Semisimple 2-categories

Ying Hong Tham

UHH

2022

Goal

$$\{\text{multifusion category}\} \xrightarrow[\simeq]{\mathcal{M}\mathrm{od}^{fin}_{s.s.}(-)} \{\text{semisimple 2-cat}\}$$

- The 2-category $\mathcal{M}\mathrm{od}^{fin}_{s.s.}(C)$ of finite semisimple module categories over a multifusion category C is semisimple.
- For any finite semisimple 2-category $\mathcal{C}$, there exists a multifusion category C such that $\mathcal{C} \simeq \mathcal{M}\mathrm{od}^{fin}_{s.s.}(C)$

Conventions

In relation to a 2-category:

- $\mathcal{C}$ (caligraphic font): 2-category;
- X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X, Y)$ for the category of morphisms from X to Y;
- $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \to Y$, we may write $\alpha \in \mathcal{C}(X, Y)(f, g)$ to indicate its sources and targets, or simply $\alpha \in \mathrm{Hom}(f, g)$ if the objects are clear

Conventions

In relation to a 1-category:

- C, A (upper case latin): category;
- a, b, f, g (lower case latin): objects in category, functor between categories;
- α, β (lower case greek): morphism in category

Conventions

We also compose morphisms from right to left: in a 2-category $\mathcal{C}$, for $\alpha \in \mathcal{C}(X, Y)(f, f'), \beta \in \mathcal{C}(Y, Z)(g, g'), \gamma \in \mathcal{C}(X, Y)(f', f'')$, we write

$$g \circ f, g \circ f', \ldots : X \to Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \to Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \to Y$$

for vertical composition of 2-morphisms.

Conventions

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X, Y)$ satisfies P.

Conventions

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X, Y)$ satisfies P.
By 2-category we always mean a weak 2-category that is furthermore locally additive over $\mathbf{k}$.
By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally $\mathbf{k}$-linear.

Review

Additive 2-category, direct sum of objects

Additive 2-category, direct sum of objects

Definition (direct sum of objects in 2-category)

A *direct sum* of two objects A_1, A_2 in $\mathcal{C}$ is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms $i_k : A_k \rightleftharpoons A_1 \boxplus A_2 : p_k$, such that

- $p_k \circ i_k \simeq \mathrm{id}_{A_k}$,
- $p_2 \circ i_1$, $p_1 \circ i_2$ are zero 1-morphisms,
- $\mathrm{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

Additive 2-category, direct sum of objects

Proposition

i_k, p_k are two-sided adjoints to each other.

Additive 2-category, direct sum of objects

Proposition

i_k, p_k are two-sided adjoints to each other.

Definition

A 1-morphism $i : X \to Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A, $i \circ - : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ is fully faithful.

Additive 2-category, direct sum of objects

Proposition

i_k, p_k are two-sided adjoints to each other.

Definition

A 1-morphism $i : X \to Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A, $i \circ - : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ is fully faithful.

Proposition

$i_k : A_k \to A_1 \boxplus A_2$ is fully faithful.

Additive 2-category, direct sum of objects

Definition (Direct sum of 2-categories)

Given 2-categories $\mathcal{C}_j$, $j \in J$, we may consider the direct sum 2-category $\mathcal{C} := \boxplus_{j \in J} \mathcal{C}_j$:

- $\operatorname{Obj} \mathcal{C} = \bigsqcup_{j \in J} \operatorname{Obj} \mathcal{C}_j$
- for $X \in \mathcal{C}_i$, $Y \in \mathcal{C}_j$, $\mathcal{C}(X, Y) = \begin{cases} \mathcal{C}_j(X, Y) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Idempotent completeness, separable monads, splittings

Idempotent completeness, separable monads, splittings

Definition

A *separable algebra* (a, μ, η) in a tensor category C is an algebra that admits an a-a-bimodule section ${}_a a_a \to {}_a a \otimes a_a$ to μ.

