

COMP 251

Algorithms & Data Structures (Winter 2021)

Disjoint Sets

School of Computer Science
McGill University

Based on Based on (Cormen *et al.*, 2002) & slides of
(Curles,2008), (Waldispuhl,2020) and (Langer,2014).

Outline

- Introduction.
- Operations.

Introduction - Motivation

- You have a set of nodes (numbered 1-9) on a network. You are given a sequence of pairwise connections between them:

3-7

8-2

1-6

5-7

4-8

3-5

Data Structure



{3-5-7}

{2-4-8}

{9}

{1-6}

Disjoint sets

Q: Are nodes 2 and 4 (indirectly) connected?

Q: Are nodes 3 and 8 connected?

Q: Are any of the paired connections redundant?

Q: How many sub-networks do we have?

Introduction - Motivation

Q: Are nodes 2 and 4 (indirectly) connected?

Q: Are nodes 3 and 8 connected?

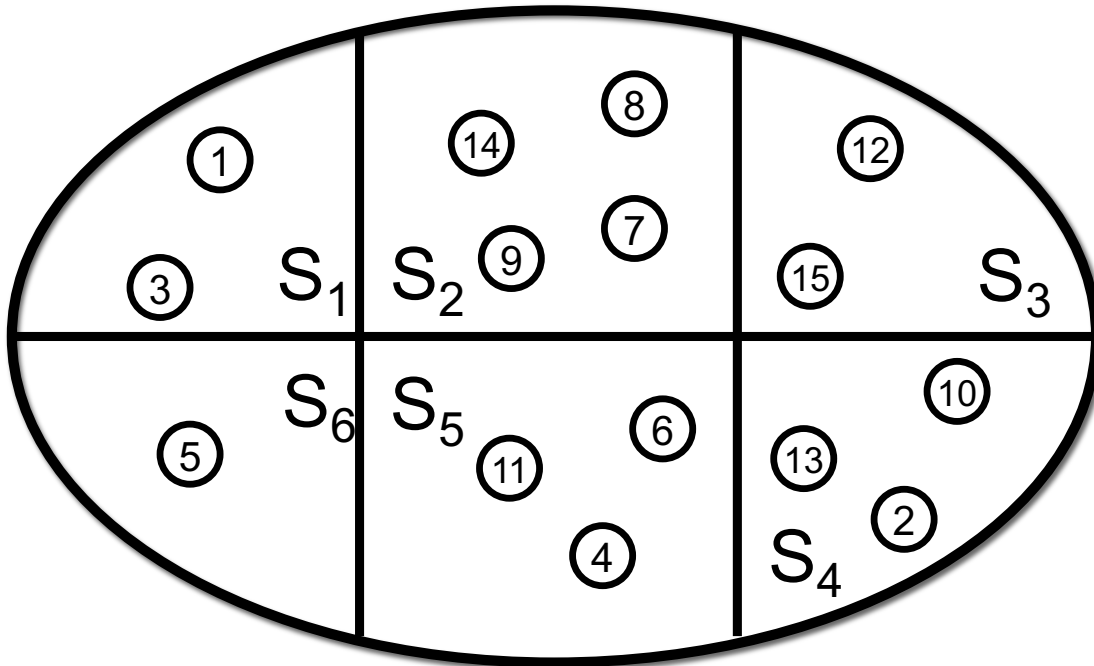
Q: Are any of the paired connections redundant?

Q: How many sub-networks do we have?

- These kind of questions arises in a huge number of areas.
 - Networks
 - Transistor interconnects
 - Compilers
 - Image segmentation
 - Graph problems (upcoming topic)
 - Etc

