上高速旋转.因受离心力的作用,水面呈抛物面形状,问当水刚要溢出容器时,水平的最低点在何处?

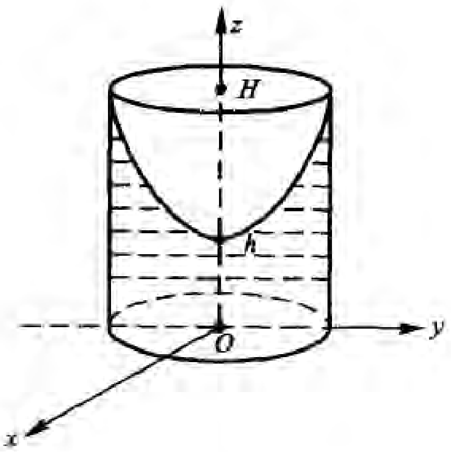
解　如图所示建立坐标系,并设水面最低点为 h.依题意有

$$\frac{2}{3}H\cdot(\pi R^2)=\iint\limits_{x^2+y^2\leqslant R^2}(h+x^2+y^2)\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\varphi\int_0^R(h+\rho^2)\rho\mathrm{d}\rho.$$

即
$$\frac{2}{3}H\cdot\pi R^2=\left(h+\frac{1}{2}R^2\right)\pi R^2,$$

于是
$$h=\frac{2}{3}H-\frac{1}{2}R^2.$$

又 $H-h=R^2$,故 $h=\frac{1}{3}H$.



(第 11 题)

习　题　6.3

(A)

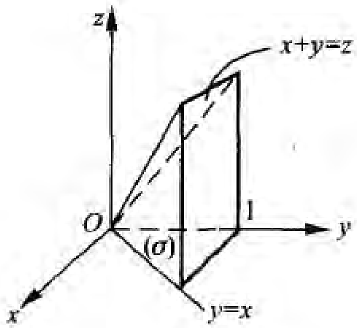
4. 计算下列三重积分.

(1) $\iiint\limits_{(V)}\mathrm{e}^x\mathrm{d}V$,$(V)$ 是由平面 $x=0,y=1,z=0,y=x$ 及 $x+y-z=0$ 所围成的闭区域;

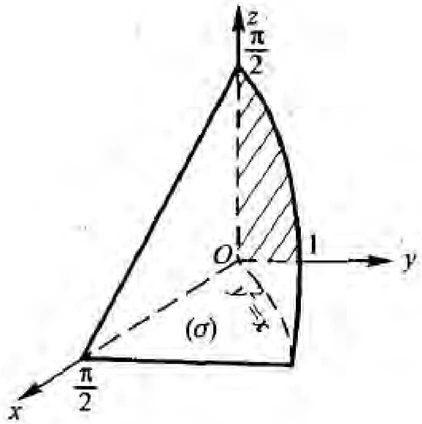
解　$\iiint\limits_{(V)}\mathrm{e}^x\mathrm{d}V=\iint\limits_{(\sigma)}\mathrm{d}\sigma\int_0^{x+y}\mathrm{e}^x\mathrm{d}z=\int_0^1\mathrm{d}y\int_0^y\mathrm{d}x\int_0^{x+y}\mathrm{e}^x\mathrm{d}z=\frac{7}{2}-\mathrm{e}.$

(2) $\iiint\limits_{(V)}y\cos(x+z)\mathrm{d}V$,$(V)$ 为由抛物面 $y=\sqrt{x}$,平面 $y=0,z=0$ 及 $x+z=$

$\frac{\pi}{2}$ 所围成的闭区域；



(第 4 题(1))



(第 4 题(2))

解
$$\iiint\limits_{(V)} y\cos(x+z)\,\mathrm{d}V = \iint\limits_{(\sigma)} \mathrm{d}\sigma \int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,\mathrm{d}z$$
$$= \int_0^{\sqrt{\frac{\pi}{2}}} \mathrm{d}y \int_{y^2}^{\frac{\pi}{2}} \mathrm{d}x \int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,\mathrm{d}z$$
$$= \frac{\pi^2}{16} - \frac{1}{2}.$$

(3) $\iiint\limits_{(V)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V$，$(V)$ 由 $z=\sqrt{x^2+y^2}$，$z=1$，$z=2$ 所围成的闭区域；

