

# VARIATION ON A VARIATIONAL PRINCIPLE

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ABSTRACT. We prove that alternating links in the thickened torus remain hyperbolic after certain augmentations.

## 1. INTRODUCTION

Alice TODO

### Definitions, Notations, and Conventions

Always assume surface  $\Sigma$  (our case  $\Sigma = \mathbb{T}^2$ ) is oriented.

define cellular decomp of surface

$V, E, F$  are set of vertices, edges, faces. No bigon faces.

$\vec{E}$  is the set of oriented edges. We may identify an oriented edge  $\vec{e}$  with the pair  $(f_{\vec{e}}, e)$ , where  $f_{\vec{e}}$  is the face to the left of  $\vec{e}$ .

When we use  $\overleftarrow{e}$  to refer to an oriented edge, it refers to the oppositely oriented edge to  $\vec{e}$ . If we construct an expression with both  $\vec{e}$  and  $\overleftarrow{e}$ , it will always be (anti-)symmetrical in the two orientations, and we assume that an arbitrary choice has been made.

Recall circle pattern.

A circle pattern is determined by the radius of the circle  $C_f$  associated to each face,  $r_f$ , and the angle that each edge subtends in adjacent faces,  $\varphi_{\vec{e}}$ . (see figure TODO) Thus determines, and is determined by a point in  $\mathfrak{X} \times \mathfrak{Q}$ , where

- $\mathfrak{X} := \mathbb{R}_+^F = \{(r_f)_{f \in F} | r_f \in \mathbb{R}_+\}$
- $\mathfrak{Q} := \mathbb{R}^{\vec{E}} = \{(\varphi_{\vec{e}})_{\vec{e} \in \vec{E}} | \varphi_{\vec{e}} \in \mathbb{R}\}$

but clearly not every point  $c \in \mathfrak{X} \times \mathfrak{Q}$  determines a circle pattern.

On  $\mathfrak{X} \times \mathfrak{Q}$ , there are several functions to consider:

- $\Phi_f = 2 \sum_{\vec{e} \in \partial f} \varphi_{\vec{e}}$ , measuring the cone angle at the center of  $C_f$
- $\theta_e = \pi - \varphi_{\vec{e}} - \varphi_{\overleftarrow{e}}$
- $l_{\vec{e}} = r_{f_{\vec{e}}} \sin \varphi_{\vec{e}}$

These fit together to give maps to the following spaces:

- $\mathfrak{F} := \mathbb{R}^F = \{(\Phi_f)_{f \in F} | \Phi_f \in \mathbb{R}\}$
- $\mathfrak{Q} := \mathbb{R}^E = \{(\theta_e)_{e \in E} | \theta_e \in \mathbb{R}\}$
- $\mathfrak{L} := \mathbb{R}^{\vec{E}} = \{(l_{\vec{e}})_{\vec{e} \in \vec{E}} | l_{\vec{e}} \in \mathbb{R}\}$

Our main argument is to deform “degenerate” circle patterns, where adjacent circles may be identical or tangent, into ones that don’t look so degenerate, hence it is convenient to extend the notion of circle pattern:

**Definition 1.1.** Let  $\Gamma$  be a graph on the torus. An *extended circle pattern* on  $\Gamma$  is  $c = ((r_f), (\varphi_{\vec{e}})) \in \mathfrak{X} \times \mathfrak{Q}$  such that  $l_{\vec{e}} = l_{\overleftarrow{e}}$  for all edges  $e \in E$ . We denote by  $\mathfrak{C}$  the set of extended circle patterns.

$$\begin{array}{ccc} \mathfrak{C} = \{l_{\vec{e}} = l_{\overleftarrow{e}}\} & \subseteq & \mathfrak{X} \times \mathfrak{Q} \xrightarrow{\Theta} \mathfrak{Q} \\ & & \downarrow \Phi \\ & & \mathfrak{F} \end{array}$$

