Note in preparation for talk for seminar on Fusion 2-Categories, Winter semester 2022, UHH

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[Almost everything here is from [1]; the only things that are new here are cleaner proofs of the main theorems as suggest by David Reutter.]

The main goal of thie note is to prove that a finite semisimple 2-category is the category of finite semisimple modules over a multifusion category, and vice versa.

That is, for a semisimple 2-category $\mathcal{C}$, there exists a multifusion category C such that

$$\mathcal{C} \simeq \mathcal{M}\mathrm{od}_{s.s.}^{fin}(C)$$

Here $\mathcal{M}\mathrm{od}_{s.s.}^{fin}(C)$, which we will abbreviate to $\mathcal{M}\mathrm{od}(C)$, stands for finite semisimple right module categories over C.

Conversely, for any mutifusion category C, $\mathcal{M}\mathrm{od}(C)$ is a finite semisimple 2-category.

0.1 Conventions

Everything is over an algebraically closed field $\mathbf{k}$ with characteristic 0.

We use different fonts/alphabets for different levels of structures:

In relation to a 2-category:

- $\mathcal{C}$ (caligraphic font): 2-category;
- X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X, Y)$ for the category of morphisms from X to Y;
- $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \to Y$, we may write $\alpha \in \mathcal{C}(X, Y)(f, g)$ to indicate its sources and targets, or simply $\alpha \in \mathrm{Hom}(f, g)$ if the objects are clear

In relation to a 1-category:

- C, A (upper case latin): category;

- a, b, f, g (lower case latin): objects in category, functor between categories;
- α, β (lower case greek): morphism in category

[We use the same type of font for 1-functors and 1-morphisms because the 1-morphisms in the 2-category $\mathcal{M}\mathrm{od}(C)$ of module categories are module functors, and we want the notation to be consistent; we use the same type of font for 2-functors and objects because 2-functors are objects in the 2-category of 2-functors]

We also compose morphisms from right to left: in a 2-category $\mathcal{C}$, for $\alpha \in \mathcal{C}(X,Y)(f,f'), \beta \in \mathcal{C}(Y,Z)(g,g'), \gamma \in \mathcal{C}(X,Y)(f',f'')$, we write

$$g \circ f, g \circ f', \ldots : X \to Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \to Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \to Y$$

for vertical composition of 2-morphisms.

We may also omit the composition symbols if the type of composition is clear (in particular for composition of 1-morphisms).

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X,Y)$ satisfies P.

By 2-category we always mean a weak 2-category that is furthermore locally additive over $\mathbf{k}$, that is, all hom-categories are additive categories over $\mathbf{k}$, and all compositions are $\mathbf{k}$-bilinear. By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally $\mathbf{k}$-linear.

1 Review

Let us recall some definitions and facts concerning semisimple 2-categories. These where covered in more detail in previous talks, so here we will simply state them without proof.

1.1 Additive 2-category, direct sums of objects

Definition 1.1 (direct sum of objects in 2-category)**.** A *direct sum* of two objects A_1, A_2 in $\mathcal{C}$ is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms $i_k : A_k \rightleftharpoons A_1 \boxplus A_2 : p_k$, such that

- $p_k \circ i_k \simeq \mathrm{id}_{A_k}$,
- $p_2 \circ i_1$, $p_1 \circ i_2$ are zero 1-morphisms,

- $\mathrm{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

Proposition 1.2. *i_k, p_k are two-sided adjoints to each other.*

Definition 1.3. A 1-morphism $i : X \to Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A, $i \circ - : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ is fully faithful.

Proposition 1.4. *$i_k : A_k \to A_1 \boxplus A_2$ is fully faithful.*

(I don't think the following was discussed, but it is a simple concept anyway)

Definition 1.5 (Direct sum of 2-categories)**.** Given 2-categories $\mathcal{C}_j$, $j \in J$, we may consider the direct sum 2-category $\mathcal{C} := \boxplus_{j \in J} \mathcal{C}_j$:

- $\mathrm{Obj}\, \mathcal{C} = \bigsqcup_{j \in J} \mathrm{Obj}\, \mathcal{C}_j$
- for $X \in \mathcal{C}_i, Y \in \mathcal{C}_j$, $\mathcal{C}(X, Y) = \begin{cases} \mathcal{C}_j(X, Y) & \text{if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

1.2 Idempotent completeness, separable monads, splittings

Definition 1.6. A *separable algebra* (a, μ, η) in a tensor category C is an algebra that admits an a-a-bimodule section ${}_a a_a \to {}_a a \otimes a_a$ to μ.

