Chapter 7 Continuous-Time Markov Chains

A continuous time stochastic process $\{X_t : t \geq 0\}$ with discrete state space S is a CTMC if

$$Pr(X_{t+s} = j \mid X_s = i, X_u = x_u, 0 \leq u < s)$$
$$= Pr(X_{t+s} = j \mid X_s = i) \quad \forall s,t \geq 0 \quad i,j,x_u \in S \quad 0 \leq u < s$$

Time homogeneous if

$$Pr(X_{t+s} = j \mid X_s = i) = Pr(X_t = j \mid X_0 = i) \ \forall \ s,t \geq 0$$

Transition matrix function

$P(t)$ is a $|S| \times |S|$ matrix function where

$$P_{ij}(t) = Pr(X_t = j \mid X_0 = i)$$

Chapman-Kolmogorov equations

$P(s+t) = P(s)P(t)$ i.e. $P_{ij}(s+t) = [P(s)P(t)]_{ij} \ \forall i,j \in S;\ s,t \geq 0$

Proof

$$P_{ij}(s+t) = Pr(X_{s+t} = j \mid X_0 = i)$$
$$= \sum_{k \in S} Pr(X_{s+t} = j \mid X_s = k, X_0 = i) Pr(X_s = k \mid X_0 = i)$$
$$\overset{MC}{=} \sum_{k \in S} Pr(X_{s+t} = j \mid X_s = k) Pr(X_s = k \mid X_0 = i)$$
$$\overset{TH}{=} \sum_{k \in S} Pr(X_t = j \mid X_0 = k) Pr(X_s = k \mid X_0 = i)$$
$$= \sum_{k \in S} Pr(X_s = k \mid X_0 = i) Pr(X_t = j \mid X_0 = k)$$
$$= [P(s)P(t)]_{ij}$$

e.g. Poisson process is a CTMC.

$S = \{0, 1, \dots\}$

$$P_{ij}(t) = \Pr(N_{s+t} = j \mid N_s = i) = \frac{\Pr(N_{s+t} = j, N_s = i)}{\Pr(N_s = i)}$$

$$= \frac{\Pr(N_{s+t} - N_s = j-i, N_s = i)}{\Pr(N_s = i)}$$ independent

$$= \frac{\Pr(N_{s+t} - N_s = j-i)\Pr(N_s = i)}{\Pr(N_s = i)}$$

$$= \Pr(N_{s+t} - N_s = j-i)$$

$$= \frac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!} \quad \text{for } j \geq i$$

$$P(t) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & e^{-\lambda t} & e^{-\lambda t}\lambda t & \cdots \\ 1 & 0 & e^{-\lambda t} & \ddots \\ \vdots & & 0 & \ddots \end{array}$$

Note CTMC holds (also TH holds since only t matters).

Holding time: Time spent in a state before transitioning to any other state

Let T_i be the random holding time for state i. For CTMC, T_i must be exponentially distributed.

<u>proof</u> Fact: The exponential distribution is the only continuous probability distribution function that is memoryless.

Let $s, t \geq 0$, if chain starts in state i, then $\{T_i > s\} = \{X_k = i, 0 \leq k \leq s\}$

$T_i > s+t \Rightarrow T_i > s \quad \forall s, t \geq 0$

Then $\Pr(T_i > s+t \mid X_0 = i) = \Pr(T_i > s+t, T_i > s \mid X_0 = i)$.

$$= \Pr(T_i > s+t \mid T_i > s, X_0 = i)\Pr(T_i > s \mid X_0 = i)$$

$$= \Pr(T_i > s+t \mid X_k = i, 0 \leq k \leq s)\Pr(T_i > s \mid X_0 = i)$$

$$\overset{MC}{=} \Pr(T_i > s+t \mid X_s = i)\Pr(T_i > s \mid X_0 = i)$$

$$\overset{TH}{=} \Pr(T_i > t \mid X_0 = i)\Pr(T_i > s \mid X_0 = i)$$

$\Rightarrow \Pr(T_i > s+t \mid T_i > s, X_0 = i) = \Pr(T_i > t \mid X_0 = i)$

not transitioned by time s.

