

Semisimple 2-categories

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Goal

$$\{\text{multifusion category}\} \xrightarrow[\simeq]{\mathcal{M}\text{od}_{s.s.}^{fin}(-)} \{\text{semisimple 2-cat}\}$$

- ▶ The 2-category $\mathcal{M}\text{od}_{s.s.}^{fin}(C)$ of finite semisimple module categories over a multifusion category C is semisimple.
- ▶ For any finite semisimple 2-category \mathcal{C} , there exists a multifusion category C such that $\mathcal{C} \simeq \mathcal{M}\text{od}_{s.s.}^{fin}(C)$

Conventions

In relation to a 2-category:

- ▶ \mathcal{C} (caligraphic font): 2-category;
- ▶ X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- ▶ f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X, Y)$ for the category of morphisms from X to Y ;
- ▶ $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \rightarrow Y$, we may write $\alpha \in \mathcal{C}(X, Y)(f, g)$ to indicate its sources and targets, or simply $\alpha \in \text{Hom}(f, g)$ if the objects are clear

Conventions

In relation to a 1-category:

- ▶ C, A (upper case latin): category;
- ▶ a, b, f, g (lower case latin): objects in category, functor between categories;
- ▶ α, β (lower case greek): morphism in category

Conventions

We also compose morphisms from right to left: in a 2-category \mathcal{C} , for $\alpha \in \mathcal{C}(X, Y)(f, f'), \beta \in \mathcal{C}(Y, Z)(g, g'), \gamma \in \mathcal{C}(X, Y)(f', f'')$, we write

$$g \circ f, g \circ f', \dots : X \rightarrow Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \rightarrow Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \rightarrow Y$$

for vertical composition of 2-morphisms.

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In general, if P is a property of a 1-category, we say that a 2-category \mathcal{C} is *locally P* if every hom-category $\mathcal{C}(X, Y)$ satisfies P .

Conventions

In general, if P is a property of a 1-category, we say that a 2-category \mathcal{C} is *locally P* if every hom-category $\mathcal{C}(X, Y)$ satisfies P . By 2-category we always mean a weak 2-category that is furthermore locally additive over \mathbf{k} . By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally \mathbf{k} -linear.

Review

Additive 2-category, direct sum of objects

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Definition (direct sum of objects in 2-category)

A *direct sum* of two objects A_1, A_2 in \mathcal{C} is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms

$i_k : A_k \Rightarrow A_1 \boxplus A_2 : p_k$, such that

- ▶ $p_k \circ i_k \simeq \text{id}_{A_k}$,
- ▶ $p_2 \circ i_1, p_1 \circ i_2$ are zero 1-morphisms,
- ▶ $\text{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

Additive 2-category, direct sum of objects

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i_k, p_k are two-sided adjoints to each other.

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Definition

A 1-morphism $i : X \rightarrow Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A , $i \circ - : \mathcal{C}(A, X) \rightarrow \mathcal{C}(A, Y)$ is fully faithful.

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Definition (Direct sum of 2-categories)

Given 2-categories \mathcal{C}_j , $j \in J$, we may consider the direct sum 2-category $\mathcal{C} := \boxplus_{j \in J} \mathcal{C}_j$:

- ▶ $\text{Obj } \mathcal{C} = \bigsqcup_{j \in J} \text{Obj } \mathcal{C}_j$
- ▶ for $X \in \mathcal{C}_i$, $Y \in \mathcal{C}_j$, $\mathcal{C}(X, Y) = \begin{cases} \mathcal{C}_j(X, Y) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Idempotent completeness, separable monads, splittings

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In other words, a separable monad over X is a separable algebra in $\mathcal{C}(X, X)$.

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Let $r \vdash l : X \rightarrow Y$ be an adjunction with unit $\eta : \text{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \text{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

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Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \rightarrow Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X .

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Under the right conditions (local idempotent completeness of \mathcal{C}), splittings are unique:

Proposition (Uniqueness of splitting)

[[1], Theorem A.3.1] In a locally idempotent complete 2-category \mathcal{C} , splittings of a separable monad are unique up to equivalence.

Idempotent completeness, 2-category

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Definition (Idempotent completion)

Let \mathcal{C} be a locally idempotent complete 2-category. The *idempotent completion of \mathcal{C}* , denoted \mathcal{C}^∇ , with:

- ▶ Objects: (X, p) separable monad in \mathcal{C} ,
- ▶ $\mathcal{C}^\nabla((X, p), (Y, q)) = q\text{-bimod-}p(\mathcal{C}(X, Y))$

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A 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ extends to a 2-functor $F^\nabla : \mathcal{C}^\nabla \rightarrow \mathcal{D}^\nabla$ that commutes with I 's.