Idempotent completeness, separable monads, splittings

Definition

A *separable algebra* (a, μ, η) in a tensor category C is an algebra that admits an a-a-bimodule section ${}_a a_a \to {}_a a \otimes a_a$ to μ.

Definition

Let (t, μ, η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $t \Rightarrow t \circ t$ to μ.

Idempotent completeness, separable monads, splittings

Definition

A *separable algebra* (a, μ, η) in a tensor category C is an algebra that admits an a-a-bimodule section ${}_a a_a \to {}_a a \otimes a_a$ to μ.

Definition

Let (t, μ, η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $t \Rightarrow t \circ t$ to μ.

In other words, a separable monad over X is a separable algebra in $\mathcal{C}(X, X)$.

Idempotent completeness, separable monads, splittings

Definition

Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Idempotent completeness, separable monads, splittings

Definition

Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Idempotent completeness, separable monads, splittings

Definition

Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition

Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Idempotent completeness, separable monads, splittings

Definition

Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition

Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Under the right conditions (local idempotent completeness of $\mathcal{C}$), splittings are unique:

Proposition (Uniqueness of splitting)

[[1], Theorem A.3.1] In a locally idempotent complete 2-category $\mathcal{C}$, splittings of a separable monad are unique up to equivalence.

Idempotent completeness, 2-category

Definition

A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Idempotent completeness, 2-category

Definition

A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition (Idempotent completion)

Let $\mathcal{C}$ be a locally idempotent complete 2-category. The *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^{\nabla}$, with:

- Objects: (X, p) separable monad in $\mathcal{C}$,
- $\mathcal{C}^{\nabla}((X,p),(Y,q)) = q\text{-bimod-}p(\mathcal{C}(X,Y))$

Idempotent completeness, 2-category

Definition

A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition (Idempotent completion)

Let $\mathcal{C}$ be a locally idempotent complete 2-category. The *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^\nabla$, with:

- Objects: (X, p) separable monad in $\mathcal{C}$,
- $\mathcal{C}^\nabla((X,p),(Y,q)) = q\text{-}\mathrm{bimod}\text{-}p(\mathcal{C}(X,Y))$

There is a natural 2-functor $I : \mathcal{C} \to \mathcal{C}^\nabla$ that is fully faithful.

Idempotent completeness, 2-category

Definition

A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition (Idempotent completion)

Let $\mathcal{C}$ be a locally idempotent complete 2-category. The *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^{\nabla}$, with:

- Objects: (X, p) separable monad in $\mathcal{C}$,
- $\mathcal{C}^{\nabla}((X,p),(Y,q)) = q\text{-}\mathrm{bimod}\text{-}p(\mathcal{C}(X,Y))$

There is a natural 2-functor $I : \mathcal{C} \to \mathcal{C}^{\nabla}$ that is fully faithful.
A 2-functor $F : \mathcal{C} \to \mathcal{D}$ extends to a 2-functor $F^{\nabla} : \mathcal{C}^{\nabla} \to \mathcal{D}^{\nabla}$ that commutes with I's.

Idempotent completeness, 2-category

Key example:
$\mathcal{B}C$: one object $*$ with endomorphism category $\mathcal{B}C(*,*)$.

$$(\mathcal{B}C)^{\nabla} = \begin{cases} \text{Obj}: & \text{separable algebras in } C \\ \text{Mor}: & (\mathcal{B}C)^{\nabla}(a,b) = b\text{-bimod-}a(C) \end{cases}$$

Idempotent completeness, 2-category

Proposition

$\mathcal{C}^\nabla$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^\nabla$ is an equivalence.

Idempotent completeness, 2-category

Proposition

$\mathcal{C}^{\nabla}$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^{\nabla}$ is an equivalence.

As a consequence, if $\mathcal{D}$ is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^{\nabla}, \mathcal{D}^{\nabla}) \simeq \mathcal{F}un(\mathcal{C}^{\nabla}, \mathcal{D})$$

Idempotent completeness, 2-category

Proposition

$\mathcal{C}^{\nabla}$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^{\nabla}$ is an equivalence.

