2.2 Limits of Sequences

DEF A sequence with elements in a set S is a function from $\mathbb{N}$ to S.

Convention If $f: \mathbb{N} \to S$ we write $\forall n \in \mathbb{N}$: $a_n = f(n)$ and denote the seq by
$(a_1, a_2, a_3 \dots)$ or (a_n), $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n=1}^{\infty}$

Sometimes, it is convinient to not state a seq at $n=1$

$(a_n)_{n\in N_0}$ $(a_n)_{n=0}^{\infty}$ $(a_n)_{n\geq n_0}$ $(a_n)_{n=n_0}^{\infty}$

Limits of Sequence

DEF Let (a_n) be a seq of real numbers and let $L \in \mathbb{R}$. We say that (a_n) converges to L if

$$\forall \varepsilon > 0, \exists \underbrace{N}_{\text{threshold index}} \in \mathbb{N}, \forall n \geq N : \underbrace{|a_n - L|}_{\text{distance between } a_n \text{ and } L} < \varepsilon$$

L is called the limit of a_n. $\lim (a_n)$, $\lim\limits_{n\to\infty} a_n$

- If the sequence (a_n) has a limit, we say that (a_n) converges.
- If the sequence (a_n) doesn't have a limit, we say that (a_n) diverges.

①. $\lim \frac{1}{n} = 0$

Let $\varepsilon > 0$. $|a_n - L| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$
If we choose $N > \frac{1}{\varepsilon}$ (which is possible by Arch. prop.)
Then $\frac{1}{n} < \varepsilon$ for $\forall n \geq N$
Thus $\forall n \geq N$: $|\frac{1}{n} - 0| < \varepsilon \Rightarrow (a_n)$ conv. to 0 i.e. $\lim (\frac{1}{n}) = 0$

② $\lim \frac{n}{n^2+1} = 0$

Let $\varepsilon > 0$. $|\frac{n}{n^2+1} - 0| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$

③ $\lim \frac{2n^2-n}{n^2-6} = 2$

Let $\varepsilon > 0$. $|\frac{2n^2-n}{n^2-6} - 2| = |\frac{2n^2-n-2(n^2-6)}{n^2-6}| = |\frac{12-n}{n^2-6}|$

Notice that $|12-n| = 12-n$ for $\forall n \geq 12$ $|n^2-6| = n^2-6$ for $n \geq 3$

$\Rightarrow \forall n \geq 12$, $|\frac{12-n}{n^2-6}| = \frac{12-n}{n^2-6} < \frac{n}{n^2-6}$

‼PROBLEM‼ $\frac{n}{n^2-6} < \frac{n}{n^2} = \frac{1}{n}$

wrong direction

<u>lemma</u> $n^2-6 > \frac{1}{2}n^2$ for $\forall n \geq 4$

$\frac{n}{n^2-6} < \frac{2}{n} < \varepsilon \iff \frac{2}{n} > \frac{1}{\varepsilon} \iff n > \frac{2}{\varepsilon}$

Let $N = \max\{12, \frac{2}{\varepsilon}\}$. Then $\forall n \geq N$, $\left|\frac{2n^2-n}{n^2-6} - 2\right| < \varepsilon$.

④ $((-1)^n)$ diverges

Assume that $((-1)^n)$ converges to L. Let $\varepsilon = 1$, then $\exists N \in \mathbb{N}\ \forall n \geq N$:
$|(-1)^n - L| < \varepsilon = 1$

<u>even</u> $|1-L| < 1 \Rightarrow 0 < L < 2$
<u>odd</u> $|-1-L| < 1 \Rightarrow -2 < L < 0$

Especially, $0 < L < 0$ contradiction

⑤ $0 < a < 1$. (a^n) converges to 0.

