Chapter 1 · Intro / The Real Numbers

Some sample proof :

① Prove $x \to x^2$ is increasing on $[0,\infty) = \{x \in \mathbb{R} : x \geq 0\} = \mathbb{R}_0^+$

i.e. Prove that $\forall\ 0 \leq x \leq y : x^2 \leq y^2$

<Proof 1> $y^2 - x^2 = \underbrace{(y-x)}_{\geq 0}\underbrace{(y+x)}_{\geq 0} \geq 0 \Rightarrow y^2 \geq x^2 \Rightarrow x^2 \leq y^2$ ■ Q.E.D.

Direct proof: Start with a true statement and we reach the statement we want to prove via a finite sequence of implecations.

<Proof 2> Let $0 \leq x \leq y \Rightarrow 0 \leq x^2 \leq xy$ and ① $0 \leq xy \leq y^2 \Rightarrow x^2 \leq xy \leq y^2$

$\Rightarrow x^2 \leq y^2$ ■

Exercise : strict increasing on $\mathbb{R}_0^+$ i.e. for $0 \leq x < y \Rightarrow x^2 < y^2$.

② Prove $0 \leq x \leq y \Rightarrow \sqrt{x} \leq \sqrt{y}$

<Proof 1> Case 1: $0 \leq x < y$

$$\sqrt{y} - \sqrt{x} = \frac{(\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x})}{\sqrt{y}+\sqrt{x}} = \frac{y-x}{\sqrt{y}+\sqrt{x}} \begin{matrix} >0 \\ >0 \end{matrix} > 0$$

$\Rightarrow$ check $\neq 0$

$\Rightarrow \sqrt{y} - \sqrt{x} > 0 \Rightarrow \sqrt{x} \leq \sqrt{y}$

Case 2: $0 = x = y \Rightarrow \sqrt{x} = 0 \leq 0 = \sqrt{y}$

In all cases, we thus have that $\sqrt{x} \leq \sqrt{y}$ ■

<Proof 2> Assume that $0 \leq x \leq y$, but that $\sqrt{x}$ is NOT less than or equal to $\sqrt{y}$ i.e. $\sqrt{x} > \sqrt{y}$

$\Rightarrow 0 \leq \sqrt{y} < \sqrt{x} \underset{①}{\Rightarrow} (\sqrt{y})^2 = y < x = (\sqrt{x})^2$

$\Rightarrow y < x$ ↯ contradiction Our assumption was wrong which proves that $\sqrt{x} \leq \sqrt{y}$ ■

Indirect proof

③ Prove: If $x, y \geq 0$, then $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$

False! $\boxed{\sqrt{x+y} \leq \sqrt{x}+\sqrt{y} \overset{①}{\Rightarrow} (\sqrt{x+y})^2 \leq (\sqrt{x}+\sqrt{y})^2 \Rightarrow x+y \leq x+y+2\sqrt{xy} \Rightarrow 0 \leq 2\sqrt{xy}}$

Instead, $0 \leq 2\sqrt{xy} \Rightarrow x+y \leq x+y+2\sqrt{xy} \overset{②}{\Rightarrow} \sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$ ∎

OR, use $\Leftrightarrow$

④ Prove $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational. Thus $\exists\ a, b \in \mathbb{N}$ s.t. $GCD(a,b)=1$ and $\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2}b = a \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ even $\Rightarrow a$ even $\Rightarrow \exists\ c \in \mathbb{N}$ s.t. $a = 2c \Rightarrow 2b^2 = a^2 = 4c^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2$ even $\Rightarrow b$ even
$\Rightarrow$ both a and b are even $\Rightarrow GCD(a,b) \neq 1$ ↯
Thus our assumption that $\sqrt{2}$ is rational is wrong
$\Rightarrow \sqrt{2}$ is irrational.

