AUGMENTED LINKS IN THE THICKENED TORUS

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Abstract. abstract goes here...

1. Introduction

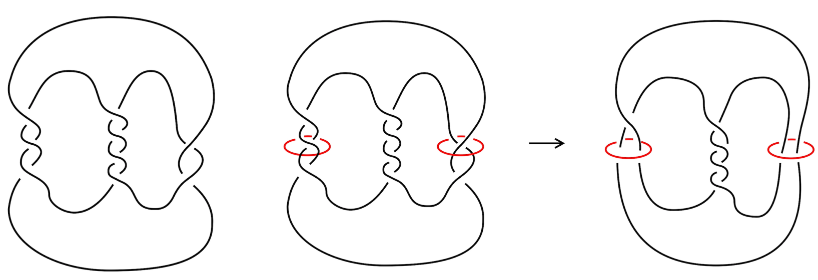
Given a twist reduced diagram of a link L, *augmentation* is a process in which an unknotted circle component is added to one or more twist regions (a single crossing or a string of bigons) of L. Due to the added circle component we can remove full twists at the twist region of L. If the twist region has an odd number of crossings then all but one crossing is removed, whereas if the twist region has an even number of crossings then all are removed. The newly obtained link diagram is called an *augmented link diagram*. See Figure 2.

Adams showed in [1] that given a hyperbolic alternating link K in S^3 the link L obtained by augmentation K is hyperbolic. In this paper we investigate if this statement holds for links in the thickened torus i.e. if L is a link obtained from augmenting a hyperbolic alternating link K in the thickened torus. In this chapter we find many families of hyperbolic links in the thickened torus which remain hyperbolic after augmentation.

2. Augmented Links

Champanerkar, Kofman and Purcell have studied alternating links in the thickened torus. They define a link in the thickened torus as a quotient of a biperiodic alternating link as follows,

Definition 2.1. [3] A *biperiodic alternating link* $\mathcal{L}$ is an infinite link which has a projection onto $\mathbb{R}^2$ which is invariant under an action of a two dimensional lattice Λ by translations, such that $L = \mathcal{L}/\Lambda$ is an alternating link in $\mathbb{T}^2 \times I$, where $I = (-1, 1)$, with the projection on $\mathbb{T}^2 \times \{0\}$. We call L a link diagram in $\mathbb{T}^2 \times I$.



(a) (b)

Figure 1. The left shows a pretzel knot before augmentation and the right shows after augmentation

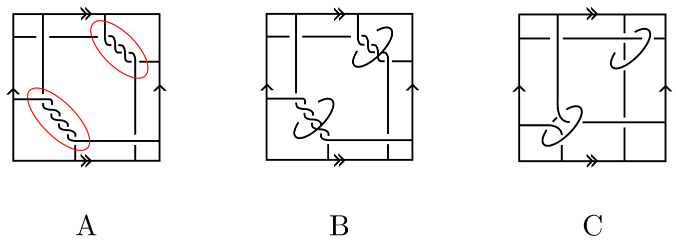


FIGURE 2. A: The top right has an odd number of twists while the bottom left has an even number of twists. B: The picture of the link on the right after augmentation twist regions circled in red. C: The link with the twists removed.

Remark 2.2. Since $\mathbb{T}^2 \times I \cong S^3 - H$, where H is a Hopf link. The complement $\mathbb{T}^2 \times I - L = S^3 - (L \cup H)$.

Champanerkar, Kofman and Purcell [3] extended the definition of prime links in S^3 for links in $\mathbb{T}^2 \times I$ called weakly prime.

Definition 2.3. A diagram of a link L is weakly prime if whenever a disk is embedded in the diagram surface meets the diagram transversely in exactly two edges, then the disk contains a simple edge of the diagram and no crossings.

Definition 2.4. A *twist region* in a link diagram $L = \mathcal{L}/\Lambda$ in $\mathbb{T}^2 \times I$, is the quotient of a twist region in the biperiodic link $\mathcal{L}$. A biperiodic link $\mathcal{L}$ is called *twist-reduced* if for any simple closed curve on the plane that intersects $\mathcal{L}$ transversely in four points, with two points adjacent to one crossing and the other two points adjacent to another crossing, the simple closed curve bounds a subdiagram consisting of a (possibly empty) collection of bigons strung end to end between these crossings. We say L is *twist-reduced* if it is the quotient of a twist-reduced biperiodic link.

Now we can define augmentation for a link in $\mathbb{T}^2 \times I$ the same way we define augmentation for links in S^3. For a link in $\mathbb{T}^2 \times I$, the crossing circles are added to the diagram projected onto $\mathbb{T}^2 \times \{0\}$. Let L be a twist reduced diagram in $\mathbb{T}^2 \times I$, we define *augmentation* as a process in which an unknotted circle component is added to one or more twist regions of L. See Figure 2

2.1. **Torihedral Decomposition of Augmented Alternating Links in Thickened Torus.** We show a method of decomposing an augmented link in the thickened torus into two torihedra. The idea is to combine methods of Menasco [7] and the use of crossing edges between each crossing of our link and Lackenby's "cut-slice-flatten" method [6] on the augmentation sites.

