TIME VALUE OF MONEY

Required Reading

- **Chapter 4**, "*Time Value of Money*" from J. Berk et al., Fundamentals of Corporate Finance, Second Canadian Edition.

Corporate Finance

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Valuation of real and financial assets

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Valuation of cash flows generated by real and financial assets:

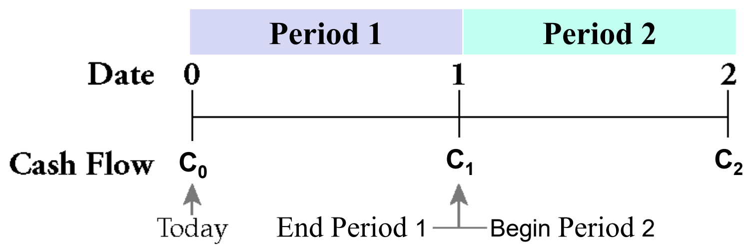
- *Size of Cash Flows*

- *Timing of Cash Flows*

- *Risk of Cash Flows*

The Timeline

- Identifying Dates on a Timeline
  - →Date 0 is today, the beginning of the first period
  - →Date 1 is the end of the first period
  - →Date 2 is the end of the second period, etc.

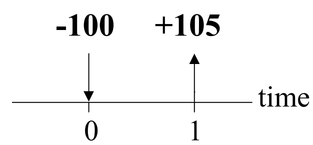


Question

Assume that the interest rate is 5% per year. How much would you be willing to pay for a bond that pays $105 in one year?

Solution

"*If I invest $100 today at the current interest rate, I can get $105 next year*"



Question (Cont)

This means that:

- $105 is the **FUTURE VALUE** of $100 received today.
- $100 is the **PRESENT VALUE** of $105 received in one year.
- I am willing to pay up to $100 dollars for a bond that pays $105 in one year.
- I would be indifferent between receiving $100 today or $105 in one year.
- $100 dollars today are worth $105 next year.

Question

Assume that the interest rate is 10% per year. If you invest \$100, how much will you have in three years? And how much will you have in 100 years?

Solution

At the end of the first year:

$$\underbrace{\$100}_{\text{Principal}} + \underbrace{\$100 \times 0.1}_{\text{Interest}} = \$100 \times (1+0.1) = \$110$$

Solution (cont)

At the end of the second year:

$$\underbrace{\underbrace{\$100 \times (1+0.1)}_{\textit{After 1 year}} \times (1+0.1)}_{\textit{After 2 years}} = \$\ 121$$

At the end of the third year:

$$\$100 \times (1+0.1) \times (1+0.1) \times (1+0.1) = \$\ 133.1$$

After 100 years:

$$\$100 \times (1+0.1)^{100} = \$1{,}378{,}061$$ **Millionaires !!!!**

Question: Why do we have $133.1 instead of $130 by the end of the third year?

Answer: Because we are earning interest on the interest earned in previous years. This is what we called **COMPOUNDING.**

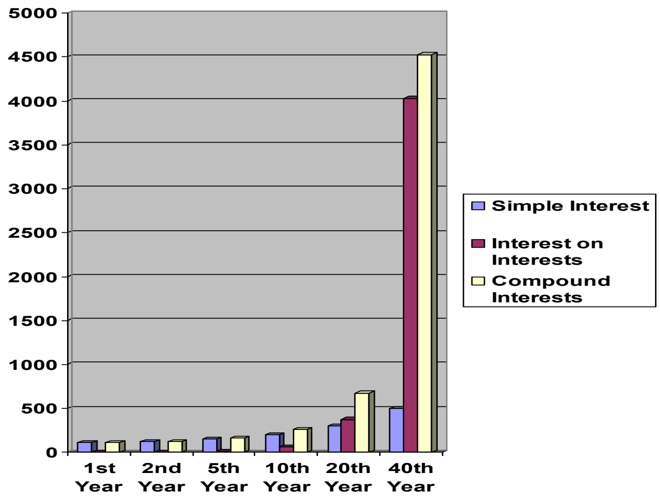
- ***Compounding*****:** The process of accumulating interest in an investment over time to earn more interest.
- ***Compound Interest*****:** Interest earned on both, the initial principal and the interest reinvested from prior periods.
- ***Simple Interest*****:** Interest earned on the original amount invested.

How Big a Deal is Compounding?

Let's go back to our example of $100 and 10% interest rate.

