Financial Derivatives FINE 448

4. Black-Scholes: Formula, Greeks, and Practical Uses

closed form

① compute volatility

② hedging

③ trading

④ extension

Daniel Andrei

McGill

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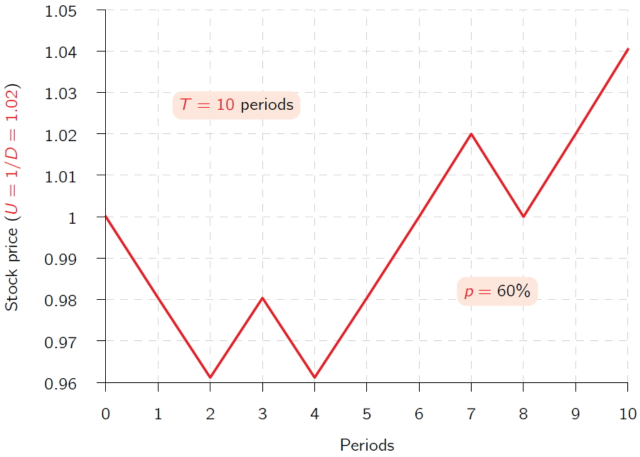
Outline

Discrete Time vs Continuous Time

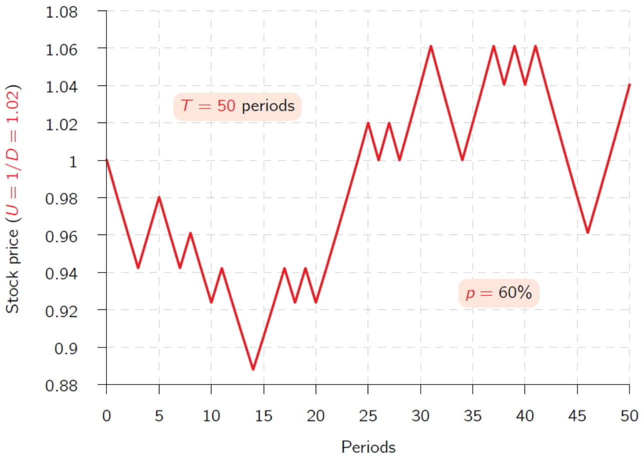
- We refer to the structure of the binomial model as **discrete time**, which means that time moves in distinct increments
- This is much like looking at a calendar and observing only months, weeks, or days
- We know that time moves forward at a rate faster than one day at a time: hours, minutes, seconds, fractions of seconds, and fractions of fractions of seconds
- When we talk about time moving in the tiniest increments, we are talking about **continuous time**.

Discrete time world	**Continuous time world**
Binomial model	Black-Scholes model

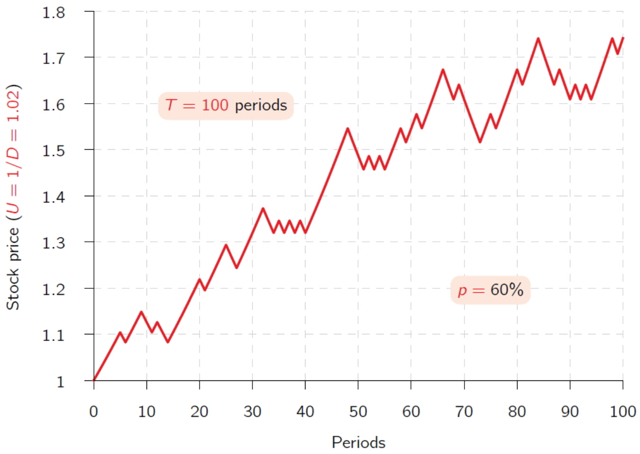
Discrete Time vs Continuous Time



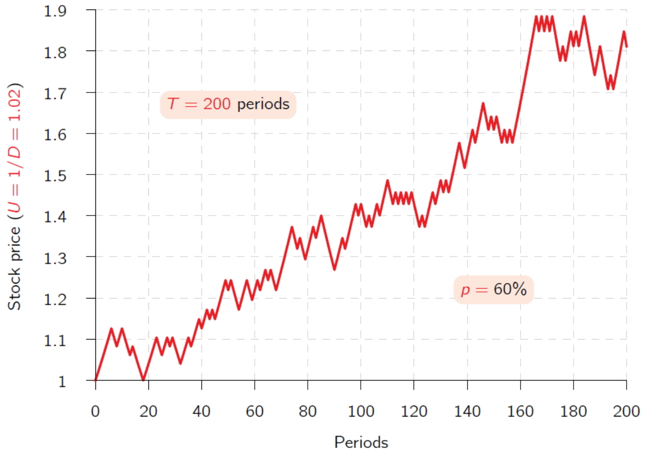
Discrete Time vs Continuous Time



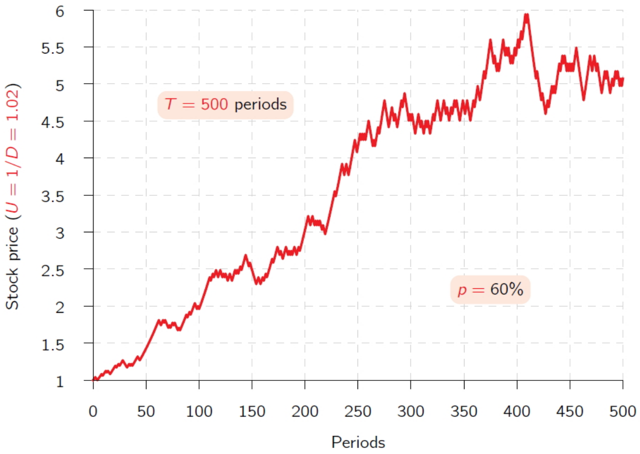
Discrete Time vs Continuous Time



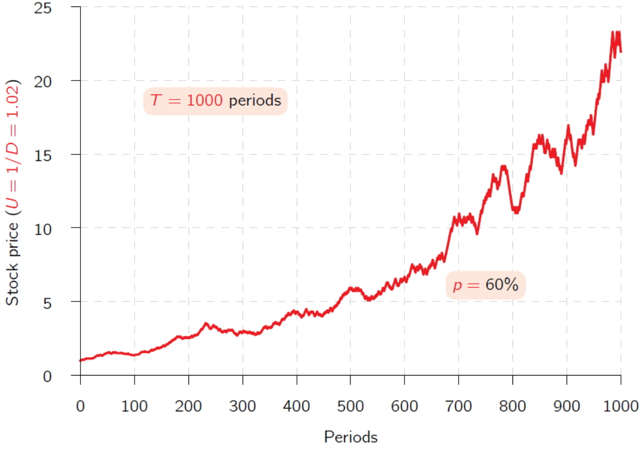
Discrete Time vs Continuous Time



Discrete Time vs Continuous Time



Discrete Time vs Continuous Time



The Limiting Case of the Binomial Formula

- An obvious objection to the binomial calculations thus far is that the stock can only have a few different values at expiration. **It seems unlikely that the option price calculation will be accurate**.
- The solution to this problem is to divide the time to expiration into more periods, generating a more realistic tree.
- To illustrate how to do this, we will re-examine the 1-year European call option, which has a \$40 strike and initial stock price \$41.
- Let there be 3 binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$. We will assume that other inputs stay the same, so $r = 0.08$ and $\sigma = 0.3$.

The Limiting Case of the Binomial Formula (cont'd)

- Since the length of the binomial period is shorter, u and d are closer to 1 than before (1.2212 and 0.8637 as opposed to 1.4623 and 0.8025 with $h = 1$).
- The risk-neutral probability of the stock price going up in a period is

$$p^* = \frac{e^{(0.08-0)\times 1/3} - 0.8637}{1.2212 - 0.8637} = 0.4568$$

- The binomial model implicitly assigns probabilities to the various nodes. The risk-neutral probability for each final period node, together with the call value, is:

Call Price in 1 Year (3 periods)	**Probability**
\$34.678	$p^{*3} = 0.0953$
\$12.814	$3p^{*2}(1-p^*) = 0.34$
\$0	$3p^*(1-p^*)^2 = 0.4044$
\$0	$(1-p^*)^3 = 0.1603$

The Limiting Case of the Binomial Formula (cont'd)

- Thus, the price of the European call option is given by

$$\begin{aligned} C_0 &= e^{0.08\times 1}(0.0953\times \$34.678 + 0.34\times \$12.814) \\ &= \$7.0739 \end{aligned}$$

- We can vary the number of binomial steps, holding fixed the time to expiration, T. The general formula is

$$C_0 = e^{-rT}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}p^{*k}(1-p^*)^{n-k}\max\left[S_0 u^k d^{n-k} - K, 0\right] \quad (1)$$

(we also need to modify u, d, and p^* at each time).

The Limiting Case of the Binomial Formula (cont'd)

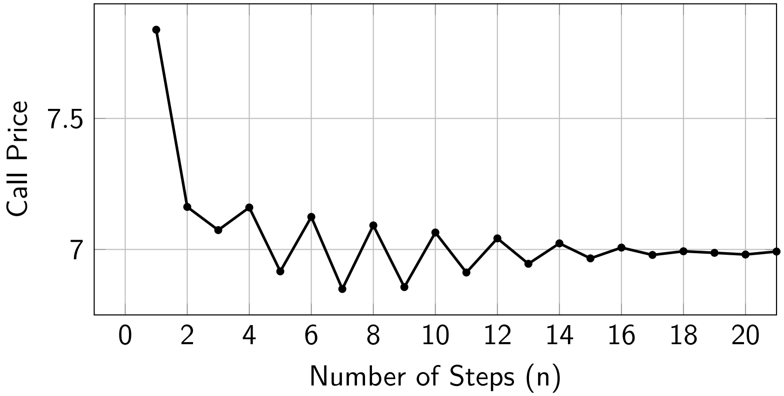
- The following table computes binomial call option prices, using the same inputs as before.

Number of steps (n)	Binomial Call Price (\$)
1	\$7.839
2	\$7.162
3	\$7.074
4	\$7.160
10	\$7.065
50	\$6.969
100	\$6.966
500	\$6.960
∞	\$6.961

- Changing the number of steps changes the option price, but once the number of steps becomes great enough we appear to approach a limiting value for the price, given by the **Black-Scholes formula**.

The Limiting Case of the Binomial Formula (cont'd)

- This can be seen graphically below:



Problem 12.2: Using the *BinomCall* function, compute the binomial approximations for the following call option: $S_0 = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 0.08$, $T = 0.25$ (3 months), and $\delta = 0$. Be sure to compute prices for $n = 8, 9, 10, 11$, and 12. What do you observe about the behavior of the binomial approximation?

N	Call Price
8	3.464
9	3.361
10	3.454
11	3.348
12	3.446

The observed values are slowly converging towards the Black-Scholes value (3.399). Please note that the binomial solution oscillates as it approaches the Black-Scholes value.

Lognormality and the Binomial Model

Binomial distribution

Distribution of successes from n independent experiment with prob of p. $X \sim Binom(n, p)$

If n is large,

$X \xrightarrow[d]{} N(np, np(1-p))$

$S_0 u^k d^{n-k}$

S_0

- The lognormal distribution is the probability distribution that arises from the assumption that continuously **compounded returns on the stock are normally distributed**. $\Leftrightarrow S_T \sim \log\text{-}N$
- In a binomial tree, as we increase the number of periods until expiration, continuously compounded returns approach a normal distribution.
- We can plot probabilities of outcomes (of stock returns) from the binomial tree for different values of n (2, 3, 6, and 10), as shown in the following figure.
- The 10-period binomial tree approaches fairly well the normal distribution.

continually compounded return

$\ln(u^k d^{n-k}) = k\ln u + (n-k)\ln d$

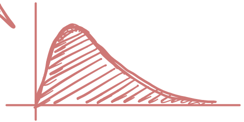
$= k(\ln u - \ln d) + n\ln d \sim N$

$\ln(S_n)$

$= \ln S_0 + k(\ln u - \ln d)$

$+ n\ln d$

$\sim N$

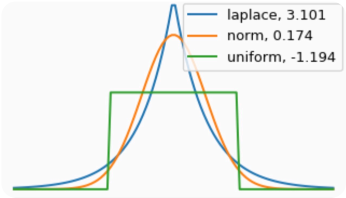


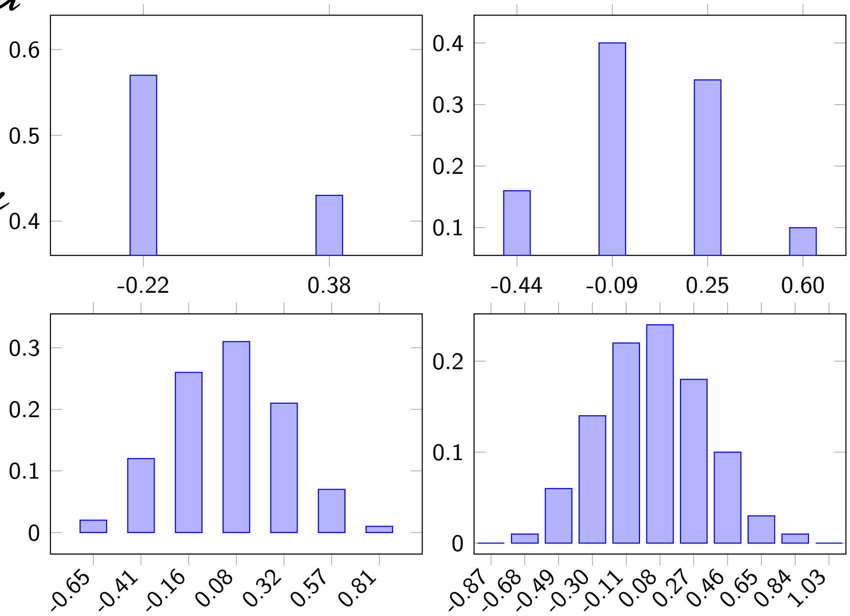
asymmetric.

However, the standard normal model doesn't always fit.

① negative skewness (more likely to have negative return)

② excess kurtosis

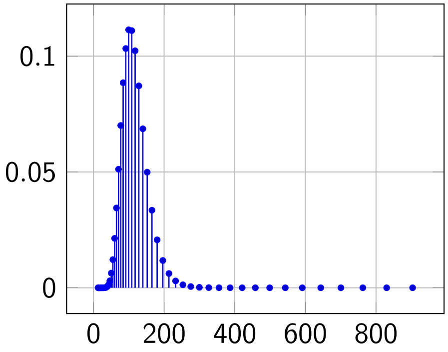




Problem 11.13: Let $S = \$100$, $\sigma = 0.30$, $r = 0.08$, $t = 1$, and $\delta = 0$. Use equation (1) to compute the probability of reaching a terminal node and the price at that node and plot the risk-neutral distribution of year-1 stock prices for $n = 50$. What is the risk-neutral probability that $S_1 < \$80$? $S_1 > \$120$?

