Tutorial 1 Jan16

① A probability (measure) is a (set) fn

$P: F \to [0,1]$ satisfying

i) $P(\phi)=0$, $P(\Omega)=P(S)=1$

ii) For disjoint events $A_1, A_2 \cdots \in F$ i.e. $A_i \cap A_j = \phi$, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} P(A_i)$$

② If $P(B)>0$, the conditional probability that A occurs given B occurs is defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

③ Law of Total Probability

Events partitioning S (or Ω) , $B_1, \cdots B_n$ with $P(B_i)>0$ for $i=1 \dots n$

Then for any A ,

$$P(A) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$$

④ Independence   random variable : function

Two events (set) A, B are independent iff $P(A\cap B) = P(A)P(B)$

⑤ Expectation

For any X , provided it exists

$$E(X) = \mu_X = \int_{\mathbb{R}} x f_X(x)\, dx .$$

⑥ Joint distribution

$$F(X,Y) = P(X \le x, Y \le y)$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$$

joint density

⑦ Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

marginal distribution (CDF)

$$F_X(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(u,y)\, dy \right) du$$

⑧ Independence of X, Y

$$\Leftrightarrow \begin{array}{l} F(x,y) = F_X(x) F_Y(y) \\ f(x,y) = f_X(x) f_Y(y) \end{array}$$

⑨ $Cov(X,Y) = E(XY) - E(X)E(Y)$

⑩ Conditional distributional function

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} \frac{f(x,v)}{f_X(x)} dv \quad f_X(x) > 0$$

⑪ Conditional density

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

⑫ Conditional expectation

$$E(Y|X=x) = \begin{cases} \sum_y y P(Y=y|X=x) \\ \int_R y f_{Y|X}(y|x) dy \end{cases}$$

⑬ Properties of expectation

i) Linearity $E(aY+b|X=x) = aE(Y|X=x)+b$

ii) $E(g(Y)|X=x) = \begin{cases} \sum_y g(y) P(Y=y|X=x) \\ \int_R g(y) f_{Y|X}(y|x) dy \end{cases}$

iii) $E(Y|X=x) = E(Y)$ iff $X \perp Y$

iv) $Y = g(X) \Rightarrow E(Y|X=x) = g(x)$ number

but $E(Y|X) = g(X)$ random variable

v) $E_Y(Y) = E_X(E_{Y|X}(Y|X))$

$Var_Y(Y) = Var_X(E_{Y|X}(Y|X)) + E_X[Var_{Y|X}(Y|X)]$

Exercise 1. A man is saving up to buy a new car at a cost of N units of money. He starts with k units where $0 < k < N$ and tries to win the remainder by the following gamble with his bank manager. He tosses a fair coin repeatedly; if it comes up heads then the manager pays him one unit, but if it comes up tails then he pays the manager one unit. He plays this game repeatedly until one of two events occurs: either he runs out of money and is bankrupted or he wins enough to buy the car. What is the probability that he is ultimately bankrupted? Observed random walk

Hint: Use the Law of Total Probability to start off. You might want to remember how to solve a recurrence relation and also to remind yourself of the Single Root Theorem in that context. This first exercise is a bit of a challenging one: yet it is fundamental and it gives a first insight into what we will be looking at in this course.

H: win T: lose

$A = \{\text{person goes bankrupt}\}$

$B = \{\text{the first toss is head}\}$

independent of k

$$P_k(A) = P_k(A|B) \cdot P(B) + P_k(A|B^c)\, P(B^c)$$
$$= P_{k+1}(A) \cdot \tfrac{1}{2} + P_{k-1}(A) \cdot \tfrac{1}{2}$$

$\Rightarrow \quad P_k = \frac{1}{2}(P_{k+1} + P_{k-1})$ recurrence relation

$$\theta^k = \tfrac{1}{2}\theta^{k+1} + \tfrac{1}{2}\theta^{k-1}$$
$$\theta^{k-1}\left(\tfrac{1}{2}\theta^2 - \theta + \tfrac{1}{2}\right) = 0$$

$\Rightarrow \quad \theta = 1$

Single Root Thm $\quad P_k = a\theta^k + bk\theta^k$

Also, we know $P_0 = 1$ and $P_N = 0$

$$\Rightarrow \begin{cases} P_0 = a = 1 \\ P_N = a + bN = 0 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -\frac{1}{N} \end{cases}$$

$$\Rightarrow P_k = a\theta^k + bk\theta^k = 1 - \frac{k}{N}$$

Exercise 2. A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from the remaining fifteen.