[TODO Check this] The usual notion of circle pattern would be restricted to those  $c \in \mathfrak{C}$  with  $\varphi_{\vec{e}} \in (0, \pi)$  and  $\theta_e \in (0, \pi)$ . One may consider deforming a circle pattern so as to have some  $\theta_e$  approach 0 or  $\pi$ . The limit  $\theta_e \rightarrow \pi$  is easy to picture, one simply gets that the two circles  $C_f, C_{f'}$  of the adjacent faces become

tangent. The limit  $\theta_e \rightarrow 0$  is a bit more complex, as the final shape of  $Q_e$  (the quadrangle associated to  $e$ ) may depend on  $\varphi_{\bar{e}}$  or  $\varphi_{\bar{e}'}$ . If we parametrize circle patterns by  $(r_f)$  and  $(\theta_e)$  as in TODO [?BS], the limiting shape would depend on the relationship between  $r_f$  and  $r_{f'}$ . As our main result depends on extending [?BS] result to such limits and beyond, we find it more convenient to describe extended circle patterns by  $(r_f)$  and  $(\varphi_{\bar{e}})$ .

We will mostly be working with extended circle patterns that “look normal”, with all  $\varphi_{\bar{e}}$  in the range  $[0, \pi)$ , so we do not dwell on the meaning of negative  $\varphi_{\bar{e}}$ 's.<sup>1</sup>

**Definition 1.2.** An extended circle pattern is said to be *face non-singular* if  $\Phi_f = 2\pi$  for all faces  $f$ ; it is said to be *vertex non-singular* if  $\sum_{e \ni v} \theta_e = 2\pi$  for all vertices  $v$ . Finally it is said to be *non-singular* if it is both.

TODO non-degenerate circle pattern?

**Definition 1.3.** Let  $f$  be a non-singular face (i.e.  $\Phi_f = 2\pi$ ), such that for all edges  $\bar{e} \in \partial f$ ,  $\varphi_{\bar{e}} \in [0, \pi)$ . We say  $f$  is *thin* if exactly two edges of  $f$  have nonzero  $\varphi_{\bullet}$ ; we say it is *thick* otherwise. We say it is *convex* if for all edges  $\bar{e} \in \partial f$ , we have  $\varphi_{\bar{e}} \in [0, \pi/2)$ .

**Definition 1.4.** Given an extended circle pattern  $c \in \underline{\mathfrak{C}}$ , an edge  $e$  is *short* if it has length 0,  $l_e = 0$ ; it is *long* otherwise.

**Definition 1.5.** Given an extended circle pattern  $c \in \underline{\mathfrak{C}}$ , a *wide path* is a sequence of faces  $f_0, f_1, \dots, f_n$  such that  $f_i, f_{i+1}$  share a long edge. A *wide cycle* is a wide path with  $f_0 = f_n$ .

**Lemma 1.6.** Let  $c$  be a face non-singular extended circular pattern such that  $\varphi_{\bar{e}} \geq 0$  for all  $\bar{e} \in \vec{E}$ . Furthermore, suppose that there is no cycle of edges  $e_0, e_1, \dots, e_n = e_0$  such that  $e_i, e_{i+1}$  belong to the same face, and  $\varphi_{\bar{e}_i} = \varphi_{\bar{e}_{i+1}} = \pi/2$ . Then  $\underline{\mathfrak{C}}$  is a manifold in a neighborhood of  $c$ .

*Proof.* We need to show that the differentials  $d(l_{\bar{e}} - l_{\bar{e}'}) = \sin \varphi_{\bar{e}} dr_{f_{\bar{e}}} + r_{f_{\bar{e}}} \cos \varphi_{\bar{e}} d\varphi_{\bar{e}} - \sin \varphi_{\bar{e}'} dr_{f_{\bar{e}'}} - r_{f_{\bar{e}'}} \cos \varphi_{\bar{e}'} d\varphi_{\bar{e}'}$  are linearly independent at  $c$ . If  $\varphi_{\bar{e}} \neq \pi/2$ , then the  $d\varphi_{\bar{e}}$  term makes  $d(l_{\bar{e}} - l_{\bar{e}'})$  linearly independent from the rest (no other differential  $d(l_{\bar{e}'} - l_{\bar{e}''})$  contains such component); likewise for  $\varphi_{\bar{e}'} \neq \pi/2$ .