Definition 1.7. Let (t, μ, η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $t \Rightarrow t \circ t$ to μ.

In other words, a separable monad over X is a separable algebra in $\mathcal{C}(X, X)$.

Definition 1.8. Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition 1.9. Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Under the right conditions (local idempotent completeness of $\mathcal{C}$), splittings are unique (see Proposition 1.10).

Proposition 1.10 (Uniqueness of splitting)**.** *[*[1]*, Theorem A.3.1] In a locally idempotent complete 2-category $\mathcal{C}$, splittings of a separable monad are unique up to equivalence.*

In particular, this holds true when $\mathcal{C}$ is locally semisimple.

Thought to self: [1] proves this by showing that a splitting is equivalent to an "Eilenberg-Moore object" and also a "Kleisli object", which are in themselves important and interesting objects characterized by universal properties. I'd like to have a more direct proof, somehow constructing an equivalence between any two splittings directly from the splitting data. I don't have a full proof, but here's an attempt. Say for a separable monad t on X, we have two adjunctions $r \vdash l : X \to Y$ and $r' \vdash l' : X \to Y'$ that split t. There is are obvious 1-morphisms $l' \circ r : Y \to Y'$, $l \circ r' : Y' \to Y$, but it is unlikely that these are equivalences. A promising candidate for an equivalence is $l' \circ_t r$, the coequalizer of $l' \circ t \circ r \rightrightarrows l' \circ r$ (here r is a left t-module from the counit: $tr \simeq rlr \Rightarrow r$; similarly l' is a right t-module). Such a coequalizer does appear in the proof of [1, Theorem A.3.1] anyway, and I've found that this thought process makes the consideration of the Eilenberg-Moore and Kleisli objects more motivated.

Now that we've defined the notion on objects, we consider the property as a global 2-category-wide property:

Definition 1.11. A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition 1.12. Let $\mathcal{C}$ be a locally idempotent complete 2-category. Define the *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^\triangledown$, to be the 2-category whose objects are given by separable monads of $\mathcal{C}$, and the hom-category of morphisms from separable monad (X,p) to (Y,q) is the category q-bimod-$p(\mathcal{C}(X,Y))$ of q-p-bimodules in $\mathcal{C}(X,Y)$.

There is a natural 2-functor $I : \mathcal{C} \to \mathcal{C}^\triangledown$ that is fully faithful.

A 2-functor $F : \mathcal{C} \to \mathcal{D}$ extends to a 2-functor $F^\triangledown : \mathcal{C}^\triangledown \to \mathcal{D}^\triangledown$ that commutes with I's.

Proposition 1.13. *$\mathcal{C}^\triangledown$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^\triangledown$ is an equivalence.*

As a consequence, if $\mathcal{D}$ is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C},\mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^\triangledown,\mathcal{D}^\triangledown) \simeq \mathcal{F}un(\mathcal{C}^\triangledown,\mathcal{D})$$

Proposition 1.14. *If $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, then $F^\triangledown : \mathcal{C}^\triangledown \to \mathcal{D}^\triangledown$ is also fully faithful.*

$(-)^\triangledown$ is a categorified version of the Karoubi completion operation. By going along the lines of [2], the definition of $\mathcal{C}^\triangledown$ is very suggestive of the idea that objects of $\mathcal{C}^\triangledown$ should be thought of as the Morita 2-category of algebras in some tensor category; indeed, let C be multifusion, and let $\mathcal{B}C$ be the 2-category with one object $*$ and endmorphisms given by C (the "delooping of C"). Then by construction, we have

$$(\mathcal{B}C)^\triangledown = \begin{cases} \text{Obj}: \text{ separable algebras in } C \\ \text{Mor}: (\mathcal{B}C)^\triangledown(a,b) = b\text{-bimod-}a(C) \end{cases}$$