(a) What is the probability that this cherry contains a stone? $P = \frac{5}{20} = \frac{1}{4}$

(b) Given that this cherry contains a stone, what is the probability that the pig consumed at least one stone?

$S = \{\text{this cherry contains a stone}\}$

$P = \{\text{The pig ate at least one stone}\}$

$$P(P|S) = 1 - P(P^c|S)$$
$$= 1 - \frac{15}{19} \times \frac{14}{18} \times \frac{13}{17} \times \frac{12}{16} \times \frac{11}{15}$$

Exercise 3. Prove that, for any set E and F, we have

$$(E \cup F)^c = E^c \cap F^c.$$

Note: This identity is called the De Morgan's law. Note that we also have $(E \cap F)^c = E^c \cup F^c$.

$\omega \in (E \cup F)^c \iff \omega \notin E \cup F \iff \omega \notin E$ and $\omega \notin F$
$\iff \omega \in E^c$ and $\omega \in F^c \iff \omega \in E^c \cap F^c$

Exercise 4. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn, c additional balls of the same colour are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $\frac{b}{(b+r+c)}$.

$A = \{$ First ball drawn was black $\}$
$B = \{$ Second ball drawn was red $\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{b}{b+r} \cdot \frac{r}{b+r+c}}{\frac{br + r(r+c)}{(b+r)(b+r+c)}} = \frac{rb}{br+r(r+c)} = \frac{b}{b+r+c}$$

$$P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$$
$$= \frac{b}{b+r} \cdot \frac{r}{b+r+c} + \frac{r}{b+r} \cdot \frac{r+c}{b+r+c}$$

Exercise 5. For a fixed event B, show that the collection $\mathbf{P}\{A|B\}$, defined for all events A, satisfies the three conditions for a probability (the so-called *Kolmogorov's axioms*). Then verify that

$$\mathbf{P}\{A|B\} = \mathbf{P}\{A|B \cap C\}\,\mathbf{P}\{C|B\} + \mathbf{P}\{A|B \cap C^c\}\,\mathbf{P}\{C^c|B\}.$$

$$\frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(C \cap B)}{P(B)} \qquad \frac{P(A \cap B \cap C^c)}{P(B \cap C^c)} \cdot \frac{P(C^c \cap B)}{P(B)}$$

$$= \frac{P(A \cap B \cap C)}{P(B)} \qquad = \frac{P(A \cap B \cap C^c)}{P(B)}$$

$$RHS = \frac{P(A \cap B \cap C + A \cap B \cap C^c)}{P(B)} = \frac{P(A \cap B)}{P(B)} = P(A|B) = LHS$$

Exercise 6. A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have won two more games than the other. Find the probability that A is the overall winner. Then find the expected number of games played.

$A = \{A \text{ wins overall}\}$

$Y = \{\# \text{ wins of } A \text{ in the first two games}\}$

$$P(A) = \overset{0}{P(A|Y=0)}\,P(Y=0) + \overset{P(A)}{P(A|Y=1)}\,P(Y=1) + \overset{1}{P(A|Y=2)}\,P(Y=2)$$

$$= 0 + P(A)\,2p(1-p) + p^2$$

$$\Rightarrow P(A) = \frac{p^2}{1-2p(1-p)}$$

$X = \{\# \text{ games played}\}$

$$E(X) = \overset{\text{over}}{E(X|Y=0)}\,P(Y=0) + E(X|Y=1)\,P(Y=1) + \overset{\text{over}}{E(X|Y=2)}\,P(Y=2)$$

$$= 2(1-p)^2 + (E(X)+2)\,2(1-p)p + 2p^2$$

$$\Rightarrow E(X) = \frac{2}{1-2(1-p)p}$$

Tutorial 2 - Jan 26

limiting distribution

$$\lim_{n\to\infty}(P^n)_{ij}=\lambda_j \quad \forall i,j$$

$\underline{\pi}$ is a stationary distribution of the tran x if

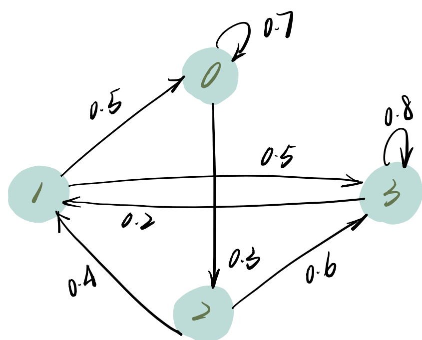
① $\pi_j \geq 0 \ \forall j$ and $\sum_j \pi_j = 1$