As a consequence, if $\mathcal{D}$ is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^{\nabla}, \mathcal{D}^{\nabla}) \simeq \mathcal{F}un(\mathcal{C}^{\nabla}, \mathcal{D})$$

Proposition

If $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, then $F^{\nabla} : \mathcal{C}^{\nabla} \to \mathcal{D}^{\nabla}$ is also fully faithful.

Idempotent completeness, 2-category

Proposition ([1]Prop 1.3.13)

For a multifusion category C, the following 2-functor is an equivalence:

$$
\begin{aligned}
(-)\text{-}\mathit{mod}(C) : (\mathcal{B}C)^{\nabla} &\to \mathcal{M}od(C) \\
a &\mapsto a\text{-}\mathit{mod}(C) \\
{}_b m_a &\mapsto m \otimes_a - \\
\varphi &\mapsto \varphi \otimes_a -
\end{aligned}
$$

Idempotent completeness, 2-category

Proof.

Essential surjectivity follows from:

-[2]Theorem 1: for a finite semisimple right module category over multifusion C, there exists a semisimple algebra a in C such that $M \simeq a\text{-}\mathrm{mod}(C)$ as right C-module categories; and

Idempotent completeness, 2-category

Proof.

Essential surjectivity follows from:
-[2]Theorem 1: for a finite semisimple right module category over multifusion C, there exists a semisimple algebra a in C such that $M \simeq a\text{-}\mathrm{mod}(C)$ as right C-module categories; and
-[4]Corollary 2.6.9: When C is multifusion over a field of characterisitc 0, a right C-module category M is separable ($\simeq a\text{-}\mathrm{mod}(C)$ for a separable algebra a) if and only if it is semisimple.

Idempotent completeness, 2-category

Proof.

Essential surjectivity follows from:

-[2]Theorem 1: for a finite semisimple right module category over multifusion C, there exists a semisimple algebra a in C such that $M \simeq a\text{-}\mathrm{mod}(C)$ as right C-module categories; and

-[4]Corollary 2.6.9: When C is multifusion over a field of characterisitc 0, a right C-module category M is separable ($\simeq a\text{-}\mathrm{mod}(C)$ for a separable algebra a) if and only if it is semisimple.

Fully faithfulness follows from [5]Prop 7.11.1. $\square$

Idempotent completeness, 2-category

Proof.

Essential surjectivity follows from:
-[2]Theorem 1: for a finite semisimple right module category over multifusion C, there exists a semisimple algebra a in C such that $M \simeq a\text{-}\mathrm{mod}(C)$ as right C-module categories; and
-[4]Corollary 2.6.9: When C is multifusion over a field of characterisitc 0, a right C-module category M is separable ($\simeq a\text{-}\mathrm{mod}(C)$ for a separable algebra a) if and only if it is semisimple.
Fully faithfulness follows from [5]Prop 7.11.1. □

This is almost one half of the main result; one still needs to prove local semisimplicity and existence of adjoints. This will follow from more results from [3],[4], which we show later.

Simple objects

Simple objects

Proposition (equivalent notions of simple-ness)

Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:

Simple objects

Proposition (equivalent notions of simple-ness)

Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:
(1) any subobject $i : Y \to X$ of X is either 0 or an equivalence;

Simple objects

Proposition (equivalent notions of simple-ness)

Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:
(1) any subobject $i : Y \to X$ of X is either 0 or an equivalence;
(2) X cannot be written as a non-trivial direct sum, i.e. if $X = \boxplus X_i$, then $X_i \simeq 0$ for all but one i;

Simple objects

Proposition (equivalent notions of simple-ness)

Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:
(1) any subobject $i : Y \to X$ of X is either 0 or an equivalence;
(2) X cannot be written as a non-trivial direct sum, i.e. if $X = \boxplus X_i$, then $X_i \simeq 0$ for all but one i;
(3) id_X is a simple object in $\mathcal{C}(X, X)$.

