1 "Examples on the Method of Moments"

Example 6:

Let $X_1, X_2, \dots, X_n \overset{i.i.d}{\sim} \text{Gamma}(\alpha, \beta)$ such that $\alpha, \beta > 0$ and both unknown.

Find the moment estimators of α & β.

Solution: we know that

$$\begin{cases} E(X) = \alpha\beta, \\ E(X^2) = \text{Var}(X) + E^2(X) = \alpha\beta^2 + (\alpha\beta)^2, \end{cases}$$

we now match the above moment with the corresponding moments of the sample,

$$\begin{cases} m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{\checkmark}{=} \alpha\beta \\ m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \overset{\checkmark}{=} \alpha\beta^2 + \alpha^2\beta^2 \end{cases}$$

Solving w.r.t. α & β

(2)

The moment estimators of α & β are:

$$\begin{cases} \hat{\alpha}_n = \dfrac{(\bar{X}_n)^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2} \\ \hat{\beta}_n = \dfrac{\bar{X}_n}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2} \end{cases}$$

Remarks: (1) Here, both $\hat{\alpha}_n$ & $\hat{\beta}_n$ are biased estimators. [Not easy to find their biases]

(2) It is easy to see that $T = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \ln(X_i)\right)$ is a suff. stat. for (α, β). But the $\hat{\alpha}_n$ & $\hat{\beta}_n$ given above are not functions of T!

(3) Using the WLLN & Slutsky's theorem, it can be seen that

$$\hat{\alpha}_n \xrightarrow{P} \alpha \quad \text{as } n \to \infty,$$
$$\hat{\beta}_n \xrightarrow{P} \beta$$

[i.e. they are both consistent estimators of α & β]

Example 7:

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} t(\nu)$, $\nu > 0$

(student's t-dist.)

with ν (unknown & positive) degrees of freedom.

Find the moment estimator of ν.

Solution: Note that $E(X) = 0$. Thus, we move up to the second moment of X.

$$E(X^2) = \frac{\nu}{\nu - 2} \xrightarrow[m_2]{\text{Set to}} m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\Rightarrow \hat{\nu}_n = \left[\frac{2}{n}\sum_{i=1}^{n} X_i^2\right] \Big/ \left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1\right]$$

Remarks: ① there is no guarantee that $\hat{\nu}_n > 0$.

② In addition, $\hat{\nu}_n$ makes sense only when $\nu > 2$.

as $\hat{\nu}_n$ is consistent only when $\nu > 4$.

this is undesirable because ν is unknown.