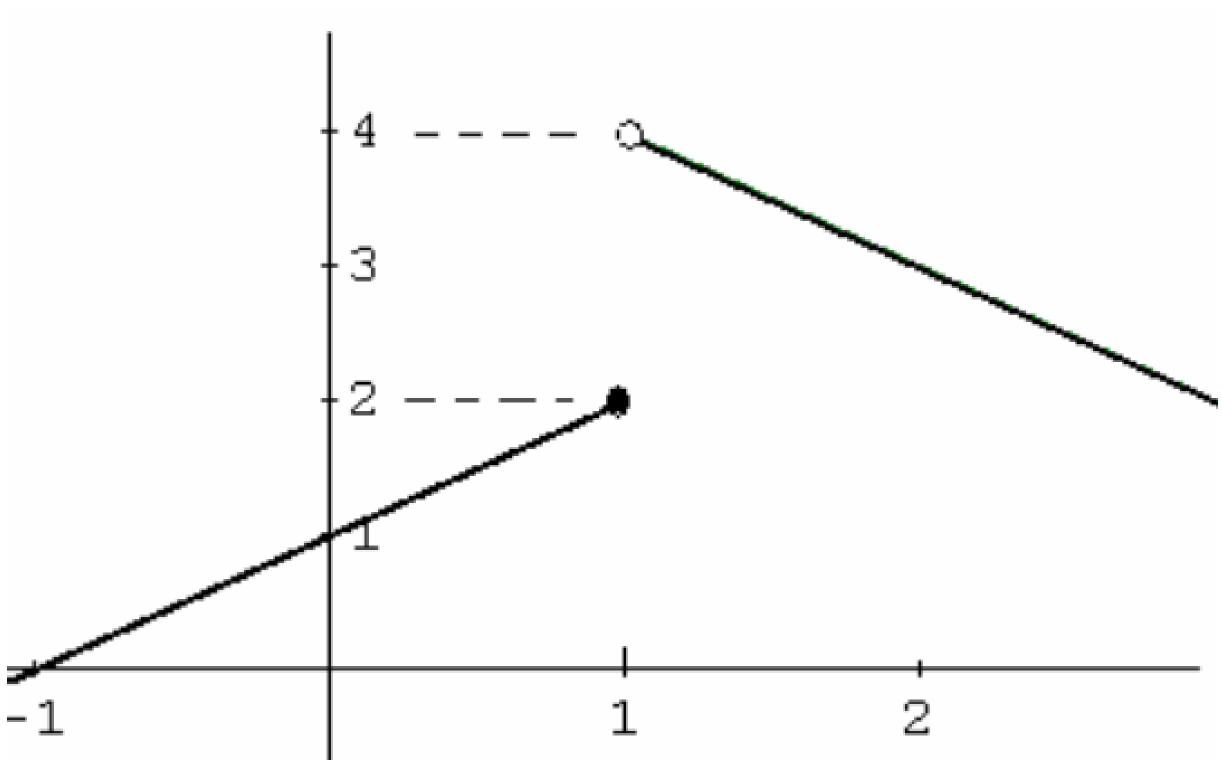
COMP 250

Lecture 35

big O

Nov. 30, 2018

Recall Calculus 1:
Limit of a continuous function



Limit of a sequence

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

What is a "limit" of a sequence ?

Informal definition:

A sequence $t(n)$ has a limit t_∞ means that $t(n)$ becomes arbitrarily close to t_∞ as $n \to \infty$.

Formal definition :

A sequence $t(n)$ has a limit t_∞ if, for any $\varepsilon > 0$, there exists an n_0 such that for any $n \geq n_0$,

$$| t(n) - t_\infty | < \varepsilon.$$

What is a “limit” of a sequence ?

Informal:

A sequence $t(n)$ has a limit t_∞ means that $t(n)$ becomes arbitrarily close to t_∞ as $n \to \infty$.

Formal: (This definition is in all Calculus 1 books.)

A sequence $t(n)$ has a limit t_∞ if, for any $\varepsilon > 0$, there exists an n_0 such that for any $n \geq n_0$,

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$$| t(n) - t_\infty | < \varepsilon.$$

Towards a formal definition of big O

Let $t(n)$ be a function that describes the time it takes some algorithm to run for an input size n.

We would like to express how $t(n)$ grows with n, as n becomes large i.e. *asymptotic* behavior.

Unlike with limits, we want to say that $t(n)$ grows like certain *simpler* functions such as

$log_2 n,\ \ n,\ \ n^2, \ldots, 2^n$, etc.

