Note in preparation for talk for seminar on Fusion 2-Categories, Winter semester 2022, UHH

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The main goal of my talk today is to prove that a finite semisimple 2-category is the category of finite semisimple modules over a multifusion category, and vice versa.

That is, for a semisimple 2-category $\mathcal{C}$, there exists a multifusion category C such that

$$\mathcal{C} \simeq \mathcal{M}od_{s.s.}^{fin}(C)$$

Here $\mathcal{M}od_{s.s.}^{fin}(C)$, which we will abbreviate to $\mathcal{M}od-C$, stands for finite semisimple right module categories over C.

Conversely, for any mutifusion category C, $\mathcal{M}od-C$ is a semisimple 2-category.

0.1 Conventions

Everything is over an algebraically closed field $\mathbf{k}$ with characteristic 0.

We use different fonts/alphabets for different levels of structures:

In relation to a 2-category:

- $\mathcal{C}$ (caligraphic font): 2-category;
- X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X,Y)$ for the category of morphisms from X to Y;
- $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \to Y$, we may write $\alpha \in \mathcal{C}(X,Y)(f,g)$ to indicate its sources and targets, or simply $\alpha \in \mathrm{Hom}(f,g)$ if the objects are clear

In relation to a 1-category:

- C, A (upper case latin): category;
- a, b, f, g (lower case latin): objects in category, functor between categories;
- α, β (lower case greek): morphism in category

[We use the same type of font for 1-functors and 1-morphisms because the 1-morphisms in the 2-category $\mathcal{M}od(C)$ of module categories are module functors, and we want the notation to be consistent; we use the same type of font for 2-functors and objects because 2-functors are objects in the 2-category of 2-functors]

We also compose morphisms from right to left: in a 2-category $\mathcal{C}$, for $\alpha \in \mathcal{C}(X,Y)(f,f'), \beta \in \mathcal{C}(Y,Z)(g,g'), \gamma \in \mathcal{C}(X,Y)(f',f'')$, we write

$$g \circ f, g \circ f', \ldots : X \to Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \to Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \to Y$$

for vertical composition of 2-morphisms.

We may also omit the composition symbols if the type of composition is clear (in particular for composition of 1-morphisms).

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X,Y)$ satisfies P.

By 2-category we always mean a weak 2-category that is furthermore locally additive over $\mathbf{k}$, that is, all hom-categories are additive categories over $\mathbf{k}$, and all compositions are $\mathbf{k}$-bilinear. By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally $\mathbf{k}$-linear.

1 Review

Let us recall some definitions and facts concerning semisimple 2-categories. These where covered in more detail in previous talks, so here we will simply state them without proof.

1.1 Additive 2-category, direct sums of objects

Definition 1.1 (direct sum of objects in 2-category)**.** A *direct sum* of two objects A_1, A_2 in $\mathcal{C}$ is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms $i_k : A_k \rightleftharpoons A_1 \boxplus A_2 : p_k$, such that

- $p_k \circ i_k \simeq \mathrm{id}_{A_k}$,
- $p_2 \circ i_1$, $p_1 \circ i_2$ are zero 1-morphisms,
- $\mathrm{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

Proposition 1.2. *i_k, p_k are two-sided adjoints to each other.*

Definition 1.3. A 1-morphism $i : X \to Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A, $i \circ - : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ is fully faithful.

Proposition 1.4. *$i_k : A_k \to A_1 \boxplus A_2$ is fully faithful.*

1.2 Idempotent completeness, separable monads, splittings

Definition 1.5. Let (t, μ, η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $\delta : t \Rightarrow t \circ t$ to μ.

Definition 1.6. Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition 1.7. Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Under the right conditions (local idempotent completeness of $\mathcal{C}$), separable splittings are unique:

Proposition 1.8 (Uniqueness of separable splitting)**.** *[*[1, Theorem A.3.1]*] In a locally idempotent complete 2-category $\mathcal{C}$, separable splittings of a separable monad are unique up to equivalence.*

In particular, this holds true when $\mathcal{C}$ is locally semisimple.

