回首2019, 我们不忘初心, 牢记使命, 锐意进取, 风雨兼程走进成电

迎接2020, 我们满怀期待, 砥砺前行, 信心百倍, 斗志昂扬共谱新篇

常微分方程疑难分析

——人间方程是清欢

微分方程的基本类型：

1.可分离变量的微分方程;

2.齐次微分方程; $y' + p(x)y = q(x)$

3.一阶线性非齐次微分方程;(*常数变易法)

4.伯努利微分方程; $y' + p(x)y = q(x)y^k \quad (k \neq 0,1)$

5.可降阶的高阶微分方程(3种);

6.二(高)阶常系数线性(非)齐次微分方程;

7.欧拉方程 (变系数).

常微分方程 非常1+8！！

微分方程的学习方法：

1.熟记微分方程的类型和解法.

2.如果不是学习过的熟悉类型,则考虑变量代换方法转化：

a 自变量代换;(如：欧拉方程 $x=e^t$)

b 函数代换;(如：伯努利方程 $z=y^{1-k}$)

c 自变量与函数相结合的代换. (如：齐次方程 $u=\dfrac{y}{x}$)

典型例题分析

年年期末都难过，年年过的都不错.

1 求下列微分方程的通解.

(1) $y' = e^y - \dfrac{2}{x}$;　　(2) $y' = \dfrac{y^2}{4} + \dfrac{1}{x^2}$;

(3) $y' \cos y = (1 + \cos x \sin y) \sin y$;

(4) $y' + x = \sqrt{x^2 + y}$;　　(4)提示：$y = x^2 u$

(5) $y'' - y = \sin^2 x$.　　$\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$

(6) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x \sin^2(xy)} - \dfrac{y}{x}$.　(7) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x \sin 2y = x e^{-x^2} \cos^2 y$.

(8) $y'' + y' - 2y = \dfrac{e^x}{1 + e^x}$.

(8) $y''+y'-2y=\dfrac{e^x}{1+e^x}$.

解 因为$y''+y'-2y=(y''+2y')-(y'+2y)$

$$=(y'+2y)'-(y'+2y).$$

则所给微分方程变为一阶线性微分方程

$$u'-u=\frac{e^x}{1+e^x}$$

它的通解为：$u=e^{-\int(-1)\mathrm{d}x}[C_1+\int\dfrac{e^x}{1+e^x}\cdot e^{\int(-1)\mathrm{d}x}\mathrm{d}x]$

$$= e^{x}[C_1 + \int \frac{e^{x}}{1+e^{x}} \cdot e^{-x}\mathrm{d}x] = e^{x}[C_1 + \int \frac{1}{1+e^{x}}\mathrm{d}x]$$

$$= e^{x}[C_1 - \int \frac{1}{1+e^{-x}}\mathrm{d}e^{-x}] = e^{x}[C_1 - \ln(1+e^{-x})]$$

所以$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{x}[C_1 - \ln(1+e^{-x})]$.

所以上式的通解为

$$y = e^{-\int 2\mathrm{d}x}[C_2 + \int e^{x}[C_1 - \ln(1+e^{-x})]e^{\int 2\mathrm{d}x}\mathrm{d}x]$$

$$= e^{-2x}[C_2 + \int e^{3x}[C_1 - \ln(1+e^{-x})]\mathrm{d}x]$$

$$\int e^{3x}[C_1 - \ln(1+e^{-x})]\mathrm{d}x = \frac{1}{3}\int [C_1 - \ln(1+e^{-x})]\mathrm{d}e^{3x} \quad = \cdots$$

$$= \frac{1}{3}e^{3x}[C_1 - \ln(1+e^{-x})] - \frac{1}{6}e^{2x} + \frac{1}{3}e^{x} - \frac{1}{3}x - \frac{1}{3}\ln(1+e^{-x}).$$

所以通解为：$y = \frac{1}{3}C_1e^{x} + C_2e^{-2x} - \frac{1}{3}e^{-x}\ln(1+e^{x}) - \frac{1}{6}$

$$+ \frac{1}{3}e^{-x} - \frac{1}{3}xe^{-x} - \frac{1}{3}e^{-2x}\ln(1+e^{-x}).$$

2 求微分方程$x^3y''=(y-xy')^2$满足初始条件$y(1)=0, y'(1)=3$的解.

