Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

Ying Hong Tham

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

2-Cob denotes the bicategory with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms.

Proposition 0.1. *Consider functors $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.*

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ be a natural transformation, and suppose it is the unit to an adjunction $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' are extended $(n+1)$-TQFTs such that Z', Z'' agree with Z on $(n-1)$- and n-manifolds, and $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, then $Z' \cong Z''$.

Proof. From the topology section below, we know that a 2-morphism of 2-Cob that realizes the attachment of an $(n+1)$-dim $(k+1)$-handle,$0 \le k \le n$, is determined by some counit ε_k of an adjunction, whose corresponding unit can be built from handles of index at most k, and thus, the value of extensions Z' of Z on 2-morphisms is completely determined by its value on the $(n+1)$-dim 0-handle, which is exactly η_0. □

1 Topology

1.1 Adjunctions from topology

For the purposes of the proof of the proposition above, only the first simple example from this section is needed, the reader may then skip to Section 1.2.

In 2-Cob, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition 0.1. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk $D^{n+1} : \varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure 1 for $n = 1$ case).

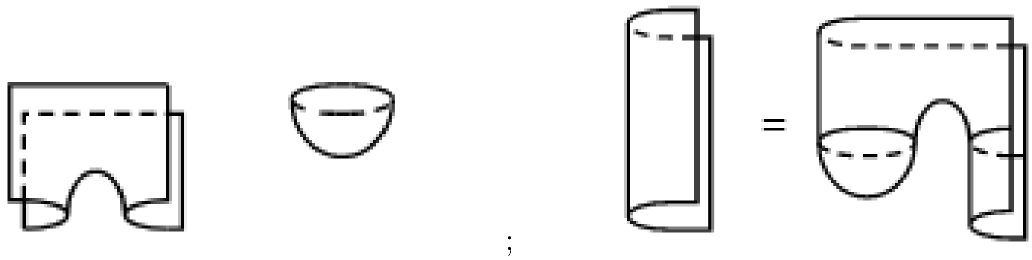

Figure 1: Counit and unit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$, along with one of the snake equations; relative cobordism goes up (stolen from [2], Figure 1.6 and 1.10, get rotated)

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n-k$. [See Figure 2]

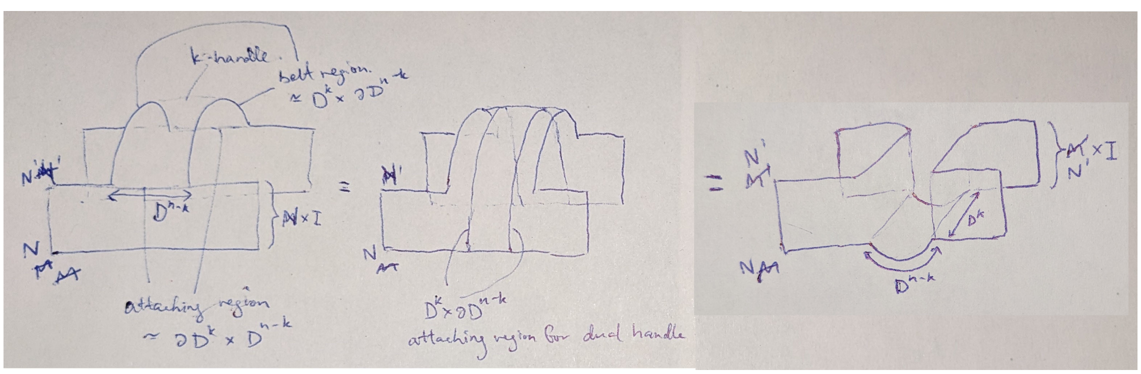

Figure 2: $M : N \to N'$ is an elementary cobordism of index k; it is built from attaching a disk D^n to $N \times I$, with attaching region $\partial D^k \times D^{n-k}$. It can also be built from the other direction, by attaching a disk D^n to $N' \times I$, with attaching region $D^k \times \partial D^{n-k}$. Thus, turning it upside-down, i.e. treated as a cobordism $\overline{M} : N' \to N$, it is an elementary cobordism of index $(n-k)$.

