DEF A subset $U \subseteq \mathbb{R}$ is called **open** if $\forall x \in U \; \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq U$.

open interval   closed interval

① $\mathbb{R}$ is open

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. Then $V_\varepsilon(x) \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ is open.

② $\emptyset$ is open

The condition for openness is satisfied since there is no $x \in \emptyset$

THM Every open interval is open

THM (a) Arbitrary unions of open sets are open

i.e. If I is an arbitrary index set and $\forall i \in I : U_i \subseteq \mathbb{R}$ is open, then $U = \bigcup_{i \in I} U_i$ is open

(b) Finite intersections of open sets are open

i.e. If $U_1, \dots U_n \subseteq \mathbb{R}$ are open, then $\bigcap_{i=1}^{n} U_i$ is open.

Remark: Infinite intersections of open sets are, in general, not open.

e.g. Let $I = \mathbb{N}$, $\forall n \in \mathbb{N}$, $U_n = (-\frac{1}{n}, \frac{1}{n}) \Rightarrow U = \bigcap_{i \in \mathbb{N}} U_i = \{0\}$ not open since $\forall \varepsilon > 0$, $V_\varepsilon(0) \nsubseteq \{0\}$.

THM A subset of $\mathbb{R}$ is open iff it's a countable union of open intervals

(not used)

DEF A subset $A \subseteq \mathbb{R}$ is called **closed** if A^c is open

$\mathbb{R} \setminus A$

THM Every closed interval is closed

① $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ closed

$\forall x \notin A \;\; \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^c$ (to be proved)

Note that $\{\frac{1}{n} : n \in \mathbb{N}\}$ not closed since every ε-neighborhood about 0 intersects an inf. many points.

② both $\emptyset$ and $\mathbb{R}$ are closed

$\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open and $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open

THM ϕ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ which are both open and closed.
(not used)

THM (a) Finite unions of closed sets are closed.

(b) Arbitrary intersections of closed sets are closed.

<u>Remark</u> There're closed subsets of $\mathbb{R}$ which are not countable unions of closed intervals. e.g. Cantor set

DEF Let $A \subseteq \mathbb{R}$. We say that a seq (x_n) is in A if $\forall n \in \mathbb{N}: x_n \in A$

THM Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a conv. seq in A
Let $x = \lim(x_n) \Rightarrow x \in A$

DEF Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a boundary point of A if
$\forall \varepsilon > 0: V_\varepsilon(x) \cap A \neq \phi$ and $V_\varepsilon(x) \cap A^c \neq \phi$

The set of all boundary points of A is called the boundary of A and is denoted as ∂A

Ex. $I = [a, \infty) \Rightarrow \partial I = \{a\}$

Let $x > a$. We know that (a, ∞) is open
$x \in (a,\infty) \Rightarrow \exists \varepsilon > 0: V_\varepsilon(x) \subseteq (a, \infty) \subseteq I$

$\Rightarrow V_\varepsilon(x) \cap I^c = \phi \Rightarrow x \notin \partial I$

Now let $x < a$

Consider $(-\infty, a)$ which is open
$x \in (-\infty, a) \Rightarrow \exists \varepsilon > 0: V_\varepsilon(x) \subseteq (-\infty, a) \subseteq I^c$

$\Rightarrow V_\varepsilon(x) \cap I = \phi \Rightarrow x \notin \partial I$

Finally, $\forall \varepsilon > 0: V_\varepsilon(a) \cap I \neq \phi$ since $a + \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I$
Similarly, $a - \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I^c \Rightarrow V_\varepsilon(a) \cap I^c \neq \phi$

$\Rightarrow a \in \partial I \Rightarrow \partial I = \{a\}$

<u>exercise</u> $\partial[a,b] = \partial[a,b) = \partial(a,b] = \partial(a,b) = \{a,b\}$

THM Let $A \subseteq R$, then

(a) A is open iff A doesn't contain any of its boundary points
i.e. $A \cap \partial A = \phi \Rightarrow \partial A \subseteq A^c$

(b) A is closed iff A contain All of its boundary points
i.e. $\partial A \subseteq A$.

DEF A subset $A \subseteq R$ is called compact if A is both closed and open.

DEF A subset $A \subseteq R$ is called sequentially compact if for all seq (x_n) in A, it holds that (x_n) has a conv. subseq (x_{n_k}) whose limit lies in A

THM A subset $A \subseteq R$ is compact iff it's sequentially compact. ?