Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

2-Cob denotes the bicategory with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms.

Proposition 0.1. *Consider functors $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.*

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ be a natural transformation, and suppose it is the unit to an adjunction $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' are extended $(n+1)$-TQFTs such that Z', Z'' agree with Z on $(n-1)$- and n-manifolds, and $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, then $Z' \cong Z''$.

Proof. From the topology section below, we know that a 2-morphism of 2-Cob that realizes the attachment of an $(n+1)$-dim $(k+1)$-handle,$0 \leq k \leq n$, is determined by some counit ε_k of an adjunction, whose corresponding unit can be built from handles of index at most k, and thus, the value of extensions Z' of Z on 2-morphisms is completely determined by its value on the $(n+1)$-dim 0-handle, which is exactly η_0. □

1 Topology

1.1 Adjunctions from topology

For the purposes of the proof of the proposition above, only the first simple example from this section is needed, the reader may then skip to Section 1.2.

In 2-Cob, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition 0.1. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk $D^{n+1} : \varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure 1 for $n = 1$ case).

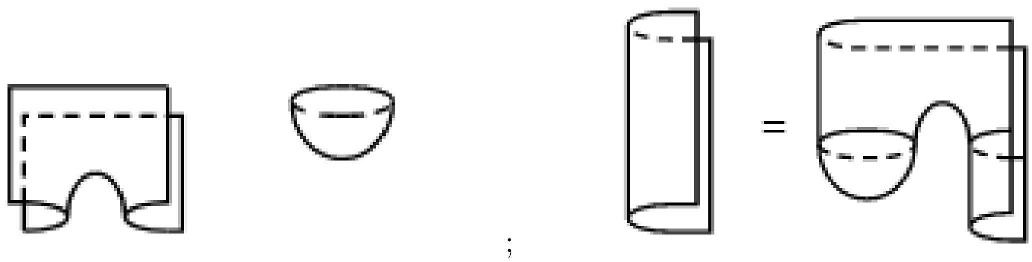

Figure 1: Counit and unit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$, along with one of the snake equations; relative cobordism goes up (stolen from [2], Figure 1.6 and 1.10, get rotated)

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n - k$. [See Figure 2]

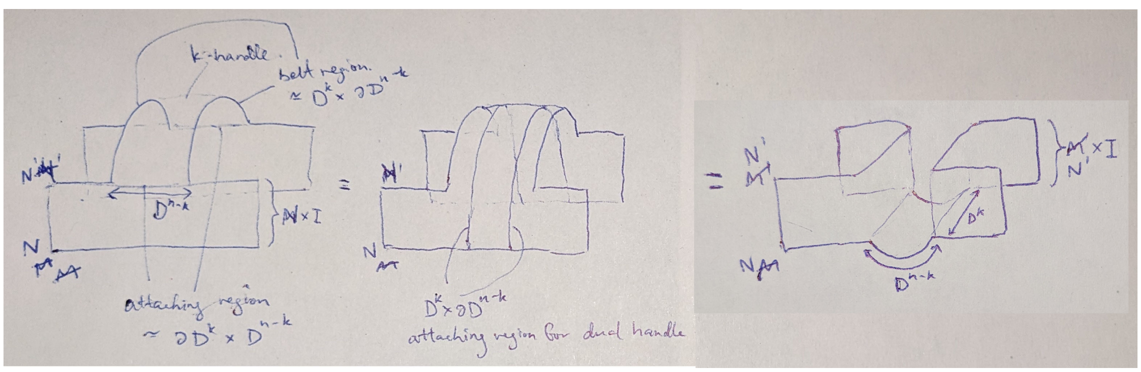

Figure 2: $M : N \to N'$ is an elementary cobordism of index k; it is built from attaching a disk D^n to $N \times I$, with attaching region $\partial D^k \times D^{n-k}$. It can also be built from the other direction, by attaching a disk D^n to $N' \times I$, with attaching region $D^k \times \partial D^{n-k}$. Thus, turning it upside-down, i.e. treated as a cobordism $\overline{M} : N' \to N$, it is an elementary cobordism of index $(n - k)$.

[See Figure 3] We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching an $(n+1)$-dimensional $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the

k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$. Similarly, we construct the unit $\eta : \mathrm{id}_N \Rightarrow M \circ \overline{M} : N \to N$ by attaching an $(n+1)$-dimensional k-handle to $\mathrm{id}_N = N \times I$; the attaching region for this $(n+1)$-dim k-handle is (the attaching region for the n-dim k-handle that defines M) $\times I$. The snake equations $\mathrm{id}_M = (\varepsilon \circ M) \cdot (M \circ \eta) : M \Rightarrow M \circ \overline{M} \circ M \Rightarrow M$ and $\mathrm{id}_{\overline{M}} = (\overline{M} \circ \eta) \cdot (\eta \circ \overline{M}) : \overline{M} \Rightarrow \overline{M} \circ M \circ \overline{M} \Rightarrow \overline{M}$ follow from the fact that these $(n+1)$-dim handles form a cancelling pair in both cases.

Here we have $M \dashv \overline{M}$, but we may very well have $\overline{M} \dashv M$; the counit $\varepsilon' : \overline{M} \circ M \Rightarrow \mathrm{id}_N$ is an $(n+1)$-dim elementary cobordism of index $(n-k+1)$. It is interesting to note that this counit is the dual cobordism to the unit $\eta : \mathrm{id}_N \Rightarrow \overline{M} \circ M$ previously described.

