Upgrading an $(n + \varepsilon)$-TQFT to an extended $(n + 1)$-TQFT

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n + 1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n - 1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n - 1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

2-Cob denotes the bicategory with closed $(n - 1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n + 1)$-dimensional relative cobordisms as 2-morphisms.

Proposition 0.1. *Consider functors* $Z(D^n) : Z(\varnothing^{n-1}) \rightleftarrows Z(S^{n-1}) : Z(\overline{D^n})$.

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ *be a natural transformation, and suppose it is the unit to an adjunction* $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' *are extended* $(n+1)$-*TQFTs such that* Z', Z'' *agree with* Z *on* $(n-1)$- *and* n-*manifolds, and* $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, *then* $Z' \cong Z''$.

Proof. From the topology section below, we know that a 2-morphism of 2-Cob that realizes the attachment of an $(n + 1)$-dim $(k + 1)$-handle, $0 \le k \le n$, is determined by some counit ε_k of an adjunction, whose corresponding unit can be built from handles of index at most k, and thus, the value of extensions Z' of Z on 2-morphisms is completely determined by its value on the $(n + 1)$-dim 0-handle, which is exactly η_0. $\qquad\square$

1 Topology

1.1 Adjunctions from topology

For the purposes of the proof of the proposition above, only the first simple example from this section is needed, the reader may then skip to Section **??**.

In 2-Cob, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition **??**. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk $D^{n+1} :$ $\varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure **??** for $n = 1$ case).
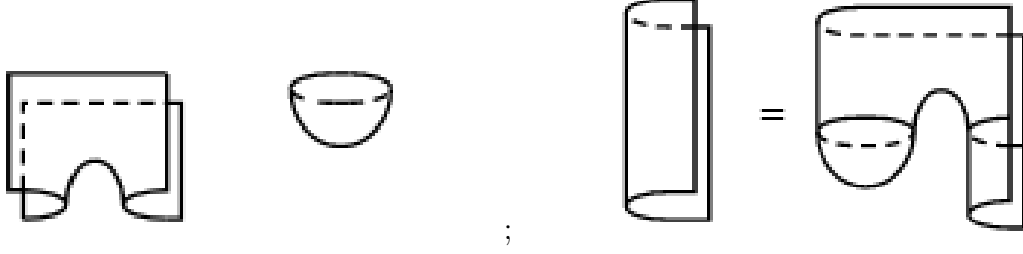

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Figure 1: Counit and unit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$, along with one of the snake equations; relative cobordism goes up (stolen from [?SPries], Figure 1.6 and 1.10, get rotated)

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n - k$. [See Figure **??**]
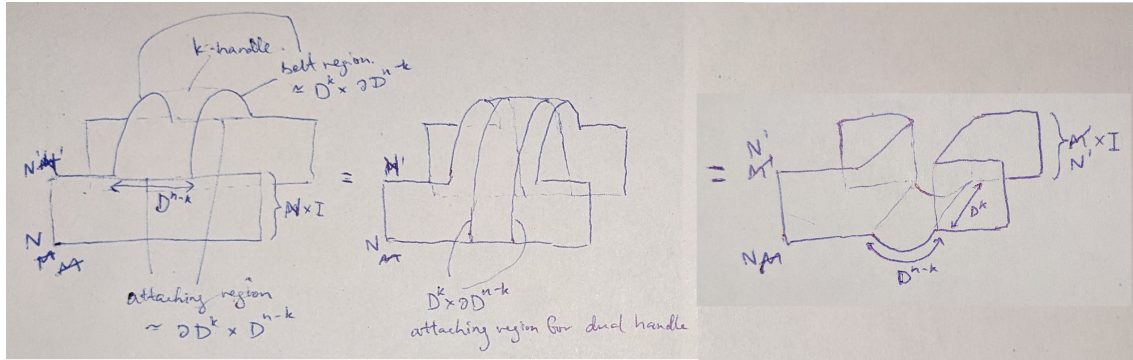

Figure 2: $M : N \to N'$ is an elementary cobordism of index k; it is built from attaching a disk D^n to $N \times I$, with attaching region $\partial D^k \times D^{n-k}$. It can also be built from the other direction, by attaching a disk D^n to $N' \times I$, with attaching region $D^k \times \partial D^{n-k}$. Thus, turning it upside-down, i.e. treated as a cobordism $\overline{M} : N' \to N$, it is an elementary cobordism of index $(n - k)$.