② $\underline{\pi} = \underline{\pi} P \quad (\pi_j = \sum_i \pi_i P_{ij})$

Exercise 1. Suppose that whether or no t it rains today depends on previous weather conditions through the last two days. Furthermore, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; finally, it is hasn't rained in the past two days, then it will rain tomorrow with probability 0.2. The process is in:

1) State 0: if it rained both today and yesterday;
2) State 1: if it rained today but not yesterday;
3) State 2: if it rained yesterday but not today;
4) State 3: if it did not rain either yesterday or today.

Construct the transition matrix for this Markov Chain. Make sure to verify the conditions for such matrix hold.



Exercise 2. Consider exercise 1 from last week's tutorial (Gambler's ruin problem). Is it a Markov Chain ? What is particular about it ? If it is a Markov Chain, write down the transition probabilities. (Note: you will not be able to express it in a matrix form if N is not fixed; however, you can still write the transition probabilities for specified states).

winning P, losing $1-P$

$P(X_n=j \mid X_{n-1}=i, X_{n-2}=m \dots) = P(X_n=j \mid X_{n-1}=i)$

$j, i, m \dots$: amount of money in each state

$\Rightarrow$ X is a Markov chain.

$P_{i,i+1} = p = 1 - P_{i,i-1} \quad i = 1 \dots N-1$

$\begin{cases} P_{0,0} = 1 \\ P_{N,N} = 1 \end{cases}$ stop and stay

$N = 3$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 1-p & 0 & p & 0 \\ 2 & 0 & 1-p & 0 & p \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Exercise 3. Consider a Markov Chain $\{X_n; n \geq 0\}$ with states $S = \{0, 1, 2\}$. The transition matrix is given by:

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

If $\mathbf{P}\{X_0 = 0\} = \mathbf{P}\{X_0 = 1\} = 1/4$, find $\mathbf{E}[X_3]$.

$$P^3 = \begin{pmatrix} 13/36 & 11/56 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{pmatrix}$$

$E(X_3) = \sum_k P(X_3 = k)k = 0 \cdot P(X_3 = 0) + 1 \cdot P(X_3 = 1) + 2 \cdot P(X_3 = 2)$

$P(X_3) = P(X_3 = 1 | X_0 = 0) \underbrace{P(X_0 = 0)}_{\frac{1}{4}} + P(X_3 = 1 | X_0 = 1) \underbrace{P(X_0 = 1)}_{\frac{1}{4}} + P(X_3 = 1 | X_0 = 2) \underbrace{P(X_0 = 2)}_{\frac{1}{2}}$

$P(X_3) = P(X_3 = 2 | X_0 = 0) \underbrace{P(X_0 = 0)}_{\frac{1}{4}} + P(X_3 = 2 | X_0 = 1) \underbrace{P(X_0 = 1)}_{\frac{1}{4}} + P(X_3 = 2 | X_0 = 2) \underbrace{P(X_0 = 2)}_{\frac{1}{2}}$

aP^3

Exercise 4. Let X be a Markov Chain with three states and let its transition matrix be given by

$$P = \begin{pmatrix} 0.95 & 0.04 & 0.01 \\ 0.15 & 0.8 & 0.05 \\ 0 & 0.25 & 0.75 \end{pmatrix}.$$

limiting

Find the long-run probabilities to be in each state, i.e. π_1, π_2, π_3. $\underset{\sim}{\pi} = (\pi_1, \pi_2, \pi_3)$

$\begin{cases} \pi_1 + \pi_2 + \pi_3 = 1 \\ \pi P = \pi \quad \text{(stationary)} \end{cases} \begin{cases} \pi_1 = 0.95\pi_1 + 0.15\pi_2 \\ \pi_2 = 0.04\pi_1 + 0.8\pi_2 + 0.25\pi_3 \end{cases}$