	Compound Interest	Simple Interest	Interest on Interest
1st Year	110	110	0
2nd Year	121	120	1
5th Year	161	150	11
10th Year	259	200	59
20th Year	673	300	373
40th Year	4526	500	4026
100th Year	1,378,061	1100	1,376,961

The Power of Compounding



Future Value of a Cash Flow

$$\boxed{FV_n = C_0 \times (1+r)^n}$$

where

***n* :** Number of periods

***r* :** Interest rate or discount rate per period

***C_0* :** Cash flow at time 0

***FV_n*:** *Future Value at time n, that is,* the value of C_0 after *n* periods when invested at a rate of *r* per period.

Present Value & Discounting

So far we have been asking the question of how much an investment would grow over time given an interest rate. We have called this process compounding.

Discounting refers to the opposite question: how much is the current value of some future cash flow?

Discounted Cash Flow (DCF) Valuation is the process of finding the present value of a future cash flow.

Let's see some examples…

Q. How much should I invest today if I want to be a millionaire in 40 years and the interest rate is 12% per year?

Answer

$$PV_0 = \frac{1{,}000{,}000}{(1+0.12)^{40}} = 10{,}747$$

Q. Dot.com has decided to go ahead and expand into a new business. The company is selling bonds that pay the holder $2,500 dollars after 5 years. The market interest rate at that time is 9%. What is current market value of the bond?

Answer

$$PV_0 = \frac{2{,}500}{(1+0.09)^{5}} = 1{,}624.83$$

Present Value of a Cash Flow

$$\boxed{PV_0 = \frac{C_n}{(1+r)^n}}$$

where

n : Number of periods

r : Interest rate or discount rate per period

C_n : Cash flow n periods from now (i.e., at time n)

PV_0 : *Present Value,* that is, what a cash flow C_n at time n is worth today.

Present Value and Future Value

$$\boxed{FV_n = PV_0 \times (1+r)^n}$$

The following statements are equivalent:

1. PV_0 is the present value of FV_n dollars received n periods from today.
2. FV_n is the future value at time n of PV_0 dollars received today.
3. You would be indifferent between receiving PV_0 dollars today or FV_n dollars in n periods.
4. FV_n dollars in n periods are worth PV_0 dollars today.

Q. Mary has purchased a painting in a yard sale for \$50. Two years later she goes to the Antique Road Show and she is told that her painting is from the 1800's. She decides to sell the painting in auction and see gets \$500,000. What is the *rate of return* of her investment?

Answer

$$500{,}000 = 50 \times (1 + \boldsymbol{r})^2$$

Solving for *r* in the above equation:

$$r = \sqrt[2]{\frac{500{,}000}{50}} - 1 = 99 \Rightarrow 9900\%$$

Not a bad return!!!

Q. Peter has \$1,000 in his bank account. The interest rate is 4% per year. How long will it take for Peter to become a millionaire?

Answer

$$1{,}000 = \frac{1{,}000{,}000}{(1.04)^{\boldsymbol{n}}} \Rightarrow (1.04)^{\boldsymbol{n}} = 1{,}000 \Rightarrow \boldsymbol{n} = 176.12 \text{ years}$$

Aside: To solve for an exponent you need to take the logarithm of both sides and use the fact that $\ln(\boldsymbol{x}^{\boldsymbol{y}}) = \boldsymbol{y}\ln(\boldsymbol{x})$

$$(1.04)^{\boldsymbol{n}} = 1000 \Rightarrow \boldsymbol{n}\ln(1.04) = \ln(1000) \Rightarrow \boldsymbol{n} = \frac{\ln(1000)}{\ln(1.04)} = 176.12$$

Combining & Comparing Cash Flows

Only values at the same point in time can be combined or compared

Example:

To compare a cash flow generated at t=0 with a cash flow generated at t=5 you need to discount the cash flow generated at t=5 by 5 periods and *then* compare the result with the cash flow generated at t=0.

Present Value of a Stream of Cash Flows

Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, C_2 at date 2, and so on, up to C_n at date n.