Mathematical Induction

The Principle of Mathematical Induction

Let $P(n)$ be a statement defined for all $n \in \mathbb{N}$. If

- $P(1)$ is true
- $\forall n \in \mathbb{N}: P(n) \to P(n+1)$ i.e. If P holds for an $n \in \mathbb{N}$, then P also holds for $n+1$

Then $P(n)$ is true for all $n \in \mathbb{N}$

① $\forall n \in \mathbb{N}: \sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}$

<u>$n=1$</u> (base case) $1 = \frac{1 \cdot 2}{2} = 1$ ✓

<u>$n \to n+1$</u> (inductive step)

Assume that $1+2+\cdots+n = \frac{n(n+1)}{2}$ for some $n \in \mathbb{N}$

Then $1+2+\cdots+n+(n+1) = \underbrace{[1+2+\cdots+n]}_{\frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+2)(n+1)}{2}$
$= \frac{(n+1)[(n+1)+1]}{2}$ ✓

$\Rightarrow 1+\cdots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Exercise ① $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

② $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

② Bernoulli's Inequality $\forall x \geq -1,\ \forall n \in \mathbb{N}: (1+x)^n \geq 1+n\cdot x$

$n=1$ $(1+x)^1 = 1+x$ ✓

$n \Rightarrow n+1$ Assume that $(1+x)^n \geq 1+nx$ for some $n \in \mathbb{N}$

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$$
$$\Rightarrow (1+x)^{n+1} \geq 1+(n+1)x \quad ✓$$

Thus, $(1+x)^n \geq 1+nx$ for $\forall x \geq -1$ and $\forall n \in \mathbb{N}$

Corollary $2^n \geq n+1$

③ $n^2 < 2^n$ for $\forall n \geq 5$

$n=5$ $5^2 = 25 < 32 = 2^5$ ✓

$n \Rightarrow n+1$ <lemma> $\forall n \geq 3,\ (n+1)^2 < 2n^2$

$\Leftrightarrow n^2+2n+1 < 2n^2$

$\Leftrightarrow n^2-2n-1 > 0$

$\Leftrightarrow (n-1)^2 > 2$ true whenever $n-1 \geq 2 \Leftrightarrow n \geq 3$

Thus, $(n+1)^2 < 2n^2 < 2\cdot 2^n = 2^{n+1}$ ✓

Thus $n^2 < 2^n$ for $\forall n \geq 5$

④ The Well-Ordering Property of $\mathbb{N}$

Every non-empty subset of S of $\mathbb{N}$ has a least element

i.e. $\exists\ s \in S\ \forall t \in S:\ s \leq t$

Note : The statement is wrong for

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3 \cdots\}$
- $\mathbb{Q} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\} = \{\frac{1}{n} : n \in \mathbb{N}\}$

proof

① Case 1: S is finite

Let $n = |S|$ number of elements, or cardinality of S

We will use induction on n

- $n=1$: Let $|S|=1$ i.e. $S = \{a_1\}$ for some $a_1 \in \mathbb{N}$, then a_1 is the least element of S.
- $n \to n+1$: Assume that every subset of $\mathbb{N}$ with n elements has a least element. Let S have $n+1$ elements, say $\{a_1, \cdots a_n, a_{n+1}\}$
  Let $S' = \{a_1, \cdots a_n\}$. By the induction hypothesis, S' has a least element, say a_j. i.e. $\forall\ 1 \le k \le n$: $a_j \le a_k$
  If $a_j < a_{n+1} \Rightarrow a_j$ is the least element of S
  If $a_{n+1} < a_j \Rightarrow a_{n+1}$ is the least ...

  This completes the induction step. Thus every finite subset of $\mathbb{N}$ has a least element.

② Case 2: S is infinite

Especially, $S \neq \emptyset$. Let $t_0 \in S$ and $S' = S \cap \{1, 2, 3, \cdots t_0\}$

Let $t \in S \setminus S'$ $\Rightarrow t > t_0$ ①

Now consider S'. S' is finite and thus has a least element s by case 1

$\Rightarrow s \le t$ for $\forall\ t \in S'$ ②

If $t \in S \setminus S'$, then $s \le \underbrace{t_0}_{\in S'} < t$

$\Rightarrow s \le t$ for all $t \in S$

Thus s is the least element of S ☒

Elementary Logic

A statement is any expression which is either true or false

P AND Q
(iff BOTH true)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P OR Q
(iff at least one true)

P	Q	$P \vee Q$
T	T	T
T	F	F
F	T	F
F	F	F

P implies Q
(P true then Q true)

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

If P is false, $P \Rightarrow Q$ always true.

