

COMP 251

Algorithms & Data Structures (Winter 2021)

Graphs – Flow Network 2

School of Computer Science
McGill University

Slides of (Comp321 ,2021), Langer (2014), Kleinberg & Tardos, 2005 & Cormen et al., 2009, Jaehyun Park' slides CS 97SI, Top-coder tutorials, T-414-AFLV Course, Programming Challenges books, slides from D. Plaisted (UNC) and Comp251-Fall McGill.

Announcements

Outline

- Graphs.
 - Introduction.
 - Topological Sort. / Strong Connected Components
 - Network Flow 1.
 - Introduction
 - Ford-Fulkerson
 - Network Flow 2.
 - Min-cuts
 - Shortest Path.
 - Minimum Spanning Trees.
 - Bipartite Graphs.

Flow Network

$G = (V, E)$ directed.

Each edge (u, v) has a **capacity** $c(u, v) \geq 0$.

If $(u, v) \notin E$, then $c(u, v) = 0$.

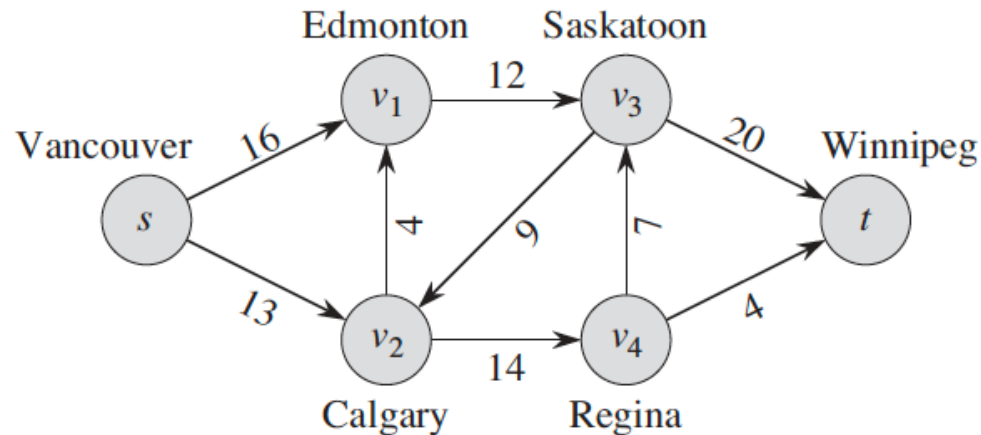
Source vertex s , **sink** vertex t , assume $s \rightsquigarrow v \rightsquigarrow t$ for all $v \in V$.

$c(u, v)$ is a non-negative integer.

If $(u, v) \in E$, then $(v, u) \notin E$.

No incoming edges in source.

No outgoing edges in sink.



Ford-Fulkerson algorithm - correctness

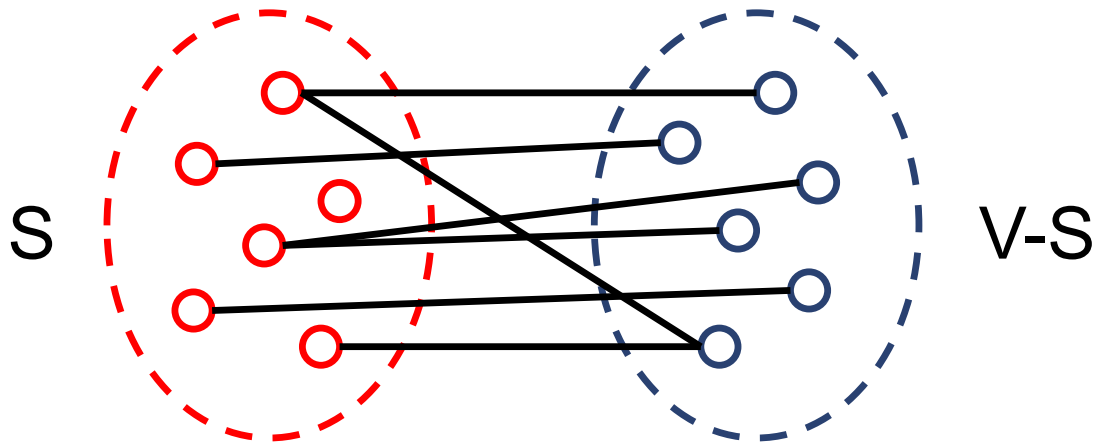
Claim: The Ford-Fulkerson algorithm terminates.

