Chapter 3 ARMA

Definition: $\{X_t\}$ is an ARMA (p,q) process if $\{X_t\}$ is stationary and if for every t, $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q}$ where $\{Z_t\} \sim WN(0, \sigma^2)$ and $(1-\phi_1 z - \cdots \phi_p z^p)$ and $(1+\theta_1 z + \cdots \theta_q z^q)$ have no common factors.

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k}$$

For AR(p) process,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$
$$\Rightarrow X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t$$

$\{X_t\}$ is an ARMA (p,q) process with constant mean μ if $\{X_t - \mu\}$ is an ARMA (p,q) process with mean 0.

Shorthand notation

$$\Phi(B) X_t = \Theta(B) Z_t \quad \text{where} \quad \Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots \phi_p z^p$$
$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots \theta_q z^q$$

$$\Rightarrow \Phi(B) X_t = B^0 X_t - \phi_1 B^1 X_t - \phi_2 B^2 X_t \cdots \phi_p B^p X_t$$
$$= X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \cdots \phi_p X_{t-p}$$

$$\Theta(B) Z_t = B^0 Z_t + \theta_1 B^1 Z_t + \theta_2 B^2 Z_t + \cdots \theta_p B^q Z_t$$
$$= Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots \theta_q Z_{t-q}$$

If $\{X_t\}$ is an AR(p), then $\Theta(z) = 1$

If $\{X_t\}$ is an MA(q), then $\Phi(z) = 1$

If $\{X_t\}$ is a white noise, then $\Theta(z) = \Phi(z) = 1$

① Stationary

Take ARMA(1,1) $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$

So $\Phi(B) = 1-\phi B$ and $\Theta(B) = 1+\theta B$

If $|\phi|<1$, let $\chi(z)$ be the power series expansion of

$$\frac{1}{\Phi(z)} = \frac{1}{1-\phi z}$$

$\frac{1}{1-x} = 1+x+x^2+\cdots$ for $-1<x<1$

$$= (z-0)^0 \frac{1}{1-\phi z}\Big|_{z=0} + (z-0)^1 \frac{\phi}{(1-\phi z)^2}\Big|_{z=0} + \cdots\cdots$$

$$= \sum_{j=0}^{\infty} z^j \phi^j$$

$$= \chi(z)$$

$$\chi(B)\Phi(B)X_t = \sum_{j=0}^{\infty}\phi^j B^j [1-\phi B] X_t = \sum_{j=0}^{\infty}[\phi^j B^j - \phi^{j+1}B^{j+1}] X_t$$

$$= X_t - \phi B X_t + \phi B X_t - \phi^2 B^2 X_t + \cdots\cdots$$

$$= X_t$$

$$\chi(B)\Phi(B)X_t = \chi(B)\Theta(B)Z_t$$

$$X_t = \chi(B)\Theta(B)Z_t = \psi(B)Z_t \quad \text{where } \psi(B) = \sum_{j=0}^{\infty}\psi_j B^j$$

$$\chi(B)\Theta(B) = (1+\phi B+\phi^2 B^2+\cdots)(1+\theta B)$$

$\Rightarrow \psi_0 = 1 \quad \psi_j = (\phi+\theta)\phi^{j-1}$ for $j\geq 1$

$\Rightarrow X_t = Z_t + (\phi+\theta)\sum_{j=1}^{\infty}\phi^{j-1} Z_{t-j}$

the MA(∞) process

$\Rightarrow \{X_t\}$ ARMA(1,1) is stationary if $\sum_{j=1}^{\infty}(\phi+\theta)\phi^{j-1}$ is summable

$\Rightarrow$ stationary if $|\phi|<1$ b.c. $\sum_{j=1}^{\infty}|\phi|^{j-1}<\infty$

If $|\phi|>1$

Consider $-\sum_{j=1}^{\infty}\phi^{-j}z^{-j} = -\sum_{j=1}^{\infty}(\phi z)^{-j} = -\sum_{j=1}^{\infty}\left(\frac{1}{\phi z}\right)^j$

$$= -\left[\frac{1}{1-\frac{1}{\phi z}} - 1\right]$$

$$= \frac{1}{1-\phi z}$$

Let $\chi(B) = -\sum_{j=1}^{\infty} \phi^{-j} B^{-j}$ summable for $|\phi|>1$

$$\chi(B)\Phi(B)X_t = \chi(B)\Theta(B)Z_t$$
$$X_t = \chi(B)\Theta(B)Z_t$$
$$X_t = -\theta\phi^{-1}Z_t - (\theta+\phi)\sum_{j=1}^{\infty}\theta^{-j-1}Z_{t+j}.$$

depend on the future.

