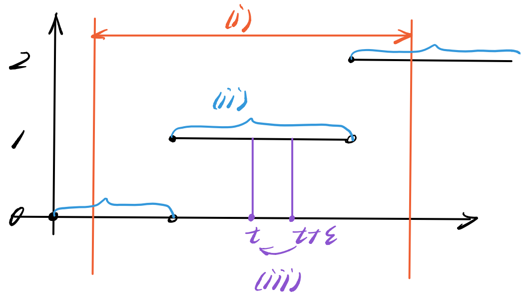
Chapter 6 · Poisson Process

Switch from discrete time to continuous time.

Couting process Number of events (injuries, infections, deaths)

Let N_t be the number of events that occur in the interval $[0,t]$. The collection $\{N_t : t \geq 0, t \in \mathbb{R}^+ \cup \{0\}\}$ is an uncountable collection of random variables (discrete-valued).

More generally, a counting process is a collection of random variables such that if $0 \leq s \leq t$, then $N_s \leq N_t$. non-decreasing



Poisson process (Most commonly used counting process)

There are at least 3 ways to define a Poisson process

(i) Model N_s directly: Number of events in fixed $[s,t]$.

(ii) Model the time between "jumps" of the process

(iii) Behavior on infinitesmal intervals.

definitions

(i) Poisson process with parameter λ is a counting process $\{N_t : t \geq 0\}$ with

(a) $N_0 = 0$

(b) $\forall t$, $N_t \sim \text{Poisson}(\lambda t)$ → only depends on the length of the interval.

stationary increments (c) $\forall s,t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t.

$$Pr(N_{t+s} - N_s = k) = Pr(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

time homogeneity

independent increments (d) For all $0 \leq q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent.

Non-overlapping intervals ⇒ counts in the two intervals are independent.

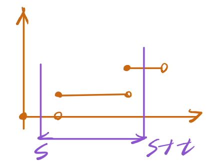
$$\begin{cases} E(N_t) = \lambda t \\ Var(N_t) = \lambda t \end{cases}$$

rate of arrivals $= \frac{E(N_t)}{t} = \frac{\lambda t}{t} = \lambda$

Translated process

Let $\{N_t : t \geq 0\}$ be Poisson process with rate λ.

$\tilde{N}_t = N_{t+s} - N_s$ # of events in $[s, t+s]$



Then $\{\tilde{N}_t : t \geq 0\}$ is a Poisson process with rate λ.

(a) $\{\tilde{N}_t : t \geq 0\}$ is a counting process

$\tilde{N}_r \leq \tilde{N}_t \quad \forall r \leq t, \quad \tilde{N}_0 = 0$

First two properties.

$N_s - N_s = 0$

(b) By stationary increments, all the properties carry through.

(ii) Let X be the arrival time of the first event.

Under a Poisson process rate λ, no arrivals in $[0,t]$ iff $X > t$.

$Pr(N_t = 0) = e^{-\lambda t}$ by $N_t \sim \text{Poisson}(\lambda t)$

$\Rightarrow$ $Pr(X \leq t) = 1 - e^{-\lambda t}$. $\qquad \{N_t = 0\} = \{X > t\}$

CDF of $\text{Exp}(\lambda)$.

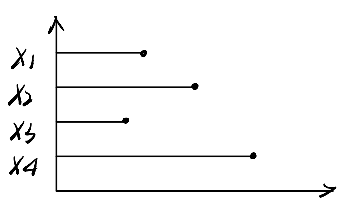
$\Rightarrow$ $X \sim \text{Exp}(\lambda)$

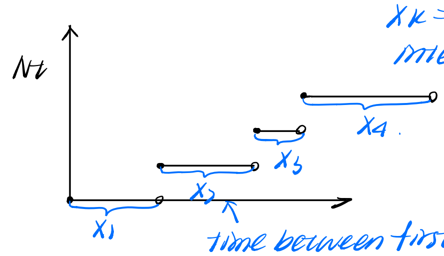
Let $X_1, X_2, \ldots$ be a sequence of iid $\text{Exp}(\lambda)$ random variables. For $t \geq 0$,

Let $N_t = \max\{n : X_1 + \cdots + X_n \leq t\}$

Let $S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$, then $N_t = \max\{n : S_n \leq t\}$

Then N_t is a Poisson process with rate λ.





$X_k = S_k - S_{k-1}$ interarrival time

time between first and second event.

