COMP 250
INTRODUCTION TO COMPUTER SCIENCE

Lecture 20 – Recursion 2 (Binary Search)

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FROM LAST CLASS

- Some examples of recursive algorithms

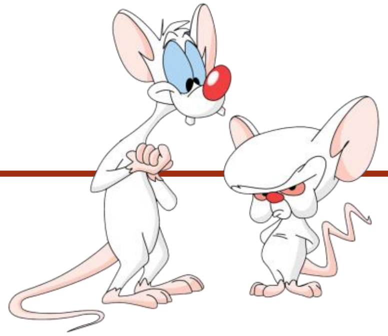
REVERSING A LIST (RECURSIVE)

```
public static void reverse(List list) {
    if(list.size()==1) {
        return;
    }
    firstElement = list.remove(0); // remove first element
    reverse(list); // now the list has n-1 elements
    list.add(firstElement); // appends at the end of the list
}
```

EXAMPLE 5 – SORTING A LIST (RECURSIVE)

```
public static void sort(List list) {
   if(list.size()==1) {
      return;
   }
   minElement = removeMinElement(list);
   sort(list); // now the list has n-1 elements
   list.add(0, minElement); // insert at the beginning of list
}
```

WHAT ARE WE GOING TO DO TODAY?



- More recursive algorithms
- Binary Search

RECALL: DECIMAL TO BINARY (ITERATIVE)

ALGORITHM
Constructing Base 2 Expansions

procedure $BinaryExpansion(n)$
$k := 0$
While $m > 0$
    $a_k := n\%2$
    n$:= n/2$
    $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$

Recall that a decimal number n requires approximately $\log_2 n$ bits for its binary representation.

DECIMAL TO BINARY (RECURSIVE)

ALGORITHM
Constructing Base 2 Expansions

procedure *BinaryExpansion*(n)
If $n > 0$:
    BinaryExpansion$(n/2)$
    print$(n\%2)$

Also in this case, there are $\log_2 n$ recursive calls

POWER (x^n) – ITERATIVE

Let x a positive integer and let n be a positive number.

x has some number of bits e.g. 32.

```
power(x, n) {
   result =1;
   for(int i=1; i<=n; i++) {
      result = result *x;
   }
   return result;
}
```

POWER (x^n) – RECURSIVE

```
power(x, n) {
   if(n==0) {
      return 1;
   } else {
      return x*power(x,n-1);
   }
}
```

POWER() – CAN WE DO BETTER?

More interesting approach using recursion:

$$x^{18} = x^9 * x^9$$

$$x^9 = x^4 * x^4 * x$$

$$x^4 = x^2 * x^2$$

POWER() – CAN WE DO BETTER?

```
power( x, n) {
   if (n == 0)
      return 1;
   else if (n == 1)
      return x;
   else{
      tmp = power(x, n/2);
         if (n%2==0)
            return tmp*tmp;     // one multiplication
         else
            return tmp*tmp*x    // two multiplications
   }
}
```

A SIMILAR IDEA CAN BE IMPLEMENTED ITERATIVELY

IDEA: Let's use the binary expansion of n, say $n = (a_{k-1}, \dots, a_1, a_0)_2$.

Note that:

$$x^n = x^{a_{k-1}2^{k-1} + \cdots + a_1 2 + a_0} = x^{a_{k-1}2^{k-1}} \cdots x^{a_1 2} \cdot x^{a_0}$$

This shows how to compute x^n: we only need to compute the values of $x, x^2, (x^2)^2 = x^4, \dots, x^{2^k}$. Once we have these terms we multiply the terms x^{2^j}, where $a_j = 1$.

EXAMPLE: x^{243}

$$n = (243)_{10} = (11110011)_2$$

Q: How many multiplications do we need?

EXAMPLE: x^{243}

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Q: How many multiplications do we need?

A: Recursive method: 5*2 + 2*1 = 12.
Iterative method: 7 + 7 = 14

The highest order bit in the recursive method is the base case, and doesn't require a multiplication.
The lowest order bit in the iterative method does not require multiplication.

EXAMPLE: x^{243}

$$n = (243)_{10} = (11110011)_2$$

Q: How many multiplications do we need?

A: $O(\log_2 n)$

OBSERVATIONS

The second approach we looked at uses fewer multiplications than the first one, and thus the second approach *seems faster.*

Q: Is this indeed the case?

A: No. Why not ?

OBSERVATIONS

Hint: Let x be a positive integer with M digits.