Now consider the set  $E'$  of edges  $e$  that have  $\varphi_{\bar{e}} = \varphi_{\bar{e}'} = \pi/2$ . Consider a maximal sequence of faces  $f_0, \dots, f_n$  with no repetitions and  $f_i, f_{i+1}$  share an edge  $e_i \in E'$ . By the condition on no cycles,  $f_0 \neq f_n$ . By face non-singularity, any face can meet at most two edges from  $E'$ , thus if  $e \in E'$ , it is contained in a unique such string of faces/edges. Thus the differentials  $d(l_{\bar{e}_i} - l_{\bar{e}_{i+1}}) = dr_{f_i} - dr_{f_{i+1}}$  are linearly independent.  $\square$

**1.1. Modifications to Extended Circle Patterns.** Let  $\Gamma$  be a graph on the torus, and let  $c$  be a non-singular extended circle pattern on  $\Gamma$ . There are two simple operations to modify  $\Gamma$ : cutting a face along a new edge, and splitting a vertex in two then joining them by a new edge. We describe how to associate an extended circle pattern  $c'$  to the new graph.

Cutting a face: Consider a face  $f$ , and suppose we added a new edge  $e'$  so that  $f$  is split into two faces  $f'$  and  $f''$ , so that  $\partial f' = \{\vec{e}_1, \dots, \vec{e}_k, \vec{e}'\}$ , and  $\partial f'' = \{\vec{e}', \vec{e}_{k+1}, \dots, \vec{e}_n\}$ . Then for the new extended circle pattern  $c'$ , we set  $r_{f'}(c') = r_{f''}(c') = r_f(c)$ ,  $\varphi_{\bar{e}'} = \pi - \varphi_{\bar{e}_1} - \dots - \varphi_{\bar{e}_k}$ ,  $\varphi_{\bar{e}'_i} = \pi - \varphi_{\bar{e}_{k+1}} - \dots - \varphi_{\bar{e}_n}$ , and  $r_{\bullet}(c') = r_{\bullet}(c)$ ,  $\varphi_{\bullet}(c') = \varphi_{\bullet}(c)$  for all other faces, edges. It is straightforward to check that  $c'$  is non-singular.

Consider a vertex  $v$  of the TODO

## 2. MAIN ?? RESULTS

TODO: make a list of images of vectors, e.g.  $\Theta_*(\frac{\partial}{\partial \varphi_{\bar{e}}}) = -\frac{\partial}{\partial \theta_e}$ .

Our general strategy for obtaining a (non-singular) circle pattern is to start with an assignment  $\underline{\theta} = (\theta_{\bullet})$  for which we know an extended circle pattern exists (say from results of [?BandS]), and a path  $\gamma$  in  $\underline{\Theta}$  starting at  $\underline{\theta}$ . We then attempt to lift  $\gamma$  to a path  $\tilde{\gamma}$  in  $\underline{\mathfrak{C}}$ , so that  $\tilde{\gamma}$  remains (face) non-singular (vertex non-singularity is already determined by  $\gamma$ ).

<sup>1</sup>Intuitively, one can visualize decreasing  $\varphi_{\bar{e}}$  from  $\varepsilon$  to  $-\varepsilon$  as moving the black vertices of the quadrangle  $Q_e$  past each other, thus flipping  $Q_e$  along its long axis and making it have negative area.

Note that since  $\sum_{e \in E} \theta_e = 2\pi|E| - \sum_{\vec{e} \in \vec{E}} \varphi_{\vec{e}} = 2\pi|E| - \sum_{f \in F} \Phi_f$ , we see that maintaining face non-singularity of  $\tilde{\gamma}$  forces  $\sum \theta_e$  to be constant. We will show that this is the only obstruction on  $\gamma$  to the lifting to such  $\tilde{\gamma}$ .

