Note in preparation for talk for seminar on Fusion 2-Categories, Winter semester 2022, UHH

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The main goal of my talk today is to prove that a finite semisimple 2-category is the category of finite semisimple modules over a multifusion category, and vice versa.

That is, for a semisimple 2-category $\mathcal{C}$, there exists a multifusion category C such that

$$\mathcal{C} \simeq \mathcal{M}od_{s.s.}^{fin}(C)$$

Here $\mathcal{M}od_{s.s.}^{fin}(C)$, which we will abbreviate to $\mathcal{M}od - C$, stands for finite semisimple right module categories over C.

Conversely, for any mutifusion category C, $\mathcal{M}od - C$ is a semisimple 2-category.

0.1 Conventions

Everything is over an algebraically closed field $\mathbf{k}$ with characteristic 0.

We use different fonts/alphabets for different levels of structures:

In relation to a 2-category:

- $\mathcal{C}, \mathcal{F}$ (caligraphic font): 2-category, functor between 2-categories
- C, X, Y (upper case latin): object of 2-category
- f, g (lower case latin): 1-morphism; we write $\mathcal{C}(X, Y)$ for the category of morphisms from X to Y
- $\eta, \varepsilon, \delta$ (lower case greek): 2-morphism; for a 2-morphism $\alpha : f \Rightarrow g : X \to Y$, we may write $\alpha \in \mathcal{C}(X, Y)(f, g)$ to indicate its sources and targets, or simply $\alpha \in \mathrm{Hom}(f, g)$ if the objects are clear

In relation to a 1-category:

- C, A (upper case latin): category, functor between categoreis
- a, b, f, g (lower case latin): objects in category
- α, β (lower case greek): morphism in category

We also compose morphisms from right to left: in a 2-category $\mathcal{C}$, for $\alpha \in \mathcal{C}(X,Y)(f,f')$, $\beta \in \mathcal{C}(Y,Z)(g,g')$, $\gamma \in \mathcal{C}(X,Y)(f',f'')$, we write

$$g \circ f, g \circ f', \ldots : X \to Z$$

for composition of 1-morphisms,

$$\beta \circ \alpha : (g \circ f) \Rightarrow (g' \circ f') : X \to Z$$

for horizontal composition of 2-morphisms,

$$\gamma \cdot \alpha : f \Rightarrow f'' : X \to Y$$

for vertical composition of 2-morphisms.

We may also omit the composition symbols if the type of composition is clear (in particular for composition of 1-morphisms).

In general, if P is a property of a 1-category, we say that a 2-category $\mathcal{C}$ is *locally* P if every hom-category $\mathcal{C}(X,Y)$ satisfies P.

1 Review

Let us recall some definitions and facts concerning semisimple 2-categories. These where covered in more detail in previous talks, so here we will simply state them without proof.

1.1 Idempotent completeness, separable monads, splittings

Definition 1.1. Let (t,μ,η) be a monad on an object X in a 2-category $\mathcal{C}$. We say t is *separable* if there is a t-t-bimodule section $\delta : t \Rightarrow t \circ t$ to μ.

Definition 1.2. Let $r \vdash l : X \to Y$ be an adjunction with unit $\eta : \mathrm{id}_X \Rightarrow rl$ and counit $\varepsilon : lr \Rightarrow \mathrm{id}_Y$. We say the adjunction $l \dashv r$ is *separable* if ε admits a section.

Clearly, if an adjunction $l \dashv r$ is separable, then the monad rl is separable.

Definition 1.3. Let (t,μ,η) be a separable monad on an object $X \in \mathcal{C}$. A *splitting* of t is a separable adjunction $r \vdash l : X \to Y$ together with an isomorphism $\psi : rl \simeq t$ as monads on X.

Under the right conditions (local idempotent completeness of $\mathcal{C}$), splittings are unique (see Proposition 1.4).

Proposition 1.4 (Uniqueness of splitting)**.** *[[1], Theorem A.3.1] In a locally idempotent complete 2-category $\mathcal{C}$, splittings of a separable monad are unique up to equivalence.*

In particular, this holds true when $\mathcal{C}$ is locally semisimple.

Thought to self: [1] proves this by showing that a splitting is equivalent to an "Eilenberg-Moore object" and also a "Kleisli object", which are in themselves important and interesting objects characterized by universal properties. I'd like to have a more direct proof, somehow constructing an equivalence between any two splittings directly from the splitting data. I don't have a full proof, but here's an attempt. Say for a separable monad t on X, we have two adjunctions $r \vdash l : X \to Y$ and $r' \vdash l' : X \to Y'$ that split t. There is are obvious 1-morphisms $l' \circ r : Y \to Y'$, $l \circ r' : Y' \to Y$, but it is unlikely that these are equivalences. A promising candidate for an equivalence is $l' \circ_t r$, the coequalizer of $l' \circ t \circ r \Rightarrow l' \circ r$ (here r is a left t-module from the counit: $tr \simeq rlr \Rightarrow r$; similarly l' is a right t-module). Such a coequalizer does appear in the proof of [1][Theorem A.3.1] anyway, and I've found that this thought process makes the consideration of the Eilenberg-Moore and Kleisli objects less of an ass-pull.

1.2 Simple objects

Proposition 1.5 (equivalent notions of simple-ness)**.** *Let $\mathcal{C}$ be a locally finite semisimple and idempotent complete 2-category, and let $X \in \mathcal{C}$ be an object. Then the following notions of X being simple are equivalent:*

- *any subobject $i : Y \to X$ of X is either 0 ($Y \simeq 0$) or (i is) an equivalence;*
- *X cannot be written as a non-trivial direct sum, i.e. if $X = \boxplus X_i$, then $X_i \simeq 0$ for all but one i;*
- id_X *is a simple object in* $\mathcal{C}(X, X)$.

Definition 1.6 ((finite) semisimple 2-category)**.** A 2-category $\mathcal{C}$ is *semisimple* if it is:

- locally semisimple,
- admits left and right adjoints for every 1-morphism,
- additive,
- idempotent complete.

It is furthermore *finite semisimple* if it is also locally finite and has finitely many equivalence classes of simple objects.

References

[1] Douglas, Christopher L., and David J. Reutter. "Fusion 2-categories and a state-sum invariant for 4-manifolds." arXiv preprint arXiv:1812.11933 (2018).