Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

Ying Hong Tham

December 23, 2022

In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

Proposition 0.1. *Consider functors* $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ *be a natural transformation, and suppose it is the unit to an adjunction* $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' *are extended* $(n+1)$*-TQFTs such that* Z', Z'' *agree with* Z *on* $(n-1)$*- and* n*-manifolds, and* $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$*, then* $Z' \cong Z''$.

The proof of this proposition occupies the rest of this article.

0.1 Adjunctions from topology

In the 2-category with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

-Consider simplest case, empty and sphere -then do adding k-handle -then in general

(In [TODO Y-product paper], we considered this construction without realizing their connection to these adjunctions.)