Introduction

Transaction cost

Broker: commision

Market maker: bid-ask spread.

Short-selling

The lender must be compensated for dividend received.

Compounding (T: # years)

Effective annual rate : $(1+r)^T$

Annual rate (compounded n times per year) : $(1+\frac{r}{n})^{nT}$

Annualized continuously compounded rate : $e^{rT} = \lim_{n\to\infty}(1+\frac{r}{n})^{nT}$

Forward

Future contracts: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today.

$$\text{payoff} \begin{cases} \text{long: } S_T - F_{0,T} \\ \text{short: } F_{0,T} - S_T \end{cases}$$

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

Suppose that $F_{0,T} > S_0e^{(r-\delta)T}$. Then an investor can execute the following trades at time 0 (**buy low and sell high**) and obtain an arbitrage profit:

Transaction	Cash Flows Time 0	Cash Flows Time T (expiration)
Buy tailed position in stock ($e^{-\delta T}$ units)	$-S_0e^{-\delta T}$	S_T
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T} > 0$

div

Suppose that $F_{0,T} < S_0e^{(r-\delta)T}$. Then an investor can execute the following trades at time 0 (**buy low and sell high**) and obtain once again an arbitrage profit:

Transaction	Cash Flows Time 0	Cash Flows Time T (expiration)
Short tailed position in stock ($e^{-\delta T}$ units)	$S_0e^{-\delta T}$	$-S_T$
Lend $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$S_0e^{(r-\delta)T}$
Long forward	0	$S_T - F_{0,T}$
Total	0	$S_0e^{(r-\delta)T} - F_{0,T} > 0$

Note: The future and forward profits differ because of the interest earned on the mark-to-market proceeds.

Option

$$\text{call} \begin{cases} \text{payoff} & C_T = \max[S_T - K, 0] \\ \text{profit} & \text{call profit} = \max[S_T - K, 0] - FV(\text{premium}) \end{cases}$$

$$\text{put} \begin{cases} \text{payoff} & P_T = \max[K - S_T, 0] \\ \text{profit} & \text{put profit} = \max[K - S_T, 0] - FV(\text{premium}) \end{cases}$$

Put-Call Parity

payoff at maturity: $C_T - P_K = S_T - K$

$\Rightarrow C_t - P_t = S_t - PV(K)$

Parity generally fails for American-style options, which may be exercised prior to maturity.

$\Rightarrow$ If the stock is paying dividend,
$C_t - P_t = [S_t - PV(Div)] - PV(K)$.

Maximum and Minimum Option Prices.

$$\text{Call: } \begin{cases} S_t \geq C_t \\ C_t \geq \max[0, S_t - PV(Div) - PV(K)] \end{cases}$$

$$\text{Put: } \begin{cases} P_t \leq K \\ P_t \geq \max[0, PV(K) - S_t + PV(Div)] \end{cases}$$

Commodity Futures

$F_{0,T} = S_0 e^{(r-\delta_l)T}$ commodity lease rate. $\begin{cases} \delta_l = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right) & \text{continuous} \\ \delta_l = \frac{1+r}{(F_{0,T}/S_0)^{1/T}} - 1 & \text{discrete} \end{cases}$

$\delta_l = \underset{-u}{\text{storage cost}} + \underset{y}{\text{convience yield}}$

$\Rightarrow F_{0,T} = S_0 e^{(r+u-y)T} = S_0 e^{(c-y)T}$ where $c = r+u$ is the cost of carry

The Forward Curve

contango (upward) – storage high
backwardation (downward) – conviency high

Basis Risk The future or forward contract don't represent exactly what is being hedged.