$\Rightarrow \pi_1 = 75/108$

$\pi_2 = 25/135$

$\pi_3 = 8/108$

Exercise 5. Suppose the rain process is modelled with a Markov process with two states: if it rains today, then the probability that it rains tomorrow is 0.5; if it doesn't rain today, the probability that it rains tomorrow is 0.3. Find the transition matrix associated to this Markov chain and calculate the limiting probability that it rains on two consecutive days.

$$P = \begin{array}{c|cc} & R & NR \\ \hline R & 0.5 & 0.5 \\ NR & 0.3 & 0.7 \end{array}$$

$$\lim_{n\to\infty} P(X_n=1, X_{n+1}=1) = \lim_{n\to\infty} P(X_n=1)P(X_{n+1}=1 \mid X_n=1)$$

$$\begin{cases} \pi_1+\pi_2=1 \\ 0.5\pi_1+0.3\pi_2=\pi_1 \end{cases} \Rightarrow \begin{cases} \pi_1 = 3/8 \\ \pi_2 = 5/8 \end{cases}$$

$$\lim_{n\to\infty} P(X_n=1, X_{n+1}=1) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16} \approx 0.1875$$

Tutorial 3 – Jan 30

1. ① regular ✓

② irreducible ✗

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.

(a) $\lim_{n\to\infty} Pr(X_n=j \mid X_{n-1}=i) \overset{TH}{=} Pr(X_1=j \mid X_0=i) = P_{ij}$

(b) $\lim_{n\to\infty} Pr(X_n=j \mid X_0=i) = \lim_{n\to\infty} (P^n)_{ij} = \pi_j$

(c) $\lim_{n\to\infty} Pr(X_{n+1}=k, X_n=j \mid X_0=i) = \lim_{n\to\infty} Pr(X_{n+1}=k \mid X_n=j)P(X_n=j \mid X_0=i)$

$\overset{TH}{=} \lim_{n\to\infty} Pr(X_1=k \mid X_0=j)P(X_n=j \mid X_0=i)$

$= P_{ij} \cdot \pi_j$

(d) $\lim_{n\to\infty} Pr(X_0=j \mid X_n=i) = \lim_{n\to\infty} \frac{P(X_n=i \mid X_0=j)P(X_0=j)}{P(X_n=i)} = \frac{\pi_i \pi_j}{\pi_i} = \pi_j$

Bayes.

Tutorial 3 - Jan 30

① If A, B are doubly stochastic, then AB is also doubly stochastic

both rows and column sum to 1.

② $P(X_n=i)=(\alpha P^n)_i = (\underset{\sim}{\lambda} P^n)_i = \lambda_i$

assume $\alpha=\underset{\sim}{\lambda}$ limiting distribution

③ TPM P is regular if $\exists n$ s.t. $P^n > 0$

④ For a square matrix W, if $W\underset{\sim}{v} = \lambda \underset{\sim}{v}$, we say that $\underset{\sim}{v}$ is a right eigenvector of W with eigenvalue λ.

$$W = V \Lambda V^{-1} \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}, \; V = (\underset{\sim}{v_1} \cdots \underset{\sim}{v_k})$$

Exercise 1. Let P be a stochastic matrix. If P is regular, is P^2 also regular? Yes If P is the transition matrix of an irreducible Markov chain, is P^2 also the transition matrix of an irreducible Markov chain ? No

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

always consider this alternating matrix as possibe counterexample

Exercise 2. A Markov chain has transition matrix P and limiting distribution $\boldsymbol{\pi}$. Further assume that $\boldsymbol{\pi}$ is the initial distribution of the chain. That is, the chain is in stationarity. Find the following:

a) $\lim_{n\to\infty} \mathbf{P}\{X_n = j | X_{n-1} = i\}$;

b) $\lim_{n\to\infty} \mathbf{P}\{X_n = j | X_0 = i\}$;

c) $\lim_{n\to\infty} \mathbf{P}\{X_{n+1} = k, X_n = j | X_0 = i\}$;

d) $\lim_{n\to\infty} \mathbf{P}\{X_0 = j | X_n = i\}$.