Preliminary Formal Definition

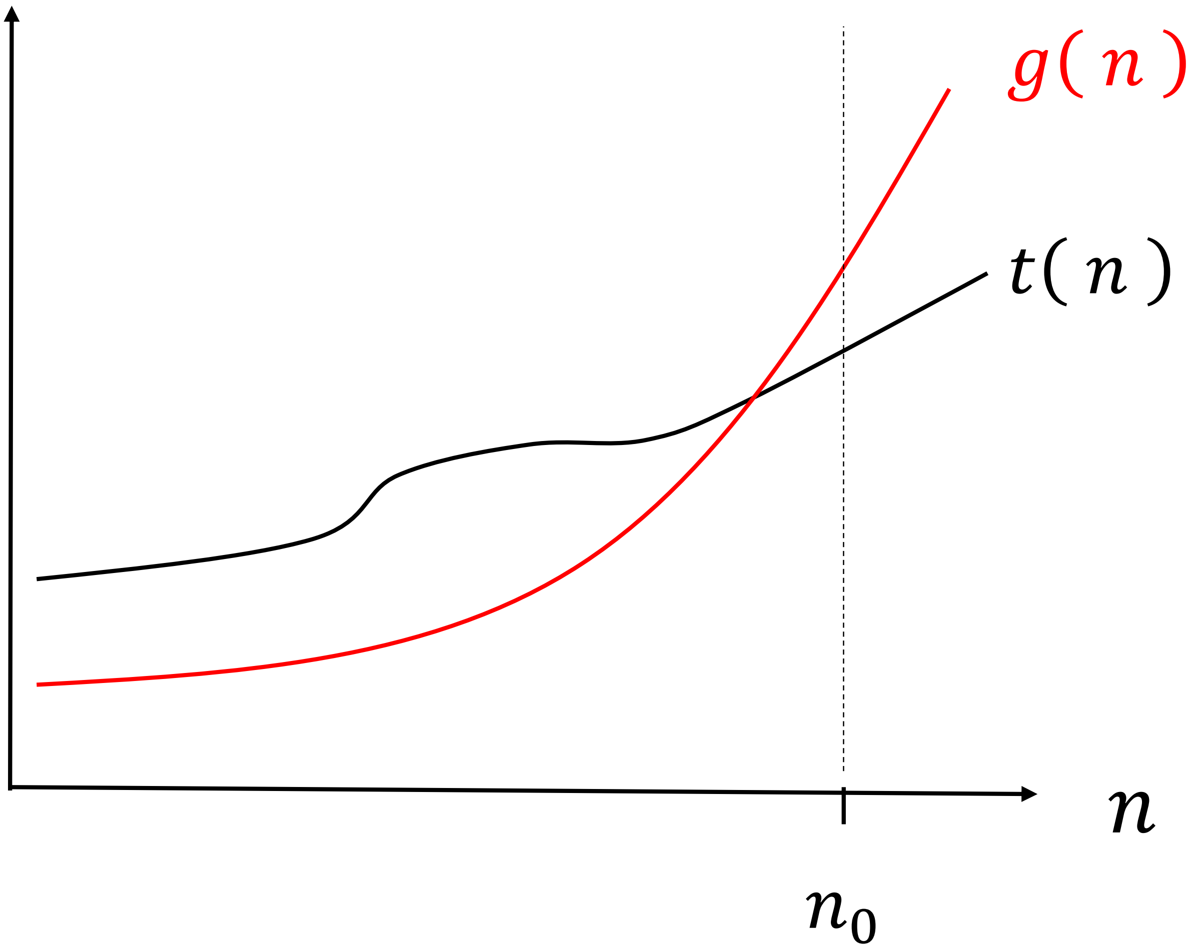
Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.
We say $t(n)$ is *asymptotically bounded above* by $g(n)$
if there exists n_0 such that, for all $n \geq n_0$,