Useful properties from definition (ii)

① Memoryless property (for $s \leq t$, $s, t \geq 0$)

If $X \sim \text{Exp}(\lambda)$, $Pr(X > s+t \mid X > s) = \dfrac{Pr(X > s+t, X > s)}{Pr(X > s)}$

$$= \frac{Pr(X > s+t)}{Pr(X > s)}$$

$$= e^{-\lambda t} = Pr(X > t).$$

Regardless of how long you have waited, the distribution of the time you still have to wait is the same as the original waiting time

③ Minimum of exponential random variables.

Let $M = \min(X_1, \cdots X_n)$, $X_i \overset{iid}{\sim} Exp(\lambda)$

(a) $Pr(M > t) = Pr(X_1 > t, X_2 > t \cdots X_n > t)$

$$= \prod_{i=1}^{n} Pr(X_i > t)$$

$$= \prod_{i=1}^{n} e^{-\lambda t}$$

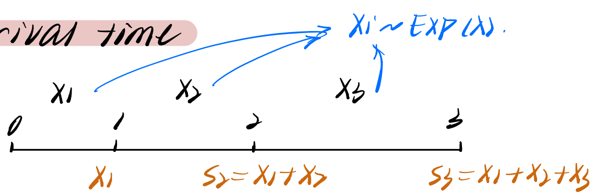
$$= e^{-n\lambda t}$$

Note: If $X_i \overset{ind.}{\sim} Exp(\lambda_i) \Rightarrow Pr(M > t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-t \sum_{i=1}^{n} \lambda_i}$

$\Rightarrow$ M is also an Exponential random variable with rate $\sum_{i=1}^{n} \lambda_i$.

(b) $Pr(M = X_k) = \dfrac{\lambda_k}{\lambda_1 + \cdots + \lambda_n} = \dfrac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$

Arrival time



The distribution of S_n (the arrival time of n^{th} event)

$S_n = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$ since $X_i \overset{iid}{\sim} Exp(\lambda)$

$f_{S_n}(t) = \dfrac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$

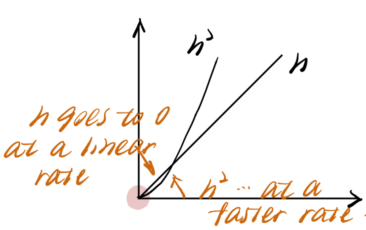
proof (i) Induction $S_n = S_{n-1} + X_n$
(ii) Moment generating function.

$$\begin{cases} E(S_n) = \dfrac{n}{\lambda} \\ Var(S_n) = \dfrac{n}{\lambda^2} \end{cases}$$

(iii) Preliminaries $f(h) = o(h)$ if $\lim_{h \to 0} \dfrac{f(h)}{h} = 0$

e.g. $f(h) = h^2$ $\quad \lim_{h \to 0} \dfrac{h^2}{h} = \lim_{h \to 0} h = 0$ ✓

$f(h) = h$ $\quad \lim_{h \to 0} \dfrac{h}{h} = 1$ ✗



- $f(h) = o(g(h))$ if $\lim_{h\to 0} \frac{f(h)}{g(h)} = 0$

e.g. $e^h = 1+h+\frac{1}{2}h^2+\frac{1}{3!}h^3+\cdots$

$= 1+h+R(h)$

$R(h) = \frac{1}{2}e^z h^2$ for some $z \in [-h,h]$

$\Rightarrow \frac{R(h)}{h} = \frac{1}{2}e^z h \to 0$ as $h\to 0$

$\Rightarrow e^h = 1+h+o(h)$.

- Let $f(h), g(h)$ be $o(h)$. Let c be a constant. Then

$f(h)+g(h) = o(h)$

$cf(h) = o(h)$

- If $f(h) = o(1)$, then $f(h)\to 0$ as $h\to 0$ because $\lim_{h\to 0} \frac{f(h)}{1} \to 0$

A Poisson process with rate λ is a counting process $\{N_t : t\geq 0\}$ s.t.

(a) $N_0 = 0$

(b) Process has stationary and independent increments (non-overlapping)

(c) $Pr(N_h = 0) = 1-\lambda h+o(h)$

(d) $Pr(N_h = 1) = \lambda h+o(h)$

(e) $Pr(N_h > 1) = o(h)$

In words, the probability of more than 1 event in a "small" interval is close to zero.

Probability of having exactly one event in a "small interval" is approximately λh.

Probability of having exactly zero event in a "small interval" is approximately $1-\lambda h$.

All three definitions are equivalent.

Assume definition (ii) holds.