Negation: $\neg P$ "NOT P"

Quantifiers & Negation

$\neg(\forall x: P(x)) = \exists x: \neg P(x)$

$\neg(\exists x: P(x)) = \forall x: \neg P(x)$

Set Theory

- $N = \{1, 2, 3 \cdots\}$ $\quad N_0 = \{0, 1, 2, 3 \cdots\}$
- $Z = \{\cdots -2, -1, 0, 1, 2 \cdots\}$ integer
- $Q = \{\frac{a}{b} : a \in Z, b \in N\}$ rational
- R real number
- $R \setminus Q$ irrational
- C complex number

- $\in$ is an element of
- $\subseteq$ subset $\quad \supseteq$ superset
- $\cup$ union $\quad \cap$ intersection

Ex. $\bigcup_{i=1}^{\infty} \{-i, i\} = Z \setminus \{0\}$

$\bigcup_{x \in R^+} \{-x, x\} = R \setminus \{0\}$

index set $\quad \bigcup_{i \in I} A_i = \{x : \exists i \in I : x \in A_i\}$

De Morgan's Law

① $A \setminus \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A \setminus B_i)$

② $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A \setminus B_i)$

Functions

DEF Let D, E be sets. A function $f: D \to E$ is a rule that assigns to each element in D a uniquely determined element in E.
If $x \in D$ and $y \in E$ is the element assigned to x by f, we write $y = f(x)$.
D is called the domain of f and E is called the co-domain of f.
The set $f(D) := \{f(x) : x \in D\}$ is called the range of f.
By construction, $f(D) \subseteq E$. But $f(D)$ may NOT equal E in general.

DEF A function $f: D \to E$ is called

(a) Injective (One-to-one) if
$\forall x, y \in D, x \neq y : f(x) \neq f(y)$ or equivalently
$\forall x, y \in D : f(x) = f(y) \Leftrightarrow x = y$

(b) Subjective (Onto) if
$f(D) = E$ i.e. every element in E is a value under f.

(c) Bijective if
f is both injective and subjective

DEF Bijective functions are invertible
There exists an inverse function $f^{-1}: E \to D$

Images and Preimages of Sets

DEF Let $A \subseteq D$, then $f(A) = \{f(x) : x \in A\}$ is called the image of A under f.
Similarly, let $B \subseteq E$. Then $f^{-1}(B) = \{x \in D : f(x) \in B\}$ is called the preimage of B under f.

well-defined even if f NOT invertible

Ex. $f: \mathbb{R} \to \mathbb{R}$ $x \to x^2$ Note that f NOT invertible!
$f(\{2\}) = \{4\}$; $f(\{-1, 2\}) = \{1, 4\}$
$f^{-1}(\{4\}) = \{-2, 2\}$; $f^{-1}(\{0, 4\}) = \{0, -2, 2\}$

What if f invertible?
$f: \mathbb{R} \to \mathbb{R}$ $x \to x^3$ $\Rightarrow$ invertible $f^{-1}(y) = \sqrt[3]{y}$

$f^{-1}(27) = 3$ | $f^{-1}(\{27\}) = \{3\}$

inverse function, number | premage, sets

THM Let $f: D \to E$ be a function

(a) Let $A, B \subseteq E$, then

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

(b) Let $A, B \subseteq D$, then

$$f(A \cup B) = f(A) \cup f(B)$$
$$f(A \cap B) \subseteq f(A) \cap f(B)$$

<u>proof</u> ① We will show that $LHS \subseteq RHS$ and $LHS \supseteq RHS \Rightarrow LHS = RHS$

$\subseteq$ Let $x \in f^{-1}(A \cup B)$ be arbitrary.

