The Crane-Yetter Invariant as an Extended TQFT

- $CY(W^4) = K^{\sigma(W)} D^{\chi(W)/2}$ [CY 1993], [CKY 1993], [Roberts 1995], [Ooguri 199
- naturally extends to TQFT
- defined similarly to Turaev-Viro, as state sum.
  - TV extends to codim 2
  - expect CY extends to codim 2 as well.
  - $CY(\Sigma^2)$ = category
- Conjecturally, Reshetikhin-Turaev is boundary theory of CY. (e.g [Barrett et al., 2007])

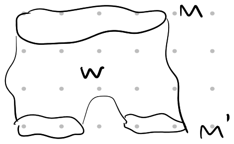
Outline

- briefly describe CY as state sum
- alternative definition in terms of skeins
- signature of 4-mfld, Wall non-additivity.
- Reshetikhin Turaev as boundary theory of CY
- skein categories / category of boundary values

Preliminaries / Background

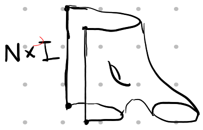
- Topology : PL ($\dim \leq 6$, smooth = PL)

Cobordisms :

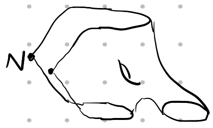


$W : M \to M'$

Relative cobord.



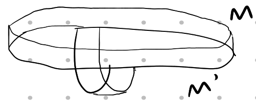
Cornered cobordism (over N)



$W : M \to_N M'$

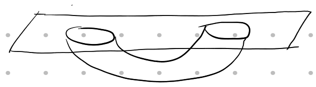
– convention : W : 4-mfld
M : 3-mfld
N : 2-mfld .

- handle decomposition : $W = W_m \circ \cdots \circ W_1$,

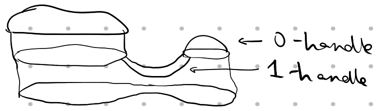
W_i = elementary 

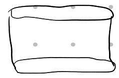
$= \mathcal{H}_k \circ \mathrm{id}_M$

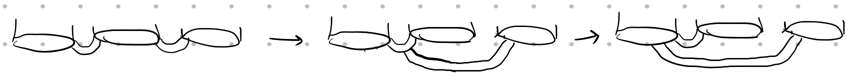
n-dim k-handle as cornered cobord. : $H_k = B^n : \partial B^k \times B^{n-k} \xrightarrow[\partial B^k \times \partial B^{n-k}]{} B^k \times \partial B^{n-k}$

$n=3,\ k=1$: 

- Any cornered cob admits handle decomp.
  - any two related by moves :
    - swap order
    - pair cancellation / creation
    - handle slide



=  = id

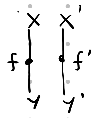


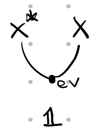
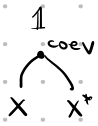
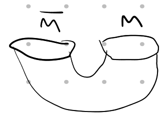
Preliminaries / Background

Algebra : Premodular categories , string diagrams / graphical calculus.

$\mathcal{A}$. eg. Vec, Rep(G), Cob

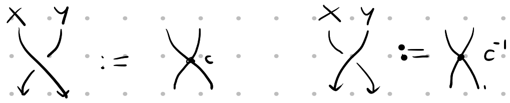
$f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$  or 

monoidal structure $\otimes$: $f' \in \mathrm{Hom}_{\mathcal{A}}(X', Y')$ $f \otimes f'$   ($\sqcup$ in Cob)

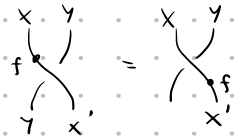
duals X^*, ev, coev  :=  ,  :=  ($\overline{M}$, opposite or't, )

– pivotal structure $\delta : X \xrightarrow{\sim} X^{**}$

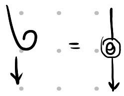
braiding $c : X \otimes Y \xrightarrow{\sim} Y \otimes X$



natural, e.g. for $f : X \to X'$,

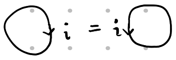


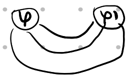
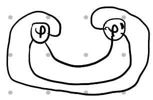
— twist
$\theta : X \xrightarrow{\sim} X$

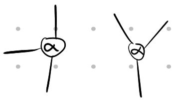
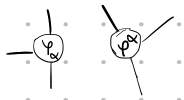
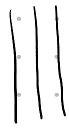
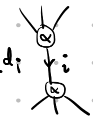