<u>solution 1</u> Let $\varepsilon > 0$. $|a^n - 0| = a^n < \varepsilon \iff \ln(a^n) < \ln\varepsilon \iff n > \frac{\ln\varepsilon}{\ln a}$

Using exponential functions or logarithms will NOT be allowed in this course b.c. the definition of a^x (x irrational) is NOT provided.

<u>solution 2</u> since $0 < a < 1$, $\frac{1}{a} > 1 \Rightarrow b = \frac{1}{a} - 1 > 0 \Rightarrow a = \frac{1}{1+b}$
$\Rightarrow a^n = \left(\frac{1}{1+b}\right)^n$

> Recall Bernoulli's inequality:
> $\forall x \geq -1, \forall n \in \mathbb{N}: (1+x)^n \geq 1+nx$

$(1+b)^n \geq 1+bn \Rightarrow \frac{1}{(1+b)^n} \leq \frac{1}{1+bn} < \frac{1}{nb} < \varepsilon$
$\iff nb > \frac{1}{\varepsilon} \iff n > \frac{1}{b\varepsilon}$

DEF Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. The set $\{x \in \mathbb{R} : |x-a| < \varepsilon\}$ is called the ε-neighborhood centered about a; in symbols: $V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$

$\Rightarrow$ CONV. of SEQ Let (a_n) be a seq, let $a \in \mathbb{R}$. We say that (a_n) converges to a if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N: a_n \in V_\varepsilon(a)$

2.3 Limit laws

THM Uniqueness of the limit

Let (a_n) be a conv. seq. Then the limit of (a_n) is uniquely determined i.e. If L_1 and L_2 are limits of (a_n), then $L_1 = L_2$.

lemma: Let $x \in \mathbb{R}$ s.t. $\forall \varepsilon > 0 : 0 \le x < \varepsilon$, then $x = 0$

Assume that $x > 0$ and let $\varepsilon = \frac{x}{2} > 0$, then $0 < \frac{x}{2} < x \Rightarrow \varepsilon < x$ ↯

Proof of theorem

Let L_1, L_2 be limits of (a_n). Let $\varepsilon > 0$, then

$\exists N_1 \in \mathbb{N} \ \forall n \ge N_1 : |a_n - L_1| < \frac{\varepsilon}{2}$

$\exists N_2 \in \mathbb{N} \ \forall n \ge N_2 : |a_n - L_2| < \frac{\varepsilon}{2}$

Let $N = \max\{N_1, N_2\}$, then $\forall n \ge N : |a_n - L_1| < \frac{\varepsilon}{2} \wedge |a_n - L_2| < \frac{\varepsilon}{2}$

Then $|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \le |L_1 - a_n| + |a_n - L_2|$

$= |a_n - L_1| + |a_n - L_2| < \varepsilon$

$\Rightarrow 0 \le |L_1 - L_2| < \varepsilon$ for $\forall \varepsilon > 0$

$\Rightarrow |L_1 - L_2| = 0$ by lemma

$\Rightarrow L_1 = L_2$

DEF A sequence which converges to 0 is called a null sequence

THM Let (a_n) be seq, $L \in \mathbb{R}$ and let (b_n) be a non-negative null seq.
If $\exists k \in \mathbb{N}$ s.t. $\forall n \ge k : |a_n - L| \le b_n$
$\Rightarrow (a_n)$ conv. to L.

Let $\varepsilon > 0$, since (b_n) conv. to 0, $\exists \tilde{N} \ \forall n \ge \tilde{N} : |b_n| = b_n < \varepsilon$
Let $N = \max\{k, \tilde{N}\}$, then $\forall n \ge N : |a_n - L| \le b_n < \varepsilon \Rightarrow |a_n - L| < \varepsilon$

Ex. $a > 1$. $\sqrt[n]{a}$ conv. to 1.

