# Fixed Income Markets

## FINE 452: Applied Quantitative Finance

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# Outline

- Bond Valuation
- The Many Definitions of Interest Rates:
  - Yield to Maturity
  - Spot Rates
  - Forward Rates
  - Bond Return
- The Term Structure of Interest Rates

# Definitions

A bondholder is entitled to a fixed set of cash payoffs: equity : residual

- Regular (usually annual or semiannual) interest payments called the bond's **coupon** until the bond matures;
- The **face value** of the bond (the bond's **principal**) at maturity.

- $\Rightarrow$ A bond can be viewed as a package of two investments:
  1. The first investment generates the regular coupon payments, $\$C_t$, till (and including) maturity date $T$.
  2. The second produces the face value, \$F, at maturity $T$.

## Bond Valuation

The bond can be valued as: coupon-paying bond.

$$
\begin{aligned}
B_0 &= PV \ \text{(coupon payments)} + PV \ \text{(payment of principal)} \\
&= \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{F}{(1+r)^T},
\end{aligned}
$$

0 1 2 3 4 5

$-B_0$ $C_t$ $C_t$ $C_t$ ... $C_t$ $C_t+F$

where r is the discount rate.

# Example 1

- Bond has
  - Face value \$100.
  - Annual coupon rate 8.5% of face value.
  - Maturity 4 years.
  - Discount rate 3%.
- bond value is

$$B_0 = \frac{8.50}{(1.03)^1} + \frac{8.50}{(1.03)^2} + \frac{8.50}{(1.03)^3} + \frac{8.50}{(1.03)^4} + \frac{100}{(1.03)^4} = \$120.44$$

# Example 2

- Bond has
  - Face value \$1000. APR
  - Semiannual coupon rate of 4.875% a year.
  - Maturity 3 years. APR
  - Semiannually compounded return (discount rate) 1.2006%.
- coupon payment every six months=$4.875/2 = 2.4375\%$ of face value=24.375
- semiannual rate of return=$1.2006/2 = .6003\%$

⇒ bond value is

$$B_0 = \sum_{t=1}^{6} \frac{24.375}{(1.006003)^t} + \frac{1000}{(1.006003)^6} = \$1107.95$$

# Zero Coupon Bonds

## Definition

Zero coupon bonds, also referred to as **stripped bonds** or **strips**, with maturity $T$ make a single payment, equal to its face value, at time $T$ (do not make intermediate coupon payments).

- $\Rightarrow$ the price of a 5-year zero coupon bond with face value \$$F$ is

$$b_5 = \frac{F}{(1+r)^5}$$

## Prices of Coupon Paying Bonds Depend on Prices of Zero Coupon Bonds by an Arbitrage Argument

Let \$$b_i$ denote the price of an $i$-period zero-coupon bond with face value \$1. The price of a $T$-period bond with coupon rate $c$ and face value \$$F$ is:

$$B_0 = (cF)\, b_1 + (cF)\, b_2 + ... + (cF + F)\, b_T$$

# Yield to Maturity

Given the bond price, we can compute

- "yield to maturity" (y).
- It is the annual return earned by buying a bond and holding it to maturity. APR.
- It is the discount rate that sets the value of future payments equal to the price. $B_0 = \sum_{t=1}^{T} \frac{Cr}{(1+r)^t} + \frac{F}{(1+r)^T}$
- It is the rate that sets the NPV of investing in the bond equal to zero. $0 = NPV = \sum_{t=1}^{T} \frac{Cr}{(1+r)^t} + \frac{F}{(1+r)^T} - B_0.$

In Example 1, solve the polynomial for $y$

$$B_0 = \frac{8.50}{(1+y)^1} + \frac{8.50}{(1+y)^2} + \frac{8.50}{(1+y)^3} + \frac{8.50}{(1+y)^4} + \frac{100}{(1+y)^4} = 120.44$$

In Example 2, solve the polynomial for $y$

$$B_0 = \sum_{t=1}^{6} \frac{24.375}{(1+\frac{y}{2})^t} + \frac{1000}{(1+\frac{y}{2})^6} = 1107.95$$

This must be done numerically, but Excel and many calculators do it automatically.

# How bond prices vary with Interest Rates, r

- Bond prices and interest rates must move in opposite directions.
- The yield to maturity (our measure of the interest rate on a bond) is defined as the discount rate that explains the bond price.

$\Rightarrow$ when bond prices fall, yields to maturity must rise. When yields to maturity rise, bond prices must fall.