Proposition 1.15 ([1, Prop 1.3.13])**.** *For a multifusion category C, the following 2-functor is an equivalence:*

$$\begin{array}{rrcl} (-)\text{-}mod(C): & (\mathcal{B}C)^{\triangledown} & \longrightarrow & \mathcal{M}od(C) \\ \mathrm{Obj}: & a & \longmapsto & a\text{-}mod(C) \\ 1\text{-}Mor: & {}_b m_a & \longmapsto & m \otimes_a - \\ 2\text{-}Mor: & \varphi & \longmapsto & \varphi \otimes_a - \end{array}$$

Proof. Essential surjectivity follows from:
-[2, Theorem 1]: for a finite semisimple right module category over multifusion C, there exists a semisimple algebra a in C such that $M \simeq a$-mod(C) as right C-module categories; and
-[4, Corollary 2.6.9]: When C is multifusion over a field of characterisitc 0, a right C-module category M is separable ($\simeq a$-mod(C) for a separable algebra a) if and only if it is semisimple.

Fully faithfulness follows from [5, Prop 7.11.1]. □

This is almost one half of the main result; one still needs to prove local semisimplicity and existence of adjoints. This will follow from more results from [3],[4], which we show later.

More details on proofs (again mostly just an outline). The first part of essential surjectivity, $M \simeq a$-mod(C) for some algebra, as proved in [2, Theorem 1], uses "internal Homs". For objects $m_1, m_2 \in M_C$ in a *right* C-module category (see note after), we define $\underline{\mathrm{Hom}}(m_1, m_2)$ to be the object representing the functor $C(m_1 \triangleleft -, m_2)$:

$$C(a, \underline{\mathrm{Hom}}(m_1, m_2)) \simeq M(m_1 \triangleleft a, m_2)$$

(Note [2] considers left C-module categories; one may consider M as ${}_{C^{mp}}M$ and define the internal homs as objects in C^{mp}. The main confusing thing is that internal homs for right C-module category compose the other way.)

In a semisimple M, simple objects do not "talk" to each other, but they do "interact" if acted upon by C; the internal hom captures that interaction. In particular, if M is indecomposable as a C-module cat, then internal hom between any pair of nonzero objects is nonzero.

The plain hom $M(m_1, m_2)$ can be recovered as $M(m_1, m_2) = C(\mathbf{1}, \underline{\mathrm{Hom}}(m_1, m_2))$. "Precomposing" or "postcomposing" by morphisms in M also gives corresponding morphisms in C: for $f : m_1 \to m_1'$, $g : m_2 \to m_2'$, composing $g \circ - \circ f : M(m_1 \triangleleft a, m_2) \to M(m_1' \triangleleft a, m_2')$ gives

$$g \circ - \circ f : \underline{\mathrm{Hom}}(m_1, m_2) \to \underline{\mathrm{Hom}}(m_1', m_2')$$

The identity $\mathrm{id} \in C(\underline{\mathrm{Hom}}(m_1, m_2), \underline{\mathrm{Hom}}(m_1, m_2))$ becomes, in M,

$$\mathrm{ev} : m_1 \triangleleft \underline{\mathrm{Hom}}(m_1, m_2) \to m_2$$

and then composing two such ev,

$$m_1 \triangleleft \underline{\mathrm{Hom}}(m_1, m_2) \triangleleft \underline{\mathrm{Hom}}(m_2, m_3) \to m_3$$

and then translating back to C, we get composition

$$\underline{\circ} : \underline{\mathrm{Hom}}(m_1, m_2) \otimes \underline{\mathrm{Hom}}(m_2, m_3) \to \underline{\mathrm{Hom}}(m_1, m_3)$$

in particular, for any $x \in M$, $\underline{\mathrm{Hom}}(x, x)$ is a monoid, and in fact an algebra: the identity $\mathrm{id} \in M(x \triangleleft \mathbf{1}, x)$ translates to the unit map for $\underline{\mathrm{Hom}}(x, x)$.