Simple objects

Proof sketch.

(1) $\Rightarrow$ (2): Contravariant statement follows from fully faithfulness of $i_k : A_k \to A_1 \boxplus A_2$.

Simple objects

Proof sketch.

(1) $\Rightarrow$ (2): Contravariant statement follows from fully faithfulness of $i_k : A_k \to A_1 \boxplus A_2$.

(2) $\Rightarrow$ (3): Contravariant statement is "identity splitting implies object splitting", uses idempotent completeness of $\mathcal{C}$ to split out objects corresponding to summands of id_X (which are separable monads) (see [1]Prop 1.3.16).

Simple objects

Proof sketch.
(1) $\Rightarrow$ (2): Contravariant statement follows from fully faithfulness of $i_k : A_k \to A_1 \boxplus A_2$.
(2) $\Rightarrow$ (3): Contravariant statement is "identity splitting implies object splitting", uses idempotent completeness of $\mathcal{C}$ to split out objects corresponding to summands of id_X (which are separable monads) (see [1]Prop 1.3.16).
(3) $\Rightarrow$ (1): for non-zero fully faithful $r : Y \to X$, with id_X simple, consider the left adjoint $l : X \to Y$, use fully faithfulness to get a preimage $\delta : \mathrm{id}_Y \Rightarrow lr$ of $\eta \circ r : r \Rightarrow rlr$. Use simplicity of id_X to get section of the unit η. Show δ is a section of the counit. Etc. (See [1]Prop 1.2.14) $\square$

Semisimple 2-category

Definition ((finite) semisimple 2-category)

Semisimple 2-category

Definition ((finite) semisimple 2-category)

A 2-category $\mathcal{C}$ is *semisimple* if it is:

- locally semisimple,
- admits left and right adjoints for every 1-morphism,
- additive,
- idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

New Stuff

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case.

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case. However, recall $\mathcal{M}\mathrm{od}(\mathrm{Vec}_{\mathbb{Z}/2})$:

$$\begin{tikzcd}
\mathrm{Vec}_{\mathbb{Z}/2} \arrow[loop left, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[rr, bend left, "\mathrm{Vec}"] & & \mathrm{Vec} \arrow[ll, bend left, "\mathrm{Vec}"] \arrow[loop right, "\mathrm{Vec}_{\mathbb{Z}/2}"]
\end{tikzcd}$$

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case. However, recall $\mathcal{M}\mathrm{od}(\mathrm{Vec}_{\mathbb{Z}/2})$:

$$\begin{tikzcd} \mathrm{Vec}_{\mathbb{Z}/2} \arrow[loop left, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[r, bend left, "\mathrm{Vec}"] & \mathrm{Vec} \arrow[l, bend left, "\mathrm{Vec}"] \arrow[loop right, "\mathrm{Vec}_{\mathbb{Z}/2}"] \end{tikzcd}$$

There can be nonzero morphisms between non-equivalent simple objects - weird!

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case. However, recall $\mathcal{M}\mathrm{od}(\mathrm{Vec}_{\mathbb{Z}/2})$:

$$\begin{tikzcd}
\mathrm{Vec}_{\mathbb{Z}/2} \arrow[loop left, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[rr, bend left, "\mathrm{Vec}"] & & \mathrm{Vec} \arrow[loop right, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[ll, bend left, "\mathrm{Vec}"]
\end{tikzcd}$$

There can be nonzero morphisms between non-equivalent simple objects - weird!
"2-Morita equivalences" between fusion categories

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$*, if* $f : A \to B, g : B \to C$ *are nonzero 1-morphisms between simple objects* A, B, C*, then* $g \circ f$ *is also nonzero.*

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.

Proof.

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, phantom, "\overset{\varepsilon}{\Longleftrightarrow}" description] \arrow[r, phantom, shift right=3, "\exists\delta"] & B \arrow[l, bend left=50, "f^*"] \arrow[loop left] \arrow[r, "g"] & C \end{tikzcd}$$

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.

