Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

Proposition 0.1. *Consider functors $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.*

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ be a natural transformation, and suppose it is the unit to an adjunction $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' are extended $(n+1)$-TQFTs such that Z', Z'' agree with Z on $(n-1)$- and n-manifolds, and $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, then $Z' \cong Z''$.

The proof of this proposition occupies the rest of this article.

0.1 Adjunctions from topology

In the 2-category with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition 0.1. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk $D^{n+1} : \varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure 1 for $n = 1$ case).

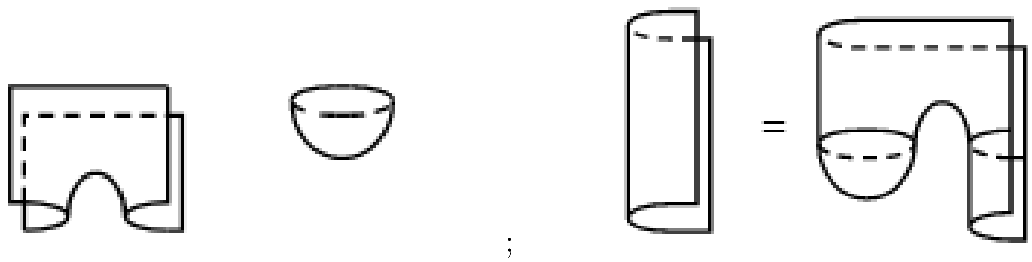

Figure 1: Counit and unit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$, along with one of the snake equations; relative cobordism goes up (stolen from [2], Figure 1.6 and 1.10, get rotated)

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n-k$. [See Figure 2]

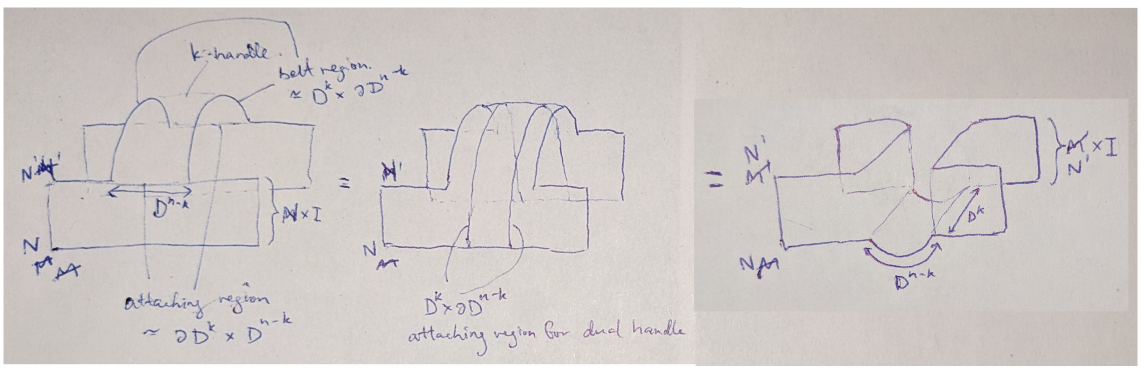

Figure 2: $M : N \to N'$ is an elementary cobordism of index k; it is built from attaching a disk D^n to $N \times I$, with attaching region $\partial D^k \times D^{n-k}$. It can also be built from the other direction, by attaching a disk D^n to $N' \times I$, with attaching region $D^k \times \partial D^{n-k}$. Thus, turning it upside-down, i.e. treated as a cobordism $\overline{M} : N' \to N$, it is an elementary cobordism of index $(n-k)$.

[See Figure 3] We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching an $(n+1)$-dimensional $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$. Similarly, we construct the unit $\eta : \mathrm{id}_N \Rightarrow M \circ \overline{M} : N \to N$ by attaching an $(n+1)$-dimensional k-handle to $\mathrm{id}_N = N \times I$; the attaching region for this $(n+1)$-dim k-handle is (the attaching region for the n-dim k-handle that defines M) $\times I$. The snake equations $\mathrm{id}_M = (\varepsilon \circ M) \cdot (M \circ \eta) : M \Rightarrow M \circ \overline{M} \circ M \Rightarrow M$ and $\mathrm{id}_{\overline{M}} = (\overline{M} \circ \eta) \cdot (\eta \circ \overline{M}) : \overline{M} \Rightarrow \overline{M} \circ M \circ \overline{M} \Rightarrow \overline{M}$ follow from the fact that these $(n+1)$-dim handles form a cancelling pair in both cases.

