Extended Crane-Yetter via Skeins

- $CY(W^4) = K^{\sigma(W)} D^{\chi(W)/2}$ [CY 1993], [CKY 1993], [Roberts 1995], [Ooguri 1993]
- naturally extends to TQFT
- defined similarly to Turaev-Viro, as state sum.
  - TV extends to codim 2
  - expect CY extends to codim 2 as well.
  - $CY(\Sigma^2)$ = category
- Conjecturally, Reshetikhin-Turaev is boundary theory of CY. (e.g [Barrett et al., 2007])

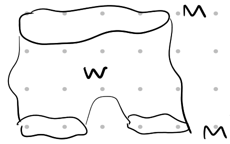
Outline

- briefly describe CY as state sum
- alternative definition in terms of skeins
- Reshetikhin Turaev as boundary theory of CY

Preliminaries / Background

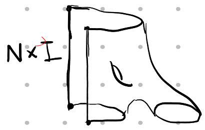
- Topology : PL (dim ≤ 6, smooth = PL)

Cobordisms :

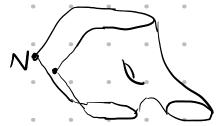


$W : M \longrightarrow M'$

Relative cobord.



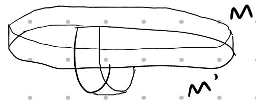
Cornered cobordism (over N)



$W : M \longrightarrow_N M'$.

– convention : W : 4-mfld
M : 3-mfld
N : 2-mfld .

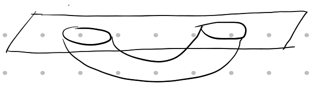
— handle decomposition : $W = W_m \circ \cdots \circ W_1$,

W_i = elementary 

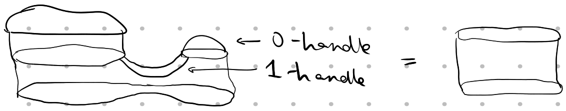
$= \mathcal{H}_k \circ id_M$

n-dim k-handle as cornered cobord. : $H_k = B^n : \partial B^k \times B^{n-k} \xrightarrow[\partial B^k \times \partial B^{n-k}]{} B^k \times \partial B^{n-k}$

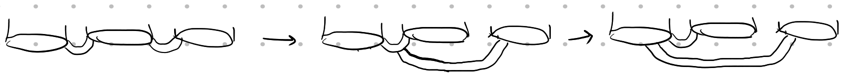
$n=3$, $k=1$:



- Any cornered cob admits handle decomp.
  - any two related by moves :
    - swap order
    - pair cancellation / creation
    - handle slide



$= id$



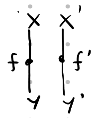
Preliminaries / Background

Algebra : Premodular categories , string diagrams / graphical calculus.

$\mathcal{A}$. eg. Vec, Rep(G), Cob

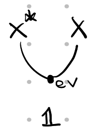
$f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ 

or 

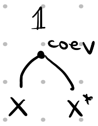
monoidal structure $\otimes$: $f' \in \mathrm{Hom}_{\mathcal{A}}(X', Y')$ $f \otimes f'$ 

($\sqcup$ in Cob)

duals X^*, ev, coev 

$:=$ 

, 

$:=$ 

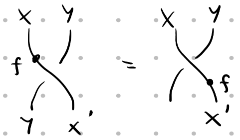
($\overline{M}$, opposite or't, 

)

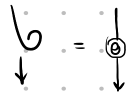
- pivotal structure $\delta : X \xrightarrow{\sim} X^{**}$

braiding $c : X \otimes Y \xrightarrow{\sim} Y \otimes X$

natural, e.g. for $f : X \to X'$,



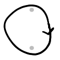
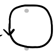
— twist
$\theta : X \xrightarrow{\sim} X$

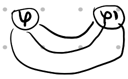
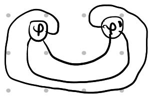


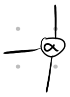
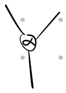
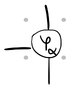
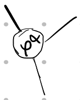
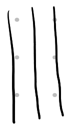
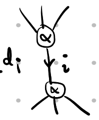
— premodular : all of above

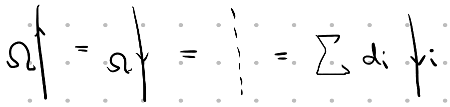
- structures compatible
- semisimple / $\mathbb{k} = \overline{\mathbb{k}}$
- finite