If $|\phi|<1$ for ARMA(1,1), we call it a causal process

If $|\phi|>1$ for ARMA(1,1), we call it a noncausal process

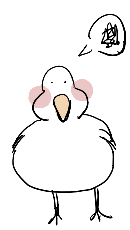
If $|\phi|=1 \Rightarrow$ nonstationary

If $|\theta|<1$, let $\xi(z)$ be the power series expansion of

$$\frac{1}{\Theta(z)} = \frac{1}{1+\theta z} = \sum_{j=0}^{\infty}(-\theta)^j z^j$$

Note $\sum_{j=0}^{\infty}|(-\theta)^j| < \infty$, then

$$\begin{aligned}\xi(z)\Theta(B) &= \left(\sum_{j=0}^{\infty}(-\theta)^j B^j\right)(1+\theta B)\\ &= \sum_{j=0}^{\infty}(-1)^j\theta^j B^j + \sum_{j=0}^{\infty}(-1)^j\theta^{j+1}B^{j+1}\\ &= \sum_{j=0}^{\infty}(-1)^j\theta^j B^j + \sum_{j=0}^{\infty}(-1)^j\theta^{j+1}B^{j+1}\\ &= 1\end{aligned}$$



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$$\xi(B)\Theta(B)Z_t = \xi(B)\Phi(B)X_t$$
$$Z_t = \pi(B)X_t$$

where $\pi(B) = \sum_{j=0}^{\infty}\pi_j B^j = (1-\theta B+(-\theta)^2+\cdots)(1-\phi B)$

$$\Rightarrow Z_t = X_t - (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j}$$

If $|\theta|<1$, then the ARMA(1,1) process is invertible, and Z_t is expressed in terms of X_s, $s\le t$

$Z_t = \sum_{j=0}^{\infty}\pi_j X_{t-j}$ where $\sum|\pi_j|<\infty$

If $|\theta|>1$, then the ARMA(1,1) process is noninvertible, and Z_t is expressed in terms of X_s, $s\ge t$

$$Z_t = -\phi\theta^{-1}X_t + (\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{-j-1}X_{t+j}.$$

The definition of ARMA(p,q) require $\{X_t\}$ to be stationary.

For ARMA$(1,1)$, a stationary solution exists if and only if $\phi_1 \neq \pm 1$.

$\Rightarrow$ equivalent to the condition that autoregressive polynomial

$\Phi(z) = 1 - \phi_1 z \neq 0$ for $z = \pm 1$

The analogous condition for the general ARMA(p,q) process is

$\Phi(z) = 1 - \phi_1 z - \cdots \phi_p z^p \neq 0$ for all complex z with $|z| = 1$

If $\Phi(z) \neq 0$ $\forall z$ on unit circle, then $\exists \delta > 0$ s.t.

$$\frac{1}{\Phi(z)} = \sum_{j=-\infty}^{\infty} \chi_j z^j \quad \text{for } 1-\delta < |z| < 1+\delta$$

and $\sum_{j=-\infty}^{\infty} |\chi_j| < \infty$. Then define $\frac{1}{\Phi(B)}$ as a linear filter with absolutely summable coefficients.

Apply the operator $\chi(B) : \frac{1}{\Phi(B)}$ to both sides of the definition

$\Rightarrow \chi(B)\Phi(B) X_t = X_t = \chi(B)\Theta(B) Z_t = \psi(B) Z_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$

$\Rightarrow X_t = \psi(B) Z_t$ is a unique solution.

Existence Uniqueness

A stationary solution $\{X_t\}$ exists and is unique iff

$\Phi(z) = 1 - \phi_1 z_1 - \phi_2 z^2 \cdots \phi_p z^p \neq 0 \quad \forall\ |z| = 1$

root $a+bi$ $\sqrt{a^2+b^2} = 1$

$\Rightarrow$ There are no roots of $\Phi(z)$ on the unit circle.

Causality

An ARMA(p,q) process is causal if $\exists \{\psi_j\}$ s.t. $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ (X_t only depends on the past).

$\Leftrightarrow \Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \neq 0 \quad \forall\ |z| \leq 1$

No roots of $\Phi(z)$ inside the unit circle.

If $\{X_t\}$ is causal, $\chi(z) = \frac{1}{\Phi(z)} = \sum_{j=0}^{\infty} \chi_j z^j$

This can only be done if the roots of $\Phi(z)$ are outside the unit circle.

If the roots $\Phi(z)$ are inside the unit circle, then we cannot obtain summable χ_j coefficients.