For $n = 50$, we have $u = 1.0450$, $d = 0.9600$, and $p^* = 0.4894$. We get the following diagram:
To obtain the required probabilities we sum all probabilities for which the final stock price is below 80 or above 120 respectively. We obtain

$$\Pr(S < 80) = 0.2006$$
$$\Pr(S > 120) = 0.2829$$



Black-Scholes Assumptions

- Assumptions about stock return distribution
  - Continously compounded returns on the stocks are normally distributed and independent over time
  - The volatility of continuously compounded returns is known and constant
  - Future dividends are known, either as dollar amount or as a fixed dividend yield
- Assumptions about the economic environment
  - The risk-free rate is known and constant
  - There are no transaction costs or taxes
  - It is possible to short-sell costlessly and to borrow at the risk-free rate

Inputs in the Binomial Model and in Black-Scholes

- There are seven inputs to the binomial model and six inputs to the Black-Scholes model:

Inputs	**Binomial Model**	**Black-Scholes**
Current price of the stock, S_0	✓	✓
Strike price of the option, K	✓	✓
Volatility of the stock, σ	✓	✓
Continuously compounded risk-free interest rate, r	✓	✓
Time to expiration, T	✓	✓
Dividend yield on the stock, δ	✓	✓
Number of binomial periods, n	✓	

Convergence from binomial tree to Black-Scholes

- See Appendix of Chapter 12 in Hull (8^{th} edition)
- The binomial price is

$$
\begin{aligned}
C_0 &= e^{-rT} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} p^{*k}(1-p^*)^{n-k} \max\left[S_0 u^k d^{n-k} - K, 0\right] && (2)\\
&= e^{-rT} \sum_{k>\alpha} \frac{n!}{(n-k)!k!} p^{*k}(1-p^*)^{n-k} \left(S_0 u^k d^{n-k} - K\right) && (3)\\
&= e^{-rT} \left(S_0 U_1 - K U_2\right) && (4)
\end{aligned}
$$

where

$$
\alpha = \frac{n}{2} - \frac{\ln(S_0/K) + (r-\delta)T}{2\sigma\sqrt{T/n}} \qquad (5)
$$

Convergence from binomial tree to Black-Scholes (cont'd)

- The term

$$U_2 = \sum_{k>\alpha} \frac{n!}{(n-k)!k!} p^{*k} (1-p^*)^{n-k} \tag{6}$$

represents the probability of the option being in the money at maturity. When $n \to \infty$, this probability tends to

$$U_2 = N\left(\frac{np^* - \alpha}{\sqrt{np^*(1-p^*)}} \right) \tag{7}$$

- Replace α, p^*, u, and d in Equation (7) to obtain:

$$U_2 = N\left(\underbrace{\frac{\ln(S_0/K) + (r - \delta - \sigma^2/2)\,T}{\sigma\sqrt{T}}}_{d_2} \right) \tag{8}$$

Convergence from binomial tree to Black-Scholes (cont'd)

- Take now the term

$$U_1 = \sum_{k>\alpha} \frac{n!}{(n-k)!k!} (p^* u)^k [(1-p^*)d]^{n-k} \tag{9}$$

and define

$$q \equiv \frac{p^* u}{p^* u + (1-p^*)d} = \frac{p^* u}{e^{(r-\delta)T/n}} \tag{10}$$

It then follows that

$$1 - q = \frac{(1-p^*)d}{e^{(r-\delta)T/n}} \tag{11}$$

and thus

$$U_1 = e^{(r-\delta)T} \sum_{k>\alpha} \frac{n!}{(n-k)!k!} q^k (1-q)^{n-k} \tag{12}$$

Convergence from binomial tree to Black-Scholes (cont'd)

- This looks like (6), except that we have q instead of p^*. When $n \to \infty$, this probability tends to

$$U_1 = e^{(r-\delta)T} N\left(\frac{nq-\alpha}{\sqrt{nq(1-q)}}\right) \tag{13}$$

- Replace q, α, p^*, u, and d in Equation (13) to obtain (proof in class):

$$U_1 = e^{(r-\delta)T} N\left(\underbrace{\frac{\ln(S_0/K) + (r-\delta+\sigma^2/2)\,T}{\sigma\sqrt{T}}}_{d_1}\right) \tag{14}$$

- We obtain the Black-Scholes formula:

$$\begin{aligned} C_0 &= e^{-rT}\left(S_0 U_1 - K U_2\right) && (15)\\ &= S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) && (16) \end{aligned}$$

Outline

Black-Scholes Formula for a European Call Option

Money to pay at maturity

$e^{-rT}[-K\,Pr(ex) + 0(1-Pr(ex))]$

$= -K e^{-rT}\,Pr(ex)$

$= -K e^{-rT}\,N(d_2)$

probability of exercising

- The Black-Scholes formula for a European call option on a stock that pays dividends at the continuous rate δ is

$$C_0(S_0, K, \sigma, r, T, \delta) = \underbrace{S_0 e^{-\delta T} N(d_1)}_{\Delta} - \underbrace{K e^{-rT} N(d_2)}_{B} \qquad (17)$$

sensitivity

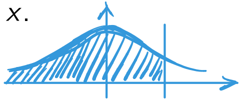
where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \qquad (18)$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad (19)$$

always smaller than d_1.

- $N(x)$ is the cumulative normal distribution function, which is the probability that a number randomly drawn from a standard normal distribution will be less than x.



excel

? NORM.S.DIST(di,1)

Black-Scholes Formula for a European Call Option (cont'd)

- Let $S_0 = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 0.08$, $T = 1$ year, and $\delta = 0$. Computing the Black-Scholes call price, we obtain

$$\begin{aligned} C_0 &= \$41 \times e^{-0} \times N(0.49898) - \$40 \times e^{-0.08} \times N(0.19898) \\ &= \$41 \times e^{-0} \times 0.6911 - \$40 \times e^{-0.08} \times 0.5789 \\ &= \$6.961 \end{aligned}$$

- Note that this result corresponds to the limit obtained from the binomial model.

Black-Scholes Formula for a European Call Option (cont'd)

- The following figure plots Black-Scholes call option prices (today and at expiration) for stock prices ranging from \$20 to \$60.

in the money

$\max[S_T - K, 0] \approx S_T - K$

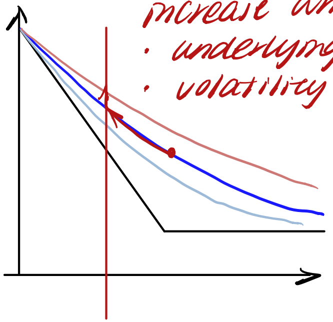
$PV(S_T - K) = P_0 - PV(K)$

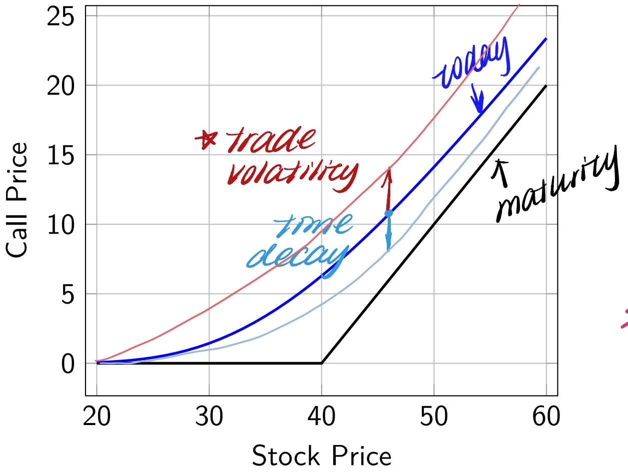
$< P_0 - K$

$\Rightarrow$ blue line is above

Put

increase when $S_t \downarrow$
- underlying $\downarrow$
- volatility $\uparrow$





slope : Δ

A Remarkable Result

- In the binomial model, we have not specified the probabilities of the stock going up and down (which would give us the expected return of the stock).
- The expected return on the stock does not appear in the Black-Scholes formula either.
- This raises a question: Consider a stock with a higher beta (and hence having a higher expected return). A call option on this stock would have a higher probability of settlement in-the-money, hence a higher option price. **Why is this not the case?**

$$C_0 = \Delta S_0 + B$$

$$1 = \frac{\Delta S_0}{C_0} + \frac{B}{C_0}$$

$$\uparrow \beta_C = \frac{\Delta S_0}{C_0}\beta_S + \frac{B}{C_0}\cdot 0$$

$$= \frac{\Delta C_0}{C_0}\beta_S \uparrow$$

higher expected return

higher payoff at maturity, but is discounted more heavily.

$$e^{-\mu T}\downarrow E(X)\uparrow = \text{same}.$$

A Remarkable Result (cont'd)

- The Black-Scholes formula (17) provides the answer. This formula shows the composition of the replicating portfolio, which in the binomial case was

$$C_0 = \Delta S_0 + B$$

- We can easily identify Δ (the position in the risky asset) and B (the dollar amount of borrowing or lending) in the Black-Scholes formula:

$$\Delta = e^{-\delta T} N(d_1) \tag{20}$$

$$B = -Ke^{-rT} N(d_2) \tag{21}$$

A Remarkable Result (cont'd)

- If β_S is the stock beta, then the option beta is

larger than one

$$\beta_{Option} = \frac{\Delta S_0}{\Delta S_0 + B}\beta_S \tag{22}$$

negative for call options

- A high stock beta implies a high option beta, so the discount rate for the expected payoff of the option is correspondingly greater.
- The net result—one of the key insights from the Black-Scholes analysis—is that beta is irrelevant: **The larger average payoff to options on high beta stocks is exactly offset by the larger discount rate.**

Black-Scholes Formula for a European Put Option

- The Black-Scholes formula for a European put option is

Pr(exercising)

$$P_0(S_0, K, \sigma, r, T, \delta) = -S_0 e^{-\delta T} N(-d_1) + Ke^{-rT} \boxed{N(-d_2)} \quad (23)$$

$\Delta = -e^{-\delta T} N(-d_1)$ negative

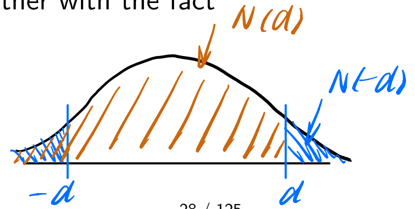
where d_1 and d_2 are given by equations (18) and (19).

- Put-call parity must hold:

$$P_0(S_0, K, \sigma, r, T, \delta) = C_0(S_0, K, \sigma, r, T, \delta) + Ke^{-rT} - Se^{-\delta T} \quad (24)$$

- This follows from the formulas (17) and (23), together with the fact that for any x, $N(-x) = 1 - N(x)$.

$$C_0 - P_0 = S_0 e^{-\delta T}(N(d_1) + N(-d_1)) - Ke^{-rT}(N(d_2) + N(-d_2))$$



Black-Scholes Formula for a European Put Option (cont'd)

- Let $S_0 = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 0.08$, $T = 1$ year, and $\delta = 0$. Computing the Black-Scholes put price, we obtain

$$\begin{aligned} P_0 &= -\$41 \times e^{-0} \times N(-0.49898) + \$40 \times e^{-0.08} \times N(-0.19898) \\ &= -\$41 \times e^{-0} \times 0.3089 + \$40 \times e^{-0.08} \times 0.4211 \\ &= \$2.886 \end{aligned}$$

- In the binomial model, if we fix the number of periods to $n = 500$, we obtain a price of \$2.885.
- Computing the price using put-call parity (equation 24) yields

$$\begin{aligned} P_0 &= \$6.961 + \$40e^{-0.08} - \$41 \\ &= \$2.886 \end{aligned}$$

Black-Scholes Formula for a European Put Option (cont'd)

- The following figure plots Black-Scholes put option prices (today and at expiration) for stock prices ranging from \$20 to \$60.

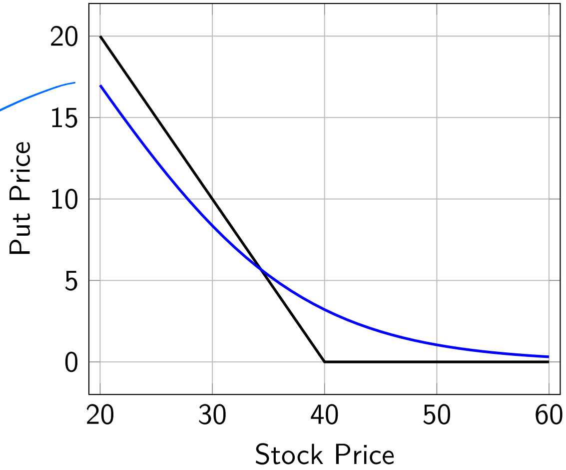
in the money

$\max[K - S_T, 0] \approx K - S_T$

$PV(K - S_T) = PV(K) - S_0$

$< K - S_0$

blue line is below



Problem 12.20: Let $S = \$100$, $K = \$90$, $\sigma = 30\%$, $r = 8\%$, $\delta = 5\%$, and $T = 1$.

a. What is the Black-Scholes call price?

	A	B	C	D
1				
2		S0	100	
3		K	90	
4		sigma (v)	0.3	
5		r	0.08	
6		Maturity (T)	1	
7		delta (d)	0.05	
8				
9		=BSCall(C2,C3,C4,C5,C6,C7)		

The Black-Scholes call price is \$17.70

b. Now price a put where $S = \$90$, $K = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 8\%$, and $T = 1$.

	A	B	C	D
1				
2		S0	90	
3		K	100	
4		sigma (v)	0.3	
5		r	0.05	
6		Maturity (T)	1	
7		delta (d)	0.08	
8				
9		=bsput(C2,C3,C4,C5,C6,C7)		

The Black-Scholes put price is \$17.70

Outline

Precursors to Black & Scholes (1973)

Most previous work on option valuation was expressed in terms of warrants and OTC options because listed options did not trade in modern markets until April 1973. In the Bible, the first book, Genesis, describes the supposed first option. Castelli (1877), Bachlier (1900), Sprenkle (1961), Ayres (1965), Boness (1964), Samuelson (1965), Baumol, Malkiel, and Quandt (1966), and Chen (1970) all produced valuation formulas of the same general form. These formulas were incomplete because they all involved arbitrary parameters and they did not include the idea of a self-financing riskless hedge. A few of these examples are presented below.

BACHLIER

His ideas were ahead of their time.

Théorie de la Spéculation.

1. **Bachlier (1900)**

$$C(S,T)=SN\left(\frac{S-X}{\sigma\sqrt{T}}\right)-XN\left(\frac{S-X}{\sigma\sqrt{T}}\right)-\sigma\sqrt{T}\left(\frac{X-S}{\sigma\sqrt{T}}\right)$$

Bachlier's formulation permits both negative underlying security and option prices and does not account for the time value of money. As will be shown this formulation gives very small option premium.