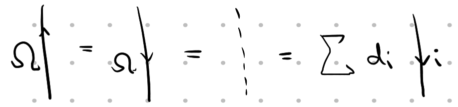
— premodular : all of above
- structures compartible
- semisimple / $\Bbbk = \overline{\Bbbk}$
- finite

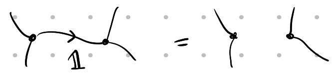
Useful conventions / lemmas :

- representatives X_i for simple objects, $i \in \mathrm{Irr}(\mathcal{A})$
- $d_i =$ 
- $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, V_1 \otimes \cdots \otimes V_k) =: \langle V_1, \dots, V_k \rangle$, $\varphi' \in \langle V_k^*, \dots, V_1^* \rangle$

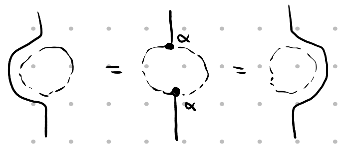
  $ev(\varphi, \varphi') :=$  $=$ 

-  $:= \sum_\alpha$  $\{\varphi_\alpha\}, \{\varphi^\alpha\}$ dual bases  $= \sum_i d_i$ 

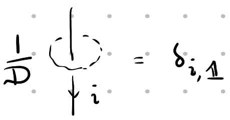


, 

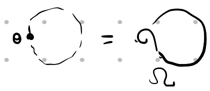
(don't draw $\mathbb{1}$)

Sliding lemma : 

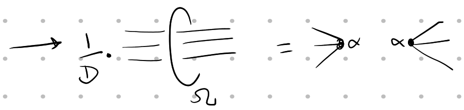
Killing lemma : [A modular]
(aka charge conservation)



$D :=$  $= \sum d_i^2$

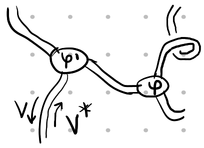
$K :=$ 

$= \sum d_i^2 \theta_i$

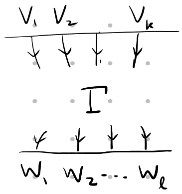


$\mathcal{A}$-colored ribbon graphs, skeins

- $\Gamma \subseteq M^3$, (framed / thickened)



  - directed edges colored w/ objects
  - vertices colored w/ morphisms
  - Theorem [RT 1990] : graph

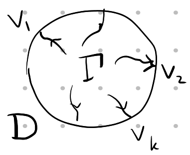


has well-defined evaluation

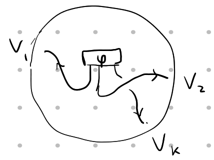
$Z_{RT}(\Gamma) \in \mathrm{Hom}_{\mathcal{A}}(V_1 \otimes \cdots \otimes V_k, W_1 \otimes \cdots \otimes W_\ell) \simeq \langle V_k^*, \dots, V_1^*, W_1, \dots, W_\ell \rangle$

- invariant w.r.t. isotopy.

  - equivalent : $\Gamma \subset D \cong B^3$



$\longmapsto \langle \Gamma \rangle_D = Z_{RT}(\Gamma) \in \langle V_1, \dots, V_k \rangle$



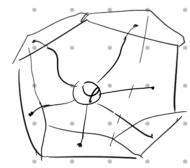
CY – triangulation/PLCW

- label $\ell : \{\text{oriented 2-cells}\} \to \mathrm{Irr}(\mathcal{A})$

$$\ell(\vec{F}) = \ell(\overleftarrow{F})^*$$

- oriented 3-cell $C \rightsquigarrow H(C,\ell)$ – "local state space"

$$H(C,\ell) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \bigotimes_{\vec{F} \in \partial C} \ell(\vec{F})) = \langle \ell(\vec{F}_1), \dots, \ell(\vec{F}_k) \rangle$$



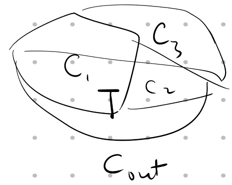
$$H(\bar{C},\ell) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \bigotimes_{\overleftarrow{F} \in \overline{\partial C}} \ell(\overleftarrow{F})) = \langle \ell(\vec{F}_k)^*, \dots, \ell(\vec{F}_1)^* \rangle$$

$$ev : H(C,\ell) \otimes H(\bar{C},\ell) \longrightarrow \mathbb{k}$$

- $\mathcal{M} \rightsquigarrow H(\mathcal{M},\ell) = \bigotimes H(C,\ell)$

- oriented 4-cell $T \longrightarrow Z(T,\ell)$ local invariant

$$Z(T,\ell) \in \bigotimes_{\vec{C} \in \partial T} H(\vec{C},\ell)$$



$$\begin{array}{c} \varphi_1 \qquad\quad \varphi_2 \\ \cap \qquad\quad\;\; \cap \\ H(C_1,\ell) \otimes H(C_2,\ell) \otimes \dots \\ \big\downarrow Z(T,\ell) \\ H(C_{out},\ell) \end{array}$$

$$\mapsto Z_{RT}(\varphi_1 \otimes \varphi_2 \otimes \dots)$$

- W^4, possibly w/ ∂.

