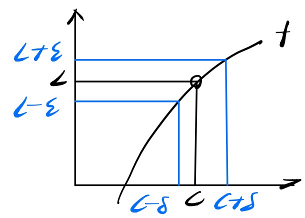
Chapter 4: Limits of functions and continuity

Goal: Define $\lim_{x \to c} f(x)$

I ε-δ definition by Weierstrass

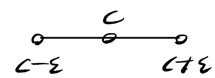


Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$. We say that L is the limit of f as x approaches c if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D \setminus \{c\}: |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\equiv \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D: 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

DEF Let $c \in \mathbb{R}$, $\varepsilon > 0$, then $V_\varepsilon^*(c) = V_\varepsilon(c) \setminus \{c\}$ is called the punctured ε-neighborhood about c



Rephrase ε-δ def

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D \setminus \{c\}: |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$$\equiv \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D: x \in V_\delta^*(c) \Rightarrow f(x) \in V_\varepsilon(L)$$

$$\equiv \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in V_\delta^*(c) \cap D: f(x) \in V_\varepsilon(L)$$

$$\equiv \forall \varepsilon > 0\ \exists \delta > 0: f(V_\delta^*(c) \cap D) \subseteq V_\varepsilon(L)$$

① $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$. LET $c \in \mathbb{R}$. SHOW THAT $\lim_{x \to c} f = 2c$.

LET $\varepsilon > 0$. LET $\delta > 0$ BE ARBITRARY <u>FOR NOW</u>.

LET $0 < |x - c| < \delta$. THEN $|f(x) - L| = |2x - 2c| = 2|x - c|$

$< 2\delta < \varepsilon \Leftrightarrow \delta < \frac{\varepsilon}{2}$

NOW LET $\delta < \frac{\varepsilon}{2}$. THEN $\forall x \in \mathbb{R}: 0 < |x - c| < \delta$ IT HOLDS THAT $|f(x) - 2c| < \varepsilon$. THUS $\lim_{x \to c} f = 2c$.

② $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. LET $c \in \mathbb{R}$. SHOW THAT $\lim_{x \to c} f = c^2$.

LET $\varepsilon > 0$. LET $\delta > 0$ BE ARBITRARY FOR NOW.

LET $0 < |x - c| < \delta$. THEN:

$|f(x) - L| = |x^2 - c^2| = |(x - c)(x + c)| = |x - c| \cdot |x + c|$

$< \delta \cdot |x + c| = \delta \cdot |(x - c) + 2c|$

$\leq \delta(|x - c| + 2|c|) < \delta(\delta + 2|c|) \quad (*)$

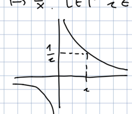
NOW LET $\delta < 1$. THEN

$(*) < \delta(1 + 2|c|) < \varepsilon \Leftrightarrow \delta < \frac{\varepsilon}{1 + 2|c|}$ > 0

NOW LET $\delta < \min\{1, \frac{\varepsilon}{1 + 2|c|}\}$

THEN IT FOLLOW FROM THE ABOVE THAT $|f(x) - L| < \varepsilon$ WHENEVER $0 < |x - c| < \delta$. THUS $\lim_{x \to c} f = c^2$.

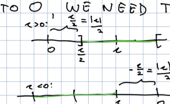
③ $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$. LET $c \in \mathbb{R} \setminus \{0\}$. SHOW THAT $\lim_{x \to c} = \frac{1}{c}$



LET $\varepsilon > 0$. LET $\delta > 0$ BE ARBITRARY FOR NOW. LET $0 < |x - c| < \delta$, WHERE $x \neq 0$. THEN:

$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c - x}{xc}\right| = \frac{|x - c|}{|x| \cdot |c|} < \frac{\delta}{|x| \cdot |c|} \quad (*)$

SINCE $\frac{1}{|x|}$ CAN BE MADE ARBITRARILY LARGE BY CHOOSING x CLOSE ENOUGH TO 0, WE NEED TO "BOUND x AWAY FROM 0"



LET $\delta < \frac{|c|}{2}$.

THEN $|x| = |(x - c) + c| = |c - (c - x)| \geq |c| - |c - x|$

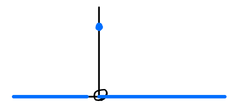
$= |c| - |x - c| > |c| - \delta > |c| - \frac{|c|}{2} = \frac{|c|}{2}$ i.e. $|x| > \frac{|c|}{2}$

THUS:

$(*) < \frac{\delta}{\frac{|c|}{2} \cdot |c|} = \frac{2\delta}{|c|^2} = \frac{2\delta}{c^2} < \varepsilon \Leftrightarrow \delta < \varepsilon \cdot \frac{c^2}{2}$

NOW LET $\delta < \min\{\frac{|c|}{2}, \frac{c^2}{2} \cdot \varepsilon\}$. THEN IT FOLLOWS FROM THE ABOVE THAT $|f(x) - L| < \varepsilon$ WHENEVER $x \in \mathbb{R} \setminus \{0\} \cap V_\delta^*(c)$. THUS $\lim_{x \to c} f = \frac{1}{c}$

④ Let $f: R \to R$, $x = \begin{cases} 1 & x=0 \\ 0 & x \neq 0 \end{cases}$



Show that $\lim_{x \to 0} f = 0 \neq 1 = f(0)$

Let $\varepsilon > 0$. Let $\delta > 0$ be arbitrary for now. Let $0 < |x-0| < \delta \iff 0 < |x| < \delta$

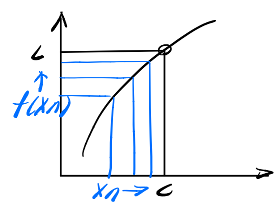
Then $|\underbrace{f(x)}_{0}-L| = |0-0| = 0 < \varepsilon$

$\neq 0$

Thus, for $\forall \delta > 0$, $|f(x)-L| < \varepsilon$ whenever $x \in V_\delta^*(0) \Rightarrow \lim_{x \to 0} f = 0$

II Sequential def.

Let $f: D \subseteq R \to R$. We say that L is the limit of f as x approaches c, in symbol $\lim_{x \to c} f = L$ if for $\forall$ seq (x_n) in $D \setminus \{c\}$ with $\lim (x_n) = c$, it holds that $\lim (f(x_n)) = L$



① $\lim_{x \to c} x^2 = c^2$

Let (x_n) be any seq in $R \setminus \{c\}$ s.t.

$\lim (x_n) = c$. Then $\lim (x_n^2) \underset{\text{limit law}}{=} [\lim (x_n)]^2 = c^2$

② $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$

Let (x_n) be any seq in $R \setminus \{0, c\}$

Sequential Criterion for Non-Existence of the Limit of a Function

Let $f: D \subseteq R \to R$. Let $c \in R$

(a) 2 sequence criterion

If $\exists (x_n), (u_n)$ in $D \setminus \{c\}$ s.t. $\lim (x_n) = \lim (u_n) = c$

and both $(f(x_n))$ and $(f(u_n))$ conv.

