

INTEREST RATES

Required Reading

- **Chapter 5**, “*Interest Rates*” from J. Berk et al., Fundamentals of Corporate Finance, Second Canadian Edition.

Effective Annual Rate (EAR)

Effective Annual Rate (EAR): The total amount of interest that will be earned at the end of one year.

Note: The Effective Annual Rate (**EAR**) is also referred to as the Annual Percentage Yield (**APY**) or the Effective Annual Yield (**EAY**).

Question

Mary takes a \$1 two-year loan at a 2-year interest of 10%. What is the interest rate that Mary pays per year?

Answer

- After two years Mary has to pay: $1.1 \times 1 = \$1.1$

- The effective annual interest rate is:

$$(1 + r)^2 \times 1 = \$1.1 \Rightarrow r = 4.88\%$$

- Notice that:

$$(1 + 0.0488)^2 = 1 + 0.1 \Rightarrow 1 + 0.0488 = (1 + 0.1)^{1/2}$$

Question

Mary takes a \$1 two-year loan at a interest of 5% per year. What is the interest rate that Mary pays over the two-year period?

Answer

- After two years Mary has to pay: $(1.05)(1.05) \times 1 = \1.1025

- Hence the interest rate over the two-year period is:

$$2 - \text{Year Rate} = \frac{1.1025 - 1}{1} = 10.25\%$$

- Notice that: $(1.05)(1.05) = 1 + 0.1025$

Adjusting the Discount Rate to Different Periods

- Cash flows can arrive at different time intervals (e.g., monthly, quarterly, semi-annual, annual, every two years, etc.)
- When computing present value or future values, you should adjust the discount rate to match the time period of the cash flows.

Next we see how to do this....

Adjusting the Discount Rate to Different Periods

An effective discount rate r for one period can be converted to an equivalent effective discount rate for n periods using the following formula:

$$\text{Equivalent } n\text{-Period Effective Rate} = (1 + r)^n - 1$$

Adjusting the Discount Rate to Different Periods

Examples

1. Quoted Rate: **0.1% per day**

$$\text{Equivalent Effective Annual Rate} = (1 + 0.001)^{365} - 1 = 44.02\%$$

2. Quoted Rate: **20% per a two-year period**

$$\text{Equivalent Effective Annual Rate} = (1 + 0.2)^{0.5} - 1 = 9.54\%$$

$$\text{Equivalent Quarterly Effective Rate} = (1 + 0.2)^{1/8} - 1 = 2.31\%$$

Problem

Your bank account pays interests monthly with an effective annual rate of 6%.

- a. What interest rate will you earn each month?
- b. If you have no money in the bank now, how much will you need to save at the end of each month to accumulate \$100,000 in 10 years?

Problem (cont)

Part a. What interest rate will you earn each month?

$$\text{Effective Monthly Rate} = (1 + 0.06)^{1/12} - 1 = 0.4868\%$$

Therefore, an effective annual rate of 6% is equivalent to an effective monthly rate of 0.4868%.

Problem (cont)

Part b. If you have no money in the bank now, how much will you need to save at the end of each month to accumulate \$100,000 in 10 years?

The proposed cash flow stream is a 120 period annuity (i.e., 10 years x 12 months) with an interest rate of 0.4868% per period (i.e., per month) and a future value of \$100,000. The problem asks for the constant payment C of this annuity. Using the future value of annuity formula and solving for C :

$$C = \frac{FV_n}{\frac{1}{r}[(1+r)^n - 1]} = \frac{100,000}{\frac{1}{0.004868}[(1 + 0.004868)^{120} - 1]} = \$615.47 \text{ per month}$$

Problem (cont)

Thus, if we save \$615.47 per month and we earn interest monthly at an effective annual rate of 6%, we will have \$100,000 in 10 years.

- Notice that **the timing in the annuity formula must be consistent for all of the inputs:**

In this problem, we had a monthly deposit, so we needed to convert our interest rate to a monthly interest rate ($r = 0.4868\%$) and then use the total number of monthly deposits ($n = 120$).

Question What is the present value on a lottery prize that is to be received in 40 equal semi-annual payments of \$125,000, with the first payment beginning in one year? Assume an EAR of 7%.

