Chapter 4 Branching process

"Population" dynamics are often modelled by branching process.

Assume that each unit in the population at time n produces offspring according to some distribution.

Each member of the population produces offspring

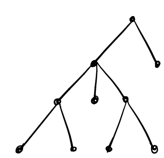
① independently

② according the same distribution

$$\underset{\sim}{a} = (a_0, a_1 \cdots\cdots)$$
$$= (Pr(X_i=0), Pr(X_i=1) \cdots\cdots)$$

where X_i = # offspring

Let Z_n be the number of population units at time n (or generation n). Then $\{Z_n\}$ is a branching process.



$\{Z_n\}$ can be represented as a Markov chain

$Z_n \in N = \{0, 1, 2 \cdots\}$

$$Pr(Z_{n+1} = i_{n+1} \mid Z_n = i_n, Z_{n-1} = i_{n-1} \cdots Z_0 = i_0) = Pr(Z_{n+1} = i_{n+1} \mid Z_n = i_n)$$

$\begin{cases} \text{If } a_0 = 0, \text{ then } Z_{n+1} \geq Z_n & \text{(Every pair produces at least one offspring)} \\ \text{If } a_0 = 1, \text{ then } Z_1 = 0 \text{ and the population is extinct at generation} \end{cases}$

For the rest of Chapter 4, we will assume $0 < a_0 < 1$

Fact: 0 is an absorbing state

Lemma: All non-zero states are transient

$$Pr(Z_{n+1}=0 \mid Z_n = i) = a_0^i$$

$$f_i = Pr(Z_n = i \text{ for some } n \geq 1 \mid Z_0 = i)$$ Probability that I return to i at all.

$$< Pr(Z_1 > 0 \mid Z_0 = i)$$

If $n=1$ and $Z_1 = 0$, you cannot get back to $Z_t = i$

$$1 - Pr(Z_1 = 0 \mid Z_0 = i) = 1 - a_0^i < 1$$

$$\begin{cases} E(X_i)=\mu \\ Var(X_i)=\sigma^2. \end{cases}$$

4.2 Possibilities for our process.

① Get absorbed

② Process grow without bound.

Mean Generation size

Let $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$ where X_i is the number of offspring for the ith member of $(n-1)^{st}$ generation

- Z_n is a sum of iid random variable $X_1, \cdots X_{Z_{n-1}}$
- Z_n is a random sum of iid random variables

$E(X_i) = \mu = \sum_{k=0}^{\infty} k a_k$ where $a_k = Pr(X_i = k)$ mean of the offspring distribution

$$\begin{aligned}
E_{Z_n}(Z_n) &= E_{Z_{n-1}} E_{Z_n|Z_{n-1}}[Z_n \mid Z_{n-1}] \\
&= \sum_{k=0}^{\infty} \{ E_{Z_n|Z_{n-1}=k}[\sum_{i=1}^{k} X_i \mid Z_{n-1}=k] \} \times Pr(Z_{n-1}=k) \\
&= \sum_{k=0}^{\infty} E(\sum_{i=1}^{k} X_i) Pr(Z_{n-1}=k) \\
&= \sum_{k=0}^{\infty} k\mu \, Pr(Z_{n-1}=k) \\
&= \mu \underbrace{\sum_{k=0}^{\infty} k \, Pr(Z_{n-1}=k)}_{E_{Z_{n-1}}(Z_{n-1})} \\
&= \mu E_{Z_{n-1}}(Z_{n-1}) \\
&= \mu [\mu Z_{n-2}(Z_{n-2})] \\
&\quad \cdots \quad \cdots \\
&= \mu^n E_{Z_0}(Z_0)
\end{aligned}$$

If $Z_0 = 1$ with probability 1, then $E_{Z_n}(Z_n) = \mu^n$.

As $n \to \infty$: $\lim_{n\to\infty} E(Z_n) = \lim_{n\to\infty} \mu^n = \begin{cases} 0 & \text{if } \mu < 1 \\ 1 & \text{if } \mu = 1 \\ \infty & \text{if } \mu > 1 \end{cases}$

The probability that the population is extinct by generation n is

$$\begin{aligned}
P(E) = P(Z_n = 0) &= 1 - P(Z_n \geq 1) = 1 - \sum_{k=1}^{\infty} P(Z_n = k) \\
&\geq 1 - \sum_{k=1}^{\infty} k P(Z_n = k) = 1 - E(Z_n) = 1 - \mu^n.
\end{aligned}$$

For subcritical process $(\mu < 1)$, $\lim_{n\to\infty} P(Z_n = 0) \geq \lim_{n\to\infty} 1 - \mu^n = 1$

$\Rightarrow P(E) = 1$ with probability 1, a subcritical process eventually goes extinct.