$$Z(W,\ell) = ev\Big(\bigotimes_{T \in W} Z(T,\ell) \Big) \in H(\partial W,\ell)$$

$$\in \bigotimes_{\substack{\text{interior} \\ \text{3-cells}}} \big(H(\vec{C},\ell) \otimes H(\overleftarrow{C},\ell)\big) \otimes \bigotimes_{\substack{\text{boundary} \\ \text{3-cell}}} H(\vec{C},\ell)$$

$$Z_{CY}(W) = D^{\chi(\mathring{W}) + \frac{1}{2}\chi(\partial W)} \sum_{\ell} \prod d_{\ell(f)}^{n_f} \, Z(W,\ell) \in H(\partial W) = \bigoplus_{\ell} H(\partial W,\ell)$$

- $H(\mathcal{M})$ too big, depends on triangulation
- $A_{\mathcal{M},\mathcal{M}'} := Z_{CY}(M \times I) : H(\mathcal{M}) \longrightarrow H(\mathcal{M}')$

$$Z_{CY}(M) = \mathrm{im}(A_{\mathcal{M},\mathcal{M}''}) = \mathrm{im}(A_{\mathcal{M}',\mathcal{M}''}) \subseteq H(\mathcal{M}'')$$

Extend to 4-mflds w/ corner

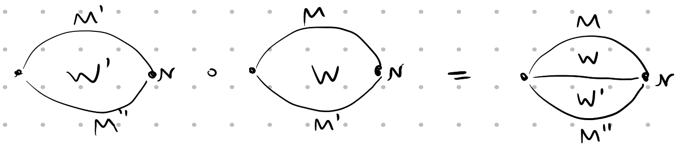
$N \subseteq \partial W$

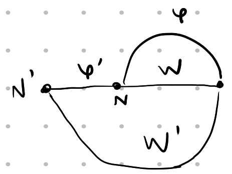
$$Z_{CY}(W;N) = D^{\chi(\mathring{W}) + \frac{1}{2}\chi(\partial W \setminus N)} \sum_{\ell} \prod_{f \notin N} d_{\ell(f)}^{n_f} \, Z(W,\ell) \in H(\partial W)$$

$W : M \xrightarrow[N]{} M'$, $W' : M' \xrightarrow[N]{} M''$

$W' \circ W = W' \cup_{M'} W$

$$Z_{CY}(W' \circ W; N) = Z_{CY}(W';N) \circ Z_{CY}(W;N) : H(M) \longrightarrow H(M'')$$





$$Z_{cy}(W' \cup W ; N') = Z_{cy}(W' ; N') \circ \left(id \otimes Z_{cy}(W ; N) \right)$$

For surfaces :

$$\hat{Z}_{cy}(N) : \begin{cases} \text{Obj} : \bigoplus (\mathcal{N}, \ell) \ , \ \mathcal{N} \text{ triangulation on } N \ , \ \ell \text{ simple labeling} \\ \text{Mor} : \mathrm{Hom}\big((\mathcal{N},\ell),(\mathcal{N}',\ell')\big) = Z_{cy}\left(N \times I \ ; (\mathcal{N},\ell),(\mathcal{N}',\ell')\right) \end{cases}$$

$$Z_{cy}(N) = \mathrm{Kar}\left(\hat{Z}_{cy}(N) \right)$$

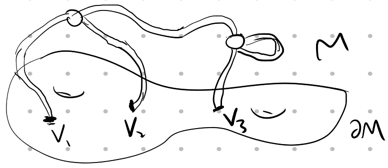
CY - skeins

- $\text{Skeins}(M^3) = \left\{ \begin{array}{l} \text{formal linear combination} \\ \text{of } \mathcal{A}\text{ - colored ribbon graphs} \\ \text{in } M \end{array} \right\} \Big/ \text{null graphs}$