Solution

$$\text{Effective Six-Month Rate} = (1 + 0.07)^{0.5} - 1 = 3.44\%$$

$$PV = \frac{1}{1 + 0.0344} \frac{125,000}{0.0344} \left(1 - \frac{1}{(1 + 0.0344)^{40}} \right) = 2,604,454$$

Annual Percentage Rate (APR)

Annual Percentage Rate (APR): Indicates the amount of simple interest earned in one year without considering the effects of compounding that may occur.

Example

- 11% APR compounded monthly
- 10% APR compounded daily
- 11% APR compounded weekly

Annual Percentage Rate (APR)

- Because the annual percentage rate (APR) does not take into account the effect of compounding:
 - the APR **does not** reflect the true amount that you will earn over one year.
 - the APR itself **cannot** be used as a discount rate.
- To discount cash flows or to calculate future values, you will need to find the appropriate effective rate implied by the quoted APR.

From an APR to an Effective Rate

- An APR is an indirect way of quoting an actual interest each compounding period:

$$\text{Effective Interest Rate per Compounding Period} = \frac{APR}{m}$$

(where ***m*** is the number of compounding period per year)

From an APR to an Effective Rate

Example

- A 11% APR compounded **monthly** implies a $\frac{11\%}{12} = 0.9166\%$ effective **monthly** rate.
- An 10% APR compounded **daily** implies a $\frac{10\%}{365} = 0.0274\%$ effective **daily** rate.
- A 11% APR compounded **quarterly** implies a $\frac{11\%}{4} = 2.75\%$ effective **quarterly** rate.

Example

1. Quoted Rate: **12% APR compounded monthly**

- This is equivalent to a 1% (i.e., $12\% / 12 = 1\%$) effective monthly rate
- This is **NOT** equivalent to a 3% (i.e., $12\% / 4 = 3\%$) effective quarterly rate

2. Quoted Rate: **12% APR compounded quarterly**

- This is equivalent to a 3% (i.e., $12\% / 4 = 3\%$) effective quarterly rate

From an APR to an EAR

- Once the effective interest earned per compounding period is computed, the equivalent effective interest rate for any other time interval can be computed.
- The *EAR* corresponding to an *APR* with *m* compounding periods per year is:

$$EAR = \left(1 + \frac{APR}{m}\right)^m - 1$$

Examples

1. Quoted Rate: **11% APR compounded monthly**

$$EAR = \left(1 + \frac{0.11}{12}\right)^{12} - 1 = 11.5718\%$$

2. Quoted Rate: **10% APR compounded daily**

$$EAR = \left(1 + \frac{0.10}{365}\right)^{365} - 1 = 10.5155\%$$

3. Quoted Rate: **11% APR compounded quarterly**

$$EAR = \left(1 + \frac{0.11}{4}\right)^4 - 1 = 11.4621\%$$

Notice the following:

Quoted Rate: 11% APR compounded monthly

$$EAR = \left(1 + \frac{0.11}{12}\right)^{12} - 1 = 11.5718\%$$

The above formula We can do this calculation in two steps:

- 1) An 11% APR compounded monthly is equivalent to an effective monthly rate of $\frac{11\%}{12} = 0.9166\%$
- 2) And if we compound 12 times a 0.9166% effective monthly rate we obtain the equivalent effective annual rate:

$$EAR = (1 + 0.9166\%)^{12} - 1 = 11.5718\%$$

Problem

Your firm is purchasing a new telephone system which will last for four years. You can purchase the system for and upfront costs of \$150,000 or you can lease the system from the manufacturer for \$4,000 paid at the end of each month. You can borrow at a 5% APR with semiannual compounding. Should you purchase the system outright or lease it?

Solution

- The cost of leasing is a 48-month annuity of \$4,000 per month. To calculate the PV of this annuity we need to calculate the *effective monthly rate*
- A 5% APR with semiannual compounding is equivalent to a 2.5% effective six-month rate. To convert a six-month discount rate into a monthly discount rate we can use:

$$1 + r_{\text{six-month}} = (1 + r_{\text{monthly}})^6 \Rightarrow r_{\text{monthly}} = (1.025)^{1/6} - 1 = 0.4124\%$$

- Using the PV (Annuity) formula:

$$PV = \frac{4,000}{0.004124} \left(1 - \frac{1}{1.004124^{48}} \right) = \$173,867$$

- Since \$173,867 > \$150,000 we should buy the system outright (if we do not have the money we are can borrow from the bank at 5% APR).

