

Semisimple 2-categories

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Goal

$$\{\text{multifusion category}\} \xrightarrow[\simeq]{\mathcal{M}\text{od}_{s.s.}^{fin}(-)} \{\text{semisimple 2-cat}\}$$

- ▶ The 2-category $\mathcal{M}\text{od}_{s.s.}^{fin}(C)$ of finite semisimple module categories over a multifusion category C is semisimple.
- ▶ For any finite semisimple 2-category \mathcal{C} , there exists a multifusion category C such that $\mathcal{C} \simeq \mathcal{M}\text{od}_{s.s.}^{fin}(C)$

Conventions

In relation to a 2-category:

- ▶ \mathcal{C} (caligraphic font): 2-category;
- ▶ X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- ▶ f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X, Y)$ for the category of morphisms from X to Y ;
- ▶ $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \rightarrow Y$, we may write $\alpha \in \mathcal{C}(X, Y)(f, g)$ to indicate its sources and targets, or simply $\alpha \in \text{Hom}(f, g)$ if the objects are clear

Conventions

In relation to a 1-category:

- ▶ C, A (upper case latin): category;
- ▶ a, b, f, g (lower case latin): objects in category, functor between categories;
- ▶ α, β (lower case greek): morphism in category

Conventions

We also compose morphisms from right to left: in a 2-category \mathcal{C} , for $\alpha \in \mathcal{C}(X, Y)(f, f'), \beta \in \mathcal{C}(Y, Z)(g, g'), \gamma \in \mathcal{C}(X, Y)(f', f'')$, we write

$$g \circ f, g \circ f', \dots : X \rightarrow Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \rightarrow Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \rightarrow Y$$

for vertical composition of 2-morphisms.

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In general, if P is a property of a 1-category, we say that a 2-category \mathcal{C} is *locally P* if every hom-category $\mathcal{C}(X, Y)$ satisfies P .

Conventions

In general, if P is a property of a 1-category, we say that a 2-category \mathcal{C} is *locally P* if every hom-category $\mathcal{C}(X, Y)$ satisfies P . By 2-category we always mean a weak 2-category that is furthermore locally additive over \mathbf{k} . By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally \mathbf{k} -linear.

Review

Additive 2-category, direct sum of objects

Definition (direct sum of objects in 2-category)

A *direct sum* of two objects A_1, A_2 in \mathcal{C} is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms

$i_k : A_k \Rightarrow A_1 \boxplus A_2 : p_k$, such that

- ▶ $p_k \circ i_k \simeq \text{id}_{A_k}$,
- ▶ $p_2 \circ i_1, p_1 \circ i_2$ are zero 1-morphisms,
- ▶ $\text{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

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Definition

A 1-morphism $i : X \rightarrow Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A , $i \circ - : \mathcal{C}(A, X) \rightarrow \mathcal{C}(A, Y)$ is fully faithful.

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Additive 2-category, direct sum of objects

Definition (Direct sum of 2-categories)

Given 2-categories \mathcal{C}_j , $j \in J$, we may consider the direct sum 2-category $\mathcal{C} := \boxplus_{j \in J} \mathcal{C}_j$:

- ▶ $\text{Obj } \mathcal{C} = \bigsqcup_{j \in J} \text{Obj } \mathcal{C}_j$
- ▶ for $X \in \mathcal{C}_i$, $Y \in \mathcal{C}_j$, $\mathcal{C}(X, Y) = \begin{cases} \mathcal{C}_j(X, Y) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Idempotent completeness, separable monads, splittings

Definition

A *separable algebra* (a, μ, η) in a tensor category \mathcal{C} is an algebra that admits an a - a -bimodule section ${}_a a_a \rightarrow {}_a a \otimes a_a$ to μ .

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In other words, a separable monad over X is a separable algebra in $\mathcal{C}(X, X)$.

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Let $r \vdash l : X \rightarrow Y$ be an adjunction with unit $\eta : \text{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \text{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

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Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \rightarrow Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X .

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Under the right conditions (local idempotent completeness of \mathcal{C}), splittings are unique:

Proposition (Uniqueness of splitting)

[[1], Theorem A.3.1] In a locally idempotent complete 2-category \mathcal{C} , splittings of a separable monad are unique up to equivalence.

Idempotent completeness, 2-category

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Definition (Idempotent completion)

Let \mathcal{C} be a locally idempotent complete 2-category. The *idempotent completion* of \mathcal{C} , denoted \mathcal{C}^∇ , with:

- ▶ Objects: (X, p) separable monad in \mathcal{C} ,
- ▶ $\mathcal{C}^\nabla((X, p), (Y, q)) = q\text{-bimod-}p(\mathcal{C}(X, Y))$

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A 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ extends to a 2-functor $F^\nabla : \mathcal{C}^\nabla \rightarrow \mathcal{D}^\nabla$ that commutes with I 's.

Idempotent completeness, 2-category

Proposition

\mathcal{C}^∇ is idempotent complete. Moreover, if \mathcal{C} is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^\nabla$ is an equivalence.

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As a consequence, if \mathcal{D} is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^\nabla, \mathcal{D}^\nabla) \simeq \mathcal{F}un(\mathcal{C}^\nabla, \mathcal{D})$$

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Proposition

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful, then $F^\nabla : \mathcal{C}^\nabla \rightarrow \mathcal{D}^\nabla$ is also fully faithful.