$\left\langle \quad \right\rangle_D = 0$

  - null graph w.r.t. $D \cong B^3 \subseteq M$
    - $\Gamma = \sum c_j \Gamma_j$,
      - Γ_i agree outside D
      - $\langle \sum c_j \Gamma_j \rangle_D = \sum c_j \langle \Gamma_j \rangle_D = 0$

- Γ defines <u>boundary value</u> on ∂M
  - $\mathbb{V} = (B, \{V_{\vec{b}}\})$



- $Z^{sk}_{\mathcal{C}\mathcal{Y}}(M;\mathbb{V}) = \{\text{Skeins in } M \text{ w/ } \partial\text{-value } \mathbb{V}\} \subseteq \mathrm{Skein}(M)$

For surface N, $\hat{Z}^{sk}_{\mathcal{C}\mathcal{Y}}(N) = \begin{cases} \text{Obj : boundary values } \mathbb{V} = (B, \{V_{\vec{b}}\}) \\ \text{Mor : } \mathrm{Hom}(\mathbb{V}, \mathbb{V}') = Z^{sk}_{\mathcal{C}\mathcal{Y}}(N \times I; \mathbb{V}^*, \mathbb{V}') \end{cases}$

$Z^{sk}_{\mathcal{C}\mathcal{Y}}(N) = \mathrm{Kar}(\hat{Z}^{sk}_{\mathcal{C}\mathcal{Y}}(N))$ Karoubi envelope / idempotent completion

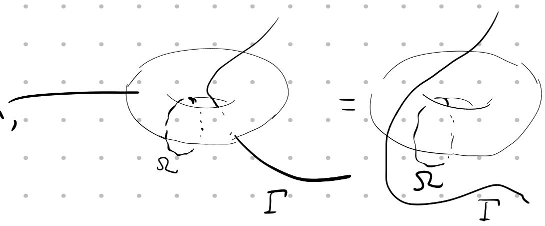
- $\mathbb{E}$:= empty configuration $= (B = \emptyset, \emptyset)$
- $\emptyset^{sk}_M$ = empty graph $\in Z^{sk}_{\mathcal{C}\mathcal{Y}}(M;\mathbb{E})$

- $W: M \xrightarrow[N]{} M'$, handle decomp $W = W_\ell \circ \dots \circ W_1$
- define $Z^{sk}_{CY}(W)$ for elementary cobordisms
  - case index k-handle:
    - $k = 0$: $Z^{sk}_{CY}(W; N)(\Gamma) := \mathcal{D} \cdot \Gamma \cup \emptyset^{sk}_{S^3}$
    - $k = 4$: $Z^{sk}_{CY}(W; N)(\Gamma' \cup \Gamma'') := Z_{RT}(\Gamma'') \cdot \Gamma'$ (Γ'' in S^3)
    - $k = 1$: isotope Γ to avoid attaching region, then, treating Γ as graph in new mfld,
      $Z^{sk}_{CY}(W; N)(\Gamma) = \mathcal{D}^{-1} \cdot \Gamma$
    - $k = 2, 3$
- inv under handle moves — $Z^{sk}_{CY}(W; N)$ indep of handle decomp.

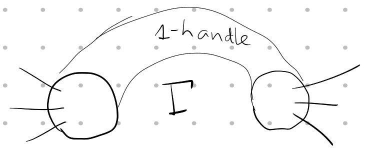
• $k = 2$: isotope Γ to avoid attaching region, then, treating Γ as graph in new mfld,

$$Z^{sk}_{\mathcal{C}}(W;N)(\Gamma) = \Gamma \cup \Gamma' ,$$

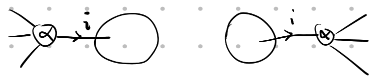
where Γ' = meridian loop of attaching region, colored by Ω.



• $k = 3$:



[add 3-handle = remove 1-handle] by sliding lemma.

$= \sum d_i$  $\xmapsto{Z^{sk}_{\mathcal{C}}(W;N)}$ 

[balls are now filled in]

Theorem : $Z_{\mathcal{C}} = Z^{sk}_{\mathcal{C}}$

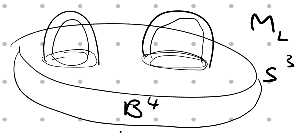
- eg. for 3-mflds, essentially just forget triangulation, get skein
  ↪ 2-mflds follows easil'
- for 4-mflds, check they agree on handles.