Since $a > 1$, $\sqrt[n]{a} > 1 \Rightarrow \sqrt[n]{a} - 1 = b > 0 \Rightarrow b + 1 = \sqrt[n]{a} \Rightarrow a = (1+b)^n \ge 1 + nb$
$\Rightarrow a \ge 1 + nb \Rightarrow b \le \frac{a-1}{n} \Rightarrow 0 \le \sqrt[n]{a} - 1 \le \frac{a-1}{n}$
<1. def of conv.> Let $\varepsilon > 0$, $|\sqrt[n]{a} - 1| = \sqrt[n]{a} - 1 \le \frac{a-1}{n} < \varepsilon \Leftrightarrow n > \frac{a-1}{\varepsilon}$
<2. previous THM> $|\sqrt[n]{a} - 1| = \sqrt[n]{a} - 1 \le \frac{a-1}{n}$
• null sequence

THM All convergent seq of real numbers is bounded i.e. if (a_n) converges $\exists M>0 : \forall n \in \mathbb{N} : |a_n| \le M$

Let $a = \lim(a_n)$. Let $\varepsilon = 1$ $\exists N \in \mathbb{N}$, $\forall n \ge N : |a_n - a| < \varepsilon = 1$.

Then $|a_n| = |(a_n - a) + a| \underset{\Delta\text{-ineq}}{\le} |a_n - a| + |a| < 1 + |a|$

i.e. $\forall n \ge N : |a_n| < 1 + |a|$

Let $M = \max\{|a_1|, |a_2| \cdots |a_{N-1}|, 1+|a|\}$

Then $\forall n \in \mathbb{N} : |a_n| \le M$

$\Rightarrow (a_n)$ is bounded

Remark The converse is false.

e.g. $((-1)^n)$ is bounded b.c. $|(-1)^n| = 1 \le 1$; However, this seq diverges.

THM Algebraic Limit Theorem

Let $(a_n), (b_n)$ be conv. seq ; let $c \in \mathbb{R}$. then:

① $(a_n + b_n)$ conv. and $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$

② $(c \cdot a_n)$ conv and $\lim(c \cdot a_n) = c \lim(a_n)$

③ $(a_n - b_n)$ conv and $\lim(a_n - b_n) = \lim(a_n) - \lim(b_n)$

④ $(a_n b_n)$ conv and $\lim(a_n b_n) = \lim(a_n) \lim(b_n)$

⑤ If $\forall n \in \mathbb{N} : b_n \ne 0$ and $\lim(b_n) \ne 0$. Then

$$\left|\frac{a_n}{b_n}\right| \text{ conv. and } \lim \left|\frac{a_n}{b_n}\right| = \frac{\lim(a_n)}{\lim(b_n)}$$

Ex. $(\frac{1}{n^k})$ is a null sequence

Induction on k:

$k=1$ $\lim(\frac{1}{n}) = 0$

$k \to k+1$ Assume that $\lim(\frac{1}{n^k}) = 0$. $\frac{1}{n^{k+1}} = \frac{1}{n} \cdot \frac{1}{n^k}$ where both $(\frac{1}{n})$ and $(\frac{1}{n^k})$ conv. $\Rightarrow$ $(\frac{1}{n} \cdot \frac{1}{n^k}) = (\frac{1}{n^{k+1}})$ conv. and

$\lim(\frac{1}{n^{k+1}}) = \lim(\frac{1}{n}) \cdot \lim(\frac{1}{n^k}) = 0 \cdot 0 = 0$

Ex. $\lim \left(\frac{2n^2-n}{n^2-6}\right) = 2$

$$\lim \frac{2n^2-n}{n^2-6} = \frac{\lim\, 2-\frac{1}{n}}{\lim\, 1-6\frac{1}{n^2}} = \frac{2-\lim \frac{1}{n}}{1-\lim 6\frac{1}{n^2}} = \frac{2-0}{1-6\cdot 0} = 2$$

Read this chain of equation backwards for full justification of the applicablity of the limit laws.

<u>Corollary</u> Let (a_n) be a null seq and let $c \in \mathbb{R}$. Then $(c \cdot a_n)$ is a null seq.