Definition 2.5. [3] A *torihedron* $\mathcal{T}$ is a cone on the torus, i.e. $\mathbb{T}^2 \times [0,1]/(\mathbb{T}^2 \times \{1\})$, with a cellular graph $G = G(\mathcal{T})$ on $\mathbb{T}^2 \times \{0\}$. An *ideal torihedron* is a torihedron with the vertices of G and the vertex $\mathbb{T}^2 \times \{1\}$ removed. Hence, an ideal torihedron is homeomorphic to $\mathbb{T}^2 \times [0,1)$ with a finite set of points (ideal vertices) removed from $\mathbb{T}^2 \times \{0\}$. We refer to the vertex $\mathbb{T}^2 \times \{1\}$ as the cone point.

For visualization purposes, we typically draw the graph $G(\mathcal{T})$ of a torihedron from the perspective of the cone point $\mathbb{T}^2 \times \{1\}$.

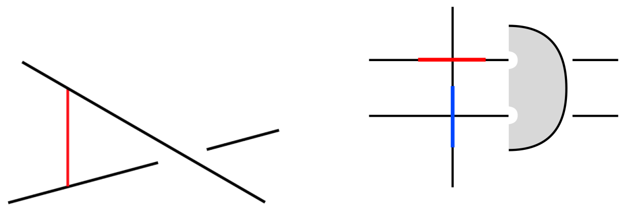


FIGURE 3. Left: The black strands are part of the link and the red strand is the crossing edge. Right: The blue and red edges represent the split crossing edges and the shaded half disk is bounded by the crossing circle

If the faces of $G(\mathcal{T})$ are disks, then $\mathcal{T}$ can be decomposed into a union of pyramids, obtained by coning each face to the cone point of $\mathcal{T}$. This also gives a decomposition of the corresponding ideal torihedron into ideal pyramids. We call these the *pyramidal decompositions* of $\mathcal{T}$ and its ideal version.

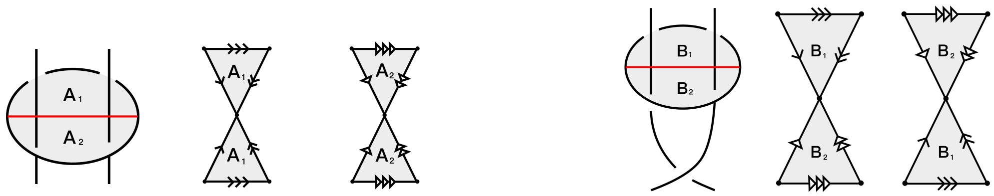
Proposition 2.6. *Let L be an augmented link in $\mathbb{T}^2 \times I$. There is a decomposition of the complement, $(\mathbb{T}^2 \times I) - L$ into two ideal torihedra.*

Proof. We will begin by assuming that there are no half twists and then arrange the link diagram of L in the following way: first place the added circle components (augmentation) perpendicular to the projection plane, $\mathbb{T}^2 \times \{0\}$ leaving the remaining part of the link parallel to the projection plane. We now place a crossing edge on each crossing of the link so that for each crossing edge, one end of the edge lies on a bottom strand while the other end lies on a top strand as in Figure 3 left.

We view the link from the point at infinity from the top. We will push the top strand to the bottom strand, splitting the crossing edge into two identical edges as in Figure 3 right. We push the link components to infinity and stretch the crossing edge so that we have flattened the link onto $\mathbb{T}^2 \times \{0\}$ except for the crossing circles which will remain perpendicular to the projection plane.

Now place a disk on each crossing circle, so that the disk is bounded by the crossing circle. We can then cut $\mathbb{T}^2 \times I$ along $\mathbb{T}^2 \times \{0\}$ and focus on the top half, $\mathbb{T}^2 \times [0, 1)$. We will follow the same method on the bottom half to obtain the second identical torihedron. The disk we place on each crossing circle is now cut in half. This half disk is now bounded by the projection plane and the semi-cricle arc of the crossing circle. We push down on the crossing circle and split the disk into two identical disks. We then push the arc of each crossing circle to infinity, collapsing them to ideal vertices. We obtain two triangular faces which represent the disk which look like a bow-tie as in Figure 4.

We repeat the steps for the bottom half of $\mathbb{T}^2 \times I$, $\mathbb{T}^2 \times (-1, 0]$. Then we get two torihedra. The graph of each will come from crossing edges and edges of the disk. Now, if there are half twists we will decompose the complement of the link the same way as if there are no half twists and we will identify the two bow-ties as in Figure 4. Finally, we obtain the complement of the link by gluing the two torihedra with the gluing information given by identifying crossing edges and triangles of the bow-tie. We glue the faces of the torihedra which do not correspond to a bow-tie with a $2\pi/n$ twist where n is the number of sides of each face as in Figure 8 clockwise or counterclockwise.