Introduction - Partition

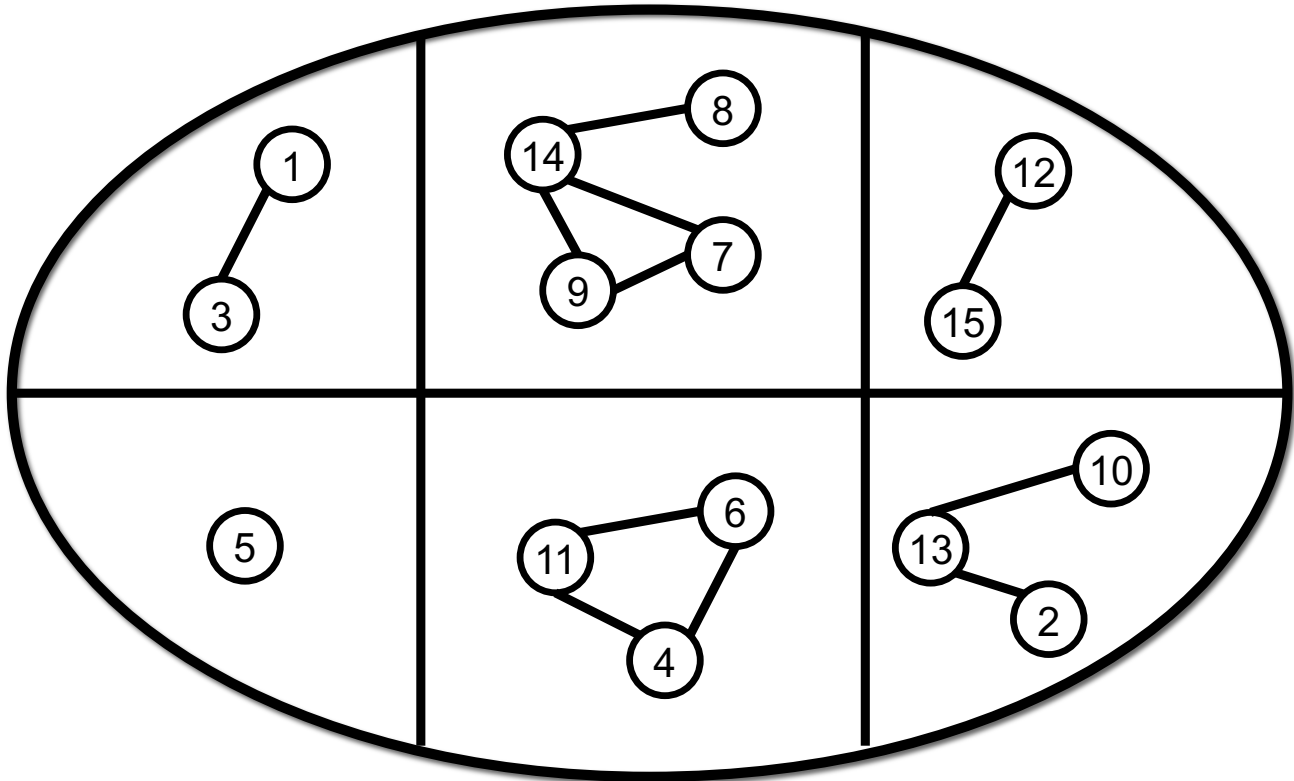
Generalization: Set of object partitioned into disjoint subsets.



$$S = S_1 \cup S_2 \cup \dots \cup S_n \left\{ \begin{array}{l} S_i \neq \emptyset \quad \forall i \in \{1, \dots, n\} \\ S_i \cap S_j = \emptyset \text{ iff } i \neq j \end{array} \right.$$

Introduction – Graph example

Connected component: Set of nodes connected by a path.



Question: Given 2 nodes A & B, are they in the same component?

Introduction – Equivalence relation

A relation that is:

- Reflexive $\forall a \in S, (a,a) \in R$
- Symmetric $\forall a,b \in S, (a,b) \in R \Rightarrow (b,a) \in R$
- Transitive $\forall a,b,c \in S, (a,b) \in R \text{ and } (b,c) \in R \Rightarrow (a,c) \in R$

Example:

For any undirected graph, the connections define an equivalence relation on vertices.

- For all $u \in V$, there is a path of length 0 from u to u .
- For all $u,v \in V$, There is a path from u to v , iff there is a path from v to u .
- For all $u,v,w \in V$, if there is a path from u to v and a path from v to w , then there is a path from u to w .

Introduction – Equivalence relation

Java

`equals()` defines an equivalence
relation on objects

[http://docs.oracle.com/javase/7/docs/api/java/lang/Object.html#equals\(java.lang.Object\)](http://docs.oracle.com/javase/7/docs/api/java/lang/Object.html#equals(java.lang.Object))