Variance

$$Var_{Z_n}(Z_n) = Var_{Z_{n-1}}[E_{Z_n|Z_{n-1}}(Z_n)] + E_{Z_{n-1}}[Var_{Z_n|Z_{n-1}}(Z_n)]$$

$$Var_{Z_{n-1}}(\mu Z_{n-1})$$
$$= \mu^2 Var_{Z_{n-1}}(Z_{n-1})$$

$$E_{Z_{n-1}}[Var_{Z_n|Z_{n-1}}(\sum_{i=1}^{Z_{n-1}} X_i \mid Z_{n-1})]$$
$$= E_{Z_{n-1}}[Z_{n-1}\alpha^2]$$
$$= \alpha^2 E_{Z_{n-1}}(Z_{n-1})$$
$$= \alpha^2 \mu^{n-1}$$

$$\Rightarrow Var(Z_n) = \mu^2 Var(Z_{n-1}) + \alpha^2\mu^{n-1}.$$

If $Z_0 = 1$ with probability 1.

$$Var(Z_0) = 0$$
$$Var(Z_1) = \mu^2 \cdot 0 + \alpha^2\mu^0 = \alpha^2$$
$$Var(Z_2) = \mu^2 \cdot \alpha^2 + \alpha^2\mu = \mu(1+\mu)\alpha^2.$$
$$Var(Z_3) = \mu^2(\mu(1+\mu)\alpha^2) + \alpha^2\mu^2 = \alpha^2\mu^2(1+\mu+\mu^2)$$
$$\cdots$$
$$Var(Z_n) = \alpha^2\mu^{n-1}\sum_{k=0}^{n-1}\mu^k.$$

	$E(Z_n)$	$Var(Z_n)$	
$\mu=1$	1	$n\alpha^2$	Critical Process
$\mu<1$	$\mu^n \xrightarrow{n\to\infty} 0$	$\frac{\alpha^2\mu^{n-1}(\mu^n-1)}{\mu-1} \xrightarrow{n\to\infty} 0$	subcritical process.
$\mu>1$	$\mu^n \xrightarrow{n\to\infty} \infty$	$\frac{\alpha^2\mu^{n-1}(\mu^n-1)}{\mu-1} \xrightarrow{n\to\infty} \infty$	supercritical process

4.3 Probability generating functions

For a discrete random variable X taking values in $\{0, 1 \cdots\}$, the probability generating function is

$$G(s) = E(s^X) = \sum_{k=0}^{\infty} s^k P(X=k)$$

A power series with coefficients summing to 1.

$$= P(X=0) + P(X=1)s + P(X=2)s^2 + \cdots$$

① $G_X(1) = \sum_{k=0}^{\infty} Pr(X=k) = 1$

② Series converges absolutely for $|s| \leq 1$.

Note: Similar to MGF (moment generating function)

$$M_X(t) = E(e^{Xt}) = E((e^t)^X)$$

e.g. Geometric distribution $X \sim Geom(p)$ failure

$$G(s) = E(s^X) = \sum_{k=1}^{\infty} s^k p(1-p)^{k-1} = sp \sum_{k=1}^{\infty} (s(1-p))^{k-1} = \frac{sp}{1-s(1-p)} \text{ for } |s|<1$$

① Find probability

$$G(0) = Pr(X=0)$$

$$G'(0) = \sum_{k=1}^{\infty} k s^{k-1} Pr(X=k) = 0 + \sum_{k=1}^{\infty} k s^{k-1} Pr(X=k) = Pr(X=1)$$

$$G''(0) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} Pr(X=k) = 2 Pr(X=2)$$

$\cdots \quad \cdots$

$$G^{(j)}(0) = \sum_{k=j}^{\infty} k(k-1)\cdots(k-j+1) s^{k-j} Pr(X=k) = j!\, Pr(X=j)$$

$$\Rightarrow Pr(X=j) = \frac{G^{(j)}(0)}{j!}$$

② Sums of independent random variables

Let $Y = X_1 + X_2 + \cdots X_n$

$$G_Y(s) = E(s^Y) = E(s^{X_1+X_2 \cdots X_n}) = E(\prod_{k=1}^{n} s^{X_k}) = \prod_{k=1}^{n} E(s^{X_k}) = \prod_{k=1}^{n} G_{X_k}(s)$$

$\Rightarrow$ If $X_1, X_2 \cdots X_n$ are iid, then

$$G_Y(s) = [G_X(s)]^n$$ where $X_1, X_2 \cdots X_n \sim F$.