Proof.

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, phantom, "\underset{\exists\delta}{\overset{\varepsilon}{\Longleftrightarrow}}"] & B \arrow[l, bend left=60, "f^*"] \arrow[r, "g"] & C
\end{tikzcd}
$$

Let $f^* : B \to A$ be a right adjoint to f.

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.

Proof.

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, phantom, "\overset{\varepsilon}{\Longleftrightarrow}"'] & B \arrow[l, bend left=50, "f^*"] \arrow[r, "g"] & C
\end{tikzcd}
$$

$\exists\delta$

Let $f^* : B \to A$ be a right adjoint to f. Since id_B is simple, $\exists$ section $\delta : \mathrm{id}_B \Rightarrow ff^*$ to counit $\varepsilon : ff^* \Rightarrow \mathrm{id}_B$.

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.

Proof.

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, phantom, shift left=1, "\varepsilon"] \arrow[r, phantom, shift right=1, "\exists\delta"'] & B \arrow[l, bend left=50, "f^*"] \arrow[r, "g"] & C \end{tikzcd}$$

Let $f^* : B \to A$ be a right adjoint to f. Since id_B is simple, $\exists$ section $\delta : \mathrm{id}_B \Rightarrow ff^*$ to counit $\varepsilon : ff^* \Rightarrow \mathrm{id}_B$. Postcomposing with g, we have $\mathrm{id}_g = (\mathrm{id}_g \circ \varepsilon) \cdot (\mathrm{id}_g \circ \delta) : g \Rightarrow gff^* \Rightarrow g$.

Schur's lemma

Proposition (Schur's Lemma, [1]Prop 1.2.19)

In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.

Proof.

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[r, phantom, "\underset{\exists\delta}{\overset{\varepsilon}{\Longleftrightarrow}}"] & {} & B \arrow[ll, bend left, "f^*"] \arrow[r, "g"] & C \end{tikzcd}$$

Let $f^* : B \to A$ be a right adjoint to f. Since id_B is simple, $\exists$ section $\delta : \mathrm{id}_B \Rightarrow ff^*$ to counit $\varepsilon : ff^* \Rightarrow \mathrm{id}_B$. Postcomposing with g, we have $\mathrm{id}_g = (\mathrm{id}_g \circ \varepsilon) \cdot (\mathrm{id}_g \circ \delta) : g \Rightarrow gff^* \Rightarrow g$. Thus if $gf = 0$, then $\mathrm{id}_g = 0$, contradicting nonzero-ness of B. $\square$

Schur's lemma - components

Definition (component of semisimple 2-category)

In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

Schur's lemma - components

Definition (component of semisimple 2-category)

In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

-transitivity from Schur's lemma, symmetry from adjoints, reflexivity from identity

Schur's lemma - components

Definition (component of semisimple 2-category)

In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

-transitivity from Schur's lemma, symmetry from adjoints, reflexivity from identity
In a finite semisimple 2-category $\mathcal{C}$, there will be finite set J of components;

Schur's lemma - components

Definition (component of semisimple 2-category)

In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

-transitivity from Schur's lemma, symmetry from adjoints, reflexivity from identity
In a finite semisimple 2-category $\mathcal{C}$, there will be finite set J of components;
full subcategory $\mathcal{C}_j$, $j \in J$: direct sum of simple objects in j

Schur's lemma - components

Definition (component of semisimple 2-category)

In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

-transitivity from Schur's lemma, symmetry from adjoints, reflexivity from identity

In a finite semisimple 2-category $\mathcal{C}$, there will be finite set J of components;

full subcategory $\mathcal{C}_j$, $j \in J$: direct sum of simple objects in j gives direct sum decomposition $\mathcal{C} \simeq \boxplus_{j \in J} \mathcal{C}_j$.

Main results

Theorem ([1]Theorem 1.4.8)

The 2-category of finite semisimple module categories of a multifusion category $\mathcal{C}$ is a finite semisimple 2-category.