reflexive `a.equals(a)` returns true

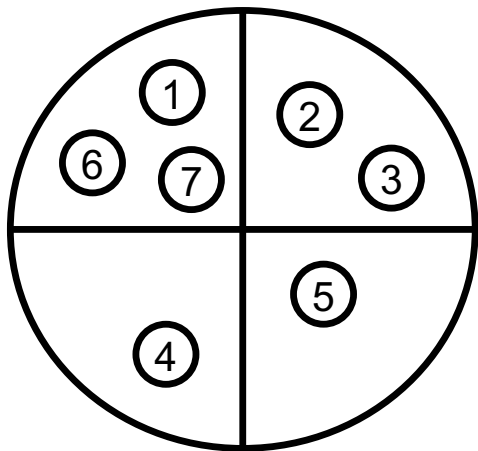
symmetric `a.equals(b) == b.equals(a)`

transitive `a.equals(b) and b.equals(c)`
 \Rightarrow `a.equals(c)`

Taken from Langer2014

Introduction – Equivalence relation

An equivalence relation is the same as a partition:

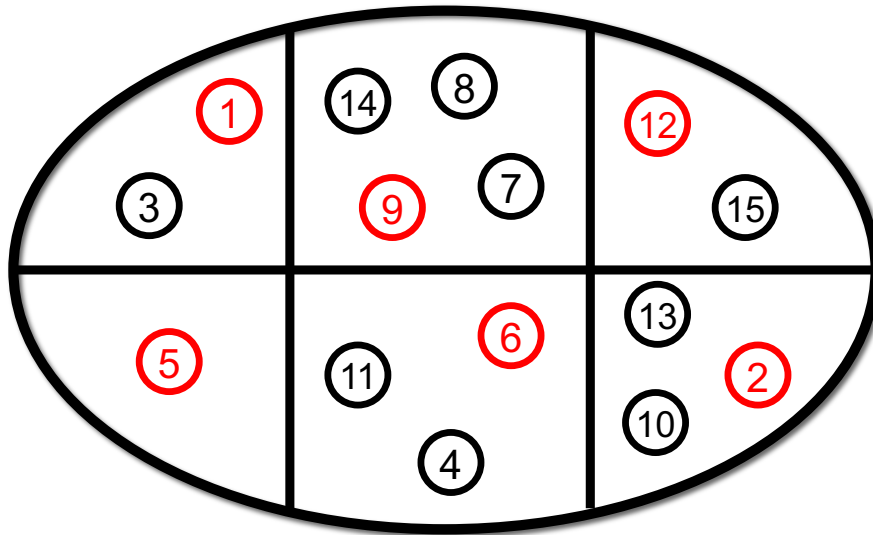


	1	2	3	4	5	6	7
1	1	0	0	0	0	1	1
2	0	1	1	0	0	0	0
3	0	1	1	0	0	0	0
4	0	0	0	1	0	0	0
5	0	0	0	0	1	0	0
6	1	0	0	0	0	1	1
7	1	0	0	0	0	1	1

i is equivalent to j if they belong to the same set.

(more constrained than general relation)

Introduction – Disjoint set ADT



i is equivalent to j if they belong to the same set.

Each set in the partition has a unique name (a representative member for convenience).

- $\text{find}(i)$ returns the representative of the set that contains i .
- $\text{sameset}(i,j)$ returns the boolean value $\text{find}(i) == \text{find}(j)$
- $\text{union}(i,j)$ merges the sets containing i and j .

Introduction - Operations

- Maintain disjoint sets:
 - $\{3, 5, 7\}, \{4, 2, 8\}, \{9\}, \{1, 6\}$
- Each set has a representative:
 - $\{3, 5, 7\}, \{4, 2, 8\}, \{9\}, \{1, 6\}$
- Find(x) returns the representative of the set containing x
 - Find(1) = 5
 - Find(4) = 8
- Union(x,y) takes the union of the two sets that contains x and y
 - Union(5,1) = $\{3, 5, 7, 1, 6\}, \{4, 2, 8\}, \{9\}$
 - Union(9,6) = $\{3, 5, 7\}, \{4, 2, 8\}, \{9, 1, 6\}$