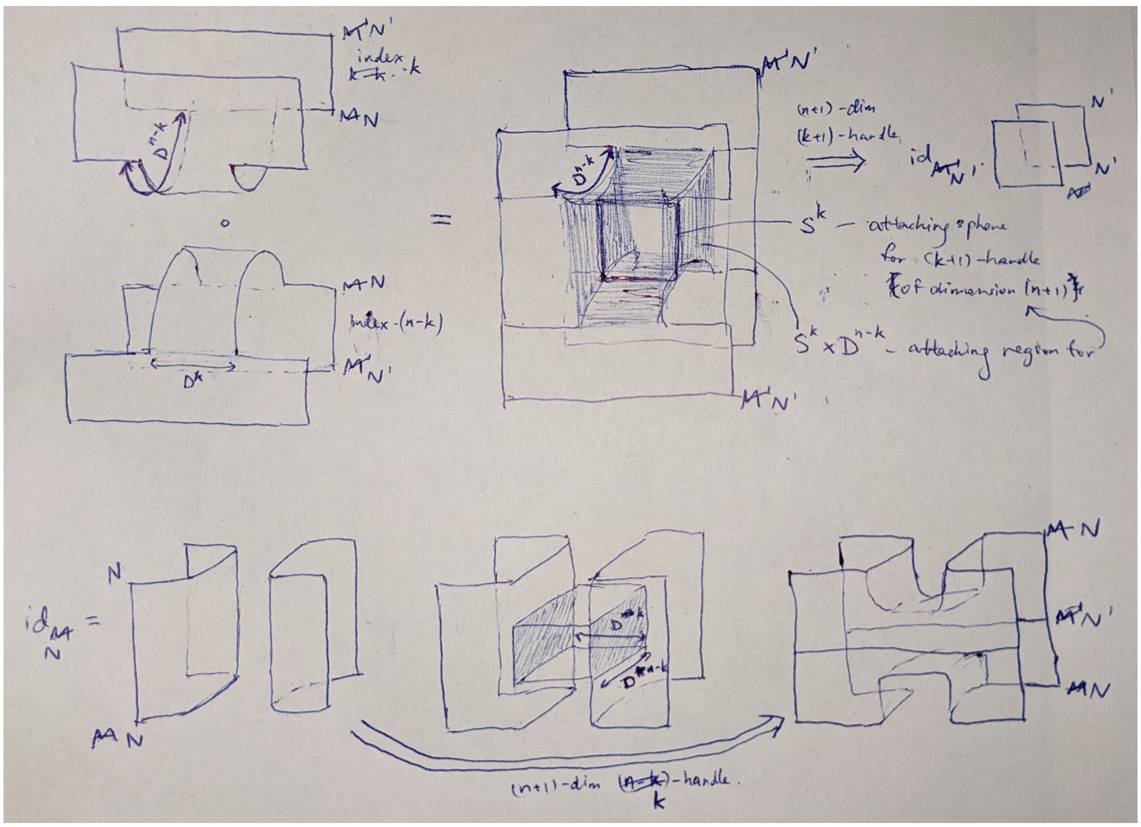

Figure 3: Counit (top) and unit (bottom) for adjunction $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k. Note the way N is drawn here looks like N' in Figure 2 and vice versa (by accident, sorry for minor confusion)

In general, we may consider the pair of n-dim cobordisms $M : N \rightleftharpoons N' : \overline{M}$. By presenting M as a composition of elementary cobordisms, we may compose the adjunctions constructed for each of these elementary cobordisms as above, and obtain an adjunction $M \dashv \overline{M}$.

Remark 1.1. In [1], we considered this construction without realizing their connection to these adjunctions; there we consider the more general case where N, N' may have (possibly different) boundary, and $M : N \to N'$ is a relative cobordism (with the boundary cobordism that is not necessarily the identity cobordism).

1.2 Producing $(n+1)$-dim k-handles from some adjunctions

Throughout this section, $0 \leq k < n$.

We show how to construct the $(n+1)$-dim $(k+1)$-handle from the counit ε_k of the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and the unit η_0 of the adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. (The 0-handle is already given by η_0, while the $(n+1)$-handle is the counit to the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$; we say a few more words about this at the end of this section.)

The process of attaching a $(k+1)$-handle to an $(n+1)$-manifold can be implemented as postcomposing by a 2-morphism. More precisely, given an $(n+1)$-manifold W presented as a 2-morphism $W : M \Rightarrow M' : N \to N'$, and an attaching region $S^k \times D^{n-k}$ in M', the $(n+1)$-manifold W' obtained from attaching a $(k+1)$-handle along the specified attaching region may be considered as a 2-morphism $W' : M \Rightarrow M'' : N \to N'$, where M'' is obtain from M' by performing surgery along the attaching region (cutting it out and gluing in $D^{k+1} \times S^{n-k-1}$); then $W' = \omega_{k+1} \cdot W$, where ω_{k+1} is a 2-morphism that we will describe below.

Our 2-morphism ω_{k+1} is of the form $\omega_{k+1} : S^k \times D^{n-k} \Rightarrow D^{k+1} \times S^{n-k-1} : S^k \times S^{n-k-1} \to \varnothing^{n-1}$. Since this is unchanged as W varies, we clearly need to make some arrangements in order to use ω_{k+1}. More specifically, we need to present M' as a composition

$$(\mathrm{id}_{N'} \sqcup \overline{S^k \times D^{n-k}}) \circ (M \backslash \overline{S^k \times D^{n-k}}) : N \to N' \sqcup S^k \times S^{n-k+1} \to N'$$

which is always possible by basic Morse theory. So we have $(M^\circ := M \backslash \overline{S^k \times D^{n-k}})$:

$$W' = \begin{tikzcd}
N \arrow[rr, bend left=50, "M"] \arrow[rr, bend left=25, "M'"] \arrow[r, "M^\circ"'] & N' \sqcup S^k \times S^{n-k+1} \arrow[u, phantom, "\Downarrow \simeq"] \arrow[r, "\overline{S^k \times D^{n-k}}"'] \arrow[r, bend right=60, "\overline{D^{k+1} \times S^{n-k-1}}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \omega_{k+1}"] & N' \sqcup \varnothing^{n-1}
\end{tikzcd}$$

Now let us describe how to construct ω_{k+1} out of ε_k and η_0, which are, as a reminder, the counit and unit of the adjunctions $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, respectively.