To that end, let  $L$  be the  $(|E| - 1)$ -plane distribution on  $\underline{\Theta}$  tangent to the level sets of  $\sum_{e \in E} \theta_e$ . The following proposition proves it (TODO avoid using ‘it’?) up to first order at a point:

**Proposition 2.1.** *Let  $c \in \underline{\mathfrak{C}}$  be a non-singular extended circle pattern on a graph with no bigons or self-loops such that there is no wide cycle of thin faces. Let  $K_c = \ker(\Phi_*|_c : T_c \underline{\mathfrak{C}} \rightarrow T_{\Phi(c)} \underline{\Phi})$ . Then  $\Theta_*(K_c) = L_{\Theta(c)}$ .*

*In other words, for any vector  $a \in T_{\Theta(c)} \underline{\Theta}$  with sum of components 0, one can vary  $c$  so that its first order change in  $\theta_\bullet$  is  $a$ , and also remains face non-singular up to first order.*

Before we prove this, it is convenient to first prove the following, which will establish some useful notation:

**Lemma 2.2.** *Let  $c \in \underline{\mathfrak{C}}$  be as in Proposition 2.1. Then  $\Phi$  is a submersion in a neighbourhood of  $c$ .*

*Proof.* We construct vectors  $\beta_f \in T_c \underline{\mathfrak{C}}$  so that  $\Phi_*(\beta_f) = \frac{\partial}{\partial \Phi_f}$ . The vector

$$(2.1) \quad \alpha_f := \frac{\partial}{\partial r_f} - \sum_{\vec{e} \in \partial f} \frac{\tan \varphi_{\vec{e}}}{r_f} \frac{\partial}{\partial \varphi_{\vec{e}}} \in T(\mathfrak{X} \times \underline{\Omega})$$

doesn’t change  $l_{\vec{e}}$ , so is in  $T_c \underline{\mathfrak{C}}$ . Intuitively,  $\alpha_f$  is like pulling the center of  $C_f$  up off the plane, increasing  $r_f$  and decreasing all  $\varphi$ ’s. Its pushforward under  $\Phi$  is simply  $\Phi_*(\alpha_f) = (-\frac{2}{r_f} \sum_{\vec{e} \in \partial f} \tan \varphi_{\vec{e}}) \frac{\partial}{\partial \Phi_f}$ .

If  $f$  is a convex face, then the  $\tan \varphi_{\vec{e}}$  are all non-negative, being 0 if and only if  $e$  is short, and because  $\Phi_f = 2\pi$ , at least three edges are long. So  $\Phi_*(\alpha_f)$  is a negative multiple of  $\frac{\partial}{\partial \Phi_f}$ ; we choose  $\beta$  so that

$$(2.2) \quad \beta_f := \beta \cdot \alpha_f; \quad \Phi_*(\beta_f) = \frac{\partial}{\partial \Phi_f}$$

If  $f$  is thick but not convex, let  $\vec{e}$  be an edge with largest  $\varphi_{\vec{e}}$ , so that  $\varphi_{\vec{e}} \geq \pi/2$ . If  $\varphi_{\vec{e}} > \pi/2$ , then since  $\tan$  is convex, and  $f$  has at least two other long edges, we see that  $\Phi_*(\alpha_f) = -\sum \frac{\tan \varphi_{\vec{e}}}{r_f} > 0$ , so we can define  $\beta_f$  as in (2.2). If  $\varphi_{\vec{e}} = \pi/2$ , we can take

$$\beta_f = \frac{1}{2} \frac{\partial}{\partial \varphi_{\vec{e}}}$$

If  $f$  is thin, we actually have that  $\alpha_f \in K_c$ , since the components sum to 0, so we need a different approach. First suppose  $f$  shares an edge  $e$  with a thick face  $f'$ , with  $\vec{e} \in \partial f$ . If  $\varphi_{\vec{e}} = \pi/2$ , we can take  $\beta_f = \frac{1}{2} \frac{\partial}{\partial \varphi_{\vec{e}}}$  as before. Otherwise, we can increase  $l_{\vec{e}}$  by varying  $\varphi_{\vec{e}}$  while holding  $r_f$  constant, and increase  $l_{\vec{e}'}$  by increasing  $r_{f'}$ . This will affect both  $\Phi_f, \Phi_{f'}$ , so we use  $\beta_{f'}$  to make  $\Phi_{f'}$  constant. We can repeat this procedure with  $f'$  set to this thin face, and  $f$  set to another thin face adjacent to it, etc.  $\square$