解法 I 设(V_1)为由 $z=\sqrt{x^2+y^2}$ 与 $z=2$ 围成的立体区域，(V_2)为由 $z=\sqrt{x^2+y^2}$ 与 $z=1$ 围成的立体，则由积分的区域可加性，得

$$\iiint\limits_{(V)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V = \iiint\limits_{(V_1)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V - \iiint\limits_{(V_2)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V$$

$$= \iint\limits_{\rho\leqslant 2} \rho\mathrm{d}\rho\mathrm{d}\varphi \int_\rho^2 \frac{1}{\rho}\mathrm{e}^z\mathrm{d}z - \iint\limits_{\rho\leqslant 1} \rho\mathrm{d}\rho\mathrm{d}\varphi \int_\rho^1 \frac{1}{\rho}\mathrm{e}^z\mathrm{d}z = 2\pi\mathrm{e}^2.$$

解法 II 如图所示$(V_1)=(V)/(V_{圆柱})$

$$\iiint\limits_{(V)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V = \iiint\limits_{(V_{圆柱})} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V + \iiint\limits_{(V_1)} \frac{\mathrm{e}^z}{\sqrt{x^2+y^2}}\mathrm{d}V$$

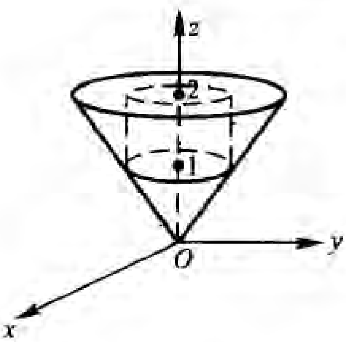
$$= \iint\limits_{\rho\leqslant 1} \rho\mathrm{d}\rho\mathrm{d}\varphi \int_1^2 \frac{1}{\rho}\mathrm{e}^z\mathrm{d}z + \iint\limits_{1\leqslant\rho\leqslant 2} \rho\mathrm{d}\rho\mathrm{d}\varphi \int_\rho^2 \frac{\mathrm{e}^z}{\rho}\mathrm{d}z$$

$$= 2\pi e^2.$$

（6）$\iiint\limits_{(V)} xy\mathrm{d}V$，$(V)$ 由 $xy = z, x + y = 1$ 与 $z = 0$ 所围成的闭区域；

解　$\iiint\limits_{(V)} xy\mathrm{d}V = \iint\limits_{(\sigma)} \mathrm{d}\sigma \int_0^{xy} xy\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^{1-x} x^2y^2\mathrm{d}y$

$$= \frac{1}{180}.$$



（第 4 题(3)）

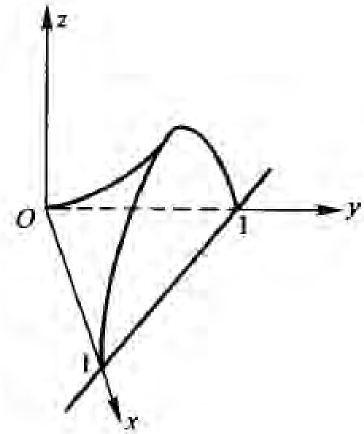
（7）$\iiint\limits_{(V)} (x^2 + y^2)\mathrm{d}V$，$(V)$ 由 $z = \sqrt{a^2 - x^2 - y^2}$，$z = \sqrt{A^2 - x^2 - y^2}$，$z = 0$ 所围成，其中 $A > a > 0$；

解　$\iiint\limits_{(V)} (x^2 + y^2)\mathrm{d}V$（采用球坐标）

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_a^A r^2\sin^2\theta r^2\sin\theta\mathrm{d}r$$

$$= \frac{4}{15}\pi(A^5 - a^5).$$

（9）$\iiint\limits_{(V)} \frac{\mathrm{d}V}{1 + x^2 + y^2}$，$(V)$ 由 $x^2 + y^2 = z^2$ 与 $z = 1$ 所围成；



（第 4 题(6)）

解　利用柱坐标，

$$\text{原式} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \rho\mathrm{d}\rho \int_\rho^1 \frac{\mathrm{d}z}{1 + \rho^2} = \pi\left(\ln 2 - 2 + \frac{\pi}{2}\right).$$

（13）$\iiint\limits_{(V)} (x + y)\mathrm{d}V$，$(V)$ 由 $x^2 + y^2 = 1, x^2 + y^2 = 4, z = 0, z = x + 2$ 所围成；