2. Sprenkle Formula (1961)

$$C(S,T)= e^{\rho T} S\, N(d_1) - (1-A)\, X\, N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{e^{\rho T}S}{X}\right) + \tfrac{1}{2}\sigma^2 T\right] \quad and \quad d_2 = d_1 - \sigma\sqrt{T}$$

Sprenkle's equation is similar to Black Scholes but without the riskless hedge or discounting. Instead rho (ρ) is the average growth rate of the share price and A corresponds to the degree of risk aversion, and both are very difficult to estimate. This formulation produces a very large option premium.

3. Boness Formula (1964)

$$C(S,T)= S\, N(d_1) - X e^{-\rho T} N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + (\rho + \tfrac{1}{2}\sigma^2)T\right] \quad and \quad d_2 = d_1 - \sigma\sqrt{T}$$

Boness accounted for the time value of money by discounting exercise proceeds by the expected rate of return on the stock. This formulation gives much better option values but requires the expected return on the stock.

4. Samuelson Formula (1965)

$$C(S,T) = S\,e^{-(\rho-\alpha)T}\,N(d_1) - X\,e^{-\alpha T}\,N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + (\rho + \tfrac{1}{2}\sigma^2)T\right] \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

Samuelson allowed for the option and the stock to have different levels of risk and thus different expected growth rates. He defined rho (ρ) and alpha (α) as the average rate of growth of the stock price and the option price, respectively. Rho and alpha are very difficult to estimate.

5. Samuelson and Merton (1969)

They treated the option price as a function of the stock price, and reasoned that the discount rate must be determined by the requirement that investors hold both the stock and the option. Their formula depended on the utility function assumed for the "representative" investor, and involved the sum of many conditional terms.

6. Thorpe and Kassouf (Beat the Market 1967)

They graphically approximated warrant prices using linear regression to determine the exponent of a non-linear approximating function. They did not realize that in a no-arbitrage situation the expected return on the hedged position must be the riskless rate. Furthermore, no direct measure of volatility entered their methodology.

7. Black and Scholes (JPE 1973)

risk-free.

$$C(S,T) = S\,N(d_1) - X\,e^{-rT}\,N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + (r + \tfrac{1}{2}\sigma^2)T\right] \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

Black & Scholes formula is similar to Boness (1964). However, their breakthrough (with the help of Merton, BS JPE, footnote 3) was to realize that the return on the hedge position must be equivalent to the return on the riskless asset. They used a two security portfolio which did not include the self-financing bond. Merton (Bell Journal, 1973) proved that a self-financing three security portfolio ($w_C C + w_S S + w_B B$) would replicate the option. This correct formula, its intuition, and other accomplishments allowed Scholes and Merton (Black had died) to win the Nobel Prize in Economics in 1996.

„This was pointed out to us by Robert Merton"

Outline

④ Market-Maker Risk and Delta Hedging

WSJ, Feb 2014: Big Banks Take Hits On Trusty Oil Hedge

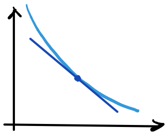
$$P_0 = \overset{<0}{\Delta} S_0 + B$$

sell put

$$-P_0 = \underset{>0}{-\Delta} S_0 - B$$

hedge by selling futures

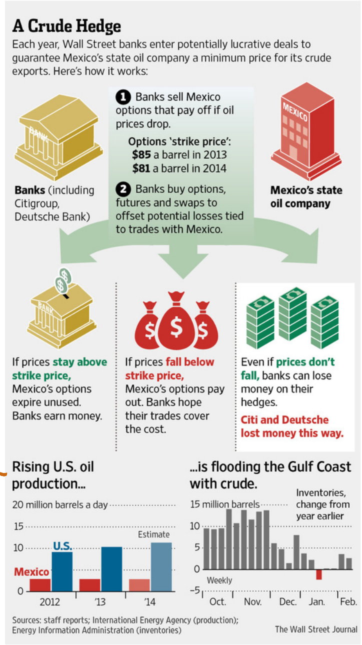
more liquid than option



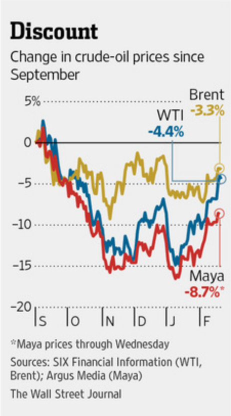
$P_{oil} \downarrow \quad |\Delta| \uparrow$

$\Leftrightarrow \quad -|\Delta| \uparrow$

$\Rightarrow$ sell oil futures.



WSJ, Feb 2014: Big Banks Take Hits On Trusty Oil Hedge



Reuters, Oct 15, 2014: Banks rush to hedge oil option deltas, accelerating rout

- Wall Street banks have scrambled to **neutralize their exposure** to big oil options trades.
- Banks have written protection to companies and they sell futures contracts to offset option deals that are **unexpectedly** in the money.
- Oil producers seek to **hedge their production** by buying put options with strikes \$75 to \$85.
- As futures prices approach the these strike levels, big banks that have sold put options (or similar hedges) to oil producers are forced to protect themselves through **delta hedging**: they sell futures to remain **market neutral**.
- The delta hedging selling was cited by several traders as a factor behind Tuesday's **rapid swoon in prices**.

Market-Maker Risk

- A **market-maker** stands ready to sell to buyers and to buy from sellers.
- Without hedging, an active market-maker will have an arbitrary position generated by fulfilling customer orders. This arbitrary portfolio has uncontrolled risk.
- Consequently, market-makers attempt to hedge the risk of their position.
- We will se here how they do so.

Market-Maker Risk (cont'd)

- Suppose a customer wishes to buy 100 European call options with maturity of 91 days. The market-maker fills this order by selling 100 call options. To be specific, suppose that $S = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 0.08$ (continuously compounded), and $\delta = 0$. We will let T denote the expiration time of the option and t the present, so time to expiration is $T - t$. Thus, $T - t = 91/365 = 0.249$.
- Suppose that the market-maker does not hedge the written option (**naked position**) and the stock has the following evolution over the next 5 days:

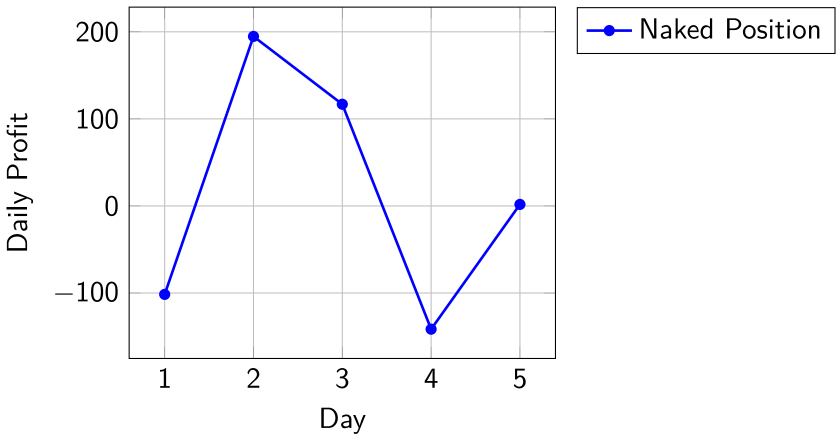
Day	**Stock ($)**
0	41
1	42.5
2	39.5
3	37
4	40
5	40

Market-Maker Risk (cont'd)

- We can measure the profit of the market-maker by **marking-to-market** the position: **if we liquidated the position today, what would be the gain or loss?**

Day	Stock ($)	Call Position ($)	Daily Profit ($)
0	41	-339.47	
1	42.5	-441.04	-101.57
2	39.5	-246.31	194.73
3	37	-129.49	116.82
4	40	-271.04	-141.55
5	40	-269.27	1.77

Market-Maker Risk (cont'd)



Delta-Hedging

$C_0 = S_0\Delta + B$

sell Δ shares $\underline{-S_0\Delta}$

$0 + B$

S tomorrow ↑ Δ ↑

sell more shares

$-C_0 = -S_0\Delta - B$

buy Δ shares $\underline{S_0\Delta}$

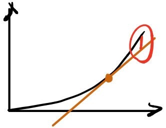
$0 + B$

S tomorrow ↓ Δ ↓

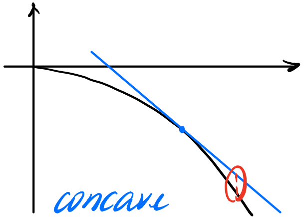
sell some shares

- Suppose the market-maker hedges the position with shares. At time 0, the delta of a call at a stock price of \$41 is 0.645.
- This suggests that a \$1 increase in the stock price should increase the value of the option by approximatively \$0.645.
- The market-maker takes an offsetting position in shares, position that hedges the fluctuations in the option price. We say that such a position is **delta-hedged**.
- Then, the market-maker rebalances the portfolio each day, by computing the new delta of the call.
- The following table summarizes delta, the number of purchased shares, the net investment, and profit for each day for 5 days (interest expenses are ignored for simplicity).

Delta-Hedging (cont'd)



convex.



concave

(Market maker usually sell options)

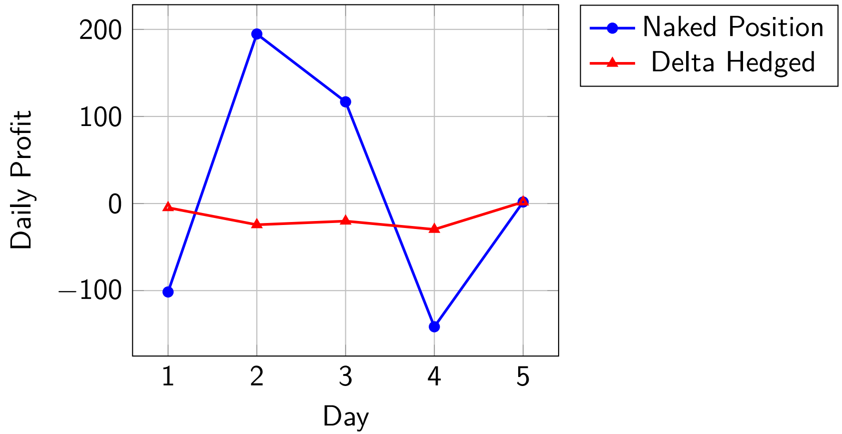
⇒ want to hedge everything

Take second derivative Delta-Gamma Hedging

Day	Stock ($)	Option Delta	Stock Position (# shares)	Daily Profit (Call)	Daily profit (Shares)	Daily profit (Total)
0	41	0.645	64.54			
1	42.5	0.730	73.03	-101.57	96.81	-4.77
2	39.5	0.548	54.81	194.73	-219.10	-24.38
3	37	0.373	37.27	116.82	-137.02	-20.20
4	40	0.581	58.06	-141.55	111.82	-29.73
5	40	0.580	58.01	1.77	0.00	1.77

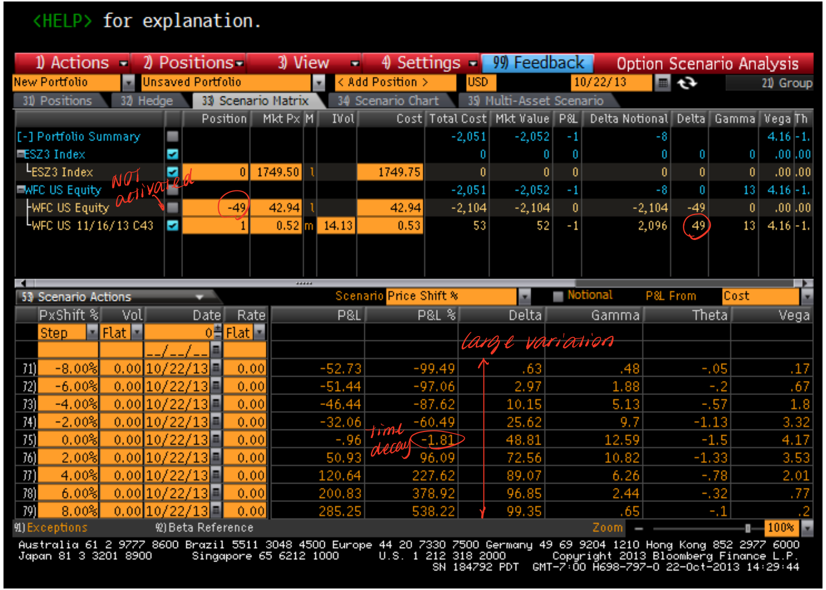
negative most of time

Delta-Hedging (cont'd)

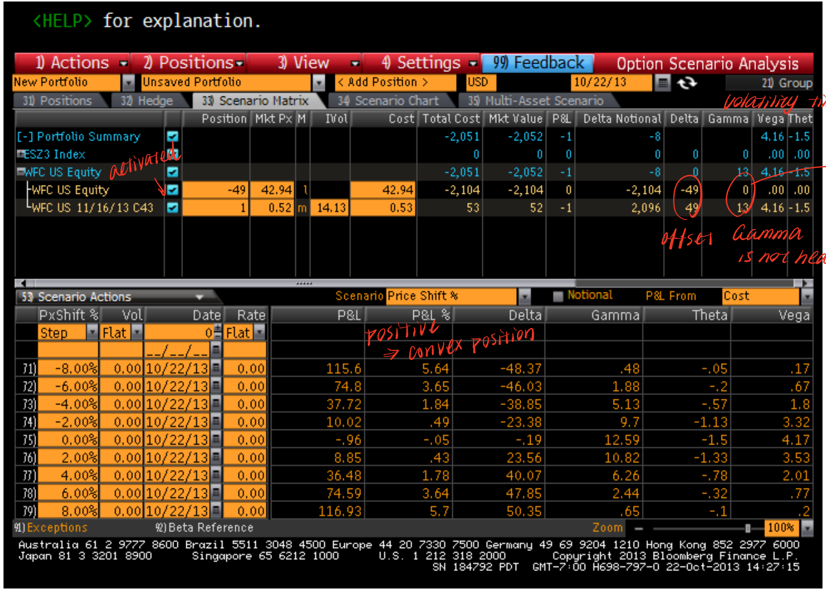


- Delta hedging prevents the position from reacting to small changes in the underlying stock. **For large changes, we need to take into account the fluctuations in delta.**

WFC US <EQUITY> OSA <GO> (not delta-hedged)



WFC US <EQUITY> OSA <GO> (delta-hedged)



<HELP> for explanation.

1) Actions | 2) Positions | 3) View | 4) Settings | 99) Feedback | Option Scenario Analysis

New Portfolio | Unsaved Portfolio | < Add Position > | USD | 10/22/13 | 21) Group

31) Positions | 32) Hedge | 33) Scenario Matrix | 34) Scenario Chart | 35) Multi-Asset Scenario

volatility time decay

	Position	Mkt Px	M	IVol	Cost	Total Cost	Mkt Value	P&L	Delta Notional	Delta	Gamma	Vega	Thet
[-] Portfolio Summary						-2,051	-2,052	-1	-8			4.16	-1.5
ESZ3 Index						0	0	0	0	0	0	.00	.00
WFC US Equity						-2,051	-2,052	-1	-8	0	13	4.16	-1.5
WFC US Equity	-49	42.94	l		42.94	-2,104	-2,104	0	-2,104	-49	0	.00	.00
WFC US 11/16/13 C43	1	0.52	m	14.13	0.53	53	52	-1	2,096	49	13	4.16	-1.5

activated

offset

Gamma is not hedged.