Recall if $\{X_t\}$ is causual

$$\Phi(B)X_t = \Theta(B)Z_t$$
$$\chi(B)\Phi(B)X_t = \chi(B)\Theta(B)Z_t$$
$$X_t = \psi(B)Z_t$$
$$\text{where } \psi(z) = \chi(z)\Theta(z) = \frac{1}{\Phi(z)}\Theta(z) = \sum_{j=0}^{\infty}\psi_j z^j$$

$\Rightarrow$ $\Phi(z)\psi(z) = \Theta(z)$

$$(1-\phi_1 z-\phi_2 z^2-\cdots\phi_p z^p)\times(1+\psi_1 z+\psi_2 z^2+\cdots) = 1+\theta_1 z+\cdots+\theta_q z^q$$

$\psi_0 = 1$ 0-degree terms

$\psi_1 - \psi_0\phi_1 = \theta_1$ 1-degree terms

$\psi_2 - \psi_1\phi_1 - \psi_0\phi_2 = \theta_2$ 2-degree terms.

$\vdots$

$\psi_j - \sum_{k=1}^{p}\phi_k\psi_{j-k} = \theta_j$ for $j = 0, 1, \cdots$

$\theta_j = 0$ for $j > q$

Invertibility

ARMA(p,q) process $\{X_t\}$ is invertible if $\exists$ constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty}|\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty}\pi_j X_{t-j}$ $\forall t$

depends on past and current X_t's.

$\Leftrightarrow$ $\Theta(z) = 1+\theta_1 z+\theta_2 z^2+\cdots+\theta_q z^q \neq 0$ $\forall |z| \leq 1$

No roots of $\Theta(z)$ on or inside the unit circle.

$$\delta(z) = \frac{1}{\Theta(z)} = \sum_{j=0}^{\infty}\theta_j z^j \quad \text{where } \delta(B)\Theta(B)Z_t = \delta(B)\Phi(B)X_t = \pi(B)X_t$$

$$\text{where } \pi(z) = \frac{\Phi(z)}{\Theta(z)}$$

$$(1+\theta_1 z+\theta_2 z^2+\cdots\theta_q z^q)\times(1+\pi_1 z+\pi_2 z^2+\cdots) = 1-\phi_1 z-\phi_2 z^2\cdots\phi_p z^p$$

$\Rightarrow$ $\pi_0 = 1$ 0 degree

$-\phi_1 = \pi_1 + \theta_1$ 1-degree

$\cdots$

$-\phi_j = \pi_j + \sum_{k=1}^{q}\theta_k\pi_{j-k}$ j-degree

where $\phi_0 = -1$, $\phi_j = 0$ for $j > p$, $\pi_j = 0$ for $j < 0$, $\theta_k = 0$ for $k > j$.

e.g. ARMA(1,1) $X_t - 0.8X_{t-1} = Z_t + 0.2Z_t$

① $\Phi(Z) = 1 - 0.8Z$ has zero at $Z = 1.25$, outside unit circle
Thus, $\exists$ a unique ARMA process that is stationary and causal.

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \phi_j$$

$$\begin{cases} \psi_0 = 1 \\ \psi_1 = \theta_1 + \phi_1\psi_0 = 0.2 + 0.8(1) = 1 \\ \psi_2 = \theta_2 + \phi_1\psi_1 = 0 + 0.8(1) = 0.8 \\ \psi_3 = \theta_3 + \phi_1\psi_2 = 0 + 0.8(0.8) = 0.8^2 \\ \dots \quad \dots \\ \psi_j = 0.8^{j-1} \end{cases}$$

② $\Theta(Z) = 1 + 0.2Z$ has zero at $Z = -5$, outside the circle
Thus, invertible

$$-\phi_j = \pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} \Rightarrow \pi_j = -\phi_j - \sum_{k=1}^{q} \theta_k \pi_{j-k}$$

$$\begin{cases} \pi_0 = 1 \\ \pi_1 = -\phi_1 - \theta_1\pi_0 = -0.8 - 0.2(1) = -1 \\ \pi_2 = -\phi_2 - \theta_1\pi_1 = 0 - 0.2(-1) = 0.2 \\ \pi_3 = -\phi_3 - \theta_1\pi_2 = 0 - 0.2(0.2) = -0.2^2 \\ \dots \\ \pi_j = -(0.2)^{j-1} \end{cases}$$

e.g. ARMA(2,1) $X_t - 0.75X_{t-1} + 0.5625X_{t-2} = Z_t + 1.25Z_{t-1}$

$$\Phi(Z) = 1 - \frac{3}{4}Z + \frac{9}{16}Z^2 \Rightarrow r = \frac{-(-\frac{3}{4}) \pm \sqrt{(-\frac{3}{4})^2 - 4(\frac{9}{16})}}{2(\frac{9}{16})} = \frac{2(1 \pm \sqrt{3}i)}{3}$$

$|r| = \sqrt{(\frac{2}{3})^2 + (\frac{2}{\sqrt{3}})^2} > 1$ outside unit circle

Thus, it is a causal process but NOT invertible since -0.8 is inside the unit circle.