We may consider $S^n : \varnothing^{n-1} \to \varnothing^{n-1}$ as the composition $\overline{D^{k+1} \times S^{n-k-1}} \circ S^k \times D^{n-k} : \varnothing^{n-1} \to S^k \times S^{n-k-1} \to \varnothing^{n-1}$.

Then ω_{k+1} is given by the composition of 2-morphisms

$$\omega_{k+1} = (\mathrm{id}_{\overline{D^{k+1} \times S^{n-k-1}}} \circ \varepsilon_k) \cdot (\mathrm{id}_{\overline{S^k \times D^{n-k}}} \circ \eta_0)$$

$$
\begin{tikzcd}
\omega_{k+1} = & S^k\times S^{n-k-1} \arrow[r, "\overline{S^k\times D^{n-k}}"] \arrow[rr, bend right=50, "\mathrm{id}_{S^k\times S^{n-k-1}}"'] \arrow[rr, phantom, shift right=5ex, "\Downarrow\varepsilon_k"] & \varnothing^{n-1} \arrow[r, "S^k\times D^{n-k}"] \arrow[rr, bend left=50, "\varnothing^n"] \arrow[rr, phantom, shift left=5ex, "\Downarrow\eta_0"] & S^k\times S^{n-k-1} \arrow[r, "\overline{D^{k+1}\times S^{n-k-1}}"] & \varnothing^{n-1}
\end{tikzcd}
$$

A few words on the $(n+1)$-handle, more generally the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$. The unit is a 2-morphism $\eta : \mathrm{id}_{S^{n-1}} \Rightarrow D^n \circ \overline{D^n}$, which is clearly an elementary cobordism of index n.

A similar phenomenon happens with ε_k, that is, η_k, the unit to the adjunction $S^k \times D^{n-k} \dashv \overline{S^k \times D^{n-k}}$, to which ε_k is the counit, is determined by handles of index at most k, and indeed, $\eta_k = S^k \times D^{n-k+1} : \varnothing^n \Rightarrow S^k \times S^{n-k} : \varnothing^{n-1} \to \varnothing^{n-1}$ is built from a 0-handle and a k-handle.

Thus, since the counit is uniquely determined by the unit, the $(n+1)$-dim k-handle, for $k > 0$, is determined by handles of lower index. This may not be very useful in the topology world, but on the algebraic side of a TQFT, this means that everything is determined by the 0-handle.

(It may be helpful to note that the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ is simply S^k times the first example but with n set to $n-k$, $D^{n-k} : \varnothing^{n-k} \rightleftharpoons S^{n-1-k} : \overline{D^{n-k}}$.)

Note ε_k is not just a k-handle, but may use lower k; e.g. for $n = 3, k = 1$, $\varepsilon_k : \overline{S^1 \times D^2} \sqcup S^1 \times D^2 \Rightarrow S^1 \times S^1 \times I$ uses one 1-handle (connecting the solid tori) then a 2-handle.

1.3 In terms of generators and relations?

Q: are there any more relations that need to be imposed on the $(n+1)$-dim handles?

In [2], he presents $2,1,0$-Cob in terms of generators and relations. More specifically, there are four 1-morphisms, tensoring them generates all 1-morphisms, then there are two types of 2-morphism generators, the invertible ones - cusps, and the non-invertible ones - the 2-dim handles; there are three types relations between 2-morphisms, those involving only invertible 2-morphisms - cusp relations and swallowtail, those involving invertible and non-invertible - cusp flip, and those involving the non-invertible 2-morphisms - handle cancelations.

It seemed to me at first that the cusp flip relation could be ignored if we define one type of 2d 1-handle in terms of the other. See Figure 4.

However, it turns out this is not enough, essentially because of symmetry. In Figure 4, we see that we could have defined the 1-handle using the last diagram instead of the second diagram; but these are not equal unless we introduce the cusp flip relations.

An alternative is to define 1-handle in a way that, I think, is more natural from a 2-dim perspective. For the 0-manifold $N = (+,-,+,-)$, there are two ways to realize a 1-morphism $D^1 \times S^0 : N \to \varnothing$, either pairing up the first two and last two points, or pairing up the first

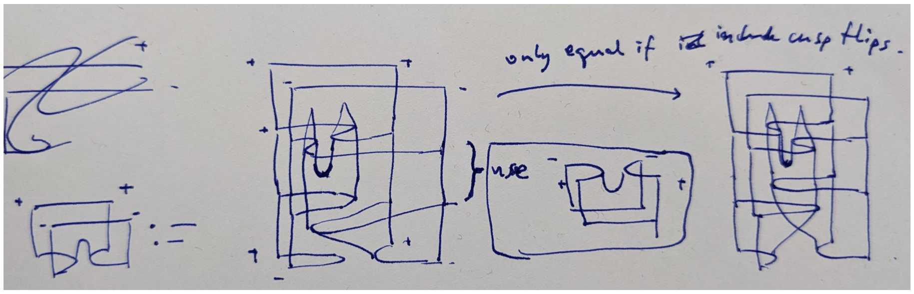

Figure 4: Defining 1-handle in terms of the other 1-handle

and last and the middle two points (by pair up I mean of course attaching D^1 to them); a 2d 1-handle is a 2-morphism between them.

This approach is cleaner, and now the consequences of the cusp flip relation is taken care of by stipulating that this 2-morphism is invariant under the $\mathbb{Z}/2$-symmetry of N swapping $+$ and $+$, $-$ and $-$.

This symmetry would not exist if the objects (0-manifolds) are ordered...

Upshot is that now we only need to types of relations of 2-morphisms, those involving the invertible ones, and those involving only handles.

So this got me to wonder, can this be the case for higher dimensions? If so, then this should prove that no additional relations are imposed after the choice of η_0 (assuming all criteria on the $(n+\varepsilon)$-theory needed to extended are met; e.g. that some of these 1-morphisms are two-sided "adjoint-able").

I want to understand this in terms of generators and relations.

