Upgrading an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT

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In this note, we show that one can promote an $(n+\varepsilon)$-TQFT to an extended $(n+1)$-TQFT by only specifying the value associated to the $(n+1)$-disk $Z(D^{n+1}) \in Z(S^n)$.

Suppose we are given an $(n+\varepsilon)$-TQFT Z, that is, it assigns a category $Z(N)$ to a closed $(n-1)$-manifold N, and a functor $Z(M) : Z(N) \to Z(N')$ to an n-dimensional cobordisms $M : N \to N'$ between $(n-1)$-manifolds.

TODO perhaps comment on requirements on Z, e.g. a natural isom for $M \simeq M'$, especially for $Z(M' \circ M) \simeq Z(M') \circ Z(M)$ that is consistent. Or say, at this point, no assumption on existence of adjointness of functors $Z(M) \dashv Z(\overline{M})$.

The empty k-manifold is denoted by $\varnothing^k$. Composition of cobordisms is written from right to left, so composition of $M : N \to N'$ and $M' : N' \to N''$ is denoted by $M' \circ M : N \to N''$.

2-Cob denotes the bicategory with closed $(n-1)$-manifolds as objects, n-dimensional cobordisms as 1-morphisms, and $(n+1)$-dimensional relative cobordisms as 2-morphisms.

Proposition 0.1. *Consider functors $Z(D^n) : Z(\varnothing^{n-1}) \rightleftharpoons Z(S^{n-1}) : Z(\overline{D^n})$.*

Let $\eta_0 : Z(\varnothing^n) \Rightarrow Z(S^n = \overline{D^n} \circ D^n) : Z(\varnothing^{n-1}) \to Z(\varnothing^{n-1})$ be a natural transformation, and suppose it is the unit to an adjunction $Z(D^n) \dashv Z(\overline{D^n})$.

Then if Z', Z'' are extended $(n+1)$-TQFTs such that Z', Z'' agree with Z on $(n-1)$- and n-manifolds, and $Z'(D^{n+1}) = Z''(D^{n+1}) = \eta_0$, then $Z' \cong Z''$.

Proof. From the topology section below, we know that a 2-morphism of 2-Cob that realizes the attachment of an $(n+1)$-dim $(k+1)$-handle,$0 \le k \le n$, is determined by some counit ε_k of an adjunction, whose corresponding unit can be built from handles of index at most k, and thus, the value of extensions Z' of Z on 2-morphisms is completely determined by its value on the $(n+1)$-dim 0-handle, which is exactly η_0. □

0.1 Adjunctions from topology

In 2-Cob, the n-dimensional cobordisms $M : N \rightleftharpoons N' : \overline{M}$ form an adjunction.

Let us first consider a simple case, which is the main setting in Proposition 0.1. Consider n-dim cobordisms $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. This can be promoted to an adjunction $D^n \dashv \overline{D^n}$ with the following unit and counit 2-morphisms: the unit is given by the $(n+1)$-disk D^{n+1} :

$\varnothing^n \Rightarrow (\overline{D^n} \circ D^n) = S^n$, and the counit is given by the $(n+1)$-disk which, as a manifold with corner $S^0 \times S^{n-1}$, is a 2-morphism $D^{n+1} = I \times D^n : D^n \circ \overline{D^n} \Rightarrow I \times S^{n-1} = \mathrm{id}_{S^{n-1}}$. This is easily checked to be an adjunction, the unit is an $(n+1)$-dimensional 0-handle, and the counit is attaching an $(n+1)$-dimensional 1-handle to $D^n \sqcup \overline{D^n}$ (see Figure 1 for $n = 1$ case).

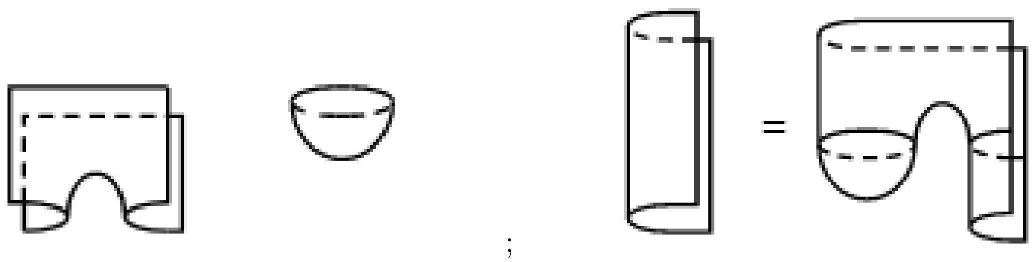

Figure 1: Counit and unit for adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, for $n = 1$, along with one of the snake equations; relative cobordism goes up (stolen from [2], Figure 1.6 and 1.10, get rotated)

