Markov chain $P(X_t=x_t \mid X_{t-1}=x_{t-1} \cdots X_0=x_0)=P(X_t=x_t \mid X_{t-1}=x_{t-1})$

Time-homogeneous $P(X_t=x_t \mid X_{t-1}=x_{t-1} \cdots)=P(X_t \mid X_{t-1})=P(X_s \mid X_{s-1}) \quad \forall s,t$

Stochastic matrix A nonnegative matrix whose row sum to one.

① n-step transition matrix $(P^n)_{ij}=P(X_n=j \mid X_0=i) \quad \forall i,j$

② distribution of X_n $\quad P(X_n=j)=(\alpha P^n)_j \quad \forall j$

③ most recent past $\quad P(X_{n+1}=j \mid X_0=i_0 \cdots X_{n-m}=i)=P(X_{m+1}=j \mid X_0=i)=(P^{m+1})_{ij}$

④ joint distribution $\quad P(X_{n_1}=i_1, X_{n_2}=i_2 \cdots X_{n_k}=i_k)=(\alpha P^{n_1})_{i_1}(P^{n_2-n_1})_{i_1 i_2}\cdots(P^{n_k-n_{k-1}})_{i_{k-1} i_k}$

Limiting distribution

$\lim_{n\to\infty}(P^n)_{ij}=\lambda_j \quad \forall i,j$

or $\lim_{n\to\infty}\alpha P^n=\lambda \quad \forall \alpha$

or $\lim_{n\to\infty}P(X_n=j)=\lambda_j$

or $\lim_{n\to\infty}P^n=\Omega=(\underset{\sim}{\lambda} \cdots \underset{\sim}{\lambda})^T$

counter-example $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $(0.5, 0.5)$ $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\forall x$

$P=\begin{pmatrix}1-p&p\\q&1-q\end{pmatrix} \Rightarrow \lambda=\left(\frac{q}{q+p}, \frac{p}{q+p}\right)$

$\frac{q}{p+q}+\frac{p}{p+q}(1-p-q)^n$ $\quad (|1-p-q|<1)$

Stationary distribution $\pi P=\pi$ $\quad$ limiting $\Rightarrow$ stationary $\pi P=\lim_{n\to\infty}\alpha P^n=\lim_{n\to\infty}\alpha P^{n+1}\ P=\pi P$

If choose $\underset{\sim}{\lambda}$ (limiting) as the initial distribution, then $X_0, X_1 \cdots$ are identically marginally distributed. $Pr(X_t=i)=\lambda_i \ \forall t$ But they're not independent!

Positive TPM: All entries are larger than 0 $(P>0)$

Regular TPM: $\exists n$ s.t. $P^n>0$ $\Rightarrow$ There exists a unique $\underset{\sim}{\pi}$ that is the stationary distribution for P and $\underset{\sim}{\pi}$ will also be a limiting distribution.

- If we can get to state j from state i ($\exists n \geq 0$, $(P^n)_{ij}>0$), the j is **accessible** from i
- j is accessible from i and i is accessible from j $\Rightarrow$ $i(j)$ **communicates** with $j(i)$.

① symmetric ② reflexive $(P^0_{ii}=1>0)$ ③ transitive

- **Communication class**: A class of states which all communicate with each other.
- **Irreducible**: If a MC only has one CC

Define $T_j=\min\{n>0: X_n=j \text{ if } X_0=j\}$ First hitting time $\quad$ < visit time >

- $Pr(T_j<\infty \mid X_0=j)=1$, then state j is a **recurrent** state $\quad$ iff $\sum_{n=0}^{\infty}P^n_{jj}=\infty$
- $Pr(T_j<\infty \mid X_0=j)<1$, then state j is a **transient** state $\quad$ iff $\sum_{n=0}^{\infty}P^n_{jj}<\infty$

All states in a CC, they're either transient or recurrent.

For a finite irreducible MC, all states are recurrent.

Closed CC: $P_{ij}=0 \ \forall i\in C, j\notin C$ $\quad$ if it consists of all recurrent states.

Canonical decomposition of the chain $\quad S=T\cup R_1\cup R_2\cdots\cup R_m$

$\leftarrow$ class property $\quad$ closed recurrent

$$\begin{array}{c|c} & T \quad R_1 \quad \cdots \quad R_m \\ \hline \begin{array}{c} T \\ R_1 \\ \vdots \\ R_m \end{array} & \begin{pmatrix} \square & ? \quad ? \quad ? \\ & P_1 \qquad 0 \\ & \ddots \\ 0 & \qquad P_m \end{pmatrix} \end{array} \rightarrow \lim_{n\to\infty}P_i^n$$

$d(i)=\gcd\{n>0: (P^n)_{ii}>0\}$. If $d(i)=1$, state i is **aperiodic**.

ergodic ① irreducible ② aperiodic ③ have finite return time (positive recurrent)

Time reversible for an irreducible chain, $\pi_i P_{ij}=\pi_j P_{ji} \ \forall i,j$ for stationary $\underset{\sim}{\pi}$

$P(X_n=i)P(X_{n+1}=j \mid X_n=i)=P(X_n=j)P(X_{n+1}=i)P(X_n=j) \Rightarrow P(X_n=i, X_{n+1}=j)=P(X_{n+1}=j, X_n=i)$

$\Rightarrow P(X_0=i_0 \cdots X_n=i_n)=P(X_0=i_n \cdots X_n=i_0)$ $\quad$ If $\exists\, f=(f_1 \cdots f_k)$ s.t. $\sum f_i=1$

that satisfies time reversibility $f_i P_{ij}=f_j P_{ji} \ \forall i,j$, then f_i is stationary. $(\sum_i)$

