Note in preparation for talk for seminar on Fusion 2-Categories, Winter semester 2022, UHH

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[Almost everything here is from [1]; the only things that are new here are cleaner proofs of the main theorems]

The main goal of my talk today is to prove that a finite semisimple 2-category is the category of finite semisimple modules over a multifusion category, and vice versa.

That is, for a semisimple 2-category $\mathcal{C}$, there exists a multifusion category C such that

$$\mathcal{C} \simeq \mathcal{M}od^{fin}_{s.s.}(C)$$

Here $\mathcal{M}od^{fin}_{s.s.}(C)$, which we will abbreviate to $\mathcal{M}od-C$, stands for finite semisimple right module categories over C.

Conversely, for any mutifusion category C, $\mathcal{M}od-C$ is a semisimple 2-category.

0.1 Conventions

Everything is over an algebraically closed field $\mathbf{k}$ with characteristic 0.

We use different fonts/alphabets for different levels of structures:

In relation to a 2-category:

- $\mathcal{C}$ (caligraphic font): 2-category;
- X, Y, F (upper case latin): object of 2-category, functor between 2-categories;
- f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X,Y)$ for the category of morphisms from X to Y;
- $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \to Y$, we may write $\alpha \in \mathcal{C}(X,Y)(f,g)$ to indicate its sources and targets, or simply $\alpha \in \mathrm{Hom}(f,g)$ if the objects are clear

In relation to a 1-category:

- C, A (upper case latin): category;

- a, b, f, g (lower case latin): objects in category, functor between categories;
- α, β (lower case greek): morphism in category

[We use the same type of font for 1-functors and 1-morphisms because the 1-morphisms in the 2-category $\mathcal{M}od(C)$ of module categories are module functors, and we want the notation to be consistent; we use the same type of font for 2-functors and objects because 2-functors are objects in the 2-category of 2-functors]

We also compose morphisms from right to left: in a 2-category $\mathcal{C}$, for $\alpha \in \mathcal{C}(X,Y)(f,f'), \beta \in \mathcal{C}(Y,Z)(g,g'), \gamma \in \mathcal{C}(X,Y)(f',f'')$, we write

$$g \circ f, g \circ f', \ldots : X \to Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \to Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \to Y$$

for vertical composition of 2-morphisms.

We may also omit the composition symbols if the type of composition is clear (in particular for composition of 1-morphisms).

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X,Y)$ satisfies P.

By 2-category we always mean a weak 2-category that is furthermore locally additive over $\mathbf{k}$, that is, all hom-categories are additive categories over $\mathbf{k}$, and all compositions are $\mathbf{k}$-bilinear. By 2-functor (sometimes just functor for simplicity) between 2-categories will always be locally $\mathbf{k}$-linear.

1 Review

Let us recall some definitions and facts concerning semisimple 2-categories. These where covered in more detail in previous talks, so here we will simply state them without proof.

1.1 Additive 2-category, direct sums of objects

Definition 1.1 (direct sum of objects in 2-category)**.** A *direct sum* of two objects A_1, A_2 in $\mathcal{C}$ is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms $i_k : A_k \rightleftharpoons A_1 \boxplus A_2 : p_k$, such that

- $p_k \circ i_k \simeq \mathrm{id}_{A_k}$,
- $p_2 \circ i_1$, $p_1 \circ i_2$ are zero 1-morphisms,

- $\mathrm{id}_{A_1 \boxplus A_2} \simeq i_1 \circ p_1 \oplus i_2 \circ p_2$

Proposition 1.2. *i_k, p_k are two-sided adjoints to each other.*

Definition 1.3. A 1-morphism $i : X \to Y$ is *fully faithful* (or (X, i) is a *subobject* of Y) if it induces fully faithful functors between hom-categories by post-composition, i.e. for all objects A, $i \circ - : \mathcal{C}(A, X) \to \mathcal{C}(A, Y)$ is fully faithful.

Proposition 1.4. *$i_k : A_k \to A_1 \boxplus A_2$ is fully faithful.*

(I don't think the following was discussed, but it is a simple concept anyway)

Definition 1.5 (Direct sum of 2-categories)**.** Given 2-categories $\mathcal{C}_j$, $j \in J$, we may consider the direct sum 2-category $\mathcal{C} := \boxplus_{j \in J} \mathcal{C}_j$:

- $\mathrm{Obj}\,\mathcal{C} = \bigsqcup_{j \in J} \mathrm{Obj}\,\mathcal{C}_j$
- for $X \in \mathcal{C}_i, Y \in \mathcal{C}_j$, $\mathcal{C}(X, Y) = \begin{cases} \mathcal{C}_j(X, Y) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

1.2 Idempotent completeness, separable monads, splittings

Definition 1.6. Let (t, μ, η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $\delta : t \Rightarrow t \circ t$ to μ.