Now consider $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k, i.e. it is obtained from $N \times I$ by attaching a k-handle. Then $\overline{M}$ is the dual elementary cobordism which is of index $n - k$. [See Figure 2]

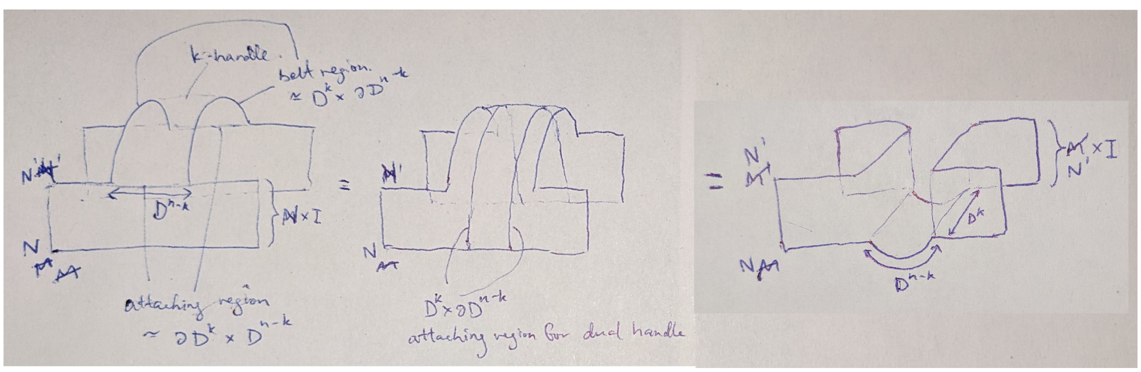

Figure 2: $M : N \to N'$ is an elementary cobordism of index k; it is built from attaching a disk D^n to $N \times I$, with attaching region $\partial D^k \times D^{n-k}$. It can also be built from the other direction, by attaching a disk D^n to $N' \times I$, with attaching region $D^k \times \partial D^{n-k}$. Thus, turning it upside-down, i.e. treated as a cobordism $\overline{M} : N' \to N$, it is an elementary cobordism of index $(n - k)$.

[See Figure 3] We construct the counit $\varepsilon : M \circ \overline{M} \Rightarrow \mathrm{id}_{N'} : N' \to N'$ by attaching an $(n+1)$-dimensional $(k+1)$-handle to $M \cup_{N'} \overline{M}$, with attaching region being essentially the k-handle in M plus the $(n-k)$-handle in $\overline{M}$; the attaching sphere is the union of the core of the k-handle in M with the co-core of the $(n-k)$-handle in $\overline{M}$. Similarly, we construct the unit $\eta : \mathrm{id}_N \Rightarrow M \circ \overline{M} : N \to N$ by attaching an $(n+1)$-dimensional k-handle to $\mathrm{id}_N = N \times I$; the attaching region for this $(n+1)$-dim k-handle is (the attaching region for the n-dim k-handle that defines $M) \times I$. The snake equations $\mathrm{id}_M = (\varepsilon \circ M) \cdot (M \circ \eta) : M \Rightarrow M \circ \overline{M} \circ M \Rightarrow M$ and $\mathrm{id}_{\overline{M}} = (\overline{M} \circ \eta) \cdot (\eta \circ \overline{M}) : \overline{M} \Rightarrow \overline{M} \circ M \circ \overline{M} \Rightarrow \overline{M}$ follow from the fact that these $(n+1)$-dim handles form a cancelling pair in both cases.

Here we have $M \dashv \overline{M}$, but we may very well have $\overline{M} \dashv M$; the counit $\varepsilon' : \overline{M} \circ M \Rightarrow \mathrm{id}_N$ is an $(n+1)$-dim elementary cobordism of index $(n-k+1)$. It is interesting to note that this counit is the dual cobordism to the unit $\eta : \mathrm{id}_N \Rightarrow \overline{M} \circ M$ previously described.