- x^2 has about ? digits.
- x^3 has about ? digits.
- $\vdots$
- x^n has about ? digits.

OBSERVATIONS

Hint: Let x be a positive integer with M digits.

- x^2 has about 2M digits.
- x^3 has about 3M digits.
- :
- x^n has about $n * \mathrm{M}$ digits.

We cannot assume that multiplication takes 'constant' time.

Taking large powers gives very large numbers and multiplications becomes more expensive.

BINARY SEARCH

SEARCHING A LIST

- Goal: find a given element in a list.
- Solution: go through all the elements in the list and check whether the element is there (*linear search*).
- Could we do this any faster if the list was sorted to begin with?

Think of how you search for a term in an index. Do you start at the beginning and then scan through to the end? (No.)

BINARY SEARCH

- Inputs:
  - A sorted list.
  - The element we are looking for (the *key*)
- IDEA: First compare the key with the element in the middle of the list
  - If the key is less than the middle element, we only need to search the first half of the list, so we continue searching on this smaller list.
  - If the key is greater than the middle element, we only need to search the second half of the list, so we continue searching on this smaller list.
  - If the key equals the middle element, we have a match – return its index.

http://www.oxfordmathcenter.com/drupal7/node/31

EXAMPLE

- Search for 25

-4	1	5	6	14	23	31	35	52	70

EXAMPLE

- Search for 25
- Look at the middle element and compare

-4	1	5	6	14	23	31	35	52	70

EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half

-4	1	5	6	14	23	31	35	52	70

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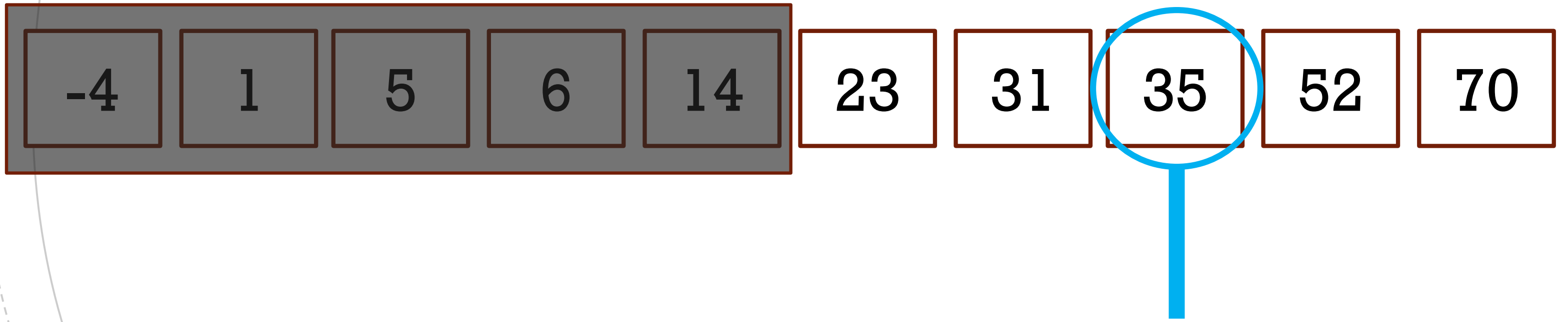

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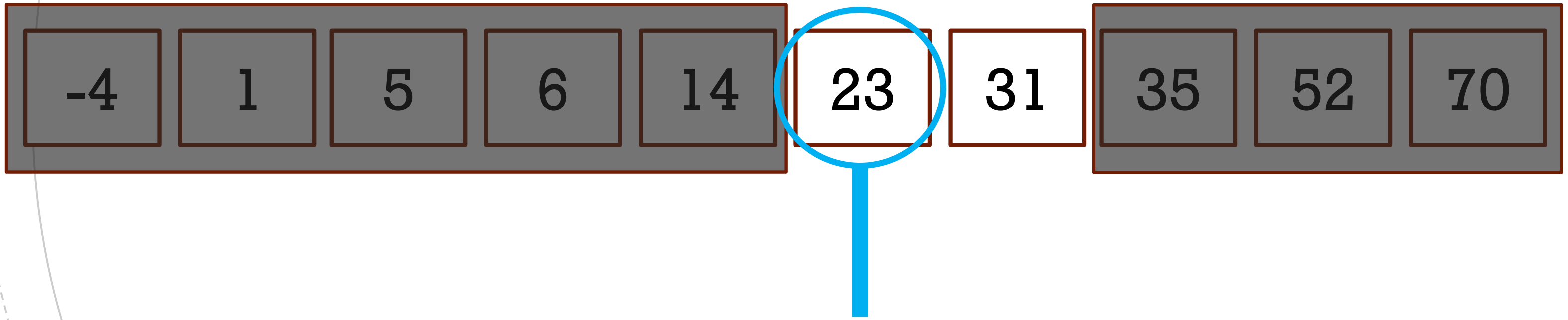

