Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

Proposition 0.1. *Consider functors $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.*

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ be a natural transformation, and suppose it is the unit to an adjunction $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' are extended $(n+1)$-TQFTs such that Z', Z'' agree with Z on $(n-1)$- and n-manifolds, and $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, then $Z' \cong Z''$.

The proof of this proposition occupies the rest of this article.

0.1 Adjunctions from topology

In the 2-category with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition 0.1. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk $D^{n+1} : \varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure 1 for $n=1$ case).

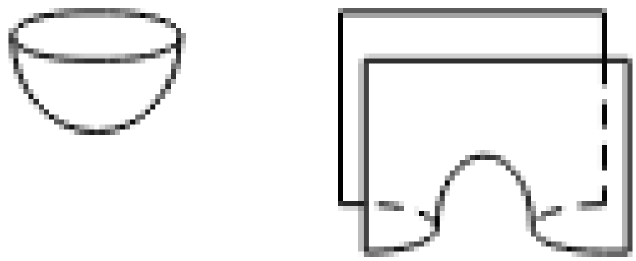

Figure 1: [from TODO Schommer-Pries] Unit and counit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$.

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n - k$.

We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching a $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$.

-then in general

(In [TODO Y-product paper], we considered this construction without realizing their connection to these adjunctions.)