COMP 250
INTRODUCTION TO COMPUTER SCIENCE

Lecture 18 – Induction

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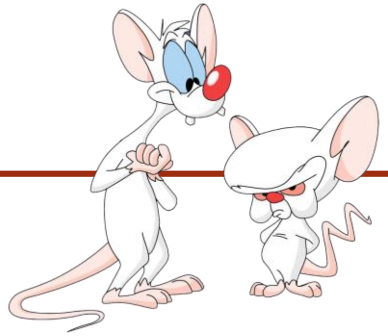
FROM LAST CLASS

- Class
- Memory allocation

WHAT ARE WE GOING TO DO TODAY?



- Induction

INDUCTION

PROOFS

For all $n \geq 1$,

$$1 + 2 + 3 + \;\ldots. + (n-1) + n = \frac{n(n+1)}{2}$$

How can we prove such a statement ?

- By "proof", we mean a formal logical argument that convincingly demonstrate the truth of a given proposition.
- Note that "convincingly" is itself not well defined.

EXAMPLE

$$1 + 2 + \ldots + (n-1) + n$$

Rewrite by considering $n/2$ pairs :

$$1 + 2 + \cdots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) \ldots + (n-1) + n$$

If n is even, then adding up the $n/2$ pairs gives

$$n/2 * (n + 1)$$

- What if n is odd?

EXAMPLE

- What if n is odd? Then, n-1 is even. So,

$$1 + 2 + \ldots + (n-1) + n$$

$$= \left(\frac{n-1}{2} * n\right) + n$$

$$= \left(\frac{n-1}{2} + 1\right) * n$$

$$= \frac{n+1}{2} * n$$

which is the same formula as before.

RECURSIVE (INDUCTIVE) DEFINITION

- Some set of elements can be define recursively/inductively.

- A recursive/inductive definition consists of the following:
  - A *base clause*

    Which one or more basic/initial element of the set.
  - One or more *inductive clauses*

    Rules on how to generate "new" elements of the set from "old" ones.
  - A *final clause*

    which simply states that no other element is part of the set.

EXAMPLE – NATURAL NUMBERS

The set of natural numbers can be defined as follows:

- *Base clause*:
  0 is a natural number

- *Inductive clause*:
  If n is a natural number, then $n+1$ is also a natural number.

- *Final clause*: Nothing else is a natural number.

MATHEMATICAL INDUCTION

Consider a statement of the form:

"For all $n \geq n_0, \quad P(n)$ is true"

where n_0 is some constant and proposition $P(n)$ has value true or false for each n.

- If n is an element of an inductively defined set, then the statement above can be proven using a technique called mathematical induction.

(WEAK) MATHEMATICAL INDUCTION

To prove a property by mathematical induction, we proceed as follows:

- *Base case*
  Show that the property holds for the basic/initial elements of the set.

- *Induction step*
  Assume the property hold for some element n. (Induction Hypothesis)
  Show that the property also holds for any element generated from n using the inductive clauses.

- *Conclusion*
  The property holds for all elements.

EXAMPLE

"For all $n \geq n_0$, $P(n)$ is true"

For all $n \geq 1$,

$$1 + 2 + 3 + \ldots . + (n-1) + n = \frac{n(n+1)}{2}$$

This is a property of natural numbers. Since this is a set that can be defined inductively, we can use mathematical induction to prove such property!

PROOF BY MATHEMATICAL INDUCTION

We need to prove the following:

- Base case:
  $P(n_0)$ is true, i.e. the property holds for n_0 which in this case is 1.