Main results

Theorem ([1]Theorem 1.4.8)

The 2-category of finite semisimple module categories of a multifusion category C is a finite semisimple 2-category.

Proof.

$\mathcal{M}\mathrm{od}(C) \simeq (\mathcal{B}C)^{\nabla}$ is idempotent complete and locally idempotent complete. $\mathcal{M}\mathrm{od}(C)$ is clearly already additive.
Locally semisimple-ness follows directly from [4]Corollary 2.5.6, and existence of adjoints for 1-morphisms follows from [3]Corollary 2.13.

$\square$

Main results

Theorem ([1]Theorem 1.4.9)

Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.

Proof.

Assume that $\mathcal{C}$ has only one component.

Main results

Theorem ([1]Theorem 1.4.9)

Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.

Proof.

Assume that $\mathcal{C}$ has only one component.
($\mathcal{C} =\simeq \boxplus_{j\in J}\mathcal{C}_j$ and $\mathcal{C}_j \simeq \mathcal{M}\mathrm{od}(C_j)$ for multifusion C_j, then $\mathcal{C} \simeq \mathcal{M}\mathrm{od}(\bigoplus_{j\in J} C_j)$.)

Main results

Theorem ([1]Theorem 1.4.9)

Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.

Proof.

Assume that $\mathcal{C}$ has only one component.
($\mathcal{C} = \simeq \boxplus_{j \in J} \mathcal{C}_j$ and $\mathcal{C}_j \simeq \mathcal{M}\mathrm{od}(C_j)$ for multifusion C_j, then $\mathcal{C} \simeq \mathcal{M}\mathrm{od}(\bigoplus_{j \in J} C_j)$.)
Fix simple object X, let $C = \mathcal{C}(X, X)$.

Main results

Theorem ([1]Theorem 1.4.9)

Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.

Proof.

Assume that $\mathcal{C}$ has only one component.
($\mathcal{C} =\simeq \boxplus_{j\in J}\mathcal{C}_j$ and $\mathcal{C}_j \simeq \mathcal{M}\mathrm{od}(C_j)$ for multifusion C_j, then $\mathcal{C} \simeq \mathcal{M}\mathrm{od}(\bigoplus_{j\in J} C_j)$.)
Fix simple object X, let $C = \mathcal{C}(X, X)$.
We show $\mathcal{C} \simeq (\mathcal{B}C)^{\nabla}$.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.
Simple object Y,

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.
Simple object Y, $\exists$ nonzero 1-morphism $f : X \to Y$;

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.
Simple object Y, $\exists$ nonzero 1-morphism $f : X \to Y$; has (nonzero) right adjoint $g : Y \to X$, with counit $\varepsilon : fg \Rightarrow \mathrm{id}_Y$.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.
Simple object Y, $\exists$ nonzero 1-morphism $f : X \to Y$; has (nonzero) right adjoint $g : Y \to X$, with counit $\varepsilon : fg \Rightarrow \mathrm{id}_Y$. id_Y is simple, so ε admits a section,

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.
Simple object Y, $\exists$ nonzero 1-morphism $f : X \to Y$; has (nonzero) right adjoint $g : Y \to X$, with counit $\varepsilon : fg \Rightarrow \mathrm{id}_Y$. id_Y is simple, so ε admits a section, hence $f \dashv g$ is a separable adjunction.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\nabla} : \mathcal{B}C \to \mathcal{C}$$

is fully faithful. Remains to show other simples are in essential image.

Simple object Y, $\exists$ nonzero 1-morphism $f : X \to Y$; has (nonzero) right adjoint $g : Y \to X$, with counit $\varepsilon : fg \Rightarrow \mathrm{id}_Y$. id_Y is simple, so ε admits a section, hence $f \dashv g$ is a separable adjunction. Thus, by uniqueness of separable splittings, Y is in the essential image of F^{∇}. $\square$

Main results

The only property of X that we used is the fact that there exists a nonzero 1-morphism from X to every simple in $\mathcal{C}$, and thus any nonzero X will do.