$\Rightarrow$ CTMC holding time are exponential

$T_i \sim Exp(q_i)$ Mostly assume $0 < q_i < \infty$

rate

never transit → absorbing

never stay → explosive.

Let $\{X_t : t \geq 0\}$ be a CTMC with ptmf $P(t)$.
Let $\{Y_n : n = 0, 1, 2 \cdots\}$ be a DTMC where $Y_n = X_{\tau_n}$ where τ_n indicate the time of the nth transition. Assume $Y_0 = X_0$.

If $\{X_t : t \geq 0\}$ is a CTMC, then $\{Y_n : n = 0, 1, 2 \cdots\}$ is a DTMC with TPM $\tilde{P}$
$\Rightarrow$ $\tilde{P}$ is a stochastic matrix, but with diagonal elements equal to 0.
$\{Y_n\}$ is often called the embedded DTMC (without self loop).

① You will always transition to new state at some point unless the state is absorbing

② Where you go depends on which state "wins" by a combination of transition intensity and chance.

Alarm clock idea

Assume $X_t = i$. For $j \neq i$, set an independent alarm clock that goes off at a random time, which has distribution $EXP(q_{ij})$.

The chain transitions to the state where alarm goes off first.

Because this is the minimum of independent $EXP(q_{ij})$ random variables, the minimum has distribution equivalent to $EXP(\sum_{j \neq i} q_{ij})$.

Let $M = \min(\tilde{T}_1, \cdots \tilde{T}_{i-1}, \tilde{T}_{i+1}, \cdots \tilde{T}_k)$ random alarm clock times.

Then, $Pr(M = \tilde{T}_l) = \dfrac{q_{il}}{\sum_{j \neq i} q_{ij}} = \tilde{P}_{il} = Pr(Y_{n+1} = l \mid Y_n = i)$ where $\tilde{P}_{ii} = 0 \ \forall i$

For a CTMC,

$$\underbrace{q_{ij}}_{\text{instaneous rate}} = \underbrace{q_i}_{\text{holding time}} \underbrace{P_{ij}}_{\text{transition probability}}$$

Characterize $\{X_t\}$ on 2 ideas:

① Holding times (time spent in this state).
② Where I transition to when I transition

Comments: $P(t)$ is not good for ① & ②.
Consider $\frac{\partial}{\partial t} P(t)$.

$\left\{\frac{\partial}{\partial t} P(t)\right\}_{ij} = \frac{\partial}{\partial t}\{(P(t))_{ij}\}$

① Assume $\{X_t : t \geq 0\}$ is CTMC and TH

② Assume $P(t)$ is differentiable

Note that $P_{ij}(0) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$

If $X_t = i$, consider the instantaneous rate of transitioning to $j \neq i$:

$$\lim_{h \to 0^+} \frac{E(\#\text{ of transitions to } j \text{ during } [t, t+h])}{h}$$

$$= \lim_{h \to 0^+} \frac{Pr(X_{t+h} = j \mid X_t = i)}{h}$$

For h sufficiently small, the number of transition to j is either 0 or 1.

$$= \lim_{h \to 0^+} \frac{Pr(X_h = j \mid X_t = i)}{h}$$

$$= \lim_{h \to 0^+} \frac{P_{ij}(h)}{h}$$

$$= \lim_{h \to 0^+} \frac{P_{ij}(h) - \underbrace{P_{ij}(0)}_{=0 \text{ b/c } j \neq i}}{h}$$

$$= P_{ij}'(0)$$

$$= q_{ij}$$

The off-diagonal entries of Q are the instantaneous transition rate

$$\frac{\partial P(t)}{\partial t} = Q = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & -(q_{12}+q_{13}) & q_{12} & q_{13} \\ 2 & q_{21} & -(q_{21}+q_{31}) & q_{23} \\ 3 & q_{31} & q_{32} & -(q_{31}+q_{32}) \end{array}$$

row sum to 0. $\Rightarrow$ Q is NOT a stochastic matrix

$$q_{ii} = \lim_{h \to 0^+} \frac{P_{ii}(h) - P_{ii}(0)}{h} \qquad (\text{let } Q = P'(0))$$

$$= \lim_{h \to 0^+} \frac{P_{ii}(h) - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{[1 - \sum_{j \neq i} P_{ij}(h)] - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{-\sum_{j \neq i} P_{ij}(h)}{h}$$