In extended $(n+1)$-Cob, there is sub-bicat, same objects, 1-morphisms, but 2-morphisms are only invertible ones.

Suppose we understand this subcat in terms of generators and relations. We can't hope to have a finite number of generators, because in general there are infinitely many $(n-1)$-manifolds, so we may consider finite number of "types" of generators or relations, e.g. it is a n-dim k-handle. That is, we have a finite collection of "types" of generating 1-morphisms, a finite collection of "types" of generating 2-morphisms, and we have a finite collection of "types" of relations among 2-morphisms, such that for an n-manifold M, presented as a composition of generating 1-morphisms in two ways, f_1, f_2, and for $W = M \times I$, presented as a composition of generating 2-morphisms in two ways, $\varphi_1, \varphi_2 : f_1 \Rightarrow f_2$, there is a finite sequence of applications of the relations that turns φ_1 into φ_2.

Now, for $1 \le k \le n$, consider one 2-morphism $\gamma_k = D^k \times D^{n-k+1} : S^{k-1} \times D^{n-k+1} \Rightarrow D^k \times S^{n-k} : S^{k-1} \times S^{n-k}$; for $k = 0, n$, we have $\gamma_0 = D^{n+1} : \varnothing^n \Rightarrow \varnothing^n : \varnothing^{n-1} \to S^{n-1}$, and $\gamma_{n+1} = D^{n+1} : \varnothing^n \Rightarrow \varnothing^n : S^{n-1} \to \varnothing^{n-1}$. Let us write $\mathcal{S}^k$ for the source $(n-1)$-manifold for γ_k.

For every 1-morphism M with target $(n-1)$-manifold of the form $N' \sqcup \mathcal{S}^k$, we add the 2-morphism $\gamma_M = \gamma_k \circ \mathrm{id}_M : M \Rightarrow M' : N \to N'$ as a generator. TODO symmetry here?

Claim: For a fixed M, all 2-morphisms γ_M that can be formed by appropriately arranging M, M' are related by invertible relations of 2-morphisms, i.e. one can "move" this piece of 2-morphism γ_k around, and it would not change the 2-morphism.

Thus, one doesn't need any new relations relating the new 2-morphism to invertible ones.

2 Profunctors

Typically, to describe the n- and $(n-1)$- part of an extended $(n+1)$-TQFT, we need to assign a category $Z(N)$ to closed $(n-1)$-manifolds and functors $Z(M) : Z(N) \to Z(N')$ for an n-dimensional cobordism between $(n-1)$-manifolds N, N'.

For skein theories (see Section 3), it is more natural to describe the functor associated to cobordisms in terms of a profunctor, that is, in this case, given any boundary value $\mathbf{V} \in \operatorname{Obj} Z(N)$ at N and $\mathbf{V}' \in \operatorname{Obj} Z(N')$ at N', we assign some vector space $\widetilde{Z(M)}(\mathbf{V}, \mathbf{V}')$. We may then convert this into a functor, but this often requiring some additional choices.

In this section, we recall some basic results about profunctors. All categories will be assumed to be small.

First, we will describe profunctors valued in Set, then profunctors valued in Vec for Vec-enriched functors.

Definition 2.1. Let $\mathcal{A}, \mathcal{B}$ be categories. A *profunctor* F from $\mathcal{A}$ to $\mathcal{B}$, denoted $F : \mathcal{A} \nrightarrow \mathcal{B}$, is a functor $F : \mathcal{B}^{\text{op}} \times \mathcal{A} \to \text{Set}$.

Remark 2.2. Equivalently, we may provide the data of $\hat{F} : \mathcal{A} \to \text{Set}^{\mathcal{B}^{\text{op}}}$, from which we may define $F(B, A) = \hat{F}(A)(B)$, and $F(b^{\text{op}} : B^{\text{op}} \to B'^{\text{op}}, a : A \to A') = \hat{F}(a)_B \circ \hat{F}(A)(b) = \hat{F}(A')(b) \circ \hat{F}(a)_{B'}$; or we may construct $\hat{F}$ from F by $\hat{F}(A)(-) = F(-, A)$, and $\hat{F}(a : A \to A')_B = F(B, a)$.

Example 2.3. From a functor $G : \mathcal{A} \to \mathcal{B}$, we may construct two profunctors $G^* : \mathcal{A} \nrightarrow \mathcal{B}, G_* : \mathcal{B} \nrightarrow \mathcal{A}$, defined by

$$\begin{array}{ccc} G^* : \mathcal{B}^{\text{op}} \times \mathcal{A} \to \text{Set} & ; & G_* : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Set} \\ (B, A) \mapsto \mathcal{B}(B, GA) & ; & (A, B) \mapsto \mathcal{B}(GA, B) \end{array}$$

(note [4] uses G^* for G_* and vice versa)

Definition 2.4. Let $F : \mathcal{A} \nrightarrow \mathcal{B}, G : \mathcal{B} \nrightarrow \mathcal{C}$ be profunctors. Their *composition* $G \circ F : \mathcal{A} \nrightarrow \mathcal{C}$ is given by (assuming it exists, e.g. when $\mathcal{B}$ is small):

$$(G \circ F)(C, A) = \int^B G(C, B) \times F(B, A) = \bigsqcup G(C, B) \times F(B, A) / \sim$$

where the equivalence relation $\sim$ identifies $(y, xb) \sim (by, x)$ for $y \in G(C, B), x \in F(B, A), b : B \to B'$ (here we write xb for $F(b, \mathrm{id}_A)(x)$ and by for $G(\mathrm{id}_C, b)(y)$).

One justification for defining composition as such is the following (simple exercise on coends):

Lemma 2.5. *The profunctor* $\mathrm{id}^*_{\mathcal{A}} = \mathcal{A}(-,-)$ *is the identity for composition of profunctors, i.e. for* $F : \mathcal{A} \nrightarrow \mathcal{B}$, $\mathrm{id}^*_{\mathcal{B}} \circ F \simeq F \simeq F \circ \mathrm{id}^*_{\mathcal{A}}$.