(a) $\lim\limits_{n\to\infty} Pr(X_n=j|X_{n-1}=i) \overset{TH}{=} Pr(X_1=j|X_0=i) = P_{ij}$

(b) $\lim\limits_{n\to\infty} Pr(X_n=j|X_0=i) = \lim\limits_{n\to\infty} (P^n)_{ij} = \pi_j$

(c) $\lim\limits_{n\to\infty} Pr(X_{n+1}=k, X_n=j | X_0=i) = \lim\limits_{n\to\infty} Pr(X_{n+1}=k|X_n=j) P(X_n=j|X_0=i)$

$\overset{TH}{=} \lim\limits_{n\to\infty} Pr(X_1=k|X_0=j) P(X_n=j|X_0=i)$

$= P_{ij} \cdot \pi_j$

(d) $\lim\limits_{n\to\infty} Pr(X_0=j|X_n=i) \underset{\text{Bayes.}}{=} \lim\limits_{n\to\infty} \dfrac{P(X_n=i|X_0=j)P(X_0=j)}{P(X_n=i)} = \dfrac{\pi_i \pi_j}{\pi_i} = \pi_j$

Exercise 3. Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}.$$

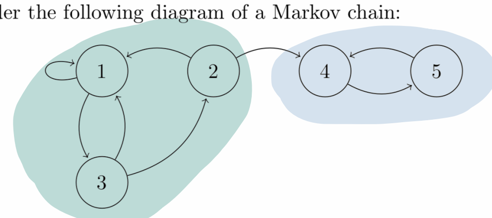
Obtain a closed form expression for P^n. Exhibit the matrix $\sum_{n=0}^{\infty} P^n$. Explain what this shows about the recurrence and transience of the states.

$$P^2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{pmatrix} \qquad P^3 = \begin{pmatrix} \frac{1}{8} & \frac{1}{8}+\frac{3}{4} \\ 0 & 1 \end{pmatrix}$$

$$P^n = \begin{pmatrix} \frac{1}{2^n} & 1-\frac{1}{2^n} \\ 0 & 1 \end{pmatrix}$$

$$\sum_{n=0}^{\infty} P^n = \begin{pmatrix} 2 & \infty \\ 0 & \infty \end{pmatrix}$$ $\Rightarrow$ 1 is transient and 2 is recurrent.

Exercise 4. Consider the following diagram of a Markov chain:



Find the communicating classes associated with the transition diagram.

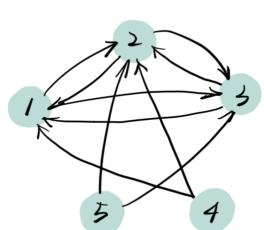
$\{1,2,3\}$
$\{4,5\}$

Exercise 5.
For each of the following transition matrices, determine the classes and specify which are recurrent and which are transient.

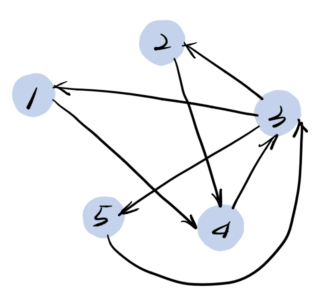
a) $P_1 = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$. $\Rightarrow P^n = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ & & \vdots & & \end{pmatrix}$ 1, 2, 3 recurrent

b) $P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$. $\Rightarrow P^n = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ & & \vdots & & \end{pmatrix}$ $\sum = \infty$ all recurrent

(a)



(b)



irreducible (one communication class)

① $T_j = \min\{n > 0 : X_n = j,\ X_0 = j\}$

j is recurrent if $P\{T_j < \infty \mid X_0 = j\} = 1$

j is transient if $P\{T_j < +\infty \mid X_0 = j\} < 1$

$$I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{otherwise} \end{cases},$$

j is recurrent if $\mathbb{E}[\sum_{n=0}^{+\infty} I_n \mid X_0 = j] = \infty$

j is transient if $\mathbb{E}[\sum_{n=0}^{+\infty} I_n \mid X_0 = j] < \infty$

↑ the process is not coming back to j

$$\sum_{n=0}^{+\infty} (P^{r+n+t})_{ii} = \sum_{n=0}^{+\infty} \sum_{\ell \in S} (P^r)_{i\ell} (P^n)_{\ell\ell} (P^t)_{\ell i}$$

$$\geq \sum_{n=0}^{+\infty} (P^r)_{ij} (P^n)_{jj} (P^t)_{ji} = +\infty \text{ if } j \text{ is recurrent}$$