$\lim (a_n) = 0 \Rightarrow \lim (c \cdot a_n) = c \lim (a_n) = c \cdot 0 = 0$

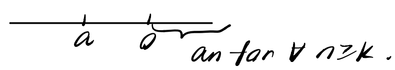
Ex. Every seq $\left(\frac{a}{n^k}\right)$ $a \in \mathbb{R}$, $k \in \mathbb{N}$ is a null seq.

Limits and Order

THM Let (a_n) be a convergent seq if $\exists k \in \mathbb{N}$ $\forall n \geq k$: $a_n \geq 0$. Then

$$\lim (a_n) \geq 0$$

Let $a = \lim (a_n)$. Assume $a < 0$



Let $\varepsilon = -a > 0$. Then $\exists \tilde{N}$, $\forall n \geq \tilde{N}$: $|a_n - a| < \varepsilon = -a$

$\Rightarrow a_n - a \leq |a_n - a| < -a \Rightarrow a_n < 0$

$\Rightarrow \forall n \geq \tilde{N}: a_n < 0$

Let $N := \max\{k, \tilde{N}\}$ Then $\forall n \geq N$: $a_n < 0 \wedge a_n \geq 0$ ↯

THM Let (a_n), (b_n) be conv. seq. If $\exists k \in \mathbb{N}$, $\forall n \geq k$: $a_n \leq b_n$
Then $\lim (a_n) \leq \lim (b_n)$

not strictly less than

e.g. $(a_n) = \left(-\frac{1}{n}\right)$ $(b_n) = \frac{1}{n}$ but $\lim (a_n) = \lim (b_n) = 0$

THM Let (b_n) be a conv. seq. Let $a, c \in \mathbb{R}$. If $\exists k \in \mathbb{N}$ $\forall n \geq k$:
$a \leq b_n \leq c$. Then $a \leq \lim (b_n) \leq c$.

i.e. $\exists k \in \mathbb{N}$ $\forall n \geq k$: $b_n \in [a, c]$ then $\lim (b_n) \in [a, c]$

<u>Remark</u> The same doesn't hold for non-closed interval

e.g. $(b_n) = \frac{1}{n}$, interval $(0,1) \Rightarrow \exists n \in \mathbb{N}$: $b_n \in (0,1)$ but $\lim (b_n) = 0 \notin (0,1]$

Squeeze THM for Sequence

Let $(a_n), (b_n), (c_n)$ be seq of real numbers s.t.

- $\exists\, k \in \mathbb{N}\ \forall\, n \geq k: a_n \leq b_n \leq c_n$
- (a_n) and (c_n) converge and have the same limit

Ex. $\lim \frac{\sin n}{n} = 0$

Note that $-1 \leq \sin n \leq 1 \Rightarrow \underset{\downarrow\, 0}{-\frac{1}{n}} \leq \frac{\sin n}{n} \leq \underset{\downarrow\, 0}{\frac{1}{n}}$ for $\forall\, n \in \mathbb{N}$

2.4 The Momentone Convergence Theorem

DEF Let (a_n) be seq ft

- $\forall\, n \in \mathbb{N}: a_n \leq a_{n+1} \Rightarrow a_n$ is increasing
- $\forall\, n \in \mathbb{N}: a_{n+1} \leq a_n \Rightarrow a_n$ is decreasing

A seq is either increasing or decreasing is called monotone

Monotone Conv. THM

- Let (a_n) be increasing and bounded from above
  Then (a_n) conv. and $\lim (a_n) = \sup \{a_n : n \in \mathbb{N}\}$
- Let (a_n) be decreasing and bounded from below
  Then (a_n) conv. and $\lim (a_n) = \inf \{a_n : n \in \mathbb{N}\}$

Euler's Number e

$\forall\, n \in \mathbb{N}: \quad a_n = (1+\frac{1}{n})^n \quad b_n = (1+\frac{1}{n})^{n+1}$