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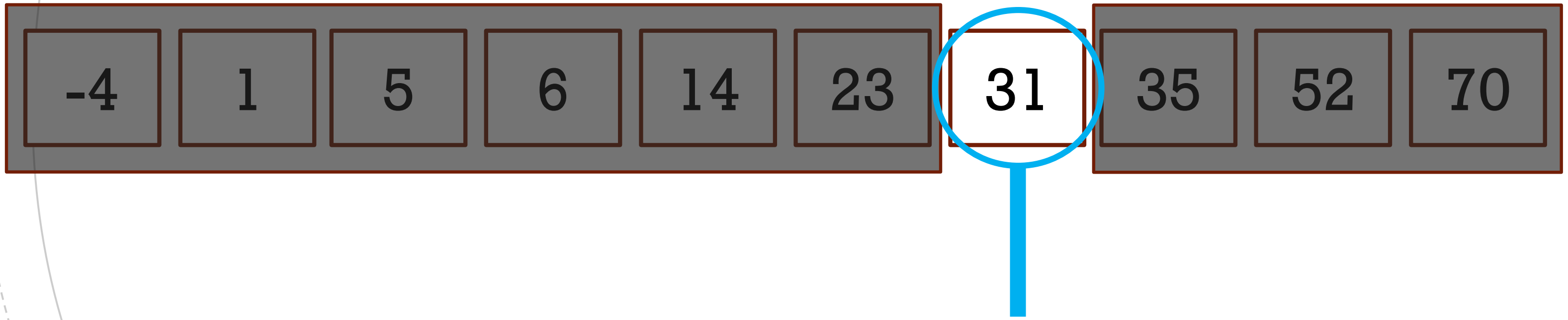

EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half

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EXAMPLE

- Search for 25
- There are no more elements in the list → the element is not there! Return -1.

-4	1	5	6	14	23	31	35	52	70

IMPLEMENT BINARY SEARCH

- Idea: keep track of the left and right indices denoting the section of the list that needs to be searched.
- What is the index of the element that we compare to the key as a function of the left and right indices?

BACK TO EXAMPLE

- Search for 25 (initialize left and right)

-4	1	5	6	14	23	31	35	52	70

left = 0

right = 9

right = size -1

BACK TO EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)

mid = (left+right)/2

mid = 4

-4	1	5	6	14	23	31	35	52	70

left = 0

right = 9

BACK TO EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update left)

mid = 4

-4	1	5	6	14	23	31	35	52	70

left = 5

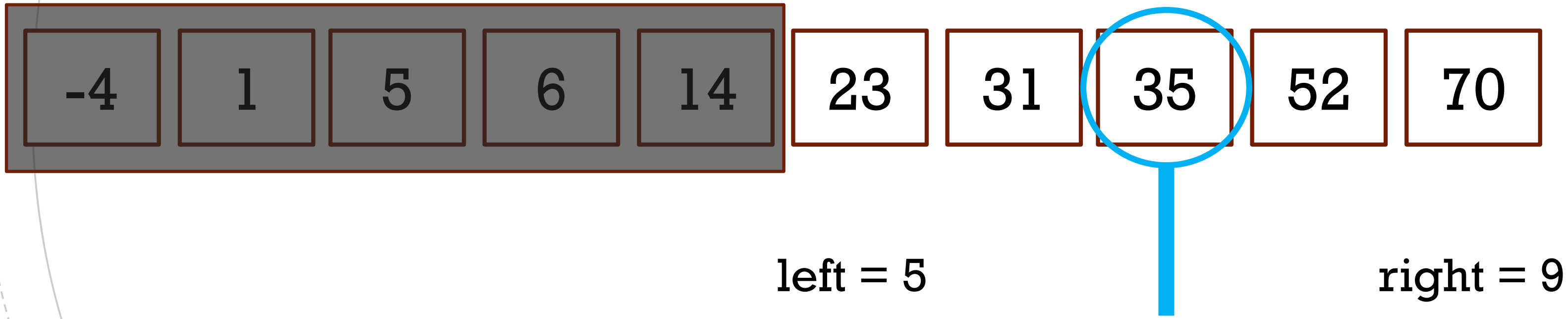
right = 9

left = mid +1

EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)

