We consider the cases $n = 2$ and $n = 3$. For $n = 2$, let $\mathcal{C}$ be a non-semisimple finite tensor category. For $n = 3$, let $\mathcal{C}$ be a non-semisimple finite ribbon category. In both cases, consider the tensor ideal $\mathcal{P}$ of projective objects in $\mathcal{C}$.

Perform a string-net construction where you admit as labels only objects of $\mathcal{P}$.

Questions:

(compute dim)

1. Compute for $n = 2$ the vector space associated to S^2 and T^2 and a surface of arbitrary genus.

2. Compute for $n = 2$ the cylinder category for the circle. Compare to the Drinfeld center.

3. Compute for $n = 3$ the vector space associated to S^3. Can we make statements about other 3-manifolds?

4. For $n = 3$, compute categories for oriented surfaces. Are they related to modules of known algebras (elliptic Drinfeld center etc.)

5. Investigate whether for $n = 2$ the construction can be extended to 3-manifolds. You are allowed to use the modified trace on $\mathcal{P}$.

6. Investigate whether for $n = 3$ the construction can be extended to 4-manifolds. You are allowed to use the modified trace on $\mathcal{P}$.

Possible variations of the problems:

1. Do not work with the monoidal ideal $\mathcal{P}$, but consider stringnets in which at least one component is colored by a projective object.

2. Replace the monoidal ideal $\mathcal{P}$ by some other monoidal ideal that also admits a modified trace.

Elaboration of variation: For $n = 2+\epsilon$: Fix a pivotal finite tensor category $\mathcal{C}$ and a tensor ideal $\mathcal{I} \subset \mathcal{C}$. Declare a string net to be $\mathcal{I}$-admissible if on each connected component of the surface there is an $\mathcal{I}$-coloured edge.

1. Show that the string net space for $\mathcal{I}$-admissible string nets on the sphere is the vector space dual of the space of two-sided modified traces on $\mathcal{I}$.

2. Compute the $\mathcal{I}$-admissible string net space on the torus. Can this be non-zero even if there are no two-sided traces?

3. Maybe look at the symplectic fermion example. This has intermediate ideals which are not too complicated. (And there is an example of an intermediate ideal which has a left and a right modified trace, but no two-sided trace.)

4. If $\mathcal{I}$ is the projective ideal, are the string net spaces related to state space of the non-ssi TFT for the Drinfeld centre of $\mathcal{C}$, or, equivalently, to the spaces Lyubasheko's modular functor for $\mathcal{Z}(\mathcal{C})$ assigns to surfaces?

Rinse and repeat for $n = 3 + \varepsilon$.