解　利用柱坐标，

$$\text{原式} = \iint\limits_{1 \le \rho \le 2} \rho\mathrm{d}\rho\mathrm{d}\varphi \int_0^{\rho\cos\varphi + 2} \rho(\sin\varphi + \cos\varphi)\mathrm{d}z$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_1^2 \rho^2(\sin\varphi + \cos\varphi)(\rho\cos\varphi + 2)\mathrm{d}z$$

$$= \frac{15}{4}\pi.$$

(14) $\iiint\limits_{(V)} \frac{z\ln(1+x^2+y^2+z^2)}{1+x^2+y^2+z^2}\mathrm{d}V,(V): x^2+y^2+z^2 \leqslant 1$;

解 用球坐标,

$$\text{原式} = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^1 \frac{r\cos\theta\ln(1+r^2)}{1+r^2}r^2\sin\theta\mathrm{d}r$$

$$= 2\pi\left[\int_0^{\pi}\sin\theta\cos\theta\mathrm{d}\theta\right]\left[\int_0^1\frac{r^3\ln(1+r^2)}{1+r^2}\mathrm{d}r\right] = 0.$$

(15) $\iiint\limits_{(V)} z(x^2+y^2)\mathrm{d}V,(V) = \{(x,y,z) \mid z \geqslant \sqrt{x^2+y^2}, 1 \leqslant x^2+y^2+z^2 \leqslant 4\}$;

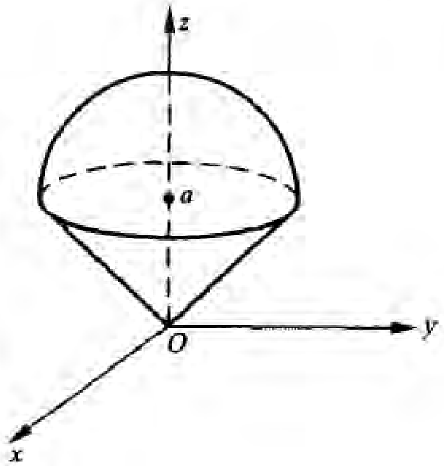
解 用球坐标,

$$\text{原式} = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_1^2 r\cos\theta\cdot r^2\sin^2\theta\cdot r^2\sin\theta\mathrm{d}r = \frac{63}{48}\pi.$$

(16) $\iiint\limits_{(V)} z\mathrm{d}V,(V) = \{(x,y,z) \mid x^2+y^2+(z-a)^2 \leqslant a^2, x^2+y^2 \leqslant z^2, a>0\}$

解 用球坐标,原式 =

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{2a\cos\theta} r\cos\theta\cdot r^2\sin\theta\mathrm{d}r = \frac{7\pi a^4}{6}.$$



(第4题(16))

5. 选用适当的坐标系计算下列累次积分.

(1) $\int_{-1}^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_{\sqrt{x^2+y^2}}^1 z^3\mathrm{d}z$ (用柱坐标)

$$= \int_0^{\pi}\mathrm{d}\varphi\int_0^1\rho\mathrm{d}\rho\int_{\rho}^1 z^3\mathrm{d}z = \frac{\pi}{12}.$$

(2) $\int_{-3}^3\mathrm{d}x\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\mathrm{d}y\int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\mathrm{d}z$

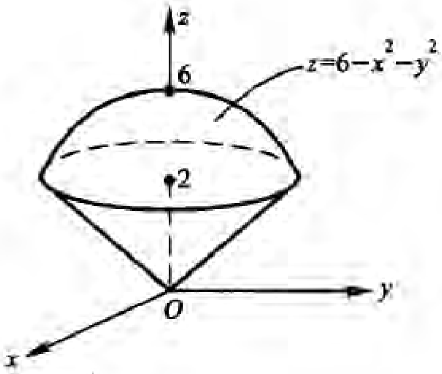
解 $(V) = \{(x,y,z) \mid z \geqslant 0, x^2+y^2+z^2 \leqslant 9\}$,于是

$$\text{原式} = \iiint\limits_{(V)} z\sqrt{x^2+y^2+z^2}\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^3 r\cos\theta\cdot r\cdot r^2\sin\theta\mathrm{d}r$$

$$=\frac{243}{5}\pi.$$

6. 求下列立体体积.

(2) 由 $z=6-x^2-y^2$ 与 $z=\sqrt{x^2+y^2}$ 所围成的立体；



(第 6 题(2))

解　$z=6-x^2-y^2$ 与锥面 $z=\sqrt{x^2+y^2}$ 的交线为 $\begin{cases}x^2+y^2=2^2,\\ z=2.\end{cases}$

用柱坐标可得所求体积为 V.