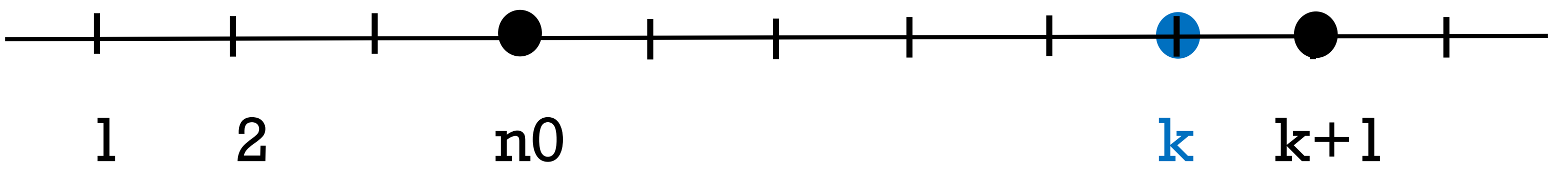
- Induction step:
  IH: Assume $P(k)$ is true, i.e. the property holds for an element k.
  Prove that $P(k+1)$ is true, i.e. the property holds for $k+1$.

Base case:

$P(n_0)$ is true.

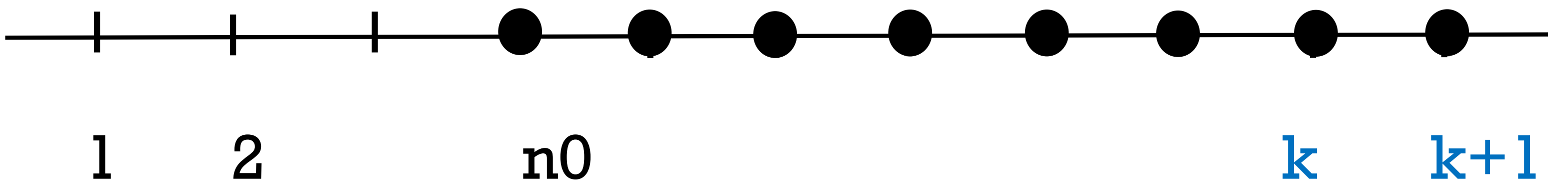
Induction step:

For any k >= n_0, if P(k) is true then P(k+1) is true.



Thus we have proved:

For any n >= n_0, P(n) is true.



BACK TO THE PROOF

For all $n \geq 1$, $1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}$

- Base case: $n = 1$, to prove

$$1 = \frac{1 * (1+1)}{2}$$

$$1 = \frac{2}{2} = 1 \quad ✓$$

BACK TO THE PROOF

- Induction step:
  IH: Assume that it holds for k, that is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

BACK TO THE PROOF

- Induction step:

  IH: Assume that it holds for k, that is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Prove it for $k + 1$:

$$1 + 2 + \cdots + k + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1), \text{ by IH}$$

$$= (k+1) * \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

✓

POSSIBLE CONFUSION

$P(k)$ has value true or false (boolean).

So, $P(k)$ *is true* means what?

EXAMPLES

"$3 = 2 + 1$" is true.

"$3 = 2 + 2$" is false.

EXAMPLES

"$3 = 2 + 1$" is true.

"$3 = 2 + 2$" is false.

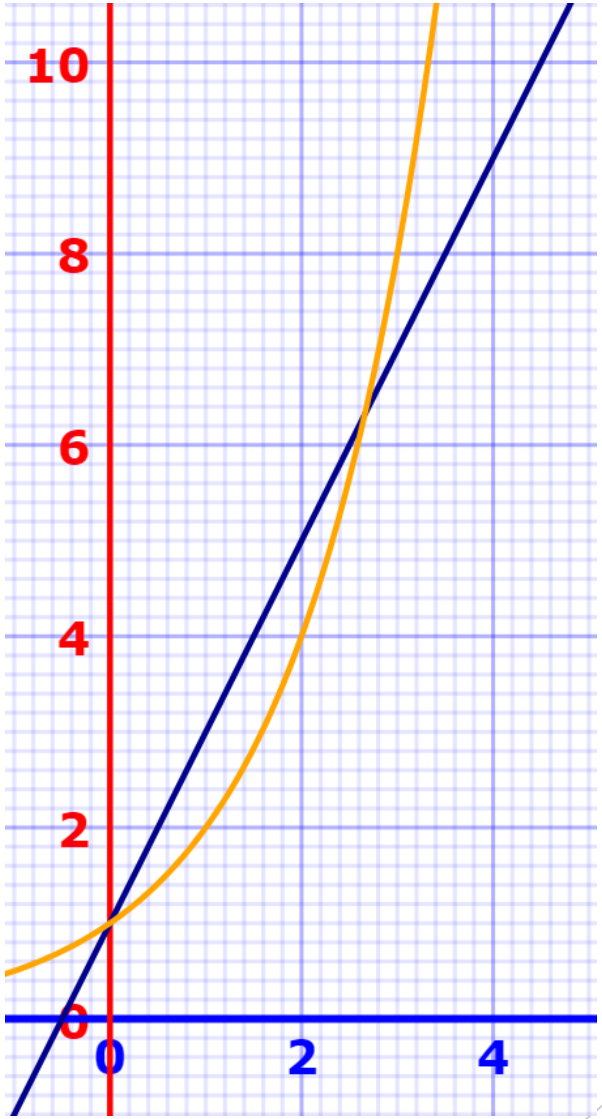
"If $3 = 2 + 2$ then $5 > 7$" is true.