Furthermore, $\underline{\mathrm{Hom}}(x, y)$ is a left module over $\underline{\mathrm{Hom}}(x, x)$. Write $a = \underline{\mathrm{Hom}}(x, x)$ for nonzero x. If M is indecomposable (or just that $\underline{\mathrm{Hom}}(x, y) \not\simeq 0$ for all nonzero y), then the functor

$$\begin{aligned} M &\to a\text{-mod}(C) \\ y &\mapsto \underline{\mathrm{Hom}}(x, y) \end{aligned}$$

is an equivalence.

Note weirdness of conflict of direction of composition like in Arabic; e.g.

$$M(m_4, m_5) \otimes \underline{\mathrm{Hom}}(m_2, m_3) \otimes \underline{\mathrm{Hom}}(m_3, m_4) \otimes M(m_1, m_2) \otimes \underline{\mathrm{Hom}}(m_5, m_6) \to$$

we should think of the composition by morphisms in M as some sort of action. Note the $M(m_4, m_5)$ could actually act on the right on the last factor. Consider the more modest example:

$$\begin{tikzcd}
M(m_2, m_3) \otimes \underline{\mathrm{Hom}}(m_1, m_2) \otimes \underline{\mathrm{Hom}}(m_3, m_4) \arrow[r, "\circ"] \arrow[dd, "\simeq"] & \underline{\mathrm{Hom}}(m_1, m_3) \otimes \underline{\mathrm{Hom}}(m_3, m_4) \arrow[d, "\underline{\circ}"] \\
 & \underline{\mathrm{Hom}}(m_1, m_4) \\
\underline{\mathrm{Hom}}(m_1, m_2) \otimes \underline{\mathrm{Hom}}(m_3, m_4) \otimes M(m_2, m_3) \arrow[r, "\circ"] & \underline{\mathrm{Hom}}(m_1, m_2) \otimes \underline{\mathrm{Hom}}(m_2, m_4) \arrow[u, "\underline{\circ}"']
\end{tikzcd}$$

This diagram commutes. It is confusing to work with these internal hom objects directly; easier to "pick coordinates", that is, use objects in C to probe the internal homs. Take $a_1, a_3 \in C$, and let

$$\begin{aligned} &\varphi_1 \in C(a_1, \underline{\mathrm{Hom}}(m_1, m_2)) \simeq M(m_1 \triangleleft a_1, m_2) \\ &\psi_2 \in M(m_2, m_3) \\ &\varphi_3 \in C(a_1, \underline{\mathrm{Hom}}(m_3, m_4)) \simeq M(m_3 \triangleleft a_3, m_4) \end{aligned}$$

Then the equality of the two ways to compose comes down to associtativity:

$$\varphi_3 \circ (\psi_2 \circ \varphi_1) = (\varphi_3 \circ \psi_2) \circ \varphi_1 \in M(m_1 \triangleleft a_1 \triangleleft a_3, m_4) \simeq C(a_1 \otimes a_3, \underline{\mathrm{Hom}}(m_1, m_4))$$

The second part of essential surjectivity, the equivalence of semisimple and separable, is more involved. However, one direction, separable module cat is semisimple, is fairly straightforward, and quite neat, as we see another reason "why" the section to the multiplication map is required to be a bimodule map. See [4, Prop 2.5.3].

We show all objects of $M \simeq a\text{-mod}(C)$ are projective. $\mathbf{1} \in C$ is simple, so is projective. This implies a separable algebra a is projective. (For $m \in M$, $C(\mathbf{1}, m) \simeq M(a, m)$). So free a-modules, $m \otimes a$, are projective. Any module $m \simeq a \otimes_a m$, and by the separability, is furthermore a summand of $a \otimes m$ by $a \otimes_a m \to (a \otimes a) \otimes_a m \simeq a \otimes m$.