Idempotent completeness, 2-category

Key example:

$\mathcal{B}C$: one object $*$ with endomorphism category $\mathcal{B}C(*, *)$.

$$(\mathcal{B}C)^{\nabla} = \begin{cases} \text{Obj : separable algebras in } C \\ \text{Mor : } (\mathcal{B}C)^{\nabla}(a, b) = b\text{-bimod-}a(C) \end{cases}$$

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Proposition

\mathcal{C}^∇ is idempotent complete. Moreover, if \mathcal{C} is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^\nabla$ is an equivalence.

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As a consequence, if \mathcal{D} is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^\nabla, \mathcal{D}^\nabla) \simeq \mathcal{F}un(\mathcal{C}^\nabla, \mathcal{D})$$

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Proposition

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful, then $F^\nabla : \mathcal{C}^\nabla \rightarrow \mathcal{D}^\nabla$ is also fully faithful.

Idempotent completeness, 2-category

Proposition ([1]Prop 1.3.13)

For a multifusion category C , the following 2-functor is an equivalence:

$$(-)\text{-mod}(C) : (\mathcal{BC})^{\nabla} \rightarrow \mathcal{Mod}(C)$$

$$a \mapsto a\text{-mod}(C)$$

$${}_b m_a \mapsto m \otimes_a -$$

$$\varphi \mapsto \varphi \otimes_a -$$

Idempotent completeness, 2-category

Proof.

Essential surjectivity follows from:

-[2] Theorem 1: for a finite semisimple right module category over multifusion C , there exists a semisimple algebra a in C such that $M \simeq a\text{-mod}(C)$ as right C -module categories; and

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This is almost one half of the main result; one still needs to prove local semisimplicity and existence of adjoints. This will follow from more results from [3],[4], which we show later.

Simple objects

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Let \mathcal{C} be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:

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- (3) id_X is a simple object in $\mathcal{C}(X, X)$.*

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Proof sketch.

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(2) \Rightarrow (3): Contravariant statement is “identity splitting implies object splitting”, uses idempotent completeness of \mathcal{C} to split out objects corresponding to summands of id_X (which are separable monads) (see [1]Prop 1.3.16).

(3) \Rightarrow (1): for non-zero fully faithful $r : Y \rightarrow X$, with id_X simple, consider the left adjoint $l : X \rightarrow Y$, use fully faithfulness to get a preimage $\delta : \text{id}_Y \Rightarrow lr$ of $\eta \circ r : r \Rightarrow rlr$. Use simplicity of id_X to get section of the unit η . Show δ is a section of the counit. Etc. (See [1]Prop 1.2.14) □

Semisimple 2-category

Definition ((finite) semisimple 2-category)

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A 2-category \mathcal{C} is *semisimple* if it is:

- ▶ locally semisimple,
- ▶ admits left and right adjoints for every 1-morphism,
- ▶ additive,
- ▶ idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

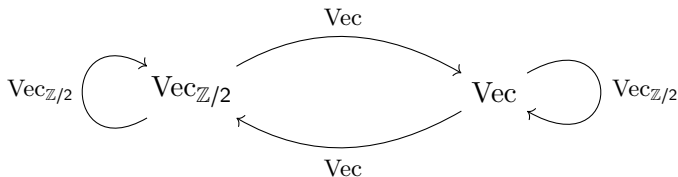
New Stuff

Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, is similar to the semisimple 1-category case.

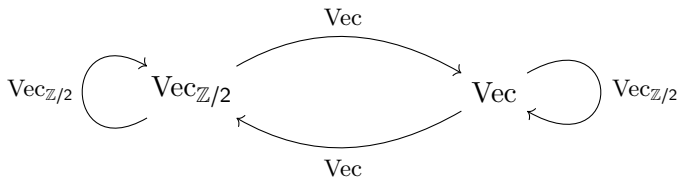
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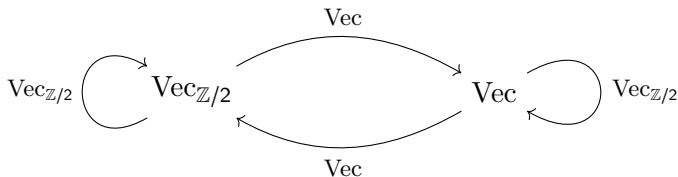
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“2-Morita equivalences” between fusion categories

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Proposition (Schur's Lemma, [1]Prop 1.2.19)

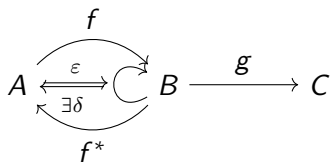
In a semisimple 2-category \mathcal{C} , if $f : A \rightarrow B, g : B \rightarrow C$ are nonzero 1-morphisms between simple objects A, B, C , then $g \circ f$ is also nonzero.