For underlying, $\Delta = 1$

$\Rightarrow$ Δ doesn't change when stock prices change.

53) Scenario Actions | Scenario Price Shift % | Notional | P&L From Cost

positive $\Rightarrow$ convex position

	PxShift %	Vol	Date	Rate	P&L	P&L %	Delta	Gamma	Theta	Vega
	Step	Flat	0	Flat						
			--/--/--							
71)	-8.00%	0.00	10/22/13	0.00	115.6	5.64	-48.37	.48	-.05	.17
72)	-6.00%	0.00	10/22/13	0.00	74.8	3.65	-46.03	1.88	-.2	.67
73)	-4.00%	0.00	10/22/13	0.00	37.72	1.84	-38.85	5.13	-.57	1.8
74)	-2.00%	0.00	10/22/13	0.00	10.02	.49	-23.38	9.7	-1.13	3.32
75)	0.00%	0.00	10/22/13	0.00	-.96	-.05	-.19	12.59	-1.5	4.17
76)	2.00%	0.00	10/22/13	0.00	8.85	.43	23.56	10.82	-1.33	3.53
77)	4.00%	0.00	10/22/13	0.00	36.48	1.78	40.07	6.26	-.78	2.01
78)	6.00%	0.00	10/22/13	0.00	74.59	3.64	47.85	2.44	-.32	.77
79)	8.00%	0.00	10/22/13	0.00	116.93	5.7	50.35	.65	-.1	.2

91) Exceptions 92) Beta Reference Zoom 100%

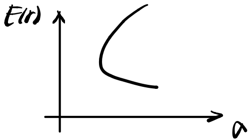
⑤ Volatility

Volatility is Crucial in Finance

measure of risk

- skewness
  (normal : 0)
- excess kurtosis
  (normal : 0)

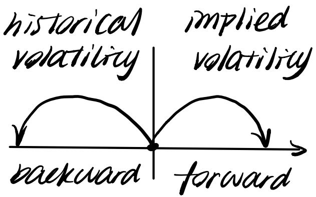
- for forecasting return
- for the pricing of derivatives
- for asset allocation (trade-off between return and risk)
- for risk management (evaluation of the risk of a portfolio)



$$E[r_i - r_f] = \beta_i E[r_m - r_f]$$

$$\downarrow$$

$$\frac{\rho_{i,m}\,\sigma_i}{\sigma_m} = \frac{Cov(r_i, r_m)}{Var(r_m)}$$



The major problem with volatility is that it is not directly observable from returns

historical

- **Unconditional volatility** is estimated as the sample standard deviation

unconditional on time.

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2} \tag{25}$$

where r_t is the log return on period t and μ is the sample mean over T periods

- However, volatility is actually not constant through time. Therefore, **conditional volatility** σ_t is a more relevant measure of risk at time t.

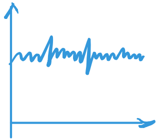
GARCH(1,1) volatility for the S&P 500 (weekly data)

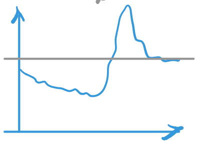
$r_t = \mu + \overbrace{\sigma_t \cdot z}^{\text{shock/deviation } \varepsilon}$ where

$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$

one step forcast

$\mu = \frac{\omega}{1-\alpha-\beta}$

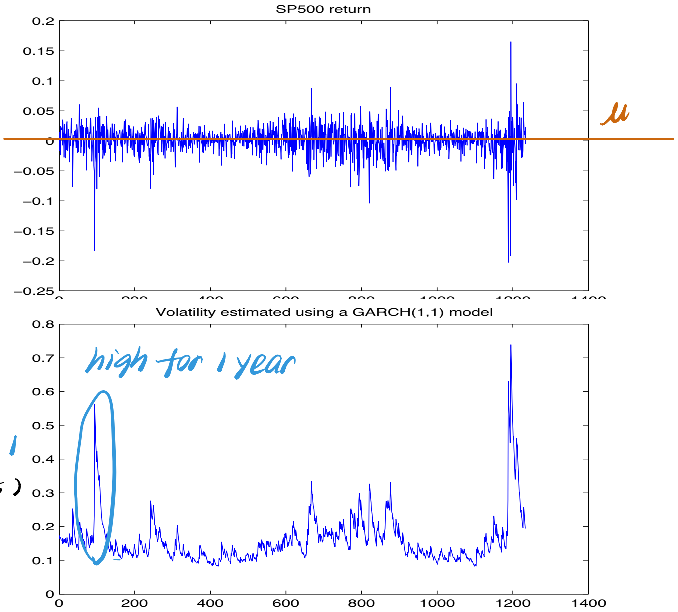




$\alpha + \beta$ close to 0

$\alpha + \beta$ close to 1

(JP morgan 0.95)



volatility is NOT constant through time (volatility clustering)

Implied Volatility

- Volatility is unobservable
- Choosing a volatility to use in pricing an option is difficult but important
- Using history of returns is not the best approach, because history is not a reliable guide to the future.
- We can invert Black-Scholes formula to obtain implied volatility
- We cannot use implied volatility to assess whether an option price is correct, but implied volatility does tell us the market's assessment of volatility

Implied Volatility (cont'd)

- Example: Let $S = \$100$, $K = \$90$, $r = 8\%$, $\delta = 5\%$, and $T = 1$. The market option price for a call option is \$18.25. What is the volatility that gives this option price?
- We must invert the following formula

goal seek.

$$\$18.25 = \text{BSCall}(100, 90, \sigma, 0.08, 1, 0.05)$$

assume normally distributed volatility

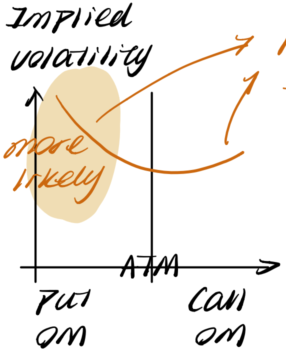
- We use the implied volatility function:

	A	B	C	D
1				
2		S0	100	
3		K	90	
4				
5		r	0.08	
6		Maturity (T)	1	
7		delta (d)	0.05	
8				
9		Call Price	18.25	
10				
11		=bscallimpvol(C2,C3,C4,C5,C6,C7,C9)		

We find that setting $\sigma = 31.73\%$ gives us a call price of \$18.25

ATM ⇒ σ around 20%. (less than 31.73%.)

But the volatility for the same volatility should be the same.



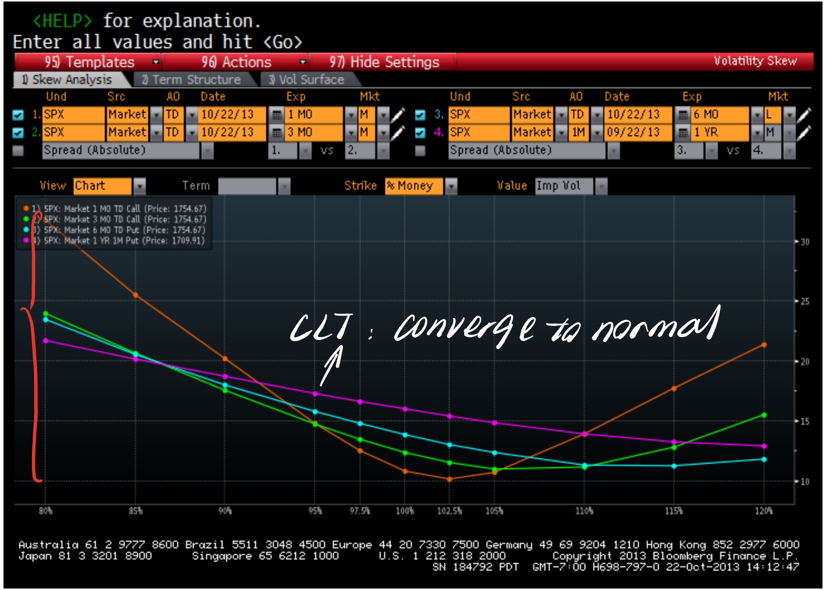
→ Higher risk
→ higher price

Implied Volatility (cont'd)

Investor is aware of it
⇒ reflected in the price.

- There is a systematic pattern of implied volatility across strike prices, called **volatility skew**
- The volatility skew is not related to whether an option is a put or a call, but rather to differences in the strike price and time to expiration
- Explaining these patterns is a challenge for option pricing theory

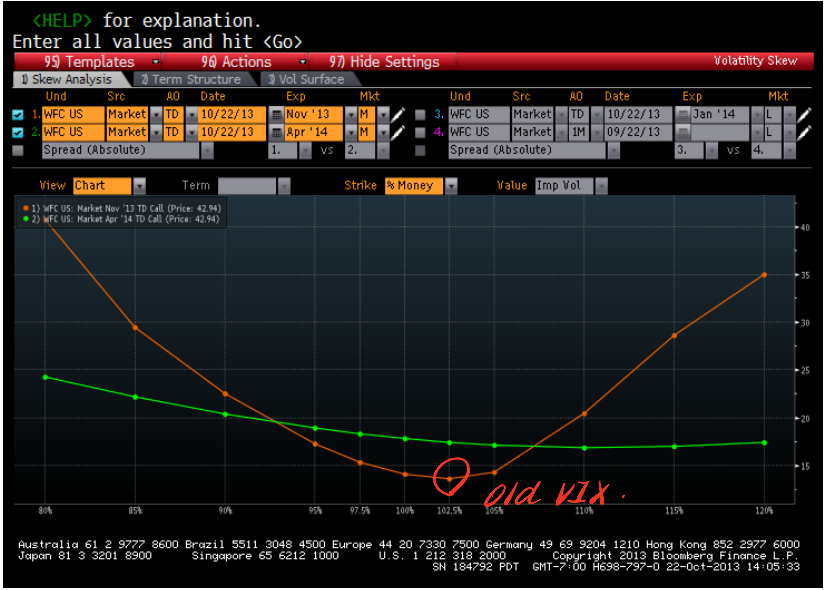
Bloomberg: SPX <INDEX> SKEW <GO>



3 times large

* a very pronounce skewness for short maturity option.

Bloomberg: WFC US <EQUITY> SKEW <GO>



Using Implied Volatility

- Implied volatility is important for a number of reasons
  - If you need to price an option for which you **cannot** observe a market price, you can use implied volatility to generate a price consistent with the price of traded options
  - Implied volatility is often used as a quick way to describe the level of option prices on a given underlying asset. Option prices are quoted sometimes in terms of volatility, rather than as a dollar price
  - Volatility skew provides a measure of how well option pricing models work
- Just as stock markets provide information about stock prices and permit trading stocks, option markets provide information about volatility, and, in effect, permit the trading of volatility.

The CBOE Volatility Index (VIX)

- In 1993, the Chicago Board Options Exchange (CBOE) introduced the CBOE Volatility Index, VIX, which was originally designed to measure the market's expectation of 30-day volatility implied by at-the-money S&P 100 Index option prices
- Ten years later in 2003, CBOE and Goldman Sachs updated the VIX to reflect a new way to measure expected volatility. The new VIX is based on the S&P 500 Index, and estimates expected volatility by averaging the weighted prices of puts and calls over a wide range of strike prices.

The CBOE Volatility Index (VIX)

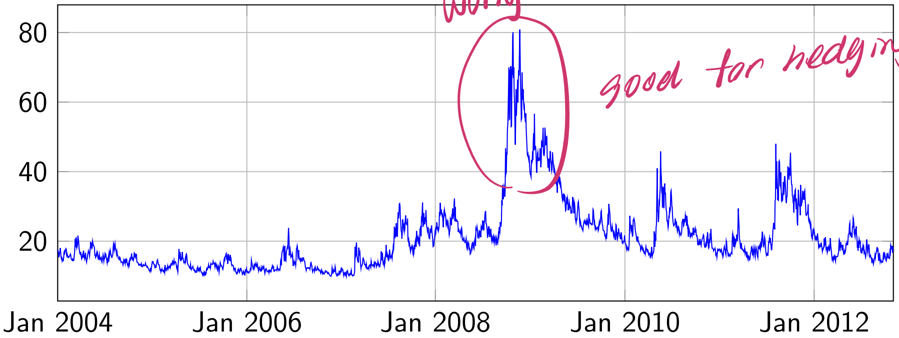
- The CBOE utilizes a wide variety of strike prices for SPX puts and calls to calculate the VIX
- VIX provides important information about investor sentiment. Since volatility often signifies financial turmoil, the VIX is often referrred to as the **investor fear gauge**.
- Investors can use VIX options and VIX futures to hedge their portfolios. The VIX is a good hedging tool because it has a strong negative correlation to the S&P 500

The CBOE Volatility Index (VIX)

worry about future

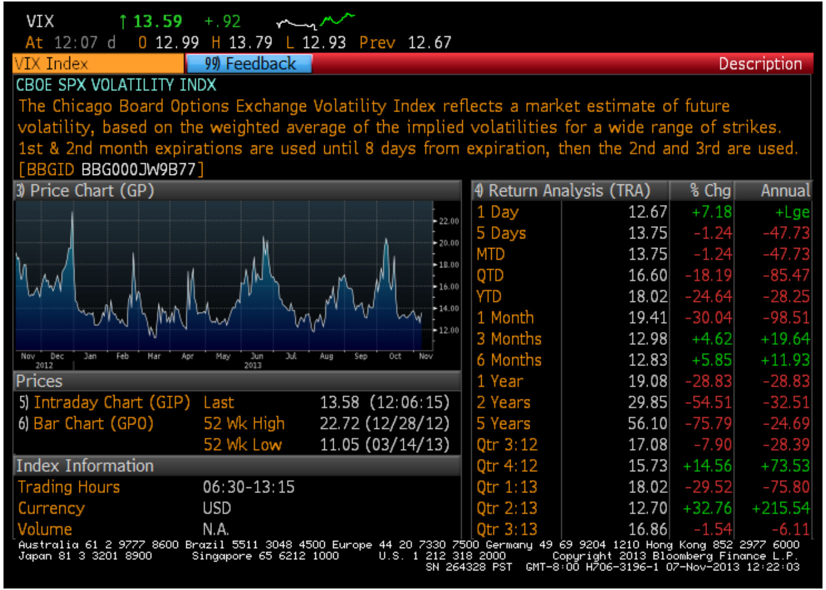
good for hedging

persistent



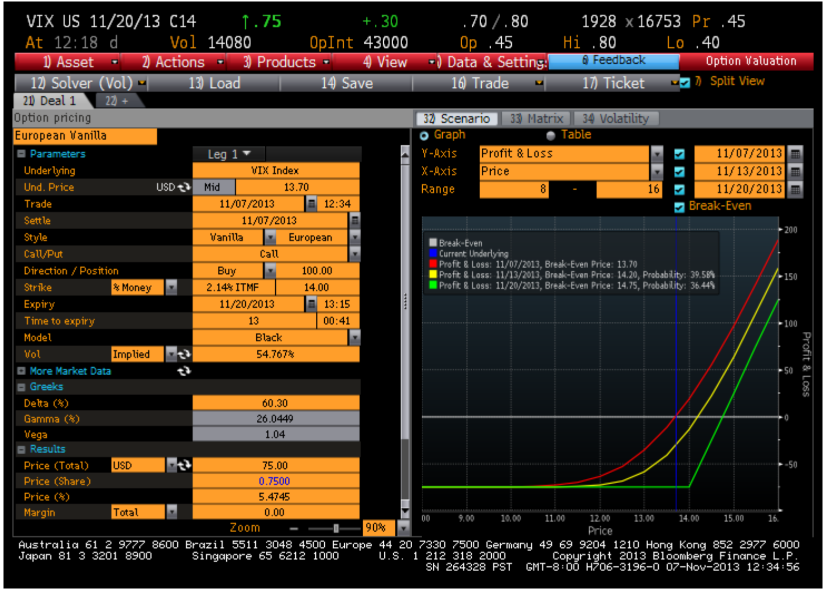
expensive!