$\Rightarrow$ (a_n) increases and is bounded from above
(b_n) decreases and is bounded from below

$\Rightarrow$ Conv. by Monotone Conv. Thm

$\Rightarrow$ $\lim (a_n) = \lim (b_n)$

2.5 Subsequences

Oct 21

DEF let $n_1, n_2, n_3 \dots \in \mathbb{N}$ s.t. $n_1 < n_2 < n_3 < \dots$ and let $(a_n) = (a_1, a_2, a_3 \dots)$ be a seq. Then $(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is called a subsequence of (a_n).

① $(a_{2n}) = (a_2, a_4, a_6 \dots)$ subseq of even indices

② $(a_{2n-1}) = (a_1, a_3, a_5 \dots)$ subseq of odd indices.

③ $(a_{n+k}) = (a_{k+1}, a_{k+2} \dots)$ k-tail of (a_n).

lemma let $n_1, n_2, n_3 \dots \in \mathbb{N}$ s.t. $n_1 < n_2 < n_3 \dots$
Then $\forall k \in \mathbb{N}: n_k \geq k$.

$\Rightarrow$ let (x_n) be a seq; let (x_{n_k}) and (x_{n_j}) be conv. subseq s.t. $\lim(x_{n_k}) \neq \lim(x_{n_j})$. Then (x_n) diverges.

EX: ① SHOW $(x_n) = ((-1)^n)$ DIVERGES.

CONSIDER THE SUBSEQ. (x_{2n}) AND (x_{2n-1}).

$x_{2n} = (-1)^{2n} = 1$

FOR ALL $n \in \mathbb{N}$.

$x_{2n-1} = (-1)^{2n-1} = -1$

$\Rightarrow (x_{2n}), (x_{2n-1})$ CONV. AND $\lim(x_{2n}) = 1 \neq -1 = \lim(x_{2n-1})$

$\Rightarrow (x_n)$ DIVERGES.

② USE SUBSEQ. TO SHOW THAT $\lim(a^n) = 0$ WHEN $0 < a < 1$.

NOTE THAT (a^n) DECREASES, SINCE $a^{n+1} = \underbrace{a}_{<1} \cdot a^n$

$\Rightarrow \forall n \in \mathbb{N}: a^{n+1} < a^n$. AND (a^n) IS BD FROM BELOW BY 0. IT FOLLOWS FROM MONOTONE CONV. THM. THAT (a^n) CONV. LET $x := \lim(a^n)$.

NOW CONSIDER THE SUBSEQ OF EVEN INDICES i.e. (a^{2n}). THEN (a^{2n}) CONV TO x BY THM. ABOVE. ON THE OTHER HAND:

$\lim(a^{2n}) = \lim(a^n \cdot a^n) = \lim(a^n) \cdot \lim(a^n) = x^2$

THUS WE NOW HAVE THAT $x = \lim(a^{2n}) = x^2$

$\Rightarrow x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow x = 0 \lor x = 1$.

HOWEVER: $\forall n \in \mathbb{N}: a^n \leq a^1 = a$

$\Rightarrow \lim(a^n) \leq a \Rightarrow x \leq a < 1$

$\Rightarrow x < 1 \Rightarrow \underline{\underline{x = 0}}$. ∎

③ LET $x_1 = 2$, $x_{n+1} = 2 - \frac{1}{x_n}$. SHOW THAT

(a) $\forall n \in \mathbb{N}: 1 < x_n \leq 2$

(b) (x_n) IS DECREASING

(c) SHOW THAT (x_n) CONV. AND DETERMINE $\lim(x_n)$.

PROOF: (a) USE INDUCTION:

$n = 1$: $x_1 = 2 \Rightarrow 1 < x_1 \leq 2$ ✓

$n \to n+1$: ASSUME THAT $1 < x_n \leq 2$ FOR SOME $n \in \mathbb{N}$.