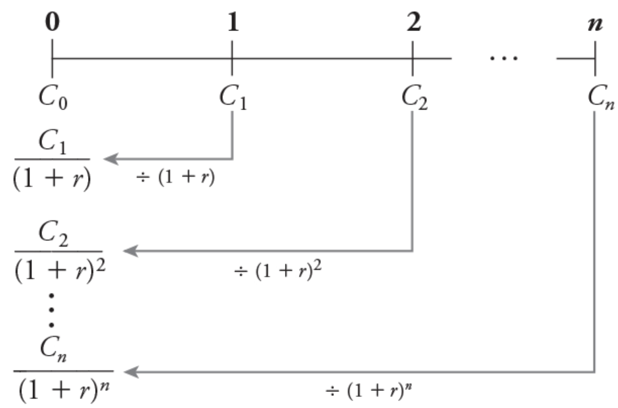
0	1	2	3		$n-1$	n	**time**
C_0	C_1	C_2	C_3	--------	C_{n-1}	C_n	**CF**

What is the present value of this stream of cash flows?

Present Value of a Stream of Cash Flows

We compute the PV of this stream of cash flows in 2 steps:

- ***First***, compute the PV of each individual cash flow.



Present Value of a Stream of Cash Flows

- ***Second***, add up the present value of each individual cash flow.

$$C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + ... + \frac{C_n}{(1+r)^n} \Leftrightarrow \sum_0^n \frac{C_t}{(1+r)^t}$$

Summing up…

Present Value of a Stream of Cash Flows

0	*1*	*2*	*3*	*....*	*n-1*	*n*	**time**
C_0	C_1	C_2	C_3	*....*	C_{n-1}	C_n	**CF**

"The present value of a stream of cash flows is the sum of the present values of each cash flow"

$$\boxed{PV_0 = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t}}$$

Q. If the interest rate is 10% per year. How much is the current market value of a bond that pays \$150 after 4 years and \$200 after 5 years.

0	1	2	3	4	5	time
?				150	200	

Answer

$$PV_0 = \frac{150}{(1+0.1)^4} + \frac{200}{(1+0.1)^5} = 226.63$$

Future Value of a Stream of Cash Flows

Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, C_2 at date 2, and so on, up to C_n at date n.

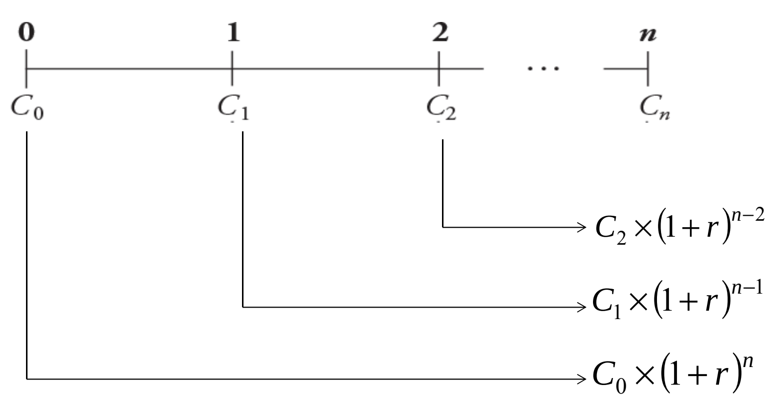
0	1	2	3		$n-1$	n	**time**
C_0	C_1	C_2	C_3	--------	C_{n-1}	C_n	**CF**

What is the future value of this stream of cash flows?

Future Value of a Stream of Cash Flows

We compute the FV of this stream of cash flows in 2 steps:

- ***First***, compute the FV of each individual cash flow.



Future Value of a Stream of Cash Flows

- ***Second***, add up the future value of each individual cash flow.

$$C_0 \times (1+r)^n + C_1 \times (1+r)^{n-1} + C_2 \times (1+r)^{n-2} + \ldots + C_n$$

(which can be also written as)

$$\sum_{t=0}^{n} C_t \times (1+r)^{n-t}$$

Summing up…

Future Value of a Stream of Cash Flows

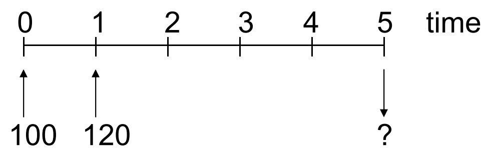
0	*1*	*2*	*3*	*....*	*n-1*	*n*	**time**
C_0	C_1	C_2	C_3	*....*	C_{n-1}	C_n	**CF**

"The future value of a stream of cash flows is the sum of the future values of each cash flow."

$$\boxed{FV_n = \sum_{t=0}^{n} C_t \times (1+r)^{n-t}}$$

Q. If the interest rate is 10% per year and you invest \$100 dollars this year and \$120 next year. How much will you have in five years?