[See Figure 3] We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching an $(n+1)$-dimensional $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the

k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$. Similarly, we construct the unit $\eta : \mathrm{id}_N \Rightarrow M \circ \overline{M} : N \to N$ by attaching an $(n+1)$-dimensional k-handle to $\mathrm{id}_N = N \times I$; the attaching region for this $(n+1)$-dim k-handle is (the attaching region for the n-dim k-handle that defines M) $\times I$. The snake equations $\mathrm{id}_M = (\varepsilon \circ M) \cdot (M \circ \eta) : M \Rightarrow M \circ \overline{M} \circ M \Rightarrow M$ and $\mathrm{id}_{\overline{M}} = (\overline{M} \circ \eta) \cdot (\eta \circ \overline{M}) : \overline{M} \Rightarrow \overline{M} \circ M \circ \overline{M} \Rightarrow \overline{M}$ follow from the fact that these $(n+1)$-dim handles form a cancelling pair in both cases.

Here we have $M \dashv \overline{M}$, but we may very well have $\overline{M} \dashv M$; the counit $\varepsilon' : \overline{M} \circ M \Rightarrow \mathrm{id}_N$ is an $(n+1)$-dim elementary cobordism of index $(n-k+1)$. It is interesting to note that this counit is the dual cobordism to the unit $\eta : \mathrm{id}_N \Rightarrow \overline{M} \circ M$ previously described.

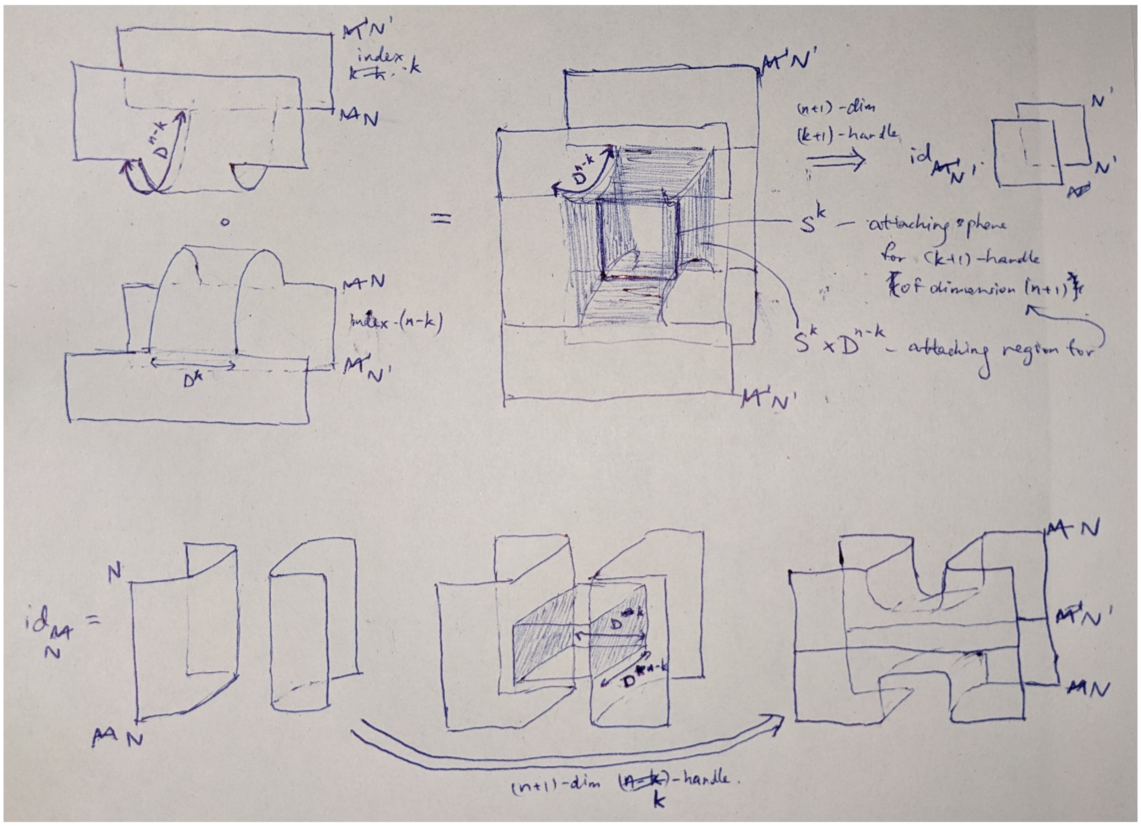

Figure 3: Counit (top) and unit (bottom) for adjunction $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k. Note the way N is drawn here looks like N' in Figure 2 and vice versa (by accident, sorry for minor confusion)

In general, we may consider the pair of n-dim cobordisms $M : N \rightleftharpoons N' : \overline{M}$. By presenting M as a composition of elementary cobordisms, we may compose the adjunctions constructed for each of these elementary cobordisms as above, and obtain an adjunction $M \dashv \overline{M}$.