② closed C which is communication class if

$P_{ij} = 0$ for all $i \in C$, $j \notin C$

③ let $\mu_j = \mathbb{E}[T_j \mid X_0 = j]$

$\rightarrow |S| < +\infty$. For an irreducible MC, $\mu_j < +\infty$,

$\exists\, \pi$ s.t. $\pi_j = \dfrac{1}{\mu_j}\ \forall j$

↘ does not imply limiting

④ Periodicity

$d(i) = \gcd\{n>0 : (P^n)_{ii} > 0\}$

state i is aperiodic if $d(i)=0$

All states in the same CC have the same period [important!]

MC is aperiodic if ① irreducible
② $d(i)=1$ $\forall i \in S$

MC is periodic if ① irreducible
② $d(i)>1$ $\forall i \in S$

MC is ergodic if ① irreducible
② aperiodic
③ all states $i \in S$ have finite return time.

Ergodic Theorem: Let $X_0, X_1 \cdots X_n$ be an ergodic MC, then $\exists!$ positive stationary distribution, which is also the limiting distribution of MC.

Tutorial 4 – Feb 6

Exercise 1. The inspection of a computer on board is made by the astronauts at times some specific times. No possibility of repairs exists. Inspection can report either of the following results: perfect functioning, minor failures that do not hinder the global functioning of the system, major failures that hinder the development of experiments, or complete failure that makes the spaceship ungovernable.

initial distribution $\alpha = (1\ 0\ 0\ 0)$

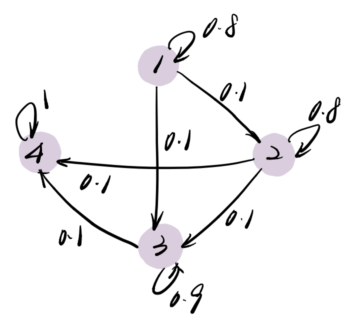
This is modelled by a Markov chain. The computer is supposed to be in perfect state at the beginning of the flight (at time $t=0$). During preceding flights, the probabilities of passage from one state to the other between two successive inspections have been estimated by the matrix P, given below:

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 & 0 \\ 0 & 0.8 & 0.1 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

though all states are aperiodic

a) Draw the diagram corresponding to this chain. Give the state space, the initial distribution and the nature of the states. Is the chain irreducible? Is it aperiodic ? Is it ergodic?
NO NO NO

b) Compute the mean number of inspections before complete failure occurs. *Hint: Let T be the random variable corresponding to the hitting time of state 4, i.e. $T := \min\{n \geq 0 : X_n = 4\}$. Find* $\mathbf{E}[T]$.



$\{1\}, \{2\}, \{3\}$ transient
$\{4\}$ recurrent

$E(T) = \sum_{i \in \{1,2,3\}} P(X_0=i) E[T \mid X_0=i] = P(X_0=1) E[T \mid X_0=1]$ start from 1.

$= \sum_y (1+E[T \mid X_0=y]) P(X_1=y \mid X_0=1)$

$e_x = E[T \mid X_0 = x]$

$$\begin{cases} e_3 = P(X_1=3 \mid X_0=3)(1+e_3) + P(X_1=4 \mid X_0=3)\cdot 1 \\ e_2 = P(X_1=2 \mid X_0=2)(1+e_2) + P(X_1=3 \mid X_0=2)(1+e_3) \\ e_1 = P(X_1=1 \mid X_0=1)(1+e_1) + P(X_1=2 \mid X_0=1)(1+e_2) + P(X_1=3 \mid X_0=1)(1+e_3) \end{cases}$$

$\Rightarrow e_1 = 15,\ e_2 = e_3 = 10 \quad \Rightarrow E(T) = 15$

Exercise 2. Consider the general two-state chain with P given by

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where p and q are not both 0. Let T_1 be the first return time to state 1 for the chain started at 1.

a) Show that $\mathbf{P}\{T_1 \geq n\} = p(1-q)^{n-2}$ for $n \geq 2$;

b) Find $\mathbf{E}[T_1]$ and verify that $\mathbf{E}[T] = 1/\pi_1$ where π_1 is the stationary probability to be in state 1.

(a) If $T_1 \ge n$, then I need to transition to 2 and stay here for at least $n-2$ steps.