Perhaps a more philosophical reason for defining composition of profunctors is from their analogy with relations on sets. In a sense, functors are to functions between sets as profunctors are to relations between sets. If $R \subseteq A \times B$ is a relation from A to B and $S \subseteq B \times C$ is a relation from B to C, their composition is $S \circ R = \{(a,c) | \exists b : (a,b) \in R, (b,c) \in S\}$.

A more practical reason is that this definition of composition is the standard (only?) trick in category of "doing everything" when/if no extra choices are allowed to be made.

Yet another reason is the following. Any object $B \in \mathcal{B}$ defines a profunctor $G \times_B F : \mathcal{A} \nrightarrow \mathcal{C}$, by $(G \times_B F)(C,A) := G(C,B) \times F(B,A)$. Any self-respecting composition $G \circ F$ should be a receptacle for this, that is, refering back to the idea that profunctors are analogs of relations, if $G(C,B)$ and $F(B,A)$ are thought of as generalized arrows from B to C and B to A, respectively, then we should be able to compose them, so we should have a map $G(C,B) \times F(B,A) \to (G \circ F)(C,A)$. Moreover, if $B, B' \in \mathcal{C}$ are related, i.e. for morphism $b : B \to B'$, $(G \times_B F)(C,A)$ and $(G \times_{B'} F)(C,A)$ should be related. Hence coend.

There's also something about Kan extensions. Profunctor $F : \mathcal{A} \nrightarrow \mathcal{B}$ can also be written $\hat{F} : \mathcal{A} \to \mathrm{Set}^{\mathcal{B}^{\mathrm{op}}}$. Let $\hat{G} : \mathcal{B} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ be another profunctor. There is the natural Yoneda embedding $Y : \mathcal{B} \to \mathrm{Set}^{\mathcal{B}^{\mathrm{op}}}$. Then the left Kan extension $L_Y(\hat{G}) : \mathrm{Set}^{\mathcal{B}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of $\hat{G}$ along Y. exists because $\mathcal{B}$ is small and $\mathrm{Set}^{\mathcal{B}^{\mathrm{op}}}$ is cocomplete. Then we define $G \circ F$ by $\widehat{G \circ F} = L_Y(\hat{G}) \circ \hat{F}$. Converting this to our definition of profunctor gives coend, not end (in particular, constructing left Kan extension uses colimit).

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Lawvere considers also $\mathcal{V}$-enriched profunctors for $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is a bicomplete symmetric monoidal closed category. Bicomplete means complete and cocomplete (all small (co)limits exist), symmetric monoidal means symmetric monoidal, and closed means there is an internal hom functor $\underline{\mathrm{Hom}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ which is adjoint to the monoidal structure: there is a natural bijection $\mathcal{V}(a, \underline{\mathrm{Hom}}(b,c)) \simeq \mathcal{V}(a \otimes b, c)$.

For our applications, we will only consider $\mathcal{V} = \mathrm{Vec}$, with tensor product as the monoidal structure.

Definition 2.6. For $\mathbf{k}$-linear categories $\mathcal{A}, \mathcal{B}$, we define the category $\mathcal{A} \hat{\boxtimes} \mathcal{B}$ with set of objects $\mathrm{Obj}\,\mathcal{A} \times \mathrm{Obj}\,\mathcal{B}$, and morphisms

$$\mathrm{Hom}(A \hat{\boxtimes} B, A' \hat{\boxtimes} B') := \mathrm{Hom}_{\mathcal{A}}(A, A') \otimes \mathrm{Hom}_{\mathcal{B}}(B, B')$$

(Lawvere [4] writes $\mathcal{A} \bigotimes \mathcal{B}$ for $\mathcal{A} \hat{\boxtimes} \mathcal{B}$.)

It is clear that this notion of tensor product is associative and symmetric.

Definition 2.7. For $\mathbf{k}$-linear categories $\mathcal{A}, \mathcal{B}$, a profunctor $F : \mathcal{A} \nrightarrow \mathcal{B}$ is a $\mathbf{k}$-linear functor $F : \mathcal{B}^{\mathrm{op}} \hat{\boxtimes} \mathcal{A} \to \mathrm{Vec}$.

Remark 2.8. Again, this is equivalent to providing $\hat{F} : \mathcal{A} \to \mathrm{Vec}^{\mathcal{B}^{\mathrm{op}}}$. Indeed, for $a : A \to A'$ in $\mathcal{A}$ and $b : B \to B'$ in $\mathcal{B}$, we see that the assignment from Remark 2.2, $F(b,a) = \hat{F}(a)_B \circ \hat{F}(A)(b)$ is bilinear in a, b. Thus, we see that the usage of tensor product in composition, i.e. the bilinearity of compositon of morphisms, forces the tensor product in Definition 2.6.

Let us further unpack Definition 2.7. Let $A, A' \in \mathcal{A}, B, B' \in \mathcal{B}$, $a \in \mathcal{A}(A, A'), b \in \mathcal{B}(B', B)$. On morphisms, we have $\mathcal{B}^{\mathrm{op}}\hat{\boxtimes}\mathcal{A}(B\hat{\boxtimes}A, B'\hat{\boxtimes}A') \to \mathrm{Vec}(F(B,A), F(B',A'))$. By $\otimes$ – Hom adjunction, we have $\mathcal{B}(B', B) \otimes F(B, A) \otimes \mathcal{C}(A, A') \to F(B', A')$. Hence, Lawvere [4] also calls these "bimodules"; in the example below, we see these are literally bimodules in certain contexts.

Composition of profunctors is exactly the same, except Cartesian product is replaced by tensor product. Because we work with small categories, the coend always exists, though the resulting vector space may be infinite...