- When the yield is equal to the bond's coupon rate, the bond sells for exactly its face value (**at par**). $B_0 = FV$
- When the yield is higher than the coupon rate, the bond sells **at a discount** to face value. $B_0 < FV$
- When the yield is lower than the coupon rate, the bond sells **at a premium**. $B_0 > FV$

$$B_0 = \sum_{t=1}^{T} \frac{CF}{(1+r)^t} + \frac{F}{(1+r)^T}$$

$$= \frac{CF}{1+r}\left(1 + \frac{1}{1+r} + \cdots + \frac{1}{(1+r)^{T-1}}\right) + \frac{F}{(1+r)^T} = \frac{CF}{1+r} \cdot \frac{1 - \frac{1}{(1+r)^T}}{1 - \frac{1}{1+r}} + \frac{F}{(1+r)^T}$$

# A Bond's Holding Period Return

A selling a bond before maturity is risky.

- A bond's yield to maturity is not the same as its holding period return.
- A bond's holding period return (HPR) from period 0 to 1 is defined as:

$$
\begin{aligned}
\text{HPR} &= \frac{C_1 + B_1}{B_0} - 1 \quad \text{risky.} \\
C_1 &= \text{Coupon payment at date 1} \\
\text{uncertain } B_1 &= \text{Bond price at date 1} \\
B_0 &= \text{Purchase price at date 0}
\end{aligned}
$$

YTM: risk-free (if assume no default)
We can determine y at date 0.

# Example

- Bond $A$ has a maturity of 10 years, face value of $\$1,000$, an annual coupon rate of 8%, and current price $B_0 = \$1,000$
- $\Rightarrow$ the bond's yield to maturity is 8%
- In a year, the bond price increases to $B_1 = \$1,065.15$ (interest rate $\downarrow$)
- $\Rightarrow$ HPR$= \frac{C_1+B_1}{B_0} - 1 = \frac{80+1065.15}{1000} - 1 = 14.5\%$

# Term Structure of Interest Rates

longer-term bond is more risky than short-term bond (inflation)

- So far, we have used the same discount rate, r, to calculate the PV of each period's cash flow in computing the PV (price) of a bond.
- In reality, short-term interest rates are different from long-term rates.

  smaller than
- The PV of a loan that pays \$1 at the end of one year is obtained by discounting the cash flow by the one-year rate of interest (**one-year spot-rate**) $r_1$

$$PV = \frac{1}{1 + r_1} = b_1 \ \text{(price of 1-year zero-coupon bond)}$$

downward sloping, long-term expectation is better than short-term.

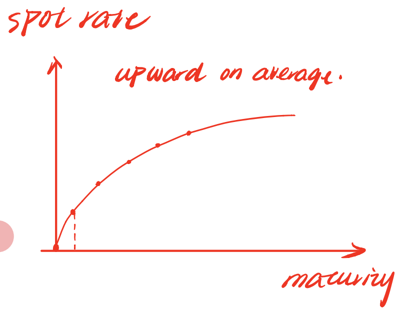
- To find the PV of a loan that pays \$1 at the end of two years, you need to discount by the two-year spot-rate $r_2$:

$$PV = \frac{1}{(1 + r_2)^2} = b_2 \ \text{(price of 2-year zero-coupon bond)}$$

- The series of spot rates $r_1$, $r_2$, ..., $r_t$, ... traces out the **term structure of interest rates**

nominal rates

TIP (real rate)



# Example

- Suppose the prices of zero-coupon bonds of maturity 1, 2, 3, 4, and 5 years are 970, 920, 850, 750, and 600. All bonds have face value of 1000.
- Using the equation $\frac{1}{(1+r_n)^n} = b_n$ where $n$ is the number of years, the spot rate $r_n$ can be calculated for all maturities

# Bond Pricing with Zero-Coupon Bond Prices or Spot Interest Rates

- In case the spot rates vary with maturity, the price of a $T$-period bond with ① coupon rate $c$ and ② face value $\$F$ is:

$$
\begin{aligned}
③\ B_0 &= (cF)\, b_1 + (cF)\, b_2 + \ldots + (cF + F)\, b_T \\
&= \frac{cF}{1 + r_1} + \frac{cF}{(1 + r_2)^2} + \ldots + \frac{cF + F}{(1 + r_T)^T}
\end{aligned}
$$

③ term structure of spot rates

Given the price, face value, and coupon rate of a bond, you can compute the yield to maturity, $y$, as the solution to the equation:

$$
B_0 = \sum_{t=1}^{T} \frac{cF}{(1 + y)^t} + \frac{F}{(1 + y)^T}
$$

# Treasury Yield Curve

source: Bloomberg 9-22-2011

upward sloping
on average.


- The **yield curve** plots the yield to maturity against the maturity of bonds.

# No Arbitrage

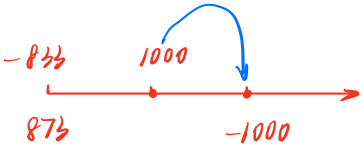
## No Arbitrage

In well-functioning markets, there is no such thing as a surefire money machine **(arbitrage)**. Arbitrage opportunities are eliminated almost instantaneously by investors who try to take advantage of them.