[See Figure **??**] We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching an $(n+1)$-dimensional $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the

k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$. Similarly, we construct the unit $\eta : \mathrm{id}_N \Rightarrow M \circ \overline{M} : N \to N$ by attaching an $(n+1)$-dimensional k-handle to $\mathrm{id}_N = N \times I$; the attaching region for this $(n+1)$-dim k-handle is (the attaching region for the n-dim k-handle that defines M) $\times I$. The snake equations $\mathrm{id}_M = (\varepsilon \circ M) \cdot (M \circ \eta) : M \Rightarrow M \circ \overline{M} \circ M \Rightarrow M$ and $\mathrm{id}_{\overline{M}} = (\overline{M} \circ \eta) \cdot (\eta \circ \overline{M}) : \overline{M} \Rightarrow \overline{M} \circ M \circ \overline{M} \Rightarrow \overline{M}$ follow from the fact that these $(n+1)$-dim handles form a cancelling pair in both cases.

Here we have $M \dashv \overline{M}$, but we may very well have $\overline{M} \dashv M$; the counit $\varepsilon' : \overline{M} \circ M \Rightarrow \mathrm{id}_N$ is an $(n+1)$-dim elementary cobordism of index $(n-k+1)$. It is interesting to note that this counit is the dual cobordism to the unit $\eta : \mathrm{id}_N \Rightarrow \overline{M} \circ M$ previously described.
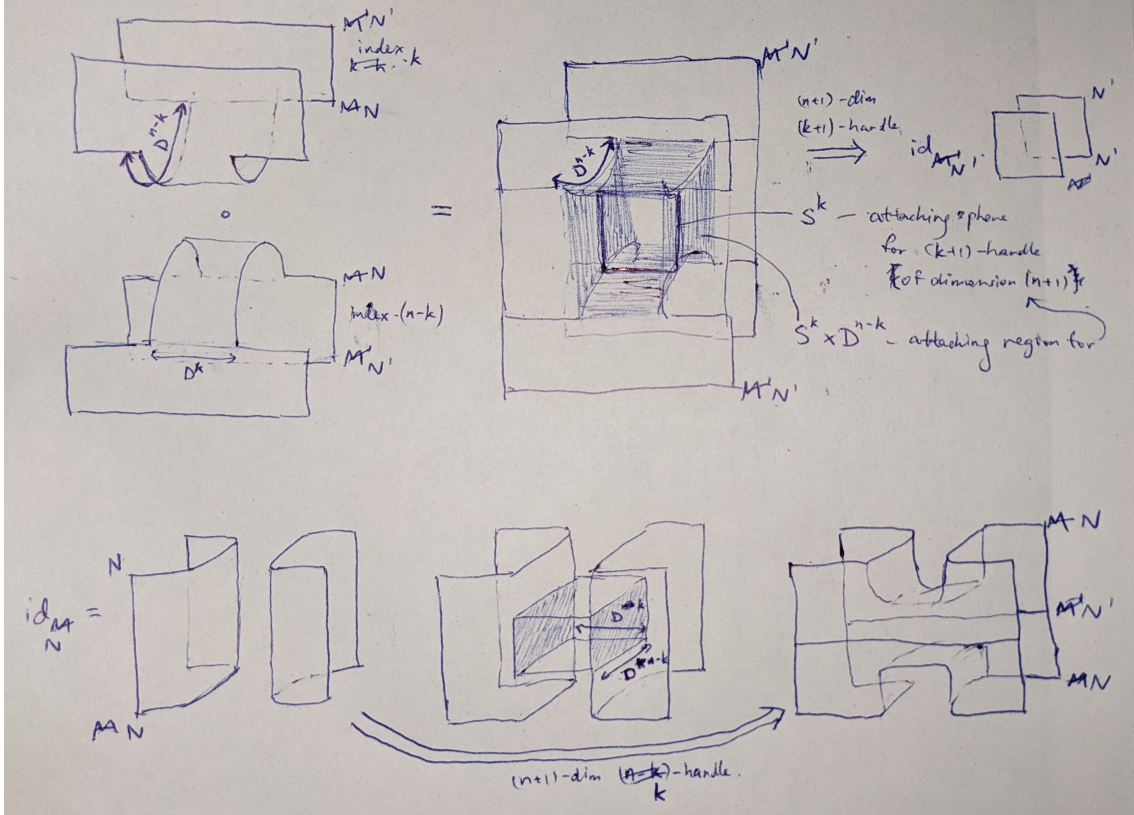

Figure 3: Counit (top) and unit (bottom) for adjunction $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k. Note the way N is drawn here looks like N' in Figure **??** and vice versa (by accident, sorry for minor confusion)

In general, we may consider the pair of n-dim cobordisms $M : N \rightleftharpoons N' : \overline{M}$. By presenting M as a composition of elementary cobordisms, we may compose the adjunctions constructed for each of these elementary cobordisms as above, and obtain an adjunction $M \dashv \overline{M}$.

Remark 1.1. In [**?Yprod**], we considered this construction without realizing their connection to these adjunctions; there we consider the more general case where N, N' may have (possibly different) boundary, and $M : N \to N'$ is a relative cobordism (with the boundary cobordism that is not necessarily the identity cobordism).

