Chapter 5 - Markov Chain Monte Carlo

Pow-law distribution

$$Pr(X=k) = \frac{k^s}{\sum_{j=1}^{\infty} j^s} \qquad k=1,2\cdots$$

if sum is finite.

$$Pr(X=k) \propto k^s$$

Conditional distribution

$X \sim g(x)$ where $g(x)$ is a pdf

$Y|X \sim f(y|x)$ where $f(y|x)$ is a pdf

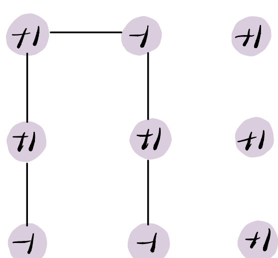
$$h(x|y) = \frac{p(x,y)}{c(y)} \qquad f(y|x)g(x)$$

$$c(y) = \int p(x,y)\, dx = \int f(y|x) g(x)\, dx$$

$$h(x|y) \propto p(x,y) = f(y|x)g(x)$$

A function of x.

e.g. Let σ be a configuration of $+1$ and -1's on the vertices of this latice



$$\text{Energy}(\sigma) = \sum_{v \sim w} -\sigma_w \sigma_v$$

all pairs

e.g. $\sigma = (+1, -1, +1, +1, +1, +1, -1, -1, +1)$

Energy (σ) = # of pairs with different signs − # of pairs of same sign.

Gibbs (or Boltzman) distribution

$$\pi(\sigma) = \frac{e^{-\beta\, \text{Eng}(\sigma)}}{\sum_{\tau} e^{-\beta\, \text{Eng}(\sigma)}}$$

sum of 2^n terms.

$\beta = 0 \Rightarrow \pi(\sigma) = \frac{1}{2^n}$ uniform over possible σ.

$\beta > 0 \Rightarrow$ favor σ with more neighbors with the same sign.

$\beta < 0 \Rightarrow$ favor σ with more neighbors with the different sign.

How to generate draws from these distribution? Markov Chain

Borel Distribution $Pr(X=x) = \frac{e^{-x\mu}(x\mu)^{x-1}}{x!}$

Goal: $E(X)$, $Pr(X \geq c)$

$X_1, \cdots\cdots X_n \overset{iid}{\sim} P_X(x)$

$\begin{cases} \overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i \\ \hat{q} = \frac{1}{n}\sum_{i=1}^{n} 1_{X_i \geq c} \end{cases}$ We know $\begin{cases} \bar{X}_n \xrightarrow{P} E(X_n) \\ \hat{q} \xrightarrow{P} q \end{cases}$ $\begin{cases} E(\bar{X}_n) = E(X_n) \\ E(\hat{q}) = q \end{cases}$

What if $X_1 \cdots X_n$ not iid? But just $X_i \sim P_X(x)$, not independent?

Strong Law of Large Numbers for Markov chain

Let $X_0, X_1 \cdots X_n$ be a Markov chain with stationary distribution $\pi(x) = P_X(x)$

Then $\frac{r(X_0) + r(X_1) + \cdots r(X_n)}{n+1} \xrightarrow{P} E_\pi(r(X)) \rightarrow \sum_j r(j)\pi_j$

for any function $r(\cdot)$ where $E_\pi(r(X)) < \infty$

Before: Given P, find π. Now: Given π, find P.

Let $\underline{\pi}$ be a discrete p.m.f. $\pi_k = Pr(X=k)$

Metropolis-Hastings Algorithm. Generate draws from pmf

① Choose any irreducible TPM T, with the same state space as $\underline{\pi}$.
② Choose $X_0 = i_0$.
③ For $n=1,2\cdots$ Propose to move from $X_{n-1}=i$ to $X_n=j$ according to T (T_{ij})
④ Decide to accept the move (set $X_n=j$) with probability

$$a(i,j) = \min\left(1, \frac{\pi_j}{\pi_i} \times \frac{T_{ji}}{T_{ij}}\right)$$

Target
move if the location is better

Proposal
if you can afford it

Otherwise, $X_n = i$

Repeat

$\Rightarrow$ $X_0 \cdots X_n$ is a Markov chain with TPM P $\begin{cases} P_{ij} = T_{ij}\,a(i,j) & \text{for } i \neq j \\ P_{ii} = 1 - \sum_{j \neq i} P_{ij} & \text{for } i = j. \end{cases}$

$P_X(x) = \pi(x)$ will be unique limiting (stationary) distribution of P.

proof

Let P be TPM of $\{X_0, X_1 \cdots\}$

For $i \neq j$, $P_{ij} = Pr(X_n = j \mid X_{n-1} = i)$

(a) Probability of proposing to move from i to j (T_{ij})

(b) Probability accepting move $(a(i,j))$

$$P_{ij} = Pr(\text{probability } j)\, Pr(\text{accepting proposal } j)$$
$$= T_{ij}\, a(i,j)$$
$$= T_{ij} \min\{1, \frac{\pi_j}{\pi_i} \frac{T_{ji}}{T_{ij}}\}$$
$$= \min\{T_{ij}, \frac{\pi_j}{\pi_i} T_{ji}\}$$

So, $\pi_i P_{ij} = \min\{\pi_i T_{ij}, \pi_j \pi_{ji}\}$

$\pi_j P_{ji} = \min\{\pi_j T_{ji}, \pi_i T_{ij}\}$

$\Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

$\Rightarrow P$ is time-reversible

$\Rightarrow \pi$ is stationary $\Rightarrow$ limiting.

For $i = j$, $l_{ii} = 1 - \sum_{j \neq i} P_{ij}$

Some R

- sample (what , probability , size , replacement=T)