varepsilon] \to \underline{\mathfrak{C}}$ that starts at $\tilde{\gamma}(0) = c$ and has constant value $\Phi \circ \tilde{\gamma}$.*

Proof. TODO. $\square$

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[1] Picture a slinky linking two rocks.