THEN: $\frac{1}{2} \leq \frac{1}{x_n} < 1 \Rightarrow -1 < -\frac{1}{x_n} \leq -\frac{1}{2}$

$\Rightarrow 1 < \underbrace{2 - \frac{1}{x_n}}_{x_{n+1}} \leq \frac{3}{2} < 2 \Rightarrow 1 < x_{n+1} \leq 2$

$\Rightarrow \forall n \in \mathbb{N}: 1 < x_n \leq 2$.

THIS ESPECIALLY MEANS THAT $\forall n \in \mathbb{N}: x_n \neq 0$.

THUS (x_n) IS WELL-DEFINED (IF EVER a_n WERE 0, THE NEXT TERM x_{n+1} WOULD BE UNDEFINED).

(b) CONSIDER $x_n - x_{n+1} = x_n - \left(2 - \frac{1}{x_n}\right) = x_n - 2 + \frac{1}{x_n}$

$= \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n - 1)^2}{x_n} \geq 0$ ($(x_n-1)^2 \geq 0$, $x_n > 1 > 0$)

THUS $\forall n \in \mathbb{N}: x_n - x_{n+1} \geq 0 \Leftrightarrow x_{n+1} \leq x_n$ $\forall n \in \mathbb{N}$

$\Rightarrow (x_n)$ IS DECREASING.

(c) (x_n) IS BD FROM BELOW BY 1, AS SHOWN IN (a). FURTHERMORE, (x_n) IS DECREASING BY (b)

$\Rightarrow$ (MONOTONE CONV. THM) (x_n) CONV. LET $x := \lim(x_n)$. RECALL THAT

$x_{n+1} = 2 - \frac{1}{x_n} \Rightarrow \underbrace{\lim(\underbrace{x_{n+1}}_{\text{1-TAIL OF } (x_n)})}_{= \lim(x_n) = x} = \lim\left(2 - \frac{1}{x_n}\right)$

$\Rightarrow x = \lim(x_{n+1}) = 2 - \frac{1}{\lim(x_n)} = 2 - \frac{1}{x}$

$\Rightarrow x = 2 - \frac{1}{x} \Rightarrow x^2 = 2x - 1 \Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0 \Rightarrow \underline{\underline{x = 1}}$

THUS (x_n) CONV. AND $\lim(x_n) = 1$ ∎

④ $\lim (1+\frac{1}{n!})^{n!}$ Oct 26

Let $x_n = (1+\frac{1}{n})^n$, then $x_{n!} = (1+\frac{1}{n!})^{n!}$

Thus $((1+\frac{1}{n!})^{n!})$ is a subseq of (x_n) and $\lim ((1+\frac{1}{n})^n) = e$

$\Rightarrow (x_{n!})$ converges to e $\Rightarrow \lim (1+\frac{1}{n!})^{n!} = e$

Similarly, $\lim (1+\frac{1}{2^n})^{2^n} = e$

Bolzano-Weierstrass

Every bounded sequence of real numbers has a convergent subsequence

e.g. $(x_n) = ((-1)^n)$

Every subsequence of (x_{2n}) converges to 1;

Every subsequence of (x_{2n+1}) converges to -1.

2.6 Cauchy Sequence

DEF A seq (x_n) is called a Cauchy sequence if $\forall \varepsilon > 0\ \exists M \in \mathbb{N}$:

$\forall n, m \geq N: |x_n - x_m| < \varepsilon$

Ex. $(\frac{1}{n})$ is a Cauchy sequence

Let $\varepsilon > 0$. $|\frac{1}{n} - \frac{1}{m}| = |\frac{1}{n} + (-\frac{1}{m})| \leq |\frac{1}{n}| + |-\frac{1}{m}| = \frac{1}{n} + \frac{1}{m}$

Let $M > \frac{2}{\varepsilon}$ and let $n, m \geq M$, then

$\frac{1}{n} \leq \frac{1}{M} = \frac{\varepsilon}{2}$; $\frac{1}{m} \leq \frac{1}{M} = \frac{\varepsilon}{2}$

$\Rightarrow \forall m, n \geq M: |\frac{1}{n} - \frac{1}{m}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

THM Every conv. seq is a Cauchy seq.