$\Rightarrow f(x) \in A \cup B \Rightarrow f(x) \in A \ \vee \ f(x) \in B$

$\Rightarrow x \in f^{-1}(A) \ \vee \ x \in f^{-1}(B)$

$\Rightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

$\supseteq$ Let $x \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow x \in f^{-1}(A) \ \vee \ x \in f^{-1}(B)$

$\Rightarrow f(x) \in A \ \vee \ f(x) \in B$

$\Rightarrow f(x) \in A \cup B$

$\Rightarrow x \in f^{-1}(A \cup B)$

$\Rightarrow f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

② Similar but more elegant

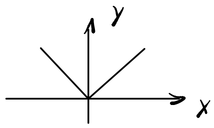
$x \in f^{-1}(A \cup B) \Leftrightarrow f(x) \in A \cup B \Leftrightarrow f(x) \in A \ \vee \ f(x) \in B$

$\Leftrightarrow x \in f^{-1}(A) \ \vee \ x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

Thus, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Ex. Some examples on functions

① The absolute function $|\ |: \mathbb{R} \to \mathbb{R} \quad x \to |x|$



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\Rightarrow$ **Triangle Inequality**

$\forall x, y \in \mathbb{R}: |x+y| \leq |x| + |y| \quad \Rightarrow \begin{cases} |x-y| \geq |x| - |y| \\ |x-y| \geq |y| - |x| \end{cases}$

② $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

f is discontinuous at All points in its domain.

1.3 Completeness

DEF Let $\phi \neq S \subseteq \mathbb{R}$. We say that

(a) S is bounded from above. If $\exists\, u \in \mathbb{R}$ s.t. $\forall x \in S: x \leq u$. Any such u is called the upper bound of S.

(b) S is bounded from below. If $\exists\, v \in \mathbb{R}$ s.t. $\forall x \in S: v \leq x$. Any such v is called a lower bound of S.

(c) S is bounded. If S is both bounded from above and below.

Ex. $S = (-\infty, 0)$. S is NOT bounded from below but bounded from above.
e.g. 0 (and any real number ≥ 0) is an upper bound.

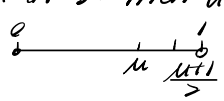
$S = [a, b]$. S is bounded.

DEF Let $\phi \neq S \subseteq \mathbb{R}$. We say that

(a) u is a maximum or greatest element of S if u is an upper bound of S and $u \in S$. We write $u = \max S$

(b) v is a minimum or least element of S if v is a lower bound of S and $v \in S$.

Ex. $S = [0, 1) \Rightarrow \min S = 0$; but S doesn't have a maximum.

<u>proof</u> Assume that u is a max of S. Then $u \in S \Rightarrow 0 \leq u < 1$ and u is an upper bound of S. Consider 

, then $u < \frac{u+1}{2} < 1$

$\Rightarrow \frac{u+1}{2} \in S$ but $u < \frac{u+1}{2}$

$\Rightarrow u$ is NOT an upper bound of S ↯

$\Rightarrow S$ doesn't have a maximum.

DEF Let $\phi \neq S \subseteq R$ ...

(a) be bounded from above. We say that $s \in R$ is the supremum or least upper bound of S, if

(i) s is an upper bound of S

(ii) $s \leq u$ for all upper bounds u of S.

(b) be bounded from below. We say that $t \in R$ is the infimum or greatest lower bound of S, if

(i) t is a lower bound of S

(ii) $v \leq t$ for all lower bounds v of S.

Ex. ① $S = [0,1)$

inf S — We'll show that $\inf S = 0$

0 is indeed the lower bound of S. Thus (i) holds.

Let t be any lower bound of S, then t is less than any elements of S. Especially $t \leq 0$. Thus (ii) holds.

sup S — We'll show that $\sup S = 1$.

1 is indeed an upper bound of S. Thus (i) holds.