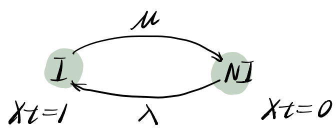
$$= -\sum_{j \neq i} P_{ij}'(h)$$

$$= -q_i$$

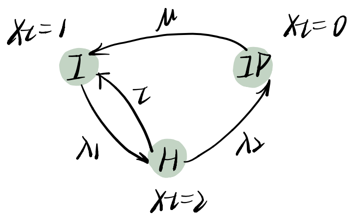
Q is the generator matrix for CTMC.

$$\begin{cases} Q_{ij} = P_{ij}'(0) = q_{ij} & \text{for } i \neq j \\ Q_{ii} = -\sum_{j \neq i} Q_{ij} = -\sum_{j \neq i} q_{ij} = -q_i \end{cases}$$

e.g.



$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} \end{array}$$



$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -\mu & \mu & 0 \\ 0 & -\lambda_1 & \lambda_1 \\ \lambda_2 & \tau & -\lambda_2-\tau \end{pmatrix} \end{array}$$

e.g. The generator for a Poisson process with parameter λ is

$$Q = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

From i, the probability that if a transition occurs at time t the process moves to a different state $j \neq i$ is

$$\lim_{h \to 0^+} P(X_{t+h} = j \mid X_t = i, X_{t+h} \neq i)$$

$$= \lim_{h \to 0^+} P(X_h = j \mid X_0 = i, X_h \neq i)$$

$$= \lim_{h \to 0^+} \frac{P(X_h = j, X_0 = i, X_h \neq i)}{P(X_0 = i, X_h \neq i)}$$

$$= \lim_{h \to 0^+} \frac{P(X_h = j \mid X_0 = i)}{P(X_h \neq i \mid X_0 = i)}$$

$$= \lim_{h \to 0^+} \frac{P_{ij}(h)/h}{[1 - P_{ii}(h)]/h}$$

$$= \frac{q_{ij}}{q_i}$$

For a CTMC,

$$q_{ij} = q_i \, P_{ij}$$

q_{ij}: instaneous rate q_i: holding time P_{ij}: transition probability

Kolmogorov Forward, Backward Equation.

A CTMC with transition function $P(t)$ and infinitesimal generator Q satisfies the forward equation

$$P'(t) = P(t)Q$$

$$P'_{ij}(t) = \sum_k P_{ik}(t)\, q_{kj} = -P_{ij}(t)\, q_j + \sum_{k\neq j} P_{ik}(t)\, q_{kj}$$

and the backward equation

$$P'(t) = Q P(t)$$

$$P'_{ij}(t) = \sum_k q_{ik} P_{kj}(t) = -q_i P_{ij}(t) + \sum_{k\neq i} q_{ik} P_{kj}(t)$$

Proof for forward

Chapman-Kolmogorov

$$\lim_{h\to 0^+} \frac{P(t+h)-P(t)}{h} = \lim_{h\to 0^+} \frac{P(t)P(h)-P(t)}{h}$$

$$P'(t) = P(t) \lim_{h\to 0} \frac{P(h)-I}{h} = P(t)Q$$

for backward

$$P(t+h) = P(h)P(t)$$

e.g. $$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} \end{array}$$

$$P'_{00}(t) = (P(t)Q)_{00} = \left(\begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} \begin{pmatrix} -q_0 & q_{01} \\ q_{10} & -q_1 \end{pmatrix} \right)_{00}$$

$$= -q_0 P_{00}(t) + q_{10} P_{01}(t)$$

$$= -\mu P_{00}(t) + (1-P_{00}(t))\, q_{10}$$

$$P'_{00}(t) = \lambda - (\lambda+\mu) P_{00}(t)$$

linear ODE.

$$P'_{11}(t) = \mu - (\lambda+\mu) P_{11}(t)$$

Boundary condition is $P_{00}(0) = P_{11}(0) = 1$

$$\Rightarrow \quad P_{00}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P_{11}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P(t) = \begin{pmatrix} P_{00}(t) & 1-P_{00}(t) \\ 1-P_{11}(t) & P_{11}(t) \end{pmatrix} = \frac{1}{\lambda+\mu} \begin{pmatrix} \mu+\lambda e^{-(\lambda+\mu)t} & \lambda-\lambda e^{-(\lambda+\mu)t} \\ \mu-\mu e^{-(\lambda+\mu)t} & \lambda+\mu e^{-(\lambda+\mu)t} \end{pmatrix}$$