- The capacities and flows are strictly positive integers.
- The sum of capacities leaving s is finite.
- Bottleneck values β are strictly positive integers.
- The flow increase by β after each iteration of the loop.
- The flow is an increasing sequence of integers that is bounded.
- How do we know that when the algorithm terminates, we have actually found a maximum flow?
 - The max-flow min-cut theorem, tells us that a flow is maximum if and only if its residual network contains no augmenting path.
 - We must first explore the notion of a **cut** of a flow network.

```
f ← 0
Gf ← G
while (there is a s-t path in Gf) {
    f . augment ( P )
    update Gf based on new f
}
```

Cuts

A graph cut is a partition of the graph vertices into two disjoint sets.



The crossing edges from S to $V-S$ are $\{ (u,v) \mid u \in S, v \in V-S \}$, also called the cut set.

Cuts in flow networks

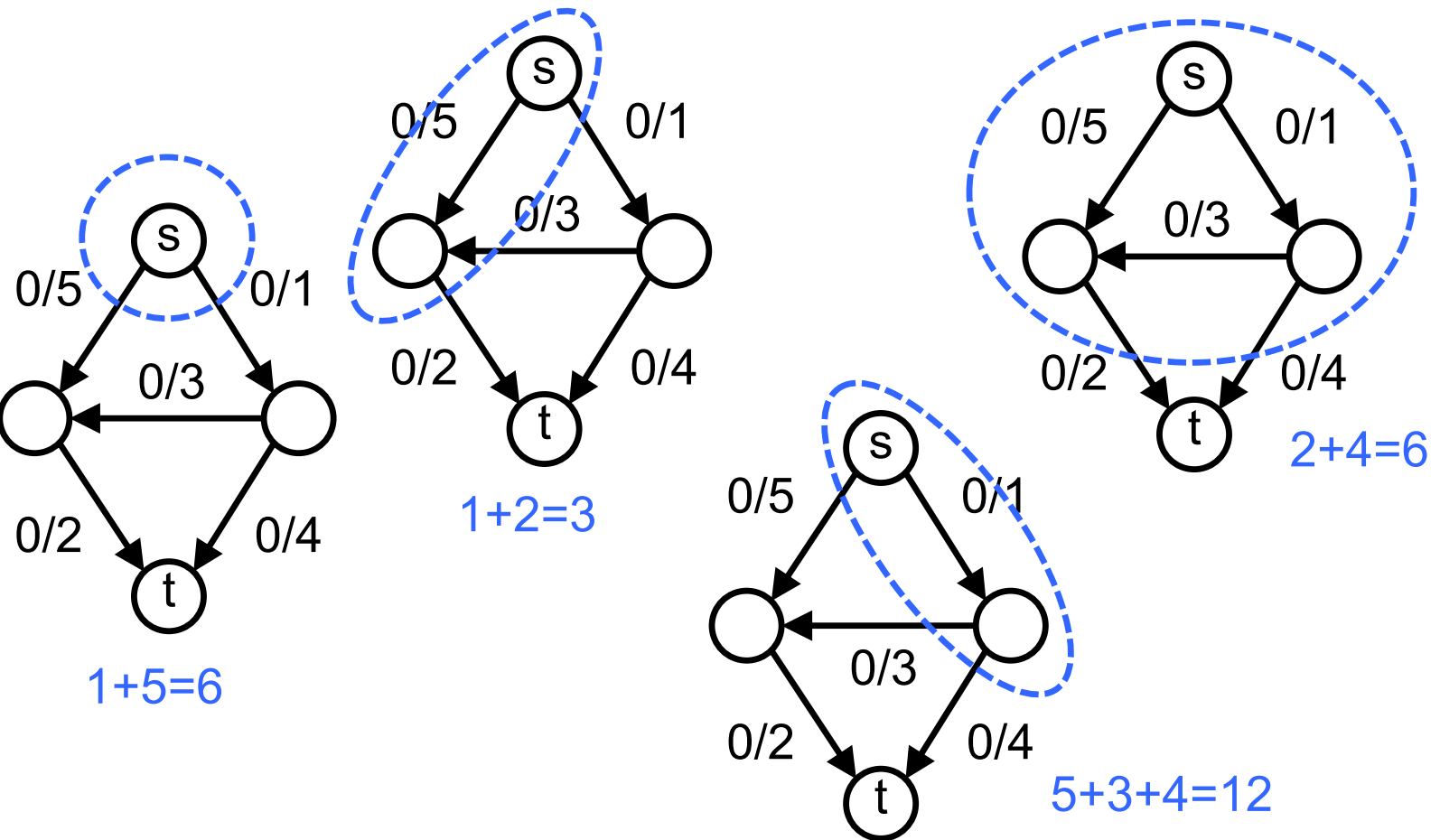
Definition: An s-t cut of a flow network is a cut A, B such that:

- $s \in A$ and $t \in B$
- $A \cup B = V$
- $A \cap B = \emptyset$

Notation: We write **cut(A,B)** the set of edges from A to B .

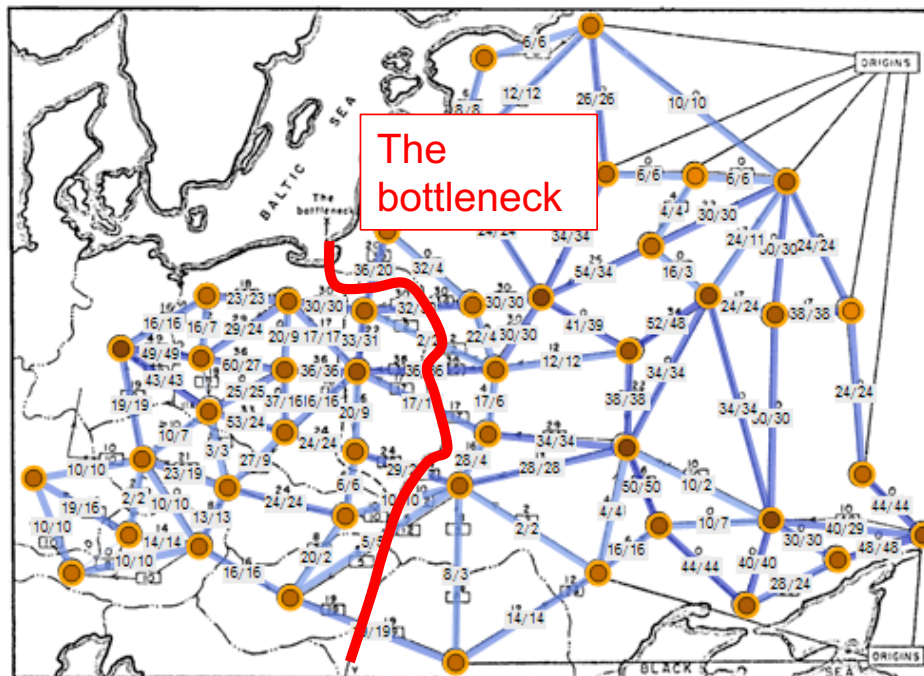
Definition: The capacity of an s-t cut is
$$\sum_{e \in \text{cut}(A,B)} c(e)$$

Cuts in flow networks



Minimum cut problem

- The minimum cut problem is to compute an (s, t) -cut whose capacity is as small as possible.
 - Intuitively, the minimum cut is the cheapest way to disrupt all flow from s to t .



