# Problem Set 4

Due date: June 22, 2015, before lecture

**Instructions:** Please hand in a single PDF-file containing all your answers and results. Show the names of the group members on top. Make use of figures and tables and always provide a short interpretation of your results. Include the source code in the appendix. The code should be well documented and readable.

## Problem 1: Interpolation of Simple Function

Read the chapter on interpolation in the book by Miranda and Fackler (2002) to get acquainted with their toolbox.

- Interpolate the function
$$f(x) = \frac{1}{1+25x^2}$$
on $x \in [-1, 1]$ using Chebychev polynomials for $n$ equidistant nodes. Increase $n$ from 5 to 15 and plot the residual by comparing the solution with the solution for some large $n$. What do you find?
*Hint:* MF assigns Chebychev nodes to Chebychev polynomials. To combine Chebychev polynomials with equidistant nodes, you
  - define the function space for Chebychev polynomials, `fspace`
  - define a vector of equidistant nodes, `x`
  - calculate the matrix of basis functions, `B`, at these nodes by `B=funbas(fspace,x)`.
  - you next calculate the function values at `x`
  - and finally get the polynomial coefficients by `c=B\y`.
- Repeat the exercise using Chebychev nodes. What do you find?

## Problem 2: Simple Optimization Problem

We now study a simple optimization problem. A household chooses consumption in two periods, $t = 0, 1$ to maximize felicity,

$$\max_{C_0, C_1} \{u(C_0) + \mathbb{E}u(C_1)\}$$

subject to the constraint that

$$\begin{aligned} W_1 &= W_0(1+r) - C_0 \\ W_0 &> 0 \text{ given.} \end{aligned}$$

$r$ is again distributed with two atoms, $r_{\min}, r^{\max}$, with probabilities $p$ and $1-p$. Since waisting resources will be not be an optimal solution, $C_1 = W_1$ and the problem simplifies to

$$\max_{C_0} \{u(C_0) + \mathbb{E}u(W_0(1+r) - C_0)\}.$$

- Let utility be given by the quadratic utility function, $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$. Write down the optimal consumption rule in period 0 to show that optimal consumption obeys certainty equivalence.

- What happens to $C_0^\star$ if the variance of $r$ increases? Does this make economic sense?
- Now, use the *CRRA* utility function, $u(c_t) = \frac{1}{1-\gamma}(c_t)^{1-\gamma}$ where $\gamma$ is the coefficient of relative risk aversion. Verify that optimal consumption is characterized by the equation

$$C_0 = \mathbb{E}\left[(W_0(1+r) - C_0)^{-\gamma}\right]^{-1/\gamma}.$$

Intuitively explain what happens to $C_0$ if the variance of $r$ increases?

- Define a grid of $W$ for $W^{\min} = 0.5$ and $W^{\max} = 50$ and set $r^{\min} = -0.08$, $r^{\max} = 0.12$, $p = 0.5$ and $\gamma = 2$. Use a Chebychev approximation with $n = 15$ interpolation nodes. Plot the solution for the linear function against the Chebychev approximation and calculate the maximum percentage error of the deviation.
- What happens to the deviation if (i) you increase $\gamma$, (ii) decrease $p$, (iii) increase the spread between $r^{\min}$ and $r^{\max}$ while preserving the mean? Notice that the linear solution is unaffected throughout.

## Problem 3: Portfolio Choice

Assume that we have a bunch of agents which differ with respect to their degree of risk aversion, $\gamma_i$. The problem of each agent $i$ is to chose $\alpha_i$ solving

$$\begin{aligned}
&\max_{\alpha} \mathbb{E}\left\{\frac{1}{1-\gamma_i}\cdot\left(1 + r^f + \alpha(r - r^f)\right)^{1-\gamma_i}\right\}\\
&\text{s.t.}\\
&0 \leq \alpha \leq 1.
\end{aligned}$$

- Show that the optimal portfolio share for each agent $i$ is the solution to

$$\mathbb{E}\left[\left(1 + r^f + \alpha(r - r^f)\right)^{-\gamma_i}\left(r - r^f\right)\right].$$

- Solve the problem for the parametrization of problem 3 and $r^f = 0.02$, $\gamma_i \in [\gamma_{\min}, \gamma^{\max}]$ for $\gamma_{\min} = 1$ and $\gamma^{\max} = 50$ using Chebychev interpolation with $n = 15$ nodes. Plot $\alpha$ against $\gamma$. What do you observe in the neighborhood of the point where the constraint on $\alpha^{\max}$ is binding?
- Now calculate the value of $\gamma$ where the constraint becomes just binding at $\alpha = 1$ by solving

$$\mathbb{E}\left[(1+r))^{1-\bar{\gamma}}\right].$$

for $\bar{\gamma}$. Define a grid for $\gamma \in [\bar{\gamma}, \gamma^{\max}]$ and solve the problem again using splines. Compare your solution to what you found above. Is this any better?

## Problem 4: Policy Function Approximation in the Neoclassical Growth Model

Consider the textbook neoclassical growth model. There is a representative household that maximizes her lifetime utility subject to a budget constraint, specifically

$$\begin{aligned}
&\max_{\{C_t, K_{t+1}\}} \left\{U(\{C_t\}) = \sum_{t=0}^{\infty}\beta^t \ln C_t\right\}\\
&\text{s.t.}\quad C_t + K_{t+1} = (1 + r_t - \delta)K_t + w_t
\end{aligned}$$

Suppose the production sector is competitive and uses a neoclassical production function in capital and labor to produce the all-purpose good. The optimization problem of the representative firm reads

$$\max_{K_t, L_t} \left\{ K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t \right\}$$

For simplicity assume $L_t = 1, \forall t$. Let $\beta = 0.96, \alpha = 0.36$ and $\delta = 0.06$.

1. Derive the first-order conditions of the household's and firm's optimization problem for given interest and wage rate.

2. Impose market clearing and derive the steady state ($C_t = C_{t+1}$).

3. Let $K^*$ denote the steady state capital stock. Construct a capital grid with $m = 9$ grid points between $0.5K^*$ and $1.5K^*$ using Chebyshev nodes.

4. Approximate the consumption policy on the capital grid by a Chebyshev polynomial of degree $n = m - 1$,

$$\tilde{C}(K) \approx \sum_{i=0}^{n} \theta_i T_i(K).$$

   Use the Euler equation to derive conditions for $\theta = \{\theta_i\}_{i=0}^{n}$. Compute $\theta$.

5. Plot your approximation of the consumption policy, $\tilde{C}(K)$, on 100 equidistant points on your capital grid.

6. Simulate the path of capital, $K_t$, for 100 periods, starting with $K_1 = 0.5K^*$. Plot the values over time. What do you observe?

7. Given your approximation of the consumption policy, compute the relative Euler equation error. I.e., for 1000 random draws on your capital grid, compute

$$E(K;\theta) = \frac{R(K;\theta)}{\tilde{C}(K)},$$

   where $R(K;\theta)$ denotes the residual of the Euler equation defined in terms of consumption, e.g., $R(K;\theta) = \tilde{C}\left(K'\left(\tilde{C}(K;\theta)\right);\theta\right) - \beta(1 + r' - \delta)\tilde{C}(K;\theta)$ with primes denoting next period variables. Report the maximum and average absolute errors.

# References

Miranda, M. J. and P. L. Fackler (2002): *Applied Computational Economics and Finance*, Cambridge: MIT Press.