For fully faithfulness, I haven't looked at the original proof, but, lest I am missing something, it seems pretty straightforward. Functors between semisimple categories are exact, in particular right exact. For $F : a\text{-mod}(C) \to b\text{-mod}(C)$, by being C-module functor, we have, for free a-modules, $F(a \otimes x) \simeq F(a) \otimes x$, and by right exactness, we have $F(m) \simeq F(a \otimes_a m) \simeq F(a) \otimes_a m$. □

1.3 Simple objects

Proposition 1.16 (equivalent notions of simple-ness). *Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:*

(1) any subobject $i : Y \to X$ of X is either 0 ($Y \simeq 0$) or (i is) an equivalence;

(2) X cannot be written as a non-trivial direct sum, i.e. if $X = \boxplus X_i$, then $X_i \simeq 0$ for all but one i;

(3) id_X *is a simple object in $\mathcal{C}(X, X)$.*

Proof idea. (1) ⇒ (2): Contravariant statement follows from fully faithfulness of $i_k : A_k \to A_1 \boxplus A_2$.

(2) ⇒ (3): Contravariant statement is "identity splitting implies object splitting", uses idempotent completeness of $\mathcal{C}$ to split out objects corresponding to summands of id_X (which are separable monads) (see [1, Prop 1.3.16]).

(3) ⇒ (1): for non-zero fully faithful $r : Y \to X$, with id_X simple, consider the left adjoint $l : X \to Y$, use fully faithfulness to get a preimage $\delta : \mathrm{id}_Y \Rightarrow lr$ of $\eta \circ r : r \Rightarrow rlr$. Use simplicity of id_X to get section of the unit η. Show δ is a section of the counit. Etc. (See [1, Prop 1.2.14]) □

1.4 Semisimple 2-category

Definition 1.17 ((finite) semisimple 2-category). A 2-category $\mathcal{C}$ is *semisimple* if it is:

- locally semisimple,
- admits left and right adjoints for every 1-morphism,

- additive,
- idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

2 New stuff

Beyond this point, we cover new stuff (i.e. not covered in previous talks).

2.1 Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, as we just recalled, is quite similar to the semisimple 1-category case. However, there is a stark difference in that there can be nonzero morphisms between non-equivalent simple objects; indeed, recall our running example of a 2-category $\mathcal{M}\mathrm{od}(\mathrm{Vec}_{\mathbb{Z}/2})$:

$$\begin{tikzcd} \mathrm{Vec}_{\mathbb{Z}/2} \arrow[loop left, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[rr, bend left, "\mathrm{Vec}"] & & \mathrm{Vec} \arrow[loop right, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[ll, bend left, "\mathrm{Vec}"] \end{tikzcd}$$

The existence of nonzero 1-morphisms between non-equivalent simple objects turns out to be a very important aspect of semisimple 2-categories since, as we shall see, it captures 2-Morita equivalences between fusion categories.

Proposition 2.1 (Schur's Lemma, [1, Prop 1.2.19]). *In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.*

Proof of Proposition 2.1. Let $f^* : B \to A$ be a right adjoint to f. Since id_B is simple, there exists a section $\delta : \mathrm{id}_B \Rightarrow ff^*$ to the counit $\varepsilon : ff^* \Rightarrow \mathrm{id}_B$. Postcomposing with g, we have $\mathrm{id}_g = (\mathrm{id}_g \circ \varepsilon) \cdot (\mathrm{id}_g \circ \delta) : g \Rightarrow gff^* \Rightarrow g$. Thus if $gf = 0$, then $\mathrm{id}_g = 0$, contradicting nonzero-ness of B. $\square$

Thus, this establishes transitivity on a weaker notion of equivalence among simple objects (namely, existence of nonzero 1-morphism); reflexivity is obvious, and symmetry is given by adjoints.

Definition 2.2 (component of semisimple 2-category). In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

In a finite semisimple 2-category $\mathcal{C}$, there will be finitely many components, say index by a set J. Consider one component $j \in J$, and consider the full subcategory $\mathcal{C}_j$ of $\mathcal{C}$ consisting of objects that are equivalent to a direct sum of simples in the component j. Clearly, this gives us a decomposition of $\mathcal{C}$ into a direct sum $\mathcal{C} \simeq \boxplus_{j\in J}\mathcal{C}_j$.

2.2 Main result

Theorem 2.3 ([1, Theorem 1.4.8])**.** *The 2-category of finite semisimple module categories of a multifusion category is a finite semisimple 2-category.*

Proof. Let C be a multifusion category.