(a) (b)

FIGURE 4. The first pictures shows gluing without half-twists the second shows gluing with half-twists

For future reference, we will denote the graph for the top and bottom torihedra by $\Gamma_T(L)$ and $\Gamma_B(L)$, respectively, where both graphs are viewed from the cone point of the top torihedron $\mathbb{T}^2 \times \{1\}$. Note that if $L = K$ is the non-augmented link, $\Gamma_T(L)$ is simply the link projection K, and in fact $\Gamma_T(K) = \Gamma_B(K)$.

□

The following figures is an example which decomposes the link (C) of Figure 2.

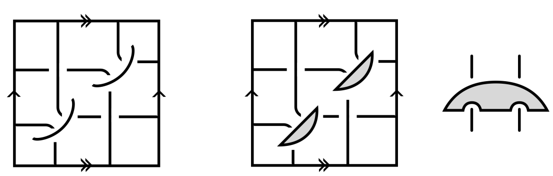


FIGURE 5. Each crossing circle bounds a disk

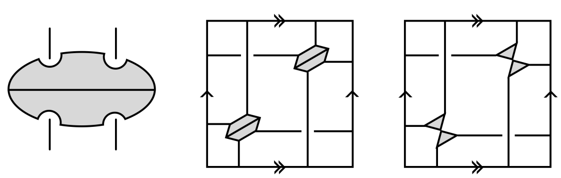


FIGURE 6. We split the disk and collapse the arc of each crossing circle to ideal vertices

TODO refer to BS about theta's

Definition 2.7. An *angled torihedron* $(\mathcal{T}, \theta^*_\bullet)$ is a torihedron $\mathcal{T}$ with an assignment $\theta^*_e \in [0, \pi]$ such that for each vertex $v \in G(\mathcal{T})$, $\sum_{e \ni v} \theta^*_e = (\deg(v) - 2)\pi$. We also denote $\theta_e = \pi - \theta^*_e$, so $\sum_{e \ni v} \theta_e = 2\pi$; we refer to θ_e as the exterior angle and θ_e as the interior angle.

We say $(\mathcal{T}, \theta^*_\bullet)$ is *degenerate* if $\theta^*_e = 0$ for some edge; we say it is *non-degenerate* otherwise.

One may ask for the pyramidal decomposition of a torihedron to "respect" angles. The following definitions make sense of this.

Definition 2.8. An *angled ideal tetrahedron* is an ideal tetrahedron with an assignment of a dihedral angle to each edge, such that

- each dihedral angle is in $[0, \pi]$;

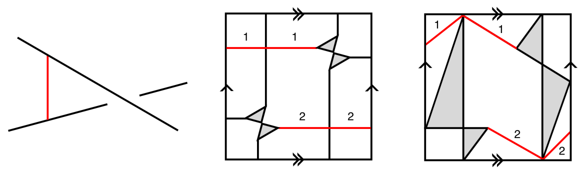


FIGURE 7. Left: The crossing arc is the edge in red. Middle: Picture of splitting the crossing edge. Right: The link component is pushed off to infinity.

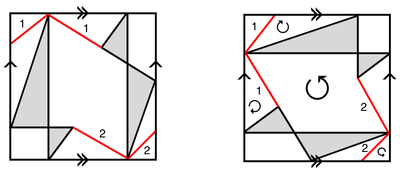


FIGURE 8. Left: The top torihedron. Right: The bottom torihdron with rotation for face gluing.

- for each tetrahedron, opposite edges have equal dihedral angles;
- the three distinct angles sum to π.

We say a angled ideal tetrahedron is *degenerate* if one dihedral angle is 0; we say it is *non-degenerate* otherwise.

Definition 2.9. A *base-angled ideal pyramid* is a pyramid whose base is an n-gon, $n \geq 3$, and each boundary edge e_i of the base face is assigned a dihedral angle $\alpha_i \geq 0$ such that the sum is $\sum \alpha_i = 2\pi$. The vertical edge e_i' that meets e_i and e_{i+1} is automatically assigned the dihedral angle $\pi - \alpha_i - \alpha_{i+1}$.

We say a base-angled ideal pyramid is *degenerate* if $\alpha_i = 0$ for some i; we say it is *non-degenerate* otherwise.

Clearly, the dihedral angles of an ideal hyperbolic pyramid make it a base-angled ideal pyramid (with $\alpha_i = \varphi_{e_i}$); it is not hard to see that the converse is true: simply consider a circumsribed polygon such that the side e_i subtends an angle of $2\alpha_i$ at the center, and take the ideal hyperbolic pyramid over it in upper-half space. Also, an angled ideal tetrahedron is simply a base-angled ideal pyramid with base a triangle, and with no preferred face.