Bloomberg: VIX <INDEX> DES <GO>



Bloomberg: UXZ3 <INDEX> DES <GO>

Bloomberg: VIX UX 11/20/13 C14 <INDEX> OV <GO>



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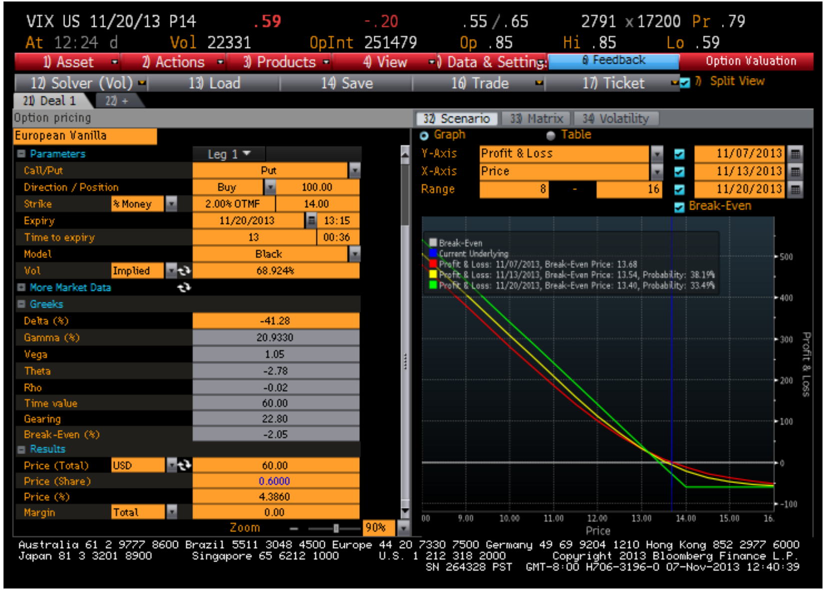

Indicators
Comparison
Date Range
1D
5D
1M
3M
6M
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1Y
2Y
5Y
Max
Interval 1W
Line
Draw
Events
VXX 37.44
YAHOO!
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Volatility Trading

- Just as stock investors think they know something about the direction of the stock market, or bond investors think they can foresee the probable direction of interest rates, so you may think you have insight into the level of future volatility
- What do you do if you simply want exposure to a stock's volatility?
- Stock options are impure: they provide exposure to both the direction of the stock price and its volatility

Volatility Trading: The Traditional Way

- Buy/sell straddles or strangles. Easy to implement, but has drawbacks:
  - Straddles and strangles yield a non-null delta once the stock price moves away from the initial ATM strike price
  - Prices need to move sharply in the case of a buy-and-hold strategy
- Option delta-hedging (avoiding sensitivity to asset price). Drawbacks:
  - The P&L generated by delta hedging an option is a function of numerous sources of risk: variance risk, volatility path dependency risk, model risk, liquidity risk, dividends risk.
  - Variance risk sometimes accounts only for 50% of the total P&L.

Advantages of Variance and Volatility Swaps

- The easy way to trade volatility is to use **volatility swaps**, sometimes called **realized volatility forward contracts**
- These products provide pure exposure to volatility (and only to volatility)
- No need to delta hedge $\Rightarrow$ allows for buy-and-hold variance strategy
- OTC products but standardized contracts with maturity similar to listed options (April 14, June 14, ...)
- Strong liquidity thanks to several investment banks providing live prices
- Either long or short positions
- On indices as well as single stocks

Volatility Swaps

- A stock volatility swap is a forward contract on annualized volatility. Its payoff at expiration is equal to

$$(\sigma_R - K_{vol}) \times N \tag{26}$$

where

  - σ_R is the realized stock volatility (quoted in annual terms) over the life of the contract
  - K_{vol} is the annualized volatility delivery price, typically quoted as a volatility, for example 30%
  - N is the notional amount of the swap in dollars per annualized volatility point, for example $N = \$250,000/(\text{volatility point})$

Volatility Swaps (cont'd)

- The holder of a volatility swap at expiration receives N dollars for every point by which the stock's realized volatility σ_R has exceeded the volatility delivery price K_{vol}
- He or she is **swapping** a fixed volatility K_{vol} for the actual (floating) future volatility σ_R
- The procedure for calculating the realized volatility should be clearly specified with respect to the following aspects:
  - How frequently the return is measured
  - Whether returns are continuously compounded or arithmetic
  - Whether the variance is measured by subtracting the mean or by simply squaring the returns
  - The period of time over which variance is measured
  - How to handle days on which trading does not occur

Variance Swaps

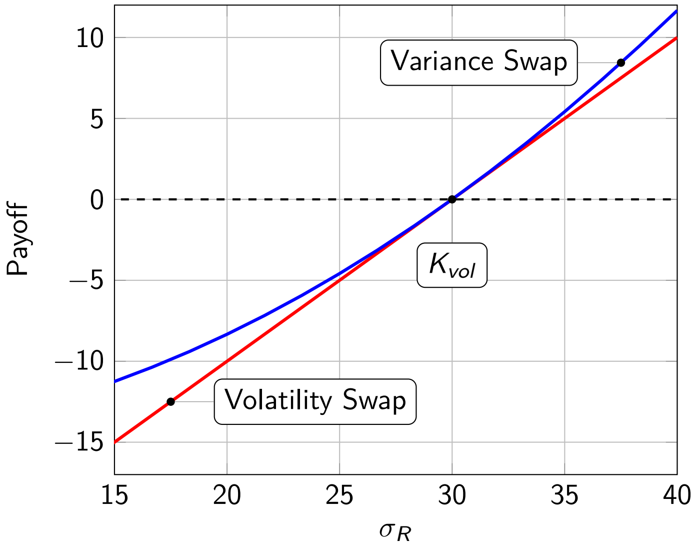
- A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$\left(\sigma_R^2 - K_{var}\right) \times N \tag{27}$$

where

  - σ_R^2 is the realized stock variance (quoted in annual terms) over the life of the contract
  - K_{var} is the delivery price for variance, for example $(30\%)^2$
  - N is the notional amount of the swap in dollars per annualized volatility point squared, for example $N = \$100,000/(\text{volatility point})^2$

Variance vs. Volatility Contracts



Replicating Variance Swaps

- If you own a portfolio of options of all strikes, weighted in inverse proportion to the square of the strike level, you will obtain an exposure to variance that is independent of stock price, just what is needed to trade variance.
- There is no simple replication strategy for synthesizing a volatility swap.
- We can approximate a volatility swap by statically holding a suitably chosen variance contract.

Outline

Option Greeks

- Option Greeks are formulas that express the change in the option price when an input to the formula changes, **taking as fixed all other inputs**.
- They are used to assess risk exposures. For example:
  - A market-making bank with a portfolio of options would want to understand its exposure to stock price changes, interest rates, volatility, maturity, etc.
  - A portfolio manager wants to know what happens to the value of a portfolio of stock index options if there is a change in the level of the stock index.
  - An options investor would like to know how interest rate changes and volatility changes affect profit and loss.

Option Greeks (cont'd)

- Before providing detailed definition of the Greeks, let's have some intuition on how changes in inputs affect option prices:

Change in input	**Change in call price**	**Change in put price**
$S_t \uparrow$	$C_t \uparrow$	$P_t \downarrow$
$\sigma \uparrow$	$C_t \uparrow$	$P_t \uparrow$
$T - t \downarrow (t \uparrow)$	C_t generally $\downarrow$	P_t ambiguous
$r \uparrow$	$C_t \uparrow$	$P_t \downarrow$
$\delta \uparrow$	$C_t \downarrow$	$P_t \uparrow$

Table 1: Changes in Black-Scholes inputs and their effect on option prices.

- An increase in the stock price (S_t) raises the chance that the call will be exercised, thus raises the call option price. Conversely, it lowers the put option price.

Option Greeks (cont'd)

- An increase in volatility raises the price of a call or put option, because it increases the expected value if the option is exercised.
- Options generally—but not always—become less valuable as time to expiration decreases, i.e., there is a **time decay**. There are exceptions, for example deep-in-the-money call options on an asset with high dividend yield and deep-in-the-money puts.
- A higher interest rate reduces the present value of the strike (to be paid by a call option holder), and thus increases the call price. The put option entitles the owner to receive the strike, whose present value is lower with a higher interest rate. Thus, a higher interest rate decreases the put price.
- A call entitles the holder to receive stock, but without dividends prior to expiration. Thus, the greater the dividend yield, the lower the call price. Conversely, a put option is more valuable when the dividend yield is greater.

Option Greeks (cont'd)

① Traders can see how sensitive are option prices to changes in market conditions.

② hedging

With the exception of the Elasticity, all the Greeks of a portfolio are calculated as a sumproduct of portfolio positions and individual greeks.

weights

- The Greeks are tools that let us to quantify these relationships:

Input	Greek	Definition	Mnemonic
S_t	Δ (Delta)	Measures the option price change when the stock price increases by \$1 first derivative	
S_t	Γ (Gamma)	Measures the change in Δ when the stock price increases by \$1 second derivative	
S_t	Ω (Elasticity)	Measures the percentage change in the option price when the stock price increases by 1%	
σ	Vega	Measures the option price change when there is an increase in volatility of 1%	vega $\leftrightarrow$ volatility
t	θ (theta)	Measures the option price change when there is a decrease in the time to maturity (increase in calendar time) of 1 day	theta $\leftrightarrow$ time
r	ρ (rho)	Measures the option price change when there is an increase in the interest rate of 1% (100 basis points) 0.01%	rho $\leftrightarrow$ r
δ	Ψ (Psi)	Measures the option price change when there is an increase in the continuous dividend yield of 1% (100 basis points)	

Option Greeks (cont'd)

- Let us come back to Table 1 and complete it with the proper signs of the Greeks:

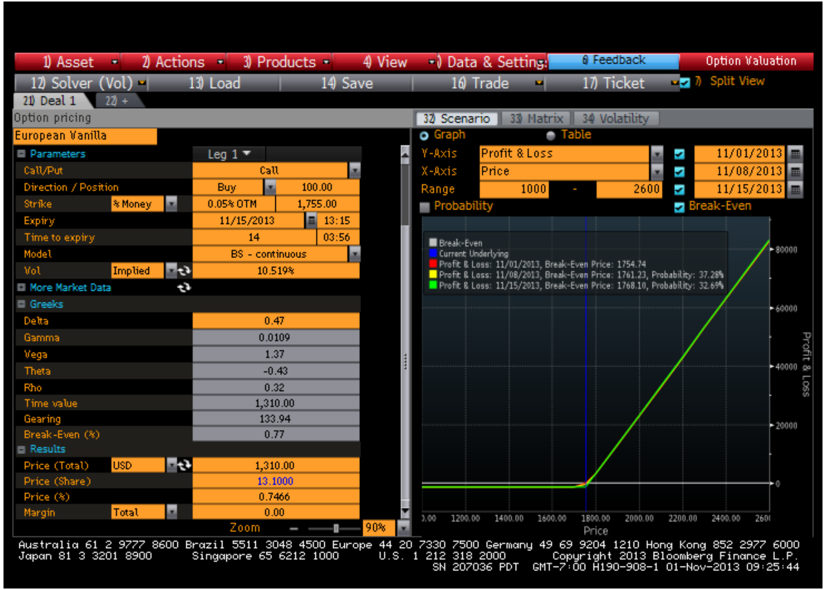
Input	**Call Option**		**Put Option**	
$S_t \uparrow$	$C_t \uparrow$	$\Delta_{Call} > 0$	$P_t \downarrow$	$\Delta_{Put} < 0$
	$\Delta_{Call} \uparrow$	$\Gamma_{Call} > 0$	$\Delta_{Put} \uparrow$	$\Gamma_{Put} > 0$
	$\Omega_{Call} \geq 1$		$\Omega_{Put} \leq 0$	
$\sigma \uparrow$	$C_t \uparrow$	$Vega_{Call} > 0$	$P_t \uparrow$	$Vega_{Put} > 0$
$t \uparrow$	C_t gen. $\downarrow$	θ_{Call} gen. < 0	P_t ambig.	θ_{Put} any
$r \uparrow$	$C_t \uparrow$	$\rho_{Call} > 0$	$P_t \downarrow$	$\rho_{Put} < 0$
$\delta \uparrow$	$C_t \downarrow$	$\Psi_{Call} < 0$	$P_t \uparrow$	$\Psi_{Put} > 0$

Option Greeks: Example

- The Greeks are mathematical derivatives of the option price formula with respect to the inputs.
- Suppose that the stock price is $S_t = \$41$, the strike price is $K = \$40$, volatility is $\sigma = 0.3$, the risk-free rate is $r = 0.08$, the time to expiration is $T - t = 1$, and the dividend yield is $\delta = 0$. The values for the Greeks are

Input	**Call Option**		**Put Option**	
S_t	$\Delta_{Call} = 0.691$	> 0	$Delta_{Put} = -0.309$	< 0
	$\Gamma_{Call} = 0.029$	> 0	$\Gamma_{Put} = 0.029$	> 0
	$\Omega_{Call} = 4.071$	≥ 1	$\Omega_{Put} = -4.389$	≤ 0
σ	$Vega_{Call} = 0.144$	> 0	$Vega_{Put} = 0.144$	> 0
t	$\theta_{Call} = -0.011$	gen. < 0	$\theta_{Put} = -0.003$	any
r	$\rho_{Call} = 0.214$	> 0	$\rho_{Put} = -0.156$	< 0
$\delta \uparrow$	$\Psi_{Call} = -0.283$	< 0	$\Psi_{Put} = 0.127$	> 0