Example 2.9. A one-object $\mathbf{k}$-linear category $\mathcal{A} = *//A$ is simply an algebra. To avoid confusion later about the order of operations, let us be pedantic: here we have $A = \mathcal{A}(*, *)$, and the composition gives a linear map $A \otimes A = \mathcal{A}(*, *) \otimes \mathcal{A}(*, *) \xrightarrow{\circ} \mathcal{A}(*, *) = A$. So for $a, a' \in A$, writing aa' means we apply the morphism a' first, and then a (since we write composition from right to left).

Functors $F : *//A \to *//B$ are simply algebra homomorphisms, and natural transformations are simply central elements of B.

On the other hand, a profunctor $F : *//A \nrightarrow *//B$ is a functor $F : (*//B)^{\mathrm{op}}\hat{\boxtimes} * //A = *//(B^{\mathrm{op}} \otimes A) \to \mathrm{Vec}$, which is equivalent to a A-B-bimodule: writing $M = F(*, *)$, for $a, a' \in A, b, b' \in B$, we have $F(b, a) : M \to M$, and $F(bb', aa') = F(b', a) \circ F(b, a')$; this means B naturally acts on the right and A acts on the left.

Alternativey, $(*//B)^{\mathrm{op}}\hat{\boxtimes}*//A = *//(B^{\mathrm{op}})\hat{\boxtimes}*//A = *//(B^{\mathrm{op}}\otimes A)$, so $F : *//(B^{\mathrm{op}}\otimes A) \to \mathrm{Vec}$ amounts to a (left) $B^{\mathrm{op}} \otimes A$-module.

Composition of profunctors amounts to relative tensor product. Given $F : *//A \nrightarrow *//B, G : *//B \nrightarrow *//C$, equivalent to bimodules ${}_AM_B, {}_BN_C$, respectively, their composition is given by $G \circ F(*_C, *_A) = \int^{b:*\to*} G(*_C, *) \otimes F(*, *_A) = N \otimes M/(bn \otimes m \sim n \otimes mb) = M \otimes_B N$.

Example 2.10. The $\mathbf{k}$-linear category $\mathcal{I} = *//\mathbf{k}$ is the unit for $\hat{\boxtimes}$. The unitors are isomorphisms $\mathcal{A}\hat{\boxtimes}\mathcal{I} \simeq \mathcal{A} \simeq \mathcal{I}\hat{\boxtimes}\mathcal{A}$, on objects $A\hat{\boxtimes}* \mapsto A \mapsto *\hat{\boxtimes}A$ and on morphisms are just the unitors for Vec, $\mathcal{A}(A, A') \otimes \mathbf{k} \simeq \mathcal{A}(A, A') \simeq \mathbf{k} \otimes \mathcal{A}(A, A')$.

3 Skein theory for $(1 + \varepsilon)$-theory

To study completion of a $(n + \varepsilon)$-theory to an extended $(n + 1)$-theory, we start by studying the simplest case, $n = 1$.

In particular, we will study the $(1 + \varepsilon)$-theory arising from skeins in 1-manifolds.

Manifolds with be assumed to be compact unless otherwise specified.

We consider oriented 0-manifolds. We often use certain 0-manifolds in particular, denoted by tuples of signs, e.g. $(+, +, -, +)$; the empty 0-manifold is denoted by $()$ or $\varnothing^0$. Note that the tuple is ordered, but the 0-manifold itself is not, in the sense that $(+, +, -, +)$ is homeomorphic to $(-, +, +, +)$; it is useful to keep an ordering of the points to keep our heads straight on the algebra.

Sometimes we will denote a singleton by simply $+$, or $-$, or pt^+, or pt^-.

Let us fix a $\mathbf{k}$-linear category $\mathcal{A}$. We will often consider the special case of $\mathcal{A} = *//A$, where A is a $\mathbf{k}$-algebra.

Let us first describe the objects of the category $Z(N)$ for an oriented 0-manifold N, so-called boundary values:

Definition 3.1. A *boundary value* $\mathbf{V}$ on an oriented 0-manifold N is an assignment of an object $V_b \in \mathcal{A}$ for every point $b \in N$, or more concisely, $\mathbf{V} : N \to \operatorname{Obj}\mathcal{A}$.

We denote by $\overline{\mathbf{V}}$ the boundary value on $\overline{N}$ (opposite orientation) with the same assignments as $\mathbf{V}$.

For an orientation-preserving diffeomorphism of 0-manifolds $f : N \simeq N'$, we may define the *push-forward* boundary value $f_*(\mathbf{V})$ of a boundary value $\mathbf{V}$ on N to be $f_*(\mathbf{V}) : n' \mapsto \mathbf{V}(f^{-1}(n'))$.

Note that this definition works for the empty 0-manifold $N = \varnothing^0$: there is only one function $\varnothing^0 \to \operatorname{Obj}\mathcal{A}$, so there is a unique boundary value on $\varnothing^0$, which we denote by $\mathbf{E}$.

Remark 3.2. Usually (for $\dim N = 1, 2$) a boundary value is a finite set of points in N with assigned objects; here we require the finite set of points of a boundary value to be all of N because: (1) if $\mathcal{A}$ has structure, e.g. rigid monoidal, we may as well assign the unit object $\mathbf{1}$ to points of N that are not in the boundary value; (2) if $\mathcal{A}$ is simply only $\mathbf{k}$-linear, say $\mathcal{A} = A-\mathrm{mod}$, we can still consider skeins as we will see below, but if we do not assign an object to a boundary point, it is unclear what such a skein should do at the boundary...

Next we define skeins in a 1-manifold. Let M be an oriented 1-manifold. Its boundary, ∂M, acquires an orientation from M, with a point $b \in \partial M$ being pt^+ if M is outwardly-oriented at b, and pt^- otherwise.