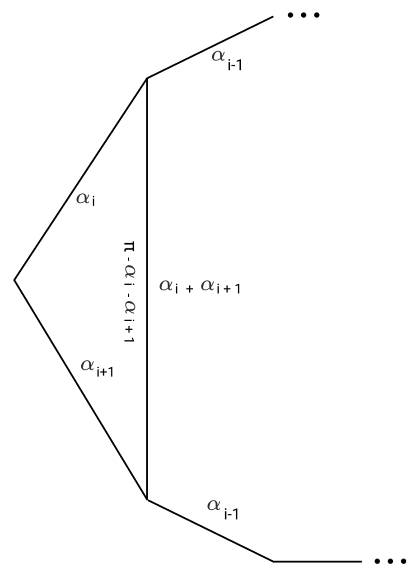
Definition 2.10. An *angle splitting* of an angled torihedron $(\mathcal{T}, \theta^*_\bullet)$ is a splitting of $\theta^*_e = \varphi_{\vec{e}} + \varphi_{\overleftarrow{e}}$ for each edge e, such that for each face f, $\sum_{\vec{e} \in \partial f} \varphi^*_{\vec{e}} = \pi$.

Equivalently, an angle splitting is a decomposition of $\mathcal{T}$ into base-angled pyramids, one for each face of $G(\mathcal{T})$, such that for each boundary edge e of $\mathcal{T}$, the dihedral angles from the two adjacent pyramids add to θ^*_e.

TODO same as [BS] "coherent angle system" or something.

TODO check the feasible flow stuff.

Lemma 2.11. *Let P_n be a base-angled ideal pyramid, and suppose we are given a decomposition of the base face into triangles by adding new edges. One gets an obvious corresponding*



triangulation of P_n, where a new face is added for each new edge. Then there is an assignment of a dihedral angle to each edge of each ideal tetrahedron in this triangulation such that

- *each tetrahedron is an angled ideal tetrahedron;*
- *the sum of dihedral angles around each new edge is π;*
- *the dihedral angles of the edges of the original base face are the same as before.*

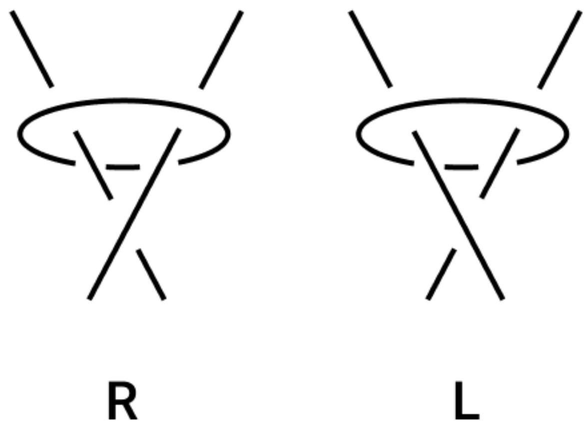
Proof. Induct on n; there is nothing to prove for the base case $n = 3$.

The proof is essentially given in Figure 2.1 below

Suppose the edges are labeled e_i, which goes between vertices v_i and v_{i+1}, and suppose e_i is assigned dihedral angle α_i. Let e' be a new edge addeed to the base face of P_n such that it separates the base face into a triangle and an $(n-1)$-gon; suppose the sides of the triangle are e_i, e_{i+1}, and e'. The new face corresponding to e' separates P_n into an ideal tetrahedron T and an ideal pyramid P_{n-1}. We assign the dihedral angle of $\pi - \alpha_i - \alpha_{i+1}$ to e' in T, and assign $\alpha_i + \alpha_{i+1}$ to e' in P_{n-1}. Clearly the sum of dihedral angles condition is satisfied in T and P_{n-1}. It remains to check that the dihedral angles assigned to the vertical edges are correct. For the vertical edge associated to v_j for $j \neq i, i+2$, there is nothing to check; for $j = i$, the dihedral angles are $\pi - \alpha_i - (\pi - \alpha_i - \alpha_{i-1})$ in T and $\pi - \alpha_{i-1} - (\alpha_i + \alpha_{i+1})$ in P_{n-1}, which sum to $\pi - \alpha_i - \alpha_{i+1}$; it is similar for $j = i + 2$. □

3. Hyperbolicity of Augmented Links

Thurston introduced a method for finding the unique hyperbolic metric for a given 3-manifold M with boundary consisting of tori [8]. The idea was to triangulate the interior of M into ideal tetrahedra and give those tetrahedra hyperbolic shapes (called shape parameters) that glue up coherently in M. The shape parameter of a tetrahedron is described by the cross-ratio of its four vertices on the sphere at infinity. Thurston had written down a system of gluing equations with shape parameters whose solutions correspond to the complete hyperbolic metric on the interior of M. Casson and Rivin separated gluing equations



into a linear and non-linear part [5]. Angle structures is the linear part of Thurston's gluing equations, and what we will use to attain hyperbolicity of complements of augmented links in the thickened torus.

Definition 3.1. Let M be an orientable 3-manifold with boundary consisting of tori. An angle structure on an ideal triangulation τ of M is an assignment of a dihedral angle to each edge of each tetrahedron, such that

- each tetrahedron is a non-degenerate angled ideal tetrahedron,
- around each edge of τ, the dihedral angles sum to 2π.