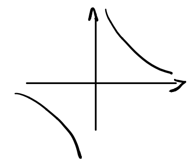
But $\lim (f(x_n)) \neq \lim (f(u_n))$. Then $\lim_{x \to c} f$ DNE.

(b) 1 sequence criterion

If $\exists (x_n)$ in $D \setminus \{c\}$ s.t. $\lim (x_n) = c$ but $(f(x_n))$ diverges

Then $\lim_{x \to c} f$ DNE.

① $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ $\quad x \mapsto \frac{1}{x}$ $\quad \Rightarrow \lim_{x\to 0} f$ DNE



Let $\forall n \in \mathbb{N}: x_n = \frac{1}{n}$. Then $\lim(x_n) = 0$ $\forall n \in \mathbb{N}: x_n \in \mathbb{R}\setminus\{0\}$.
But $(f(x_n)) = (n)$ diverges. $\Rightarrow$ 1-seq criterion

② The Dirichlet function

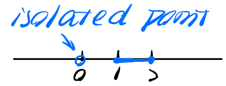
$$f: \mathbb{R} \to \mathbb{R} \quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Let $c \in \mathbb{R}$ be arbitrary, $\lim_{x\to c} f$ DNE.

Uniqueness of the limit of a function

Ex. Let $D = [1,2] \cup \{0\}$ and let $f: D \to \mathbb{R}$. $f \equiv 0$ (f is constantly 0)

Let $a \in \mathbb{R}$ be arbitrary. Show that $\lim_{x\to 0} f = a$ i.e. Any real number is a limit of f at 0.



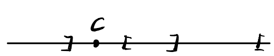
① using seq def. We need to show that

$\forall (x_n)$ in $D\setminus\{0\}$ with $\lim(x_n) = 0$ it follows that $\lim(f(x_n)) = a$

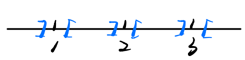
This is true since no such sequence (x_n) exists

DEF
- We say that $c \in \mathbb{R}$ is a limit point of D if $\exists (x_n)$ in $D\setminus\{c\}$ s.t. $\lim(x_n) = c$
- We say that $c \in \mathbb{R}$ is a isolated point of D if $\nexists (x_n)$ in $D\setminus\{c\}$ s.t. $\lim(x_n) = c$

THM Let $D \subseteq \mathbb{R}$. $c \in D$ is an isolated point of D iff $\exists \varepsilon > 0: V_\varepsilon(c) \cap D\setminus\{c\} = \emptyset$



① $D = \mathbb{N}$



All points in $\mathbb{N}$ are isolated

② $D = (0,1]$

Every point in D is a limit point and so is 0. $\lim(\frac{1}{n}) = 0$

THM Let $f: D \to \mathbb{R}$, $c \in \mathbb{R}$ s.t. $\lim_{x \to c} f$ exists

(a) If c is a limit point of D, then $\lim_{x \to c} f$ is uniquely def.

i.e. if $\lim_{x \to c} L_1$ and $\lim_{x \to c} L_2$, then $L_1 = L_2$

(b) If c is an isolated point of D, then any $a \in \mathbb{R}$ is a limit of f at c.

THM The ε-δ definition and the sequential def are equivalent.

Nov 16

THM: ALGEBRAIC LIMIT LAWS

LET $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$, c A LIMIT POINT OF D. ASSUME THAT $\lim_{x\to c} f$ AND $\lim_{x\to c} g$ EXIST. THEN

(a) $\lim_{x\to c} (f+g) = \lim_{x\to c} f + \lim_{x\to c} g$

(b) $\lim_{x\to c} (f-g) = \lim_{x\to c} f - \lim_{x\to c} g$

(c) $\forall k \in \mathbb{R}: \lim_{x\to c} (k \cdot f) = k \cdot \lim_{x\to c} f$

(d) $\lim_{x\to c} (f \cdot g) = \lim_{x\to c} f \cdot \lim_{x\to c} g$

(e) IF $\forall x \in D: g(x) \neq 0$ AND $\lim_{x\to c} g \neq 0$, THEN

$$\lim_{x\to c} \frac{f}{g} = \frac{\lim_{x\to c} f}{\lim_{x\to c} g}$$

PROOF: (a) LET (x_n) BE ANY SEQ IN $D\setminus\{c\}$ S.T. $\lim(x_n) = c$. THEN:

$$\lim_{x\to c} (f+g) \overset{\text{SEQ. DEF.}}{=} \lim\left((f+g)(x_n)\right) = \lim\left(f(x_n) + g(x_n)\right)$$

$$\overset{\text{LIMIT LAW FOR SEQ}}{=} \lim(f(x_n)) + \lim(g(x_n))$$

$$\overset{\text{SEQ. DEF}}{=} \lim_{x\to c} f + \lim_{x\to c} g$$

$$\Rightarrow \lim_{x\to c} (f+g) = \lim_{x\to c} f + \lim_{x\to c} g.$$

(e) LET (x_n) BE ANY SEQ IN $D\setminus\{c\}$ S.T. $\lim(x_n) = c$.

THEN: $\lim_{x\to c} \frac{f}{g} = \lim\left(\frac{f(x_n)}{g(x_n)}\right) = \frac{\lim(f(x_n))}{\lim(g(x_n))}$

$g(x_n) \neq 0$

$\neq 0$, SINCE $\lim(g(x_n)) = \lim_{x\to c} g \neq 0$

$$= \frac{\lim_{x \to a} f}{\lim_{x \to a} g}$$ THIS IS WHAT WE HAD TO SHOW.

(b) - (d): EXERCISE ■

THM: SQUEEZE THEOREM

LET $f, g, h : D \subseteq \mathbb{R} \to \mathbb{R}$ | LET a LIMIT PT OF D. LET $\forall x \in D: f(x) \leq g(x) \leq h(x)$

AND $\lim_{x \to a} f = \lim_{x \to a} h = L$

THEN $\lim_{x \to a} g$ EXISTS AND $\lim_{x \to a} g = L$.

PROOF: LET (x_n) BE ANY SEQ IN $D \setminus \{a\}$ S.T. $\lim (x_n) = a$.