Introduction - Operations

- Maintain disjoint sets:
 - {3, 5, 7}, {4, 2, 8}, {9}, {1, 6}
- Each set has a representative:
 - {3, 5, 7}, {4, 2, 8}, {9}, {1, 6}
- Q: Are nodes 2 and 4 (indirectly) connected?
 - $\text{Find}(2) == \text{Find}(4)$
- Q: Are nodes 3 and 8 connected?
 - $\text{Find}(3) == \text{Find}(8)$
- Q: How many sub-networks do we have?
 - Number of representatives.
- Q: Are any of the paired connections redundant?
 - $\text{Connections} - (|S1-1| + |S2-1| + |S3-1| + |S4-1|) = 6 - (2+2+0+1) = 1$

previous

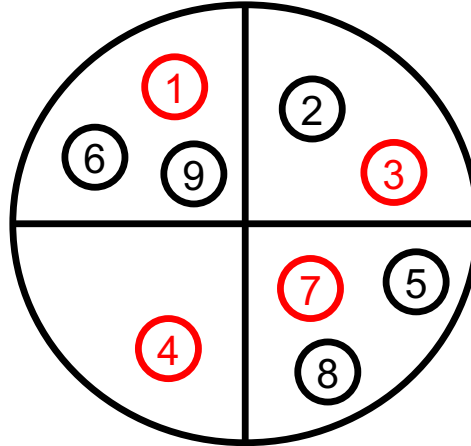
Outline

- Introduction.
- Operations.

Operations – find

Rep[]

1	1
2	3
3	3
4	4
5	7
6	1
7	7
8	7
9	1

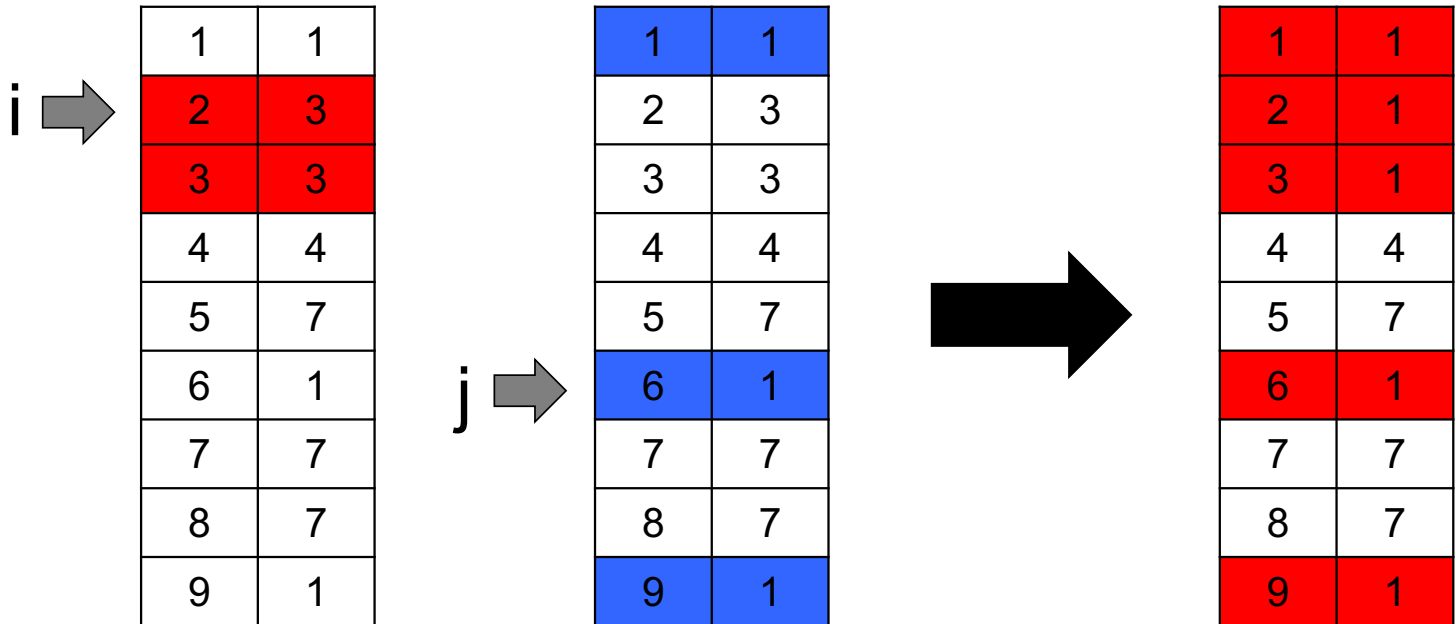


Let $\text{Rep}[i] \in \{ 1, 2, \dots, n \}$ be the representative of the set containing i .
 $\text{find}(i) \{ \text{return rep}[i]; \}$

Operations – union

- $\text{union}(i,j)$ merges the sets containing i and j .