Definition 3.3. Let M be an oriented 1-manifold, and let $f : \partial M \simeq N$ be a parametrization of its boundary by a given 0-manifold N. Let $\mathbf{V}$ be a boundary value on N, and let $\mathbf{V}' = f_*^{-1}(\mathbf{V})$. An *$\mathcal{A}$-colored graph with boundary value* $\mathbf{V}$, denoted (Γ, φ), is an oriented finite graph Γ with an embedding in M, with edge orientations agreeing with M and such that every vertex of Γ that is mapped into the interior of M has degree 2, and φ is an assignment of an object φ_e for each edge e of Γ such that the assignment to the edges meeting the boundary agrees with the boundary value $\mathbf{V}'$, and a morphism $\varphi_v : \varphi_e \to \varphi_{e'}$ for every internal vertex of Γ with an incoming edge e and outgoing edge e'.

Note that for the empty manifold, the only graph is the empty graph (which has a unique $\mathcal{A}$-coloring), while for non-empty 1-manifold M, graphs must be surjective onto M (from the boundary value condition and internal vertex condition).

Remark 3.4. We could add an additional layer of flexibility by specifying that the boundary value $\mathbf{V}'$ not be exactly the push-forward of $\mathbf{V}$ along f^{-1}, but instead have an isomorphism $\varphi_n : \mathbf{V}'(n) \simeq \mathbf{V}(n)$ for each $n \in N$. However, it doesn't seem to add anything to the theory, only additional symbols.

We often denote an $\mathcal{A}$-colored graph simply by Γ.

Definition 3.5. Let (Γ, φ) be an $\mathcal{A}$-colored graph in an oriented 1-manifold M. Let D be a 1-dimension embedded disk, i.e. an interval, in M, such that ∂D does not meet any internal vertex of Γ. Let X, X' be the objects assigned to the edges e, e', respectively, that contain or meet ∂D, with e oriented into D, and e' oriented out of D. Let $e_0 = e, e_1, \ldots, e_k = e'$ be the edges of Γ that meet D, and let v_i be the vertex between e_{i-1} and e_i.

We define the *evaluation of* (Γ, φ) *in* D to be the composition

$$\langle(\Gamma, \varphi)\rangle_D = \varphi_{v_k} \circ \cdots \circ \varphi_{v_1} \in \mathrm{Hom}_{\mathcal{A}}(X, X')$$

Definition 3.6. Let $\Gamma_1, \ldots, \Gamma_k$ be $\mathcal{A}$-colored graphs in M with boundary value $\mathbf{V}$, and let $c_1, \ldots, c_k \in \mathbf{k}$. Let D be a 1-disk in M. We say that the formal $\mathbf{k}$-linear combination $\sum c_i \cdot \Gamma_i$ is *null (with respect to D)* if Γ_i agree outside of D, and $\sum c_i \langle \Gamma_i \rangle_D = 0$.

Note that $c \cdot \Gamma - c\Gamma$ is null, where $c\Gamma$ is the $\mathcal{A}$-colored graph where one of the morphisms in Γ is scaled by c.

Definition 3.7. The *skein module* $\mathrm{Skein}(M; \mathbf{V})$ of M with boundary value $\mathbf{V}$ is the space of formal linear combinations of $\mathcal{A}$-colored graphs in M with boundary value $\mathbf{V}$ module null graphs. A *skein* is an element of a skein module.

For the empty 1-manifold $\varnothing^1$, as we mentioned, there is a unique $\mathcal{A}$-colored graph, and there are no relations from null graphs because there are no disks $D \to \varnothing^1$; thus, $\mathrm{Skein}(\varnothing^1; \mathbf{E}) = \mathbf{k}$, where recall $\mathbf{E}$ is the unique boundary value on the empty 0-manifold $\varnothing^0$.

Diffeomorphism of 1-manifolds give isomorphisms on skein modules (proof is straightforward):

Proposition 3.8. *Let M, M' be 1-manifolds, and let $\varphi : \partial M \to N, \varphi' : \partial M' \to N'$ be parametrizations of their boundaries. Let $f : M \simeq M', g : N \simeq N'$ be diffeomorphisms that are compatible with the parametrizations, i.e. $g \circ \varphi = \varphi' \circ f|_{\partial M}$. Let $\mathbf{V}$ be a boundary value on N, and let $\mathbf{V}' = g_*(\mathbf{V})$ be its push-forward.*

For an $\mathcal{A}$-colored graph Γ in M with boundary value $\mathbf{V}$, the diffeomorphism f sends it to an $\mathcal{A}$-colored graph in M' with boundary value $\mathbf{V}'$, which we denote by $f_*(\Gamma)$.

Then this operation defines an isomorphism $f_* : \mathrm{Skein}(M; \mathbf{V}) \to \mathrm{Skein}(M'; \mathbf{V}')$.

Note that it may be convenient to define push-forward of boundary values/skeins for embeddings of manifolds, and at first glance this makes sense; however, because of our stipulation that boundary values assign an object to *every* point, this is not well-defined, unless we specify beforehand what we should assign to points that were not hit by the embedding, e.g. a unit object of $\mathcal{A}$ (see Remark 3.2).

Finally, we can give the full definition of the category $Z(N)$:

Definition 3.9. The *skein category* $Z(N)$ associated to an oriented 0-manifold N is the category whose objects are boundary values $\mathbf{V}$, and morphism are given by

$$\mathrm{Hom}_{Z(N)}(\mathbf{V}, \mathbf{V}') = \mathrm{Skein}(N \times I; \overline{\mathbf{V}} \cup \mathbf{V}')$$

Composition is given by gluing two copies of $N \times I$ and identifying it with $N \times I$. More precisely, for $\mathcal{A}$-colored graphs Γ, Γ' representing elements in $\mathrm{Hom}_{Z(N)}(\mathbf{V}, \mathbf{V}'), \mathrm{Hom}_{Z(N)}(\mathbf{V}', \mathbf{V}'')$ respectively, the $\mathcal{A}$-colored graph $\Gamma' \circ \Gamma$ has underlying graph in $N \times I \cup N \times I$ given by the images of Γ and Γ' and forgetting their boundary vertices at the points where they meet.