THEN: $\forall n \in \mathbb{N}: f(x_n) \leq g(x_n) \leq h(x_n)$ AND:

$\lim (f(x_n)) = \lim_{x \to a} f = L$ AND $\lim (g(x_n)) = \lim_{x \to a} g = L$

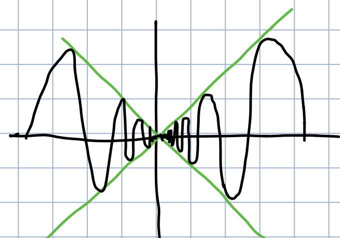
$\Rightarrow \lim (f(x_n)) = \lim (h(x_n)) = L$

IT NOW FOLLOWS FROM THE SQUEEZE THM FOR SEQUENCES THAT $\lim (g(x_n))$ EXISTS AND EQUALS L. THUS $\lim_{x \to a} g = \lim (g(x_n))$ EXISTS AND EQUALS L. THIS IS WHAT WE HAD TO SHOW ■

EX: SHOW THAT $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$

NOTE: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto x \cdot \sin\left(\frac{1}{x}\right)$ i.e. $D = \mathbb{R} \setminus \{0\}$;

0 IS A LIMIT POINT OF D.



NOTE THAT $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ FOR ALL $x \in \mathbb{R}\setminus\{0\}$

$\Rightarrow$ (1) $-x \leq x \sin\left(\frac{1}{x}\right) \leq x$ IF $x>0$, AND

$-x \geq x \sin\left(\frac{1}{x}\right) \geq x$ IF $x<0$

i.e. (2) $x \leq x \sin\left(\frac{1}{x}\right) \leq -x$ IF $x<0$

NOTE THAT $|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x<0 \end{cases}$. THUS (1) AND (2)

CAN BE REWRITTEN AS FOLLOWS:

(1)' $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$ IF $x>0$, AND

(2)' $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$ IF $x<0$

$\Rightarrow$ (3) $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$ FOR ALL $x \neq 0$

CLAIM: $\lim_{x\to 0} |x| = 0$.

PROOF: LET $\varepsilon>0$ AND LET $\delta := \varepsilon$. LET $|x-0| = |x| < \delta$.

THEN $\big| |x| - 0 \big| = |x| < \delta = \varepsilon$. IT NOW FOLLOWS

FROM THE $\varepsilon-\delta$ DEF. OF THE LIMIT OF A FUNCTION

THAT $\lim_{x\to 0} |x| = 0$.

NOW: (3) $-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$, WHERE

$$\lim_{x\to 0} |x| = 0 \quad \text{AND} \quad \lim_{x\to 0} (-|x|) \overset{\text{ALG. LIMIT LAWS}}{=} -\lim_{x\to 0} |x| = 0$$

$$\overset{\text{SQUEEZE THM}}{\Rightarrow} \quad \lim_{x\to 0} \left(x \cdot \sin\left(\frac{1}{x}\right)\right) = 0. \quad \blacksquare$$

4.3. CONTINUITY

DEF: CONTINUITY (LIMIT VERSION)

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. WE SAY THAT f IS

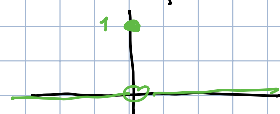
CONTINUOUS AT c, IF $\lim_{x\to c} f = f(c)$

EX: ① $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, $a \in \mathbb{R}$ ARBITRARY. SHOW THAT f IS CONTINUOUS AT a.

PROOF: $\lim\limits_{x \to a} x^2 = a^2$ (AS SEEN IN PREV. EXAMPLE)

$= f(a)$

$\Rightarrow$ f IS CONT. AT ALL $a \in \mathbb{R}$.

② 

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} 1 & \text{IF } x = 0 \\ 0 & \text{IF } x \neq 0 \end{cases}$$

SHOW THAT f IS DISCONTINUOUS (I.E. NOT CONTINUOUS) AT 0.

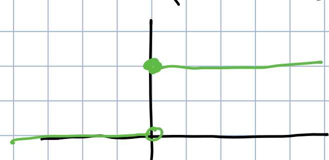
PROOF: LET'S FIRST DET. $\lim\limits_{x \to 0} f$.

LET (x_n) BE ANY SEQ IN $\mathbb{R} \setminus \{0\}$. THEN $f(x_n) = 0$ SINCE $x_n \neq 0$. $\Rightarrow \lim(f(x_n)) = \lim(0) = 0$

i.e. $\lim\limits_{x \to 0} f = 0$.

BUT: $\lim\limits_{x \to 0} f = 0 \neq 1 = f(0)$

THUS f IS DISCONT. AT 0.

③ 

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{IF } x \geq 0 \\ 0 & \text{IF } x < 0 \end{cases}$$

WE'LL SHOW THAT $\lim\limits_{x \to 0} f$ DNE.

LET $\forall n \in \mathbb{N}: x_n := \frac{1}{n}$ AND LET $\forall n \in \mathbb{N}: u_n := -\frac{1}{n}$.

THEN $\lim(x_n) = \lim(u_n) = 0$ AND

$\lim(f(\underbrace{x_n}_{>0})) = \lim(1) = 1$ AND

$\lim(f(\underbrace{u_n}_{<0})) = \lim(-1) = 0$

SINCE $\lim (f(x_n)) \neq \lim (f(u_n))$ WE CONCLUDE FROM THE 2-SEQ. CRITERION THAT $\lim\limits_{x \to 0} f$ DNE.

SINCE $\lim\limits_{x \to 0} f$ DNE, IT ESPECIALLY FOLLOWS THAT $\lim\limits_{x \to 0} f$ DOES NOT EQUAL $f(0)$. THUS f IS DISCONT. AT 0.

REMARK: A FUNCTION $f: D \to \mathbb{R}$, $c \in D$ CAN FAIL TO BE CONT. AT c IN 2 DIFFERENT WAYS:

1. $\lim\limits_{x \to c} f$ EXISTS, BUT DIFFERS FROM $f(c)$; OR:
2. $\lim\limits_{x \to c} f$ DNE.

DEF: CONTINUITY (ε-δ VERSION)

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. WE SAY THAT f IS CONTINUOUS AT c IF:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

REMARK: THE ε-δ DEF OF CONT. CAN BE EXPRESSED IN VARIOUS, LOGICALLY EQUIVALENT, WAYS:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in V_\delta(c) \cap D : f(x) \in V_\varepsilon(f(c))$$

$$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 : f\left(V_\delta(c) \cap D\right) \subseteq V_\varepsilon(f(c))$$

THM: THE LIMIT VERSION AND THE ε-δ VERSION OF CONTINUITY ARE EQUIVALENT.

Nov 18

LET'S RECALL THE LIMIT AND THE ε-δ VERSION OF THE DEFINITION OF CONTINUITY. LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D$. WE SAY THAT f IS CONTINUOUS AT a IF

- LIMIT VERSION: $\lim_{x \to a} f = f(a)$

- ε-δ VERSION: $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D: |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

THERE'S A THIRD STANDARD WAY OF DEFINING CONTINUITY:

- SEQUENTIAL VERSION: $\forall (x_n)$ IN D WITH $\lim (x_n) = a$ IT HOLDS THAT $\lim (f(x_n)) = f(a)$.

WE NEED TO SHOW THAT ALL 3 VERSION OF THE DEF. OF CONT. ARE EQUIVALENT.

THM: THE LIMIT, ε-δ AND SEQ. VERSIONS OF THE DEFINITION OF CONTINUITY ARE EQUIVALENT.