Theorem 3.2. [5] *Let M be a 3-manifold admitting an angle structure. Then M is hyperbolic.*

For a hyperbolic link K in $\mathbb{T}^2 \times I$, we produce sufficient conditions on augmentations such that the resulting link obtained from augmenting K is hyperbolic. The idea is to start with a graph from the torihedral decomposition of the link K which will give us a graph on each torihedron with $\pi/2$ edges [3]. Then by results of [2] there exist a corresponding right-angled circle pattern. We then consider the augmented link L and its torihedral decomposition from Proposition 2.6 with a corresponding "degenerate" circle pattern. We deform this degenerate circle pattern into a "proper" circle pattern which will give us a polyhedral decomposition of $(\mathbb{T}^2 \times I) - L$ with angles of the torihedra in our torihedral decomposition. Which we can further decompose into tetrahedra with angles satisfying conditions of an angle structure.

Definition 3.3. We say an augmentation is *right-augmented* if, when both strands are (locally) oriented such that they cross the augmentation disk in the same direction, the crossing is positive/a right-handed half-twist. See Figure **??**. We say an augmentation is *left-augmented* if it is not right-augmented.

We can recover L from the link diagram of K together with labels at vertices indicating left- or right-augmentation.

Lemma 3.4. *Let G be a 4-valent graph on $\mathbb{T}^2$ with no bigons or self-loops. Then for any subset $F' \subseteq F$ of faces,*

$$2\chi(F') \leq |E'|$$

where $E' = \cup_{f\in F'}\partial f$ is the set of edges that meets some face of F', and $\chi(F') = \sum_{f\in F'}\chi(f)$ is the sum of the Euler characteristics of faces in F'.

Proof. We induct on $\chi(F')$ (over all possible such graphs G); for $F' = \emptyset$, there is nothing to prove.

Call a vertex or edge *interior* if it does not meet $F'' = F\backslash F'$. Suppose there is an interior vertex v. Make the modification as in Figure **??** (later we will choose which way to "resolve" the vertex). This decreases $chi(F')$ by one and $|E'|$ by 2. However, this process may produce bigons; if it does, repeatedly eliminate bigons by either performing the modification as in Figure **??** if the two vertices of the bigon are distinct, or remove the bigon completely if the two vertices of the bigon are the same. Each of these steps also decreases $chi(F')$ by one and $|E'|$ by 2 (note in the second case, after removing the bigon, the operation merges two distinct faces or joins a face to itself; this adds an S^1, thus does not change the (total) euler characteristic of the two faces or the one face).

We must check that this process terminates with a graph which satisfies the hypotheses set out in the lemma. It is clear that 4-valency is maintained at each step, and obviously there will not be any bigons at the end. We need to ensure that eliminating bigons does not create self-loop edges. Here we sepcify how to choose which way to resolve the internal vertex v. If among the four faces touching v, one of them has greater than three sides (Case 1), then resolve v so that this face is not merged; if all four faces are triangles (Case 2), simply resolve in an arbitrary manner. After the modification of the internal vertex, at most two bigons are created; eliminate those two bigons. Now we find that all bigons will be connected end-to-end (i.e. forming a single twist region) - this is because in Case 1, only one bigon was present after smoothing v, which would result in at most two bigons that would be connected; and if we were in Case 2, we find that the face between those eliminated bigons (coming from merging the two triangles across v) will become a bigon too, thus connecting the potential bigons on either ends.

Thus, one can eliminate bigons in an order so that they belong in one twist region; in particular, the (not necessarily distinct) faces on either side are not bigons, so eliminating the bigon does not produce a self-loop edge.

Now suppose G has no interior vertices. We perform some reductions. Remove all $f \in F'$ that have non-positive Euler characteristic, since having them only makes the inequality easier to prove. Now consider some $f \in F'$. If it has at least two boundary (i.e. non-interior) edges, we can remove f from F', which decreases $chi(F')$ by 1 and $|E'|$ by at least 2. Thus we remove all such f.

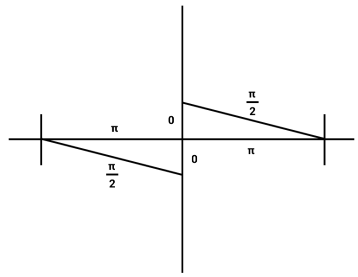
Now suppose $f \in F'$ has three contiguous interior edges e_1, e_2, e_3. (e.g. if f has at least four sides). Let v_1, v_2 be the vertices between these three interior edges. Since v_1 is not inteior, the face across v_1 from f is not in F'; likewise with v_2. But this shows that the face across e_2 from f has two boundary edges, contradicting our reduction.