From Proposition 1.15, we already see that $\mathcal{M}\mathrm{od}(C) \simeq (\mathcal{B}C)^{\triangledown}$ is idempotent complete and locally idempotent complete. $\mathcal{M}\mathrm{od}(C)$ is clearly already additive.

Locally semisimple-ness follows directly from [4, Corollary 2.5.6], and existence of adjoints for 1-morphisms follows from [3, Corollary 2.13]. $\square$

Proof, slightly more detailed. Locally semisimple-ness: One of the main results of [4] is [4, Theorem 2.5.5], which states that the relatie Deligne tensor product of separable bimodule categories is also separable. Separable-ness is the stand in for semisimple-ness for arbitrary fields, and again by [4, Corollary 2.6.9], is equivalent to semisimple-ness in our case of char $\mathbf{k} = 0$. We have, for separable algebras a, b, b-bimod-$a(C) \simeq b$-mod$(C) \boxtimes_C$ mod-$a(C)$ is separable, hence semisimple.

Existence of adjoints for 1-morphisms: functors between semisimple 1-categories are automatically exact, hence admit both right and left adjoints. Then the results follows from [3, Corollary 2.13], which states that the adjoints can be promoted to respect the bimodule structures. (As stated in [3], it only says there exists adjoints that respect bimodule structures, but the proof actually implies more, that the given adjoint can be promoted.) $\square$

Proof, even more details. For bimodule categories ${}_BM_C$ and ${}_CN_E$, we need to show that $M \boxtimes_C N$ is separable as a left A-module category and separable as a right E-module category. We just discuss for B.

Suppose $M = a$-mod(C) and $N =$ mod-$a'(C)$; then $M \boxtimes_C N \simeq a$-bimod-$a'(C) \simeq$ mod-$a'(M)$. Here mod-$a'(M)$ are module objects in M over a', that is, $m \in M$ with action $m \triangleleft a' \to m$. (In [4] they call these algebra objects for the monad $T = - \triangleleft a'$.)

Now $M =$ mod-$b(B)$ also. C does not act on B, but acts on b-modules in B, in particular on b.

One checks that $b' := b \triangleleft a'$ is the algebra object B that gives $M \boxtimes_C N$ as a left B-module category, i.e. $M \boxtimes_C N \simeq$ mod-$b'(B)$. Constructing the algbera map on b' and proving the rest is not hard, and generally quite natural, but is a bit confusing because you have to keep track of exactly where the morphisms are taking place.

The separability of b' comes from combining the sections for b and a' in an obvious way (obvious once you understand b' better).

For existence of adjoints for 1-morphisms: these can be constructed explicitly. Consider $x \in b\text{-bimod-}a(C)$,

$$x \otimes_a - : a\text{-mod}(C) \to b\text{-mod}(C)$$

is (right) exact; let $G : b\text{-mod}(C) \to a\text{-mod}(C)$ be its right adjoint. Write $A = a\text{-mod}(C), B = b\text{-mod}(C)$. We must have $A(a, G(n)) \simeq B(x, n) \simeq C(\mathbf{1}, \underline{\mathrm{Hom}}_B(x, n))$. This suggests that we (1) find a left a-module structure on $\underline{\mathrm{Hom}}_B(x, n)$, and (2) define $G(n) = \underline{\mathrm{Hom}}_B(x, n)$ so that

$$A(a, G(n)) = A(a, \underline{\mathrm{Hom}}_B(x, n)) \simeq C(\mathbf{1}, \underline{\mathrm{Hom}}_B(x, n)) \simeq B(x, n)$$

From the right a-module structure on x, $x \otimes a \to x$, we have

$$\underline{\mathrm{Hom}}_B(x, n) \to \underline{\mathrm{Hom}}_B(x \otimes a, n) \simeq {}^*a \otimes \underline{\mathrm{Hom}}_B(x, n)$$

which gives

$$a \otimes \underline{\mathrm{Hom}}_B(x, n) \to \underline{\mathrm{Hom}}_B(x, n)$$

Now we check the adjunction for free modules $a \otimes c$:

$$A(a \otimes c, G(n)) \simeq C(c, \underline{\mathrm{Hom}}_B(x, n)) \simeq B(x \triangleleft c, n)$$

then by right exactness, it's good on arbitrary a-modules $m \simeq a \otimes_a m$.