PROOF: LIMIT VERSION IS EQ. TO ε-δ VERSION:

"$\Leftarrow$" LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D$ BE CONT. BY ε-δ.

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D: |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

THUS WE HAVE ESPECIALLY

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D \setminus \{a\}: |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

$\Rightarrow \lim_{x \to a} f = f(a) \Rightarrow f$ IS CONT. AT a BY LIMIT VERSION.

"$\Rightarrow$" LET $f: D \to \mathbb{R}$, $a \in D$ BE CONT. AT a BY LIMIT VERSION i.e. $\lim_{x \to a} f = f(a)$. THUS

(*) $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D \setminus \{a\}: |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

IF $x = a$ WE HAVE THAT $|x - a| = 0 < \delta$ AND

$|f(x)-f(c)| = 0 < \varepsilon$. THUS " $\underbrace{\underbrace{|x-c| < \delta}_{\text{TRUE}} \Rightarrow \underbrace{|f(x)-f(c)| < \varepsilon}_{\text{TRUE}}}_{\text{TRUE}}$ "

IS A TRUE STATEMENT.

COMBINING THIS WITH (*) YIELDS:

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D: |x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$

THUS f IS CONT. AT c BY ε-δ.

LIMIT VERSION IS EQ. TO SEQUENTIAL VERSION:

EXERCISE.

THUS ALL 3 VERSIONS ARE EQUIVALENT ∎

EX: 1 $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x^2}$, $c \in \mathbb{R}\setminus\{0\}$.

SHOW THAT f IS CONT. AT c.

PROOF: WE'VE SEEN BEFORE THAT

$\lim_{x \to c} \frac{1}{x^2} = \frac{1}{c^2} = f(c)$. THUS f IS CONT AT c BY THE LIMIT DEF. OF CONT.

2 $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. SHOW THAT f IS CONT. AT 0.

PROOF: WE'LL USE THE ε-δ DEF. OF CONT.

LET $\varepsilon > 0$ AND LET $\delta := \varepsilon$. LET $|x-0| < \delta$. THEN

$|f(x)-f(0)| = \big| |x| - 0 \big| = |x| < \delta = \varepsilon$

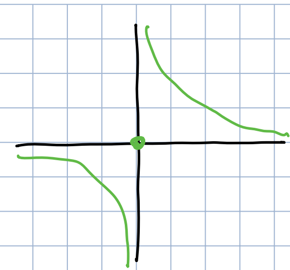
THUS $|x-0| < \delta \Rightarrow |f(x)-f(0)| < \varepsilon$

$\Rightarrow f$ IS CONT. AT 0.

3 $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{x} & \text{IF } x \neq 0 \\ 0 & \text{IF } x = 0 \end{cases}$

SHOW THAT f IS DISCONTINUOUS AT 0.

PROOF:



LET $\forall n \in \mathbb{N}: x_n := \frac{1}{n}$. THEN $\lim (x_n) = 0$ BUT $\forall n \in \mathbb{N}$: $(f(x_n)) = (n)$. THUS $(f(x_n))$ DIVERGES AND THUS ESPECIALLY DOESN'T CONVERGE TO $f(0) = 0$.

THUS f IS DISCONT AT 0 BY SEQ. VERSION OF CONT.

GENERALIZING THE LAST EXAMPLE YIELDS:

THM: SEQ. CRITERION FOR DISCONTINUITY:

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $c \in D$. IF $\exists (x_n)$ IN D WITH $\lim (x_n) = c$ S.T.

- $(f(x_n))$ DIVERGES, OR
- $(f(x_n))$ CONVERGES BUT $\lim (f(x_n)) \neq f(c)$

THEN f IS DISCONTINUOUS AT c.

PROOF: IMMEDIATE FROM THE SEQ. VERSION OF CONT. $\blacksquare$

THM: ALGEBRAIC CONT. THM

LET $f, g: D \subseteq \mathbb{R} \to \mathbb{R}$, LET $c \in D$; LET f, g BE CONT. AT c. THEN:

(a) $f + g$ IS CONT. AT c

(b) $\forall k \in \mathbb{R}$: kf IS CONT. AT c

(c) $f - g$ IS CONT AT c

(d) $f \cdot g$ IS CONT AT c

(e) IF $\forall x \in D : g(x) \neq 0$ THEN $\frac{f}{g}$ IS CONT. AT c

PROOF: (a) f, g CONT AT $c \Rightarrow \lim_{x \to c} f = f(c)$ AND $\lim_{x \to c} g = g(c)$. THUS $\lim_{x \to c} (f+g) = \lim_{x \to c} f + \lim_{x \to c} g$

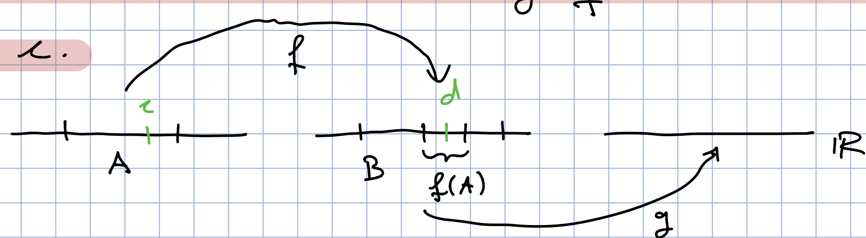
$$= f(c) + g(c) = (f+g)(c) \text{ i.e.}$$

$\lim_{x \to c} (f+g) = (f+g)(c) \Rightarrow f+g$ IS CONT. AT c.

(b) - (e) : EXERCISE ∎

THM: LET $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $f(A) \subseteq B$. LET $c \in A$, $d := f(c)$; LET f BE CONT. AT c AND LET g BE CONT. AT d. THEN $g \circ f : A \to \mathbb{R}$ IS CONT. AT c.

PROOF:



LET (x_n) BE A SEQ. IN A WITH $\lim (x_n) = c$.

f IS CONT. AT c. THUS $\lim (f(x_n)) = f(c) = d$.

THUS $(f(x_n))$ IS A SEQ. IN B S.T. $\lim (f(x_n)) = d$;

g IS CONT. AT d; THUS $\lim (g(f(x_n))) = g(d) = g(f(c))$

$\Rightarrow$ FOR ANY SEQ. (x_n) IN A WITH $\lim (x_n) = c$, IT HOLDS THAT $\lim ((g \circ f)(x_n)) = (g \circ f)(c)$

THUS $g \circ f$ IS CONT. AT c BY SEQ. VERSION OF CONT. ∎

EXERCISE: RE-PROVE THIS USING ε-δ VERSION OF CONT.