解 令$u=y-xy'$,则原方程可化为

$$-x^2u'=u^2 \Rightarrow -\frac{\mathrm{d}u}{u^2}=\frac{\mathrm{d}x}{x^2} \Rightarrow \frac{1}{u}=c-\frac{1}{x}$$

由初始条件$y(1)=0, y'(1)=3$知, $c=\frac{2}{3}$.

即 $$\frac{1}{y-xy'}=\frac{2}{3}-\frac{1}{x}=\frac{2x-3}{3x} \Rightarrow y'=\frac{y}{x}-\frac{3}{2x-3}$$

未能尽考第一,但求青春无悔.

由一阶线性非齐次微分方程公式可得:

$$y = e^{\int \frac{1}{x}\mathrm{d}x}\left(\int \frac{3}{3-2x} e^{-\int \frac{1}{x}\mathrm{d}x}\mathrm{d}x + c\right)$$

$$= x\left[\ln\left|\frac{x}{2x-3}\right| + C_1\right]$$

由初始条件$y(1) = 0, \Rightarrow C_1 = 0.$

$$y = x\ln\left|\frac{x}{2x-3}\right|$$

3 求微分方程$x^2yy''=(y-xy')^2$的通解. $x^3y''=(y-xy')^2.$

解 将所给的微分方程改写为:

$$x^2yy''=y^2-2xyy'+x^2(y')^2$$

$$\Rightarrow x^2yy''-x^2(y')^2=y^2-2xyy'$$

两边同除以x^2y^2得 $$\frac{yy'-(y')^2}{y^2}=\frac{1}{x^2}-\frac{2}{x}\left(\frac{y'}{y}\right)$$

$$\Rightarrow \left(\frac{y'}{y}\right)'+\frac{2}{x}\left(\frac{y'}{y}\right)=\frac{1}{x^2}.$$

令$u=\dfrac{y'}{y}$则所给的微分方程为 $u'+\dfrac{2}{x}u=\dfrac{1}{x^2}.$

$$u = e^{-\int \frac{2}{x}\mathrm{d}x}\left(\int \frac{1}{x^2} e^{\int \frac{2}{x}\mathrm{d}x}\mathrm{d}x + C_1\right)$$

$$= \frac{1}{x^2}\left(\int \frac{1}{x^2} e^{\int \frac{2}{x}\mathrm{d}x}\mathrm{d}x + C_1\right) = \frac{C_1}{x^2} + \frac{1}{x}$$

$$\Rightarrow \frac{y'}{y} = \frac{C_1}{x^2} + \frac{1}{x}, i.e. \quad \Rightarrow \ln y = \int \left(\frac{C_1}{x^2} + \frac{1}{x}\right)\mathrm{d}x$$

$$= -\frac{C_1}{x} + \ln x + \ln C_2,$$

原方程的通解 $y = e^{-\frac{C_1}{x} + \ln x + \ln C_2} = C_2 x e^{-\frac{C_1}{x}}.$

4 设有微分方程$y'+y=f(x)$.其中 $f(x)=\begin{cases} 2, & 0\le x\le 1, \\ 0, & x>1, \end{cases}$ 试求一连续函数 $y=y(x)$,使适合条件$y(0)=0$,且在$(0,1),(1,+\infty)$内满足上述微分方程.

解 由题意知: 当$0\le x\le 1$时, $y'+y=2$, 即有

$$y=e^{-\int dx}\left(\int 2e^{\int dx}dx+C_1\right)=2+C_1e^{-x}$$ 且$y(0)=0\Rightarrow C_1=-2$.所以 $y=2-2e^{-x}$.

又当$x>1$时, $y'+y=0$,即有$y=C_2e^{-x}$,且由y的连续性知, $\lim\limits_{x\to 1^+} y=\lim\limits_{x\to 1^-} y$,

即有$2-2e^{-x}=C_2e^{-x}$, $\Rightarrow C_2=2(e-1)\Rightarrow y=2(e-1)e^{-x}$.