mid = (left+right)/2
= 7

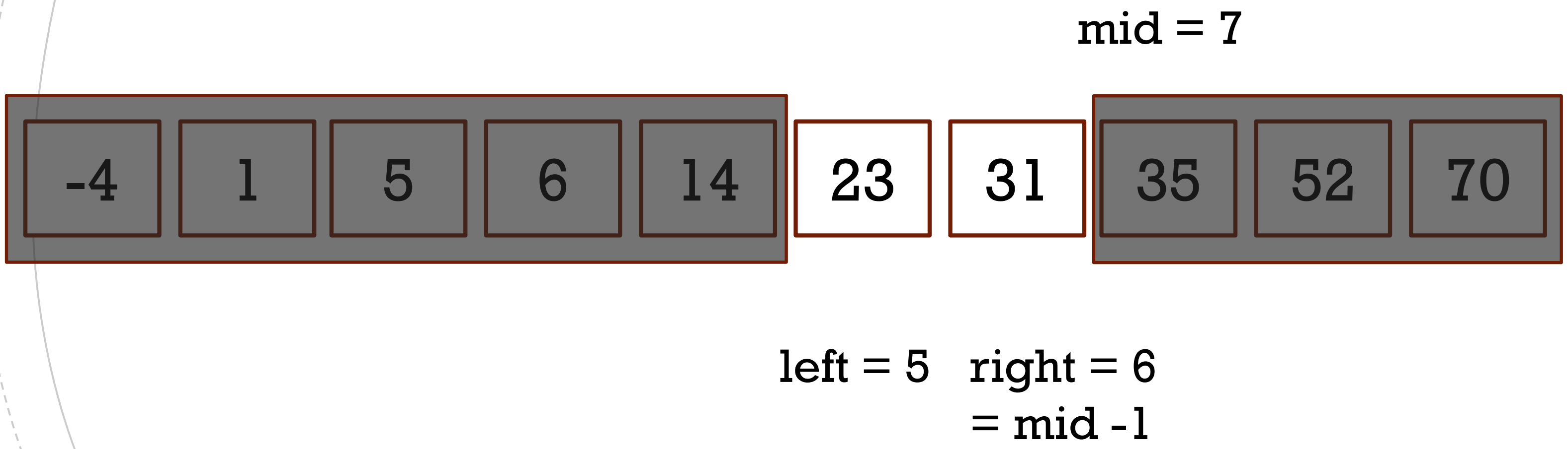


left = 5

right = 9

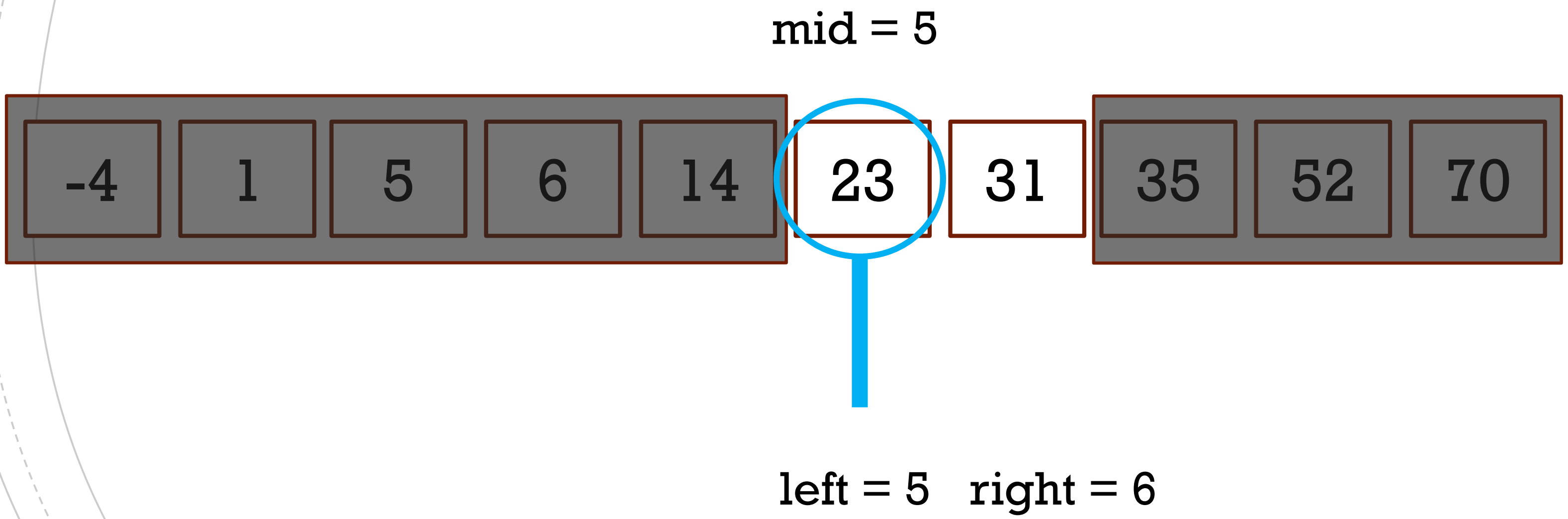
EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update right)