If this is a mystery to you, then I strongly advise you to take MATH 240 or MATH 318 (logic).

EXAMPLE 2

- Prove the following statement:

  For all $n \geq 3$, $2n + 1 < 2^n$.



EXAMPLE 2

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

- Note: P(n) is false for n= 1, 2.

  But that has nothing to do with what we need to prove.

EXAMPLE 2

<u>Statement:</u> For all $n \geq 3$, $2n + 1 < 2^n$.

Proof: (by mathematical induction)

- Base case ($n = 3$):

$$2 * 3 + 1 = 7 < 8 = 2^3$$

✓

<u>Statement:</u> For all $n \geq 3$, $2n + 1 < 2^n$.

EXAMPLE 2

- Induction step:

IH: Assume $2 * k + 1 < 2^k$.

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- Induction step:

IH: Assume $2 * k + 1 < 2^k$.

Prove it for $k + 1$:

$$2 * (k + 1) + 1$$
$$= 2 * k + 1 + 1$$

EXAMPLE 2

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

- Induction step:

IH: Assume $2 * k + 1 < 2^k$.

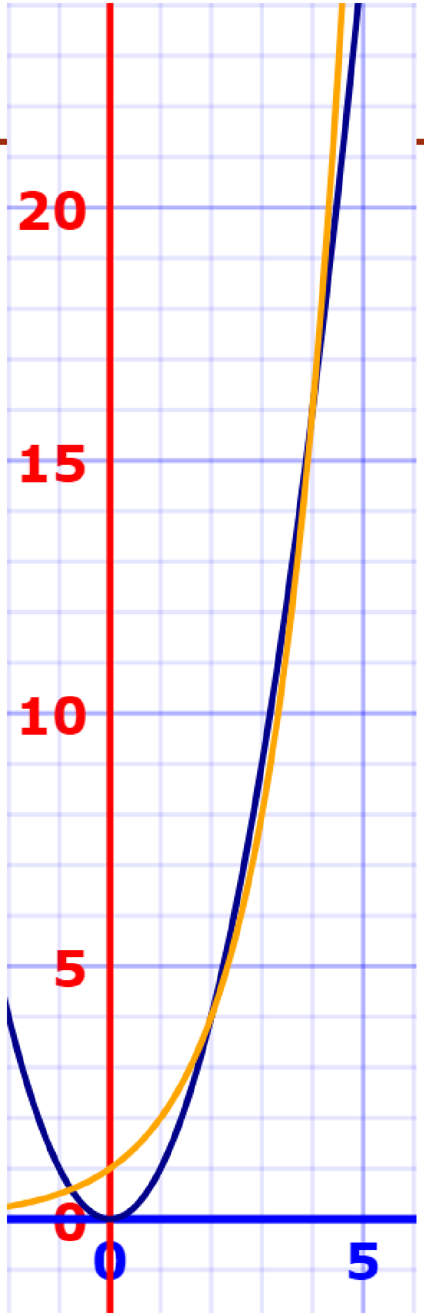
Prove it for $k + 1$:

$$2 * (k + 1) + 1$$
$$= 2 * k + 1 + 1$$
$$< 2^k + 1, \text{ by IH}$$
$$< 2^k + 2^k, \text{ for } k \geq 3$$
$$= 2^{k+1}$$

✓

EXAMPLE 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.