Example: $\text{union}(2,6)$



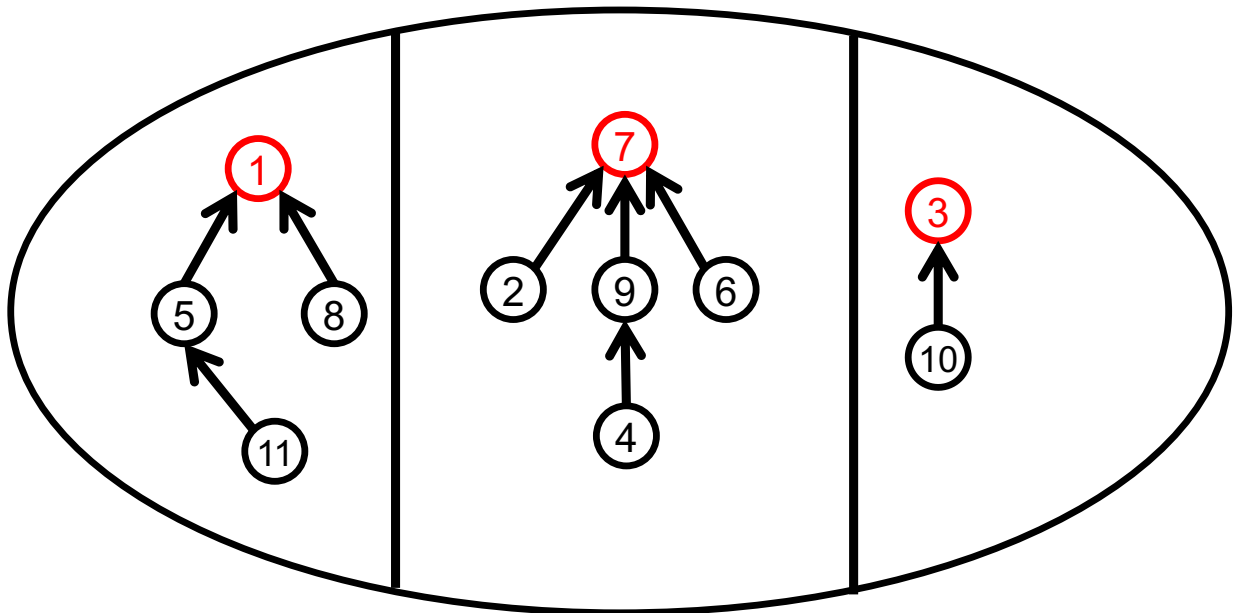
Operations – union

```
union(i,j) {  
    if rep[i] != rep[j] {  
        prevrep_i = rep[i];  
        for (k=1; k<=n; k++) {  
            if rep[k] == prevrep_i {  
                rep[k] = rep[j];  
            }  
        }  
    }  
}
```

- store value of rep[i] because it may change during the execution of the algorithm.
- $O(n)$ running time... slow.

Union – forest & tree representation

- Represent the disjoint sets by a forest of rooted trees.
 - Roots are the representative (i.e. $\text{find}(i) == \text{findroot}(i)$).
 - Each node points to its parent.

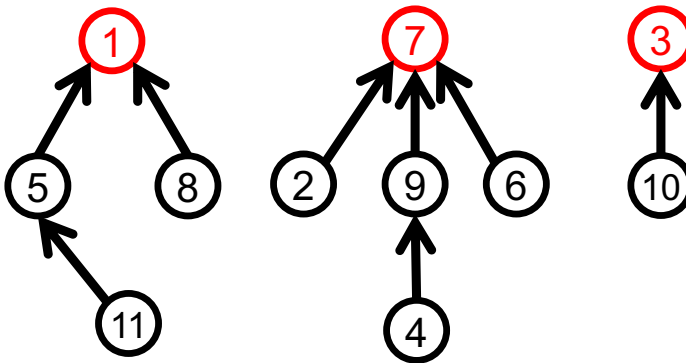


Note:

The tree structure does not necessarily represent the relationship between the stored objects.

Union – table representation

p[]	
1	1
2	7
3	3
4	9
5	1
6	7
7	7
8	1
9	7
10	3
11	5



- Non-root nodes hold index of their parent.
- Root nodes store their own value.