Main results

The only property of X that we used is the fact that there exists a nonzero 1-morphism from X to every simple in $\mathcal{C}$, and thus any nonzero X will do. Taking, say, $X = \boxplus X_i$, where the sum is over equivalence classes of simples, would result in a multifusion $C = \mathcal{C}(X, X)$.

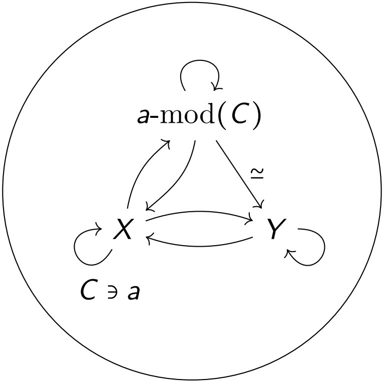
Main results

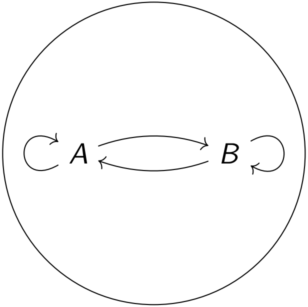
The only property of X that we used is the fact that there exists a nonzero 1-morphism from X to every simple in $\mathcal{C}$, and thus any nonzero X will do. Taking, say, $X = \boxplus X_i$, where the sum is over equivalence classes of simples, would result in a multifusion $C = \mathcal{C}(X, X)$.

We can also avoid the first step of taking only one component of $\mathcal{C}$, take X with at least one object from each $\mathcal{C}_j$ in its direct sum decomposition.

Main results

Every object contains within them the data to reconstruct the whole 2-category.





Example

$C = \mathrm{Vec}_G$ for a finite group G.

Example

$C = \mathrm{Vec}_G$ for a finite group G.
e: trivial algebra $\mathbf{k}_e$ $\quad e\text{-}\mathrm{mod}(C) \simeq C_C$

Example

$C = \mathrm{Vec}_G$ for a finite group G.
e: trivial algebra $\mathbf{k}_e$ $\quad e\text{-}\mathrm{mod}(C) \simeq C_C$
f: the group algebra $\mathbf{k}[G] = \bigoplus \mathbf{k}_g$ $\quad f\text{-}\mathrm{mod}(C) \simeq \mathrm{Vec}$
(right action on Vec = forget G-grading)

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.
For $f\text{-}\mathrm{mod}(C)$, an endofunctor is given by some $m \in f\text{-}\mathrm{bimod}\text{-}f(C)$; write $m = \bigoplus m_g$.

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.
For $f\text{-}\mathrm{mod}(C)$, an endofunctor is given by some $m \in f\text{-}\mathrm{bimod}\text{-}f(C)$; write $m = \bigoplus m_g$. The right f-action on m makes all the m_g isomorphic in a coherent manner.

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.
For $f\text{-}\mathrm{mod}(C)$, an endofunctor is given by some $m \in f\text{-}\mathrm{bimod}\text{-}f(C)$; write $m = \bigoplus m_g$. The right f-action on m makes all the m_g isomorphic in a coherent manner. For $h \in G$, conjugation (left action by h and right action by h^{-1}) gives an action of G on m_e; determines m completely.

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.
For $f\text{-}\mathrm{mod}(C)$, an endofunctor is given by some $m \in f\text{-}\mathrm{bimod}\text{-}f(C)$; write $m = \bigoplus m_g$. The right f-action on m makes all the m_g isomorphic in a coherent manner. For $h \in G$, conjugation (left action by h and right action by h^{-1}) gives an action of G on m_e; determines m completely.
Thus, $\mathcal{M}\mathrm{od}(C)(f, f) \simeq f\text{-}\mathrm{bimod}\text{-}f(C) \simeq \mathsf{Rep}(G)$.