EX: ① ALL POLYNOMIALS ARE CONT. AT ALL $c \in \mathbb{R}$.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

PROOF: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ IS CONT. AT ALL $c \in \mathbb{R}$.
BY ALG. CONT. THEOREM (PRODUCTS), IT FOLLOWS THAT $\forall n \in \mathbb{N}$: x^n IS CONT. AT ALL $c \in \mathbb{R}$. THUS FOR ANY $a_n \in \mathbb{R}$, $a_n x^n$ IS CONT. AGAIN, BY ALG. CONT. THM (SUM), $a_n x^n + \dots + a_0$ IS CONT AT ALL $c \in \mathbb{R}$.

② IN THE TUTORIAL TODAY, YOU WILL SEE THAT

$\sqrt{}: \mathbb{R}^+_0 \to \mathbb{R}$, $x \mapsto \sqrt{x}$ IS CONT AT ALL $c \geq 0$.

BY THE COMPOSITION THM, WE HAVE e.g. THAT $\sqrt{\underbrace{x^2+1}_{\geq 0}}$ IS CONT. AT ALL $c \in \mathbb{R}$.

CONTINUITY AND TOPOLOGY

DEF: LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ S.T. f IS CONT. AT ALL $c \in D$. WE THEN SAY THAT f IS CONTINUOUS (ON D)

Q: LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ BE CONTINUOUS. LET $A \subseteq D$.

- IF A IS OPEN, IS $f(A)$ ALSO OPEN?
- IF A IS CLOSED, IS $f(A)$ ALSO CLOSED?
- IF A IS BOUNDED, IS $f(A)$ ALSO BOUNDED?

THE ANSWER TO ALL OF THESE QUESTIONS IS "NO", IN GENERAL. HOWEVER:

IF A IS BOTH CLOSE AND BOUNDED (I.E. COMPACT) THEN $f(A)$ IS AGAIN CLOSED + BOUNDED, AS WE WILL SEE NEXT CLASS.

EX: ① $f: \mathbb{R} \to \mathbb{R}$, $\underbrace{x \mapsto x^2}_{\text{CONT.}}$; $A := \underbrace{]-1,1[}_{\text{OPEN}}$. THEN $f(A) = \underbrace{[0,1[}_{\text{NOT OPEN}}$

② $f: \mathbb{R} \to \mathbb{R}$, $\underbrace{x \mapsto \frac{1}{x}}_{\text{CONT.}}$; $A := \underbrace{[1, \infty[}_{\text{CLOSED}}$. THEN $f(A) = \underbrace{]0,1]}_{\text{NOT CLOSED}}$

③ $f: \mathbb{R} \to \mathbb{R}$, $\underbrace{x \mapsto \frac{1}{x}}_{\text{CONT}}$; $A := \underbrace{]0,1]}_{\text{BOUNDED}}$. THEN $f(A) = \underbrace{[1, \infty[}_{\text{UNBOUNDED}}$

HOWEVER:

THM: PRESERVATION OF COMPACTNESS

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ BE CONTINUOUS. LET $A \subseteq D$ BE COMPACT. THEN $f(A)$ IS COMPACT.

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RECALL:

- A SUBSET $A \subseteq \mathbb{R}$ IS CALLED <u>COMPACT</u> IF IT IS BOTH CLOSED AND BOUNDED.
- A SUBSET $A \subseteq \mathbb{R}$ IS CALLED <u>SEQUENTIALLY COMPACT</u> IF <u>EVERY</u> SEQUENCE (x_n) IN A HAS A CONVERGENT SUBSEQUENCE WHOSE LIMIT IS IN A.

<u>THM</u>: $A \subseteq \mathbb{R}$ IS COMPACT IFF IT IS SEQUENTIALLY COMPACT.

A PROOF OF THIS WAS POSTED ON MYCOURSES.

<u>REMARK</u>: THE THM ABOVE ALSO HOLDS IN $\mathbb{R}^n$ (BUT DOESN'T HOLD IN MORE GENERAL SPACES e.g. METRIC SPACES)

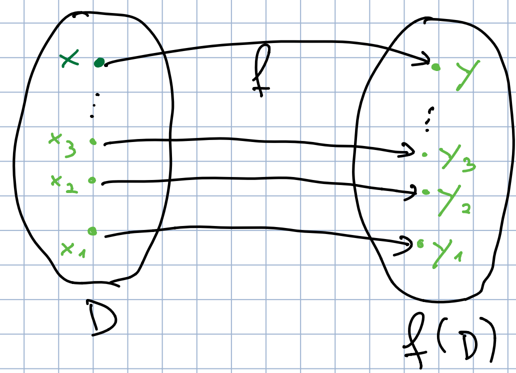
<u>THM</u>: PRESERVATION OF COMPACTNESS UNDER CONTINUOUS MAPS.

LET $D \subseteq \mathbb{R}$ BE COMPACT AND LET $f: D \to \mathbb{R}$ BE CONTINUOUS. THEN $f(D)$ IS COMPACT.

<u>PROOF</u>: SINCE D IS COMPACT, IT IS SEQUENTIALLY COMPACT. WE WILL PROVE THAT $f(D)$ IS SEQ. COMPACT WHICH WILL ALSO PROVE BY THE THM ABOVE THAT $f(D)$ IS COMPACT.

LET (y_m) BE AN ARBITRARY SEQUENCE IN $f(D)$. SINCE (y_m) IS A SEQ IN $f(D)$ THERE EXISTS AT LEAST ONE $x_n \in D$ WITH $f(x_n) = y_m$ FOR ALL $n \in \mathbb{N}$. FOR ALL $n \in \mathbb{N}$ SELECT ONE SUCH x_n AND CONSIDER

THE SEQUENCE (x_n).



SINCE D IS SEQ. COMPACT, (x_n) HAS A CONV. SUBSEQ. (x_{n_k}) WHOSE LIMIT x LIES IN D.

LET $y := f(x) \in f(D)$.

WE NOW HAVE THE FOLLOWING: $\exists$ SUBSEQ (x_{n_k}) OF (x_n) WITH $x = \lim (x_{n_k})$, WHERE $x \in D$. SINCE f IS CONT. AT x WE HAVE THAT $(f(x_{n_k})) = (y_{n_k})$ CONV. TO $f(x) = y$.

THUS THE SUBSEQUENCE (y_{n_k}) OF (y_n) CONV. TO $y = f(x)$, WHICH MEANS THAT $y \in f(D)$.

THUS $f(D)$ IS SEQUENTIALLY COMPACT AND THUS COMPACT ∎

DEF: LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$. A POINT $x_0 \in D$ IS CALLED

- AN ABSOLUTE (OR GLOBAL) MAXIMUM, IF

$$\forall x \in D: f(x) \leq f(x_0)$$

- AN ABSOLUTE (OR GLOBAL) MINIMUM, IF

$\forall x \in D: f(x) \geq f(x_0)$

THM: EXTREME VALUE THM

LET $D \subseteq \mathbb{R}$ BE COMPACT AND LET $f: D \to \mathbb{R}$ BE CONTINUOUS. THEN f HAS BOTH AN ABSOLUTE MAX AND AN ABSOLUTE MIN (IN D).