Let u be any upper bound of S, then $u \geq 0$. Assume that $u \in S$ i.e. $0 \leq u < 1$. Then $0 \leq u < 1 \Rightarrow 0 \leq u < \frac{u+1}{2} < 1$

$\Rightarrow \frac{u+1}{2} \in S$ and $u < \frac{u+1}{2}$ $\lightning$

$\Rightarrow u \geq 1$. Thus (ii) holds.

THM Let $\phi \neq S \subseteq R$

(a) If S has a maximum s, then s is also the supremum of S.

(b) If S has a minimum t, then t is also the infimum of S.

Proof (a) Let $s = \max S \Rightarrow s$ is an upper bound of $S \Rightarrow$ (i) holds.

Let u be any upper bound of S, then $\forall x \in S: x \leq u$

Especially, $s \leq u \Rightarrow$ (ii) holds

Completion R is defined as the completion of Q i.e. as the smallest set of numbers that contains Q; But, unlike Q, contains a supremum for any non-empty subset of R which is bounded from above.

Axiom of Completeness Let $\phi \neq S \subseteq R$, S is bounded from above. Then S has a supremum in R.

Remark If $S \subseteq R$, $S \neq \phi$, S is bounded from below. Then $-S = \{-x : x \in S\}$ is bounded from above and thus has a supremum $\Rightarrow \inf S = -\sup(-S)$

Consequently, every $\phi \neq S \subseteq R$ is bounded from below has an infimum in R.

1.4 Consequence of Completeness

Archimedean Property of $\mathbb{R}$

Let $x \in \mathbb{R}$ be arbitrary, then $\exists n \in \mathbb{N}$ s.t. $n > x$.

Proof Assume $\exists x \in \mathbb{R} : \forall n \in \mathbb{N}$ s.t. $n \leq x$. Then let $S = \{x \in \mathbb{R} : \forall n \in \mathbb{N} : n \leq x\}$

S is non-empty by our assumption. Furthermore, S is bounded from below e.g. 0 is a lower bound of S since $\forall n \in \mathbb{N} : n > x$ if $x \leq 0$.

By completeness of $\mathbb{R}$, S has an infimum.

Let $t = \inf S \Rightarrow t+1$ is NOT a lower bound for S.

$\Rightarrow \exists x \in S : x < t+1$

$\Rightarrow x - 1 < t$ since t is a lower bound, $x-1 \notin S$.

$\Rightarrow \exists n \in \mathbb{N} : n > x-1 \Rightarrow \underbrace{n+1}_{\in \mathbb{N}} > x \Rightarrow x \notin S$ ↯

Remark $\Rightarrow$ $\mathbb{R}$ doesn't contain any infinite number.

Corollary Let $0 < x$ Then $\exists n \in \mathbb{N} : \frac{1}{n} < x$.

$\exists n \in \mathbb{N} : n > \frac{1}{x} \Rightarrow \frac{1}{n} < x$.

Density of $\mathbb{Q}$ in $\mathbb{R}$

THM Any interval (a,b), $a, b \in \mathbb{R}$, $a < b$, contains at least one rational number. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof Case 1 $0 \leq a < b$

idea of the proof If we pick n s.t. $\frac{1}{n} < b-a$, then at least one of the rational numbers $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots$ should be in (a,b), since the distance between consecutive points in $\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n} \dots\}$ is $\frac{1}{n}$ which is less than the width of (a,b).

By Archimedean property, $\exists n \in \mathbb{N} : n > \frac{1}{b-a} \Leftrightarrow \frac{1}{n} < b-a$. Consider the set $\{\frac{k}{n} : k \in \mathbb{N}_0\} \subseteq \mathbb{Q}$. This set is unbounded; Thus $\exists k_0 \in \mathbb{N}$ s.t. $\frac{k_0}{n} > a$

The set $S = \{k \in \mathbb{N}_0 : \frac{k}{n} > a\}$ is thus non-empty. We'll show that S has a least element: If $0 \in S$ that 0 is the least element of S. If $0 \notin S$ then $S \subseteq \mathbb{N}$ and therefore has a least element. In any case, S has a least element k_1.