Operations – Find & Union

```
find(i) {  
    if p[i] == i {  
        return i;  
    } else {  
        return find(p[i]);  
    }  
}
```

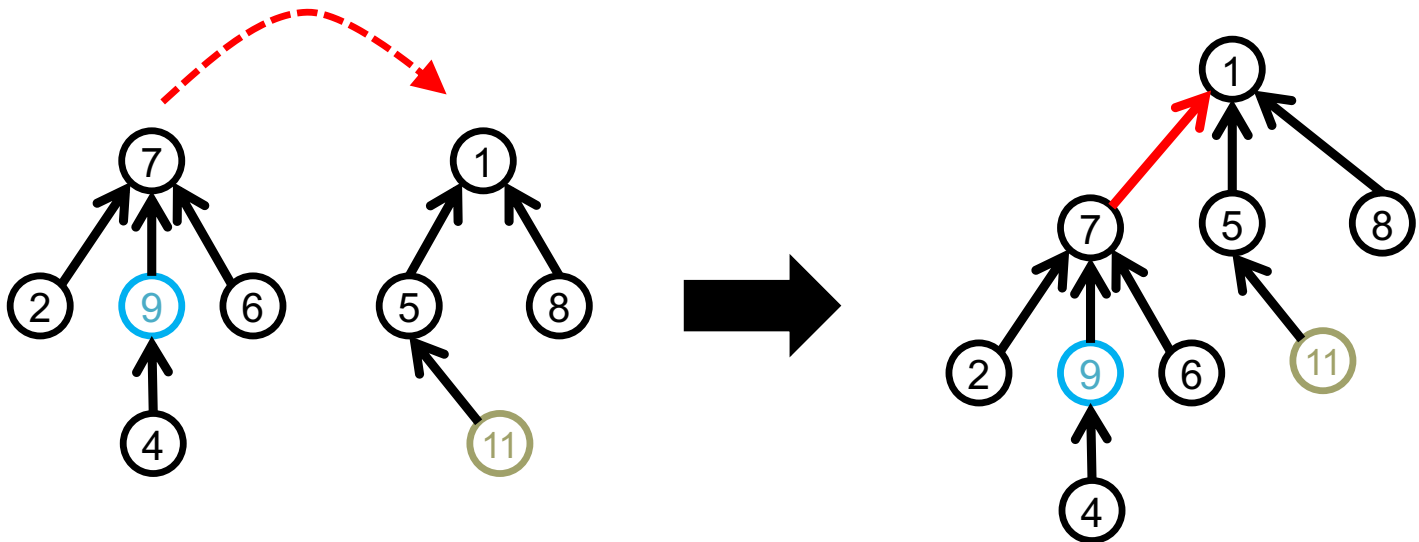
Remark: Arbitrarily
merge the set on i
into the set of j.

```
union(i,j) {  
    if find(i) != find(j) {  
        p[find(i)] = find(j);  
    }  
}
```

Operations – Union

union(9,11)

Root of the tree of 11 becomes the parent of the root of the tree of 9.



Operations – Union - > Worst case

union(1,2)

union(1,3)

union(1,4)

...

union(1,n)

Then, find(1) is $O(n)$...



Union – Heuristic – Definitions

- The **depth** of a node is the number of edges from the node to the tree's root node. A root node will have a depth of 0.
- The **height** of a node is the number of edges on the *longest path* from the node to a leaf. A leaf node will have a height of 0.