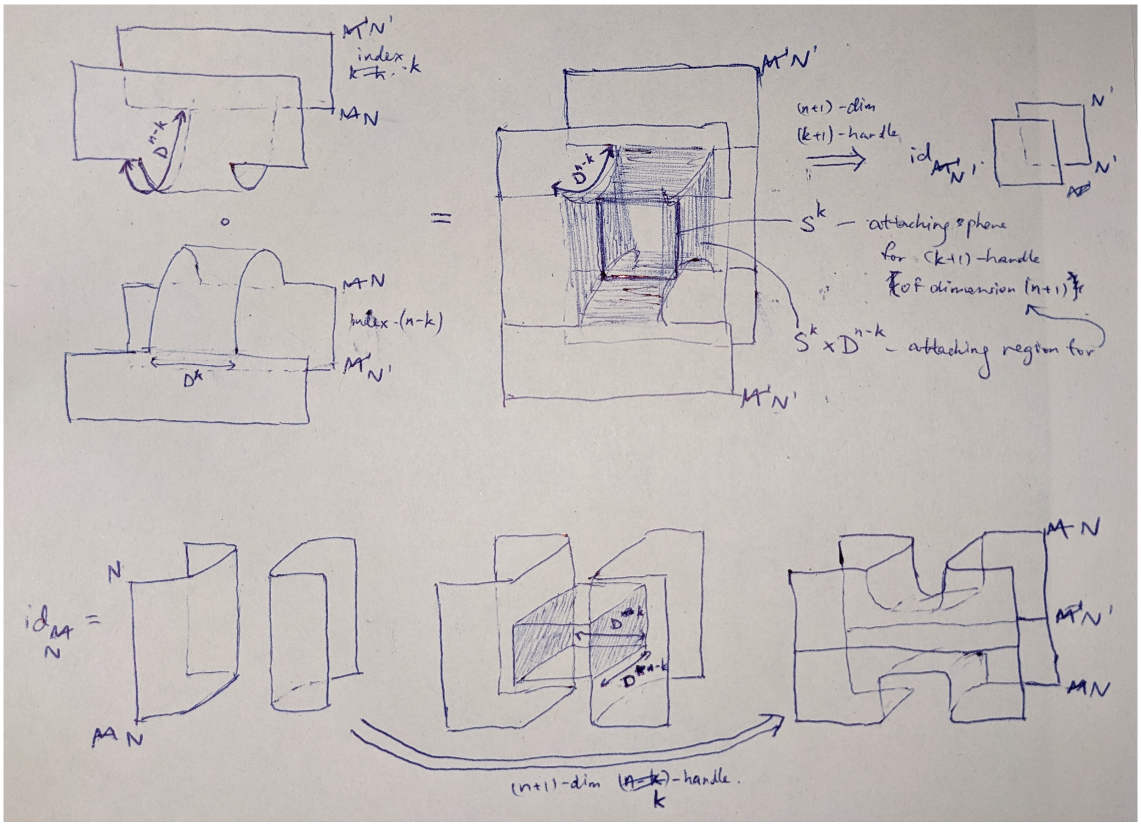

Figure 3: Counit (top) and unit (bottom) for adjunction $M : N \rightleftharpoons N' : \overline{M}$, where M is an elementary cobordism of index k. Note the way N is drawn here looks like N' in Figure 2 and vice versa (by accident, sorry for minor confusion)

In general, we may consider the pair of n-dim cobordisms $M : N \rightleftharpoons N' : \overline{M}$. By presenting M as a composition of elementary cobordisms, we may compose the adjunctions constructed for each of these elementary cobordisms as above, and obtain an adjunction $M \dashv \overline{M}$.

Remark 0.2. In [1], we considered this construction without realizing their connection to these adjunctions; there we consider the more general case where N, N' may have (possibly different) boundary, and $M : N \to N'$ is a relative cobordism (with the boundary cobordism that is not necessarily the identity cobordism).

0.2 Producing $(n+1)$-dim k-handles from some adjunctions

Throughout this section, $0 \le k < n$.

We show how to construct the $(n+1)$-dim $(k+1)$-handle from the counit ε_k of the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and the unit η_0 of the adjunction $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$. (The 0-handle is already given by η_0, while the $(n+1)$-handle is the counit to the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$; we say a few more words about this at the end of this section.)