Definition 1.7. Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition 1.8. Let (t, μ, η) be a separable monad on an object $X \in \mathcal{C}$. A *(separable) splitting* of t is a (separable) adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Under the right conditions (local idempotent completeness of $\mathcal{C}$), separable splittings are unique:

Proposition 1.9 (Uniqueness of separable splitting)**.** *[[1, Theorem A.3.1]] In a locally idempotent complete 2-category $\mathcal{C}$, separable splittings of a separable monad are unique up to equivalence.*

In particular, this holds true when $\mathcal{C}$ is locally semisimple.

Thought to self: [1] proves this by showing that a splitting is equivalent to an "Eilenberg-Moore object" and also a "Kleisli object", which are in themselves important and interesting objects characterized by universal properties. I'd like to have a more direct proof, somehow constructing an equivalence between any two splittings directly from the splitting data. I don't have a full proof, but here's an attempt. Say for a separable monad t on X, we have two adjunctions $r \vdash l : X \to Y$ and $r' \vdash l' : X \to Y'$ that split t. There is are obvious 1-morphisms $l' \circ r : Y \to Y'$, $l \circ r' : Y' \to Y$, but it is unlikely that these are equivalences. A promising candidate for an equivalence is $l' \circ_t r$, the coequalizer of $l' \circ t \circ r \Rightarrow l' \circ r$ (here r is a left t-module from the counit: $tr \simeq rlr \Rightarrow r$; similarly l' is a right t-module). Such a coequalizer does appear in the proof of [1, Theorem A.3.1] anyway, and I've found that this thought process makes the consideration of the Eilenberg-Moore and Kleisli objects more motivated.

Now that we've defined the notion on objects, we consider the property as a global 2-category-wide property:

Definition 1.10. A 2-category $\mathcal{C}$ is *idempotent complete* if every separable monad admits a splitting.

Definition 1.11. Let $\mathcal{C}$ be a locally semisimple (more generally locally idempotent complete) 2-category. Define the *idempotent completion of* $\mathcal{C}$, denoted $\mathcal{C}^\triangledown$, to be the 2-category whose objects are given by separable monads of $\mathcal{C}$, and the hom-category of morphisms from separable monad (X, p) to (Y, q) is the category q-mod-$p(\mathcal{C}(X, Y))$ of q-p-bimodules in $\mathcal{C}(X, Y)$.

There is a natural 2-functor $I : \mathcal{C} \to \mathcal{C}^\triangledown$ that is fully faithful.

A 2-functor $F : \mathcal{C} \to \mathcal{D}$ extends to a 2-functor $F^\triangledown : \mathcal{C}^\triangledown \to \mathcal{D}^\triangledown$ that commutes with I's.

This is a categorified version of the Karoubi completion operation. By going along the lines of [2], the definition of $\mathcal{C}^\triangledown$ is very suggestive of the idea that objects of $\mathcal{C}^\triangledown$ should be thought of as the Morita 2-category of algebras in some tensor category; indeed, this will be made precise in the proof of the main result.

We can already the essence of this idea in the simple example below. Let C be multifusion, and let $\mathcal{B}C$ be the 2-category with one object $*$ and endmorphisms given by C. Then by construction, we have

$$(\mathcal{B}C)^\triangledown = \begin{cases} \text{Obj}: \text{ separable algebras in } C \\ \text{Mor}: (\mathcal{B}C)^\triangledown(a, b) = b\text{-bimod-}a(C) \end{cases}$$

Proposition 1.12. *$\mathcal{C}^\triangledown$ is idempotent complete. Moreover, if $\mathcal{C}$ is already idempotent complete, then $I : \mathcal{C} \simeq \mathcal{C}^\triangledown$ is an equivalence.*

As a consequence, if $\mathcal{D}$ is idempotent complete, then we have equivalences of 2-categories

$$\mathcal{F}un(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}un(\mathcal{C}^\triangledown, \mathcal{D}^\triangledown) \simeq \mathcal{F}un(\mathcal{C}^\triangledown, \mathcal{D})$$

1.3 Simple objects

Proposition 1.13 (equivalent notions of simple-ness)**.** *Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be a nonzero object. Then the following notions of X being simple are equivalent:*

(1) any subobject $i : Y \to X$ *of* X *is either 0* $(Y \simeq 0)$ *or (i is) an equivalence;*

(2) X *cannot be written as a non-trivial direct sum, i.e. if* $X = \boxplus X_i$*, then* $X_i \simeq 0$ *for all but one* i*;*

(3) id_X *is a simple object in* $\mathcal{C}(X, X)$*.*

Proof idea. (1) ⇒ (2): Contravariant statement follows from fully faithfulness of $i_k : A_k \to A_1 \boxplus A_2$.