Question: As an investor do you prefer to receive a 48% APR compounded monthly or a 50% APR compounded semiannually?

- a. 48% APR compounded monthly
- b. 50% APR compounded semiannually

Solution

48% APR compounded monthly is equivalent to:

$$1 + EAR = \left(1 + \frac{0.48}{12}\right)^{12} \Rightarrow EAR = 60.10\%$$

50% APR compounded semiannually

$$1 + EAR = \left(1 + \frac{0.50}{2}\right)^2 \Rightarrow EAR = 56.25\%$$

Continuous Compounding

If interest is compounded continuously in time (that is, if $m \rightarrow \infty$) then:

$$1 + EAR = e^{APR}$$

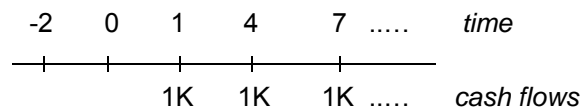
Example

- Quoted Rate: **10% APR continuously compounded**

$$EAR = e^{0.1} - 1 = 10.5171\%$$

Problem

What is the value of a bond that pays \$1,000 every three years starting next year if the interest is a 10% APR compounded continuously.



Solution

First we calculate the effective annual rate:

$$EAR = e^{APR} - 1 = e^{0.1} - 1 = 10.51\%$$

Second we calculate the effective three-year rate:

$$\text{Effective 3-Year Rate} = (1 + EAR)^3 - 1 = 1.1051^3 - 1 = 34.98\%$$

Third using the perpetuity formula we calculate the PV at $t=-2$:

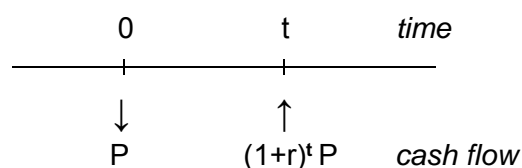
$$PV_{-2} = 1,000 / 0.3498 = \$2858.29$$

Finally calculate the PV at $t=0$:

$$PV_0 = PV_{-2}(1 + EAR)^2 = 2,858.29 \times 1.1051^2 = 3,491.13$$

LOAN TYPES

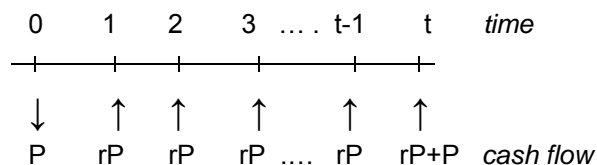
- 1. Pure Discount Loans:** The borrower receives money today and repays a single lump sum at maturity (e.g., T-Bills).



where P is the principal, r the interest rate and t is the maturity period.

LOAN TYPES

- 2. Interest-Only Loans:** The borrower receives money today and pays interest each period and repays the entire principal at some point in the future (e.g., Bonds issued by the governments and corporations).



where P is the principal, r is the interest rate and t is the maturity period.

Example (Interest only loans)

Suppose you owe a principal amount of \$3,000 bearing an annual interest rate of 10%. Prepare an amortization schedule showing the annual payment on the basis of an interest only loan.

Time	Interest Paid	Principal Paid	Total Payment	Principal Left
0	-----	-----	-----	3,000
1	300	0	300	3,000
2	300	0	300	3,000
3	300	3,000	3,300	0

LOAN TYPES

3.1. Amortized Loans in which the borrower pays the interest plus some fixed amount.

Period	Principal paid in the period	Interest paid in the period
0	-P	0
1	P/t	r P
2	P/t	r P(1-1/t)
.....
i	P/t	$rP \left(1 - \frac{i-1}{t} \right)$
.....
t	P/t	r P/t

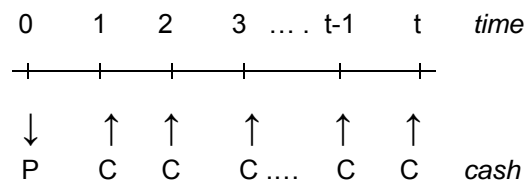
Example (Amortized Loans: Interest plus a fixed amount)

Suppose you owe a principal amount of \$3,000 bearing an effective annual interest rate of 10%. Prepare an amortization schedule showing the annual payment, on the basis that you want to amortize 1/3 of the principal every year.