The unit (on free modules) and counit are:

$$\begin{aligned}
&\eta \in A(a \otimes m, \underline{\mathrm{Hom}}_B(x, x \otimes m)) \simeq C(m, \underline{\mathrm{Hom}}_B(x, x \otimes m) \simeq B(x \otimes m, x \otimes m) \ni \mathrm{id} \\
&\varepsilon \in B(x \otimes_a \underline{\mathrm{Hom}}_B(x, n), n) \xleftarrow{\mathrm{ev}} B(n, n) \ni \mathrm{id}
\end{aligned}$$

For ε, we have to check that $\mathrm{ev} : x \otimes \underline{\mathrm{Hom}}_B(x, n) \to n$ coequalizes the a-actions:

$$\begin{tikzcd}
B(x \otimes a \otimes \underline{\mathrm{Hom}}_B(x,n), n) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & C(a \otimes \underline{\mathrm{Hom}}_B(x,n), \underline{\mathrm{Hom}}_B(x,n)) \arrow[d, "\simeq"] \\
C(\underline{\mathrm{Hom}}_B(x,n), \underline{\mathrm{Hom}}_B(x \otimes a, n)) & C(\underline{\mathrm{Hom}}_B(x,n), {}^*a \otimes \underline{\mathrm{Hom}}_B(x,n)) \arrow[l, "\simeq"'] \\
C(\underline{\mathrm{Hom}}_B(x,n), \underline{\mathrm{Hom}}_B(x,n)) \arrow[u, "\text{right } a\text{-action on } x"] \arrow[uur, "\text{left} a\text{-action on } \underline{\mathrm{Hom}}_B(x,n)"'] &
\end{tikzcd}$$

□

Theorem 2.4 ([1, Theorem 1.4.9])**.** *Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.*

Proof. For simplicity, let us assume that $\mathcal{C}$ has only one component. Indeed, if there are multiple components, then from our previous discussion, we have a direct sum decomposition $\mathcal{C} =\simeq \boxplus_{j \in J} \mathcal{C}_j$, and if $\mathcal{C}_j \simeq \mathcal{M}\mathrm{od}(C_j)$ for multifusion C_j, then $\mathcal{C} \simeq \mathcal{M}\mathrm{od}(\bigoplus_{j \in J} C_j)$.

Take a simple object X, and let $C = \mathcal{C}(X, X)$. Since id_X is simple, C is in fact fusion.

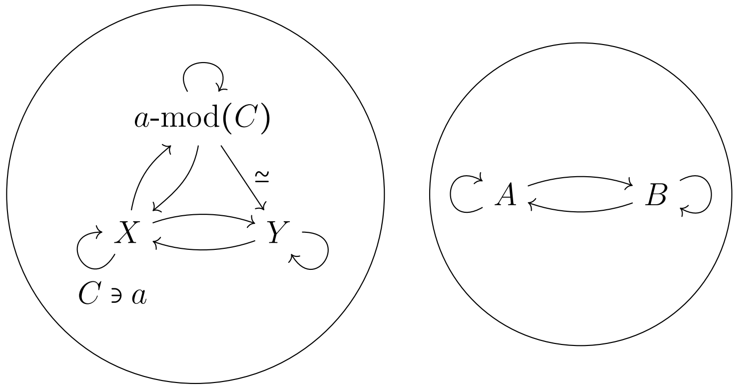

Figure 1: Abstract picture of finite semisimple 2-categories, showing the simples separated into components; within each component, every simple can be recovered from the endomorphism category of other simples.

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{B}C &\to \mathcal{C} \\ * &\mapsto X \end{aligned}$$

which is fully faithful by construction. Then

$$F^{\triangledown} : (\mathcal{B}C)^{\triangledown} \to \mathcal{C}$$

is fully faithful. It remains to show that $F^{\triangledown}$ is essentially surjective, in particular, that any simple is in the essential image of $F^{\triangledown}$.