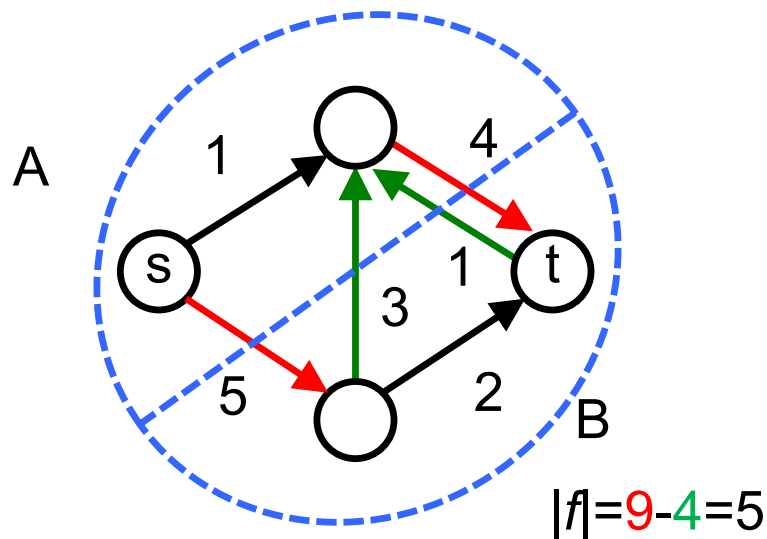
How to cut supplies if cold war turns into real war!

Flow through a cut

Claim: Given a flow network. Let f be a flow and A, B be a s-t cut. Then, the net flow is defined to be:

$$|f| = \sum_{e \in \text{cut}(A,B)} f(e) - \sum_{e \in \text{cut}(B,A)} f(e)$$

Notation: $|f| = f^{\text{out}}(A) - f^{\text{in}}(A)$



Flow through a cut – Upper bound

Claim: For any network flow f , and any s-t cut(A,B)

$$|f| \leq \sum_{e \in \text{cut}(A,B)} c(e)$$

Proof:

$$\begin{aligned} |f| &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &\leq f^{\text{out}}(A) \\ &\leq \sum_{e \in \text{cut}(A,B)} c(e) \end{aligned}$$

- The asymmetry between the definitions of flow and capacity of a cut is intentional and important.
- Right-hand side is independent of any particular flow f
- The value of every flow is upper-bounded by the capacity of every cut
- The value of a maximum flow in a network is bounded from above by the capacity of a minimum cut of the network.

Flow through a cut – Observations

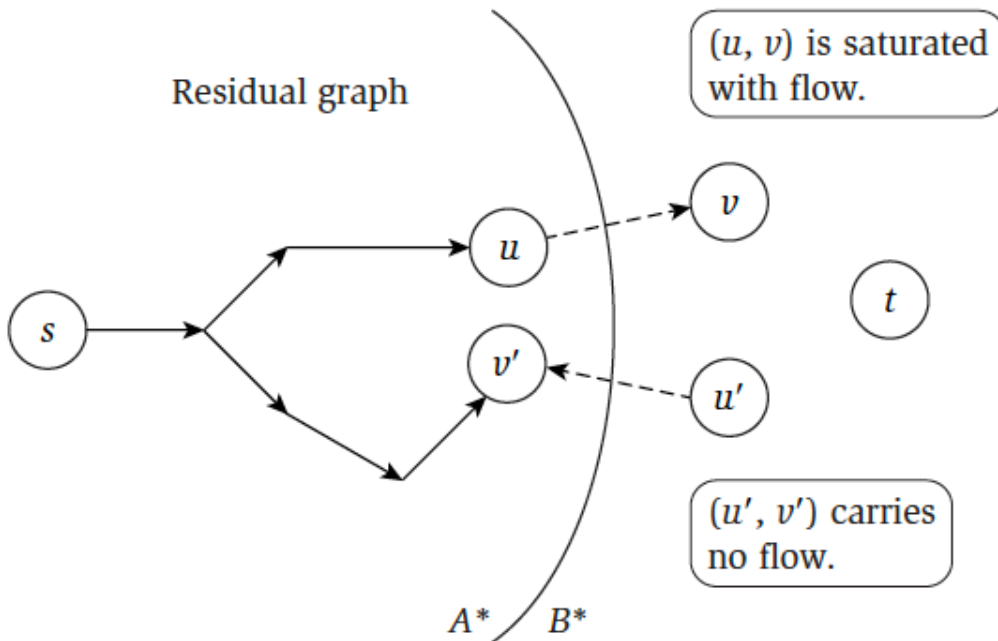
- Some cuts have greater capacities than others.
- Some flows are greater than others.
- **But every flow must be \leq capacity of every s-t cut.**
 - if we exhibit any s-t cut in G of some value c^* , we know that there cannot be an s-t flow in G of value greater than c^* .
 - Conversely, if we exhibit any s-t flow in G of some value v^* , we know that there cannot be an s-t cut in G of value less than v^* .
- Thus, the value of the maximum flow is less than capacity of the minimum cut.
 - The max-flow min-cut theorem, says that the value of a maximum flow is in fact equal to the capacity of a minimum cut.