Thus, we now have a graph whose faces are triangles, and each face has exactly one boundary edge. Now the problem is solved with a simple counting argument: each f corresponds to one boundary edge and two interior edges, but since the interior edges are shared by two faces, they should count as half an edge; so

$$|E'| = \sum_{f\in F'} 1 + 1/2 + 1/2 = 2|F'| = 2\chi(F')$$

and we are done. □

The following theorem is quoted from [2] (see also [4])



Theorem 3.5 (Feasible Flow Theorem)**.** *Let (N, X) be a directed graph, with a* lower capacity bound a_x *and* upper capacity bound b_x *for each directed edge $x \in X$, with $-\infty \leq a_x \leq b_x \leq \infty$. A* feasible flow *is a function $\varphi : X \to \mathbb{R}$ such that $\varphi_x \in [a_x, b_x]$ and Kirchhof's current law is satisfied (i.e. flow in = flow out at every vertex).*

A feasible flow exists if and only if, for every proper nonempty subset $N' \subset N$,

$$\sum_{x \in ex(N')} b_x \geq \sum_{x \in in(N')} a_x$$

where $ex(N'), in(N')$ refer to edges leaving, entering N', respectively.

Theorem 3.6. *Let K be a weakly prime, alternating link whose diagram has no bigons. Let L be a link obtained from augmenting K. Then L is hyperbolic.*

Proof. By [3, Theorem 7.5], (as discussed in Proposition 2.6,) $\mathbb{T}^2 \times I - K$ can be decomposed into two torihedra $\mathcal{T}_T$ and $\mathcal{T}_B$, whose graphs are $\Gamma_T(K), \Gamma_B(K)$; viewed from the top cone point $\mathbb{T}^2 \times \{1\}$, they are both the same as the projection graph of K. We make them non-degenerate angled torihedra by assigning $\theta^* = \pi/2$ for all edges.

We obtain an angle-splitting by applying the Feasible Flow theorem (Theorem 3.5) as follows: Consider the directed graph whose vertex set is $E \cup F \cup \{\otimes\}$, where E, F are the set of edges, faces in $\Gamma_T(K)$, and $\otimes$ is some abstract vertex. There is a directed edge

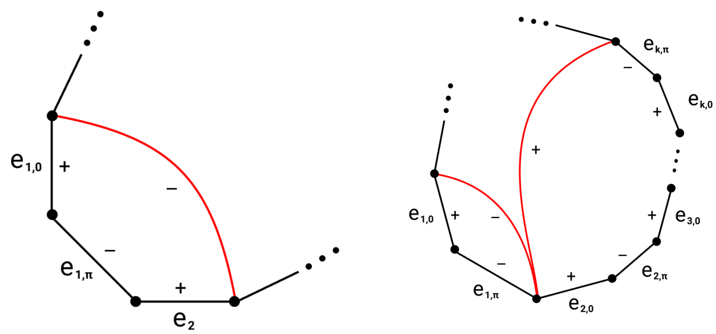
- $\otimes \to f$ for each face $f \in F$, with capacity interval $[\pi, \infty)$,
- $f \to e$ for each edge $e \in \partial f$, with capacity interval $[\varepsilon, \infty)$ for some $\varepsilon > 0$ to be set later,
- $e \to \otimes$ for each edge e, with capacity interval $(-\infty, \pi/2]$.

By Lemma 3.4, $2|F'| = 2\chi(F') \leq |E'|$, and taking $\varepsilon < \pi/|\text{max face size}|$, the feasible flow condition is satisfied, so a feasible flow exists. Since $2|F| = |E|$, the capacity interval restrictions on the flow at $\otimes$ is sharp, so out-edges at $\otimes$ have flow π and in-edges at $\otimes$ have flow $\pi/2$. Then the flow $f \to e$ gives us $\varphi_{\vec{e}}$, where f is the face to the left of $\vec{e}$. (we adapted this argument from [2]).

By Proposition 2.6, $\mathbb{T}^2 \times I - L$ can be obtained by gluing two torihedra $\mathcal{T}_T(L), \mathcal{T}_B(L)$ with graphs $\Gamma_T(L), \Gamma_B(L)$. We make them degenerate angled torihedra by assigning θ^*'s to edges of the bow-ties as in Figure 3, and assign $\pi/2$ to all other edges. It is easy to check that upon gluing, each edge has sum of dihedral angles (θ^*) equals 2π.

Furthermore, we can obtain an angle-splitting of $\mathcal{T}_T(L)$ (and similarly $\mathcal{T}_B(L)$) by modifying the angle-splitting for $\mathcal{T}_T(K)$; this is shown in Figure **??**.

Now we have a decomposition of the two torihedra into degenerate base-angled pyramids. However, we need the pyramids to be non-degenerate, so we first need to modify the angles and graph on the torihedra to make all θ^* nonzero.



Consider a face f of $\Gamma_T(L)$ that is not in a bow-tie. Suppose the corresponding face $\bar{f}$ of Γ, the projection graph (which is equal to $\Gamma_T(K)$), had vertices $v_1, \ldots, v_n$ in counter-clockwise order. We label the edges of f by $e_{i,0}$, $e_{i,\pi}$, or e_i, depending on whether the θ^* is 0, π, or $\pi/2$ respectively, with i non-decreasing from 1 to n, adjacent edges having the same i if and only if they belong to the same bow-tie. For sake of concreteness, suppose that if a vertex v_i is right-augmented, then the augmentation circle intersects $\bar{f}$ (everything is similar if it is left-augmented vertices' circles that intersect $\bar{f}$). In other words, locally, f meets two of the edges of the bow-tie corresponding to a right-augmented vertex v_i (which would be labeled $e_{i,0}, e_{i,\pi}$ in counter-clockwise order), but only meets one of the edges of the bow-tie corresponding to a left-augmented vertex.