(2) ⇒ (3): Contravariant statement is "identity splitting implies object splitting", uses idempotent completeness of $\mathcal{C}$ to split out objects corresponding to summands of id_X (which are separable monads) (see [1, Prop 1.3.16]).

(3) ⇒ (1): for non-zero fully faithful $r : Y \to X$, with id_X simple, consider the left adjoint $l : X \to Y$, use fully faithfulness to get a preimage $\delta : \mathrm{id}_Y \Rightarrow lr$ of $\eta \circ r : r \Rightarrow rlr$. Use simplicity of id_X to get section of the unit η. Show δ is a section of the counit. Etc. (See [1, Prop 1.2.14]) □

1.4 Semisimple 2-category

Definition 1.14 ((finite) semisimple 2-category)**.** A 2-category $\mathcal{C}$ is *semisimple* if it is:

- locally semisimple,
- admits left and right adjoints for every 1-morphism,
- additive,
- idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

2 New stuff

Beyond this point, we cover new stuff. Thus, the writing will be more speech like, and written to reflect more or less the order of presentation during the talk (at least how I plan to present it).

2.1 Schur's lemma

The equivalence between notions of a simple object in a seimsimple 2-category, as we just recalled, is quite similar to the semisimple 1-category case. However, there is a stark difference in that there can be nonzero morphisms between non-equivalent simple objects; indeed, recall our running example of a 2-category $\mathcal{M}od(\mathrm{Vec}_{\mathbb{Z}/2})$:

$$
\begin{tikzcd}
\mathrm{Vec}_{\mathbb{Z}/2} \arrow[loop left, "\mathrm{Vec}_{\mathbb{Z}/2}"] \arrow[rr, bend left, "\mathrm{Vec}"] & & \mathrm{Vec} \arrow[ll, bend left, "\mathrm{Vec}"] \arrow[loop right, "\mathrm{Vec}_{\mathbb{Z}/2}"]
\end{tikzcd}
$$

The existence of nonzero 1-morphisms between non-equivalent simple objects turns out to be a very important aspect of semisimple 2-categories since, as we shall see, it captures 2-Morita equivalences between fusion categories.

Proposition 2.1 (Schur's Lemma, [1, Prop 1.2.19])**.** *In a semisimple 2-category $\mathcal{C}$, if $f : A \to B, g : B \to C$ are nonzero 1-morphisms between simple objects A, B, C, then $g \circ f$ is also nonzero.*

Proof of Proposition 2.1. Let $f^* : B \to A$ be a right adjoint to f. Since id_B is simple, there exists a section $\delta : \mathrm{id}_B \Rightarrow ff^*$ to the counit $\varepsilon : ff^* \Rightarrow \mathrm{id}_B$. Postcomposing with g, we have $\mathrm{id}_g = (\mathrm{id}_g \circ \varepsilon) \cdot (\mathrm{id}_g \circ \delta) : g \Rightarrow gff^* \Rightarrow g$. Thus if $gf = 0$, then $\mathrm{id}_g = 0$, contradicting nonzero-ness of B. □

Thus, this establishes transitivity on a weaker notion of equivalence among simple objects (namely, existence of nonzero 1-morphism); reflexivity is obvious, and symmetry is given by adjoints.

Definition 2.2 (component of semisimple 2-category)**.** In a semisimple 2-category $\mathcal{C}$, we say two simple objects A, B belong to the same *component* if there exists a nonzero 1-morphism $f : A \to B$.

In a finite semisimple 2-category $\mathcal{C}$, there will be finitely many components, say index by a set J. Consider one component $j \in J$, and consider the full subcategory $\mathcal{C}_j$ of $\mathcal{C}$ consisting of objects that are equivalent to a direct sum of simples in the component j. Clearly, this gives us a decomposition of $\mathcal{C}$ into a direct sum $\mathcal{C} \simeq \boxplus_{j \in J} \mathcal{C}_j$.

2.2 Main result

Theorem 2.3 ([1, Theorem 1.4.8])**.** *The 2-category of finite semisimple module categories of a multifusion category is a finite semisimple 2-category.*

Proof. Let C be a multifusion category.

Hmm see Prop 1.3.13, should I put that in the discussion of idempotent completion? □

Theorem 2.4 ([1, Theorem 1.4.9])**.** *Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.*

Proof. □

References

[1] Douglas, Christopher L., and David J. Reutter. "Fusion 2-categories and a state-sum invariant for 4-manifolds." arXiv preprint arXiv:1812.11933 (2018).

[2] Ostrik, Victor. "Module categories, weak Hopf algebras and modular invariants." Transformation groups 8, no. 2 (2003): 177-206.