$$t(n) \leq g(n).$$

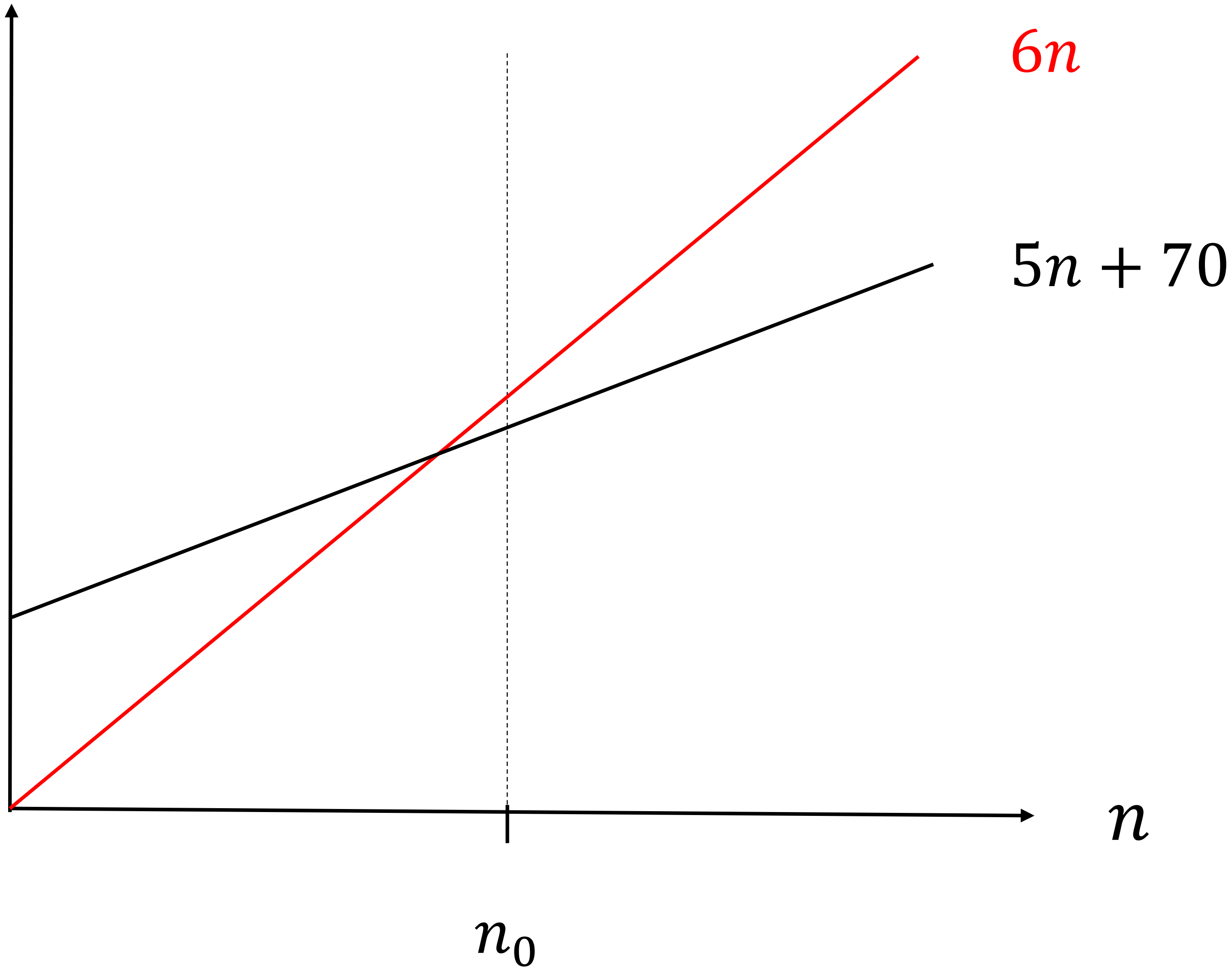
This is not yet a formal definition *of big O.*

How to visualize: "… there exists n_0 such that,

for all $n \geq n_0$, $t(n) \leq g(n)$" ?



Example



Claim: $5n + 70$ is *asymptotically bounded above* by $6n$.

Proof:

(State definition) We want to show there exists an n_0 such that, for all $n \geq n_0$, $5n + 70 \leq 6n$.

Claim: $5n + 70$ is asymptotically bounded above by $6n$.

Proof:

(State definition) We want to show there exists an n_0 such that, for all $n \geq n_0$, $5n + 70 \leq 6n$.

$$5n + 70 \leq 6n$$

$$\Longleftrightarrow \quad 70 \leq n$$

Symbol "$\Longleftrightarrow$" means "if and only if" i.e. logical equivalence

Claim: $5n + 70$ is asymptotically bounded above by $6n$.

Proof:

(State definition) We want to show there exists an n_0 such that, for all $n \geq n_0$, $5n + 70 \leq 6n$.

$$
\begin{aligned}
& 5n + 70 \leq 6n \\
\Longleftrightarrow \quad & 70 \leq n
\end{aligned}
$$

So we could use $n_0 = 70$.