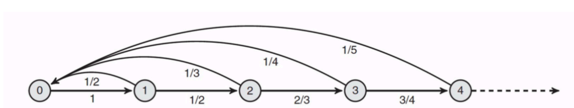
(b) For a random variable X (positive and discrete), we have

$$E(X) = \sum_{n=1}^{\infty} P(X \ge n)$$

$$E(X) = \sum_{n=1}^{\infty} n P(X=n) = P(X=1) +$$
$$P(X=2) + P(X=3)$$
$$P(X=3) + P(X=3) + P(X=3)$$
$$\ldots \quad \ldots$$

$$E(T_1) = \sum_{n=1}^{\infty} P(T_1 \ge n) = \underset{P(T_1 \ge 0)}{1} + \sum_{n=2}^{\infty} p(1-q)^{n-2} = 1 + \frac{p}{1-(1-q)} = \frac{p+q}{q} = \frac{1}{\pi_1}$$

Exercise 3. Consider the infinite Markov chain on the non-negative integers described by the figure below.



a) Show that the chain is both irreducible and aperiodic.

b) Show that the chain is recurrent by computing the first return time to 0 for the chain started at 0.

c) Show that the chain is null recurrent.

Tutorial 5 – Feb 13

① State i is absorbing if $P_{ii}=1$

Markov chain is absorbing if $\exists$ state i absorbing.

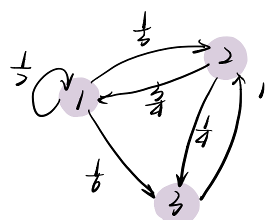
② Suppose $S=\{1,\dots,n\}$ with t transient state and h absorbing state $(n=t+h)$

Define TPM: $P=\begin{pmatrix} Q & \\ 0 & I \end{pmatrix}$

$P^n=\begin{pmatrix} Q^n & \\ 0 & \end{pmatrix}$

Exercise 5. Consider the Markov chain with transition matrix given by P as below:

$$P=\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}.$$



1 CC

irreducible

a) Show that the chain is irreducible and aperiodic. Is its TPM regular?

b) Suppose the process starts in state 1. Find the probability that it is in state 3 after two steps.

c) Find the matrix which is the limit of $\mathbf{P}^n$ as $n\to\infty$.

(a) Aperiodic: Since all states share the same period property in a CC, then check 1 state is sufficient: $d(1)=1$.

(b). $P(X_3=3\mid X_0=1)=(P^2)_{13}=\sum_{i=1}^{3}P_{1i}P_{i3}=P_{11}P_{13}+P_{12}P_{23}+P_{13}P_{33}=\frac{1}{6}.$

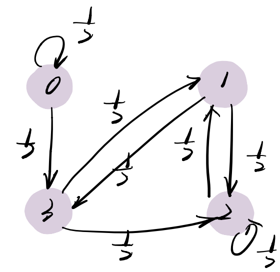
(c). If the chain is regular $\Rightarrow$ The limit exists.

$$\lim_{n\to\infty}P^n=\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix}\begin{matrix}\to\lambda\\ \to\lambda\\ \to\lambda\end{matrix} \Rightarrow \begin{cases} \frac{1}{2}\pi_1+\frac{3}{4}\pi_2=\pi_1 \\ \frac{1}{3}\pi_1+\pi_3=\pi_2 \\ \frac{1}{6}\pi_1+\frac{1}{4}\pi_2=\pi_3 \\ \pi_1+\pi_2+\pi_3=1 \end{cases} \Rightarrow \lambda=(\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{6})$$

Exercise 2. Consider the following transition matrix associated with a Markov chain $\{X_n\}$:

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

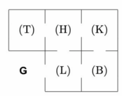
Q R M



Classify the states and compute the matrix $(I-Q)^{-1}R$, where Q is the submatrix of P with the transient states and R is the matrix with the transition probabilities from the transient states to the recurrent ones. Interpret the entries of $(I-Q)^{-1}R$.

$$(I-Q)^{-1}R = (1-\tfrac{1}{2})^{-1}(0\ 0\ \tfrac{1}{2}) = (0\ 0\ 1)$$

Exercise 3. Consider the image below, which represents a house with toilets (T), a hall (H), a kitchen (K), a bedroom (B), a living room (L) and a garden (G).