Time	Interest Paid	Principal Paid	Total Payment	Principal Left
0	-----	-----	-----	3,000
1	300	1,000	1,300	2,000
2	200	1,000	1,200	1,000
3	100	1,000	1,100	0

LOAN TYPES

3.2. Amortized Loans in which the borrower pays a fixed amount each period (e.g., mortgages).



where **P** is the principal, **r** is the interest rate and **t** is the maturity period.

Example (Amortized Loans: Fixed Amount)

Suppose you owe a principal amount of \$3,000 bearing an effective annual interest rate of 10%. Prepare an amortization schedule showing the annual payment, on the basis of a three-year loan with constant annual payments.

Sol.

$$3,000 = \frac{C}{0.1} \left(1 - \frac{1}{1.1^3} \right) \Rightarrow C = \$1,206.34$$

Time	Interest Paid	Principal Paid	Total Payment	Principal Left
0	-----	-----	-----	3,000.0
1	300.0	906.3	1,206.3	2,093.7
2	209.4	997.0	1,206.3	1,096.7
3	109.7	1,096.7	1,206.3	0.0

Example Suppose you owe a principal amount of \$2,000 bearing an effective annual interest rate of 5%. If you must pay two annual payments of \$1,075.6 calculate the amount of principal paid (*i.e.*, amortized) during the first year.

Time	Interest Paid	Principal Paid	Total Payment	Principal Left
0	-----	-----	-----	2,000.0
1	100.0	975.6	1,075.6	1,024.4
2	51.2	1,024.4	1,075.6	0.0

Application: Mortgages

- Mortgages are annuities (usually) with monthly payments.
- In Canada mortgages rates are typically quoted as APR with semi-annual compounding.
- Financial Institutions offer mortgages with rates fixed for various periods. However, the payments on conventional mortgages are calculated to maturity.
- Partially amortized mortgages and balloon or bullet payments (A balloon or bullet payment is the amount that the borrower needs to pay to liquidate the loan before maturity.)

Problem

Suppose that we have a \$100,000 commercial mortgage with a 12% annual rate compounded semiannually and a 20- year amortization. Further suppose that the mortgage has a five-year balloon.

Part a. Calculate the monthly payment.

Part b. Calculate the five-year balloon payment.

Part a. Calculate the monthly payment.

- A 12% APR with semiannual compounding is equivalent to a 6% effective six-month rate. To find the monthly discount rate:

$$r_{\text{monthly}} = (1 + r_{\text{six-month}})^{1/6} - 1 = (1.06)^{1/6} - 1 = 0.98\%$$

- The mortgage is a 240-month annuity with present value of \$100,000 per month with a 0.98% monthly rate and a coupon C:

$$100,000 = \frac{C}{0.0098} \left(1 - \frac{1}{1.0098^{240}} \right) \Rightarrow C = \$1,084.40 \text{ per month}$$

Part b. Calculate the five-year balloon payment.

The balloon is the present value in year five of the remaining 180 payments (i.e., 15 years x 12 months). Note that this stream of payments is an 180-month annuity with C=\$1,084.40 and monthly rate of 0.98%. Hence:

$$\text{Balloon} = \frac{1,084.40}{0.0098} \left(1 - \frac{1}{1.0098^{15 \times 12}} \right) = \$91,528$$

The Determinants of Interest Rates

Fundamentally, interest rates are determined in the market based on individuals' willingness to borrow and lend. Some of the main factors that influence interest rates are:

- Expectations of future growth and productivity
- Government policy
- Inflation

Real Versus Nominal Interest Rates and Cash-Flows

Q. If we invest \$10 at a 10% interest rate and the inflation rate is 15% . How much is this dollar *really* worth in a year?

$$(10 \times 1.1) / 1.15 = \$9.565$$

In *real terms* is really worth less than \$10!!

Q. What is the real return in our investment?

$$(9.565 - 10) / 10 = -4.35\%$$

Real Versus Nominal Interest Rates and Cash-Flows

- To convert nominal cash flows into real cash flows:

$$\text{Real CF} = \frac{\text{Nominal CF}}{1+i}$$

where i is the expected inflation.

- To convert nominal interest rates r into real ones r_r :

$$1 + r_r = \frac{1 + r}{1 + i} \Rightarrow r_r = \frac{r - i}{1 + i} \approx r - i$$

Real Versus Nominal Interest Rates and Cash-Flows

RULE

(be consistent)

- Discount nominal CF by nominal discount rates
- Discount real CF by real discount rates

Example

What is the present value of \$1.1 received in one year if the nominal interest rate is 10% and the inflation 15%?