We'll show that $\frac{k_1}{n} \in (a,b)$. By construction of S, we have that $\frac{k_1}{n} > a$. Since k_1 is minimal with this property, it also follows that $\frac{k_1 - 1}{n} \leq a$.

$\Rightarrow \frac{k_1}{n} \leq a + \frac{1}{n} < a + (b-a) = b$

$\Rightarrow \frac{k_1}{n} < b \Rightarrow \underset{\in \mathbb{Q}}{\frac{k_1}{n}} \in (a,b)$

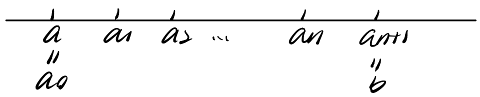
Case 2: $a < b \le 0$

$\Rightarrow 0 \le -b < -a$. By case 1 $\exists r \in \mathbb{Q}$ s.t. $-b < r < -a \Rightarrow a < \underbrace{-r}_{\in \mathbb{Q}} < b$

Case 3: $a < \underbrace{0}_{\in \mathbb{Q}} < b$

Corollary Any interval (a,b) $a,b \in \mathbb{R}$ $a<b$ contains infinitely many rationals.

Assume that (a,b), $a<b$ contains only finitely many, say n, rational numbers. Divide (a,b) into $n+1$ subintervals of equal width i.e.
Let $a_i = a + \frac{b-a}{n+1} i$, $0 \le i \le n+1$.



By density theorem, each of the subintervals (a_i, a_{i+1}), $0 \le i \le n$ contains at least one rational number r_i.

Note that the subintervals above are <u>pairwise disjoint</u> i.e. Each two of them have empty intersection. Thus the r_i are pairwise distinct i.e. No two of them are the same. Thus (a,b) contains at least $n+1$ rational numbers $r_0, r_1 \cdots r_n$ ↯

Thus (a,b) has to contain infinitely many rational numbers.

Ex. Revisit completeness of $\mathbb{R}$

(a) $S = \{x \in \mathbb{R} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \Rightarrow S$ is bounded; $\sup(S) = \sqrt{2}$

(b) $S = \{x \in \mathbb{Q} : x^2 < 2\} \Rightarrow$ NO supremum.

Assume that $s = \sup(S)$, $s \in \mathbb{Q}$

① Case 1: $s^2 > 2$. Then $\exists r \in \mathbb{Q} : \sqrt{2} < r < s$ by density of $\mathbb{Q}$ in $\mathbb{R}$
$\Rightarrow r^2 > 2$, $r < s \Rightarrow r$ is an upper bound of S which is smaller than s ↯

② Case 2: $s^2 < 2$. Then $\exists r \in \mathbb{Q} : 0 < s < r < \sqrt{2}$
$\Rightarrow s^2 < r^2 < 2 \Rightarrow r \in S$ and $s < r$ i.e. s is not an upper bound of S ↯

③ Case 3: $s^2 = 2$ impossible since $\sqrt{2}$ irrational.

$\Rightarrow S$ doesn't have a supremum in $\mathbb{Q}$.

This reflects that $\mathbb{Q}$ is incomplete. The "missing" supremum is NOT rational; $\mathbb{Q}$ has a "hole" where the sup should be. By adding ALL "missing" superema of bounded sets of rational numbers as irrational numbers, we obtain the completion of $\mathbb{Q}$, called $\mathbb{R}$.

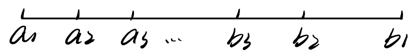
DEF Let $S_1, S_2, S_3 \cdots$ be sets. We say that these sets are **nested** if $S_1 \supseteq S_2 \supseteq \cdots$

THM Nested Interval Property on $\mathbb{R}$

Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be nested closed and bounded intervals. Then all of these intervals have at least one point in common i.e. $\bigcap_{j \in \mathbb{N}} I_j \neq \phi$.

Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \cdots$

Since $I_1 \supseteq I_2 \supseteq \cdots$ We have that $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \geq b_2 \geq b_3 \geq \cdots$



i.e. The a's are increasing and the b's are decreasing. Furthermore, $\forall n \in \mathbb{N}: a_n \leq b_n$

lemma $\forall n, k \in \mathbb{N}: a_n \leq b_k$

① case 1: $n \leq k$ Then $a_n \leq a_k \leq b_k \Rightarrow a_n \leq b_k$

② case 2: $n > k$ Then $a_n \leq b_n \leq b_k \Rightarrow a_n \leq b_k$

$\Rightarrow \forall n, k \in \mathbb{N}: a_n \leq b_k$

Especially, $\forall n \in \mathbb{N}: a_n \leq b_1$. Let $S = \{a_1, a_2, a_3 \cdots\}$, then S is bounded from above (by b_1), and thus has a sup. Let $a = \sup\{a_1, a_2 \cdots\}$

Similarly, $\forall k \in \mathbb{N}: a_1 \leq b_k$. Let $S' = \{b_1, b_2 \cdots\}$, then S' is bounded from below (by a_1), and thus has a inf. Let $b = \inf\{b_1, b_2 \cdots\}$

lemma 2 $\forall k \in \mathbb{N}: a \leq b_k$

Fix a $k \in \mathbb{N}$. By lemma 1, $\forall n \in \mathbb{N}: a_n \leq b_k \Rightarrow b_k$ is an upper bound of $S = \{a_1, a_2 \cdots\}$. Since a is the least upper bound of S, we must have that $\forall k \in \mathbb{N}: a \leq b_k$.

lemma 3 $a \leq b$

By lemma 2, $\forall k \in \mathbb{N}: a \leq b_k \Rightarrow a$ is a lower bound of $S' = \{b_1, b_2 \cdots\}$ Since b is the largest lower bound of S', we must have that $a \leq b$.

We have now shown that $a_1 \leq a_2 \cdots \leq a \leq b \leq \cdots \leq b_2 \leq b_1$.

$\Rightarrow [a,b] \subseteq I_1 = (a_1, b_1)$, $[a,b] \subseteq I_2 = (a_2, b_2) \cdots$

$\Rightarrow \forall n \in \mathbb{N}: [a,b] \subseteq I_n$

$\Rightarrow \underbrace{[a,b]}_{\neq \phi} \subseteq \bigcap_{j \in \mathbb{N}} I_j$

$\Rightarrow \bigcap_{j \in \mathbb{N}} I_j \neq \phi$

Remark ① $[a,b] = \bigcap_{j \in N} I_j$ Note: It's possible that $a=b$ i.e. $\bigcap_{j\in N} I_j$ single point

② If I_j is NOT all closed and bounded, $\bigcap_{j \in N} I_j$ might be empty

e.g. $I_n = (0, \frac{1}{n}]$ $I_1 \supseteq I_2 \supseteq \cdots$ nested but $\bigcap_{n \in N} I_n = \phi$.

Assume $x \in \bigcap_{n\in N} I_n$. Then $x > 0$. By Archimedean property,

$\exists n \in N : n > \frac{1}{x} \Leftrightarrow \frac{1}{n} < x \Rightarrow x \notin (0, \frac{1}{n}) \Rightarrow x \notin \bigcap_{n \in N} I_n$ ↯

e.g. $\bigcap_{n \in N} [0, \frac{1}{n}] = \{0\}$

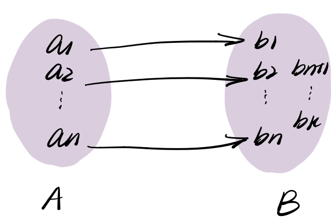
1.5 Cardinality

DEF Let S be a finite set. The cardinality of S, in symbol $|S|$, is defined to be the number of elements of S.

Cardinality and Function

Let A, B be finite sets, $|A| = n$, $|B| = k$.

- If $|A| < |B|$ i.e. $n < k$. Then there exists an injective map from A to B.