Remark 1.1. In [1], we considered this construction without realizing their connection to these adjunctions; there we consider the more general case where N, N' may have (possibly different) boundary, and $M : N \to N'$ is a relative cobordism (with the boundary cobordism that is not necessarily the identity cobordism).

1.2 Producing $(n+1)$-dim k-handles from some adjunctions

Throughout this section, $0 \le k < n$.

We show how to construct the $(n+1)$-dim $(k+1)$-handle from the counit ε_k of the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and the unit η_0 of the adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. (The 0-handle is already given by η_0, while the $(n+1)$-handle is the counit to the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$; we say a few more words about this at the end of this section.)

The process of attaching a $(k+1)$-handle to an $(n+1)$-manifold can be implemented as postcomposing by a 2-morphism. More precisely, given an $(n+1)$-manifold W presented as a 2-morphism $W : M \Rightarrow M' : N \to N'$, and an attaching region $S^k \times D^{n-k}$ in M', the $(n+1)$-manifold W' obtained from attaching a $(k+1)$-handle along the specified attaching region may be considered as a 2-morphism $W' : M \Rightarrow M'' : N \to N'$, where M'' is obtain from M' by performing surgery along the attaching region (cutting it out and gluing in $D^{k+1} \times S^{n-k-1}$); then $W' = \omega_{k+1} \cdot W$, where ω_{k+1} is a 2-morphism that we will describe below.

Our 2-morphism ω_{k+1} is of the form $\omega_{k+1} : S^k \times D^{n-k} \Rightarrow D^{k+1} \times S^{n-k-1} : S^k \times S^{n-k-1} \to \varnothing^{n-1}$. Since this is unchanged as W varies, we clearly need to make some arrangements in order to use ω_{k+1}. More specifically, we need to present M' as a composition

$$(\mathrm{id}_{N'} \sqcup \overline{S^k \times D^{n-k}}) \circ (M \backslash \overline{S^k \times D^{n-k}}) : N \to N' \sqcup S^k \times S^{n-k+1} \to N'$$

which is always possible by basic Morse theory. So we have $(M^\circ := M \backslash \overline{S^k \times D^{n-k}})$:

$$W' = \begin{tikzcd}[column sep=large] N \arrow[rr, bend left=50, "M"] \arrow[rr, bend left=25, "M'"] \arrow[r, "M^\circ"'] & N' \sqcup S^k \times S^{n-k+1} \arrow[u, phantom, "\Downarrow\simeq"] \arrow[r, "\overline{S^k \times D^{n-k}}"'] \arrow[r, bend right=60, "\overline{D^{k+1} \times S^{n-k-1}}"'] \arrow[r, phantom, bend right=30, "\Downarrow\omega_{k+1}"] & N' \sqcup \varnothing^{n-1} \end{tikzcd}$$

Now let us describe how to construct ω_{k+1} out of ε_k and η_0, which are, as a reminder, the counit and unit of the adjunctions $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, respectively.

We may consider $S^n : \varnothing^{n-1} \to \varnothing^{n-1}$ as the composition $\overline{D^{k+1} \times S^{n-k-1}} \circ S^k \times D^{n-k} : \varnothing^{n-1} \to S^k \times S^{n-k-1} \to \varnothing^{n-1}$.

