Chapter 2

healthy $\qquad$ injured & performing $\qquad$ out

$$\cdots \; 0 \to 0 \to \cdots \; 1 \; \cdots \to 2 \to$$

$\{X_t : t \in \{0, 1 \cdots\}\}$, $X_t \in \{0, 1, 2\}$

$$P(X_n = 1) = \sum_{k=0}^{2} P(X_n = 1 \mid X_{n-1} = k) \, P(X_{n-1} = k)$$

$$P(X_n = 1) = \sum_{(x_0, x_1 \cdots x_{n-1}) \in \{0,1,2\}^n} P(X_n = 1 \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2} \cdots X_0 = x_0) \, P(X_{n-1} = x_{n-1} \cdots X_0 = x_0)$$

Too complex!

Markov Chain

A stochastic process $\{X_t : t \in I\}$ where

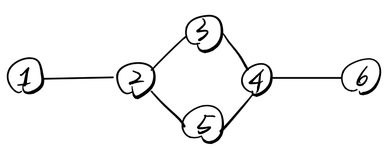
$$P(X_t = x_t \mid X_{t-1} = x_{t-1} \cdots X_0 = x_0) = P(X_t = x_t \mid X_{t-1} = x_{t-1})$$

$\{X_t : t \in \{0, 1 \cdots n \cdots\}\}$, $X_t \in S$ state space (discrete set of values)

- Time-homogeneous Markov Chain

$$P(X_t \mid X_{t-1}, \cdots X_0) = P(X_t \mid X_{t-1}) = P(X_s \mid X_{s-1}) \quad \forall s, t$$

e.g.



$P(X_{t+1} = 2 \mid X_t = 1) = 1$

$P(X_{t+1} = 1 \mid X_t = 2) = \frac{1}{3}$

3	2	$\frac{1}{3}$
5	2	$\frac{1}{3}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

stochastic matrix

$$P_{ij} = P(X_t = j \mid X_{t-1} = i)$$

e.g. Not TH

$$P(X_t = 5 \mid X_{t-1} = 2) = \begin{cases} \frac{1}{3} & t \text{ even} \\ \frac{1}{2} & t \text{ odd} \end{cases}$$

$$P(X_t \mid X_{t-1}) = e^{-\lambda t / 5}$$

Given $X_0 = x_0$, $P(X_t = j \mid X_{t-1} = i)$
If state space is discrete and finite, then we need to specify k^2 possibilities for k states. $\Rightarrow$

Transition Probability Matrix (TPM)

If P is a TPM, it is a $k \times k$ matrix for $|S| = k$ where
$$P_{ij} = P(X_t = j \mid X_{t-1} = i)$$

P is a stochastic matrix : Rows sum to 1.

$$\sum_j P_{ij} = \sum_j P(X_1 = j \mid X_0 = i) = \sum_j \frac{P(X_1 = j, X_0 = i)}{P(X_0 = i)} = \frac{P(X_0 = i)}{P(X_0 = i)} = 1$$

e.g. $S = \{0, 1\}$ $X_t \in S$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} (1-p) & p \\ q & (1-q) \end{pmatrix} \end{array}$$

$P(X_4 = 1 \mid X_0 = 0) = ?$

① $P(X_1 = 1 \mid X_0 = 0) = p$

② $P(X_2 = 1 \mid X_0 = 0)$
$$= \underbrace{P(X_2 = 1 \mid X_1 = 1, X_0 = 0)}_{1-q \text{ (TH)}} \overset{p}{P(X_1 = 1 \mid X_0 = 0)} + \underbrace{P(X_2 = 1 \mid X_1 = 0, X_0 = 0)}_{p \text{ (TH)}} \overset{1-p}{P(X_1 = 0 \mid X_0 = 0)}$$
$$= (1-q)p + p(1-p)$$
going starting

③ $P(X_3 = 1 \mid X_0 = 0)$
$$= \underbrace{P(X_3 = 1 \mid X_2 = 1, X_0 = 0)}_{1-q \text{ (TH)}} \overset{(1-q)p + p(1-p)}{P(X_2 = 1 \mid X_0 = 0)} + \underbrace{P(X_3 = 1 \mid X_2 = 0, X_0 = 0)}_{p \text{ (TH)}} \overset{1-(1-q)p - p(1-p)}{P(X_2 = 0 \mid X_0 = 0)}$$

$$\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = \begin{pmatrix} (1-p)^2 + pq & (1-p)p + p(1-q) \\ (1-p)q + (1-q)p & pq + (1-q)^2 \end{pmatrix}$$

$$\Rightarrow \underbrace{P^2}_{\text{2-step PTM}} = \begin{pmatrix} P(X_2 = 0 \mid X_0 = 0) & P(X_2 = 1 \mid X_0 = 0) \\ P(X_2 = 0 \mid X_0 = 1) & P(X_2 = 1 \mid X_0 = 1) \end{pmatrix}$$

$$P^2 P = \begin{pmatrix} P(X_2 = 0 \mid X_0 = 0) & P(X_2 = 1 \mid X_0 = 0) \\ P(X_2 = 0 \mid X_0 = 1) & P(X_2 = 1 \mid X_0 = 1) \end{pmatrix} \begin{pmatrix} P(X_3 = 0 \mid X_2 = 0) & P(X_3 = 1 \mid X_2 = 0) \\ P(X_3 = 0 \mid X_2 = 1) & P(X_3 = 1 \mid X_2 = 1) \end{pmatrix}$$

$\boxed{\text{Note}}$ $P^n = P^{(n)}$ n-step TPM $\quad Pr(X_{n+m} = i_{n+m} \mid X_m = i_m) = (P^n)_{i_m, i_{n+m}}$ NOT $(P_{i_m, i_{n+m}})^n$

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)} \Leftrightarrow P^{n+m} = P^n \cdot P^m$$

$\boxed{\text{Note}}$ $$Pr(X_n = i) = \sum_{l=1}^{k} \overset{P_{li}^{(n)}}{Pr(X_n = i \mid X_0 = l)}\, Pr(X_0 = l)$$

$\alpha = (Pr(X_0 = 1), \cdots Pr(X_0 = k))$ vector of initial state probabilities

$\Rightarrow \alpha P^n = (Pr(X_n = 1) \cdots Pr(X_n = k))$

$S = \{1, \cdots, k\}$

Joint probability distribution of $(X_{t_1}, X_{t_2} \cdots X_{t_m})$ $t_1 < t_2 \cdots < t_m$

$$Pr(X_{t_m} = i_m \mid X_{t_{m-1}} = i_{m-1}) \cdots Pr(X_2 = i_2 \mid X_1 = i_1)\, Pr(X_{t_1} = t_1)$$

$$P(X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_{k-1}} = i_{k-1}, X_{n_k} = i_k)$$
$$= (\alpha P^{n_1})_{i_1} (P^{n_2 - n_1})_{i_1 i_2} \cdots (P^{n_k - n_{k-1}})_{i_{k-1} i_k}$$