Let Y be a simple object in $\mathcal{C}$. Since $\mathcal{C}$ only has one component, there exists a nonzero 1-morphism $f : X \to Y$; it has a (nonzero) right adjoint $g : Y \to X$, with counit $\varepsilon : fg \Rightarrow \mathrm{id}_Y$. Since Y is simple, id_Y is simple, so ε admits a section, hence $f \dashv g$ is a separable adjunction. Thus, by uniqueness of separable splittings, Y is in the essential image of $F^{\triangledown}$. □

In the proof above, the only property of X that we used is the fact that there exists a nonzero 1-morphism from X to every simple in $\mathcal{C}$, and thus any X will do. Taking, say, $X = \boxplus X_i$, where the sum is over equivalence classes of simples, would result in a multifusion $C = \mathcal{C}(X, X)$.

We can also avoid the first step of taking only one component of $\mathcal{C}$, as long as we are careful to take X that has at least one object from each component in its direct sum decomposition.

Remark 2.5. Every object (assuming $\mathcal{C}$ has one component) contains the information need to reconstruct every other object; given any nonzero X, every object is constructed from the endomorphisms of X.

This is reminiscent of the metaphor of Indra's Net from Mahayana Buddhism, where, in the realm of the god Indra, there is a vast net that stretches infinitely in all directions; at each crossing there is a jewel that perfectly reflects, and in the reflection is every other jewel, and within them is again reflected every other jewel and so on. The metaphor is meant to capture the concept of the interconnectedness of all phenomena.

A concrete "picture" of Indra's Net is the universal cover of $\mathbf{T}^3 \# \mathbf{T}^3$: at every lattice point in $\mathbb{Z}^3 \subset \mathbb{R}^3$, cut out a ball; at each ball, invert the exterior into itself; repeat this process infinitely.

Example 2.6. Consider $C = \mathrm{Vec}_G$ for a finite group G. We will focus on two of its algebras, the trivial algebra $\mathbf{k}_e$ and the group algebra $\mathbf{k}[G] = \bigoplus \mathbf{k}_g$, which we denote by e and f, respectively, as objects of C.

Clearly e-mod(C) is simply C itself as a right module category over itself. For f, it is clear that a module over f in C is determined by any of its components (as a G-graded vector space), thus f-mod$(C) \simeq \mathrm{Vec}$, with right C-module action given by forgeting the grading on C.

Next we study the functor categories. Clearly the C-endofunctors of C_C is C itself. For f-mod(C), an endofunctor is given by an f-f-bimodule in C. Let $m \in f$-bimod-$f(C)$; write $m = \bigoplus m_g$. The right f-action on m makes all the m_g isomorphic in a coherent manner. For $h \in G$, conjugation (left action by h and right action by h^{-1}) gives an action of G on m_e; this action completely determines the left action on other components m_g as well. Thus, $\mathcal{M}\mathrm{od}(C)(f, f) \simeq \mathrm{Rep}(G)$, where we simply write f for f-mod(C) for simplicity.

For functor categories between them, we see that $\mathcal{M}\mathrm{od}(C)(e, f) \simeq f$-bimod-$e(C) \simeq f$-mod$(C) \simeq \mathrm{Vec}$ and $\mathcal{M}\mathrm{od}(C)(f, e) \simeq e$-bimod-$f(C) \simeq$ mod-$f(C) \simeq \mathrm{Vec}$.

Thus we recover the running example of $\mathcal{M}\mathrm{od}(\mathrm{Vec}_{\mathbb{Z}/2})$; we note that in the simple case of $G = \mathbb{Z}/2$, $\mathrm{Vec}_{\mathbb{Z}/2} \simeq \mathrm{Rep}(\mathbb{Z}/2)$, but here we see a difference between the endomorphism categories of the two objects we considered.

Of course, there are many other algebras. For a subgroup $H \subseteq G$, we can consider the group algebra $\mathbf{k}[H]$, which we denote by h as an object in C. The category of modules h-mod(C) is simply the direct sum $\bigoplus_{H \backslash G} \mathrm{Vec}$, as the components m_g of an h-module must be the same within each H-orbit. The endomorphism category seems a lot more complicated.

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