*Proof.* NEW PROOF NEW PROOF NEW PROOF NEW PROOF NEW PROOF NEW PROOF We construct a vector  $u_e \in K_c \subseteq T_c \underline{\mathfrak{C}}$  for each edge  $e$  and show that  $\{\Theta_*(u_e)\}_{e \in E}$  spans an  $|E| - 1$  dimensional space, thus must be equal  $L_{\Theta(c)}$ .  $u_e$  will have the following property: if  $\Theta_*(u_e) = \sum_{e' \in E} a^{e'} \frac{\partial}{\partial \theta_{e'}}$ , then

- $a^e = 1$ ;
- $a^{e'} \leq 0$  for all  $e' \neq e$ ;
- $\sum_{e' \in E} a^{e'} = 0$  (follows directly from  $u_e \in K_c$ )

To illustrate the main idea of the construction, first suppose that  $e$  is an edge between two convex faces  $f, f'$  (so  $\vec{e} \in \partial f, \vec{e} \in \partial f'$ ). Observe that

$$(2.3) \quad \alpha_e = \frac{1}{2r_f \cos \varphi_{\vec{e}}} \frac{\partial}{\partial \varphi_{\vec{e}}} + \frac{1}{2r_{f'} \cos \varphi_{\vec{e}}} \frac{\partial}{\partial \varphi_{\vec{e}'}}$$

increases  $\Phi_f$  and  $\Phi_{f'}$ . We can compensate using  $\beta_f$  from (2.2), so we consider

$$w_e = \alpha_e - \frac{1}{\cos \varphi_{\vec{e}}} \beta_f - \frac{1}{\cos \varphi_{\vec{e}'}} \beta_{f'} \in K_c$$

For  $\vec{e} \in \partial f \cup \partial f' \setminus \{\vec{e}, \overleftarrow{e}\}$ , the  $\frac{\partial}{\partial \varphi_{\vec{e}'}}$  component only appears in  $\beta_f$  or  $\beta_{f'}$ , which is positive by construction (this requires  $f, f'$  to be convex faces); thus,  $d\theta_{e'}(w_e) \geq 0$  for such  $e'$ , and is equal 0 if and only if  $e'$  is short. Finally, since  $w_e \in K_c$ , it leaves the sum  $\sum_{e \in E} \theta_e$  constant, so  $d\theta_e(w_e) < 0$ , so we can take

$$(2.4) \quad u_e := (d\theta_e(w_e))^{-1} \cdot w_e$$

□

*Proof of Proposition 2.1.* We construct a vector  $u_e \in K_c \subseteq T_c \mathfrak{C}$  for each edge  $e$  and show that  $\{\Theta_*(u_e)\}_{e \in E}$  spans an  $|E| - 1$  dimensional space, thus must be equal  $L_{\Theta_c}$ .

For long edges  $e$ ,  $u_e$  will have the following property: if  $\Theta_*(u_e) = \sum_{e' \in E} a^{e'} \frac{\partial}{\partial \theta_{e'}}$ , then

- $a^e = 1$ ;
- $a^{e'} \leq 0$  for all  $e' \neq e$ ;
- $a^{e'} = 0$  for short edges  $e'$ .
- $\sum_{e' \in E} a^{e'} = 0$  (follows directly from  $u_e \in K_c$ )

Furthermore, these  $u_e$ 's collectively satisfy the following connectivity property: consider the graph  $G_{\text{long}}$  whose vertex set is the set of long edges, and we connect two long edges  $e, e'$  by an edge if there is some  $e''$  such that  $a^e, a^{e'}$  are both nonzero in  $\Theta_*(u_{e''})$ ; then  $G_{\text{long}}$  is connected.