It is straightforward to check that $Z(pt^+) \simeq \mathcal{A}$ and $Z(pt^-) \simeq \mathcal{A}^{\mathrm{op}}$ (in fact these are isomorphisms).

TODO clarify geometry $Z(---<---)\hat{\boxtimes}Z(---<---) \to Z(---<---)$.

We see that for the empty 0-manifold, $Z(\varnothing^0) = */\!/\mathbf{k}$.

Proposition 3.10. $Z(N_1 \sqcup N_2) \simeq Z(N_1)\hat{\boxtimes}Z(N_2)$

Proof. There is a natural bijection for the objects $\mathrm{Obj}\, Z(N_1 \sqcup N_2) = \{N_1 \sqcup N_2 \to \mathrm{Obj}\,\mathcal{A}\} \simeq \{N_1 \to \mathrm{Obj}\,\mathcal{A}\} \times \{N_1 \to \mathrm{Obj}\,\mathcal{A}\} = \mathrm{Obj}\, Z(N_1) \times \mathrm{Obj}\, Z(N_2)$. We denote by $\mathbf{V}_1 \sqcup \mathbf{V}_2$ the object of $Z(N_1 \sqcup N_2)$ corresponding to $(\mathbf{V}_1, \mathbf{V}_2)$ under the bijection above.

Fix objects $\mathbf{V}_1, \mathbf{V}_1' \in \mathrm{Obj}\, Z(N_1)$, $\mathbf{V}_2, \mathbf{V}_2' \in \mathrm{Obj}\, Z(N_2)$. For $f_1 = (\Gamma_1, \varphi_1) \in Z(N_1)(\mathbf{V}_1, \mathbf{V}_1')$, $f_2 = (\Gamma_2, \varphi_2) \in Z(N_2)(\mathbf{V}_2, \mathbf{V}_2')$, we have the morphism

$$f_1 \otimes f_2 \in Z(N_1)(\mathbf{V}_1, \mathbf{V}_1') \otimes Z(N_2)(\mathbf{V}_2, \mathbf{V}_2') = Z(N_1)\hat{\boxtimes}Z(N_2)(\mathbf{V}_1 \hat{\boxtimes} \mathbf{V}_2, \mathbf{V}_1' \hat{\boxtimes} \mathbf{V}_2')$$

and also the morphism

$$f_1 \sqcup f_2 = (\Gamma_1 \sqcup \Gamma_2, \varphi_1 \sqcup \varphi_2) \in Z(N_1 \sqcup N_2)(\mathbf{V}_1 \sqcup \mathbf{V}_2, \mathbf{V}_1' \sqcup \mathbf{V}_2')$$

where $\varphi_1 \sqcup \varphi_2$ is the $\mathcal{A}$-coloring that restricts to φ_1, φ_2 on Γ_1, Γ_2 respectively. It is straightforward to check that the latter assigment $f_1 \sqcup f_2$ is bilinear, thus inducing a map

$$Z(N_1)\hat{\boxtimes}Z(N_2)(\mathbf{V}_1 \hat{\boxtimes} \mathbf{V}_2, \mathbf{V}_1' \hat{\boxtimes} \mathbf{V}_2') \to Z(N_1 \sqcup N_2)(\mathbf{V}_1 \sqcup \mathbf{V}_2, \mathbf{V}_1' \sqcup \mathbf{V}_2')$$

and that this map is an isomorphism. It is also clear that this isomorphism is functorial, thus gives the desired isomorphism $Z(N_1 \sqcup N_2) \simeq Z(N_1)\hat{\boxtimes}Z(N_2)$. ☐

Note that we did not rely on any non-emptiness assumptions on N_1 or N_2. Applying this to $\varnothing^0$, we have

$$Z(N) = Z(N \sqcup \varnothing^0) \simeq Z(N)\hat{\boxtimes}Z(\varnothing^0) \simeq Z(N)\hat{\boxtimes} */\!/\mathbf{k}$$

which is in agreement with the fact that $*/\!/\mathbf{k}$ is the unit for $\hat{\boxtimes}$.

Definition 3.11. For a cobordism $M : N \to N'$, we define the profunctor $Z(M) : Z(N) \nrightarrow Z(N')$ defined by

$$(\mathbf{V}, \mathbf{V}') \mapsto \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}')$$

Proposition 3.12. *For cobordisms $M : N \to N', M' : N' \to N''$, $Z(M' \circ M) \simeq Z(M') \circ Z(M)$.*

Proof. We need to show that, for boundary values $\mathbf{V}, \mathbf{V}''$ on N, N'' respectively,

$$\int^{\mathbf{V}'} \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \simeq \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$$

The obvious concatenation of graphs in M, M' to form a graph in $M' \circ M$ gives a map $\Phi_{\mathbf{V}'} : \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \to \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$ for each $\mathbf{V}'$. It is easy to see that the sum of these maps, $\sum_{\mathbf{V}'} \Phi_{\mathbf{V}'} : \bigoplus_{\mathbf{V}'} \mathrm{Skein}(M'; \overline{\mathbf{V}'} \cup \mathbf{V}'') \otimes \mathrm{Skein}(M; \overline{\mathbf{V}} \cup \mathbf{V}') \to \mathrm{Skein}(M' \circ M; \overline{\mathbf{V}} \cup \mathbf{V}'')$, factors through the coend.

That this map is surjective is obvious; for injectivity, see notes on excision (or see [3]). ☐

References

[1] Kwon, Alice, and Ying Hong Tham. The Y-Product. arXiv preprint arXiv:2209.14251 (2022).

[2] Schommer-Pries, Christopher John. The classification of two-dimensional extended topological field theories. University of California, Berkeley, 2009.

[3] Kirillov Jr, Alexander, and Ying Hong Tham. "Factorization homology and 4D TQFT." arXiv preprint arXiv:2002.08571 (2020).

[4] Lawvere, F. William. "Metric spaces, generalized logic, and closed categories." Rendiconti del seminario matématico e fisico di Milano 43 (1973): 135-166.