PROOF: IT FOLLOWS FROM PRESERVATION OF COMPACTNESS THAT $f(D)$ IS COMPACT AND THUS CLOSED AND BOUNDED.

SINCE $f(D)$ IS BOUNDED, IT HAS A SUPREMUM M AND AN INFIMUM m. WE HAVE:

$\forall x \in D: \quad m = \inf(f(D)) \leq f(x) \leq M = \sup(f(D))$

RECALL THAT THE SUPREM AND THE INFIMUM OF ANY NON-EMPTY, BOUNDED SET ARE BOTH BOUNDARY POINTS OF THE SET. HERE:

$m, M \in \partial(f(D))$. HOWEVER, SINCE $f(D)$ IS COMPACT, IT IS CLOSED, AND THUS CONTAINS ALL OF ITS BOUNDARY POINTS.

$\Rightarrow \quad m, M \in \underbrace{\partial(f(D))}_{\subseteq f(D)} \quad \Rightarrow \quad m, M \in f(D)$

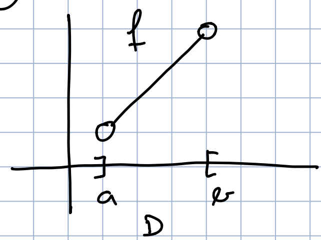
THUS THERE EXIST $x_0, x_1 \in D$ WITH $f(x_0) = m$ AND $f(x_1) = M$. THEN $\forall x \in D$:

$f(x_0) = m \leq f(x) \leq M = f(x_1)$

THUS x_0 IS AN ABSOLUTE MIN OF f ON D AND x_1 IS AN ABSOLUTE MAX OF f ON D. ∎

REMARK: ALL CONDITIONS IN THIS THM (i.e. D IS CLOSED AND BOUNDED, f IS CONT ON D) ARE ESSENTIAL.

EX: (1) D NOT CLOSED BUT BOUNDED, f CONT.

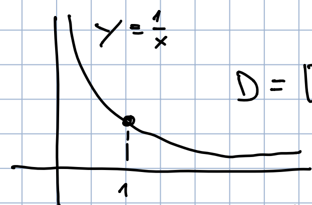


NOTE THAT $f(D)$ HAS NEITHER A MAX NOR A MIN.

$\sup(f(D)) = f(b) \notin f(D)$, AND

$\inf(f(D)) = f(a) \notin f(D)$

(2) D CLOSED BUT NOT BOUNDED, f CONT.

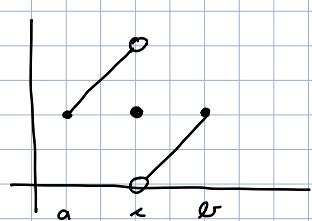


$D = [1, \infty[$ CLOSED + UNBOUNDED

f DOESN'T HAVE AN ABS. MIN:

$f(D) =]0, 1]$ WITH $\inf(f(D)) = 0 \notin f(D)$

(3) D IS COMPACT BUT f IS DISCONT.

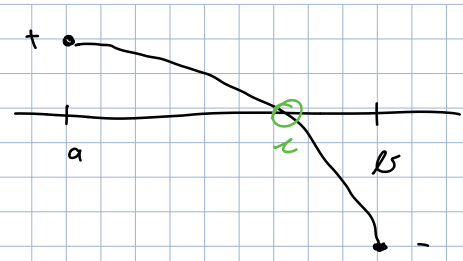


f HAS NEITHER MAX NOR MIN.

THM: LOCALIZATION OF ROOTS (BOLZANO)

LET $a, b \in \mathbb{R}$, $a < b$ AND LET $f : [a, b] \to \mathbb{R}$ CONT S.T. $f(a)$ AND $f(b)$ HAVE OPPOSITE SIGNS (i.e. EITHER $f(a) > 0$ AND $f(b) < 0$ OR VICE VERSA).

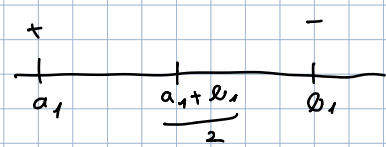
THEN $\exists c \in]a, b[$ S.T. $f(c) = 0$.



PROOF: WE WILL USE THE NESTED INTERVAL PROPERTY.

LET $a_1 := a$, $b_1 := b$ AND LET $I_1 = [a_1, b_1]$

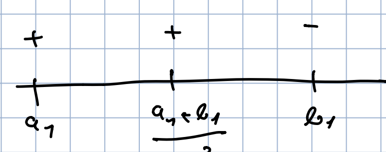
ASSUME W.L.O.G. THAT $f(a_1) > 0$ AND $f(b_1) < 0$ (OTHERWISE, CONSIDER $-f$ INSTEAD OF f. NOTE THAT THE ROOTS OF f AND $-f$ ARE EXACTLY THE SAME).



DIVIDE I_1 INTO 2 SUBINTERVALS OF EQUAL WIDTH.

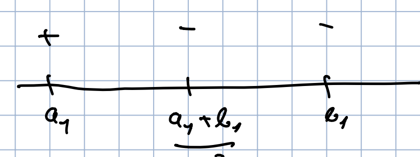
IF $f\left(\frac{a_1+b_1}{2}\right) = 0$ WE SET $c := \frac{a_1+b_1}{2}$ AND WE'RE DONE.

IF $f\left(\frac{a_1+b_1}{2}\right) > 0$:



SET $a_2 := \frac{a_1+b_1}{2}$, $b_2 := b_1$

IF $f\left(\frac{a_1+b_1}{2}\right) < 0$:



SET $a_2 := a_1$, $b_2 := \frac{a_1+b_1}{2}$

IN ANY CASE, LET $I_2 := [a_2, b_2]$

DIVIDE I_2 INTO 2 SUBINTERVALS OF EQUAL WIDTH AND SELECT ONE OF THEM AS I_3 IN THE

SAME FASHION AS ABOVE.

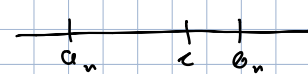
WE NOW HAVE THE FOLLOWING:

EITHER THIS ALGORITHM STOPS AFTER FINITELY MANY STEPS IN CASE $f\left(\frac{a_n+b_n}{2}\right)=0$ FOR SOME $n \in \mathbb{N}$ IN WHICH CASE WE'VE FOUND OUR ROOT, OR WE OBTAIN A NESTED SEQ. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ OF CLOSED + BD INTERVALS.

BY NESTED INTERVAL PROPERTY, $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$

LET $c \in \bigcap_{n\in\mathbb{N}} I_n$. WE WILL SHOW THAT $f(c)=0$.