③ Moments

$$G_X^{(1)}(s) = \sum_{k=0}^{\infty} k s^{k-1} Pr(X=k)$$

$$\Rightarrow G_X^{(1)}(1) = \sum_{k=0}^{\infty} k Pr(X=k) = E(X).$$

$$G_X^{(2)}(1) = \sum_{k=0}^{\infty} k(k-1) Pr(X=k) = E_X(X^2) - E(X)$$

Properties of PGF

① (a) $G(1)=1$

(b) $P(X=k) = \dfrac{G^{k}(0)}{k!}$ for $k \geq 0$

(c) $E(X) = G'(1)$

(d) $Var(X) = G''(1) + G'(1) - G'(1)^2$

② If X and Y are random variables s.t. $G_X(s) = G_Y(s)$ for all s, then X and Y have the same distribution.

③ If X and Y are independent, then $G_{X+Y}(s) = G_X(s) G_Y(s)$.

4.4 Extinction is forever!

PGF for Z_n : # of population units at generation n.

$$G_n(s) = \sum_{k=0}^{\infty} s^k Pr(Z_n=k)$$

Z_n

Let $G_X(s) = \sum_{k=0}^{\infty} s^k a_k$ be PGF of the offspring distribution.

So $G_n(s) = E(s^{\sum_{i=1}^{Z_{n-1}} X_i})$ by $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$

$= E_{Z_{n-1}}[E(s^{\sum_{i=1}^{Z_{n-1}} X_i}) \mid Z_{n-1})]$

Then, $E(s^{\sum_{i=1}^{Z_{n-1}} X_i}) \mid Z_{n-1}=z) = E(s^{\sum_{i=1}^{z} X_i}) = \prod_{i=1}^{z} G_X(s) = [G_X(s)]^z.$

$\Rightarrow G_n(s) = E_{Z_{n-1}}[G_X(s)^{Z_{n-1}}]$ If $|s| \le 1$, $G_X(s) \le 1$.

$= E_{Z_{n-1}}[u^{Z_{n-1}}]$

$= G_{n-1}(u)$

$= G_{n-1}(G_X(s))$

$G_0(s) = \sum_{k=0}^{\infty} s^k Pr(Z_0 = k) = s$ if $Z_0 = 1$ with probability 1.

$G_1(s) = G_0(G_X(s)) = G_X(s)$

$G_2(s) = G_1(G_X(s)) = G_X(G_X(s)) = G_X(G_1(s))$

$G_3(s) = G_2(G_X(s)) = G_1(G_X(G_X(s))) = G_X(G_X(G_X(s))) = G_1(G_2(s))$

... ...

$G_n(s) = G_{n-1}(G_X(s)) = \underbrace{G_X(G_X(G_X}_{n\ G_X\text{'s}}(\cdots))) = G_X(G_{n-1}(s))$

Theorem For a branching process, let $G_X(s)$ be the PGF for the offspring distribution. The probability of extinction ever is the smallest positive root of

$$s = G_X(s).$$

Important

If $\mu \le 1$ (subcritical / critical), the extinction probability is equal to 1.

e.g. Geometric (# of failures)

$$a_k = (1-p)^k p$$

$$A_X(s) = \sum_{k=0}^{\infty} s^k (1-p)^k p = p \sum_{k=0}^{\infty} s^k (1-p)^k = \frac{p}{1-s(1-p)}$$

$$\mu = A_X'(s) = \frac{p(1-p)}{(1-s(1-p))^2} \Big|_{s=1} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$$

$$\begin{cases} \text{if } p \geq \frac{1}{2}, & \mu \leq 1 \\ \text{if } p < \frac{1}{2}, & \mu > 1 \quad \text{super critical process} \end{cases}$$

$$s = \frac{p}{1-s(1-p)}$$

$$\Rightarrow \frac{p}{1-s(1-p)} - s = 0$$

$$\Rightarrow s^2(1-p) - s + p = 0$$

$$\Rightarrow s = \frac{1 \pm \sqrt{1+4p^2-4p}}{2(1-p)} = \frac{1 \pm (1-2p)}{2(1-p)}$$

$$\Rightarrow \text{smaller positive root} \begin{cases} \frac{p}{1-p} & p < \frac{1}{2} \\ 1 & p \geq \frac{1}{2} \end{cases}$$