Useful conventions / lemmas :

- representatives X_i for simple objects, $i \in \mathrm{Irr}(\mathcal{A})$
- $d_i =$  $i =$ i 
- $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, V_1 \otimes \cdots \otimes V_k) =: \langle V_1, \dots, V_k \rangle$, $\varphi' \in \langle V_k^*, \dots, V_1^* \rangle$

  $ev(\varphi, \varphi') :=$  $=$ 

-   $:= \sum_\alpha$   $\{\varphi_\alpha\}, \{\varphi^\alpha\}$ dual bases  $= \sum_i d_i$ 



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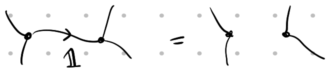
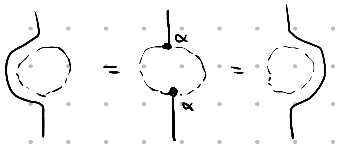


(don't draw $\mathbb{1}$)

Sliding lemma:

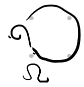


Killing lemma : [A modular]
(aka charge conservation)

$\frac{1}{D}$ 

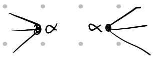
$= \delta_{i,\mathbb{1}}$

$D :=$  $= \sum d_i^2$

$K := \theta \cdot$  $=$ 

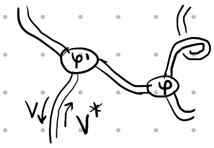
$= \sum d_i^2 \theta_i$

$\rightarrow \frac{1}{D} \cdot$ 

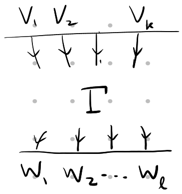
$=$ 

$\mathcal{A}$-colored ribbon graphs, skeins

- $\Gamma \subseteq M^3$, (framed / thickened)



  - directed edges colored w/ objects
  - vertices colored w/ morphisms
  - Theorem [RT 1990]: graph

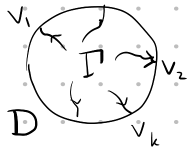


has well-defined evaluation

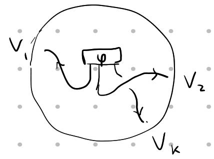
$Z_{RT}(\Gamma) \in \mathrm{Hom}_{\mathcal{A}}(V_1 \otimes \cdots \otimes V_k, W_1 \otimes \cdots \otimes W_l) \simeq \langle V_k^*, \dots, V_1^*, W_1, \dots, W_l \rangle$

- invariant w.r.t. isotopy.

  - equivalent: $\Gamma \subset D \cong B^3$



$\mapsto \langle \Gamma \rangle_D = Z_{RT}(\Gamma) \in \langle V_1, \dots, V_k \rangle$



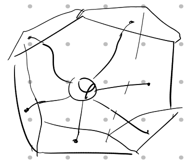
CY - triangulation/PLCW

- label $\ell : \{\text{oriented 2-cells}\} \to \mathrm{Irr}(\mathcal{A})$

$$\ell(\vec{F}) = \ell(\overleftarrow{F})^*$$

- oriented 3-cell $C \rightsquigarrow H(C,\ell)$ - "local state space"

$$H(C,\ell) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \bigotimes_{\vec{F} \in \partial C} \ell(\vec{F})) = \langle \ell(\vec{F}_1), \dots, \ell(\vec{F}_k) \rangle$$



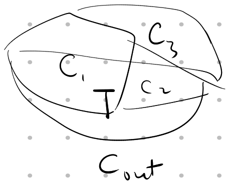
$$H(\bar{C},\ell) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, \bigotimes_{\overleftarrow{F} \in \overline{\partial C}} \ell(\overleftarrow{F})) = \langle \ell(\vec{F}_k)^*, \dots, \ell(\vec{F}_1)^* \rangle$$

$$ev : H(C,\ell) \otimes H(\bar{C},\ell) \longrightarrow \mathbb{k}$$

- $\mathcal{M} \rightsquigarrow H(\mathcal{M},\ell) = \bigotimes H(C,\ell)$

- oriented 4-cell $T \longrightarrow Z(T,\ell)$ local invariant

$$Z(T,\ell) \in \bigotimes_{\vec{c} \in \partial T} H(\vec{c},\ell)$$



$$\begin{array}{c} \varphi_1 \quad\quad\quad \varphi_2 \\ \cap \quad\quad\quad\;\; \cap \\ H(C_1,\ell) \otimes H(C_2,\ell) \otimes \dots \\ \big\downarrow Z(T,\ell) \\ H(C_{out},\ell) \end{array}$$

$$\mapsto Z_{RT}(\varphi_1 \otimes \varphi_2 \otimes \dots)$$

- W^4, possibly w/ ∂.