Example

L = framed link in S^3

W_L = attach 2-handle to each component

$M_L = \partial W_L$ = 3-mfld obtained from S^3 under surgery along L



$$\overline{M_L} \xrightarrow{W_L} \emptyset$$

- $W_L : \emptyset \xrightarrow{\text{0-handle}} S^3 \xrightarrow[\text{along } L]{\text{2-handles}} M_L$
- $W_L : \overline{M_L} \xrightarrow[\text{dual 2-handles}]{\text{attach}} \overline{S^3} \xrightarrow{\text{4-handle}} \emptyset$

  - <u>meridians</u> of attaching region of dual 2-handles are exactly the <u>attaching spheres</u> of original 2-handles (= link components)

$$\text{So}\quad \begin{tikzcd} \phi^{sk}_{\overline{M}_L} \arrow[r, mapsto] & \underset{\text{in } \overline{S^3}}{\Omega L} \arrow[r, mapsto, "\text{4-handle}"'] \arrow[d, leftrightarrow] & Z_{RT}(\Omega \overline{L}) \\ & \underset{\text{in } S^3}{\Omega \overline{L}} \arrow[ur, mapsto, bend right, "\text{4\_handle}"'] & \end{tikzcd}$$

– essentially gives the Reshetikhin–Turaev invariant of M_L : [$\mathcal{A}$ modular]

$$Z_{RT}(M_L) = K^{-\sigma(L)} \cdot D^{(-|L|-1)/2} \cdot Z_{RT}(\Omega \overline{L})$$

usually ΩL, convention issue

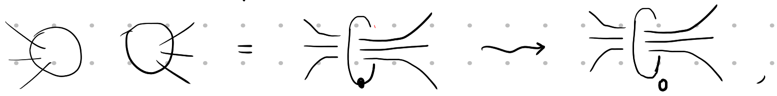
$$\rightarrow \quad Z^{sk}_{CY}(W_L)\left(\frac{1}{Z_{RT}(M_L)} \cdot \phi^{sk}_{\overline{M}_L}\right) = K^{\sigma(W_L)} \cdot D^{\chi(W_L)/2}$$

Theorem $\mathcal{A}$ modular. Define $\varepsilon_M = \frac{1}{Z_{RT}(\overline{M})} \cdot \phi_M^{sk} \in Z_{CY}^{sk}(M) \cong \Bbbk$.

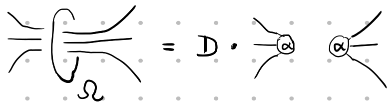
For cobord. $W : M \to M'$,

$$Z_{CY}^{sk}(W)(\varepsilon_M) = K^{\sigma(W)} \cdot D^{\chi(W)/2} \cdot \varepsilon_{M'}$$

- uses fact that Z_{CY}^{sk}(closed 3-mfld) is 1-dim
- key observation to generalize example: trade 1-handle for appropriate 2-handle in dual handle decomp,



use killing lemma



[Akbulut, 1977], [Gompf, Stipsicz, 1999]

$$Z^{sk}_{CY}(W' \cup W) = Z^{sk}_{CY}(W') \circ Z^{sk}_{CY}(W)$$

reflects additivity of signature & Euler characteristic

$$\sigma(W \cup W') = \sigma(W) + \sigma(W') \quad \text{(Novikov additivity)}$$

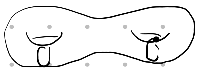
$$\chi(W \cup W') = \chi(W) + \chi(W') \; - \cancel{\chi(W \cap W')}^{\,0}$$

[Wall, 1969] : when glue 4-mflds along 3-mfld w/ ∂, signature doesn't add :

$$W : M_1 \xrightarrow[N]{} M_2 \,, \quad W' : M_2 \xrightarrow[N]{} M_3$$

$$\sigma(W \cup W') = \sigma(W) + \sigma(W') - \sigma(V; L_1, L_2, L_3)$$

- $V = H_1(N; \mathbb{R})$, $L_j = \ker(H_1(N; \mathbb{R}) \longrightarrow H_1(M_j; \mathbb{R}))$