Value of flow in Ford-Fulkerson

- Ford-Fulkerson terminates when there is no augmenting path in the residual graph G_f .
- Let A be the set of vertices reachable from s in G_f , and $B=V-A$.
- A, B is a s - t cut in G_f .
- A, B is an s - t cut in G (G and G_f have the same vertices).
- $|f| = f^{\text{out}}(A) - f^{\text{in}}(A)$
- We want to show: $|f| = \sum_{e \in \text{cut}(A, B)} c(e)$
- And in particular:

$$\textcircled{1} \quad f^{\text{out}}(A) = \sum_{e \in \text{cut}(A, B)} c(e) \qquad \textcircled{2} \quad f^{\text{in}}(A) = 0$$

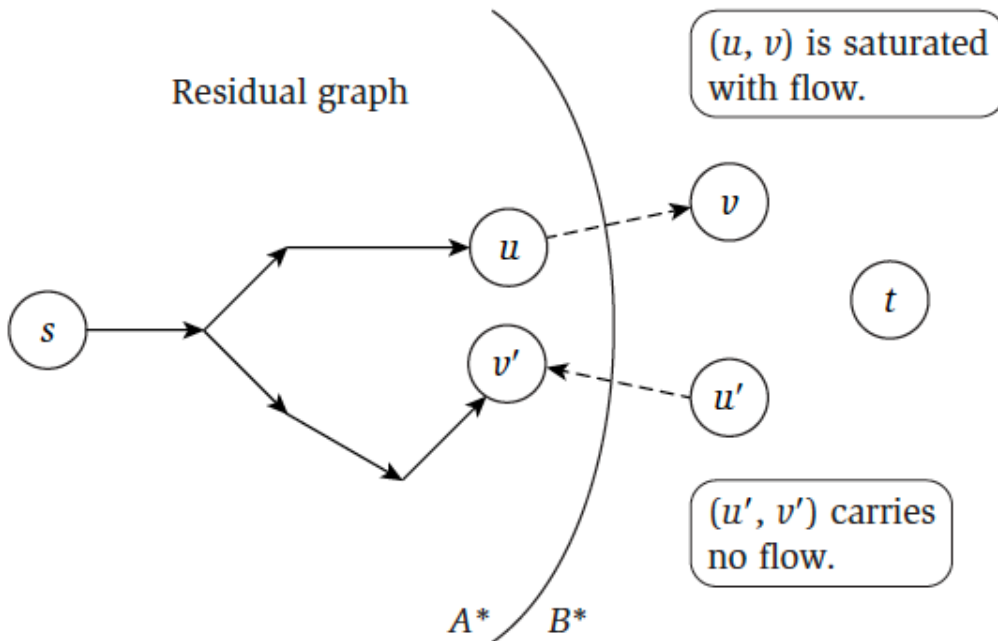
Value of flow in Ford-Fulkerson



Suppose that $e = (u, v)$ is an edge in G for which $u \in A^*$ and $v \in B^*$. We claim that $f(e) = c_e$. For if not, e would be a forward edge in the residual graph G_f , and since $u \in A^*$, there is an s - u path in G_f ; appending e to this path, we would obtain an s - v path in G_f , contradicting our assumption that $v \in B^*$.