Suppose, after cyclically reindexing, $v_1, \ldots, v_k$ is a maximally contiguous subsequence of left-augmented vertices of $G(K)$ around the face $\bar{f}$; the edges around f would start $e_{1,0}, e_{1,\pi}, e_{2,0}, e_{2,\pi}, \ldots$. We add new edges across f as follows. (See Figure 3; the + and - signs will be explained later.)

First suppose $k = n$; then we do nothing.

Next suppose there is only one such maximal contiguous subsequence. If $k = 1$, we add an edge that goes across $e_{1,0}, e_{1,\pi}, e_2$ (in the sense that the new edge separates the edges of f into two sets one of them being those three edges; since $n \geq 3$, this edge is new). If $k \geq 2$, we add an edge across $e_{1,0}, e_{1,\pi}$ and another edge across $e_{2,0}, e_{2,\pi}, e_{3,0}, \ldots, e_{k,\pi}$ (these two edges do not form a bigon because we've ruled out $k = n$).

Finally, if there are multiple such maximal contiguous subsequences, we just add edges as above for each contiguous subsequence. The only caveat is that if the procedure calls to add a new edge that would form a bigon with the existing edges, we just don't add it.

This way we obtain a new graph $\Gamma_T(L)'$, which defines a new torihedron $\mathcal{T}_T(L)'$. We make it angled using the angles from $\mathcal{T}_T(L)$ for old edges, and putting π for all new edges.

Now we deform the θ^* based on Figure 3, increasing/decreasing by some small $\varepsilon' > 0$ if the edge is labeled $+/-$. Note some edges may be labeled twice, in which case we perform both increasing/decreasing (e.g. if it is labeled + and $-$, the θ^* is not changed). It is easy to see that the sum of θ^* around a vertex remains unchanged. Furthermore, all the edges with θ^* originally equal 0, i.e. all $e_{i,0}$'s, now have positive θ^*, (it receives only one label +, because the other face it meets is a bow-tie).

Thus, we may now apply the feasible flow theorem in a similar manner, and obtain an angle-splitting for $\mathcal{T}_T(L)'$. Alternatively, we can directly get an angle-splitting for $\mathcal{T}_T(L)'$ using the angle-splitting for $\mathcal{T}_T(L)$. We reuse the $+/-$ assignments from Figure 3. For $e = e_{i,0}$, we increase $\varphi_{\vec{e}}, \varphi_{\overleftarrow{e}}$ by $\varepsilon'/2$ each; for $e = e_{i,\pi}$, we decrease them by $\varepsilon'/2$ each. For other edges, we increase/decrease $\varphi_{\vec{e}}$ by ε', where $\vec{e}$ is the oriented edge corresponding to the side on which the $+/-$ sign appears in Figure 3; so for example, if an edge e receives both $+$ and $-$, then one of $\varphi_{\vec{e}}, \varphi_{\overleftarrow{e}}$ increases while the other decreases, thus θ^*_e remains constant.

Now we address $\mathcal{T}_B(L)$. In the gluing of $\mathcal{T}_T(L)$ to $\mathcal{T}_B(L)$, non-bow-tie faces of $\Gamma_T(L)$ are identified with non-bow-tie faces of $\Gamma_B(L)$. Under this identification, we add the same edges to $\Gamma_B(L)$, thus obtaining the new torihedron $\mathcal{T}_B(L)'$ with graph $\Gamma_B(L)'$. We perform the same deformations of θ^*'s (or φ's).

We need to check that upon gluing $\mathcal{T}_T(L)'$ to $\mathcal{T}_B(L)'$, the sum of dihedral angles around each edge is still 2π. This was clearly true before deforming, as the new edges of $\Gamma_T(L)'$ only gets identified with the unique corresponding edge of $\Gamma_B(L)'$, and they are both labeled with $\theta^* = \pi$. To see that the deformation does not change these sums, note that in the identification of faces of $\Gamma_T(L)'$ to $\Gamma_B(L)'$, an edge with $\theta^* = 0$ is identified with an edge with $\theta^* = \pi$, so an increase in the former would be conterbalanced by a decrease in the latter. It is also easy to see this is the case for the other edges. TODO hmm more detail?

Finally, for each face of $\Gamma_T(L)'$ that has more than three sides, we arbitrarily decompose it into triangles and apply Lemma 2.11 to obtain a triangulation of $\mathcal{T}_T(L)'$ into non-degenerate angled tetrahedra; perform the corresponding decomposition for faces of $\Gamma_B(L)'$ and obtain a triangulation of $\mathcal{T}_B(L)'$ into non-degenerate angled tetrahedra. Upon gluing, this gives an angle structure on the triangulation of $\mathbb{T}^2 \times I - L$ □

4. From Extended Circle Patterns to polyhedra

In this section, we describe how to obtain a decomposition of a torihedron into hyperbolic ideal pyramids from a non-singular extended circle pattern.