The process of attaching a $(k+1)$-handle to an $(n+1)$-manifold can be implemented as postcomposing by a 2-morphism. More precisely, given an $(n+1)$-manifold W presented as a 2-morphism $W : M \Rightarrow M' : N \to N'$, and an attaching region $S^k \times D^{n-k}$ in M', the $(n+1)$-manifold W' obtained from attaching a $(k+1)$-handle along the specified attaching region may be considered as a 2-morphism $W' : M \Rightarrow M'' : N \to N'$, where M'' is obtain from M' by performing surgery along the attaching region (cutting it out and gluing in $D^{k+1} \times S^{n-k-1}$); then $W' = \omega_{k+1} \cdot W$, where ω_{k+1} is a 2-morphism that we will describe below.

Our 2-morphism ω_{k+1} is of the form $\omega_{k+1} : S^k \times D^{n-k} \Rightarrow D^{k+1} \times S^{n-k-1} : S^k \times S^{n-k-1} \to \varnothing^{n-1}$. Since this is unchanged as W varies, we clearly need to make some arrangements in order to use ω_{k+1}. More specifically, we need to present M' as a composition

$$(\mathrm{id}_{N'} \sqcup \overline{S^k \times D^{n-k}}) \circ (M \backslash \overline{S^k \times D^{n-k}}) : N \to N' \sqcup S^k \times S^{n-k+1} \to N'$$

which is always possible by basic Morse theory.

Now let us describe how to construct ω_{k+1} out of ε_k and η_0, which are, as a reminder, the counit and unit of the adjunctions $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ and $D^n : \varnothing^{n-1} \rightleftharpoons S^{n-1} : \overline{D^n}$, respectively.

We may consider $S^n : \varnothing^{n-1} \to \varnothing^{n-1}$ as the composition $\overline{D^{k+1} \times S^{n-k-1}} \circ S^k \times D^{n-k} : \varnothing^{n-1} \to S^k \times S^{n-k-1} \to \varnothing^{n-1}$.

Then ω_{k+1} is given by the composition of 2-morphisms

$$\omega_{k+1} = (\mathrm{id}_{\overline{D^{k+1} \times S^{n-k-1}}} \circ \varepsilon_k) \cdot (\mathrm{id}_{\overline{S^k \times D^{n-k}}} \circ \eta_0)$$

$$\begin{tikzcd}
S^k \times S^{n-k-1} \arrow[r, "\overline{S^k \times D^{n-k}}"] \arrow[rr, bend right=60, "\mathrm{id}_{S^k \times S^{n-k-1}}"'] & \varnothing^{n-1} \arrow[r, "S^k \times D^{n-k}"] \arrow[rr, bend left=60, "\varnothing^n"] \arrow[d, phantom, "\Downarrow \varepsilon_k"] & S^k \times S^{n-k-1} \arrow[r, "\overline{D^{k+1} \times S^{n-k-1}}"] \arrow[u, phantom, "\Downarrow \eta_0"] & \varnothing^{n-1}
\end{tikzcd}$$

A few words on the $(n+1)$-handle, more generally the adjunction $\overline{D^n} : S^{n-1} \rightleftharpoons \varnothing^{n-1} : D^n$. The unit is a 2-morphism $\eta : \mathrm{id}_{S^{n-1}} \Rightarrow D^n \circ \overline{D^n}$, which is clearly an elementary cobordism of index n.

A similar phenomenon happens with ε_k, that is, η_k, the unit to the adjunction $S^k \times D^{n-k} \dashv \overline{S^k \times D^{n-k}}$, to which ε_k is the counit, is determined by handles of index at most k, and indeed, $\eta_k = S^k \times D^{n-k+1} : \varnothing^n \Rightarrow S^k \times S^{n-k} : \varnothing^{n-1} \to \varnothing^{n-1}$ is built from a 0-handle and a k-handle.

Thus, since the counit is uniquely determined by the unit, the $(n+1)$-dim k-handle, for $k > 0$, is determined by handles of lower index. This may not be very useful in the topology world, but on the algebraic side of a TQFT, this means that everything is determined by the 0-handle.

[It may be helpful to note that the adjunction $S^k \times D^{n-k} : \varnothing^n \rightleftharpoons S^k \times S^{n-k-1} : \overline{S^k \times D^{n-k}}$ is simply S^k times the first example but with n set to $n-k$, $D^{n-k} : \varnothing^{n-k} \rightleftharpoons S^{n-1-k} : \overline{D^{n-k}}$.]

References

[1] Kwon, Alice, and Ying Hong Tham. The Y-Product. arXiv preprint arXiv:2209.14251 (2022).

[2] Schommer-Pries, Christopher John. The classification of two-dimensional extended topological field theories. University of California, Berkeley, 2009.