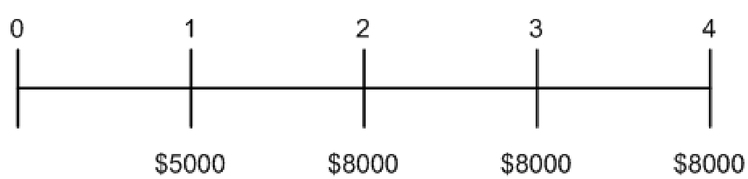
Solution



$$FV_5 = 100 \times (1+0.1)^5 + 120 \times (1+0.1)^4 = 336.74$$

Example: PV and FV of a stream of cash flows

Assuming an interest rate of 6% per year calculate **(a)** the present value and **(b)** the future value of the following stream of cash flows:



Example: PV and FV of a stream of cash flows

(a) The present value of the stream of cash flows is the sum of the present values of each cash flow:

$$\boldsymbol{PV}_0 = \frac{5000}{1.06} + \frac{8000}{1.06^2} + \frac{8000}{1.06^3} + \frac{8000}{1.06^4} = \$24{,}890.66$$

Example: PV and FV of a stream of cash flows

(b) Two **equivalent** ways of calculating the future value of the stream of cash flows*:*

1. *As the sum of the future values of each cash flow:*

$$FV_4 = 5000\times 1.06^3 + 8000\times 1.06^2 + 8000\times 1.06 + 8000$$
$$= \$31{,}423.88$$

2. *As the future value of the present value of the stream of cash flows calculated in part **(a)**:*

$$FV_4 = PV_0 \times (1+r)^4 = \$24{,}890.66\times 1.06^4$$
$$= \$31{,}423.88$$

Example: PV and FV of a stream of cash flows

Hence, the example shows that you can compute the future value of a stream of cash flows directly:

$$\boxed{FV_n = \sum_{t=0}^{n} C_t \times (1+r)^{n-t}}$$

or you can first compute the PV of the stream of cash flows and them move this PV to the future:

$$\boxed{FV_n = PV_0 \times (1+r)^n}$$

$$PV_0 = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t}$$

Relation between the Present Value and the Future Value of a Stream of Cash Flows

0	1	2	3	....	n-1	n	**time**
C_0	C_1	C_2	C_3	....	C_{n-1}	C_n	**CF**

$$\boxed{FV_n = PV_0 \times (1+r)^n}$$

- The stream of cash flows is worth PV_0 today.
- The stream of cash flows is worth FV_n in n periods
- An investor would be indifferent between receiving the stream of cash flows **or** PV_0 dollars today **or** FV_n dollars in n periods.

The Net Present Value (NPV)

The Net Present Value (NPV) of an investment is the present value of the stream of cash flows associated to the investment.

NPV = PV(Benefits) - PV(Costs) = PV(Benefits - Costs)

- If the NPV > 0 the investment increases value and should be taken.

Example: Net Present Value –NPV– of an Investment Opportunity

Q. If you invest $1,000 today, you will receive $500 at the end of each of the next three years. If you could otherwise earn 10% per year on your money, should you undertake the investment opportunity?

Solution

0	1	2	3	time
-1000	500	500	500	

$$NPV = -1{,}000 + \frac{500}{1.1} + \frac{500}{1.1^2} + \frac{500}{1.1^3} = \$243.43$$

Since the NPV is positive the *PV* of the benefits (that is, \$500 at t=1,2, and 3) exceed the *PV* of the costs (that is, \$1,000 at t=0), we should make the investment.

Perpetuity

Perpetuity: A stream of equal cash flows that occurs at regular intervals and last forever.

- The timeline for a perpetuity:

0	1	2	3	...
	C	C	C	

- **Note**: The ***first cash flow*** does not occur immediately; it arrives at time 1.

Deriving the PV of an Perpetuity

$$PV_0(\text{Perpetuity}) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots\ldots$$

- Multiplying both sides by (1+r):

$$(1+r)PV_0(\text{Perpetuity}) = C + \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots\ldots$$

- Subtracting the second expression from the first one and rearranging:

$$\boxed{PV_0(\text{Perpetuity}) = \frac{C}{r}}$$

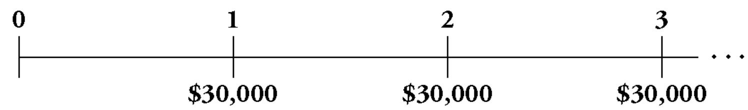
Question

You want to endow an annual graduation party at your alma mater. You want the event to be a memorable one, so you budget $30,000 per year forever for the party.