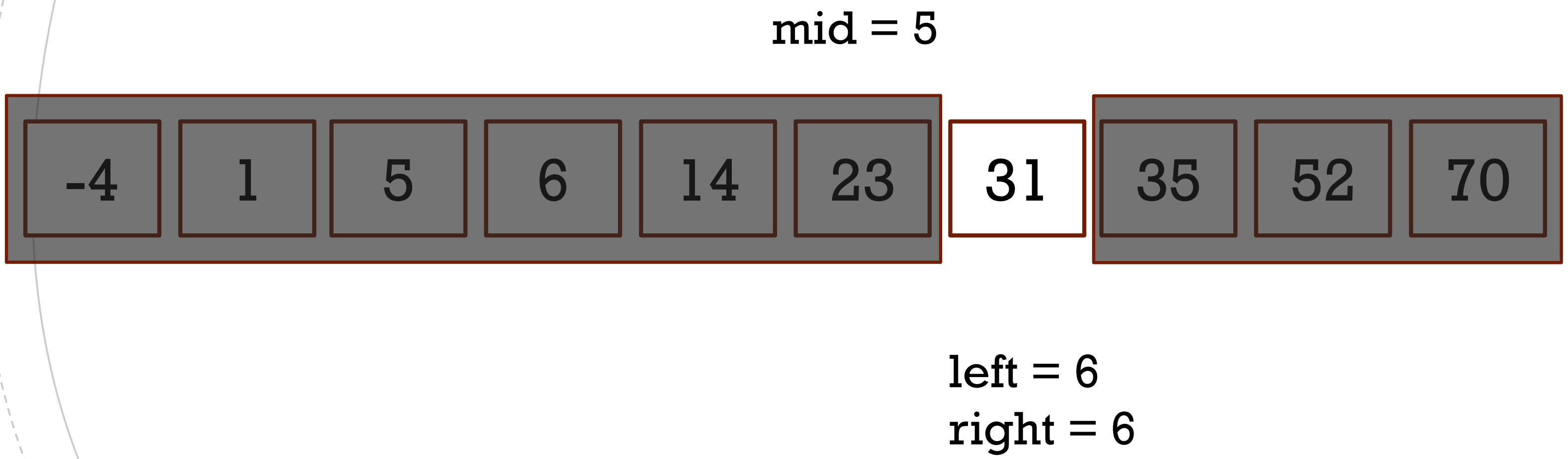
EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)