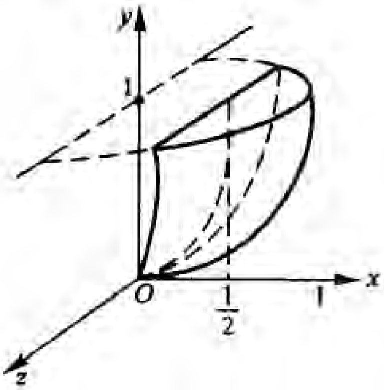
$$V=\iiint\limits_{(V)}\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\varphi\int_0^2\rho\mathrm{d}\rho\int_\rho^{6-\rho^2}\mathrm{d}z=\frac{32}{3}\pi.$$

(3) 由 $(x^2+y^2+z^2)^2=a^3z\ (a>0)$ 所围成的立体；

解　曲面 $(x^2+y^2+z^2)^2=a^3z$ 关于 xOy，yOz 平面均对称，且位于 xOy 平面上方 $(z\geqslant 0)$ 的闭曲面. 用球坐标，则所求体积

$$V=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{a\sqrt[3]{\cos\theta}}r^2\sin\theta\mathrm{d}r=\frac{\pi a^3}{3}.$$

(4) 由 $x=\sqrt{y-z^2}$，$\frac{1}{2}\sqrt{y}=x$ 与 $y=1$ 所围立体体积；



(第 6 题(4))(a)

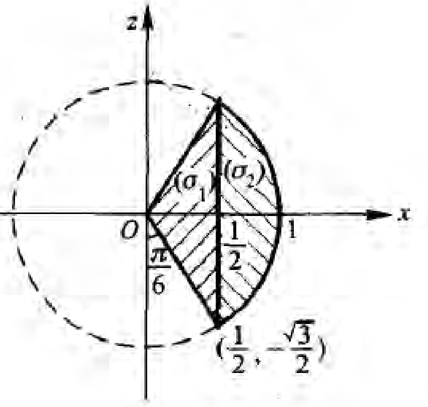
解　如图(a)所示对称轴为 y 轴的抛物面 $x=\sqrt{y-z^2}$（即 $x^2+z^2=y$，$x\geqslant 0$）与母线平行于 z 轴的抛物柱面 $x=\frac{1}{2}\sqrt{y}$（即 $y=4x^2$，$x\geqslant 0$）的交线 $\begin{cases}y=\frac{4}{3}z^2,\\ x=\frac{1}{\sqrt{3}}|z|\end{cases}$ 在 xOz 平面的投影如图(b)所示为 $|z|=\sqrt{3}x$.

故所求立体体积 $V=\iint\limits_{(\sigma_1)}\mathrm{d}x\mathrm{d}z\int_{x^2+z^2}^{4x^2}\mathrm{d}y+\iint\limits_{(\sigma_2)}\mathrm{d}x\mathrm{d}z\int_{x^2+z^2}^{1}\mathrm{d}y$.

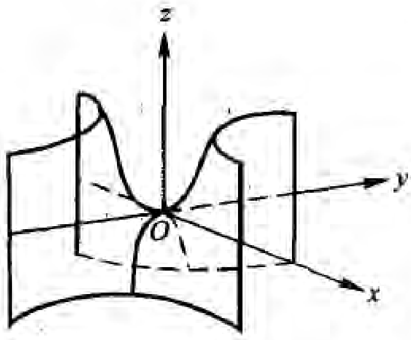
即
$$V=\int_0^{\frac{1}{2}}\mathrm{d}x\int_{-\sqrt{3}x}^{\sqrt{3}x}\mathrm{d}z\int_{x^2+z^2}^{4x^2}\mathrm{d}y+\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\mathrm{d}\varphi\int_{\frac{1}{2\cos\varphi}}^{1}\rho\mathrm{d}\rho\int_{\rho^2}^{1}\mathrm{d}y$$

$$= \frac{\sqrt{3}}{16} + \left(\frac{\pi}{6} - \frac{3\sqrt{3}}{16}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

（5）由 $z = \dfrac{xy}{a}, x^2 + y^2 = ax(a > 0)$ 与 $z = 0$ 所围成的立体；



（第 6 题(4)）(b)