In addition, for short edges  $e$ ,  $u_e$  have the following property: again writing  $\Theta_*(u_e) = \sum_{e' \in E} a^{e'} \frac{\partial}{\partial \theta_{e'}}$ , one has

- $a^e = 1$ ;
- $a^{e'} = 0$  for other short edges  $e'$ .
- $\sum_{e' \in E} a^{e'} = 0$  (follows directly from  $u_e \in K_c$ )

Let us first suppose we have constructed such  $u_e$ , and show that these properties ensure that  $\{\Theta_*(u_e)\}_{e \in E}$  spans an  $|E| - 1$  dimensional space. This is a simple exercise in linear algebra, but we show it for completeness.

Put the  $\Theta_*(u_e)$ 's into a  $E \times E$  matrix, denoted  $M$ , so that the  $e$ -th row corresponds to  $u_e$ . By virtue of  $u_e \in K_c$ , we have that  $(1 \ 1 \ \dots \ 1)^T$  is in the null space of  $M$ ; our goal is to show that it spans the null space.

Suppose  $b = (b_e)^T$  is in the null space of  $M$ . Reorder  $E$ , if necessary, so that long edges come before short edges. Then by the last property of  $u_e$ 's for long edges, the top right block of long-by-short entries are 0. Thus the vector  $b_{\text{long}} := (b_e)_{e \in \text{long}}^T$  is in the null space of the submatrix  $M_{\text{long}}$  of long-by-long entries.

Let  $|b_e|$  be the largest among components of  $b_{\text{long}}$ ; rescale  $b$  so that  $b_e = 1$ . The  $e$ -th component of  $M_{\text{long}} b_{\text{long}}$  is  $1 - \sum_{e' \neq e} a^{e'} b_{e'}$ , where  $a^{e'}$  are the components of  $\Theta_*(u_e)$ . Since  $\sum_{e' \neq e} |a^{e'}| = 1$ , this can be 0 if and only if for all  $e'$  with  $a^{e'} < 0$ , we have exactly  $b_{e'} = 1$ . By connectedness of  $G_{\text{long}}$ , this implies  $b_{\text{long}} = (1 \ \dots \ 1)^T$ .

Returning to the full  $M$ , it is now easy to see that for a short edge  $e$ , by looking at the  $e$ -th row of  $M$ , we have  $b_e = \sum a^{e'} = 1$ , so we are done.

Now we construct the  $u_e$ 's. First suppose that  $e$  is a long edge between two convex faces  $f, f'$  (so  $\vec{e} \in \partial f, \overleftarrow{e} \in \partial f'$ ). Observe that  $\alpha_e$  from (2.3) increases  $\Phi_f$  and  $\Phi_{f'}$ , and we can compensate using  $\beta_f$  from (2.2), so we consider

$$w_e = \alpha_e - \frac{1}{\cos \varphi_{\vec{e}}} \beta_f - \frac{1}{\cos \varphi_{\overleftarrow{e}}} \beta_{f'} \in K_c$$

For  $\vec{e} \in \partial f \cup \partial f' \setminus \{\vec{e}, \overleftarrow{e}\}$ , the  $\frac{\partial}{\partial \varphi_{\vec{e}'}}$  component only appears in  $\beta_f$  or  $\beta_{f'}$ , which is positive by construction (this requires  $f, f'$  to be convex faces); thus,  $d\theta_{e'}(w_e) \geq 0$  for such  $e'$ , and is equal 0 if and only if  $e'$  is short. Finally, since  $w_e \in K_c$ , it leaves the sum  $\sum_{e \in E} \theta_e$  constant, so  $d\theta_e(w_e) < 0$ , so we can take

$$(2.5) \quad u_e := (d\theta_e(w_e))^{-1} \cdot w_e$$

In general, we consider a wide path of faces  $f_0, \dots, f_n$ , such that all faces except  $f_0$  and  $f_n$  are thin, and  $f_i$  are distinct except possibly  $f_0 = f_n$ .<sup>2</sup> Let  $e_i$  be the long edge between  $f_i$  and  $f_{i+1}$ , and orient so that  $\vec{e}_i \in \partial f_i, \overleftarrow{e}_i \in \partial f_{i+1}$ .