1.2 Producing $(n+1)$-dim k-handles from some adjunctions

Throughout this section, $0 \le k < n$.

We show how to construct the $(n+1)$-dim $(k+1)$-handle from the counit ε_k of the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and the unit η_0 of the adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. (The 0-handle is already given by η_0, while the $(n+1)$-handle is the counit to the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$; we say a few more words about this at the end of this section.)

The process of attaching a $(k+1)$-handle to an $(n+1)$-manifold can be implemented as postcomposing by a 2-morphism. More precisely, given an $(n+1)$-manifold W presented as a 2-morphism $W : M \Rightarrow M' : N \to N'$, and an attaching region $S^k \times D^{n-k}$ in M', the $(n+1)$-manifold W' obtained from attaching a $(k+1)$-handle along the specified attaching region may be considered as a 2-morphism $W' : M \Rightarrow M'' : N \to N'$, where M'' is obtain from M' by performing surgery along the attaching region (cutting it out and gluing in $D^{k+1} \times S^{n-k-1}$); then $W' = \omega_{k+1} \cdot W$, where ω_{k+1} is a 2-morphism that we will describe below.

Our 2-morphism ω_{k+1} is of the form $\omega_{k+1} : S^k \times D^{n-k} \Rightarrow D^{k+1} \times S^{n-k-1} : S^k \times S^{n-k-1} \to \varnothing^{n-1}$. Since this is unchanged as W varies, we clearly need to make some arrangements in order to use ω_{k+1}. More specifically, we need to present M' as a composition

$$(\mathrm{id}_{N'} \sqcup \overline{S^k \times D^{n-k}}) \circ (M \backslash \overline{S^k \times D^{n-k}}) : N \to N' \sqcup S^k \times S^{n-k+1} \to N'$$

which is always possible by basic Morse theory. So we have $(M^\circ := M \backslash \overline{S^k \times D^{n-k}})$:

Now let us describe how to construct ω_{k+1} out of ε_k and η_0, which are, as a reminder, the counit and unit of the adjunctions $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, respectively.

We may consider $S^n : \varnothing^{n-1} \to \varnothing^{n-1}$ as the composition $\overline{D^{k+1} \times S^{n-k-1}} \circ S^k \times D^{n-k} : \varnothing^{n-1} \to S^k \times S^{n-k-1} \to \varnothing^{n-1}$.

Then ω_{k+1} is given by the composition of 2-morphisms

$$\omega_{k+1} = (\mathrm{id}_{\overline{D^{k+1} \times S^{n-k-1}}} \circ \varepsilon_k) \cdot (\mathrm{id}_{\overline{S^k \times D^{n-k}}} \circ \eta_0)$$

$$\omega_{k+1} = \quad S^k \times S^{n-k-1} \xrightarrow{\overline{S^k \times D^{n-k}}} \varnothing^{n-1} \xrightarrow{S^k \times D^{n-k}} S^k \times S^{n-k-1} \xrightarrow{\overline{D^{k+1} \times S^{n-k-1}}} \varnothing^{n-1}$$

with $\varnothing^n$ above, η_0, ε_k, and $\mathrm{id}_{S^k \times S^{n-k-1}}$.

A few words on the $(n+1)$-handle, more generally the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$. The unit is a 2-morphism $\eta : \mathrm{id}_{S^{n-1}} \Rightarrow D^n \circ \overline{D^n}$, which is clearly an elementary cobordism of index n.

A similar phenomenon happens with ε_k, that is, η_k, the unit to the adjunction $S^k \times D^{n-k} \dashv \overline{S^k \times D^{n-k}}$, to which ε_k is the counit, is determined by handles of index at most k, and indeed, $\eta_k = S^k \times D^{n-k+1} : \varnothing^n \Rightarrow S^k \times S^{n-k} : \varnothing^{n-1} \to \varnothing^{n-1}$ is built from a 0-handle and a k-handle.

Thus, since the counit is uniquely determined by the unit, the $(n+1)$-dim k-handle, for $k > 0$, is determined by handles of lower index. This may not be very useful in the topology world, but on the algebraic side of a TQFT, this means that everything is determined by the 0-handle.

[It may be helpful to note that the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ is simply S^k times the first example but with n set to $n-k$, $D^{n-k} : \varnothing^{n-k} \rightleftharpoons S^{n-1-k} : \overline{D^{n-k}}$.]

References

[1] Kwon, Alice, and Ying Hong Tham. The Y-Product. arXiv preprint arXiv:2209.14251 (2022).

[2] Schommer-Pries, Christopher John. The classification of two-dimensional extended topological field theories. University of California, Berkeley, 2009.