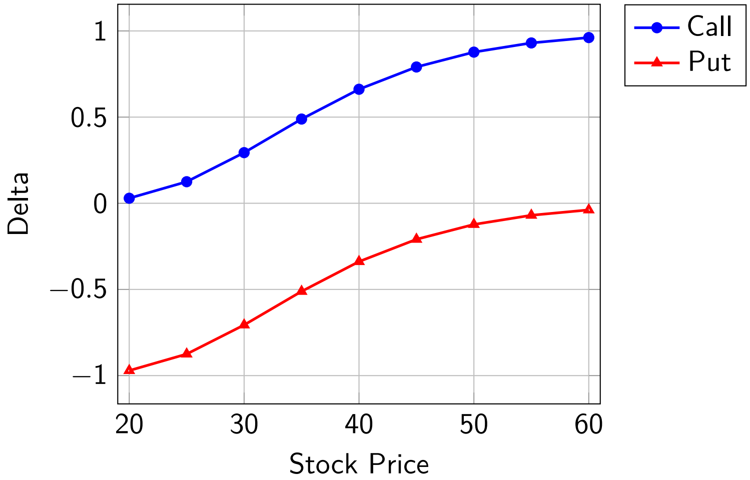
Bloomberg: SPX 11/16/13 C1755 <INDEX> OV <GO>



Bloomberg: Greeks Convention

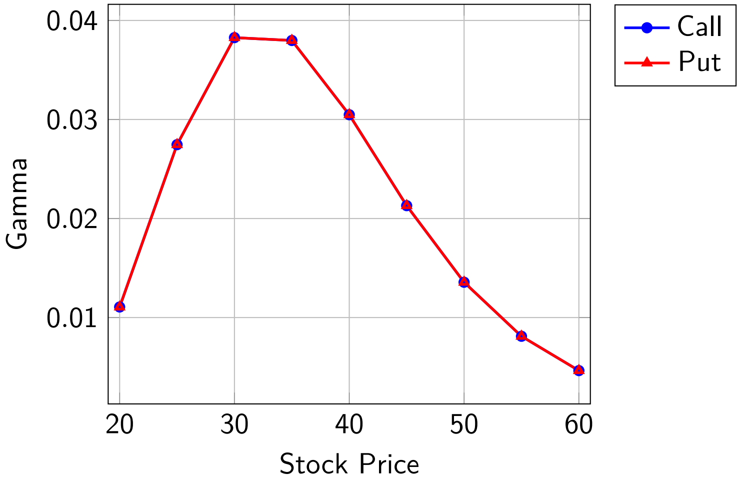
Delta (Δ)

Measures the change in the option price for a \$1 change in the stock price:



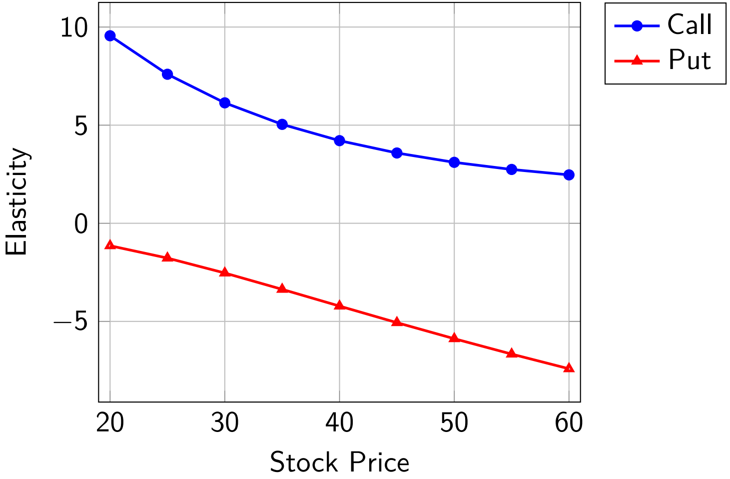
Gamma (Γ)

Measures the change in delta when the stock price changes:



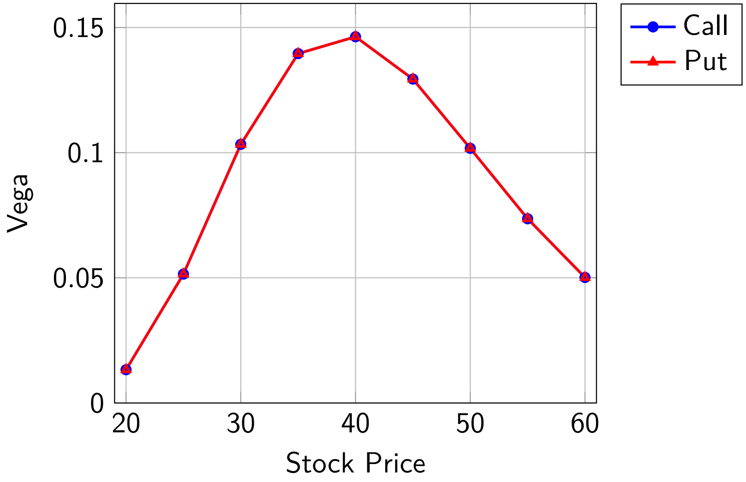
Elasticity (Ω)

Measures the percentage change in the option price relative to the percentage change in the stock price:



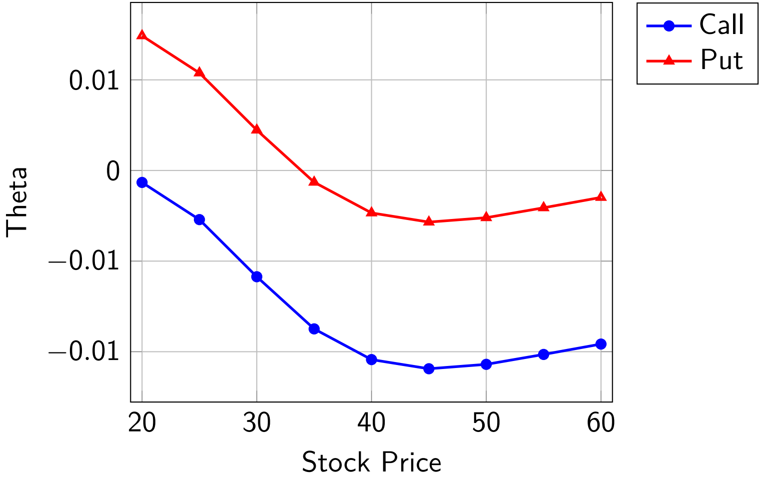
Vega

Measures the change in the option price when volatility changes (divide by 100 for a change per percentage point):



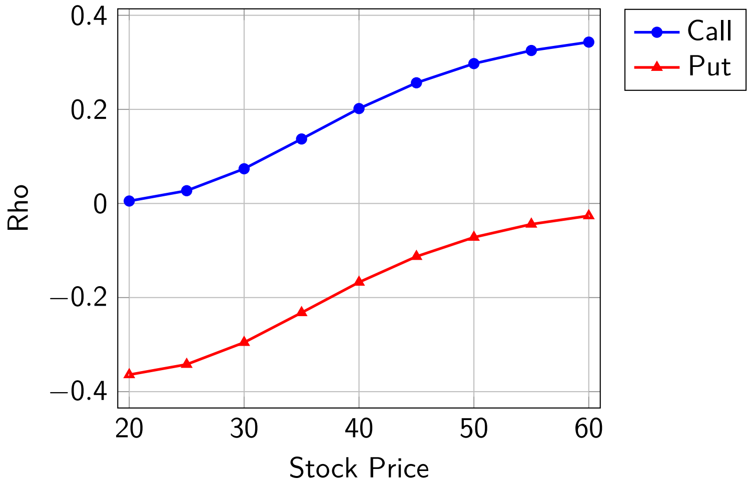
Theta (θ)

Measures the change in the option price with respect to calendar time, t, holding fixed the maturity date T. To obtain per-day theta, divide by 365.



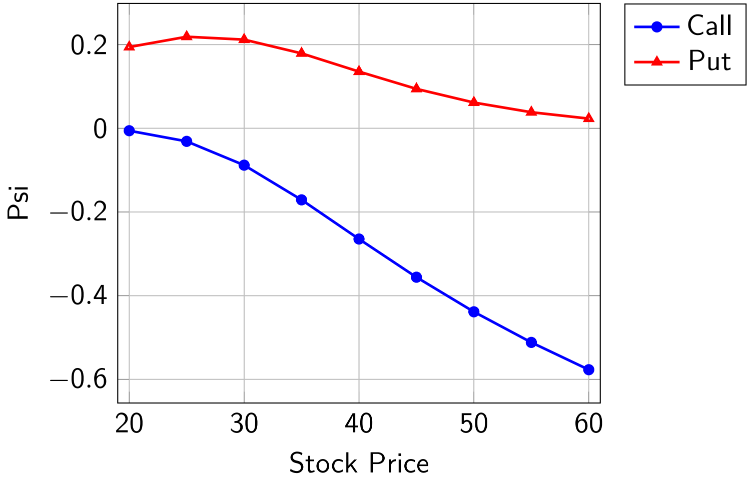
Rho (ρ)

Measures the change in the option price when the interest rate changes (divide by 100 for a change per percentage point, or by 10,000 for a change per basis point):



Psi (Ψ)

Measures the change in the option price when the continuous dividend yield changes (divide by 100 for a change per percentage point):



Gamma-Neutrality

① liquid

② can be any financial instrument written on the same underlying that has a nonzero gamma and that is different from the original call.

- Gamma hedging is the construction of options positions that are hedged such that the total gamma of the position is zero.
- We cannot do this using just the stock, because the gamma of the stock is zero (the delta of a stock is constant and equal to 1).
- Hence, we must acquire another option in an amount that offsets the gamma of the written call.
- Let us go back to our previous example of delta-hedging. In addition to the 91-day call, consider a 45-strike 120-day call.
- The ratio of the gamma of the two options is

$$\frac{\Gamma_{K=40,T-t=91}}{\Gamma_{K=45,T-t=120}} = \frac{0.0606}{0.0540} = 1.1213 \qquad (28)$$

- Thus, we need to buy 1.1213 of the 45-strike options for every 40-strike option we have sold.

Gamma-Neutrality (cont'd)

- The Greeks resulting form this position are in the last column of the following table:

	40-Strike Call	**45-Strike Call**	**Total Position**
Price (\$)	3.395	1.707	-1.481
Delta (Δ)	0.645	0.381	-0.218
Gamma (Γ)	0.061	0.054	0
Theta (θ)	-0.018	-0.014	0.002

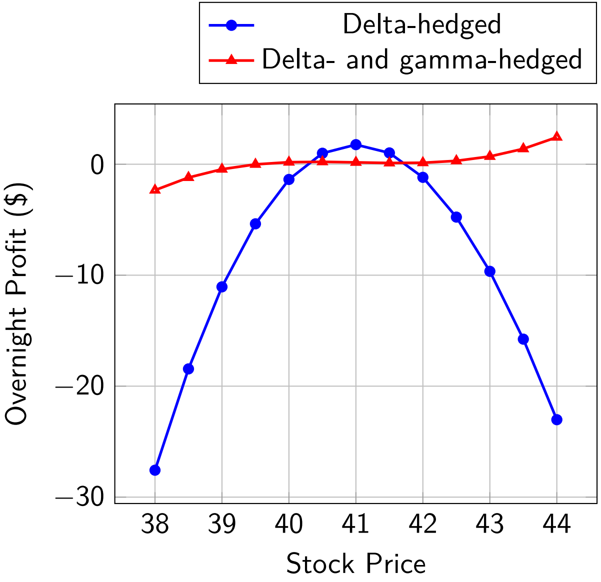
- Since delta is -0.218, we need to buy 21.8 shares of stock to be both delta- and gamma-hedged.

PS3 Q1

Gamma-Neutrality (cont'd)

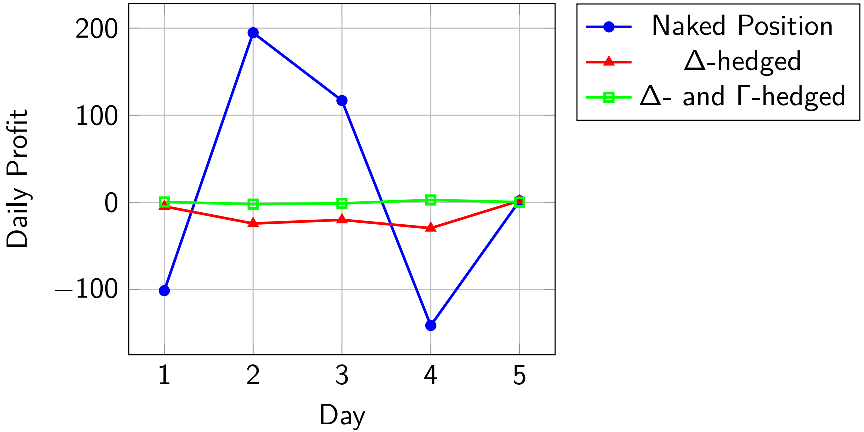
- We can compare the delta-hedged position with the delta- and gamma-hedged position. The delta-hedged position has the problem that large moves always cause losses.
- The delta- and gamma-hedged position loses less if there is a large move down, and can make money if the stock price increases.
- This can be seen from the following figure. It compares the 1-day holding period profit for delta-hedged position described earlier and delta- and gamma hedged position.