Answer

\$1.1 dollars is a nominal CF and therefore it should be discounted with nominal interest rates:

$$PV = \frac{CF_{\text{Nominal}}}{1+r} = \frac{1.1}{1+0.1} = \$1$$

Example

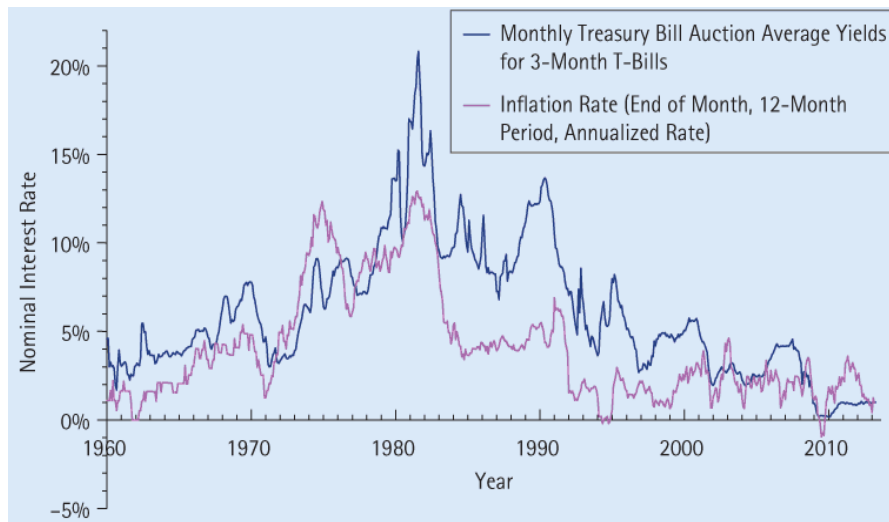
You are going to receive \$40 in real terms in one year. What is the present value if the nominal interest rate is 20% and the inflation 15%?

Answer

\$40 dollars is a real CF and therefore it should be discounted with real interest rates:

$$PV = \frac{CF_{\text{real}}}{1+r_{\text{real}}} = \frac{CF_{\text{real}}}{1+r / (1+i)} = \frac{40}{1+0.2 / (1+0.15)} = \$38.33$$

Interest Rates & Inflation 1960–2013

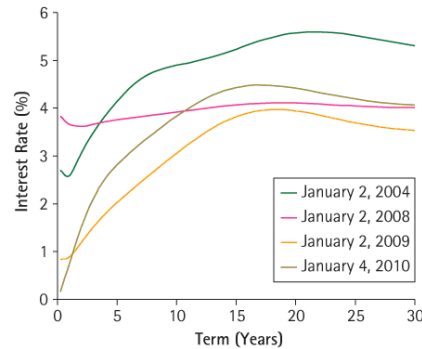


The Yield Curve

- **Term Structure:** The relationship between the investment term and the interest rate
- **Yield Curve:** A plot of bond yields as a function of the bonds' maturity date
- **Risk-Free Interest Rate:** The interest rate at which money can be borrowed or lent without risk over a given period.

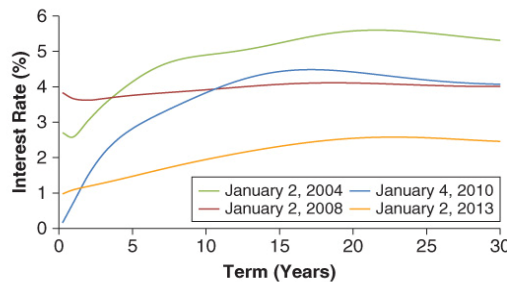
Term Structure of Risk-Free Interest Rates for January, 2004, 2008, 2009, and 2010