Let $e_n = P(Z_n = 0)$ denote the probability that the population goes extinct by generation n. We have

$$e_n = P(Z_n=0) = G_n(0) = G_X(G_{n-1}(0)) = G_X(P(Z_{n-1}=0)) = G_X(e_{n-1})$$

$\Rightarrow \lim_{n\to\infty} e_n = \lim_{n\to\infty} G_X(e_{n-1})$ Note: $e = \lim_{n\to\infty} e_n$

↓ continuous

$\Rightarrow e = G_X(\lim_{n\to\infty} e_{n-1}) = G_X(e)$

$\Rightarrow$ e is a root of $s = G_X(s)$.

Let x be a positive solution of $s = G_X(s)$. We need to show that $e \le x$.

Since $G(s) = \sum_{k=0}^{\infty} s^k P(X=k)$ is an increasing function on $[0,1]$ and $0<x$

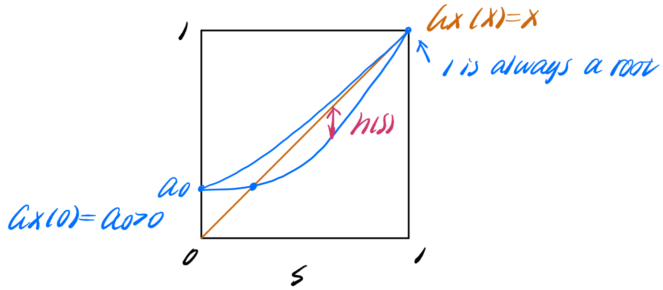
$$e_1 = P(Z_1=0) = G_1(0) = G_X(0) \quad \text{if } Z_0 = 1$$
$$\le G_X(x) = x$$

$G_X(s)$ is nondecreasing.

By induction, assuming $e_k \le x$, for all $k<n$

$$e_n = P(Z_n=0) = G_n(0) = G_X(G_{n-1}(0)) = G_X(e_{n-1}(0))$$
$$\le G_X(x) = x$$

$\Rightarrow$ Take limits $\lim_{n\to\infty} e_n \le \lim_{n\to\infty} x \Rightarrow e \le x$.



$G''_X(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} P(X=k) \ge 0 \Rightarrow G_X(s)$ is convex

$G'_X(s) = \sum_{k=0}^{\infty} k s^{k-1} P(X=k)$ increasing function of s

$\Rightarrow G'_X(s) \le G_X'(1) = \mu$

Let $h(s) = s - G_X(s)$ At s, the distance between s and $G_X(s)$ $\begin{cases} \text{positive} & \text{below} \\ 0 & \text{on the line} \\ \text{negative} & \text{above} \end{cases}$

$h'(s) = 1 - G_X'(s)$

$\underline{\mu \le 1}$ $\begin{cases} h(1) = 0 \\ h(s) < 0 \text{ for } s < 1 \end{cases} \Rightarrow G_X(s) > s \Rightarrow e = 1$

$\mu = G'_X(s) \le 1 \Rightarrow h'(s) \ge 0 \Rightarrow h(s)$ increasing

$\underline{\mu > 1}$ $\begin{cases} h(0) = 0 - a_0 < 0 \\ h(s) = s - G_X(s) \\ h(1) = 0 \end{cases}$ $\exists$ some $t \in (0,1)$ s.t. $h(t) = 0$

$\mu = G'_X(s) > 1 \Rightarrow h'(s) < 0 \Rightarrow h(s)$ increasing

e.g. Let $a = (\overset{0}{\tfrac{1}{4}}, \overset{1}{\tfrac{1}{4}}, \overset{2}{\tfrac{1}{2}})$, $\mu_X = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4} > 1$

$G_X(s) = \sum_{k=0}^{\infty} s^k \Pr(X=k) = \frac{1}{4} + \frac{1}{4}s + \frac{1}{2}s^2$

$G_X(s) - s = \frac{1}{4} - \frac{3}{4}s + \frac{1}{2}s^2 = 0$

$\Rightarrow (2s-1)(s-1) = 0 \quad s = \{\frac{1}{2}, 1\}$

Not the probability $X_i = 0$, which is $\frac{1}{4}$.

Probability of extinction $e = \frac{1}{2} = \sum_{n=1}^{\infty} \Pr(T_n = 0)$ where

$T_n = \min\{n : Z_n = 0\}$