Gamma-Neutrality (cont'd)

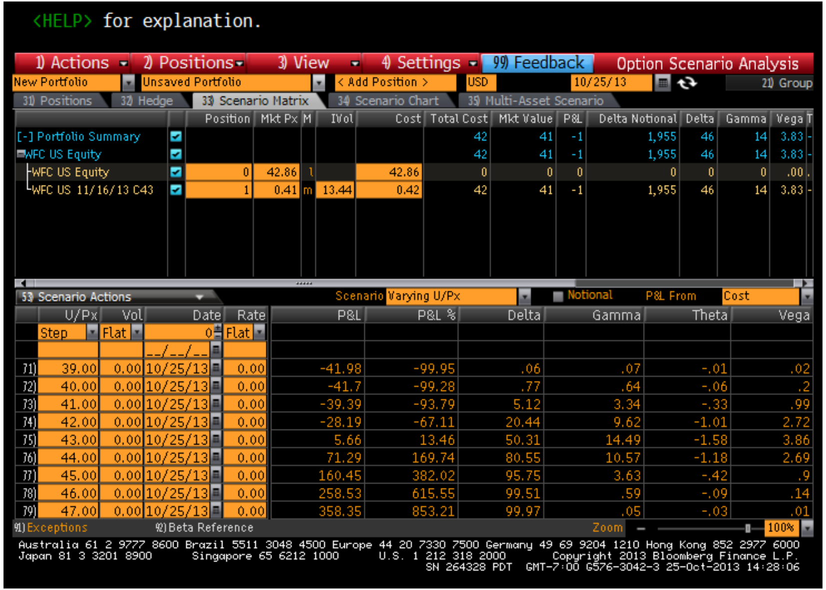


Gamma-Neutrality (cont'd)

- Delta-gamma hedging prevents the position from reacting to large changes in the underlying stock:



No Hedge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1) Actions 2) Positions 3) View 4) Settings 99) Feedback Option Scenario Analysis
New Portfolio Unsaved Portfolio < Add Position > USD 10/25/13 21) Group
31) Positions 32) Hedge 33) Scenario Matrix 34) Scenario Chart 35) Multi-Asset Scenario
Position Mkt Px M IVol Cost Total Cost Mkt Value P&L Delta Notional Delta Gamma Vega
[-] Portfolio Summary 42 41 -1 1,955 46 14 3.83
WFC US Equity 42 41 -1 1,955 46 14 3.83
WFC US Equity 0 42.86 l 42.86 0 0 0 0 0 0 .00
WFC US 11/16/13 C43 1 0.41 m 13.44 0.42 42 41 -1 1,955 46 14 3.83
53) Scenario Actions Scenario Varying U/Px Notional P&L From Cost
U/Px Vol Date Rate P&L P&L % Delta Gamma Theta Vega
Step Flat 0 Flat
71) 39.00 0.00 10/25/13 0.00 -41.98 -99.95 .06 .07 -.01 .02
72) 40.00 0.00 10/25/13 0.00 -41.7 -99.28 .77 .64 -.06 .2
73) 41.00 0.00 10/25/13 0.00 -39.39 -93.79 5.12 3.34 -.33 .99
74) 42.00 0.00 10/25/13 0.00 -28.19 -67.11 20.44 9.62 -1.01 2.72
75) 43.00 0.00 10/25/13 0.00 5.66 13.46 50.31 14.49 -1.58 3.86
76) 44.00 0.00 10/25/13 0.00 71.29 169.74 80.55 10.57 -1.18 2.69
77) 45.00 0.00 10/25/13 0.00 160.45 382.02 95.75 3.63 -.42 .9
78) 46.00 0.00 10/25/13 0.00 258.53 615.55 99.51 .59 -.09 .14
79) 47.00 0.00 10/25/13 0.00 358.35 853.21 99.97 .05 -.03 .01
91) Exceptions 92) Beta Reference Zoom 100%
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Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2013 Bloomberg Finance L.P.
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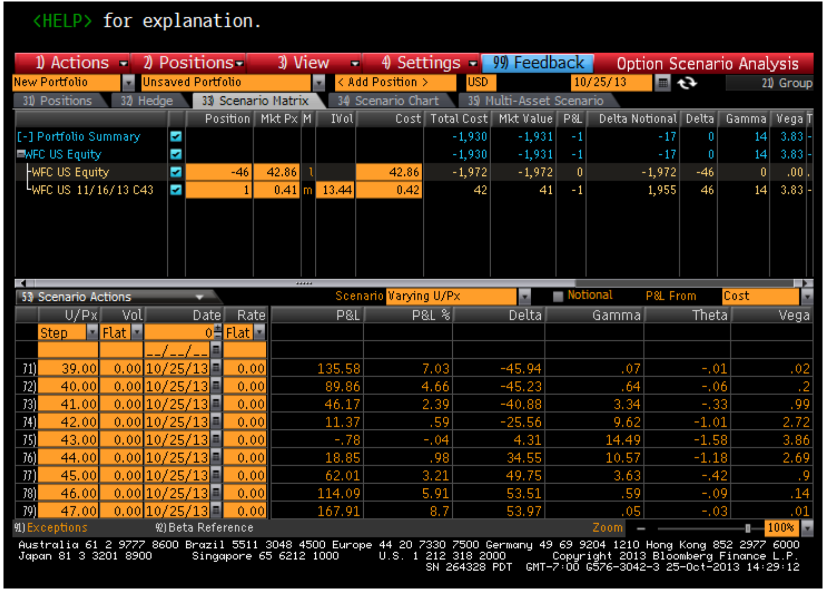
Delta-Hedged



<HELP> for explanation.

1) Actions | 2) Positions | 3) View | 4) Settings | 99) Feedback | Option Scenario Analysis

New Portfolio | Unsaved Portfolio | < Add Position > | USD | 10/25/13 | 21) Group

31) Positions | 32) Hedge | 33) Scenario Matrix | 34) Scenario Chart | 35) Multi-Asset Scenario

	Position	Mkt Px	M	IVol	Cost	Total Cost	Mkt Value	P&L	Delta Notional	Delta	Gamma	Vega
[-] Portfolio Summary						-1,930	-1,931	-1	-17	0	14	3.83
WFC US Equity						-1,930	-1,931	-1	-17	0	14	3.83
WFC US Equity	-46	42.86	l		42.86	-1,972	-1,972	0	-1,972	-46	0	.00
WFC US 11/16/13 C43	1	0.41	m	13.44	0.42	42	41	-1	1,955	46	14	3.83

53) Scenario Actions | Scenario Varying U/Px | Notional | P&L From Cost

	U/Px	Vol	Date	Rate	P&L	P&L %	Delta	Gamma	Theta	Vega
	Step	Flat	0	Flat						
			--/--/--							
71)	39.00	0.00	10/25/13	0.00	135.58	7.03	-45.94	.07	-.01	.02
72)	40.00	0.00	10/25/13	0.00	89.86	4.66	-45.23	.64	-.06	.2
73)	41.00	0.00	10/25/13	0.00	46.17	2.39	-40.88	3.34	-.33	.99
74)	42.00	0.00	10/25/13	0.00	11.37	.59	-25.56	9.62	-1.01	2.72
75)	43.00	0.00	10/25/13	0.00	-.78	-.04	4.31	14.49	-1.58	3.86
76)	44.00	0.00	10/25/13	0.00	18.85	.98	34.55	10.57	-1.18	2.69
77)	45.00	0.00	10/25/13	0.00	62.01	3.21	49.75	3.63	-.42	.9
78)	46.00	0.00	10/25/13	0.00	114.09	5.91	53.51	.59	-.09	.14
79)	47.00	0.00	10/25/13	0.00	167.91	8.7	53.97	.05	-.03	.01

91) Exceptions | 92) Beta Reference | Zoom 100%

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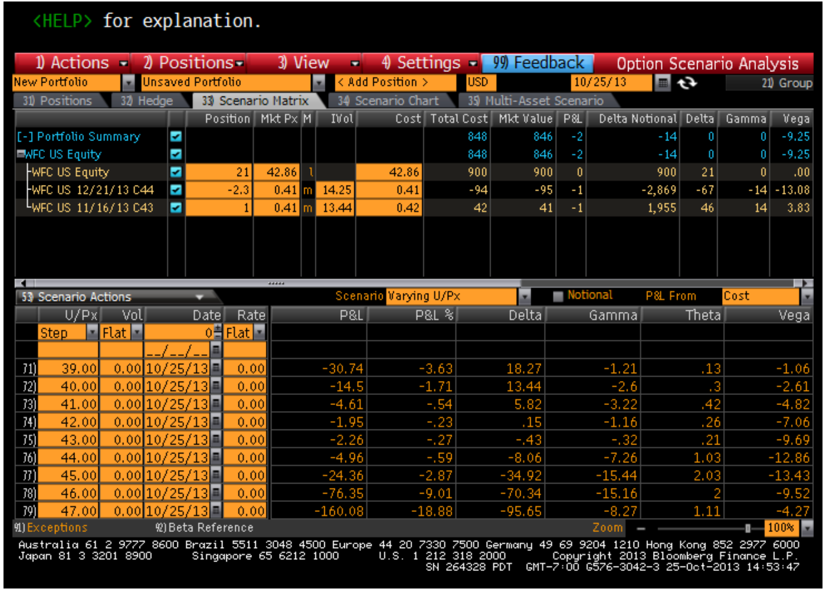
Delta-Gamma-Hedged



<HELP> for explanation.

1) Actions | 2) Positions | 3) View | 4) Settings | 99) Feedback | Option Scenario Analysis

New Portfolio | Unsaved Portfolio | < Add Position > | USD | 10/25/13 | 21) Group

31) Positions | 32) Hedge | 33) Scenario Matrix | 34) Scenario Chart | 35) Multi-Asset Scenario

	Position	Mkt Px	M	IVol	Cost	Total Cost	Mkt Value	P&L	Delta Notional	Delta	Gamma	Vega
[-] Portfolio Summary						848	846	-2	-14	0	0	-9.25
WFC US Equity						848	846	-2	-14	0	0	-9.25
WFC US Equity	21	42.86	l		42.86	900	900	0	900	21	0	.00
WFC US 12/21/13 C44	-2.3	0.41	m	14.25	0.41	-94	-95	-1	-2,869	-67	-14	-13.08
WFC US 11/16/13 C43	1	0.41	m	13.44	0.42	42	41	-1	1,955	46	14	3.83

53) Scenario Actions | Scenario Varying U/Px | Notional | P&L From Cost

	U/Px	Vol	Date	Rate	P&L	P&L %	Delta	Gamma	Theta	Vega
	Step	Flat	0	Flat						
			--/--/--							
71)	39.00	0.00	10/25/13	0.00	-30.74	-3.63	18.27	-1.21	.13	-1.06
72)	40.00	0.00	10/25/13	0.00	-14.5	-1.71	13.44	-2.6	.3	-2.61
73)	41.00	0.00	10/25/13	0.00	-4.61	-.54	5.82	-3.22	.42	-4.82
74)	42.00	0.00	10/25/13	0.00	-1.95	-.23	.15	-1.16	.26	-7.06
75)	43.00	0.00	10/25/13	0.00	-2.26	-.27	-.43	-.32	.21	-9.69
76)	44.00	0.00	10/25/13	0.00	-4.96	-.59	-8.06	-7.26	1.03	-12.86
77)	45.00	0.00	10/25/13	0.00	-24.36	-2.87	-34.92	-15.44	2.03	-13.43
78)	46.00	0.00	10/25/13	0.00	-76.35	-9.01	-70.34	-15.16	2	-9.52
79)	47.00	0.00	10/25/13	0.00	-160.08	-18.88	-95.65	-8.27	1.11	-4.27

91) Exceptions | 92) Beta Reference | Zoom 100%

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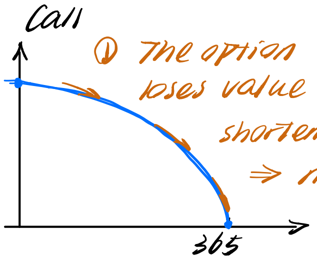
Calendar Spreads

Long long maturity
Short short maturity

- To protect against a stock price increase when selling a call, you can simultaneously buy a call option with the same strike and greater time to expiration.
- This purchased calendar spread exploits the fact that the written near-to-expiration option exhibits greater time decay than the purchased far-to-expiration option, and therefore is profitable if the stock price does not move.



after x days

① The option loses value as maturity shortens (time decay) ⇒ negative theta

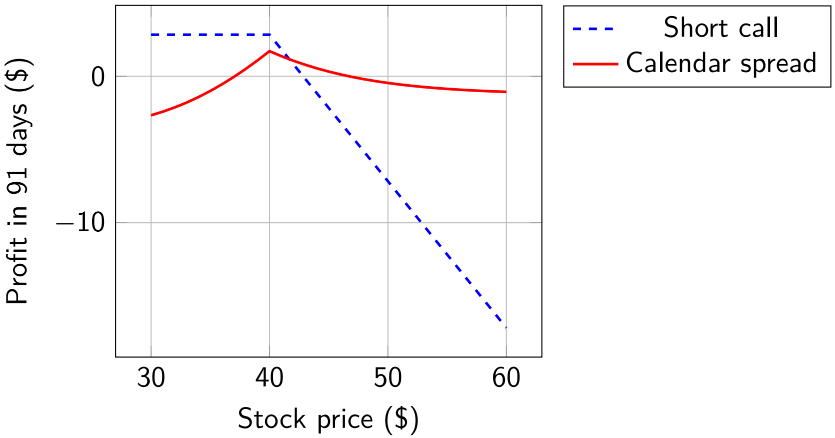
② The option price go down faster and faster ⇒ theta is stronger for short maturity options

Calendar Spreads (cont'd)

- Suppose you sell a 40-strike call with 91 days to expiration and buy a 40-strike call with 1 year to expiration. Assume a stock price of \$40, $r = 8\%$, $\sigma = 30\%$, and $\delta = 0$.
- The premiums are \$2.78 for the 91-day call and \$6.28 for the 1-year call.
- Theta is more negative for the 91-day call (-0.0173) than for the 1-year call (-0.0104). Thus, if the stock price does not change over the course of 1 day, the position will make money since the written option loses more value than the purchased option.

Calendar Spreads (cont'd)

- The profit diagram for this position for a holding period of 91 days is displayed below



Outline

Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon

- Consider the investment under uncertainty problem from the binomial option pricing section.
- A project requires an initial investment of \$100. Thus, $K = 100$.
- The project is expected to generate a perpetual cash flow stream, with a first cash flow \$18 in one year, expected to grow at 3% annually. Assume a discount rate of 15%.

$$\Rightarrow \begin{cases} \text{Perpetual growing annuity} \Rightarrow PV = \frac{\$18}{0.15-0.03} = \$150 \\ \text{Static NPV} = \$150 - \$100 = \$50 \\ \text{Cont. compounded div. yield } \delta = \ln\left(\frac{\$18}{\$150} + 1\right) = 0.1133 \end{cases} \tag{29}$$

- The cont. compounded risk-free rate is $r = 6.766\%$. The cash flows of the project are normally distributed with a volatility of $\sigma = 50\%$.

Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon (cont'd)

- The above example assumes that we must start the project by year 2.
- Suppose instead that the project can be started at any time and then will live forever.
- The project is then a **perpetual call option**

Valuing Perpetual Options

- Calls and puts that never expire ar known as **perpetual options**.
- Perpetual American options always have the same time to expiration, namely infinity.
- Because time to expiration is constant, the option exercise problem will look the same today, tomorrow, and forever.
- Thus, the price at which it is optimal to exercise the option is constant.
- The optimal exercise strategy entails picking the exercise barrier that maximizes the value of the option, and then exercising the option the first time the stock price reaches that barrier.