Term (Years)	Date			
	Jan. 2004	Jan. 2008	Jan. 2009	Jan. 2010
0.25	2.6990	3.8327	0.8310	0.1597
1	2.5875	3.6641	0.8908	0.7381
2	3.0484	3.6220	1.1881	1.5069
3	3.4756	3.6651	1.5043	2.0958
4	3.8278	3.7159	1.7811	2.5152
5	4.1399	3.7588	2.0239	2.8189
6	4.4059	3.7942	2.2449	3.0579
7	4.6099	3.8252	2.4539	3.2665
8	4.7494	3.8545	2.6571	3.4624
9	4.8379	3.8843	2.8571	3.6513
10	4.8969	3.9157	3.0533	3.8315
11	4.9469	3.9485	3.2424	3.9981
12	5.0023	3.9819	3.4191	4.1453
13	5.0696	4.0145	3.5779	4.2682
14	5.1490	4.0444	3.7135	4.3637
15	5.2363	4.0701	3.8218	4.4307
16	5.3252	4.0902	3.9007	4.4701
17	5.4089	4.1041	3.9500	4.4845
18	5.4815	4.1113	3.9713	4.4776
19	5.5385	4.1123	3.9681	4.4540
20	5.5774	4.1078	3.9448	4.4181
21	5.5973	4.0988	3.9065	4.3746
22	5.5993	4.0866	3.8584	4.3272
23	5.5855	4.0724	3.8050	4.2794
24	5.5590	4.0577	3.7507	4.2337
25	5.5232	4.0435	3.6986	4.1920
26	5.4814	4.0308	3.6514	4.1556
27	5.4368	4.0205	3.6107	4.1251
28	5.3922	4.0132	3.5778	4.1010
29	5.3497	4.0092	3.5531	4.0832
30	5.3109	4.0089	3.5368	4.0714



Term Structure of Risk-Free Interest Rates for January, 2004, 2008, 2010, and 2013

Term (Years)	Date			
	Jan. 2004	Jan. 2008	Jan. 2010	Jan. 2013
0.25	2.6990	3.8327	0.1597	0.97651
1	2.5875	3.6641	0.7381	1.09294
2	3.0484	3.6220	1.5069	1.18849
3	3.4756	3.6651	2.0958	1.27885
4	3.8278	3.7159	2.5152	1.37361
5	4.1399	3.7588	2.8189	1.47261
6	4.4059	3.7942	3.0579	1.57306
7	4.6099	3.8252	3.2665	1.67218
8	4.7494	3.8545	3.4624	1.76806
9	4.8379	3.8843	3.6513	1.85971
10	4.8969	3.9157	3.8315	1.94685
11	4.9469	3.9485	3.9981	2.02958
12	5.0023	3.9819	4.1453	2.10808
13	5.0696	4.0145	4.2682	2.18242
14	5.1490	4.0444	4.3637	2.25245
15	5.2363	4.0701	4.4307	2.31774
16	5.3252	4.0902	4.4701	2.37761
17	5.4089	4.1041	4.4845	2.43123
18	5.4815	4.1113	4.4776	2.47767
19	5.5385	4.1123	4.4540	2.51606
20	5.5774	4.1078	4.4181	2.54569
21	5.5973	4.0988	4.3746	2.56612
22	5.5993	4.0866	4.3272	2.57727
23	5.5855	4.0724	4.2794	2.57946
24	5.5590	4.0577	4.2337	2.57347
25	5.5232	4.0435	4.1920	2.56053
26	5.4814	4.0308	4.1556	2.54225
27	5.4368	4.0205	4.1251	2.52059
28	5.3922	4.0132	4.1010	2.49776
29	5.3497	4.0092	4.0832	2.47612
30	5.3109	4.0089	4.0714	2.45814



Example

Yield Curve (Jan-08)

Term	Yield
1 year	3.6641%
2 years	3.6220%
3 years	3.6651%

- If on Jan-08, if you invest \$1,000 in a risk-free zero coupon bond that matures in 1 year, you would get the following amount in 1 year:

$$1,000 \times (1 + 0.036641) = \$1,036.64$$

- If on Jan-08, you invest \$1,000 in a risk-free zero coupon bond that matures in 2 year, you would get the following amount in 2 years:

$$1,000 \times (1 + 0.036220)^2 = \$1,073.75$$

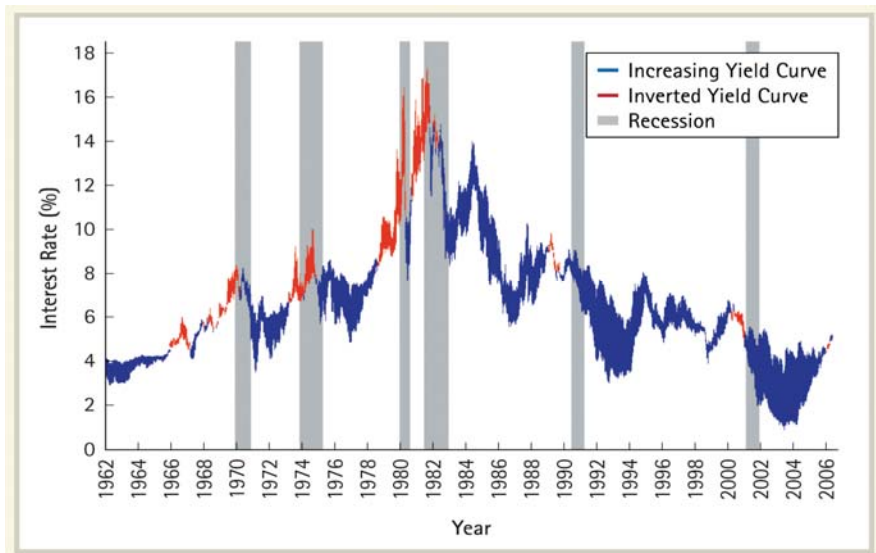
- If on Jan-08, you invest \$1,000 in a risk-free zero coupon bond that matures in 3 year, you would get the following amount in 3 years:

$$1,000 \times (1 + 0.036651)^3 = \$1,114.03$$

Yield Curve Shapes

- **Overnight rate** The rate at which banks can borrow cash reserves on an overnight basis from the Bank of Canada.
- The Bank of Canada determines very short-term interest rates through its influence on the overnight rate.
- *Shape of the yield curve:*
 - **Increasing Yield Curve:** If interest rates are expected to rise, long-term interest rates will tend to be higher than short-term rates to attract investors.
 - **Inverted Yield Curve:** If interest rates are expected to fall, long-term rates will tend to be lower than short-term rates to attract borrowers.

Yield Curve and Recessions



McGill Introduction to Finance (MGCR 341) – Prof. de Motta

53

Present Value of a Cash Flow Stream and Term Structure of Discount Rates

Rule: Cash flows should be discounted using the discount rate that is appropriate for their horizon.

$$PV = C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} + \dots + \frac{C_n}{(1+r_n)^n} = \sum_{t=0}^n \frac{C_t}{(1+r_t)^t}$$

where r_t is the effective rate per period between periods 0 and t (e.g., if cash-flows are annual, r_3 would be the effective *annual* rate that you would earn between year 0 and year 3).

McGill Introduction to Finance (MGCR 341) – Prof. de Motta

54

Example

Calculate the PV of a risk-free five-year annuity of \$1,000 per year given the yield curve for January 2010 (see previous slide).

Solution

- From the 2010 yield curve, we see that the interest rates are: 0.7381%, 1.5069%, 2.0958%, 2.5152% and 2.8189%, for terms of 1, 2, 3, 4 and 5 years, respectively.
- Discount each cash flow by the corresponding interest rate:

$$PV = \frac{1000}{1.0073} + \frac{1000}{1.0150^2} + \frac{1000}{1.0209^3} + \frac{1000}{1.0251^4} + \frac{1000}{1.0281^5} = \$4678.52$$

Opportunity Cost of Capital

- Risk and Interest Rates: Most borrowers have some risk of default, so investors require a higher rate of return from them.

Interest Rates on Five-Year Loans for Various Borrowers, October 2012

Borrower	Interest Rate
Government of Canada	1.31%
IBM (Canada)	1.90%
Bank of Nova Scotia	2.21%
National Bank of Canada	2.36%
Bell Canada	2.38%
Rogers Communications	3.53%
Sherritt International	6.38%

Opportunity Cost of Capital

- When calculating the NPV of an investment we need to use a discount rate that takes into account the timing and riskiness of the cash flows.

Cost of Capital: The best available expected return offered in the market on an investment of comparable risk and term to the cash flow being discounted.

Example

- Suppose the US government owns your firm \$1,000 to be paid in five years and the US government T-notes pay 4.94%. What is the PV of the \$1,000?

$$PV = \frac{1000}{1.0494^5} = \$785.77$$

- Suppose instead that Goodyear Co. owns your firm \$1,000 to be paid in five years and that Goodyear Co. pays 8.5% on its debt. What is the PV of the \$1,000?

$$PV = \frac{1000}{1.085^5} = \$665.05$$