Let Ξ = "full multicurve" on N 

$$P_\Xi := D^{-g}\cdot\Omega\Xi \in \mathrm{End}_{Z(N)}(\mathbb{E})$$

$$(\mathbb{E}, P_\Xi) \in Z^{sk}_{CY}(N)$$

For M w/ $\partial M = N$,

$M_\Xi := M \cup$ handlebody

(glue so Ξ is null-htpic)

$$\varepsilon_{M,\Xi} := \frac{1}{Z_{RT}(\overline{M_\Xi})}\cdot P_\Xi \circ \phi^{sk}_M$$

$$\left(Z^{sk}_{CY}(M;(\mathbb{E},P_\Xi)) \cong \Bbbk\right)$$

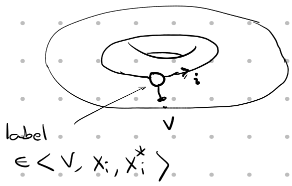
Theorem

$$Z^{sk}_{CY}(W;N)(\varepsilon_{M_1,\Xi}) = K^{\sigma(W)-\sigma(V;L_1,L_2,[\Xi])}\cdot D^{(\chi(W)-(1-g))/2}\cdot \varepsilon_{M_2,\Xi}$$

Relation to Reshetikhin–Turaev

$$Z^{sk}_{cy}(H;\mathbb{V}) \cong \bigoplus_{i_1,\dots,i_g} \langle \underbrace{V_1,\dots,V_k}_{\mathbb{V}},\ X_{i_1}, X^*_{i_1}, \dots, X_{i_g}, X^*_{i_g} \rangle \cong Z_{RT}(\partial H;\mathbb{V})$$

e.g. $g=1$,

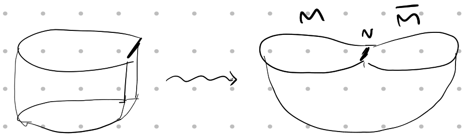


RT should be an extended 3-2-1-TQFT.
- has "anomaly"
- fix "framing": for every closed surface N, choose handlebody H_N w/ identification $\partial H_N \simeq N$

$$Z^{cy}_{RT}(N,\mathbb{V}) := Z_{cy}(H_N;\mathbb{V})$$

For M, $\partial M = N$, consider

$$M \times I : M \cup_N \overline{M} \longrightarrow \emptyset$$



(forget corner)

- defines skein pairing

$$ev^{sk} : Z^{sk}_{CY}(M; \mathbb{V}) \otimes Z^{sk}_{CY}(\overline{M}; \mathbb{V}^*) \longrightarrow \mathbb{k}$$

$$\varphi \otimes \varphi' \longmapsto Z^{sk}_{CY}(M \times I)(\varphi \cup \varphi')$$

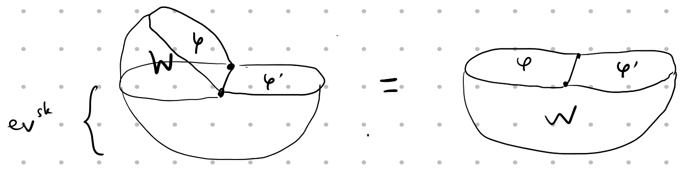
- Prop : skein pairing is non-degenerate.

$$\rightarrow \quad Z^{CY}_{RT}(\overline{N}, \mathbb{V}^*) \simeq Z^{CY}_{RT}(N, \mathbb{V})^*$$

$W : M \to_N M'$ $\xleftrightarrow{\text{skein pairing}}$ $W : M \cup_N \overline{M'} \to \emptyset$

· For $\varphi \in Z^{sk}_{CY}(M; \mathbb{V})$, $\varphi' \in Z^{sk}_{CY}(M'; \mathbb{V})$,

$$ev^{sk}\left(Z^{sk}_{CY}(W;N)(\varphi),\ \varphi'\right) = Z^{sk}_{CY}(W)(\varphi \cup \varphi')$$



For 3-mfld, as cobord. $M : N \to N'$,

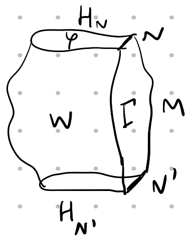
w/ $\mathcal{A}$-colored Γ in M w/ ∂-vals $\mathbb{V}^*, \mathbb{V}'$,

$$Z^{\mathcal{A}}_{RT}(M, \Gamma) : Z^{\mathcal{A}}_{RT}(N, \mathbb{V}) \longrightarrow Z^{\mathcal{A}}_{RT}(N', \mathbb{V}')$$

$$\varphi \longmapsto K^{-\sigma(W)} \cdot D^{-\chi(W)/2} \quad Z^{sk}_{\mathcal{A}}(W; N')(\Gamma \cup \varphi)$$

where W is some 4-mfld

$$\partial W = \overline{H_N} \cup_N M \cup_{N'} H_{N'}$$



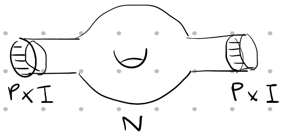
Theorem: Z_{RT}^{CY} agrees w/ RT.