EXAMPLE 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Proof: (by mathematical induction)

- Base case ($n = 5$):

$$5^2 = 25 < 32 = 2^5$$ ✓

EXAMPLE 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.

- Induction step.

  What should we assume?

  What do we need to prove?

EXAMPLE 3

Statement: For all $n \geq 5, \quad n^2 < 2^n$.

- Induction step.

What should we assume? $k^2 < 2^k$ for a $k \geq 5$

What do we need to prove? $(k+1)^2 < 2^{(k+1)}$

EXAMPLE 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.

- Induction step.

  IH: $k^2 < 2^k$ for a $k \geq 5$

$$(k+1)^2 = k^2 + 2k + 1$$

EXAMPLE 3

Statement: For all $n \geq 5, \quad n^2 < 2^n$.

- Induction step.

  IH: $k^2 < 2^k$ for a $k \geq 5$

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< 2^k + 2k + 1, \text{ by IH} \\
&< 2^k + 2^k, \text{ by Example 2} \\
&= 2^{k+1}
\end{aligned}
$$

✓

(STRONG) MATHEMATICAL INDUCTION

- Sometimes one would like to assume the induction hypothesis not only for the previous element, but also for smaller elements. This leads to a logically equivalent proof method called *strong (or complete) mathematical induction*.
- To prove a property by strong mathematical induction, we proceed as follows:
  - Induction step
    Assume the property hold *for all elements* less than or equal to n. (Induction Hypothesis)
    Show that the property also holds for any element generated from n using the inductive clauses.
  - Conclusion
    The property holds for all elements.

FIBONACCI NUMBERS

- The Fibonacci sequence is one of the most common example of a recursively-defined set.

- Consider the following sequence of numbers:

  $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

  Let f_n denote the nth Fibonacci number. How can we define the sequence above?

HISTORY

Originally developed by an Italian mathematician to model… what?

1. Pasta lengths
2. Falling objects
3. Rabbit populations

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FIBONACCI NUMBERS – INDUCTIVE DEFINITION

- Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

- Base clause:
  $f_0 = f_1 = 1$ are Fibonacci numbers.

- Inductive clause:
  If f_{n-1} and f_{n-2} are Fibonacci numbers, then $f_n = f_{n-1} + f_{n-2}$ is a Fibonacci number.

EXAMPLE 4

Statement: For all $n \geq 0$, $\quad f_n \leq \left(\frac{7}{4}\right)^n$

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Proof: (by strong mathematical induction)

- Induction step

  IH: Let k be ≥ 0, and assume that for any number i such that $0 \leq i < k$ then

$$f_i \leq \left(\frac{7}{4}\right)^n$$

EXAMPLE 4

Statement: For all $n \geq 0$, $f_n \leq \left(\frac{7}{4}\right)^n$

Proof: (by strong mathematical induction)

- Induction step

  IH: Let k be ≥ 0, and assume that for any number i such that $0 \leq i < k$ then

$$f_i \leq \left(\frac{7}{4}\right)^n$$

To show: $f_k \leq \left(\frac{7}{4}\right)^k$

EXAMPLE 4

There are 3 possible cases:

1. $k = 0$

   $f_0 = 1$ and $\left(\frac{7}{4}\right)^0 = 1$, so the claim holds.

2. $k = 1$

   $f_1 = 1$ and $\left(\frac{7}{4}\right)^1 > 1$, so the claim holds.

EXAMPLE 4

There are 3 possible cases:

3. $k > 1$

$$f_k = f_{k-1} + f_{k-2}$$

$$\leq \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2}, \textbf{ by IH}$$

$$= \left(\frac{7}{4}\right)^{k-2}\left(1 + \frac{7}{4}\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{11}{4}\right)$$

$$= \left(\frac{7}{4}\right)^{k-2}\left(\frac{44}{16}\right)$$

$$< \left(\frac{7}{4}\right)^{k-2}\left(\frac{49}{16}\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}\right)^{2}$$

$$= \left(\frac{7}{4}\right)^{k}$$

- Recursive algorithms