Here we have $M \dashv \overline{M}$, but we may very well have $\overline{M} \dashv M$; the counit $\varepsilon' : \overline{M} \circ M \Rightarrow \mathrm{id}_N$ is an $(n+1)$-dim elementary cobordism of index $(n-k+1)$. It is interesting to note that this counit is the dual cobordism to the unit $\eta : \mathrm{id}_N \Rightarrow \overline{M} \circ M$ previously described.

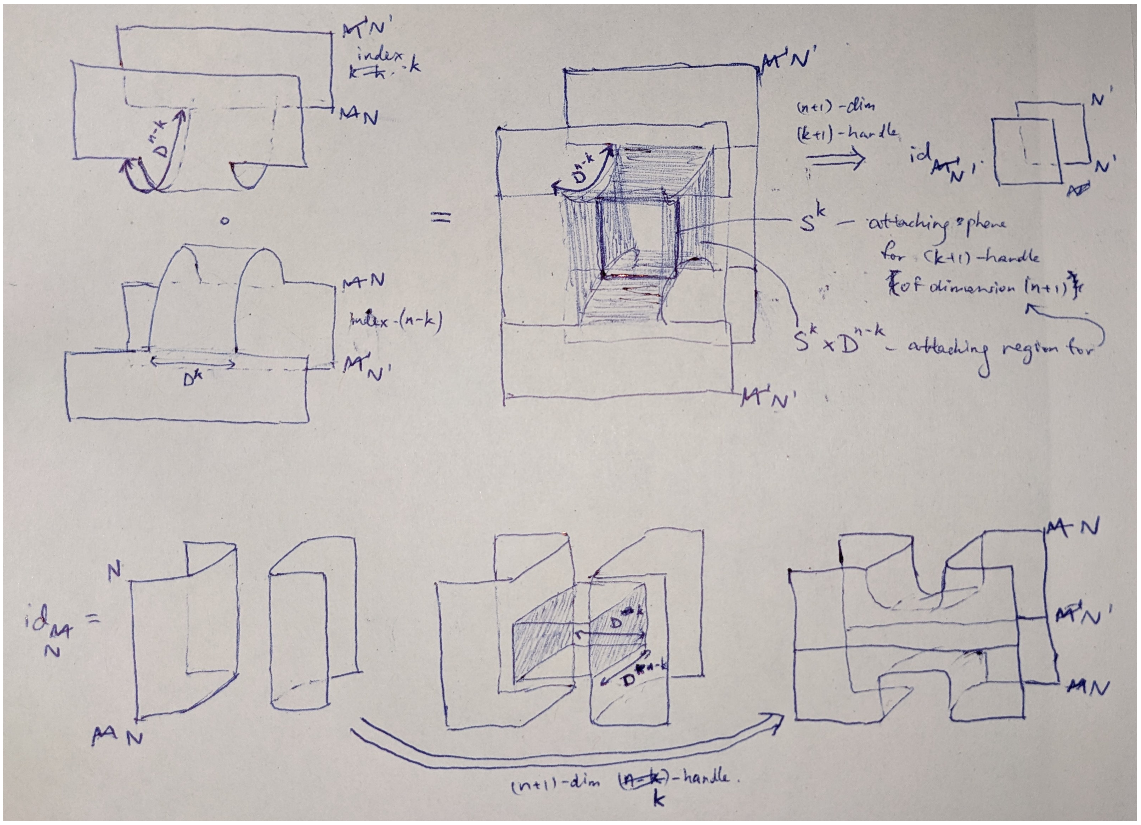

Figure 3: Counit (top) and unit (bottom) for adjunction $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k. Note the way N is drawn here looks like N' in Figure 2 and vice versa (by accident, sorry for minor confusion)

In general, we may consider the pair of n-dim cobordisms $M : N \rightleftharpoons N' : \overline{M}$. By presenting M as a composition of elementary cobordisms, we may compose the adjunctions constructed for each of these elementary cobordisms as above, and obtain an adjunction $M \dashv \overline{M}$.

Remark 0.2. In [1], we considered this construction without realizing their connection to these adjunctions; there we consider the more general case where N, N' may have (possibly different) boundary, and $M : N \to N'$ is a relative cobordism (with the boundary cobordism that is not necessarily the identity cobordism).

For our applications, we will consider in detail the adjunction $\varnothing^{n-1} : \varnothing^n \rightleftharpoons S^k \times S^{n-1-k} : S^k \times D^{n-k}$.

References

[1] Kwon, Alice, and Ying Hong Tham. The Y-Product. arXiv preprint arXiv:2209.14251 (2022).

[2] Schommer-Pries, Christopher John. The classification of two-dimensional extended topological field theories. University of California, Berkeley, 2009.