$f: \forall\, 1 \le j \le n : f(a_j) = b_j$

However, there doesn't exist a subjective map $f: A \to B$

bc. $|f(A)| \le |A| = n < k \Rightarrow f(A) \subsetneq B$.

- Conversely, if $|A| > |B|$, then $\nexists$ injective map $A \to B$, but $\exists$ subjective.
- $|A| = |B| \Rightarrow \exists$ bijective $A \to B$

DEF Let A, B be sets (finite or infinite)

- We say that A and B have the same cardinality, $|A| = |B|$ or $A \sim B$ if there $\exists$ a bijective map $f: A \to B$
- We say that $|A| < |B|$ if $\nexists$ subjective map $f: A \to B$.

Ex. $\mathbb{N}_0 = \{0, 1, 2, 3 \dots\} \supsetneq \mathbb{N} = \{1, 2, 3 \dots\}$
• true superset of $\mathbb{N}$

But $|\mathbb{N}_0| = |\mathbb{N}|$. Note that this cannot occur in the case of finite sets

Let $f: \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto n+1$

• injective: $f(n) = f(k) \Rightarrow n+1 = k+1 \Rightarrow n = k$

• subjective: let $n \in \mathbb{N}$, then $n-1 \in \mathbb{N}_0$ and $f(n-1) = n$

$\Rightarrow$ bijective

$\Rightarrow |\mathbb{N}_0| = |\mathbb{N}|$

Ex. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$ $\quad |\mathbb{Z}| = |\mathbb{N}|$

Consider the following enumeration on $\mathbb{Z}$:

$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \dots$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \dots$

$$f: \mathbb{N} \to \mathbb{Z}: n \mapsto a_n \quad (n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

DEF • A set S is called **Countably Infinite** if $|S| = |\mathbb{N}|$

• **Countable** if S is either finite or countbly infinite

• **uncountable** if it's not countable

remark The name countable comes from the fact that the elements of such a set can be enumerated or counted.
i.e. can be written in an ordered list $a_1, \dots a_n$ in the finite case or $a_1, a_2 \dots$ in the infinite case.

THM Let $A \subseteq \mathbb{N}$. Then A is countable i.e. A is finite or countably infinite

If A is finite, we're done.
Assume that A is infinite. Recall that every non-empty subset on $\mathbb{N}$ has a least number, by the well-ordering property of $\mathbb{N}$.
Define $a_1 = \min A \quad a_2 = \min A \setminus \{a_1\} \quad \dots \quad \dots$
By construction, $a_2 \neq a_1$, $a_3 \neq a_1, a_2 \dots$ i.e. They're pairwise distinct
Now let $k \in A$ be arbitrary. A can contain at most k elements $\leq k \Rightarrow$ One of the numbers $a_1, \dots a_k$ needs to equal k. $\Rightarrow a_1, a_2 \dots$ is an ordered list, containing all elements of A exactly once i.e. obtained an enumeration of A.
Thus if we define $f: \mathbb{N} \to A$ by $\forall n \in \mathbb{N}: f(n) = a_n$, then f is bijective $\Rightarrow |A| = |\mathbb{N}|$

THM Let $f: N \to A$ be surjective, then A is countable

For each $a \in A$, let $n_a = \min \underbrace{f^{-1}(\{a\})}_{\neq \varnothing \text{ since } f \text{ surjective}}$ This min exists by well-ordering property of N.

Consider $S = \{n_a : a \in A\}$, then $S \subseteq N$.

Consider $\underbrace{f|_S}_{f \text{ restricted to } S.} : S \subseteq N \to A$. We'll show that $f|_S : S \to A$ is bijective

<u>injective</u> if $f(n_a) = f(n_b) \Rightarrow a = b \Rightarrow n_a = n_b$ ✓

<u>surjective</u> let $a \in A$. Then $n_a \in S$ and $f(n_a) = a$

Thus, $f|_S : S \to A$ is bijective $\Rightarrow |S| = |A|$

Since $S \subseteq N$, it's countable. Since $|S| = |A|$, the same hold for A.

THM Q is countably infinite

THM R is uncountable