（第 6 题(5)）

解 如图所示，x 轴和 y 轴是马鞍面 $z = \dfrac{xy}{a}$ 上的两条直线，则所求立体由两个曲顶柱体 (V_1) 和 (V_2) 构成. 其中 (V_1) 位于第 1 卦限，底为半圆 $\begin{cases} z = 0, \\ 0 \leqslant y \leqslant \sqrt{ax - x^2}, \end{cases}$ 顶为马鞍面；(V_2) 位于第八卦限，底为半圆 $\begin{cases} z = 0, \\ -\sqrt{ax - x^2} \leqslant y \leqslant 0, \end{cases}$ 顶为马鞍面. 故所求体积

$$V = V_1 + V_2 = \iiint\limits_{(V_1)} \mathrm{d}V + \iiint\limits_{(V_2)} \mathrm{d}V$$

$$= \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{a\cos\varphi} \rho \mathrm{d}\rho \int_0^{\frac{\rho^2 \sin\varphi\cos\varphi}{a}} \mathrm{d}z - \int_{-\frac{\pi}{2}}^{0} \mathrm{d}\varphi \int_0^{a\cos\varphi} \rho \mathrm{d}\rho \int_0^{\frac{\rho^2 \sin\varphi\cos\varphi}{a}} \mathrm{d}z$$

$$= \frac{a^3}{12}.$$

（7）由 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$ 与 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ 所围成的立体 $(a > 0, b > 0, c > 0)$；

解 由双叶双曲面 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$ 与椭圆柱面 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ 围成的立体关

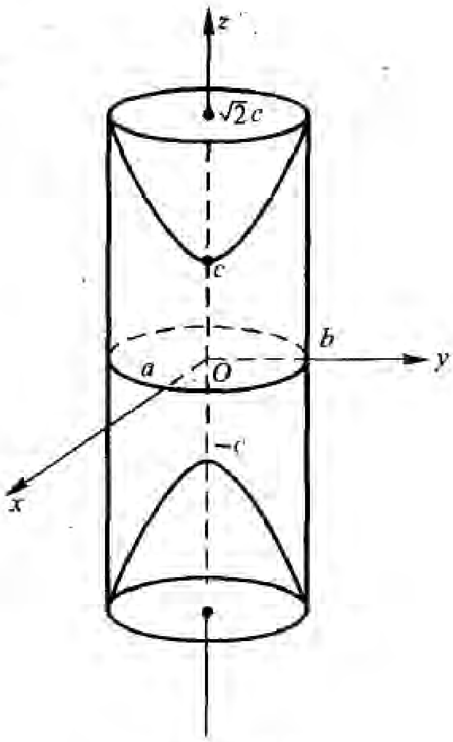
于 xOy 平面对称.

解法 I　作变换 $x=a\rho\cos\varphi, y=b\rho\sin\varphi, z=z$，则 $\dfrac{\partial(x,y,z)}{\partial(\rho,\varphi,z)}=ab\rho$. 故所求体积

$$V=2\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 ab\rho\mathrm{d}\rho\int_0^{c\sqrt{1+\rho^2}}\mathrm{d}z$$

$$=\frac{4}{3}\pi abc(2\sqrt{2}-1).$$

$\left(注: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1, z\geqslant 0 \text{ 变为 } c\sqrt{\rho^2+1}=z\right)$



（第 6 题(7)）

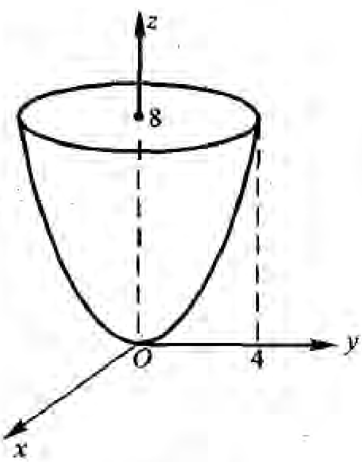
解法 II　$V=2\displaystyle\int_0^c\mathrm{d}z\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1}\mathrm{d}\sigma+$

$$2\int_c^{\sqrt{2}c}\mathrm{d}z\iint\limits_{\frac{z^2}{c^2}-1\leqslant\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1}\mathrm{d}\sigma$$

$$=2\int_0^c\pi ab\mathrm{d}z+2\int_c^{\sqrt{2}c}\pi ab\left[1-\left(\frac{z^2}{c^2}-1\right)\right]\mathrm{d}z$$

$$=\frac{4}{3}\pi abc(2\sqrt{2}-1).$$

7. 计算 $\displaystyle\iiint\limits_{(V)}(x^2+y^2)\mathrm{d}V$，其中 (V) 为平面曲线 $\begin{cases}y^2=2z,\\x=0\end{cases}$ 绕 z 轴旋转一周形成的曲面与平面 $z=8$ 所围立体.