Symbol "$\Longleftrightarrow$" means "if and only if" i.e. logical equivalence

We would like to express formally how some function $t(n)$ grows with n, as n becomes large.

We would like to compare the function $t(n)$ with *simpler* functions , $g(n)$, such as $log_2 n,\ n,\ \ n^2, \dots, 2^n$, etc.

Formal Definition of Big O

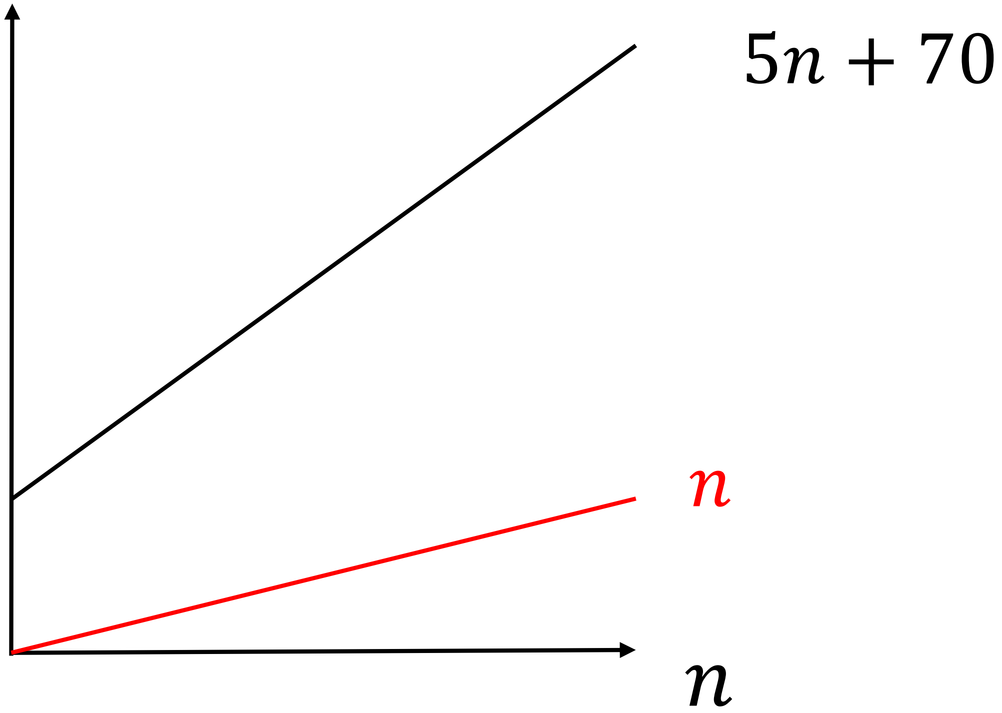
Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.

$g(n)$ will be a simple function, but this is not required in the definition.

We say $t(n)$ is $O(\ g(n)\ )$ if there exist *two* positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\ n\ ) \leq \ c\ \ g(\ n\ ).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.



Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\leq\; ?$$

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\,n\,) \leq \; c \; g(\,n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\leq\; 5\,n + 70n, \quad \text{if } n \geq 1$$

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\,n\,) \leq \; c\;\; g(\,n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\leq\; 5\,n + 70n, \quad \text{if } n \geq 1$$

$$= \; 75\,n$$

c $\qquad$ n_0

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\,n\,) \leq \; c \;\; g(\,n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\le \qquad\qquad ?$$

We can come up with a tighter bound for c by using a larger n_0.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\leq\; 5\,n + 6n, \qquad \text{if } n \geq 12$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\leq\; 5\,n + 6n, \qquad \text{if } n \geq 12$$

$$= \; 11\,n$$

So take $c = 11$, $n_0 = 12$.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 3:

$$5\,n + 70 \ \leq \qquad\qquad ?$$

We can come up with a tighter bound for c by using a larger n_0.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 3:

$$5\,n + 70 \;\leq\; 5\,n + n, \qquad\qquad n \geq 70$$

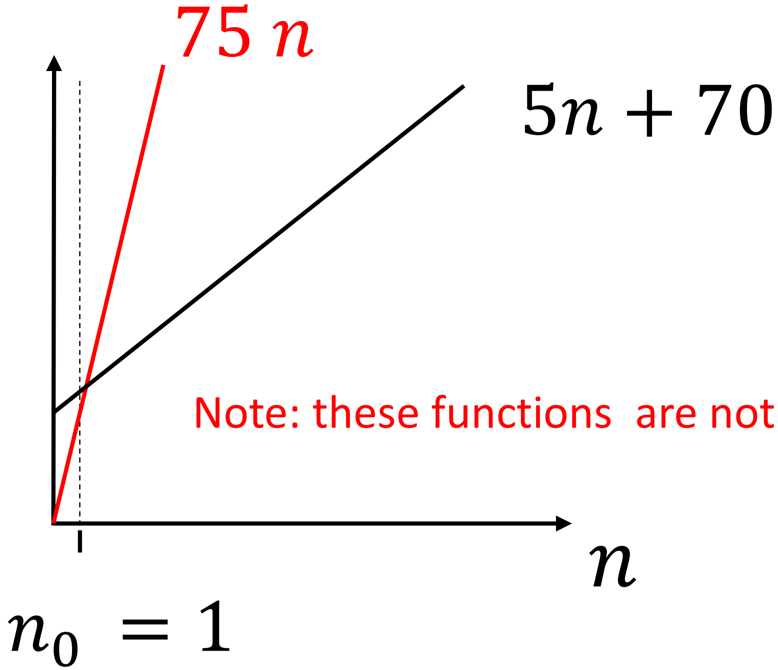
Claim: $5\,n + 70$ is $O(\,n\,)$.

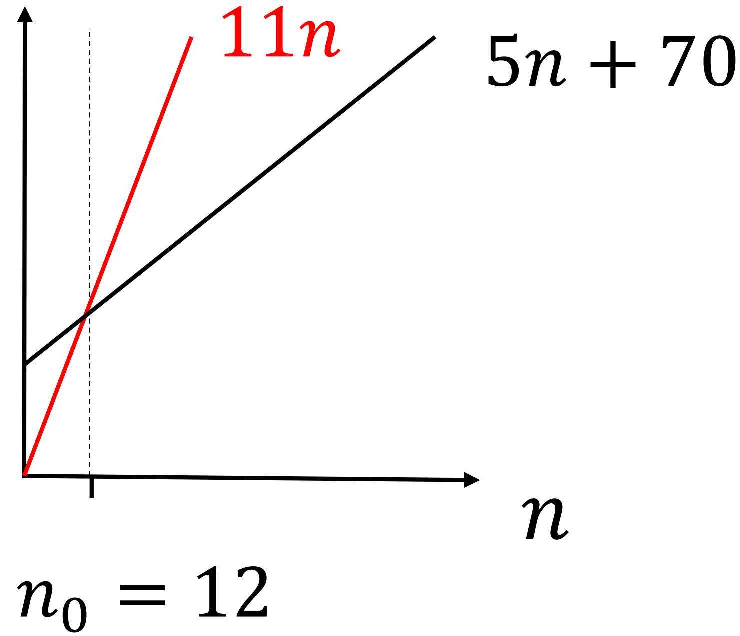
Proof 3:

$$5\,n + 70 \;\le\; 5\,n + n, \qquad\qquad n \ge 70$$

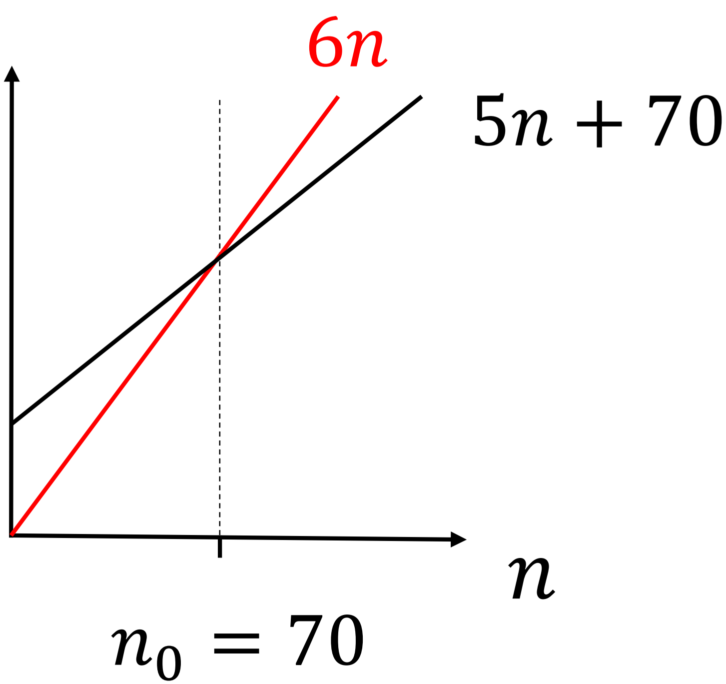
$$= \;6\,n$$

So take $c = 6,\;\; n_0 = 70$.