Then ω_{k+1} is given by the composition of 2-morphisms

$$\omega_{k+1} = (\mathrm{id}_{\overline{D^{k+1} \times S^{n-k-1}}} \circ \varepsilon_k) \cdot (\mathrm{id}_{\overline{S^k \times D^{n-k}}} \circ \eta_0)$$

$$\omega_{k+1} = \begin{tikzcd} S^k \times S^{n-k-1} \arrow[r, "\overline{S^k \times D^{n-k}}"] \arrow[rr, bend right=50, "\mathrm{id}_{S^k \times S^{n-k-1}}"'] & \varnothing^{n-1} \arrow[r, "S^k \times D^{n-k}"] \arrow[rr, bend left=50, "\varnothing^n"] \arrow[d, phantom, "\Downarrow \varepsilon_k"] & S^k \times S^{n-k-1} \arrow[r, "\overline{D^{k+1} \times S^{n-k-1}}"] \arrow[u, phantom, "\Downarrow \eta_0"] & \varnothing^{n-1} \\ & {} & & \end{tikzcd}$$

A few words on the $(n+1)$-handle, more generally the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$. The unit is a 2-morphism $\eta : \mathrm{id}_{S^{n-1}} \Rightarrow D^n \circ \overline{D^n}$, which is clearly an elementary cobordism of index n.

A similar phenomenon happens with ε_k, that is, η_k, the unit to the adjunction $S^k \times D^{n-k} \dashv \overline{S^k \times D^{n-k}}$, to which ε_k is the counit, is determined by handles of index at most k, and indeed, $\eta_k = S^k \times D^{n-k+1} : \varnothing^n \Rightarrow S^k \times S^{n-k} : \varnothing^{n-1} \to \varnothing^{n-1}$ is built from a 0-handle and a k-handle.

Thus, since the counit is uniquely determined by the unit, the $(n+1)$-dim k-handle, for $k > 0$, is determined by handles of lower index. This may not be very useful in the topology world, but on the algebraic side of a TQFT, this means that everything is determined by the 0-handle.

[It may be helpful to note that the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ is simply S^k times the first example but with n set to $n-k$, $D^{n-k} : \varnothing^{n-k} \rightleftharpoons S^{n-1-k} : \overline{D^{n-k}}$.]

Note ε_k is not just a k-handle, but may use lower k; e.g. for $n = 3, k = 1$, $\varepsilon_k : \overline{S^1 \times D^2} \sqcup S^1 \times D^2 \Rightarrow S^1 \times S^1 \times I$ uses one 1-handle (connecting the solid tori) then a 2-handle.

2 Profunctors

Typically, to describe the n- and $(n-1)$- part of an extended $(n+1)$-TQFT, we need to assign a category $Z(N)$ to closed $(n-1)$-manifolds and functors $Z(M) : Z(N) \to Z(N')$ for an n-dimensional cobordism between $(n-1)$-manifolds N, N'.

For skein theories (see Section 3), it is more natural to describe the functor associated to cobordisms in terms of a profunctor, that is, in this case, given any boundary value $\mathbf{V} \in \mathrm{Obj}\, Z(N)$ at N and $\mathbf{V}' \in \mathrm{Obj}\, Z(N')$ at N', we assign some vector space $\widetilde{Z(M)}(\mathbf{V}, \mathbf{V}')$. We may then convert this into a functor, but this often requiring some additional choices.

In this section, we recall some basic results about profunctors.

First, we will describe profunctors valued in Set, then profunctors valued in Vec for Vec-enriched functors.

Definition 2.1. Let $\mathcal{A}, \mathcal{B}$ be categories. A *profunctor* F from $\mathcal{A}$ to $\mathcal{B}$, denoted $F : \mathcal{A} \nrightarrow \mathcal{B}$, is a functor $F : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Set}$.

Example 2.2. From a functor $G : \mathcal{A} \to \mathcal{B}$, we may construct two profunctors $G^* : \mathcal{A} \nrightarrow$

$\mathcal{B}, G_* : \mathcal{B} \nrightarrow \mathcal{A}$, defined by

$$\begin{array}{rcl} G^* : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Set} & ; & G_* : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Set} \\ (B, A) \mapsto \mathcal{B}(B, GA) & ; & (A, B) \mapsto \mathcal{B}(GA, B) \end{array}$$

Definition 2.3. Let $F : \mathcal{A} \nrightarrow \mathcal{B}, G : \mathcal{B} \nrightarrow \mathcal{C}$ be profunctors. Their *composition* $G \circ F : \mathcal{A} \nrightarrow \mathcal{C}$ is given by (assuming it exists, e.g. when $\mathcal{B}$ is small):

$$(G \circ F)(C, A) = \int^B G(C, B) \times F(B, A) = \bigsqcup G(C, B) \times F(B, A)/ \sim$$

where the equivalence relation $\sim$ identifies $(y, F(f, \mathrm{id}_A)(x)) \sim (G(\mathrm{id}_C, f)(y), x)$ for $y \in G(C, B), x \in F(B, A), f : B \to B'$.