BY CONSTRUCTION $\forall n \in \mathbb{N}: |a_n - c| \leq |a_n - b_n| = |I_n|$



AND $\forall n \in \mathbb{N}: |b_n - c| \leq |a_n - b_n| = |I_n|$

WHERE $|I_n| = \dfrac{b-a}{2^{n-1}} = 2(b-a)\cdot\dfrac{1}{2^n}$

THUS $\forall n \in \mathbb{N}: \quad |a_n - c| \leq \underbrace{\underbrace{2(b-a)}_{\text{CONST.}}\cdot\underbrace{\frac{1}{2^n}}_{\text{NULL SEQ.}}}_{\text{NULL SEQ.}}$

$\Rightarrow \lim (a_n) = c$. FOR THE SAME REASON, $\lim (b_n) = c$.

f IS CONT AND THUS ESPICIALLY CONT AT c.

THUS $\lim (f(a_n)) = f(c)$ AND $\lim (f(b_n)) = f(c)$

BY CONSTRUCTON: $\forall n \in \mathbb{N}$ $f(a_n) > 0$ AND $f(b_n) < 0$

$\Rightarrow \lim \big(\underbrace{f(a_n)}_{>0} \big) \geq 0$ AND $\lim \big(\underbrace{f(b_n)}_{<0} \big) \leq 0$

$\quad \| \qquad\qquad\qquad\qquad\qquad \|$

$f(c) \qquad\qquad\qquad\qquad\quad f(c)$

THUS $f(c) \geq 0$ AND $f(c) \leq 0 \Rightarrow \underline{f(c) = 0}$

WE HAVE FOUND THE ROOT WE WERE LOOKING FOR. $\blacksquare$

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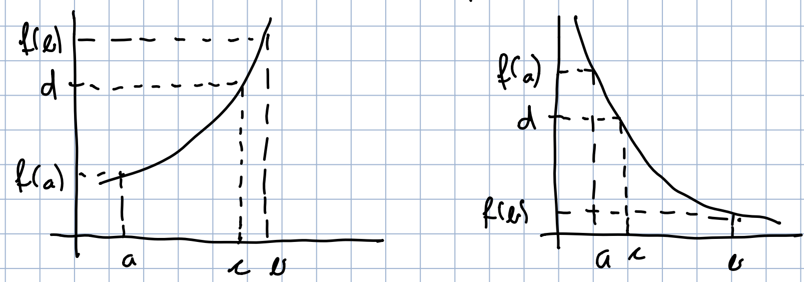
THM: THE INTERMEDIATE VALUE THM (BOLZANO)

LET $a, b \in \mathbb{R}$, $a < b$, AND LET $f : [a,b] \to \mathbb{R}$ BE CONTINUOUS.
LET d BE A REAL NUMBER BETWEEN $f(a)$ AND $f(b)$ i.e.
EITHER IT HOLDS THAT $f(a) < d < f(b)$ OR $f(b) < d < f(a)$.
THEN $\exists c \in]a,b[$ WITH $f(c) = d$.



PROOF: CONSIDER THE FUNCTION $g : [a,b] \to \mathbb{R}$, $g(x) := f(x) - d$
THEN g IS CONT AND:
- IF $f(a) < d < f(b)$ THEN $g(a) = f(a) - d < 0$ AND
  $g(b) = f(b) - d > 0$
- IF $f(b) < d < f(a)$ THEN $g(a) = f(a) - d > 0$ AND
  $g(b) = f(b) - d < 0$

IN EITHER CASE, f ASSUMES VALUES OF OPPOSITE SIGN AT a AND b. BY LOCALIZATION OF ROOTS THM
$\exists c \in]a,b[$ WITH $g(c) = 0 = f(c) - d \Rightarrow f(c) = d$ ∎

UNIFORM CONTINUITY

LET'S RECALL THE DEF OF. CONT. OF $f : D \to \mathbb{R}$ AT $c \in D$
(USING u INSTEAD OF x):
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall u \in D : |u - c| < \delta \Rightarrow |f(u) - f(c)| < \varepsilon$
IF f IS CONT. ON D (i.e. AT ALL $c \in D$) WE GET:

$\forall c \in D\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall u \in D: |u - c| < \delta \Rightarrow |f(u) - f(c)| < \varepsilon$

SINCE c IS A VARIABLE IN THIS CONTEXT, WE WILL WRITE x INSTEAD OF c:

$$\forall x \in D\ \forall \varepsilon > 0\ \exists \underbrace{\delta = \delta(x,\varepsilon)}\ > 0 \ \overset{\forall u \in D}{\underline{|}} : |u - x| < \delta \Rightarrow |f(u) - f(x)| < \varepsilon$$

δ WILL DEPEND, IN GENERAL ON BOTH x AND ε

$$\equiv \forall x \in D\ \forall \varepsilon > 0\ \exists \delta = \delta(x,\varepsilon) > 0 \ \overset{\forall u \in D}{\underline{|}} : |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

IN GENERAL, $\delta = \delta(x, \varepsilon)$ WILL DEPEND ON BOTH ε (UNAVOIDABLE) AND x. THERE ARE CASES WHERE ONE WOULD LIKE TO BE ABLE TO FIND A δ THAT DOES NOT DEPEND ON x (e.g. MATH 243 INTEGRATION) IN SUCH A CASE (WHERE δ DOES NOT DEPEND ON x) WE HAVE:

$$\forall x \in D\ \underbrace{\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0}_{\text{DOES NOT DEPEND ON } x}\ \forall u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

$$\equiv \forall \varepsilon > 0\ \exists \delta > 0\ \forall x, u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

WE DEFINE:

DEF: A FUNCTION $f: D \subseteq \mathbb{R} \to \mathbb{R}$ IS SAID TO BE UNIFORMLY CONTINUOUS (ON D) IF

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon.$$

THM: SEQUENTIAL CRITERION FOR THE ABSENCE OF UNIFORM CONTINUITY

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$. f IS NOT UNIFORMLY CONT. IFF $\exists \varepsilon > 0$ AN SEQ. $(x_n), (u_n)$ IN D S.T.

$\lim (x_n - u_n) = 0$ AND $\forall n \in \mathbb{N}: |f(x_n) - f(u_n)| \geq \varepsilon$

PROOF: " $\Rightarrow$ " LET f NOT BE UNIFORMLY CONT. THEN

$\neg(\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon)$

$\equiv\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists x, u \in D: \neg(|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon)^{(*)}$

RECALL: $P \Rightarrow Q \equiv \neg P \vee Q$ THUS

$\neg(P \Rightarrow Q) \equiv \neg(\neg P \vee Q) \equiv P \wedge \neg Q$ IN $(*)$

$\equiv\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists x, u \in D: |x - u| < \delta \wedge |f(x) - f(u)| \geq \varepsilon$

SELECT ONE SUCH ε, LET $n \in \mathbb{N}$ BE ARBITRARY

AND LET $\delta := \frac{1}{n}$. THEN $\exists x_n, u_n \in D$:

$|x_n - u_n| < \delta = \frac{1}{n}$ AND $|f(x_n) - f(u_n)| \geq \varepsilon$

CONSIDER (x_n) AND (u_n). THEN:

$\forall n \in \mathbb{N}: |x_n - u_n| < \frac{1}{n} \Rightarrow \lim (x_n - u_n) = 0$

AND : $\forall n \in \mathbb{N}: |f(x_n) - f(u_n)| \geq \varepsilon$. THIS IS WHAT WE HAD TO SHOW.