① $Pr(N_h = 0) = e^{-\lambda h}$ since $N_t \sim \text{Poisson}(\lambda t)$

$$= 1 - \lambda h + \frac{1}{2}\lambda h^2 - \cdots\cdots$$

$$= 1 - \lambda h + R(h)$$

$$= 1 - \lambda h + o(h)$$

$Pr(N_h = 1) = e^{-\lambda h}(\lambda h)$

$$= (1 - \lambda h + R(h))(\lambda h)$$

$$= \lambda h - \lambda^2 h^2 + \lambda h R(h)$$

$$= \lambda h - \underbrace{\lambda^2 h^2 + \lambda h(o(h))}_{o(h)}$$

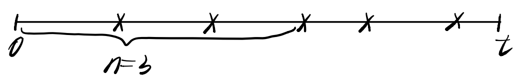
$$= \lambda h + o(h)$$

$\Rightarrow$ definition (iii) holds.

② $X_1 \cdots X_n$ be the time between events

$$S_n = \sum_{i=1}^{n} X_i$$

$N_t \geq n \iff S_n \leq t$



$Pr(N_t \geq n) = Pr(S_n \leq t)$ for $n = 1, 2 \cdots$

$$1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

Take derivative $\frac{\partial}{\partial t}$ on both sides

$$\sum_{j=0}^{n-1} -\left(\frac{\lambda^j}{j!}\left(j t^{j-1} e^{-\lambda t} - \lambda e^{-\lambda t} t^j\right)\right)$$

$$= \sum_{j=0}^{n-1} -\left(\frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!} - \frac{\lambda^{j+1} t^j e^{-\lambda t}}{j!}\right)$$

$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$ $\Rightarrow$ PDF of Gamma(n, λ)

informal proof

$\Rightarrow$ S_n has the same distribution as a sum of n iid $EXP(\lambda)$ random variables.

$\Rightarrow$ definition (ii)

Thinning and Superposition

Assume that events occur according to a Poisson process (λ).

Also assume that when an event occur, it is of type A with prob p_A and of type B with prob $(1-p_A)$.

What if we look at type A and type B separately?

$$\Pr(A_t = a, B_t = b) = \Pr(A_t = a, B_t = b, N_t = a+b)$$
$$= \Pr(A_t = a, B_t = b \mid N_t = a+b) \cdot \Pr(N_t = a+b)$$
$$= \frac{(a+b)!}{a!\,b!} p_A^{\,a} (1-p_A)^b \cdot \frac{e^{-\lambda t} (\lambda t)^{a+b}}{(a+b)!}$$
$$= \left(\frac{e^{-p_A \lambda t} p_A^{\,a} t^a}{a!} \right) \cdot \left(\frac{e^{-(1-p_A)\lambda t} (1-p_A)^b t^b}{b!} \right)$$

PDF for Poisson(λp_A) PDF for Poisson$(\lambda(1-p_A))$

$\Rightarrow$ A_t and B_t are independent and are both Poisson

Thinned Poisson Process Thinning

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. Assume that each arrival, independent of other arrivals, is marked as a *type* k event with probability p_k, for $k = 1, \ldots, n$, where $p_1 + \cdots + p_n = 1$. Let $N_t^{(k)}$ be the number of type k events in $[0, t]$. Then, $\left(N_t^{(k)}\right)_{t\geq 0}$ is a Poisson process with parameter λp_k. Furthermore, the processes

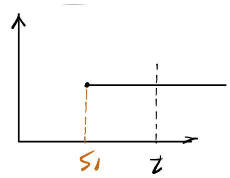
$$\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$$

are independent. Each process is called a *thinned Poisson process*.

$\Rightarrow$ Opposite direction (sum of Poisson (λ_i)).

Superposition Process Superposition

Assume that $\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$ are n independent Poisson processes with respective parameters $\lambda_1, \ldots, \lambda_n$. Let $N_t = N_t^{(1)} + \cdots + N_t^{(n)}$, for $t \geq 0$. Then, $(N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.



Uniform distribution

① One event

Now fix a time t, condition on $N_t = 1$ for a $PP(\lambda)$

Let S_1 be the time of that event in the interval $[0, t]$.

$$Pr(S_1 \le s \mid N_t = 1) = \frac{Pr(S_1 \le s, N_t = 1)}{Pr(N_t = 1)} = \frac{Pr(N_s = 1)\, Pr(N_t - N_s = 0 \mid N_s = 1)}{Pr(N_t = 1)}$$

non-overlapping

Put a restriction on S_1, $0 \le S_1 \le t$.

$$= \frac{Pr(N_s = 1)\, Pr(N_t - N_s = 0)}{Pr(N_t = 1)}$$

$$= \frac{\frac{e^{-\lambda s}(\lambda s)}{1!} \cdot e^{-\lambda(t-s)}}{\frac{e^{-\lambda t}(\lambda t)}{1!}}$$