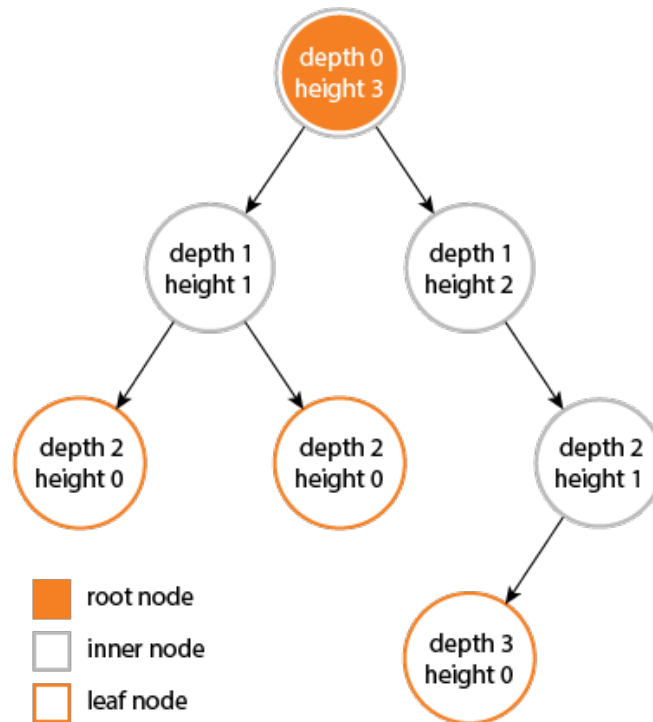
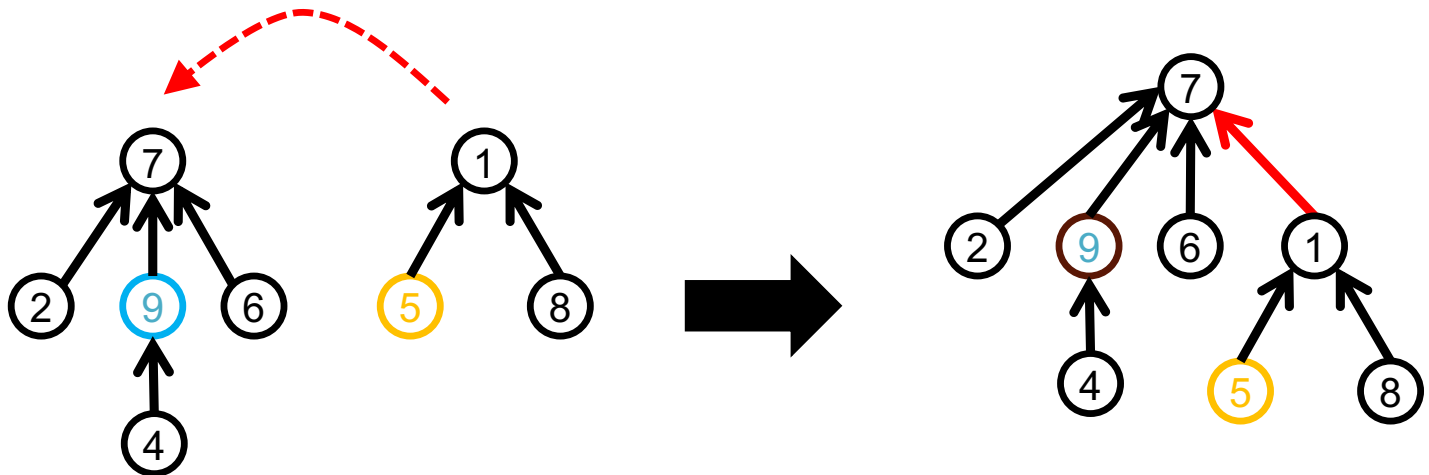


Image and definitions taken from:
<https://stackoverflow.com/>

Union – Heuristic – by size

Heuristic to control the height to the trees after merging.

Idea: Merge tree with smaller number of nodes into the tree with the largest number of nodes (In practice, we can also use rank which is an upper bound on the height of nodes).

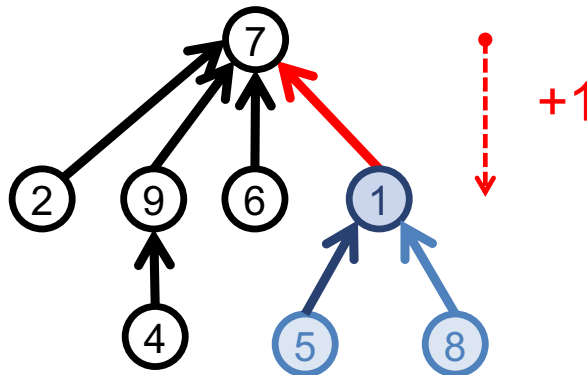


Union – Heuristic – by size

Claim: The depth of any node is at most $\log n$.

Proof:

- If union causes the depth of a node to increase, then this node must belong to the smallest tree.



- Thus, when the depth increases, the size of the (merged) tree containing this node will at least double.
- But we can double the size of a tree at most $\log n$ times.

Union – Heuristic – by height

Idea: Merge tree with smaller height into tree with larger height.

Claim: The height of trees obtained by union-by-height is at most $\log n$.

Corollary: An union-by-height tree of height h has at least $n_h \geq 2^h$ nodes.

Proof (Corollary):

- Base case: a tree of height 0 has one node.
- Induction: (hypothesis) $n_h \geq 2^h$. Show $n_{h+1} \geq 2^{h+1}$.