We will construct  $u_{e_i}$  for  $i = 1, \dots, n$ .

<sup>2</sup>Picture a closed accordian.

We want to increase the lengths of  $e_i$ 's all together. Let  $E' = \{\vec{e}_i, \overleftarrow{e}_i | i = 0, 1, \dots, n-1\}$ . Consider

$$x := \sum_{i=0, \dots, n-1} \alpha_{e_i} = \sum_{\vec{e} \in E'} \frac{1}{2r_{f_{\vec{e}}} \cos \varphi_{\vec{e}}} \frac{\partial}{\partial \varphi_{\vec{e}}}$$

with  $\alpha_e$  from (2.3). It is clear that  $x$  increases all oriented  $e_i$  lengths  $l_{\vec{e}}$  equally fast, i.e.  $dl_{\vec{e}_i}(x) = dl_{\overleftarrow{e}_i}(x) = 1/2$  for all  $i$ , so  $x \in T_c \underline{\mathfrak{C}}$ . It is also clear that for the thin faces  $f_i$ ,  $x$  makes one of the angles  $\varphi_{e_{i-1}^{\leftarrow}}, \varphi_{\vec{e}_i}$  increase and the other decrease, both at the same rate, thus  $d\Phi_{f_i}(x) = 0$ . However, the vector  $x$  increase  $\Phi_{f_0}$  and  $\Phi_{f_n}$ , so it is compensated using  $\beta_f$ 's, so that

$$y := x - \frac{1}{r_{f_0} \cos \varphi_{\vec{e}_0}} \beta_{f_0} - \frac{1}{r_{f_n} \cos \varphi_{e_{n-1}^{\leftarrow}}} \beta_{f_n} \in K_c$$

Before we modify  $y$  to give the desired  $u_{e_i}$ , we note that for  $\vec{e} \in (\cup \partial f_i) \setminus E'$ , the  $\varphi_{\vec{e}}$ -component in  $y$  is non-positive, and is 0 if and only if  $e$  is short; so  $\Theta_*(-y)$  already satisfies three of the four properties desired in  $\Theta_*(u_e)$ .

Recall that for thin face  $f$ ,  $\alpha_f$  from (2.1) is in  $K_c$ , and we have

$$\Theta_*\left(\frac{r_{f_i}}{\tan \varphi_{\vec{e}_i}} \alpha_{f_i}\right) = \frac{\partial}{\partial \theta_{e_i}} - \frac{\partial}{\partial \theta_{e_{i-1}}}$$

Write  $\bar{\alpha}_{f_i} = \frac{r_{f_i}}{\tan \varphi_{\vec{e}_i}} \alpha_{f_i}$ . Then it is easy to see that using  $\bar{\alpha}_{f_i}$ 's, we can “push around” the  $\theta_{e_i}$ -components in  $\Theta_*(y)$  so only one is nonzero. More precisely, if  $\Theta_*(y) = \sum_e y^e \frac{\partial}{\partial \theta_e}$ , the image under  $\Theta_*$  of the vector

$$\bar{u}_{e_0} := y - y^{e_{n-1}} \bar{\alpha}_{f_{n-1}} - (y^{e_{n-2}} + y^{e_{n-1}}) \bar{\alpha}_{f_{n-2}} - \dots - (y^{e_1} + \dots + y^{e_{n-1}}) \bar{\alpha}_{f_1}$$

would have  $\theta_{e_i}$ -components 0 for all  $i$  except 0; for  $i = 0$ , the  $\theta_{e_0}$ -component is now  $y^{e_0} + \dots + y^{e_{n-1}}$ , which is nonzero because  $\sum_{e' \in E} y^{e'} = 0$  and  $\sum_{e' \in E \setminus E'} y^{e'} > 0$ . Therefore, we can take

$$u_{e_0} := (d\theta_{e_0}(\bar{u}_{e_0}))^{-1} \cdot \bar{u}_{e_0}; \quad u_{e_i} = u_{e_{i-1}} + \bar{\alpha}_{f_i}$$

We still need to show the collective connectivity property of the  $u_e$ 's. It is easy to see that the long edges of a convex face are connected in  $G_{\text{long}}$  (for example,  $u_{e_0}$  defined above clearly does it for  $f_0$ ). Furthermore, all the  $\Theta_*(u_{e_i})$  have most components equal, so  $e_i$  are in the same connected component of  $G_{\text{long}}$ . Thus connectivity of  $G_{\text{long}}$  follows from the wide-path-connectivity of faces.