(A^*, B^*) is indeed an s - t cut. It is clearly a partition of V . $t \notin A^*$ by the assumption that there is no s - t path in the residual graph; hence $t \in B^*$ as desired.

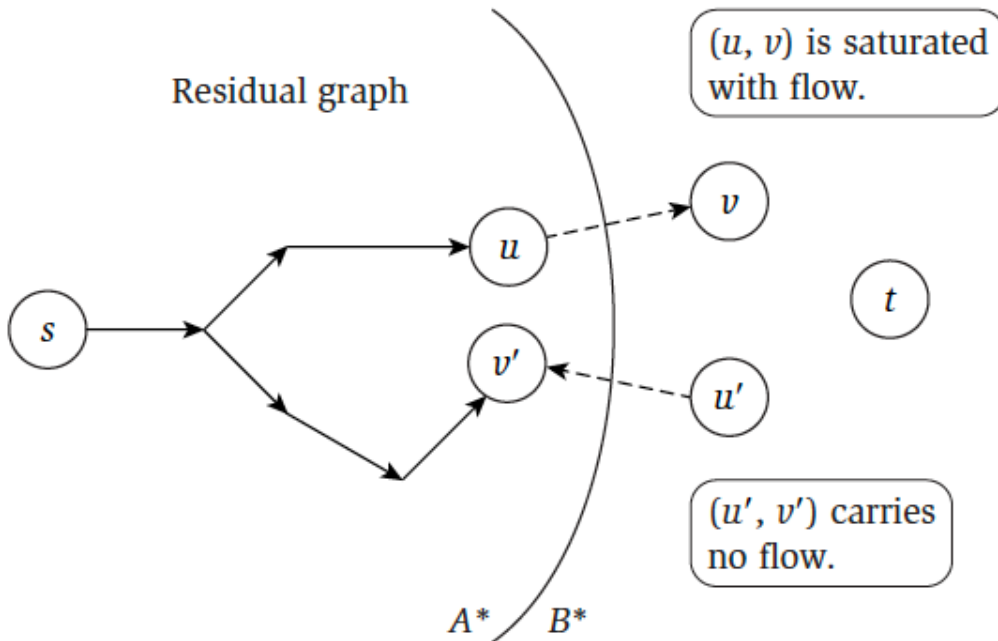
Value of flow in Ford-Fulkerson



Suppose that $e' = (u', v')$ is an edge in G for which $u' \in B^*$ and $v' \in A^*$. We claim that $f(e') = 0$. For if not, e' would give rise to a backward edge $e'' = (v', u')$ in the residual graph G_f , and since $v' \in A^*$, there is an s - v' path in G_f ; appending e'' to this path, we would obtain an s - u' path in G_f , contradicting our assumption that $u \in B^*$.

(A^*, B^*) is indeed an s - t cut. It is clearly a partition of V . $t \notin A^*$ by the assumption that there is no s - t path in the residual graph; hence $t \in B^*$ as desired.

Value of flow in Ford-Fulkerson



So all edges out of A^* are completely saturated with flow, while all edges into A^* are completely unused.

Max flow = Min cut

- Ford-Fulkerson terminates when there is no path s-t in the residual graph G_f
- This defines a cut in A,B in G (A = nodes reachable from s)

- $|f| = f^{out}(A) - f^{in}(A)$

$$= \sum_{e \in cut(A,B)} f(e) - \sum_{e \in cut(B,A)} f(e)$$

- Ford-Fulkerson flow = $\sum_{e \in cut(A,B)} c(e) - 0$
= capacity of cut(A,B)