$$Z(W,\ell) = ev\Big(\bigotimes_{T \in W} Z(T,\ell) \Big) \in H(\partial W,\ell)$$

$$\in \bigotimes_{\substack{\text{interior} \\ \text{3-cells}}} \big(H(\vec{c},\ell) \otimes H(\overleftarrow{c},\ell)\big) \otimes \bigotimes_{\substack{\text{boundary} \\ \text{3-cell}}} H(\vec{c},\ell)$$

$$Z_{cy}(W) = \mathcal{D}^{\chi(\mathring{W}) + \frac{1}{2}\chi(\partial W)} \sum_{\ell} \prod d_{\ell(f)}^{n_f} \, Z(W,\ell) \in H(\partial W) = \bigoplus_{\ell} H(\partial W,\ell)$$

$$Z_{CY}(W) = \mathcal{D}^{\chi(\mathring{W}) + \frac{1}{2}\chi(\partial W)} \sum_{\ell} \prod d_{\ell(f)}^{n_f} \, Z(W, \ell) \in H(\partial W) = \bigoplus_{\ell} H(\partial W, \ell)$$

Extend to 4-mflds w/ corner

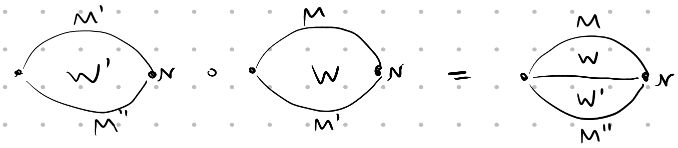
$N \subseteq \partial W$

$$Z_{CY}(W; N) = \mathcal{D}^{\chi(\mathring{W}) + \frac{1}{2}\chi(\partial W \setminus N)} \sum_{\ell} \prod_{f \notin N} d_{\ell(f)}^{n_f} \, Z(W, \ell) \in H(\partial W)$$

$W : M \xrightarrow[N]{} M'$, $W' : M' \xrightarrow[N]{} M''$

$W' \circ W = W' \cup_{M'} W$

$$Z_{CY}(W' \circ W; N) = Z_{CY}(W'; N) \circ Z_{CY}(W; N) : H(M) \longrightarrow H(M'')$$



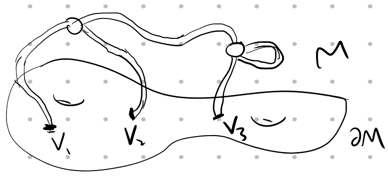
CY - skeins

- $\text{Skeins}(M^3) = \left\{ \begin{matrix} \text{formal linear combination} \\ \text{of } \mathcal{A}\text{-colored ribbon graphs} \\ \text{in } M \end{matrix} \right\} \Big/ \text{null graphs}$

$\Big\langle \quad \Big\rangle_D = 0$

- null graph w.r.t. $D \cong B^3 \subseteq M$
  - $\Gamma = \sum c_j \Gamma_j$, $\cdot$ Γ_i agree outside D
    $\cdot$ $\langle \sum c_j \Gamma_j \rangle_D = \sum c_j \langle \Gamma_j \rangle_D = 0$

- Γ defines boundary value on ∂M
  - $\mathbb{V} = (B, \{V_{\vec{b}}\})$



- $Z^{sk}_{CY}(M;\mathbb{V}) = \{\text{Skeins in } M \text{ w/ } \partial\text{-value } \mathbb{V}\} \subseteq \text{Skein}(M)$

For surface N, $\hat{Z}^{sk}_{CY}(N) = \begin{cases} \text{Obj} : \text{boundary values } \mathbb{V} = (B, \{V_{\vec{b}}\}) \\ \text{Mor} : \text{Hom}(\mathbb{V}, \mathbb{V}') = Z^{sk}_{CY}(N \times I ; \mathbb{V}^*, \mathbb{V}') \end{cases}$

$Z^{sk}_{CY}(N) = \text{Kar}(\hat{Z}^{sk}_{CY}(N))$ Karoubi envelope / idempotent completion