Finally, for short edges  $e$ , we simply define  $u_e$  as in (2.5) (here we do not worry about whether faces are thin or convex, as the non-positivity condition of  $a^{e'}$  is absent).  $\square$

**Proposition 2.3.** *Let  $c \in \underline{\mathfrak{C}}$  be as in Proposition 2.1. Given a smooth path  $\gamma : [0, \infty) \rightarrow \Theta$  starting at  $\gamma(0) = \Theta(c)$  and has constant value  $(\sum \theta_e) \circ \gamma$ , there exists  $\varepsilon > 0$  and a lift  $\tilde{\gamma} : [0, \varepsilon] \rightarrow \underline{\mathfrak{C}}$  of  $\gamma|_{[0, \varepsilon]}$  along  $\Theta$  that starts at  $\tilde{\gamma}(0) = c$  and has constant value  $\Phi \circ \tilde{\gamma}$ .*

*Proof.* By Lemma 2.2,  $\Phi$  is a submersion at  $c$ , hence is a submersion in a neighborhood  $U \in \underline{\mathfrak{C}}$  of  $c$ . Thus, the kernels  $K_{c'} = \ker(\Phi_*|_{c'} : T_{c'} \underline{\mathfrak{C}} \rightarrow T_{\Phi(c')} \Phi)$  form an  $|E|$ -plane distribution  $K \subset T \underline{\mathfrak{C}}|_U$  over  $U$ . By Proposition 2.1, the bundle map  $\Theta_*|_K : K \rightarrow L$  is full rank at  $c$ , hence it is full in a possibly smaller neighborhood, which we redefine  $U$  to be.

Since  $\sum_{e \in E} \theta_e = 2\pi|E| - \sum_{f \in F} \Phi_f$ , the fullness of  $\Phi_*$  also shows that  $\sum \theta$  has no critical points in  $U$ ; for  $s$  in some small interval around  $(\sum \theta)(c)$ , denote, by overloading notation,  $L_s = (\sum \theta)^{-1}(s)$ ,  $U_s = U \cap \Theta^{-1}(L_s)$ . We may consider  $K$  as a family of  $|E|$ -plane distributions  $K_s$  over  $U_s$ .

The vector  $\rho := \sum_{f \in F} r_f \frac{\partial}{\partial r_f}$  corresponds to scaling all radii by the same factor, and is clearly in  $T \underline{\mathfrak{C}}$ . It is obviously mapped to zero under  $\Phi_*$  and  $\Theta_*$ , so  $\text{span}\{\rho\} = \ker(\Theta_*|_K) \subset K$ . Thus, over  $U$ , we may split  $K = \text{span}\{\rho\} \oplus K'$  (say as orthogonal decomposition w.r.t. some smooth metric on  $\underline{\mathfrak{C}}$ ), so that  $\Theta_*|_{c'} : K'|_{c'} \simeq L_{\Theta(c')}$ .

In other words,  $K'|_{U_s}$  is a horizontal ( $|E| - 1$ )-plane distribution that determines an Ehresmann connection of the fibre bundle  $U_s \rightarrow L_s$  (after being appropriately restricted). Therefore, a short path  $\gamma$  starting at  $\Theta(c)$  with constant  $\sum \theta$ , i.e. a path in  $L_s$ ,  $s = (\sum \theta)(c)$ , can be lifted to a path  $\tilde{\gamma}$  in  $U_s$ , so that  $\tilde{\gamma}$  is always tangent to  $K'$ , hence  $\Phi(\tilde{\gamma})$  is constant.  $\square$

TODO Note: no face can have its edges glued together; prevented by fact of alternating links  
TODO just use Feasible Flow Theorem (see BS)

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