$Pr(S_1 \ge s \mid N_t = 1) = \frac{s}{t} \Rightarrow S_1 \sim Unif(0, t)$

② More than 1 event

dependent! $(S_1 < \cdots S_n)$

Let $S_1, \cdots S_n$ be arrival times of a $PP(\lambda)$. Condition on $N_t = n$, then the joint distribution of $(S_1 \cdots S_n)$ is the same as the distribution of the order statistics of n iid $Unif[0, t]$ random variables, i.e.

$$P(S_1, \cdots S_n) = \frac{n!}{t^n} \quad \text{for } 0 < S_1 < S_2 \cdots < S_n < t$$

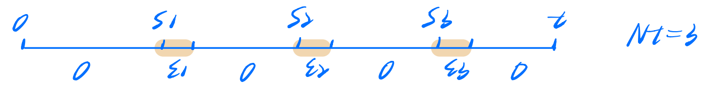
Proof For $s_1, \cdots s_n$

$$P(S_1, \cdots S_n \mid N_t = n)$$

$$= \lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_n \to 0} \frac{Pr(s_1 < S_1 < s_1 + \epsilon_1, \cdots s_n < S_n < s_n + \epsilon_n \mid N_t = n)}{\epsilon_1 \epsilon_2 \cdots \epsilon_n}$$

$$\frac{Pr(N_{s_1+\epsilon_1} - N_{s_1} = 1, \cdots N_{s_n+\epsilon_n} - N_{s_n} = 1, N_t = n)}{Pr(N_t = n)}$$

We can make ϵ_i small enough to avoid overlapping.



$$\frac{e^{-\lambda(s_1-0)} \lambda\epsilon_1 e^{-\lambda\epsilon_1} e^{-\lambda(s_2-s_1-\epsilon_1)} \lambda\epsilon_2 \cdots e^{-\lambda(t-s_n-\epsilon_n)}}{\frac{(t\lambda)^n e^{-\lambda t}}{n!}}$$

$$= \frac{\lambda^n e^{-\lambda t} \epsilon_1 \cdots \epsilon_n}{\frac{(t\lambda)^n e^{-\lambda t}}{n!}}$$

$$= n! \frac{\epsilon_1 \cdots \epsilon_n}{t^n}$$

$$\lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \cdots \lim_{\epsilon_n \to 0} \frac{\frac{n! \, \epsilon_1 \cdots \epsilon_n}{t^n}}{\epsilon_1 \cdots \epsilon_n}$$

$$= \frac{n!}{t^n} \quad \text{for } 0 < s_1 < s_2 \cdots s_n < t$$

Example 6.16

Accidents occur at an intersection according to a Poisson process with rate 2 per week.

$\frac{3}{4}$ of the accidents involve alcoho and all accidents are all assumed to be independent.

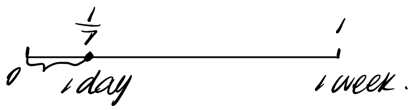
(a) What is the probability of 3 accidents involving alcohol in the next week

Alcohol accidents occur according to a Poisson process with rate $2 \cdot \frac{3}{4} = 1.5$ per week.

$\Rightarrow$ t is equal to 1 week

$$\Rightarrow Pr(N_1 = 3) = \frac{e^{-1.5}\ 1.5^3}{3!} = 0.125$$

(b) What's the probability of at least one accident tomorrow?



$$Pr(N_{\frac{1}{7}} \geq 1) = 1 - Pr(N_{\frac{1}{7}} = 0) = 1 - e^{-2 \cdot \frac{1}{7}} = 0.249$$

(c) If I observed 6 accidents in the month of February 2019 (4 weeks). What's the expected date of the 6th accident?

$t = 4$, $N_4 = 6$

$$Pr(X_6 < s) = Pr(\max(X_1, \cdots X_6) < s) \text{ if } X_1, X_2 \cdots X_6 \overset{iid}{\sim} Unif(0,t)$$

$$= Pr(X_1 < s, X_2 < s \cdots X_6 < s)$$

$$= \prod_{i=1}^{6} Pr(X_i < s)$$

$$= \prod_{i=1}^{6} \frac{s}{4}$$

$$= \frac{s^6}{4}$$

$$\Rightarrow Pr(s) = \frac{\partial}{\partial s} \frac{s^6}{4} = \frac{6}{4} s^5 = 1.5 s^5$$

$$\Rightarrow E(S_6) = \int_0^4 s \cdot 1.5 s^5 \, ds = 1.5 \int_0^4 s^6 \, ds = \frac{24}{7}$$ 3 weeks and 3 days.