Computing the min-cut

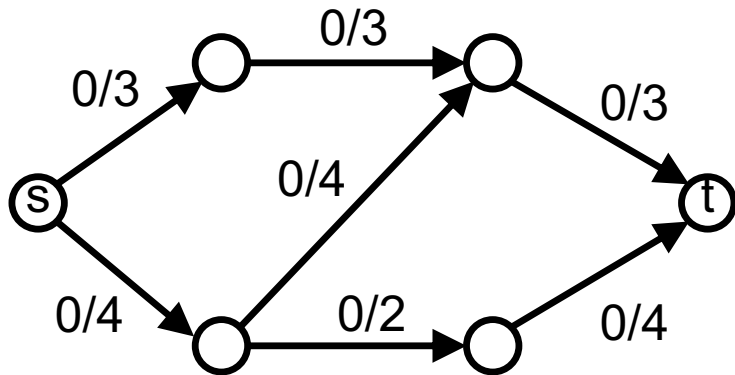
Q: Given a flow network, how can we compute a minimum cut?

Answer:

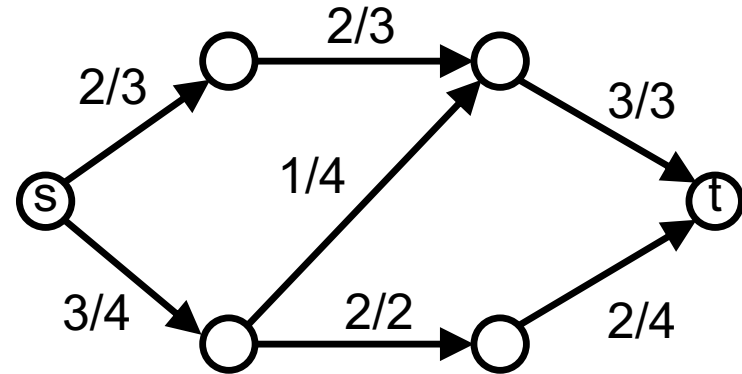
- Run Ford-Fulkerson to compute a maximum flow (it gives us G_f)
- Run BFS or DFS of s .
- The reachable vertices define the set A for the cut

Computing the min-cut – Ford-fulkerson

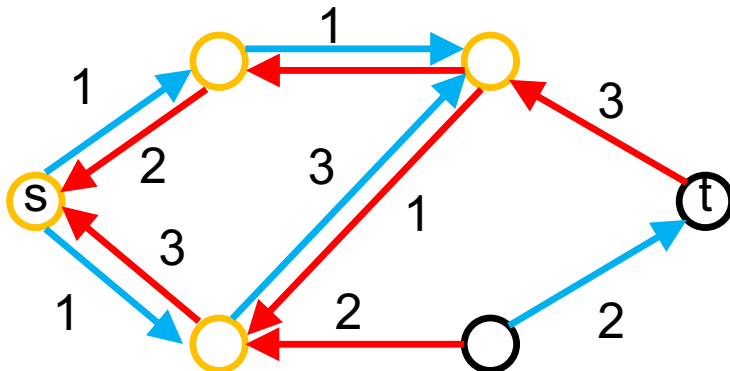
(1) Initial flow net G



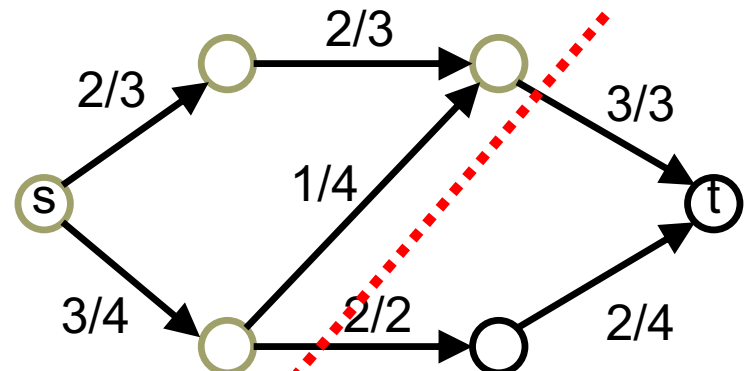
(2) Compute max flow (FF)



(3) Compute G_f and vertices accessible from s



(4) Vertices accessible from s in G_f determine the min cut



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