- clear from taking $W = W_L$
- skein pairing $\longrightarrow$ gluing axiom.

Skein categories $Z^{sk}_{CY}(N)$

(see [Cooke, 2019], [Walker 2006]) [Kirillov-Tham, 2020]

- excision.



- glue collar nghd $\longrightarrow N'$

  - "bimodule over $P\times I$"

  - $Z^{sk}_{CY}(P\times I) \boxtimes Z^{sk}_{CY}(N) \boxtimes Z^{sk}_{CY}(P\times I) \longrightarrow Z^{sk}_{CY}(N)$

Theorem: $$\mathcal{Z}_{Z(P\times I)}\left(Z^{sk}_{CY}(N)\right) \simeq Z^{sk}_{CY}(N')$$

Sketch of pf :

- for $\mathcal{C}$-bimodule cat $\mathcal{M}$, center :

[Gelaki et. al., 2009] $\quad \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) = \begin{cases} \text{Obj}: (X, \gamma), \ X \in \text{Obj}\,\mathcal{M}, \ \gamma \text{ half-braiding on } X \\ \text{Mor}: f: (X,\gamma) \to (X', \gamma') \text{ intertwines half-braidings} \end{cases}$

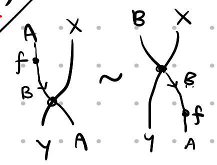
(e.g. $\mathcal{A}$ as $\mathcal{A}$-bimod, $\mathcal{Z}_{\mathcal{A}}(\mathcal{A})$ is Drinfeld center)

- horizontal trace :

$$hTr_{\mathcal{C}}(\mathcal{M}) = \begin{cases} \text{Obj}: \text{ same as } \mathcal{M} \\ \text{Mor}: \text{Hom}_{hTr}(X,Y) = \bigoplus_{A \in \mathcal{C}} \text{Hom}_{\mathcal{M}}(A \triangleright X, Y \triangleleft A) \Big/ \end{cases}$$

[Beliakova et. al, 2017]

$$= \int^{A} \text{Hom}(A \triangleright X, Y \triangleleft A)$$



for $f: B \to A$

Lemma: $\mathcal{C}$ pivotal multifusion

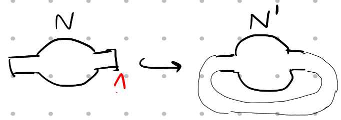
$$\mathrm{Hom}_{hTr(\mathcal{M})}(X, Y) \cong \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{M}}(X_i \triangleright X, Y \triangleleft X_i)$$

Prop: $\mathcal{C}$ pivotal multifusion, $\mathcal{M}$ finite s.s,

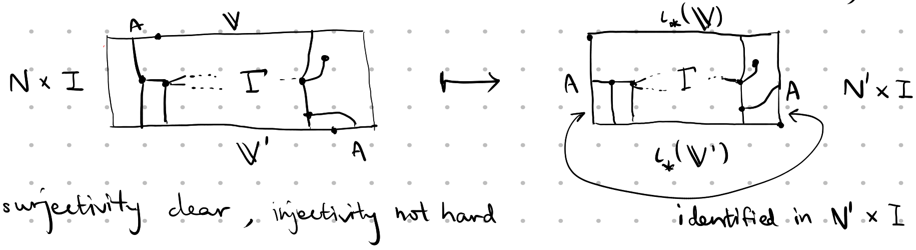
$$Z_{\mathcal{C}}(\mathcal{M}) \simeq \mathrm{Kar}(hTr_{\mathcal{C}}(\mathcal{M}))$$

$h\mathrm{Tr}(\iota_*) : h\mathrm{Tr}_{\hat{Z}(P\times I)}(\hat{Z}(N)) \simeq \hat{Z}(N')$ $\qquad [\hat{Z} = \hat{Z}^{sk}_{CY}]$

inclusion $\iota : N \hookrightarrow N'$ induces functor $\iota_* : \hat{Z}(N) \to \hat{Z}(N')$



- essentially surjective: clear
- extends to $h\mathrm{Tr}$: for $\Gamma \in \mathrm{Hom}(A \triangleright \mathbb{V}, \mathbb{V}' \triangleleft A)$, $A \in \hat{Z}(P\times I)$, $\mathbb{V}, \mathbb{V}' \in \hat{Z}(N)$,