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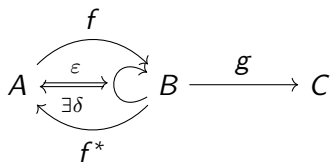


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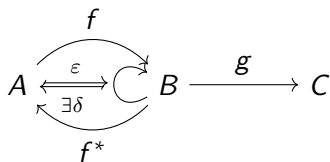
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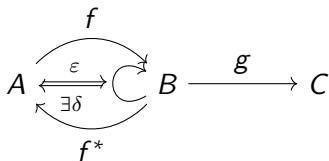
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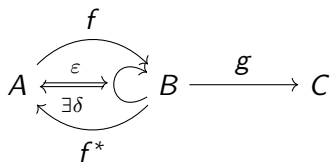
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Let $f^* : B \rightarrow A$ be a right adjoint to f . Since id_B is simple, \exists section $\delta : \text{id}_B \Rightarrow ff^*$ to counit $\varepsilon : ff^* \Rightarrow \text{id}_B$. Postcomposing with g , we have $\text{id}_g = (\text{id}_g \circ \varepsilon) \cdot (\text{id}_g \circ \delta) : g \Rightarrow gff^* \Rightarrow g$. Thus if $gf = 0$, then $\text{id}_g = 0$, contradicting nonzero-ness of B . \square

Schur's lemma - components

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full subcategory \mathcal{C}_j , $j \in J$: direct sum of simple objects in j gives direct sum decomposition $\mathcal{C} \simeq \bigoplus_{j \in J} \mathcal{C}_j$.

Main results

Theorem ([1]Theorem 1.4.8)

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Proof.

$\text{Mod}(\mathcal{C}) \simeq (\mathcal{BC})^\nabla$ is idempotent complete and locally idempotent complete. $\text{Mod}(\mathcal{C})$ is clearly already additive.

Locally semisimple-ness follows directly from [4]Corollary 2.5.6, and existence of adjoints for 1-morphisms follows from [3]Corollary 2.13.



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Theorem ([1]Theorem 1.4.9)

Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.

Proof.

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($\mathcal{C} \simeq \boxplus_{j \in J} \mathcal{C}_j$ and $\mathcal{C}_j \simeq \mathcal{M}\text{od}(\mathcal{C}_j)$ for multifusion \mathcal{C}_j , then $\mathcal{C} \simeq \mathcal{M}\text{od}(\oplus_{j \in J} \mathcal{C}_j)$.)

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Assume that \mathcal{C} has only one component.

($\mathcal{C} \simeq \boxplus_{j \in J} \mathcal{C}_j$ and $\mathcal{C}_j \simeq \mathcal{M}\text{od}(C_j)$ for multifusion C_j , then $\mathcal{C} \simeq \mathcal{M}\text{od}(\oplus_{j \in J} C_j)$.)

Fix simple object X , let $C = \mathcal{C}(X, X)$.

Main results

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We show $\mathcal{C} \simeq (\mathcal{BC})^\nabla$.

Main results

Proof (Cont.)

Consider the inclusion 2-functor

$$\begin{aligned} F : \mathcal{BC} &\rightarrow \mathcal{C} \\ * &\mapsto X \end{aligned}$$

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Simple object Y , \exists nonzero 1-morphism $f : X \rightarrow Y$; it has a (nonzero) right adjoint $g : Y \rightarrow X$, with counit $\varepsilon : fg \Rightarrow \text{id}_Y$.

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Simple object Y , \exists nonzero 1-morphism $f : X \rightarrow Y$; it has a (nonzero) right adjoint $g : Y \rightarrow X$, with counit $\varepsilon : fg \Rightarrow \text{id}_Y$. id_Y is simple, so ε admits a section, hence $f \dashv g$ is a separable adjunction. Thus, by uniqueness of separable splittings, Y is in the essential image of F^∇ . □

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