（第 7 题）

解　依题意，(V) 为旋转抛物面 $z=\dfrac{1}{2}(x^2+y^2)$ 及 $z=8$ 围成如图所示. 故

$$原式=\iint\limits_{x^2+y^2\leqslant 16}\mathrm{d}\sigma\int_{\frac{1}{2}(x^2+y^2)}^8(x^2+y^2)\mathrm{d}z$$

$$=\int_0^{2\pi}\mathrm{d}\varphi\int_0^4\rho\mathrm{d}\rho\int_{\frac{\rho^2}{2}}^8\rho^2\mathrm{d}z$$

$$=\frac{4\times 16^2\pi}{3}=\frac{1\,024\,\pi}{3}.$$

8. 证明抛物面 $z=x^2+y^2+1$ 上任一点处的切平面与曲面 $z=x^2+y^2$ 所围立体的体积恒为一常数值.

解 $z=x^2+y^2+1$ 上过 $P_0(x_0,y_0,z_0)(z_0=1+x_0^2+y_0^2)$ 处的切平面方程为

$$z=2x_0x+2y_0y+1-x_0^2-y_0^2,$$

则切平面与抛物面 $z=x^2+y^2$ 所围立体体积为

$$V=\iint\limits_{(x-x_0)^2+(y-y_0)^2\leqslant 1} d\sigma\int_{x^2+y^2}^{2x_0x+2y_0y+1-x_0^2-y_0^2}dz$$

$$=\iint\limits_{(x-x_0)^2+(y-y_0)^2\leqslant 1}[(x-x_0)^2+(y-y_0)^2+1]d\sigma=\int_0^{2\pi}d\varphi\int_0^1(\rho^2+1)\rho d\rho=\frac{3}{2}\pi.$$

与 P_0 无关的常数其中 $x=x_0+\rho\cos\varphi,y=y_0+\rho\sin\varphi$,则$\dfrac{\partial(x,y)}{\partial(\rho,\varphi)}=\rho.$

(B)

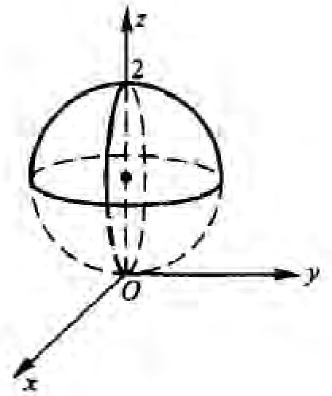
1. 计算下列三重积分

(1) $\iiint\limits_{(V)}\dfrac{1}{\sqrt{x^2+y^2+z^2}}dV,(V)=\{(x,y,z)\mid x^2+y^2+(z-1)^2\leqslant 1,z\geqslant 1,y\geqslant 0\}$;

解 用球坐标.则平面 $z=1$ 方程为 $r\cos\theta=1$,则

$$原式=\int_0^{\pi}d\varphi\int_0^{\frac{\pi}{4}}d\theta\int_{\frac{1}{\cos\theta}}^{2\cos\theta}\frac{1}{r}\cdot r^2\sin\theta dr$$

$$=\frac{\pi}{6}(7-4\sqrt{2}).$$



(第1题(1))

(2) $\iiint\limits_{(V)}\left|\sqrt{x^2+y^2+z^2}-1\right|dV,(V)$ 由 $z=\sqrt{x^2+y^2}$ 与 $z=1$ 围成;

解 $(V)=(V_1)\cup(V_2),(V_1)=\{(x,y,z)\mid x^2+y^2+z^2\geqslant 1,\sqrt{x^2+y^2}\leqslant z\leqslant 1\},(V_2)=\{(x,y,z)\mid x^2+y^2+z^2\leqslant 1,z\geqslant\sqrt{x^2+y^2}\}$.