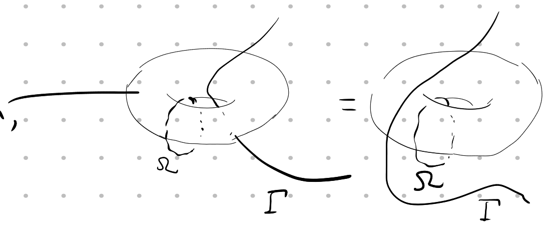
- $\mathbb{E}$:= empty configuration $= (B = \emptyset, \emptyset)$
- $\emptyset^{sk}_M$ = empty graph $\in Z^{sk}_{CY}(M;\mathbb{E})$

- $W : M \xrightarrow[N]{} M'$, handle decomp $W = W_\ell \circ \dots \circ W_1$
- define $Z^{sk}_{CY}(W)$ for elementary cobordisms
  - case index k-handle:
    - $k = 0$: $Z^{sk}_{CY}(W;N)(\Gamma) := \mathcal{D} \cdot \Gamma \cup \emptyset^{sk}_{S^3}$
    - $k = 4$: $Z^{sk}_{CY}(W;N)(\Gamma' \cup \Gamma'') := Z_{RT}(\Gamma'') \cdot \Gamma'$
      (Γ'' in S^3)
    - $k = 1$: isotope Γ to avoid attaching region, then, treating Γ as graph in new mfld,
      $Z^{sk}_{CY}(W;N)(\Gamma) = \mathcal{D}^{-1} \cdot \Gamma$
    - $k = 2, 3$
- inv under handle moves — $Z^{sk}_{CY}(W;N)$ indep of handle decomp.

• $k = 2$: isotope Γ to avoid attaching region, then, treating Γ as graph in new mfld,

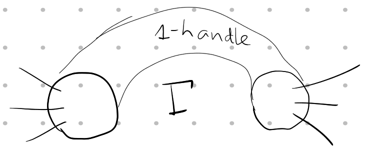
$$Z^{sk}_{\mathcal{C}_Y}(W;N)(\Gamma) = \Gamma \cup \Gamma' ,$$

where Γ' = meridian loop of attaching region, colored by Ω.



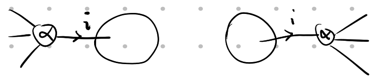
by sliding lemma.

• $k = 3$:



[add 3-handle = remove 1-handle]

$= \sum d_i$



$\xmapsto{Z^{sk}_{\mathcal{C}_Y}(W;N)}$

[balls are now filled in]

Theorem : $Z_{CY} = Z_{CY}^{sk}$

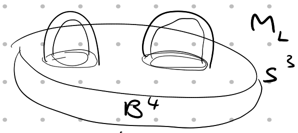
- eg. for 3-mflds, essentially just forget triangulation, get skein
  - ↪ 2-mflds follows easi'l
- for 4-mflds, check they agree on handles.

Example

L = framed link in S^3

W_L = attach 2-handle to each component

$M_L = \partial W_L$ = 3-mfld obtained from S^3 under surgery along L



$$\begin{array}{c} \overline{M_L} \\ \big\downarrow W_L \\ \emptyset \end{array}$$

- $W_L : \emptyset \xrightarrow[\text{0-handle}]{} S^3 \xrightarrow[\text{2-handles along } L]{} M_L$
- $W_L : \overline{M_L} \xrightarrow[\text{attach dual 2-handles}]{} \overline{S^3} \xrightarrow[\text{4-handle}]{} \emptyset$

  - meridians of attaching region of dual 2-handles are exactly the attaching spheres of original 2-handles (= link components)

So

$$\begin{tikzcd} \phi^{sk}_{\overline{M}_L} \arrow[r, mapsto] & \underset{\text{in } \overline{S^3}}{\Omega L} \arrow[r, mapsto, "\text{4-handle}"'] \arrow[d, leftrightarrow] & Z_{RT}(\Omega \overline{L}) \\ & \underset{\text{in } S^3}{\Omega \overline{L}} \arrow[ur, mapsto, bend right, "\text{4-handle}"'] & \end{tikzcd}$$