Cross hedging The use of a derivative on one asset to hedge another asset.

$$\text{profit} = (S_t - S_{t-1}) + N(F_t - F_{t-1})$$

$$\Delta S = a + h \times \Delta F + \varepsilon$$

$$\begin{cases} h^* = \rho \frac{\sigma_S}{\sigma_F} & \text{minimum variance hedge ratio} \\ R^2 = \rho^2 & \text{hedge effectiveness} \end{cases}$$

$N^* = \frac{h^* Q_A}{Q_F}$ (tailing the hedge) $N^* = \frac{h^* Q_A S_0}{Q_F F_{0,T}}$

Binomial Option Pricing

<u>One-step</u> T : maturity n: # period h: T/n.

replicating portfolio $\begin{cases} \Delta = e^{-\delta h} \dfrac{C_i^u - C_i^d}{S_0(u-d)} \\ B = e^{-rh} \dfrac{C_i^d u - C_i^u d}{u-d} = e^{-rh}(C_i^u - \Delta S_0 u e^{\delta h}) \end{cases}$ sensitivity of the option to a change in the stock price

$$C_0 = \Delta S_0 + B$$
$$= e^{-rh}\left(C_i^u \frac{e^{(r-\delta)h} - d}{u-d} + C_i^d \frac{u - e^{(r-\delta)h}}{u-d} \right)$$

risk-neutral probability

<u>No Arbitrage</u> $d < e^{(r-\delta)h} < u$.

Uncertainty in the Binomial

$\begin{cases} u = e^{(r-\delta)h + \sigma\sqrt{h}} \\ d = e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$

$\Rightarrow$ guaranteed to satisfy no arbitrary restriction.

$\sigma_h = \sigma\sqrt{h}$ ↗ divide the year into n periods of length h $h = \frac{1}{n}$

annualized std. of the continuously compounded stock return.

Note: zero volatility doesn't mean that prices are fixed; it means that prices are known in advance.

Risk Neutral Pricing

periods n, the price of an European call option

$$C_0 = e^{-rT} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \max[S_0 u^k d^{n-k} - K, 0]$$

At no point are we assuming that investors are risk-neutral. Rather, risk-neutral pricing is an interpretation of the formulas

Black-Scholes for a European call and put option on a stock that pays dividends at the continuous rate δ

$$C_0 = \underbrace{S_0 e^{-\delta T} N(d_1)}_{\substack{\Delta \\ \text{sensitivity}}} \underbrace{- K e^{-rT} N(d_2)}_{B}$$

probability of exercising

money to pay at maturity

$$P_0 = \underbrace{-e^{-\delta T} N(-d_1)}_{\substack{\Delta \\ \text{sensitivity}}} S_0 + \underbrace{K e^{-rT} N(-d_2)}_{B}$$

probability of exercising

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Put-Call Parity must hold: $P_0 = C_0 + K e^{-rT} - S e^{-\delta T} \quad (N(x) = 1 - N(-x))$

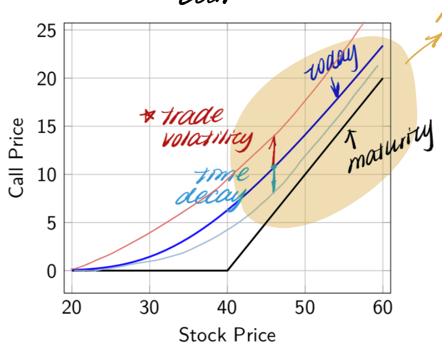
Beta is irrelevant

$$\beta_{option} = \frac{\Delta S_0}{\Delta S_0 + B} \beta_{stock} \Rightarrow \beta_{option} > \beta_{stock}$$

larger than one

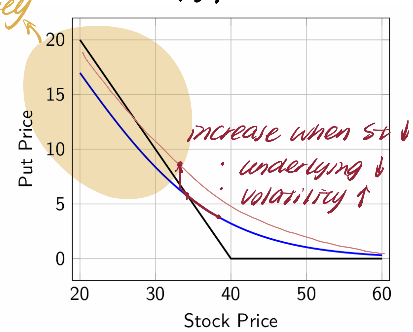
The larger average payoff to option on high beta stocks is exactly offset by the larger discount rate.

Call



in the money

Put



$\max\{S_T - K, 0\} \approx S_T - K$

$PV(S_T - K) = P_0 - PV(K) < P_0 - K$

blue line above

$\max\{K - S_T, 0\} \approx K - S_T$

$PV(K - S_T) = PV(K) - S_0 > K - S_0$

blue line below

Delta Hedging

The market-maker takes an offsetting position in shares, position that hedge the fluctuations in the option prices.