If the university earns 8% per year on its investments, and if the first party is in one year's time, how much will you need to donate to endow the party?

Solution

Timeline:



Using the PV of a perpetuity formula

$$\boldsymbol{PV}_0 = \frac{\boldsymbol{C}}{\boldsymbol{r}} = \frac{30{,}000}{0.08} = \$375{,}000$$

If you donate $375,000 today, and if the university invests it at 8% per year forever, then the graduates will have $30,000 every year for their graduation party.

Annuity

Annuity: A stream of equal cash flows arriving at regular intervals over a specified period of time.

- The timeline for an annuity:

- **Note**: The ***first cash flow*** does not arrive immediately; it arrives at time 1.

Deriving the PV of an Annuity

$$PV_0(\text{Annuity}) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^n}$$

- Multiplying both sides by (1+r):

$$(1+r)PV_0(\text{Annuity}) = C + \frac{C}{1+r} + \dots + \frac{C}{(1+r)^{n-1}}$$

- Subtracting the second expression from the first one and rearranging:

$$\boxed{PV_0(\text{Annuity}) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right)}$$

Question: You won the Quebec lotto and for the next 65 years you are going to receive a check for \$1,000,000 (*starting next year*). If the interest rate is 7% how much is the present value of your price?

Solution

$$PV = \frac{1{,}000{,}000}{0.07}\left(1 - \frac{1}{(1+0.07)^{65}}\right) = 14{,}109{,}940$$

Future Value of an Annuity

Using the relation between the present value and the future value of a stream of cash flows:

$$\boldsymbol{FV_n}(\text{Annuity}) = \boldsymbol{PV}_0(\text{Annuity}) \cdot (1+\boldsymbol{r})^{\boldsymbol{n}}$$

or equivalently (using the PV of an annuity formula):

$$\boxed{\boldsymbol{FV}_{\boldsymbol{n}}(\text{Annuity}) = \frac{\boldsymbol{C}}{\boldsymbol{r}}\left((1+\boldsymbol{r})^{\boldsymbol{n}} - 1\right)}$$

Example: Future Value of an Annuity

Q. A Ferrari will cost \$40,000 in 10 years. Peter has decided that he can save \$2,000 per year. If the interest rate is 10%, will Peter be able to buy his dream car?

Solution

$$FV_{10} = \frac{2{,}000}{0.1}\left((1.1)^{10} - 1\right) = 31{,}875$$

Not in 10 years…

How many types of problems can you have?

There are five variables:

$$PV_0(\text{Annuity}) ; FV_n(\text{Annuity}) ; C ; n ; r$$

Example: Solving for the Cash Flows

Q: Peter, a McGill graduate, has just had a baby girl. Peter has decided that he wants to send his baby to McGill. For this, he will need to have $100,000 when she turns 18. The interest rate is 10% per year and Peter is making a constant deposit every year starting with her fist birthday. How much should Peter deposit each year?

Solution

$$100{,}000 = \frac{C}{0.1}\left((1+0.1)^{18} - 1\right) \rightarrow C = 2193.02$$

Example: Solving for the Number of Periods in a Savings Plan

Q. You are saving to make a down payment on a house. You have \$10,050 saved already, and can afford to save an additional \$5,000 per year at the end of each year. If you earn 7.25% per year on your savings, how long will it take you to save \$60,000?

Solution

$$10{,}050 \times 1.0725^{\boldsymbol{n}} + \frac{5{,}000}{0.0725}\left(1.0725^{\boldsymbol{n}} - 1\right) = 60{,}000$$

Solving the above equation yields ***n*=7 years**

Aside. Solving the equation from previous example:

$$10{,}050 \times 1.0725^{\boldsymbol{n}} + \frac{5{,}000}{0.0725}\left(1.0725^{\boldsymbol{n}} - 1\right) = 60{,}000 \Leftrightarrow$$

$$1.0725^{\boldsymbol{n}} = \frac{60{,}000 + \dfrac{5{,}000}{0.0725}}{10{,}050 + \dfrac{5{,}000}{0.0725}} = 1.6322 \Leftrightarrow$$