Suppose the mouse moves in such a way that it chooses one of the adjacent room to the room it starts at (with the same probability) and runs there at times $n = 1, 2, \ldots$. The owners installed two traps - one in the bedroom, another one in the kitchen. If the mouse enters one of this room, then it is caught and never ever run into another room. If the mouse enters the garden, then it is free and it escapes the house. If the mouse starts in the toilets, what is the probability that it escapes the flat before it is caught?

Q

$$\begin{array}{c|ccc|ccc} & T & H & L & K & B & G \\ \hline T & & 1 & & & & 0 \\ H & \frac{1}{3} & & \frac{1}{3} & \frac{1}{3} & & \\ L & & \frac{1}{3} & & & \frac{1}{3} & \frac{1}{3} \\ \hline K & & & & 1 & & \\ B & & & & & 1 & \\ G & & & & & & 1 \end{array}$$

R

$$(I-Q)^{-1}R = \begin{array}{c} \\ T \\ H \\ L \end{array} \begin{array}{c} \begin{array}{ccc} K & B & G \end{array} \\ \begin{pmatrix} 3/5 & 1/5 & 1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & 2/5 & 2/5 \end{pmatrix} \end{array}$$

Exercise 5. Let $\{X_n\}$ be a Markov chain with transition probabilities given by the matrix P below and stationary distribution π also given below.

$$P = \begin{pmatrix} 0 & p_1 & q_1 \\ q_2 & 0 & p_2 \\ p_3 & q_3 & 0 \end{pmatrix}$$

and

$$\pi = \left(\frac{1-p_2q_3}{3-p_2q_3-p_3q_1-p_1q_2}, \frac{1-p_3q_1}{3-p_2q_3-p_3q_1-p_1q_2}, \frac{1-p_1q_2}{3-p_2q_3-p_3q_1-p_1q_2}\right)$$

Is the chain time reversible? $\pi_i P_{ij} = \pi_j P_{ji} \quad \forall\, i,j \in S$

As we are approaching the midterm, if time permits after the above exercises and the usual revision of class material, I will briefly summarised what we have covered so far and answer your questions.

$$\pi_1 P_{12} = \frac{P_1(1-q_3P_2)}{3-P_2q_3-P_3q_1-P_1q_2}$$

$$\pi_2 P_{21} = \frac{P_2(1-P_3q_1)}{3-P_2q_3-P_3q_1-P_1q_2}$$

For arbitrary $P_1, P_2, P_3, q_1, q_2, q_3$ NOT time reversible.

Exercise 4. Consider a branching process with offspring distribution $\mathbf{a} = (a, b, c)$, where $a+b+c = 1$. Let P be the transition probability matrix. Exhibit the first three rows of P.

1st row: $P_{0,j} = \begin{cases} 1 & \text{for } j=0 \\ 0 & \text{otherwise} \end{cases}$

2nd row: $P_{1j} = \begin{cases} a & j=0 \\ b & j=1 \\ c & j=2 \\ 0 & \text{otherwise} \end{cases}$

3rd row:
$$\begin{array}{cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \\ 2\,(& a^2 & 2ab & b^2+2ac & 2bc & c^2 & 0 & \cdots \ ) \end{array}$$

1) Def: state i is said to be absorbing if $P_{ii} = 1$.

Markov chain is absorbing if $\exists$ state i absorb

2) Suppose $S = \{1, \cdots, i\}$ with t transient states, h absorbing states $\Big\}$ $t + h = n$

Decompose TPM: $P = \begin{pmatrix} Q^{t\times t} & R^{t\times h} \\ 0 & I^{h\times h} \end{pmatrix}$

only transient state will disappear

$$P^n = \begin{pmatrix} \underline{Q^n} & (Q^{n-1} + Q^{n-2} + \cdots + Q + I)R \\ 0 & I \end{pmatrix}$$

limiting matrix here

$$\xrightarrow[\text{Lemma}]{n\to\infty} \begin{pmatrix} 0 & (I-Q)^{-1}R \\ 0 & I \end{pmatrix}, \quad (I-Q)^{-1}R =: U$$

$i \in T$, $j \in R$

$$U_{ij} = \mathbb{P}(X_\tau = j \mid X_0 = i)$$

$$\tau = \min\{n > 0 : X_n \notin T\}$$

T = set of transients states.