One justification for defining composition as such is the following:

Lemma 2.4. *The identity profunctor* id^* *is the identity for composition of profunctors, i.e. for* $F : \mathcal{A} \nrightarrow \mathcal{B}$, $\mathrm{id}^*_{\mathcal{B}} \circ F \simeq F \simeq F \circ \mathrm{id}^*_{\mathcal{A}}$.

It is a basic exercise on coends to verify this.

Perhaps a more philosophical reason for defining composition of profunctors is from their analogy with relations on sets. In a sense, functors are to functions between sets as profunctors are to relations between sets. If $R \subseteq A \times B$ is a relation from A to B and $S \subseteq B \times C$ is a relation from B to C, their composition is $S \circ R = \{(a, c) | \exists b : (a, b) \in R, (b, c) \in S\}$.

A more practical reason is that this definition of composition is the standard (only?) trick in category of "doing everything" when/if no extra choices are allowed to be made.

3 Skein theory for $(1 + \varepsilon)$-theory

To study completion of a $(n + \varepsilon)$-theory to an extended $(n + 1)$-theory, we start by studying the simplest case, $n = 1$.

In particular, we will study the $(1 + \varepsilon)$-theory arising from skeins in 1-manifolds.

Manifolds with be assumed to be compact unless otherwise specified.

We consider oriented 0-manifolds, denoted by tuples of signs, e.g. $(+, +, -, +)$; the empty 0-manifold is denoted by $()$ or $\varnothing^0$. Note that the tuple is ordered, but the 0-manifold itself is not, in the sense that $(+, +, -, +)$ is homeomorphic to $(-, +, +, +)$; it is useful to keep an ordering of the points to keep our heads straight on the algebra.

Sometimes we will denote a singleton by simply $+$, or $-$, or pt^+, or pt^-.

Let us fix a $\mathbf{k}$-linear category $\mathcal{A}$. We will often consider the special case of $\mathcal{A} = \ast // A$, where A is a $\mathbf{k}$-algebra.

Let us first describe the objects of the category $Z(N)$ for an oriented 0-manifold N, so-called boundary values:

Definition 3.1. A *boundary value* $\mathbf{V}$ on an oriented 0-manifold N is an assignment of an object $V_b \in \mathcal{A}$ for every point $b \in N$.

We denote by $\overline{\mathbf{V}}$ the boundary value on $\overline{N}$ (opposite orientation) with the same assignments as $\mathbf{V}$.

Note that usually (for $\dim N = 1, 2$) a boundary value is a finite set of points in N with assigned objects; here we require the finite set of points of a boundary value to be all of N because: (1) if $\mathcal{A}$ has structure, e.g. rigid monoidal, we may as well assign the unit object $\mathbf{1}$ to points of N that are not in the boundary value; (2) if $\mathcal{A}$ is simply only $\mathbf{k}$-linear, say $\mathcal{A} = A - \mathrm{mod}$, we can still consider skeins as we will see below, but if we do not assign an object to a boundary point, it is unclear what such a skein should do at the boundary...

Next we define skeins in a 1-manifold. Let M be an oriented 1-manifold. Its boundary, ∂M, acquires an orientation from M, with a point $b \in \partial M$ being pt^+ if M is outwardly-oriented at b, and pt^- otherwise.

Definition 3.2. Let M be an oriented 1-manifold. Let $\mathbf{V}$ be a boundary value on ∂M. An $\mathcal{A}$-colored graph (Γ, φ) is an embedded oriented graph Γ in M, with edge oriented in accordance with M and is surjective on M, and φ is an assignment of an object φ_e for each edge e of Γ such that the assignment to the edges meeting the boundary agrees with the boundary value $\mathbf{V}$, and a morphism $\varphi_v : \varphi_e \to \varphi_{e'}$ for every internal vertex of Γ with an incoming edge e and outgoing edge e'.

Note finiteness of the graph is automatic.

We often denote an $\mathcal{A}$-colored graph simply by Γ.