Lemma 4.1. *Suppose we have the graph of a torihedron. Given a non-singular extended circle pattern $c \in \underline{\mathfrak{C}}$ on the graph, such that all $\varphi_{\vec{e}} \in (0, \pi)$, there exists a decomposition of the torihedron into base-angled ideal pyramids (TODO make sure it's been defined) such that*

- *each interior edge of the torihedron has dihedral angles sum to 2π;*
- *each boundary edge e of the torihedron has dihedral angles sum to $\pi - \theta_e$.*

Proof. For each directed edge $\vec{e} \in \vec{E}$, construct the isosceles triangle $T_{\vec{e}}$ with equal sides of length $r_{f_{\vec{e}}}$ and angle $2\tilde{\varphi}_{\vec{e}}$ subtended between them, where $\tilde{\varphi}_{\vec{e}} = \varphi_{\vec{e}}$ if $\varphi_{\vec{e}}$ if it is acute, and $=$ $\pi - \varphi_{\vec{e}}$ otherwise.

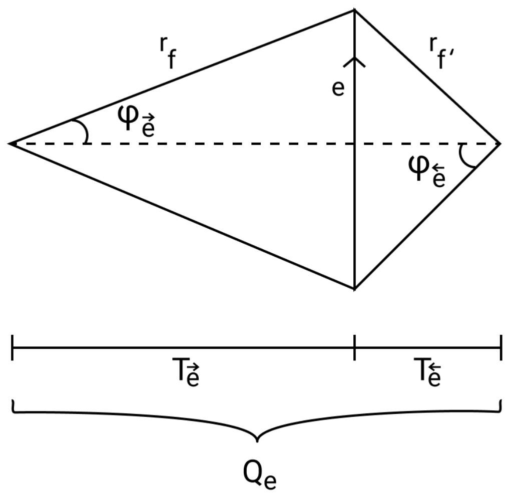
Let $f \in F$ be a face. We construct a Euclidean polygon Pol_f as follows. If $\varphi_{\vec{e}} \leq \pi/2$ for all $\vec{e} \in \partial f$, i.e. if f is convex, then the $T_{\vec{e}}$ for $\vec{e} \in \partial f$ fit together into Pol_f. If not, suppose $\varphi_{\vec{e}} > \pi/2$ for $\vec{e} = \vec{e_1}$, and $\leq \pi/2$ for $\vec{e} = \vec{e_2}, \ldots \vec{e_k} \in \partial f$. Then put $T_{\vec{e_2}}, \ldots, T_{\vec{e_k}}$ together as above, creating a $(k+1)$-gon, then subtract $T_{\vec{e_1}}$ from it to form Pol_f.

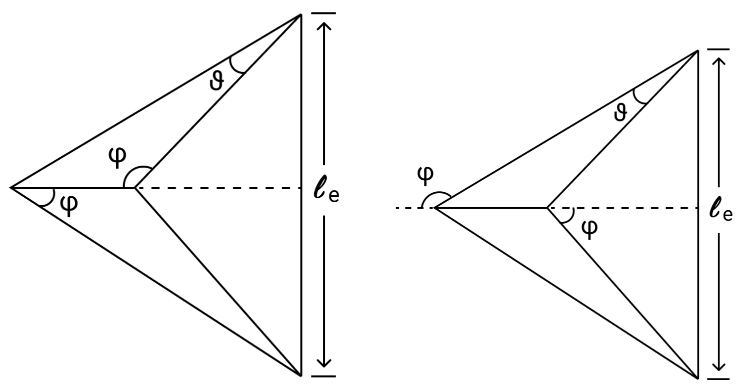
Thus, to each face, we associate the Euclidean polygon Pol_f. If v is the vertex of Pol_f between e_i and e_{i+1}, then the angle at v is $\pi - \varphi_{\vec{e_i}} - \varphi_{\vec{e_{i+1}}}$. Then the non-vertex-singularity of c guarantees that the sum of angles at v of Pol_f, for faces f containing v, is 2π.

View the Euclidean plane as the boundary of the upper-half space. Then Pol_f supports an ideal hyperbolic pyramid P_f.

Clearly, these P_f's, as abstract ideal tetrahedra, glue together into the torihedron (TODO rephrase). We need to check that the angles around each edge of the torihedron have the appropriate angles.

Consider an interior edge e of the pyramidal decomposition of the torihedron. It corresponds to a vertex v of the graph. Note that the dihedral angle of the vertical edge at v of P_f is simply the angle at v of Pol_f; these sum to 2π over $f \ni v$.





For a boundary edge e, the dihedral angles at e of the P_f's containing it are $\varphi_{\vec{e}}$ and $\varphi_{\overleftarrow{e}}$, which sum to θ_e by definition.

□

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