② ACF and PACF

Assume $\{X_t\}$ is causal, then $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ where

$$\sum_{j=0}^{\infty} \psi_j z^j = \frac{\Theta(z)}{\Phi(z)}, \ |z| \le 1 \quad \text{and} \quad \psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k} \quad \text{where}$$

$$\theta_0 = 1, \ \theta_j = 0 \text{ for } j > q, \ \psi_j = 0 \text{ for } j < 0$$

Because $\{X_t\}$ is a linear process, $\gamma(h) = E(X_{t+h} X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$ if X_t has zero mean

e.g. ARMA (1,1) $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$ with $|\phi| < 1$ for causal $\{X_t\}$

$$\psi_0 = 1$$

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta + \phi \psi_{j-1}$$

$$\Rightarrow \psi_j = (\phi + \theta) \phi^{j-1} \quad \text{for } j \ge 1$$

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2 [1 + (\theta+\phi)^2 \sum_{j=0}^{\infty} \phi^{2j}]$$

$$= \sigma^2 [1 + \frac{(\theta+\phi)^2}{1-\phi^2}]$$

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+1} \psi_j = \sigma^2 [\psi_0 \psi_1 + \sum_{j=1}^{\infty} \psi_j \psi_{j+1}]$$

$$= \sigma^2 [\phi + \theta + (\phi+\theta)^2 \sum_{j=1}^{\infty} \phi^{j-1} \phi^{j+1-1}]$$

$$= \sigma^2 [\phi + \theta + \frac{(\phi+\theta)^2 \phi}{1-\phi^{2j}}] \qquad \phi \sum_{j=0}^{\infty} \phi^{2j}$$

for $|h| > 1$,

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

but

$$\psi_{j+h} = (\phi+\theta) \phi^{j+h-1} = \phi^{h-1} (\phi+\theta) \phi^{j+1-1}$$

$$= \phi^{h-1} \psi_{j+1}$$

$$\Rightarrow \gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma^2 \sum_{j=0}^{\infty} \psi_j \phi^{h-1} \psi_{j+1} = \phi^{h-1} \gamma(1).$$

The PACF for an ARMA(p,q) process is defined by

$\alpha(0) = 1$ and $\alpha(h) = \phi_{hh}$, $h \geq 1$ where

ϕ_{hh} is the last component of $\underset{\sim}{\phi}_h = \Gamma_h^{-1} \underset{\sim}{\gamma}_h$ where

$\Gamma_h = (\gamma(i-j))_{i,j=1}^{h}$ and $\underset{\sim}{\gamma}_h = [\gamma(1), \cdots \gamma(h)]^T$

Partial autocorrelation at lag h is $(\underset{\sim}{\phi}_h)_h = \phi_{hh}$

Sample PACF

For $X_1, \cdots X_n$, $\hat{\alpha}(0) = 1$ and $\hat{\alpha}(h) = \hat{\phi}_{hh}$ where

$\hat{\phi}_{hh}$ is the last component of $\hat{\underset{\sim}{\phi}}_h = \hat{\Gamma}_h^{-1} \hat{\underset{\sim}{\gamma}}_h$

Most calculations must be done numerically.

PACF for AR(p)

For the causal AR(p) process, we know that the best linear predictor of X_{n+1} given $X_1, \cdots X_n$ is

$$\hat{X}_{n+1} = \phi_1 X_n + \cdots \phi_h X_{n+1-p}$$

Since ϕ_{hh} of X_t is ϕ_p if $h = p$, and 0 if $h > p$

Thus, we only need to calculate $\alpha(h)$ for AR(p) process for $h = 1, \cdots p-1$

PACF measures the correlation between X_{n+1} and X_1, adjusting for all $X_n, X_{n-1} \cdots X_2$.

③ Forcasting

Assume a causal ARMA process, $\Phi(B) X_t = \Theta(B) Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$

Predict X_{n+1} using $X_n, \cdots X_1$

$\Rightarrow$ Use innovation algorithm, but transform X_t in the following way.