且(V_1)与(V_2)除边界外无其他的交点,于是

$$\iiint\limits_{(V)}\left|\sqrt{x^2+z^2+y^2}-1\right|dV=\iiint\limits_{(V_1)}(\sqrt{x^2+y^2+z^2}-1)dV+$$

$$\iiint_{(V_2)}(1-\sqrt{x^2+y^2+z^2})\,\mathrm{d}V$$

$$=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_1^{\frac{1}{\cos\theta}}(r-1)r^2\sin\theta\,\mathrm{d}r+\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1(1-r)r^2\sin\theta\,\mathrm{d}r$$

$$=\frac{\pi}{6}(\sqrt{2}-1).$$

(3) $\iiint_{(V)}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,\mathrm{d}V,(V)=\left\{(x,y,z)\,\middle|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1,a>0,\right.$ $\left.b>0,c>0\right\}$.

解　令 $x=ar\sin\theta\cos\varphi,y=br\sin\theta\sin\varphi,z=cr\cos\theta$，则 $(V):0\leqslant r\leqslant 1,0\leqslant\varphi\leqslant 2\pi,0\leqslant\theta\leqslant\pi$，且 $\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right|=abcr^2\sin\theta$.

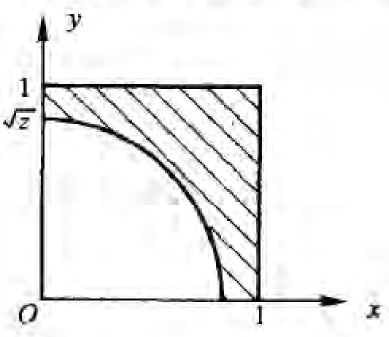
于是原积分 $=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^1(abcr^2\sin\theta)\sqrt{1-r^2}\,\mathrm{d}r=\frac{\pi^2}{4}abc.$

2. 将累次积分 $\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^{x^2+y^2}f(x,y,z)\,\mathrm{d}z$ 分别化为先对 x 和先对 y 的累次积分.

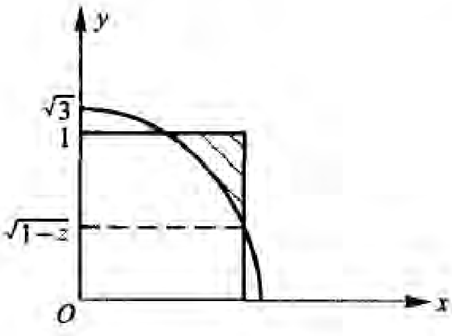
解　设 (V) 由抛物面 $z=x^2+y^2,x=0,y=0,z=0,x=1,y=1$ 围成，于是

$$\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^{x^2+y^2}f(x,y,z)\,\mathrm{d}z=\iiint_{(V)}f(x,y,z)\,\mathrm{d}V.$$

与 xOy 面平行的平面 $z=z$ 与 (V) 的截面为 (σ_z)，则 $0\leqslant z\leqslant 1$ 时如图(a)所示，$1\leqslant z\leqslant 2$ 时，(σ_z) 如图(b)所示



(第 2 题(a))



(第 2 题(b))

故

$$\iiint_{(V)}f(x,y,z)\,\mathrm{d}V=\int_0^1\mathrm{d}z\iint_{(\sigma_z)}f(x,y,z)\,\mathrm{d}\sigma+\int_1^2\mathrm{d}z\iint_{(\sigma_z')}f(x,y,z)\,\mathrm{d}\sigma$$

$$= \int_0^1 dz\left[\int_0^{\sqrt{z}} dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)\,dx + \int_{\sqrt{z}}^1 dy \int_0^1 f(x,y,z)\,dx\right] +$$

$$\int_1^2 dz \int_{\sqrt{z-1}}^1 dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)\,dx.$$

又 $\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x,y,z)\,dz$

$= \int_0^1 dx \iint\limits_{(\sigma_x)} f(x,y,z)\,d\sigma$(交换二重积分次序)

$= \int_0^1 dx\left[\int_0^{x^2} dz \int_0^1 f(x,y,z)\,dy + \int_{x^2}^{1+x^2} dz \int_{\sqrt{z-x^2}}^1 f(x,y,z)\,dy\right]$,其中 (σ_x) 如图(c)所示.

3. 设 $F(t) = \iiint\limits_{(V)} x\ln(1+x^2+y^2+z^2)\,dV$,$(V)$ 由 $x^2+y^2+z^2 \leqslant t^2$ 与 $\sqrt{y^2+z^2} \leqslant x$ 确定,求 $\frac{dF(t)}{dt}$.

解 令 $x = r\cos\theta, y = r\sin\theta\cos\varphi, z = r\sin\theta\sin\varphi$.