Limiting distribution

π is a limiting distribution of a CTMC if $\forall\, i, j \in S'$

$$\lim_{t\to\infty} P_{ij}(t) = \pi_j$$

Stationary distribution

$\pi = \pi P(t) \quad \forall\, t \geq 0 \quad$ or $\quad \pi_j = \sum_i \pi_i P_{ij}(t) \quad \forall\, j,\ \forall\, t \geq 0.$

e.g. Binary chain

$$\lim_{t\to\infty} P_{00}(t) = \frac{\lambda}{\lambda+\mu} \qquad \lim_{t\to\infty} P_{01}(t) = 1 - P_{00}(t) = \frac{\mu}{\lambda+\mu}$$

$$\lim_{t\to\infty} P_{10}(t) = \frac{\lambda}{\lambda+\mu} \qquad \lim_{t\to\infty} P_{11}(t) = \frac{\mu}{\lambda+\mu}$$

$$\Omega = \frac{1}{\lambda+\mu}\begin{pmatrix} \lambda & \mu \\ \lambda & \mu \end{pmatrix}$$

$\Rightarrow \pi = \left(\frac{\lambda}{\lambda+\mu}, \frac{\mu}{\lambda+\mu} \right)$ limiting distribution

Periodicity

If $P_{ij}(t) > 0$ for some $t > 0$, then $P_{ij}(t) > 0 \ \forall\, t > 0$

Time homogenity

(Note this is not necessarily true for embedded matrix)

Let $\{X_t : t \geq 0\}$ be a finite irreducible CTMC with $P(t)$ as tpmf, then $\exists$ a unique stationary distribution that is the limiting distribution.

If π is stationary distribution for $\{X_t : t \geq 0\}$, i.e. $\pi = \pi P(t)$, then (iff)

$$\pi Q = 0$$

Proof

$\Rightarrow$ Assume $\pi = \pi P(t) \ \forall\, t \geq 0$. Take derivatives of both sides with respect to t.

$$0 = \pi Q \quad (\text{Choose } t = 0)$$

$\Leftarrow$ Assume $\pi Q = 0$. Then $0 = \pi Q P(t) = \pi P'(t)$ backward kolmogrov

So $\pi P(t)$ is a constant with respect to t b.c. $\pi P'(t)$ is the derivative of $\pi P(t)$.

$\Rightarrow \pi P(t) = \pi P(0) = \pi \quad \forall\, t \geq 0.$

• $P(0) = I$

Two state chain $\quad Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$

$\pi Q = 0$

$$\begin{cases} -\mu\pi_0 + \lambda \overset{(1-\pi_0)}{\pi_1} = 0 \\ \mu\pi_0 - \lambda\pi_1 = 0 \end{cases} \Rightarrow \quad \pi_0 = \frac{\lambda}{\lambda+\mu} \quad \pi_1 = \frac{\mu}{\lambda+\mu}$$

$\{X_t : t \geq 0\}$ is a CTMC. $S = \{1 \dots k\}$

Assume a single absorbing state a

T be the set of $k-1$ non-absorbing states (transient) if a is accessible from each of those states.

$$Q = \begin{array}{c|c|c} & a & T \\ \hline a & 0 & 0 \\ \hline T & & V \end{array}$$

The expected time till we reach the absorbing state given that we started in state i can be obtained from

$$F = -V^{-1} \quad \text{Fundamental matrix}$$

where $a_i = \sum_j F_{ij}$ is the expected time to absorption

<u>proof</u> Let F_{ij} be the expected time to state j for a chain started in state i. Assume $j \neq i$. Assume the chain first goes to $k \neq i, j$.

$$\Pr(\text{move to } k \text{ first} \mid \text{started in } i) = \frac{q_{ik}}{q_i}$$