THM Every Cauchy seq. is bounded

THM Every Cauchy seq of real numbers converges

A sequence of real numbers conv. iff it's a Cauchy sequence

DEF A seq (x_n) is called **contractive** if Oct 28

$\exists\ 0 < c < 1$ s.t. $\forall n \in \mathbb{N}: |x_{n+2} - x_{n+1}| \leq c\,|x_{n+1} - x_n|$

THM Every contractive sequence converges.

Ex. $x_1 = 2$, $x_{n+1} = 2 + \frac{1}{x_n}$ for $\forall n \in \mathbb{N}$

(i) $\forall n \in \mathbb{N}: x_n \geq 2$ (by induction)

(ii) (x_n) is contractive

$$|x_{n+2} - x_{n+1}| = \left|\left(2 + \frac{1}{x_{n+1}}\right) - \left(2 + \frac{1}{x_n}\right)\right| = \left|\frac{x_n - x_{n+1}}{x_n x_{n+1}}\right| \leq \frac{1}{4}|x_n - x_{n+1}|$$

(iii) (x_n) conv.

(iv) $\lim(x_{n+1}) = \lim\left(2 + \frac{1}{x_n}\right) = 2 + \frac{1}{\lim(x_n)} = \lim(x_n)$

let $x = \lim(x_n)$ then $x = 2 + \frac{1}{x} \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow x = 1 \pm \sqrt{2}$

by (i), $\forall n \in \mathbb{N}: x_n \geq 2 \Rightarrow \lim(x_n) = x \geq 2$

but $1 - \sqrt{2} < 2 \Rightarrow x = 1 + \sqrt{2}$

Sequences diverge to infinity

DEF Let (x_n) be a sequence of real numbers

- We say that (x_n) diverges to $-\infty$ in symbol: $\lim(x_n) = +\infty$
  if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N: x_n > M$
- We say that (x_n) diverges to $-\infty$ in symbol: $\lim(x_n) = -\infty$
  if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N: x_n < M$

① $\lim(n) = +\infty$

Let $M \in \mathbb{R}$. By Arch Prop, $\exists N \in \mathbb{N}$ s.t. $N > M$, then $\forall n \geq N$, $n \geq N > M$

i.e. $\forall n \geq N: n > M$

$\Rightarrow (n)$ diverges to $+\infty$

② $\lim(-n) = -\infty$

Let $M \in \mathbb{R}$. By Arch Prop, $\exists N \in \mathbb{N}$ s.t. $N > -M$ then $\forall n \geq N$, $n \geq N > -M$

i.e. $\forall n \geq N: n > -M \Leftrightarrow -n < M$

$\Rightarrow (n)$ diverges to $-\infty$

③ let $a > 1$, $\lim(a^n) = +\infty$

$a > 1 \Rightarrow 0 < \frac{1}{a} < 1 \Rightarrow \lim\left(\left(\frac{1}{a}\right)^n\right) = \lim\left(\frac{1}{a^n}\right) = 0$

let $M \in \mathbb{R}$. If $M \leq 0$ then $\forall n \in \mathbb{N}: a^n > M$

Now let $M > 0$. let $\varepsilon = \frac{1}{M}$ Then $\exists N \in \mathbb{N}\ \forall n \geq N$: $\left|\frac{1}{a^n} - 0\right| = \frac{1}{a^n} < \varepsilon = \frac{1}{M}$

$\Rightarrow \forall n \geq N: \frac{1}{a^n} < \frac{1}{M} \Leftrightarrow \forall n \geq N: a^n > M$