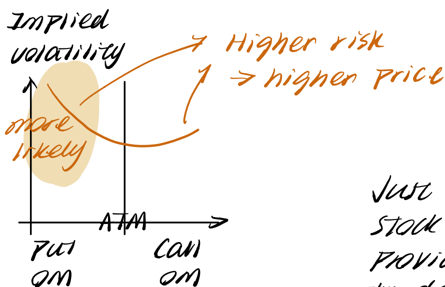
large moves always causes loses

Implied Volatility

Unconditional volatility is estimated as the sample standard deviation

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t-\mu)^2} \qquad \text{historical volatility}$$

$\Rightarrow$ Invert Black-Scholes formula to obtain implied volatility



Volatility skew

Just as stock markets provide information about stock prices and permit trading stocks, option markets provide information about volatility and permit trading volatility

① Traders can see how sensitive are option prices to changes in market conditions.

② hedging

With the exception of the Elasticity, all the Greeks of a portfolio are calculated as a sumproduct of portfolio positions (weights) and individual greeks.

Option Greeks (cont'd)

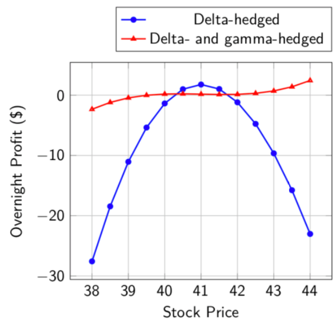
- The Greeks are tools that let us to quantify these relationships:

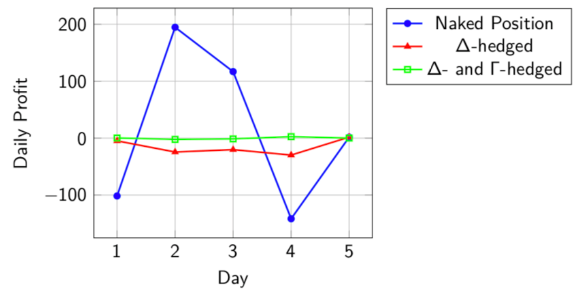
Input	Greek	Definition	Mnemonic
S_t	Δ (Delta)	Measures the option price change when the stock price increases by \$1 first derivative	
S_t	Γ (Gamma)	Measures the change in Δ when the stock price increases by \$1 second derivative	
S_t	Ω (Elasticity)	Measures the percentage change in the option price when the stock price increases by 1%	
σ	Vega	Measures the option price change when there is an increase in volatility of 1%	vega ↔ volatility
t	θ (theta)	Measures the option price change when there is a decrease in the time to maturity (increase in calendar time) of 1 day	theta ↔ time
r	ρ (rho)	Measures the option price change when there is an increase in the interest rate of 1% (100 basis points) 0.01%	rho ↔ r
δ	Ψ (Psi)	Measures the option price change when there is an increase in the continuous dividend yield of 1% (100 basis points)	

Delta - Gamma Hedging

Stock + 2 Option on the same underlying with non-zero gamma

losses less when there is large move down, and can make money if the stock price increases.



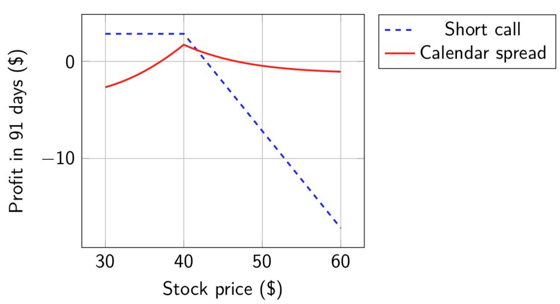


Calendar Spreads

long long maturity, short short maturity

To protect against a stock price increase when selling a call, you can simultaneously buy a call option with the same strike and greater time to expire

Written near-to-expiration option exhibits greater time decay than the purchased far-to-expiration option, and therefore is profitable if the stock prices doesn't move.



Portfolio Insurance

Stock + put (synthetically)

Maintaining a position in the underlying asset so that the delta of the position is equal to the delta of the required position.