Idempotent completeness, 2-category

$$(\mathcal{B}C)^{\nabla} = \begin{cases} \text{Obj} : \text{separable algebras in } C \\ \text{Mor} : (\mathcal{B}C)^{\nabla}(a, b) = b\text{-bimod-}a(C) \end{cases}$$

Idempotent completeness, 2-category

Proposition ([1]*Prop 1.3.13)

For a multifusion category C , the following 2-functor is an equivalence:

$$\begin{aligned}(-)\text{-mod}(C) : (BC)^{\nabla} &\rightarrow \mathcal{M}od(C) \\ a &\mapsto a\text{-mod}(C) \\ {}_b m_a &\mapsto m \otimes_a - \\ \varphi &\mapsto \varphi \otimes_a -\end{aligned}$$

Proof.

Essential surjectivity follows from:

- [2]*Theorem 1: for a finite semisimple right module category over multifusion C , there exists a semisimple algebra a in C such that $M \simeq a\text{-mod}(C)$ as right C -module categories; and
- [4]*Corollary 2.6.9: When C is multifusion over a field of characteristic 0, a right C -module category M is separable

Simple objects

Proposition (equivalent notions of simple-ness)

Let \mathcal{C} be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:

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- (2) X cannot be written as a non-trivial direct sum, i.e. if $X = \boxplus X_i$, then $X_i \simeq 0$ for all but one i ;*

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- (3) id_X is a simple object in $\mathcal{C}(X, X)$.*

Semisimple 2-category

Definition ((finite) semisimple 2-category)

A 2-category \mathcal{C} is *semisimple* if it is:

- ▶ locally semisimple,
- ▶ admits left and right adjoints for every 1-morphism,
- ▶ additive,
- ▶ idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

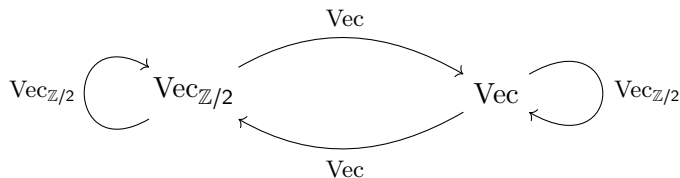
New Stuff

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case.

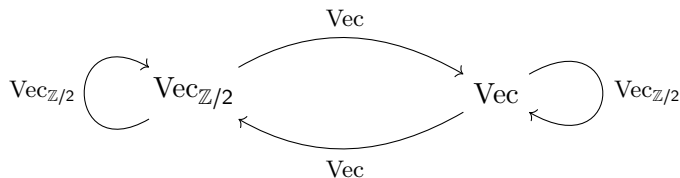
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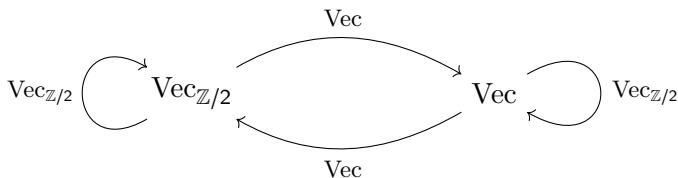
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Such nonzero morphisms turn out to be an important aspect of semisimple 2-categories since they capture “2-Morita equivalences” between fusion categories.

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Proposition (Schur's Lemma, [1]*Prop 1.2.19)

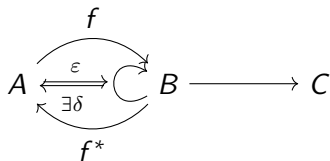
In a semisimple 2-category \mathcal{C} , if $f : A \rightarrow B, g : B \rightarrow C$ are nonzero 1-morphisms between simple objects A, B, C , then $g \circ f$ is also nonzero.

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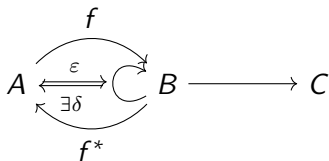


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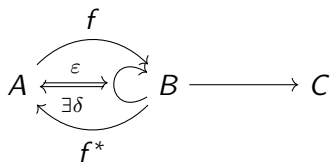
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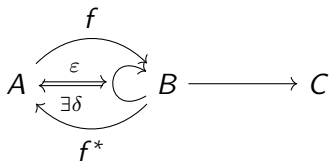
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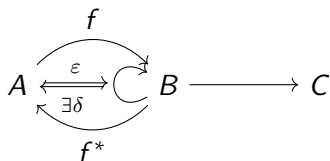
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Schur's lemma - components

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In a semisimple 2-category \mathcal{C} , we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \rightarrow B$.

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full subcategory \mathcal{C}_j , $j \in J$: direct sum of simple objects in j
gives direct sum decomposition $\mathcal{C} \simeq \bigoplus_{j \in J} \mathcal{C}_j$.

Main results

Theorem ([1]*Theorem 1.4.8)

The 2-category of finite semisimple module categories of a multifusion category \mathcal{C} is a finite semisimple 2-category.

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Proof.

$\text{Mod}(\mathcal{C}) \simeq (\mathcal{BC})^\nabla$ is idempotent complete and locally idempotent complete. $\text{Mod}(\mathcal{C})$ is clearly already additive.

Locally semisimple-ness follows directly from [4]*Corollary 2.5.6, and existence of adjoints for 1-morphisms follows from [3]*Corollary 2.13.



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