Note: these functions are not to scale.



So, different combinations of n and c will satisfy the definition that $t(n)$ is O($g(n)$).

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

Claim: $8\,n^2 \;-17n+46$ is $O(n^2)$.

Proof (1):

$$8\,n^2 \;-17n+46$$

$$\leq\; 8\,n^2 \qquad\quad +46\,n^2, \qquad n \geq\; 1$$

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

$$\leq 8\,n^2 \qquad + 46\,n^2, \qquad n \geq 1$$

$$\leq 54\,n^2$$

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

$$\leq 8\,n^2 \qquad + 46\,n^2, \qquad n \geq 1$$

$$\leq 54\,n^2$$

So take $c = 54$, $n_0 = 1$.

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (2):

$$8\,n^2 - 17n + 46$$

Claim: $8n^2 - 17n + 46$ is $O(n^2)$.

Proof (2):

$$8n^2 - 17n + 46$$

$$\leq 8n^2, \qquad n \geq 3$$

since $17 * 3 = 51$

So take $c = 8$, $n_0 = 3$.

What does $O(1)$ mean?

We say $t(n)$ is $O(1)$, if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(n) \leq c.$$

So it just means that $t(n)$ is bounded.

Note: $t(n)$ has a finite number of values for $n < n_0$.

Never write $O(\,3n\,)$, $O(\,5\,log_2 n\,)$, etc.

Instead, write $O(\,n\,)$, $O(log_2 n\,)$, etc.

Why? The point of big O notation is to *avoid dealing with these constant factors*.

It is still *technically* correct to write the above.
We just *never* do it.

"Tight Bounds"

Big O is about *upper* bounds.

If $t(n)$ is $O(n)$, then can we say that $t(n)$ is also $O(n^2)$?

“Tight Bounds”

Big O is about *upper* bounds.

If $t(n)$ is $O(n)$, then can we say that $t(n)$ is also $O(n^2)$?

According to the formal definition, yes we can.

But when we ask for “tight bounds” on $t(n)$, we want the simple function $g(n)$ with the smallest growth rate.

(More on this next class.)

Incorrect Proofs

In MATH 240 (for CS) or MATH 235 (for Math/CS), you will learn how to *write* proofs.

Here are some typical mistakes that you might make.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Incorrect Proof:

$$\begin{aligned} 5n + 70 &\leq cn \\ 5n + 70n &\leq cn, \quad n \geq 1 \\ 75n &\leq cn \end{aligned}$$

$$\text{Thus,} \quad c = 75, \quad n_0 = 1$$

Q: Why is this *proof* incorrect ?

Claim: $5\,n + 70$ is $O(\,n\,)$.

Incorrect Proof: (for a correct proof, see earlier)

$$
\begin{aligned}
5n + 70 &\le cn \\
5n + 70n &\le cn, \quad n \ge 1 \\
75n &\le cn
\end{aligned}
$$

$$\text{Thus,} \quad c = 75, \quad n_0 = 1$$

Q: Why is this *proof* incorrect ?

A: Because we don't know how lines are logically related.

Another Example of an Incorrect Proof

Claim: for all $n \geq 0$, $2n^2 \leq (n+1)^2$.

Proof:

$$
\begin{aligned}
2n^2 &\leq (n+1)^2 \\
&\leq (n+n)^2, \quad \text{when } n > 0 \\
&= 4n^2
\end{aligned}
$$

Since $2n^2 \leq 4n^2$, we are done.

Unfortunately, the claim is false! (Take $n = 3$)

Claim: for all $n \geq 0$, $2n^2 \leq (n+1)^2$.

Proof:

$$
\begin{aligned}
2n^2 &\leq (n+1)^2 \\
&\leq (n+n)^2, \quad \text{when } n > 0 \\
&= 4n^2
\end{aligned}
$$

It is incorrect to assume what you are trying to prove.

Announcements

- Quiz 5 today
- TODO next week (Mon & Tues)
  - Big Omega & Big Theta
  - Best cases and Worst Cases
  - Limit rules (Calculus 1 revisited)