Absorbing state $P_{ii}=1$ $\quad$ **Absorbing chain** $\exists$ at least one absorbing state.

$$P=\begin{pmatrix} \underset{t\times t}{Q} & \underset{t\times k}{R} \\ \underset{k\times t}{0} & \underset{k\times k}{I} \end{pmatrix}$$

transient (t) absorbing (k)

$$\lim_{n\to\infty}P^n=\begin{pmatrix}\lim_{n\to\infty}Q^n & \lim_{n\to\infty}\sum_{k=1}^{n}Q^k R \\ 0 & I\end{pmatrix}=\begin{pmatrix}0 & (I-Q)^{-1}R \\ 0 & I\end{pmatrix}$$

Fundamental matrix $F=(I-Q)^{-1}$ where F_{ij} = expected visits to state j starting in i.

expected # steps to reach some absorbing state $a_i=\sum_{k\in T}F_{ik}$, row sum of F.

Conditional Probability $P(A|B) = P(A \cap B) / P(B)$

Law of Total Probability $P(A) = P(A \cap B) + P(A \cap B^c) = P(A|B)P(B) + P(A|B^c)P(B^c)$

Conditional expectation $E(Y|X=x) = \sum_y y\, P(Y=y|X=x)$

Law of Iterated Expectation $E_Y(Y) = E_X[E_{Y|X}[Y|X]]$

Law of Iterated Variance $Var_Y(Y) = Var_X[E_{Y|X}(Y|X)] + E_X[Var_{Y|X}(Y|X)]$

Binomial distribution $B(x; n; p) = \binom{n}{x} p^x (1-p)^{n-x}$ where $\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$

Geometric distribution $Geom(p) = (1-p)^k p$

The sum of geometric series $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

The sum of harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Limit Theorem for Finite Irreducible Markov Chain

Let $X_0, X_1, \cdots$ be a finite irreducible Markov chain. Then

$\mu_j < \infty$ and $\exists$ a unique, positive stationary distribution π s.t.

$\pi_j = \frac{1}{\mu_j} \quad \forall j$

doesn't imply π_j is limiting

Furthermore, for all state i

$\pi_j = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} P_{ij}^m$

Note: If $|S| = \infty$, then $E(T_j | X_0 = j)$ could be infinite, even if j is recurrent

- Positive recurrent: recurrent j s.t. $E(T_j | X_0 = j) = \infty$
- Null recurrent: $E(T_j | X_0 = j) = \infty$

$\sum_{n=0}^{\infty} (T_j = n | X_0 = j)\, n = \infty$

transient $Pr(T_j = \infty) > 0$

$Pr(T_j = n) > 0$ as $n \to \infty$ but $Pr(T_j = n) \cdot n \not\to 0$ as $n \to \infty$

For a chain starting at transient i, define $I_n = \begin{cases} 1 & X_n = j \\ 0 & \text{otherwise} \end{cases}$

$\sum_{n=0}^{\infty} I_n$ = # visits to j

$E(\sum_{n=0}^{\infty} I_n) = \sum_{n=0}^{\infty} E(I_n) = \sum_{n=0}^{\infty} Pr(X_n = j | X_0 = i) = \sum_{n=0}^{\infty} (P^n)_{ij}$

if j is transient, then $\sum_{n=0}^{\infty} (Q^n)_{ij} = (I-Q)^{-1}_{ij}$

$\sum_{n=0}^{\infty} (P^{r+n+t})_{ii}$

$= \sum_{n=0}^{\infty} \sum_{l} (P^r)_{il} (P^n)_{ll} (P^t)_{li}$

$\geq \sum_{n=0}^{\infty} (P^r)_{ij} (P^n)_{jj} (P^t)_{ji}$

$= (P^r)_{ij} \underbrace{\sum_{n=0}^{\infty} (P^n)_{jj}}_{j \text{ recurrent}} (P^t)_{ji} = \infty$

Let $i, j \in C$. i, j commute $\Rightarrow \exists\, r, s$ s.t. $P_{ij}^r > 0$, $P_{ji}^s > 0$

Then $P_{ii}^{r+s} = \sum_k P_{ik}^r P_{ki}^s \geq P_{ij}^r P_{ji}^s > 0 \Rightarrow$ $r+s$ is a return time for i.

Assume $P_{jj}^n > 0$ for some n.

$P_{ii}^{r+n+s} \geq P_{ij}^r P_{jj}^n P_{ji}^s > 0 \Rightarrow r+n+s$ is also a return time for i.

So $d(i)$ is a divisor of both $r+s$ and $r+n+s$

So $d(i)$ is a divisor of $\{n>0 : P_{jj}^n\}$ where

$d(j)$ is the biggest divisor of $\{n > 0 : P_{jj}^n > 0\}$

$\Rightarrow d(i) \leq d(j)$

Reverse i and j, then $d(i) = d(j)$

How many times will we visit state j?

Let $I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{otherwise} \end{cases}$

$E[\sum_{n=0}^{\infty} I_n] = \sum_{n=0}^{\infty} E(I_n) = \sum_{n=0}^{\infty} (P^n)_{jj}$

- If j is recurrent, $\sum_{n=0}^{\infty} Pr(X_n = j | X_0 = j) = \infty$

  $f_j = Pr(T_j < \infty | X_0 = j)$

  If j is recurrent, then $f_j = 1$.

  Finally I get back to state j in finite time

- If j is transient, let Y_j be a random variable equal to the number I hit j before I never hit it again. The probability of never hitting again.

  $Y_j \sim Geometric(1-f_j)$

  $E(Y_j) = \frac{1}{1-f_j} < \infty$