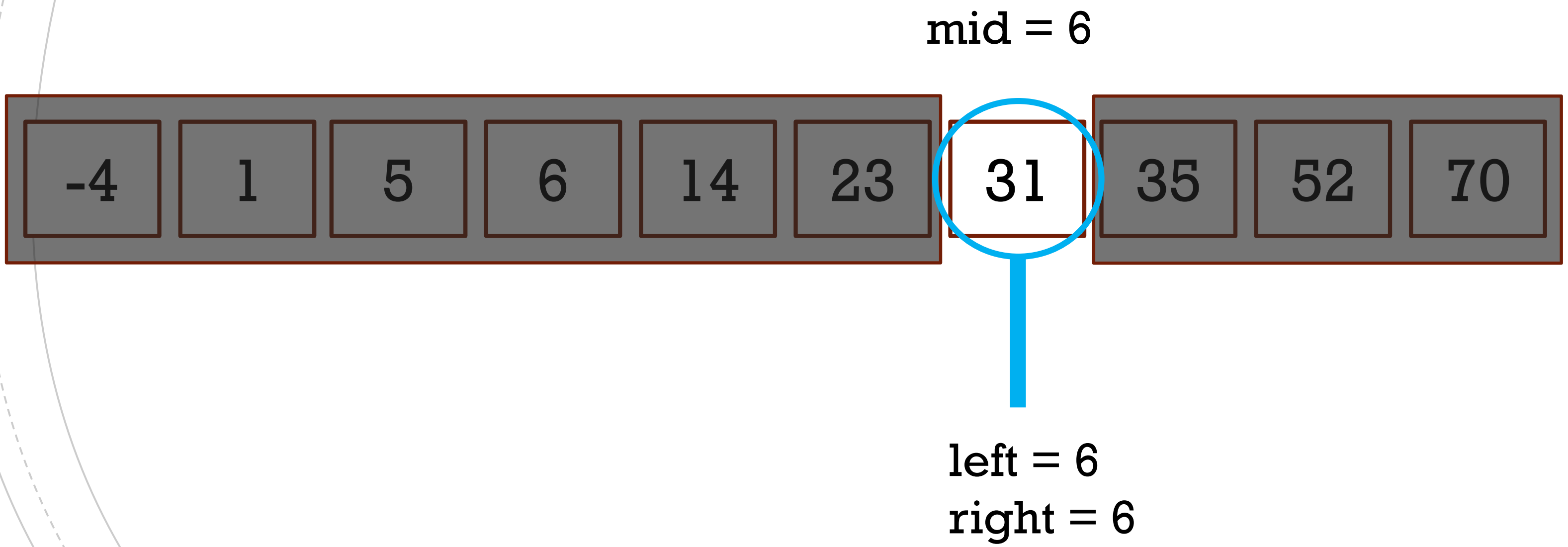
EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update left)



EXAMPLE

- Search for 25
- Look at the middle element and compare (compute mid)



EXAMPLE

- Search for 25
- Look at the middle element and compare
- If not equal: discard half of the list and keep searching on the other half (update right)

mid = 6

-4	1	5	6	14	23	31	35	52	70

right= 5 left = 6

EXAMPLE

- Search for 25
- There are no more elements in the list (right<left)
  → the element is not there! Return -1.

-4	1	5	6	14	23	31	35	52	70

right= 5   left = 6

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
    left = 0                               initialize left and right
    right = list.size() - 1
    while(low <= high){ // until there are elements to search

                  …

    }
    return -1              // key not in list
}
```

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
   left = 0                             initialize left and right
   right = list.size() - 1
   while(low <= high){ // until there are elements to search
      mid = (left + right)/2    // compute mid
      if(list[mid]==key)        // compare element with key
         return mid
      else {

            // update either left or right


      }
   }
   return -1             // key not in list
}
```

BINARY SEARCH (ITERATIVE)

```
binarySearch(list, key){
   left = 0                              initialize left and right
   right = list.size() - 1
   while(low <= high){ // until there are elements to search
      mid = (left + right)/2    // compute mid
      if(list[mid]==key)        // compare element with key
         return mid
      else {
         if(key<list[mid])
            right = mid -1      // update right
         else
            left = mid + 1      // update left
      }
   }
   return -1             // key not in list
}
```

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key){
   left = 0
   right = list.size() - 1
   while(low <= high){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            right = mid -1
         else
            left = mid + 1
      }
   }
   return -1
}
```

What should change?

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
while(low <= high){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            right = mid -1
         else
            left = mid + 1
      }
   }
   return -1
}
```

Pass left and right as parameters to the method

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
if(low <= high){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            right = mid -1
         else
            left = mid + 1
      }
   }
   return -1
}
```

Replace the while with an if

BINARY SEARCH (RECURSIVE)

```
binarySearch(list, key, left, right){
if(low <= high){
      mid = (left + right)/2
      if(list[mid]==key)
         return mid
      else {
         if(key<list[mid])
            binarySearch(list, key, left, mid-1)
         else
            binarySearch(list, key, mid+1, right)
      }
   }
   return -1
}
```

Add recursive calls

OBSERVATIONS

Q: How many times through the while loop ? (iterative)

How many recursive calls? (recursive)

A:

OBSERVATIONS

Q: How many times through the while loop ? (iterative)

How many recursive calls? (recursive)

A: Worst case: the element cannot be found. Then, worst time is $O(log_2 n)$ where n is size of the list. Why? Because each time we are approximately halving the size of the list.

- Mergesort
- Quicksort