– essentially gives the Reshetikhin–Turaev invariant of M_L : [A modular]

$$Z_{RT}(M_L) = K^{-\sigma(L)} \cdot D^{(-|L|-1)/2} \cdot Z_{RT}(\Omega \overline{L})$$

usually ΩL, convention issue

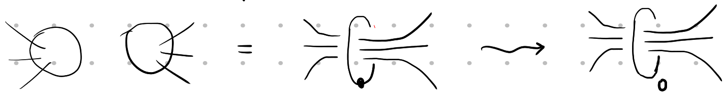
$$\to \quad Z^{sk}_{CY}(W_L)\left(\frac{1}{Z_{RT}(M_L)} \cdot \phi^{sk}_{\overline{M}_L}\right) = K^{\sigma(W_L)} \cdot D^{\chi(W_L)/2}$$

Theorem $\mathcal{A}$ modular. Define $\varepsilon_M = \frac{1}{Z_{RT}(\overline{M})} \cdot \phi^{sk}_M \in Z^{sk}_{CY}(M) \cong \Bbbk$.

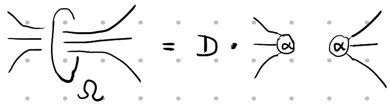
For cobord. $W: M \to M'$,

$$Z^{sk}_{CY}(W)(\varepsilon_M) = K^{\sigma(W)} \cdot D^{\chi(W)/2} \cdot \varepsilon_{M'}$$

- uses fact that Z^{sk}_{CY}(closed 3-mfld) is 1-dim
- key observation to generalize example: trade 1-handle for appropriate 2-handle in dual handle decomp,



use killing lemma

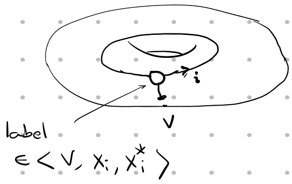


[Akbulut, 1977], [Gompf, Stipsicz, 1999]

Relation to Reshetikhin-Turaev

$$Z^{sk}_{cy}(H;\mathbb{V}) \cong \bigoplus_{i_1,\dots,i_g} \langle \underbrace{V_1,\dots,V_k}_{\mathbb{V}},\ X_{i_1}, X^*_{i_1}, \dots, X_{i_g}, X^*_{i_g} \rangle \cong Z_{RT}(\partial H;\mathbb{V})$$

e.g. $g=1$,



RT should be an extended 3-2-1-TQFT.
- has "anomaly"
- fix "framing": for every closed surface N, choose handlebody H_N w/ identification $\partial H_N \simeq N$

$$Z^{cy}_{RT}(N,\mathbb{V}) := Z_{cy}(H_N;\mathbb{V})$$

For 3-mfld, as cobord. $M : N \longrightarrow N'$,

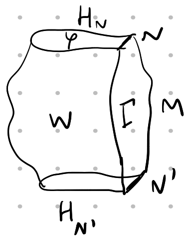
w/ $\mathcal{A}$-colored Γ in M w/ ∂-vals $\mathbb{V}^*, \mathbb{V}'$,

$$Z^{\mathcal{C}}_{RT}(M, \Gamma) : Z^{\mathcal{C}}_{RT}(N, \mathbb{V}) \longrightarrow Z^{\mathcal{C}}_{RT}(N', \mathbb{V}')$$

$$\varphi \longmapsto K^{-\sigma(W)} \cdot D^{-\chi(W)/2} \quad Z^{sk}_{\mathcal{C}}(W; N')(\Gamma \cup \varphi)$$

where W is some 4-mfld

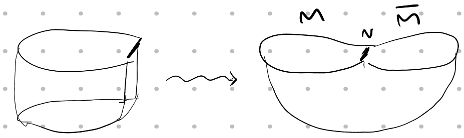
$$\partial \overline{W} = \overline{H}_N \underset{N}{\cup} M \underset{N'}{\cup} H_{N'}$$



Theorem: Z^{CY}_{RT} agrees w/ RT.

- clear from taking $W = W_L$
- skein pairing $\longrightarrow$ gluing axiom.

For M, $\partial M = N$, consider



$$M \times I : M \cup_N \overline{M} \longrightarrow \emptyset$$

(forget corner)

- defines skein pairing

$$ev^{sk} : Z_{cy}^{sk}(M; \mathbb{V}) \otimes Z_{cy}^{sk}(\overline{M}; \mathbb{V}^*) \longrightarrow \mathbb{k}$$

$$\varphi \otimes \varphi' \longmapsto Z_{cy}^{sk}(M \times I)(\varphi \cup \varphi')$$

- Prop : skein pairing is non-degenerate.

$$\rightarrow \quad Z_{RT}^{cy}(\overline{N}, \mathbb{V}^*) \simeq Z_{RT}^{cy}(N, \mathbb{V})^*$$

Thank You!