- $\Rightarrow$ a dollar tomorrow cannot be worth less than a dollar the day after tomorrow. In other words, **a one-year zero-coupon bond cannot be worth less than a two-year zero-coupon bond**, i.e. it must be that

time value of money

$$b_1 = \frac{1}{1+r_1} \geqslant \frac{1}{(1+r_2)^2} = b_2$$

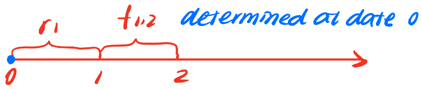



- **Reason:** Suppose $r_1 = .20$ and $r_2 = .07 \Rightarrow b_1$=.833 and $b_2$=.873. Buy a one-year strip for .833×1000=\$833 that pays \$1000 next year. Borrow the PV of \$1000 at $r_2$=.07, i.e. borrow $PV = \frac{1000}{(1.07)^2} = \$873 \Rightarrow$ make a sure immediate profit of 873−833=\$40

shortsell the more expensive one, use the proceeds to buy the cheaper one.

# Forward Interest Rates

- You want to arrange today for borrowing or lending in the future.
- $\Rightarrow$ no cost today, and no risk in the cash flows
- Example: arrange to borrow \$100 in a year to be repaid in two years at the forward rate $f_{1,2}$



# Determination of Forward Interest Rates

Forward interest rates are determined by spot interest rates through an arbitrage argument

- **First Strategy**: Invest \$1 in a 2-year zero-coupon bond $\Rightarrow$ earn $(1+r_2)^2$ at the end of 2 years
- **Second Strategy**: Invest \$1 in a 1-year zero-coupon bond, earn $(1+r_1)$ in one year and then reinvest the proceeds at the forward rate $\Rightarrow$ earn $(1+r_1)(1+f_{1,2})$ at the end of two years
- Both strategies require the same initial investment and both are risk free
- $\Rightarrow$ to avoid arbitrage, both strategies should offer the same payoff:

$$\begin{aligned}(1+r_2)^2 &= (1+r_1)(1+f_{1,2}) \\ \Rightarrow \quad 1+f_{1,2} &= \frac{(1+r_2)^2}{(1+r_1)} = \frac{b_1}{b_2}\end{aligned}$$

- More generally:

$$\boxed{f_{i,i+1} = \frac{b_i}{b_{i+1}} - 1}$$

# Example:

- Suppose the zero-coupon bond prices are $b_1 = 0.9346$, $b_2 = 0.8654$, $b_3 = 0.7939 \Rightarrow$

$$
\begin{aligned}
r_1 &= \frac{1}{b_1} - 1 = \frac{1}{0.9346} - 1 = 7.0\% \\
r_2 &= \frac{1}{(b_2)^{1/2}} - 1 = \frac{1}{(0.8654)^{1/2}} - 1 = 7.5\% \\
r_3 &= \frac{1}{(b_3)^{1/3}} - 1 = \frac{1}{(0.7939)^{1/2}} - 1 = 8.0\% \\
f_{1,2} &= \frac{b_1}{b_2} - 1 = \frac{0.9346}{0.8654} - 1 = 8.0\% \\
f_{2,3} &= \frac{b_2}{b_3} - 1 = \frac{0.8654}{0.7939} - 1 = 9.0\%
\end{aligned}
$$

# No Arbitrage Restriction on Forward Rates

## No Arbitrage

To rule out arbitrage, forward rates cannot be negative.

- We saw earlier that absence of arbitrage requires $b_i \geqslant b_{i+1}$
- This implies that

$$\begin{aligned} b_i &\geqslant b_{i+1} \\ \Rightarrow & \quad f_{i,i+1} = \frac{b_i}{b_{i+1}} - 1 \geqslant 0 \end{aligned}$$

upward sloping

# 1. Expectations Theory

demand = supply

- Expectations Theory: In equilibrium, investment in a series of short-maturity bonds must offer the same expected return as an investment in a single long-maturity bond:

$$(1 + r_{1,t})\,(1 + E_t r_{1,t+1}) = (1 + r_{2,t})^2$$

not known at time t

$-1 \qquad (1+r_{2,t})^2$

$-1 \qquad 1+r_{1,t} \qquad (1+r_{1,t})(1+r_{1,t+1})$

- ⇒the only reason for an upward (downward) sloping term structure is that investors expect short-term interest rates to rise (fall)

# 2. Risk

- The expectations theory leaves out risk
- Greater exposure of long-term bonds to changes in interest rates $\Rightarrow$ higher volatility of prices of long term bonds than prices of short term bonds
- $\Rightarrow$ long term bonds are more risky and investors demand extra return for holding them.

# 3. Inflation

- Uncertainty about future inflation makes it risky to invest in long term bonds.
- If you buy a 20-year strip, you know exactly how much money you will have at year 20 , but you don't know what that money will buy.
- Exposure to inflation risk can be reduced by investing short-term and rolling over the investment
- $\Rightarrow$ if inflation is an important source of risk for long term investors, issuers must offer some extra return if they want investors to lend long

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