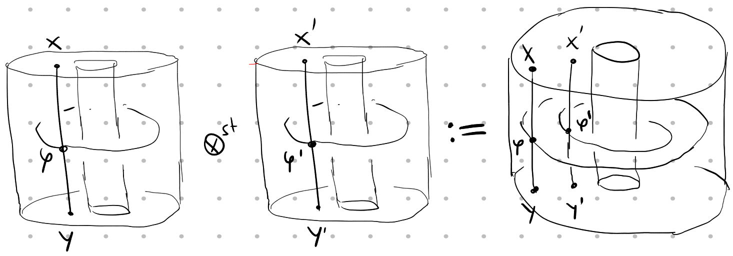
- surjectivity clear, injectivity not hard

$\square$

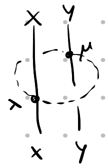
- "gluing allows more graphs ⟶ half-braiding"

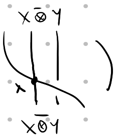
$Z^{sk}_{\mathcal{A}}(\mathrm{Ann}) \simeq \mathcal{Z}(\mathcal{A})$

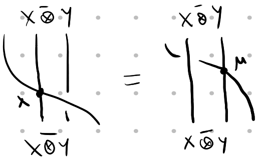
- as monoidal categories? not for usual $\otimes$ on $\mathcal{Z}(\mathcal{A})$!
- $\otimes^{st}$ on $Z(\mathrm{Ann})$ given by



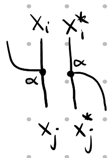
Reduced tensor product on $Z(A)$ [Tham, 2020], [Wasserman, 2020]

$$X \bar{\otimes}_{\lambda,\mu} Y := \mathrm{im}\Big(Q_{\lambda,\mu} := \frac{1}{D}$$  $$\Big)$$

$$(X,\lambda)\ \bar{\otimes}\ (Y,\mu) := \Big(X \bar{\otimes}_{\lambda,\mu} Y,$$  $$\Big)$$

$\Big($  $\Big)$

— unit object : $\bar{\mathbb{1}} = \left(\bigoplus X_i X_i^*, \Gamma \right)$

$\Gamma = \sum_{i,j} \sqrt{d_i}\sqrt{d_j}$ 

Theorem:
$$\left(\mathcal{Z}(\mathcal{A}), \bar{\otimes} \right) \simeq Z^{sk}_{Cy}(\mathrm{Ann}) \quad \text{as pivotal multifusion categories}$$

- for $\mathcal{A}$ modular, $\quad \mathcal{Z}(\mathcal{A}) \simeq \mathcal{A} \boxtimes \mathcal{A}^{bop}$

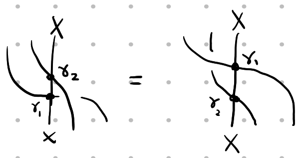
$$(X_i \boxtimes X_j^*) \;\bar{\otimes}\; (X_k \boxtimes X_\ell^*) = \begin{cases} X_i \boxtimes X_\ell^* & j = k \\ 0 & j \neq k \end{cases}$$

- for $\mathcal{A} = \mathrm{Rep}(G)$, $\quad \mathcal{Z}(\mathcal{A}) \simeq \{ G\text{-equivariant bundles over } G \}$

$\bar{\otimes}$ = fibrewise tensor product

Other surfaces [KT, 2020]

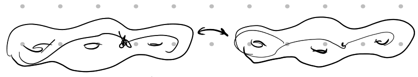
- $Z^{sk}_{Cy}(B^2) \simeq \mathcal{A}$
- $Z^{sk}_{Cy}(S^2) \simeq \mathcal{Z}^{M\ddot{u}}(\mathcal{A})$
- $Z^{sk}_{Cy}(T^2 \setminus pt.) \simeq \mathcal{Z}^{el}(\mathcal{A}) := \begin{cases} \text{Obj}: (X, \gamma_1, \gamma_2) \text{ s.t.} \\ \text{Mor}: f: (X,\gamma_1,\gamma_2) \to (X', \gamma_1', \gamma_2') \text{ intertwine } \gamma_1, \gamma_1', \ \gamma_2, \gamma_2' \end{cases}$



[Tham 2019]

- $\left(Z^{sk}_{Cy}(Ann), \otimes^{st}\right) \simeq \left(\mathcal{Z}(\mathcal{A}), \overline{\otimes}\right)$
- $\mathcal{A}$ modular : $Z^{sk}_{Cy}\left(\substack{\text{closed}\\ \text{surface}}\right) \simeq \text{Vec}$

$\Rightarrow Z^{sk}_{Cy}\left(\substack{\text{closed}\\ \text{3-mfld}}\right) \simeq \Bbbk$



Heegaard

Thank You!