Thought to self: [1] proves this by showing that a splitting is equivalent to an "Eilenberg-Moore object" and also a "Kleisli object", which are in themselves important and interesting objects characterized by universal properties. I'd like to have a more direct proof, somehow constructing an equivalence between any two splittings directly from the splitting data. I don't have a full proof, but here's an attempt. Say for a separable monad t on X, we have two adjunctions $r \vdash l : X \to Y$ and $r' \vdash l' : X \to Y'$ that split t. There is are obvious 1-morphisms $l' \circ r : Y \to Y'$, $l \circ r' : Y' \to Y$, but it is unlikely that these are equivalences. A promising candidate for an equivalence is $l' \circ_t r$, the coequalizer of $l' \circ t \circ r \Rightarrow l' \circ r$ (here r is a left t-module from the counit: $tr \simeq rlr \Rightarrow r$; similarly l' is a right t-module). Such a coequalizer does appear in the proof of [1, Theorem A.3.1] anyway, and I've found that this thought process makes the consideration of the Eilenberg-Moore and Kleisli objects more motivated.

Now that we've defined the notion on objects, we consider the property as a global 2-category-wide property:

Definition 1.9. A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition 1.10. Let $\mathcal{C}$ be a locally semisimple (more generally locally idempotent complete) 2-category. Define the *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^\nabla$, to be the 2-category whose objects are given by separable monads of $\mathcal{C}$, and the hom-category of morphisms from separable monad (X,p) to (Y,q) is the category q-mod-$p(\mathcal{C}(X,Y))$ of q-p-bimodules in $\mathcal{C}(X,Y)$.

There is a natural 2-functor $I:\mathcal{C}\to\mathcal{C}^\nabla$ that is fully faithful.

A 2-functor $F:\mathcal{C}\to\mathcal{D}$ extends to a 2-functor $F^\nabla:\mathcal{C}^\nabla\to\mathcal{D}^\nabla$ that commutes with I's.

This is a categorified version of the Karoubi completion operation. By going along the lines of [2], the definition of $\mathcal{C}^\nabla$ is very suggestive of the idea that objects of $\mathcal{C}^\nabla$ should be thought of as the Morita 2-category of algebras in some tensor category; indeed, this will be made precise in the proof of the main result.

Proposition 1.11. *$\mathcal{C}^\nabla$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I:\mathcal{C}\simeq\mathcal{C}^\nabla$ is an equivalence.*

As a consequence, if $\mathcal{D}$ is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C},\mathcal{D})\simeq\mathcal{F}un(\mathcal{C}^\nabla,\mathcal{D}^\nabla)\simeq\mathcal{F}un(\mathcal{C}^\nabla,\mathcal{D})$$

1.3 Simple objects

Proposition 1.12 (equivalent notions of simple-ness)**.** *Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X\in\mathcal{C}$ be an object. Then the following notions of X being simple are equivalent:*

(1) any subobject $i:Y\to X$ of X is either 0 ($Y\simeq 0$) or (i is) an equivalence;

(2) X cannot be written as a non-trivial direct sum, i.e. if $X=\boxplus X_i$, then $X_i\simeq 0$ for all but one i;

(3) id_X *is a simple object in $\mathcal{C}(X,X)$.*

Proof idea. (1) $\Rightarrow$ (2): Contravariant statement follows from fully faithfulness of $i_k:A_k\to A_1\boxplus A_2$.

(2) $\Rightarrow$ (3): Contravariant statement is "identity splitting implies object splitting", uses idempotent completeness of $\mathcal{C}$ to split out objects corresponding to summands of id_X (which are separable monads) (see [1, Prop 1.3.16]).

(3) $\Rightarrow$ (1): for non-zero fully faithful $r:Y\to X$, with id_X simple, consider the left adjoint $l:X\to Y$, use fully faithfulness to get a preimage $\delta:\mathrm{id}_Y\Rightarrow lr$ of $\eta\circ r:r\Rightarrow rlr$. Use simplicity of id_X to get section of the unit η. Show δ is a section of the counit. Etc. (See [1, Prop 1.2.14]) $\square$

Definition 1.13 ((finite) semisimple 2-category)**.** A 2-category $\mathcal{C}$ is *semisimple* if it is:

- locally semisimple,
- admits left and right adjoints for every 1-morphism,

- additive,
- idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

References

[1] Douglas, Christopher L., and David J. Reutter. "Fusion 2-categories and a state-sum invariant for 4-manifolds." arXiv preprint arXiv:1812.11933 (2018).

[2] Ostrik, Victor. "Module categories, weak Hopf algebras and modular invariants." Transformation groups 8, no. 2 (2003): 177-206.