则
$$F(t) = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} d\theta \int_0^t r\cos\theta\ln(1+r^2)\cdot r^2\sin\theta\,dr$$

$$= \frac{\pi}{2}\int_0^t r^3\ln(1+r^2)\,dr,$$

故
$$\frac{dF(t)}{dt} = \frac{\pi}{2}t^3\ln(1+t^2).$$

4. 设 f 为连续函数,求函数 $F(t) = \iiint\limits_{(V)} f(x^2+y^2+z^2)\,dV$ 的导数 $F'(t)$,其中 $(V) = \{(x,y,z) \mid x^2+y^2+z^2 \leqslant t^2\}$.

解 用球坐标变换,$F(t) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^t f(r^2) r^2\sin\theta\,dr$

$$= 4\pi\int_0^t r^2 f(r^2)\,dr.$$

由于 f 为连续函数,故 $F'(t) = \frac{d}{dt}\left[4\pi\int_0^t r^2 f(r^2)\,dr\right] = 4\pi t^2 f(t^2)$.

5. 设 $f(x)$ 连续,$(V) = \{(x,y,z) \mid 0 \leqslant z \leqslant h, x^2+y^2 \leqslant t^2\}$,

$$F(t) = \iiint\limits_{(V)} [z^2 + f(x^2+y^2)]\,dV,$$

求 $\frac{dF}{dt}$ 和 $\lim\limits_{t\to0^+}\frac{F(t)}{t^2}$.

解　用柱坐标，$F(t)=\int_0^{2\pi}\mathrm{d}\varphi\int_0^t\rho\mathrm{d}\rho\int_0^h[z^2+f(\rho^2)]\mathrm{d}z$

$$=2\pi\int_0^t\rho\left[\frac{1}{3}z^3+zf(\rho^2)\right]_0^h\mathrm{d}\rho$$

$$=2\pi\int_0^t\rho\left[\frac{1}{3}h^3+hf(\rho^2)\right]\mathrm{d}\rho.$$

于是，$\frac{\mathrm{d}F}{\mathrm{d}t}=2\pi ht\left[\frac{1}{3}h^2+f(t^2)\right]$.

$$\lim_{t\to0^+}\frac{F(t)}{t^2}\overset{\frac{0}{0}}{=\!=}\lim_{t\to0^+}\frac{2\pi ht\left[\frac{1}{3}h^2+f(t^2)\right]}{2t}=\pi h\left[\frac{1}{3}h^2+f(0)\right].$$

6. 计算三重积分 $\iiint\limits_{(V)}(x+y+z)^2\mathrm{d}V$，其中$(V)$：$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant1$.

解　$\iiint\limits_{(V)}(x+y+z)^2\mathrm{d}V=\iiint\limits_{(V)}(x^2+y^2+z^2)\mathrm{d}V+\iiint\limits_{(V)}(2xy+2xz+2yz)\mathrm{d}V$.

由于(V)关于 xOy 平面对称，而$(xz+yz)$关于 z 为奇函数，则 $2\iiint\limits_{(V)}(xz+yz)\mathrm{d}V=0$，类似的可知$\iiint\limits_{(V)}xy\mathrm{d}V=0$，从而$\iiint\limits_{(V)}(2xy+2xz+2yz)\mathrm{d}V=0$.

又
$$\iiint\limits_{(V)}x^2\mathrm{d}V=\int_{-a}^a x^2\mathrm{d}x\iint\limits_{\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant1-\frac{x^2}{a^2}}\mathrm{d}\sigma=\int_{-a}^a x^2\left[\pi bc\left(1-\frac{x^2}{a^2}\right)\right]\mathrm{d}x$$

$$=\frac{4}{15}\pi a^3bc.$$

类似可得$\iiint\limits_{(V)}y^2\mathrm{d}V=\frac{4}{15}\pi ab^3c$，$\iiint\limits_{(V)}z^2\mathrm{d}V=\frac{4}{15}\pi abc^3$.

故　$\iiint\limits_{(V)}(x+y+z)^2\mathrm{d}V=\frac{4}{15}\pi abc(a^2+b^2+c^2)$.

习　题　6.4

(A)

1. 求下列曲线所围成的均匀薄板的质心坐标.

(1) $ay=x^2, x+y=2a\ (a>0)$；

(2) $x=a(t-\sin t), y=a(1-\cos t)\quad(0\leqslant t\leqslant2\pi, a>0)$与 x 轴；

(3) $\rho=a(1+\cos\varphi)\quad(a>0)$.