Example : (Page 21, lecture notes)

Let $X_{(n)} = \max(X_1, X_2, \dots, X_n)$.

$$P(X_{(n)} \le x) = P(X_1 \le x, \dots, X_n \le x) \underset{\substack{\text{are} \\ \text{iid}}}{\overset{X_i\text{'s}}{=\!=\!=}} [P(X_1 \le x)]^n$$

$$= \left(\frac{x}{\theta}\right)^n$$

$$\Rightarrow f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} & ; \ 0 < x < \theta \\ 0 & ; \ \text{elsewhere} \end{cases}$$

$$\Rightarrow E\{X_{(n)}\} = \int_0^\theta \frac{xn}{\theta}\left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{n+1} \cdot \theta$$

$$\Rightarrow E\{\hat{\theta}_n\} = E\left\{\frac{n+1}{n} \cdot X_{(n)}\right\} = \theta \Rightarrow \hat{\theta}_n \text{ is an unbiased estimator of } \theta.$$

Since $\hat{\theta}_n$ is "unbiased", we try to use theorem 9.1 on page 20 of the notes.

The next step is to find $Var(\hat{\theta}_n)$.

Note that $E\{X_{(n)}^2\} = \frac{n}{n+1} \cdot \theta^2$. thus,

$$Var(\hat{\theta}_n) = \left(\frac{n+1}{n}\right)^2 Var(X_{(n)}) = \left(\frac{n+1}{n}\right)^2 \left\{\frac{n}{n+1}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2\right\},$$

$$\Rightarrow Var(\hat{\theta}_n) = \frac{n\theta^2}{n(n+2)}$$

since $\lim_{n\to\infty} Var(\hat{\theta}_n) = 0$ $\xRightarrow[\text{thm } 9.1]{\text{By}}$ $\hat{\theta}_n$ is a consistent estimator of θ.

$\boxed{}$

Example: (Page 22, lecture notes)

Note that: since X_i's are iid
$$Var(\hat{\sigma}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(|X_i|) = \frac{Var(|X_1|)}{n}$$

we can see that $\lim_{n\to\infty} Var(\hat{\sigma}_n) = 0$.

③

It is tempting to conclude that $\hat{\sigma}_n$ is consistent. However, the above conclusion is not correct. the reason is because $\hat{\sigma}_n$ is a "biased" estimator of σ:

$$E(\hat{\sigma}_n) = E\left\{\frac{1}{n}\sum_{i=1}^{n}|X_i|\right\} \underset{\text{are iid}}{\overset{X_i\text{'s}}{=\!=}} E(|X_1|)$$

$$= \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\Rightarrow \underbrace{E(\hat{\sigma}_n) = \sqrt{\frac{2}{\pi}}\,\sigma}$$ $\Rightarrow$ $\hat{\sigma}_n$ is a biased estimator of σ.

In fact, as $n \to \infty$,

$$\hat{\sigma}_n \xrightarrow{P} \sqrt{\frac{2}{\pi}}\,\sigma$$

Example : (Page 21, lecture notes) ... continued.

Consider $\tilde{\theta}_n = 2\bar{X}_n$

$$E(\tilde{\theta}_n) = 2E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = 2E(X_1)$$

$$= 2 \times \frac{0+\theta}{2} = \theta \Rightarrow \underbrace{E(\tilde{\theta}_n) = \theta}$$

$\Rightarrow \tilde{\theta}_n$ is an unbiased estimator of θ.

Also, $\quad Var(\tilde{\theta}_n) = 4\,Var(\bar{X}_n) = \dfrac{4\,Var(X_1)}{n}$

$$= \frac{4}{n} \times \frac{(\theta - 0)^2}{12}$$

$\Rightarrow Var(\tilde{\theta}_n) = \dfrac{\theta^2}{3n}$

And, $\lim\limits_{n\to\infty} Var(\tilde{\theta}_n) = 0 \quad (*)$. Since $\tilde{\theta}_n$ is an unbiased estimator of θ, $(*)$ implies that $\tilde{\theta}_n$ is a consistent estimator of θ.

$\therefore$ In this example, both $\tilde{\theta}_n$ & $\hat{\theta}_n$ are unbiased estimators of θ, and are also consistent estimators of θ.

Let us now compare them using efficiency:

$$\text{eff}(\hat{\theta}_n, \tilde{\theta}_n) = \frac{\theta^2/3n}{\theta^2/n(n+2)} = \frac{n+2}{3} \geq 1$$

$$\forall\, n \geq 1$$

this means that $\hat{\theta}_n = \frac{n+1}{n} \cdot X_{(n)}$ is more efficient than $\tilde{\theta}_n = 2\bar{X}_n$ in estimating θ.

Example: (Page 24, lecture notes)

It is easy to show that,

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n) \xrightarrow{P} \sigma^2$$

as $n \to \infty$,

i.e. the sample variance S_n^2 is a consistent estimator of σ^2. Using the result on Page 23 of the notes,

$$S_n^2 \xrightarrow{P} \sigma^2 \quad \Longrightarrow \quad \sqrt{S_n^2} = S_n \xrightarrow{P} \sqrt{\sigma^2} = \sigma$$

as $n \to \infty$ as $n \to \infty$

$\Rightarrow$ S_n is a consistent estimator of σ.