Valuing Perpetual Options (cont'd)

- First, define h_1 and h_2:

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{30}$$

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{31}$$

- Perpetual American Call:

$$C_{perpetual} = (H_c - K)\left(\frac{S}{H_c}\right)^{h_1}, \text{ where } H_c = K\frac{h_1}{h_1 - 1} \tag{32}$$

- Perpetual American Put:

$$P_{perpetual} = (K - H_p)\left(\frac{S}{H_p}\right)^{h_2}, \text{ where } H_p = K\frac{h_2}{h_2 - 1} \tag{33}$$

Real Options Revisited: Evaluating Projects with an Infinite Investment Horizon (cont'd)

- Using continuously compounded imputs, we compute:

$$\text{CallPerpetual}(\$150, \$100, 0.5, 0.06766, 0.1133) = \{\$63.4, \$245.7\} \tag{34}$$

- When the project value is \$150, the option value is \$63.4 and the optimal investment trigger is \$245.7.
- In other words, we invest when the project is worth \$245.7, more than twice the investment cost.
- If we invest immediately, the project is worth \$50.
- The ability to wait increases that value by \$13.4.

The Option to Abandon

- Firms worry that new projects will not pay off
- Having the option to abandon a project that does not pay off can be valuable
- The option to abandon takes on the characteristics of a put option:
  - V is the remaining value on a project if it continues to the end of its life
  - L is the liquidation (abandonment) value
  - Payoff from owning an abandonment option:

$$\text{Payoff} = \begin{cases} 0 & \text{if } V > L \\ L - V & \text{if } V \leq L \end{cases}$$

- Having a option to abandon a project can make otherwise unacceptable projects acceptable

Collars in Acquisitions: Valuing an Offer

The Northrop Grumman—TRW merger

In July 2002, Northrop Grummann and TRW agreed that Northrop would pay \$7.8 billion for TRW. The number of Northrop Grumman shares to be exchanged for each TRW share is

$$\begin{aligned} 0.5357 \text{ shares} &\quad \text{if } S_{NG} \leq \$112 \\ \$60/S_{NG} \text{ shares} &\quad \text{if } \$112 < S_{NG} < \$138 \\ 0.4348 \text{ shares} &\quad \text{if } S_{NG} \geq \$138 \end{aligned}$$

where S_{NG} is the average Northrop Grumman price over the 5 days preceding the close of the merger.

Suppose that TRW shareholders were certain the merger would occur at time $T = 5/12$. Assume a risk-free rate of 1.5%, and the volatility Northrop Grumman shares is 36%. Northrop Grumman pays no dividends. The closing price of Northrop Grumman is \$120.

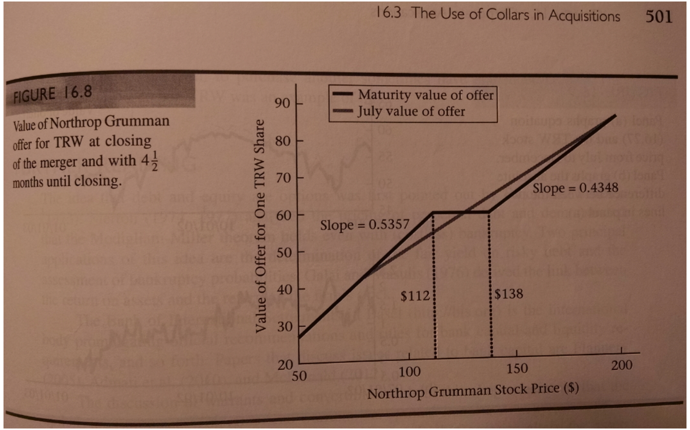
How would TRW shareholders value the Northrop offer?

Collars in Acquisitions: Valuing an Offer

- The offer is equivalent to:
  1. Buying 0.5357 shares of Northrop Grumman
  2. Selling 0.5357 112-strike calls (the Black-Scholes price is \$15.6/call)
  3. Buying 0.4348 138-strike calls (the Black-Scholes price is \$5.22/call)
- To understand this, plot the value of the Northrop Grumman offer for one TRW share, as a function of the Northrop Grumman share price (you should obtain a floating collar offer—see page 501 in McDonald and next slide)
- The value of TRW shares would then be

$$0.5357 \times 120 - 0.5357 \times 15.6 + 0.4348 \times 5.22 = \$58.2 \tag{35}$$

Collars in Acquisitions: Valuing an Offer



Collars in Acquisitions: Valuing an Offer

- The theoretical value of a TRW share under the terms of the offer is greater than the market price of a TRW share
- This is what we would expect to see, since in order to induce the target company to accept an offer, the acquirer generally has to offer a price greater than the perceived value of the target as a stand-alone company
- The difference between the two values declines toward zero as the merger is likely to take place or diverges if the merger is cancelled for some reason
- Risk arbitrageurs take positions in the two stocks in order to speculate on the success or failure of the merger
- Mitchell and Pulvino (*Characteristics of Risk and Return in Risk Arbitrage*, Journal of Finance, 2001) examine the historical returns earned by risk arbitrageurs

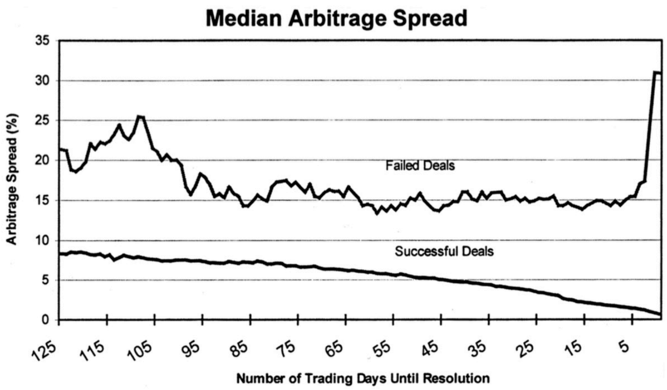


Figure 1. This figure plots the median arbitrage spread versus time until deal resolution. The arbitrage spread is defined to be the offer price minus the target price divided by the target price. For failed deals, the deal resolution date is defined as the date of the merger termination announcement. For successful deals, the resolution date is the consummation date.

Portfolio Insurance

- A portfolio manager is often interested in acquiring a put option on his or her portfolio
- The option can be created synthetically
- This involves maintaining a position in the underlying asset so that the delta of the position is equal to the delta of the required option
- There are two reasons why it might be more attractive to create the required put option synthetically than to buy it in the market:
  - Option markets do not always have the liquidity to absorb the trades required by managers of large funds
  - Fund managers often require strike prices and exercise dates that are different from those available in exchange-traded option markets

Portfolio Insurance

Example

A portfolio is worth \$90 million. To protect against market downturns the managers of the portfolio require a 6-month European put option on the portfolio with a strike of \$87 million.

The risk-free rate is 9% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is 25% per annum. The S&P 500 stands at 900.

The portfolio is considered to mimic the S&P 500 fairly closely.

Create the required option synthetically.

Portfolio Insurance

- In this case, $S_0 = 90$ million, $K = 87$ million, $r = 0.09$, $\delta = 0.03$, $\sigma = 0.25$, and $T = 0.5$

This means that adding that put option to your portfolio would change your portfolio delta to 0.6785

- The delta of the put option is

$$-e^{-\delta T} N(-d_1) = -0.3215 \tag{36}$$

risky asset position : 0.6785

- This shows that 32.15% of the portfolio (\$28.94 million) should be sold initially and invested in risk-free assets to match the delta of the required option.
- The amount of the portfolio should be monitored frequently:
  - If the portfolio reduces to \$88 million after 1 day, the delta of the required option changes to -0.3681 and further 4.66% of the original portfolio (\$4.10 million) should be sold and invested in the risk-free asset price ↓ sell ↓
  - If the portfolio increases to \$92 million, delta changes to -0.2787 and 4.29% (\$3.94 million) should be repurchased price ↑ buy
- Calculations are more involved if the portfolio's beta (β) is not 1.

1987 market crash

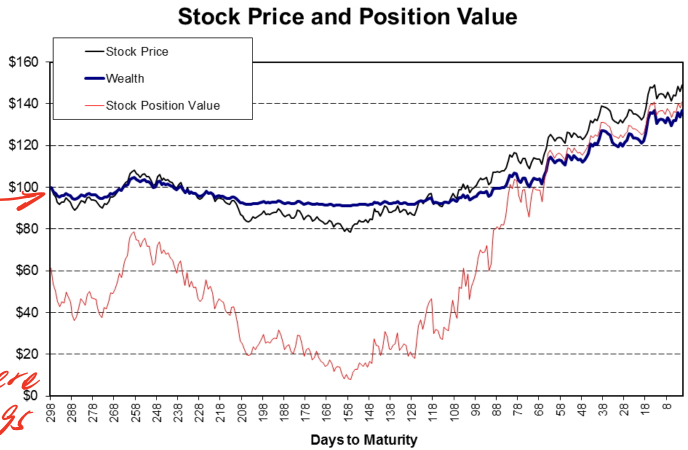
earn interest

risky	risk-free	total
59.7	28.95	88.65
62.42	28.95	91.36

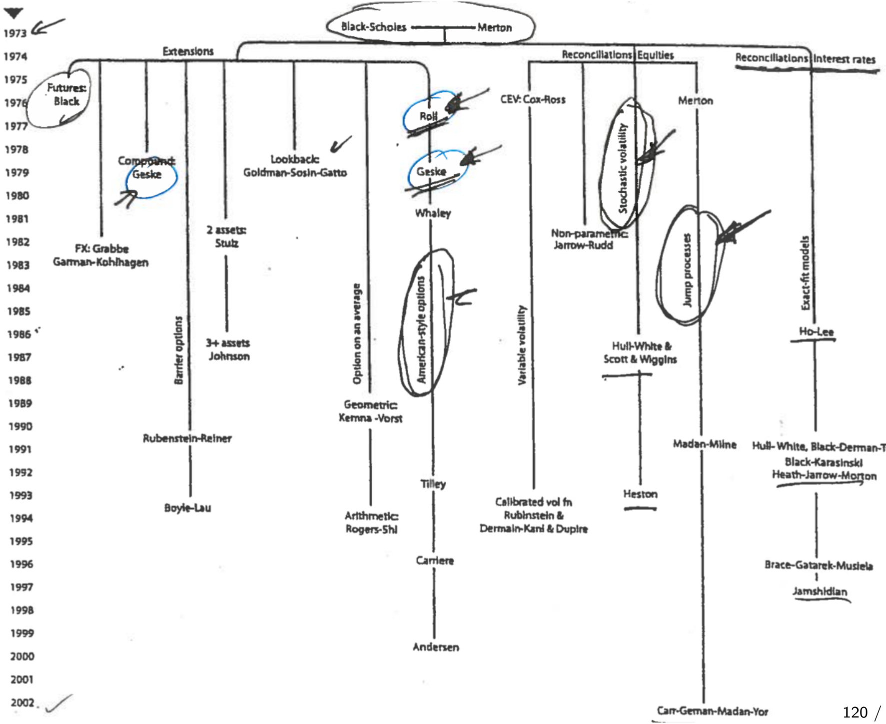
(insured)

Example: Option-Based Portfolio Insurance (OBPI) strategy over 298 trading days, for an initial investment of \$100, with a floor of \$95.

less volatile than the risky asset and doesn't go below the floor that here is fixed at \$95 ?



Outline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Black-Scholes
Merton
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Reconciliations Equities
Reconciliations Interest rates
Futures: Black
Compound: Geske
FX: Grabbe Garman-Kohlhagen
Barrier options
Rubenstein-Reiner
Boyle-Lau
2 assets: Stulz
3+ assets Johnson
Lookback: Goldman-Sosin-Gatto
Option on an average
Geometric: Kemna -Vorst
Arithmetic: Rogers-Shi
Roll
Geske
Whaley
American-style options
Tilley
Carriere
Andersen
CEV: Cox-Ross
Variable volatility
Calibrated vol fn Rubinstein & Dermain-Kani & Dupire
Non-parametric: Jarrow-Rudd
Stochastic volatility
Hull-White & Scott & Wiggins
Heston
Merton
Jump processes
Madan-Milne
Carr-Geman-Madan-Yor
Exact-fit models
Ho-Lee
Hull- White, Black-Derman-Toy
Black-Karasinski
Heath-Jarrow-Morton
Brace-Gatarek-Musiela
Jamshidian

Outline

Appendix: Formulas for Option Greeks

- Delta (Δ) measures the change in the option price for a \$1 change in the stock price:

$$\Delta_{Call} = \frac{\partial C}{\partial S} = e^{-\delta(T-t)}N(d_1) \tag{37}$$

$$\Delta_{Put} = \frac{\partial P}{\partial S} = -e^{-\delta(T-t)}N(-d_1) \tag{38}$$

- Gamma (Γ) measures the change in delta when the stock price changes:

$$\Gamma_{Call} = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}} \tag{39}$$

$$\Gamma_{Put} = \frac{\partial^2 P}{\partial S^2} = \Gamma_{Call} \tag{40}$$

Appendix: Formulas for Option Greeks (cont'd)

- Elasticity (Ω) measures the percentage change in the option price relative to the percentage change in the stock price:

$$\Omega_{Call} = \frac{S_t \Delta_{Call}}{C_t} \tag{41}$$

$$\Omega_{Put} = \frac{S_t \Delta_{Put}}{P_t} \tag{42}$$

- Vega measures the change in the option price when volatility changes (divide by 100 for a change per percentage point):

$$\text{Vega}_{Call} = \frac{\partial C}{\partial \sigma} = S e^{-\delta(T-t)} N'(d_1) \sqrt{T-t} \tag{43}$$

$$\text{Vega}_{Put} = \frac{\partial P}{\partial \sigma} = \text{Vega}_{Call} \tag{44}$$

Appendix: Formulas for Option Greeks (cont'd)

- Theta (θ) measures the change in the option price with respect to calendar time, t, holding fixed the maturity date T. To obtain per-day theta, divide by 365.

$$\theta_{Call} = \frac{\partial C}{\partial t} = \delta S e^{-\delta(T-t)} N(d_1) - rKe^{-r(T-t)} N(d_2) - \frac{Ke^{-r(T-t)} N'(d_2)\sigma}{2\sqrt{T-t}} \tag{45}$$

$$\theta_{Put} = \frac{\partial P}{\partial t} = \theta_{Call} + rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)} \tag{46}$$

- Rho (ρ) measures the change in the option price when the interest rate changes (divide by 100 for a change per percentage point, or by 10,000 for a change per basis point):

$$\rho_{Call} = \frac{\partial C}{\partial r} = (T-t) Ke^{-r(T-t)} N(d_2) \tag{47}$$

$$\rho_{Put} = \frac{\partial P}{\partial r} = -(T-t) Ke^{-r(T-t)} N(-d_2) \tag{48}$$

Appendix: Formulas for Option Greeks (cont'd)

- Psi (Ψ) Measures the change in the option price when the continuous dividend yield changes (divide by 100 for a change per percentage point):

$$\Psi_{Call} = \frac{\partial C}{\partial \delta} = -(T-t)Se^{-\delta(T-t)}N(d_1) \tag{49}$$

$$\Psi_{Put} = \frac{\partial P}{\partial \delta} = (T-t)Se^{-\delta(T-t)}N(-d_1) \tag{50}$$