Let
$$W_t = \begin{cases} \dfrac{X_t}{\sigma} & \text{for } t = 1, \cdots m \\ \dfrac{\Phi(B) X_t}{\sigma} = \dfrac{\Theta(B) Z_t}{\sigma} & \text{for } t > m \end{cases}$$

where $m = \max(p, q)$

Define $\theta_0 = 1$, $\theta_j = 0$ for $j > q$

Assume $p \geq 1$ and $q \geq 1$ (not an AR(p) and MA(q))

But can still allow for zero coefficients for θ_k or ϕ_j.

$k(i,j) = E(W_i W_j)$ ACF for zero mean

(i) For $1 \leq i, j \leq m$

$$k(i,j) = E\left(\frac{X_i}{\sigma} \frac{X_j}{\sigma}\right) = \frac{1}{\sigma^2} \gamma_X(i,j)$$

(ii) For $\underset{a}{\min(i,j)} \leq m < \underset{b}{\max(i,j)} \leq 2m$

$$\begin{aligned} k(i,j) &= E(W_a W_b) \\ &= E\left(\frac{X_a}{\sigma} \frac{\Phi(B) X_b}{\sigma}\right) \\ &= \frac{1}{\sigma^2} E[X_a (X_b - \phi_1 X_{b-1} - \phi_2 X_{b-2} - \cdots \phi_p X_{b-p})] \\ &= \frac{1}{\sigma^2} [\gamma(i-j) - \sum_{r=1}^{p} \phi_r \gamma(|i-j| - r)] \end{aligned}$$

(iii) For $\min(i,j) > m$

$$\begin{aligned} k(i,j) &= E\left(\frac{\Phi(B) X_i}{\sigma} \frac{\Phi(B) X_j}{\sigma}\right) \\ &= E\left(\frac{\Theta(B) Z_i}{\sigma} \frac{\Theta(B) Z_j}{\sigma}\right) \\ &= \frac{1}{\sigma^2} E[(\sum_{r=0}^{q} \theta_r Z_{i-r})(\sum_{s=0}^{q} \theta_s Z_{j-s})] \\ &= \frac{1}{\sigma^2} E\left(\sum_{s=0}^{q} \sum_{r=0}^{q} \theta_r \theta_s Z_{i-r} Z_{j-s}\right) \rightarrow 0 \text{ unless } i-r = j-s \\ &= \frac{1}{\sigma^2} \sum_{r=0}^{q} \theta_r \theta_{r+|i-j|} \underset{E(Z_r^2)}{\sigma^2} \\ &= \sum_{r=0}^{q} \theta_r \theta_{r+|i-j|} \end{aligned}$$

(iv) For $\overset{a}{\min}(i,j) \le m \le 2m < \overset{b}{\max}(i,j)$

$$\begin{aligned} \kappa(i,j) &= E\left(\frac{X_a}{\sigma}\ \frac{\Phi(B)X_b}{\sigma}\right) \\ &= E\left(\frac{X_a}{\sigma}\ \frac{\Theta(B)Z_b}{\sigma}\right) \\ &= \frac{1}{\sigma^2} E[X_a(Z_b + Z_{b-1} + \cdots + Z_{b-q})] \\ &= 0 \end{aligned}$$

$\rightarrow b-q > q$

None of Z_i is correlated with X_j because they all come from the future

Apply the innovation algorithm to the $\{W_n\}$ values.

$$\begin{cases} \hat{W}_{n+1} = \sum_{j=1}^{n} \theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}) & \text{for } 1 \le n < m \\ \hat{W}_{n+1} = \sum_{j=1}^{q} \theta_{nj}(W_{n+1-j} - \hat{W}_{n+1-j}) & n \ge m \end{cases}$$

because $\kappa(r,s) = 0$ for $r > m$ and $|r-s| > q$

Recall W_t is a linear combination of X_t

$\Rightarrow$ X_t is a linear combination of W_t.

$\Rightarrow$ The best linear predictor of any random variable Y using $X_1, \cdots X_n$ is the same as the BLP of Y using $W_1, \cdots W_n$.

Using the linearity of P_n, we have

$$\begin{cases} \hat{W}_t = \sigma^{-1} \hat{X}_t \\ \hat{W}_t = \sigma^{-1} [\hat{X}_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}] \end{cases}$$

$\Rightarrow$ $X - \hat{X}_t = \sigma(W - \hat{W}_t)$

$$\Rightarrow \quad \hat{X}_{n+1} = \begin{cases} \sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & \text{for } n < m \\ \phi_1 X_n + \cdots \phi_p X_{n-p} + \sum_{j=1}^{q} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) \end{cases}$$