Heuristic – supplemental material

The base case $k=0$ is easy, since a tree of height 0 always has just $1=2^0$ node (the root). Suppose the claim is true for $h=k$. Now consider a union-by-height tree of height $k+1$. There must have been a union that brought two trees together and increased the height of one of them from k to $k+1$. Let those two trees (at the time of that union) be T_1 and T_2 . We know that both T_1 and T_2 were of height k before the union. [Why? If one of them were of height less than k , then union-by-height would have changed the root of that shorter one to make it point to the root of the taller one, and the height of the unioned tree would still be k . But it's not; the unioned tree is of height $k+1$.] Now we can apply the induction hypothesis: the trees T_1 and T_2 each have at least 2^k nodes. Thus, the unioned tree has at least $2^k + 2^k = 2^{k+1}$ nodes.

Taken from Langer 2014

Union – running time

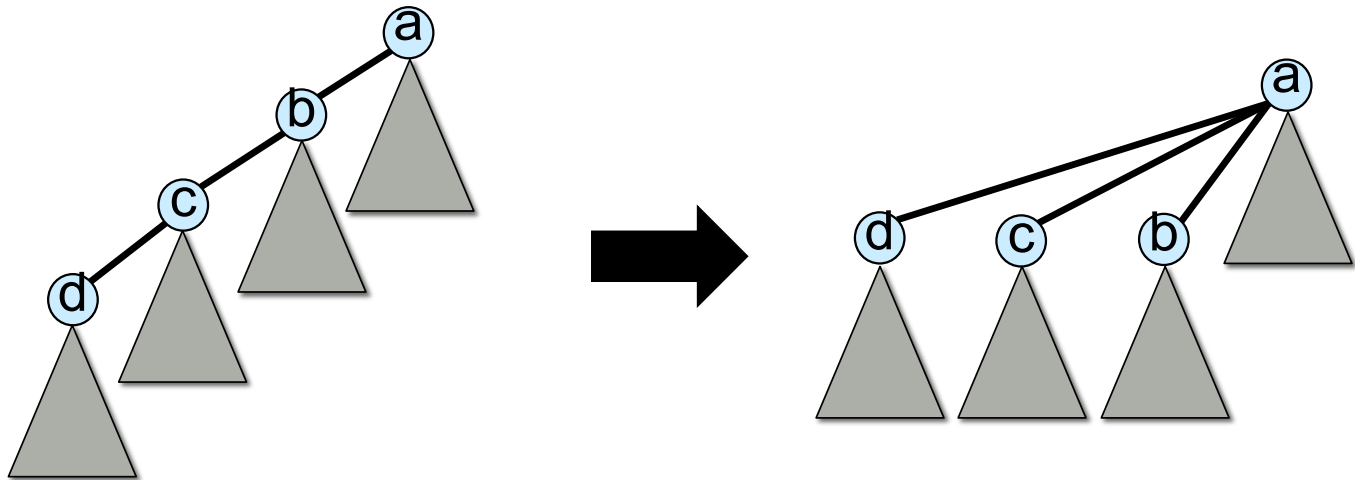
		find ^{union} (i)	union ^{find} (i,j)
Quick Union {	find	$O(1)$	$O(n)$
	Union by size	$O(\log n)$	$O(\log n)$
	Union by height	$O(\log n)$	$O(\log n)$

↑
Quick union makes 2 calls to find.

Note: These are worst case complexities.

union-find – heuristics – path compression

- Find path = nodes visited during the execution of find() on the trip to the root.
- Make all nodes on the find path direct children of root.



union-find – heuristics – path compression

```
find(i) {  
    if p[i] == i {  
        return i;  
    } else {  
        p[i] = find(p[i]);  
        return p[i];  
    }  
}
```

union-find – running time

- Use union by size and path compression.
- **m union or find operations** take $O(m \alpha(n))$.

What is $\alpha(n)$?

n	$\alpha(n)$
0 - 2	0
3	1
4 - 7	2
8 - 2047	3
2048 – $A_4(1)$	4

Where $A_4(1) \gg 10^{80} !!$

union-find – running time

- Use union by size and path compression.
- Huge Practical problem.
 - 10^{10} edges connecting 10^9 nodes.
 - The heuristics reduces time from 3.000 years to 1 minute
 - Supercomputer wont help much
 - Good algorithm makes solution possible
 - Good algorithms makes it possible to solve problems that could not otherwise be addressed.