" $\Leftarrow$ " LET $\varepsilon > 0$, (x_n), (u_n) S.T. $\lim (x_n - u_n) = 0$ AND

$\forall n \in \mathbb{N}: |f(x_n) - f(u_n)| \geq \varepsilon$.

ASSUME THAT f IS UNIF. CONT. THEN

$\exists \delta > 0\ \forall x, u \in D: |x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$

$\lim (x_n - u_n) = 0 \Rightarrow \exists N \in \mathbb{N}\ \forall n \geq N: |(x_n - u_n) - 0| = |x_n - u_n| < \delta$

THUS $|f(x_n) - f(u_n)| < \varepsilon$. BUT IT SHOULD BE $\geq \varepsilon$! $\lightning$

THUS f IS NOT UNIF. CONT. $\blacksquare$

EX: 1 LET $a > 0$. SHOW THAT x^2 IS UNIF. CONT. ON $[-a, a]$.

<u>PROOF:</u> LET $\varepsilon > 0$; LET δ BE ARBITRARY FOR NOW.
LET $x, u \in [-a, a]$ WITH $|x-u| < \delta$. THEN

$$|x^2 - u^2| = |(x-u)(x+u)| = |x-u| \cdot |x+u| < \delta \cdot |x+u|$$
$$\leq \delta\,(\underbrace{|x|}_{\leq a} + \underbrace{|u|}_{\leq a}) \leq 2a \cdot \delta < \varepsilon \iff \delta < \underbrace{\frac{\varepsilon}{2a}}_{\text{DOES NOT DEPEND ON } x \text{ OR } u.}$$

LET $\delta < \frac{\varepsilon}{2a}$. THEN, WHENEVER
$|x-u| < \delta$, $x, u \in [-a, a]$, IT HOLDS THAT $|x^2 - u^2| < \varepsilon$
$\Rightarrow$ x^2 IS UNIF. CONT. ON $[-a, a]$.

(2) SHOW THAT x^2 IS <u>NOT</u> UNIF. CONT. ON $[0, \infty[$.

<u>PROOF:</u> LET $\forall n \in \mathbb{N}$: $x_n := n + \frac{1}{n}$, $u_n := n$

THEN: $\lim (x_n - u_n) = \lim (\frac{1}{n}) = 0$.

AND: $|x_n^2 - u_n^2| = |(n + \frac{1}{n})^2 - n^2| = \cancel{n^2} + 2\cdot n \cdot \frac{1}{n} + \frac{1}{n^2} - \cancel{n^2}$
$= 2 + \frac{1}{n^2} > 2$

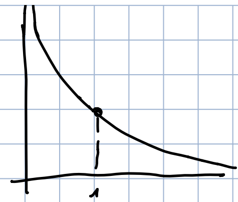
LET $\varepsilon := 2$, THEN $\forall n \in \mathbb{N}$: $|x_n^2 - u_n^2| > \varepsilon = 2$.
THUS x^2 IS <u>NOT</u> UNIF. CONT. ON $[0, \infty[$.

<u>REMARK</u>: OBVIOUSLY, EVERY UNIF. CONT. FUNCTION IS CONTINUOUS. HOWEVER, (2) SHOWS THAT THE CONVERSE DOES <u>NOT</u> HOLD. x^2 IS CONT. BUT <u>NOT</u> UNIF. CONT. ON $[0, \infty[$.

WE SAY THAT THE CONDITION OF UNIFORM CONTINUITY IS STRICTLY STRONGER THAN CONTINUITY.

(3) SHOW THAT $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ IS <u>NOT</u> UNIF. CONT. ON $]0, 1]$

||



LET $x_n := \frac{2}{n}$, $u_n := \frac{1}{n}$ FOR ALL $n \in \mathbb{N}$.

THEN $|x_n - u_n| = \left|\frac{2}{n} - \frac{1}{n}\right| = \frac{1}{n}$

$\Rightarrow \lim (x_n - u_n) = 0$.

AND: $\left|f(x_n) - f(u_n)\right| = \left|\frac{1}{\frac{2}{n}} - \frac{1}{\frac{1}{n}}\right| = \left|\frac{n}{2} - n\right| = \frac{n}{2} \geq \frac{1}{2}$

LET $\varepsilon := \frac{1}{2}$. THEN $\forall n \in \mathbb{N}: |f(x_n) - f(u_n)| \geq \frac{1}{2}$

$\Rightarrow f$ IS NOT UNIF. CONT. ON $[0, \infty[$.

(4) SHOW THAT $\sqrt{x}$ IS UNIF. CONT. ON ALL INTERVALS $[a, \infty[$, WHERE $a > 0$.

LET $\varepsilon > 0$, LET $\delta > 0$ BE ARBITRARY FOR NOW.

LET $x, u \in [a, \infty[$ WITH $|x - u| < \delta$. THEN:

$$\left|\sqrt{x} - \sqrt{u}\right| = \left|\frac{(\sqrt{x} - \sqrt{u})(\sqrt{x} + \sqrt{u})}{\sqrt{x} + \sqrt{u}}\right| = \frac{|x - u|}{\sqrt{x} + \sqrt{u}} < \frac{\delta}{\underbrace{\sqrt{x}}_{\geq \sqrt{a}} + \underbrace{\sqrt{u}}_{\geq \sqrt{a}}}$$

$$\leq \underbrace{\frac{\delta}{2\sqrt{a}}}_{\substack{\text{DOES NOT} \\ \text{DEPEND ON} \\ x \text{ OR } u}} < \varepsilon \iff \delta < 2\sqrt{a} \cdot \varepsilon$$

IF $\delta < 2\sqrt{a} \cdot \varepsilon$ WE HAVE THAT $|\sqrt{x} - \sqrt{u}| < \varepsilon$, WHENEVER $|x - u| < \delta$. $\Rightarrow \sqrt{x}$ IS UNIF. CONT. ON $[a, \infty[$.

<u>REMARK</u>: NOTE THAT THE PROOF ABOVE CANNOT BE APPLIED TO $[0, \infty[$, SINCE WE WOULD BE DIVIDING BY 0! IT IS TEMPTING TO CONCLUDE FROM THAT THAT $\sqrt{x}$ IS NOT UNIF. CONT. ON $[0, \infty[$.

HOWEVER, THIS CONCLUSION WOULD BE WRONG!
$\sqrt{x}$ <u>IS</u> UNIF. CONT. ON $[0, \infty[$. BUT THIS REQUIRES
USING A DIFFERENT METHOD.