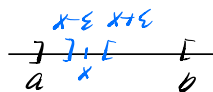
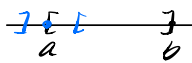


DEF A subset $U \subseteq \mathbb{R}$ is called **open** if $\forall x \in U \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq U$.



open interval



closed interval

① \mathbb{R} is open

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. Then $V_\varepsilon(x) \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ is open.

② \emptyset is open

The condition for openness is satisfied since there is no $x \in \emptyset$.

THM Every open interval is open

THM (a) Arbitrary unions of open sets are open

i.e. If I is an arbitrary index set and $\forall i \in I : U_i \subseteq \mathbb{R}$ is open, then $U = \bigcup_{i \in I} U_i$ is open

(b) Finite intersections of open sets are open

i.e. If $U_1, \dots, U_n \subseteq \mathbb{R}$ are open, then $\bigcap_{i=1}^n U_i$ is open.

Remark Infinite intersections of open sets are, in general, not open.

e.g. Let $I = \mathbb{N}$, $\forall n \in \mathbb{N}$, $U_n = (-\frac{1}{n}, \frac{1}{n}) \Rightarrow U = \bigcap_{i \in \mathbb{N}} U_i = \{0\}$ not open since $\forall \varepsilon > 0$, $V_\varepsilon(0) \not\subseteq \{0\}$.

THM A subset of \mathbb{R} is open iff it's a countable union of open intervals (not used)

DEF A subset $A \subseteq \mathbb{R}$ is called **closed** if A^c is open.

• $\mathbb{R} \setminus A$

THM Every closed interval is closed

① $A = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\}$ closed

$\forall x \notin A \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^c$ (to be proved)

Note that $\{ \frac{1}{n} : n \in \mathbb{N} \}$ not closed since every ε -neighborhood about 0 intersects an inf. many points.

② both \emptyset and \mathbb{R} are closed

$\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open and $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open

THM \emptyset and \mathbb{R} are the only subsets of \mathbb{R} which are both open and closed.
(not used)

THM (a) Finite unions of closed sets are closed.

(b) Arbitrary intersections of closed sets are closed.

Remark There're closed subsets of \mathbb{R} which are not countable unions of closed intervals. e.g. Cantor set

DEF Let $A \subseteq \mathbb{R}$. We say that a seq (x_n) is in A if $\forall n \in \mathbb{N}: x_n \in A$

THM Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a conv. seq in A
Let $x = \lim(x_n) \Rightarrow x \in A$

DEF Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a boundary point of A if
 $\forall \varepsilon > 0: V_\varepsilon(x) \cap A \neq \emptyset$ and $V_\varepsilon(x) \cap A^c \neq \emptyset$

The set of all boundary points of A is called the boundary of A
and is denoted as ∂A

Ex. $I = [a, \infty) \Rightarrow \partial I = \{a\}$

Let $x > a$. We know that (a, ∞) is open

$$x \in (a, \infty) \Rightarrow \exists \varepsilon > 0: V_\varepsilon(x) \subseteq (a, \infty) \subseteq I$$

$$\Rightarrow V_\varepsilon(x) \cap I^c = \emptyset \Rightarrow x \notin \partial I$$

Now let $x = a$

Consider $(-\infty, a)$ which is open

$$x \in (-\infty, a) \Rightarrow \exists \varepsilon > 0: V_\varepsilon(x) \subseteq (-\infty, a) \subseteq I^c$$

$$\Rightarrow V_\varepsilon(x) \cap I = \emptyset \Rightarrow x \notin \partial I$$

Finally, $\forall \varepsilon > 0: V_\varepsilon(a) \cap I \neq \emptyset$ since $a + \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I$
Similarly, $a - \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I^c \Rightarrow V_\varepsilon(a) \cap I^c \neq \emptyset$

$$\Rightarrow a \in \partial I \Rightarrow \partial I = \{a\}$$

exercise $\partial [a, b] = \partial [a, b) = \partial (a, b] = \partial (a, b) = \{a, b\}$

THM Let $A \subseteq \mathbb{R}$, then

(a) A is open iff A doesn't contain any of its boundary points
i.e. $A \cap \partial A = \emptyset \Rightarrow \partial A \subseteq A^c$

(b) A is closed iff A contains all of its boundary points
i.e. $\partial A \subseteq A$.

DEF A subset $A \subseteq \mathbb{R}$ is called **compact** if A is both closed and open.

DEF A subset $A \subseteq \mathbb{R}$ is called sequentially compact if for all seq (x_n) in A , it holds that (x_n) has a conv. subseq (x_{n_k}) whose limit lies in A .

THM A subset $A \subseteq \mathbb{R}$ is compact iff it's sequentially compact. (?)