Example

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself.
For $f\text{-}\mathrm{mod}(C)$, an endofunctor is given by some $m \in f\text{-}\mathrm{bimod}\text{-}f(C)$; write $m = \bigoplus m_g$. The right f-action on m makes all the m_g isomorphic in a coherent manner. For $h \in G$, conjugation (left action by h and right action by h^{-1}) gives an action of G on m_e; determines m completely.
Thus, $\mathcal{M}\mathrm{od}(C)(f,f) \simeq f\text{-}\mathrm{bimod}\text{-}f(C) \simeq \mathrm{Rep}(G)$.
(One can also check that C-module structure on $m : \mathrm{Vec} \to \mathrm{Vec}$ amounts to G-action on $m(\mathbf{k})$.)
Thus we have:

$$\mathcal{M}\mathrm{od}(\mathrm{Vec}_G) \simeq \mathcal{M}\mathrm{od}(\mathrm{Rep}(G))$$

Example

For functor categories between them, we see that

$$\mathcal{M}\mathrm{od}(C)(e, f) \simeq f\text{-}\mathrm{bimod}\text{-}e(C) \simeq f\text{-}\mathrm{mod}(C) \simeq \mathrm{Vec}$$

$$\mathcal{M}\mathrm{od}(C)(f, e) \simeq e\text{-}\mathrm{bimod}\text{-}f(C) \simeq \mathrm{mod}\text{-}f(C) \simeq \mathrm{Vec}$$

Example

For functor categories between them, we see that

$$\mathcal{M}\mathrm{od}(C)(e, f) \simeq f\text{-}\mathrm{bimod}\text{-}e(C) \simeq f\text{-}\mathrm{mod}(C) \simeq \mathrm{Vec}$$

$$\mathcal{M}\mathrm{od}(C)(f, e) \simeq e\text{-}\mathrm{bimod}\text{-}f(C) \simeq \mathrm{mod}\text{-}f(C) \simeq \mathrm{Vec}$$

Note for $G = \mathbb{Z}/2$, $\mathrm{Vec}_{\mathbb{Z}/2} \simeq \mathsf{Rep}(\mathbb{Z}/2)$, don't see difference in endomorphism categories.

Example

For functor categories between them, we see that

$$\mathcal{M}\mathrm{od}(C)(e,f) \simeq f\text{-}\mathrm{bimod}\text{-}e(C) \simeq f\text{-}\mathrm{mod}(C) \simeq \mathrm{Vec}$$

$$\mathcal{M}\mathrm{od}(C)(f,e) \simeq e\text{-}\mathrm{bimod}\text{-}f(C) \simeq \mathrm{mod}\text{-}f(C) \simeq \mathrm{Vec}$$

Note for $G = \mathbb{Z}/2$, $\mathrm{Vec}_{\mathbb{Z}/2} \simeq \mathrm{Rep}(\mathbb{Z}/2)$, don't see difference in endomorphism categories.
Of course, there are many other objects in $\mathcal{M}\mathrm{od}(C)$, e.g. for a subgroup $H \subseteq G$, have group algebra $\mathbf{k}[H]$.

End!

Douglas, Christopher L., and David J. Reutter. “Fusion 2-categories and a state-sum invariant for 4-manifolds.” arXiv preprint arXiv:1812.11933 (2018).

Ostrik, Victor. “Module categories, weak Hopf algebras and modular invariants.” Transformation groups 8, no. 2 (2003): 177-206.

C. L. Douglas, C. Schommer-Pries, and N. Snyder. The balanced tensor product of module categories. Kyoto J. Math., 2017. arXiv:1406.4204.

C. L. Douglas, C. Schommer-Pries, and N. Snyder. Dualizable tensor categories. Mem. Amer. Math. Soc., 2017. arXiv:1312.7188.

P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor Categories. American Mathematical Society, 2015. Available online at
http://www.math.mit.edu/ etingof/egnobookfinal.pdf.