Taking logarithms on both sides:

$$\boldsymbol{n} \times \ln(1.0725) = \ln(1.632) \Leftrightarrow \quad \boldsymbol{n} = \frac{\ln(1.6322)}{\ln(1.0725)} = 7$$

Example: Solving for the Interest Rate

Q. Suppose you have an investment opportunity that requires a \$1,952 investment today and will pay \$1,000 in each of the next 3 years. What is the interest rate, *r*, that would make the *NPV* of this investment opportunity equal to zero?

a) 10%
b) 25%
c) 30%

Solution

We have an initial \$1,952 investment followed by a three-year annuity with \$1,000 per year, hence the *NPV* of the investment opportunity is:

$$\boldsymbol{NPV} = -1{,}952 + \frac{1{,}000}{\boldsymbol{r}}\left(1 - \frac{1}{(1+\boldsymbol{r})^3}\right)$$

Example: Solving for the Interest Rate

Solution (cont.)

Setting this NPV equal to zero we get the following equation:

$$\boldsymbol{NPV} = -1{,}952 + \frac{1{,}000}{\boldsymbol{r}}\left(1 - \frac{1}{(1+\boldsymbol{r})^3}\right) = 0$$

To solve for *r*, you can either use a financial calculator, excel, or trial an error. In this case, because it is a multiple choice question, we can just plug-in the three possible solutions: 10%, 25%, and 30%. When you do so it can be verified that the correct answer is **b. 25% :**

$$\boldsymbol{NPV} = -1{,}952 + \frac{1{,}000}{0.25}\left(1 - \frac{1}{(1+0.25)^3}\right) = 0$$

Growing Perpetuity

- ***Growing Perpetuity:*** A stream of cash flows that occurs at regular intervals and grows at a constant rate *g* forever.

0	1	2	3	4	
	C	C(1+g)	$C(1+g)^2$	$C(1+g)^3$	- - - -

$$\boxed{PV_0(\text{Growing Perpetuity}) = \frac{C}{r-g}}$$

Example: Growing Perpetuity

Q. In return for an loan of \$1 million, the company has agreed to pay the bank \$100,000 the first year, increase the amount by 4% each year, and continue to make these payments forever. What is the rate of return for the bank assuming that the company fulfills its commitments.

Solution

$$0 = -1{,}000{,}000 + \frac{100{,}000}{r - 0.04}$$

Solving for r in the above equation yields ***r*=14%**

Growing Annuity

Growing Annuity: A stream of *n* growing cash flows paid at regular intervals.

0	1	2	3	...	n
	C	C(1+g)	$C(1+g)^2$		$C(1+g)^{n-1}$

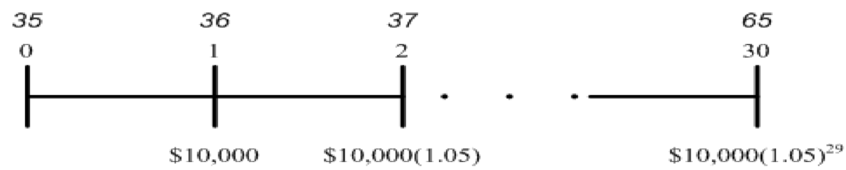
$$PV_0(\text{Growing Annuity}) = \frac{C}{r-g}\left(1-\left(\frac{1+g}{1+r}\right)^{n}\right)$$

Example: Growing Annuity

Q. Ellen is currently 35 and is planning to start saving for her retirement. Although \$10,000 is the most she can save in the first year, she expects her salary to increase each year so that she will be able to increase her savings by 5% per year. With this plan, if she earns 10% per year on her savings, how much will Ellen have saved at age 65?

Solution

Ellen's savings plan timeline:



Hence stream of cash involves a 30-year growing annuity with a growth rate of 5% and an initial cash flow of $10,000, whose PV is:

$$\boldsymbol{PV} = \frac{10{,}000}{0.1-0.05}\left(1-\left(\frac{1+0.05}{1+0.1}\right)^{30}\right) = \$150{,}463$$

Solution (cont)

The question asks for the *FV*, that is, how much money will Ellen has in 30 years when she turns 65. To calculate the *FV* we just need to bring forward the *PV* by 30 years:

$$\boldsymbol{FV}_{30} = \boldsymbol{PV}_0(1+\boldsymbol{r})^{30} = 150{,}463 \times 1.10^{30} = \$2.625 \text{ million}$$

Hence Ellen will have $2.625 million at age 65.