Definition 3.3. Let (Γ, φ) be an $\mathcal{A}$-colored graph in an oriented 1-manifold M. Let D be a 1-dimension embedded disk, i.e. an interval, in M, such that ∂D does not meet any internal vertex of Γ. Let X, X' be the objects assigned to the edges e, e', respectively, that contain or meet ∂D, with e oriented into D, and e' oriented out of D. Let $e_0 = e, e_1, \dots, e_k = e'$ be the edges of Γ that meet D, and let v_i be the vertex between e_{i-1} and e_i.

We define the *evaluation of* (Γ, φ) *in* D to be the composition

$$\langle(\Gamma, \varphi)\rangle_D = \varphi_{v_k} \circ \cdots \circ \varphi_{v_1} \in \mathrm{Hom}_{\mathcal{A}}(X, X')$$

Definition 3.4. Let $\Gamma_1, \dots, \Gamma_k$ be $\mathcal{A}$-colored graphs in M with boundary value $\mathbf{V}$, and let $c_1, \dots, c_k \in \mathbf{k}$. Let D be a 1-disk in M. We say that the formal $\mathbf{k}$-linear combination $\sum c_i \cdot \Gamma_i$ is *null (with respect to D)* if Γ_i agree outside of D, and $\sum c_i \langle \Gamma_i \rangle_D = 0$.

Note that $c \cdot \Gamma - c\Gamma$ is null, where $c\Gamma$ is the $\mathcal{A}$-colored graph where one of the morphisms in Γ is scaled by c.

Definition 3.5. The *skein module* $\mathrm{Skein}(M; \mathbf{V})$ of M with boundary value $\mathbf{V}$ is the space of formal linear combinations of $\mathcal{A}$-colored graphs in M with boundary value $\mathbf{V}$ module null graphs. A *skein* is an element of a skein module.

Finally, we can give the full definition of the category $Z(N)$:

Definition 3.6. The *skein category* $Z(N)$ associated to an oriented 0-manifold N is the category whose objects are boundary values $\mathbf{V}$, and morphism are given by

$$\mathrm{Hom}_{Z(N)}(\mathbf{V}, \mathbf{V}') = \mathrm{Skein}(N \times I; \overline{\mathbf{V}} \cup \mathbf{V}')$$

It is straightforward to check that $Z(pt^+) \simeq \mathcal{A}$ and $Z(pt^-) \simeq \mathcal{A}^{\mathrm{op}}$ (in fact these are isomorphisms).

Definition 3.7. For $\mathbf{k}$-linear categories $\mathcal{B}, \mathcal{C}$, we define the category $\mathcal{B} \hat{\boxtimes} \mathcal{C}$ as the additive completion of the category whose objects are pairs of objects B_i, C_i from $\mathcal{B}, \mathcal{C}$, denoted $B_i \hat{\boxtimes} C_i$, and morphisms are

$$\mathrm{Hom}(B_i \hat{\boxtimes} C_i, B_j \hat{\boxtimes} C_j) := \mathrm{Hom}_{\mathcal{B}}(B_i, B_j) \otimes \mathrm{Hom}_{\mathcal{C}}(C_i, C_j)$$

Proposition 3.8. $Z(N_1 \sqcup N_2) \simeq Z(N_1) \hat{\boxtimes} Z(N_2)$

Definition 3.9. For a cobordism $M : N \to N'$, we define the profunctor $Z(M) : Z(N) \nrightarrow Z(N')$ defined by

$$(\mathbf{V}, \mathbf{V}') \mapsto \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}')$$

Proposition 3.10. *For cobordisms $M : N \to N', M' : N' \to N''$, $Z(M' \circ M) \simeq Z(M') \circ Z(M)$.*

Proof. We need to show that, for boundary values $\mathbf{V}, \mathbf{V}''$ on N, N'' respectively,

$$\int^{\mathbf{V}'} \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \simeq \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$$

The obvious concatenation of graphs in M, M' to form a graph in $M' \circ M$ gives a map $\Phi_{\mathbf{V}'} : \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \to \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$ for each $\mathbf{V}'$. It is easy to see that the sum of these maps, $\sum_{\mathbf{V}'} \Phi_{\mathbf{V}'} : \bigoplus_{\mathbf{V}'} \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \to \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$, factors through the coend.

That this map is surjective is obvious; for injectivity, see notes on excision (or see [3]). ☐

References

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[3] Kirillov Jr, Alexander, and Ying Hong Tham. "Factorization homology and 4D TQFT." arXiv preprint arXiv:2002.08571 (2020).