Case ①: k is the absorbing state $\Rightarrow T_{ij} = 0$

Case ②: $k \neq a$, $F_{ij} = F_{kj}$ memoryless

$$i \neq j \qquad F_{ij} = \sum_{\substack{k \neq i \\ k \in T}} \underbrace{\frac{q_{ik}}{q_i}}_{\text{probability of transiting to } k \text{ first}} \underbrace{F_{kj}}_{\text{expected time from } k \text{ to } j}$$

$$= \frac{1}{q_i} \sum_{\substack{k \neq i \\ k \in T}} V_{ik} F_{kj} \quad i\text{th row of } V * j\text{th column of } F$$

$$= \frac{1}{q_i} \left[(VF)_{ij} - V_{ii} F_{ij} \right]$$

$$F_{ij} = \frac{1}{q_i}\left[(VF)_{ij} + q_i F_{ij} \right]$$

$$= \frac{(VF)_{ij}}{q_i} + F_{ij}$$

$$\Rightarrow \quad (VF)_{ij} = 0$$

transition to absorbing state

$$F_{ii} = \frac{q_{ia}}{q_i}\left(\frac{1}{q_i}\right) + \sum_{\substack{k \in T \\ k \neq i}}^{\text{not to } a} \frac{q_{ik}}{q_i}\left(\frac{1}{q_i} + F_{ki}\right)$$

holding time

$$= \frac{q_{ia}}{q_i^2} + \frac{1}{q_i^2} \sum_{\substack{k \in T \\ k \neq i}}^{\text{not } a} q_{ik} + \frac{1}{q_i} \sum_{\substack{k \in T \\ k \neq i}} q_{ik} F_{ki}$$

$$= \frac{1}{q_i^2} \sum_{\substack{k \in T \\ k \neq i}} q_{ik} +$$

$Q_{ii} = -q_i$

$$= \frac{-Q_{ii}}{q_i^2} + \frac{1}{q_i}\left[(VF)_{ii} - V_{ii} F_{ii} \right]$$

$-Q_{ii} = q_i$

$$= \frac{1}{q_i} + \frac{(VF)_{ii}}{q_i} + F_{ii}$$

$$\Rightarrow \quad (VF)_{ii} = -1$$

Therefore, $(VF)_{ij} = \begin{cases} 0 & i \neq j \\ -1 & i = j \end{cases}$ $\quad \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & \ddots \ -1 \end{pmatrix}$

$$\Rightarrow \quad VF = -I$$

Absorption time for a chain started in i =

Given that I started in i, the total amount of time that I spend in states that are not a

$$a_i = \sum_j F_{ij}$$

Birth-and-Death Process

Transitions only occur to neighboring states. Birth occurs from i to $i+1$ at the rate λ_i. Deaths occur from i to $i-1$ at rate μ_i.

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\
\mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\
0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\
0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}.
$$

Stationary Distribution for Birth-and-Death Process

For a birth-and death process with birth rates λ_i and death rates μ_i, for $i = 1, 2, \ldots$, assume that

$$\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} < \infty$$

Then, the unique stationary distribution π is

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \quad \text{for } k = 1, 2, \ldots$$

where

$$\pi_0 = \left(\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \right)^{-1}$$

TABLE 7.1 Types of Birth-and-Death Processes

Type	Birth Rate	Death Rate
Pure birth	λ_i	$\mu_i = 0$
Poisson process	$\lambda_i = \lambda$	$\mu_i = 0$
Pure death	$\lambda_i = 0$	μ_i
Linear process	$\lambda_i = i\lambda, i > 0$	$\mu_i = i\mu$
Yule process	$\lambda_i = \lambda i, i, \lambda > 0$	$\mu_i = 0$
Linear with immigration	$\lambda_i = i\lambda + \alpha, i, \alpha > 0$	$\mu_i = i\mu$

Example 7.22 (Yule process) The Yule process arises in biology to describe the growth of a population where each individual gives birth to an offspring at a constant rate λ independently of other individuals. Let X_t denote the size of the population at time t. If $X_t = i$, then a new individual is born when one of the i members of the population gives birth, which occurs at rate $i\lambda$. A Yule process is a birth-and-death process with birth rate $\lambda_i = i\lambda$ and death rate $\mu_i = 0$. In a Yule process, all states are transient and no limiting distribution exists.