综上所述, $y=\begin{cases} 2(1-e^{-x}), 0\le x\le 1, \\ 2(e-1)e^{-x}, x>1. \end{cases}$

5 求微分方程 $y''+(x+e^{2y})y'^3=0$的通解.

分析：原方程不是 y 的线性方程, 可将x看成 y的函数.

解 $y'=\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}=\dfrac{1}{x'}$,

$x'(y)$

$$y''=\frac{dy'}{dx}=\frac{dy'}{dy}\cdot\frac{dy}{dx}=\frac{d}{dy}\left(\frac{1}{x'}\right)\cdot\frac{1}{x'}=-\frac{x''}{x'^2}\cdot\frac{1}{x'}=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3},$$

代入方程得 $-\dfrac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}+(x+e^{2y})\dfrac{1}{\left(\dfrac{dx}{dy}\right)^3}=0,$

即 $\dfrac{d^2x}{dy^2}-x=e^{2y}$, ……

通解是否就是所有的解?

例6 求方程 $\frac{\mathrm{d}y}{\mathrm{d}x}=(1-y^2)\tan x$的通解.

解 当 $y^2\neq 1$时,分离变量得 $\frac{\mathrm{d}y}{1-y^2}=\tan x\mathrm{d}x,$

两边积分得 $\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|=-\ln|\cos x|,$

解得:$y=\frac{C-\cos^2 x}{C+\cos^2 x},$

当 $y^2=1$时,即$y=\pm 1$也是原方程的解.

$C=0$时,$y=-1.$

但$y=1$时不包含在通解之中.

7.找出所有可微函数$f:(0,+\infty)\to(0,+\infty)$,对于这样一个函数,存在$a>0$,

使得对于$\forall x>0$,有$f'(\frac{a}{x})=\frac{x}{f(x)}$.

分析：$\forall x>0$,有$f'(\frac{a}{x})=\frac{x}{f(x)}\Rightarrow f'(\frac{a}{x})f(x)=x \Rightarrow f'(x)f(\frac{a}{x})=\frac{a}{x}$

$$\therefore [f(\frac{a}{x})f(x)]'=f'(x)f(\frac{a}{x})+f'(\frac{a}{x})f(x)(-\frac{a}{x^2}) =\frac{a}{x}+x(-\frac{a}{x^2})=0.$$

解 设$g(x)=f(\frac{a}{x})f(x)$

$\because g'(x)=f'(x)f(\frac{a}{x})+f'(\frac{a}{x})f(x)(-\frac{a}{x^2})=0$ $\therefore g(x)=b$,b为一常数.

即$g(x)=f(x)f(\frac{a}{x})=f(x)[\frac{a}{x}\frac{1}{f'(x)}]=b$ 即:$\frac{f'(x)}{f(x)}=\frac{a}{bx}$

即:$\ln f(x)=\frac{a}{b}\ln x+\ln c\Rightarrow f(x)=cx^{\frac{a}{b}}(c>0)$.

令$x=a,\Rightarrow c^2a^{\frac{a}{b}}=b\Rightarrow f(x)=\sqrt{b}(\frac{x}{\sqrt{a}})^{\frac{a}{b}},b>0$.

8 求方程$y''+4y'+5y=8\cos x$在$(-\infty,0)$内有界的特解.

解 此方程属 $f(x)=e^{\lambda x}[P_l(x)\cos\omega x+P_n(x)\sin\omega x]$型.

$$(\lambda=0,\ \omega=1,\ P_l(x)=8,\ P_n(x)=0).$$

其特征方程为: $r^2+4r+5=0,\ \Rightarrow r_{1,2}=-2\pm i.$

对应齐次方程的通解为 $Y=e^{-2x}(C_1\cos x+C_2\sin x).$

$\lambda+i\omega=i$ 不是特征根, $\therefore k=0.\ \therefore\ y^*=A\cos x+B\sin x.$

将 y^* 代入原方程并比较系数可得其特解:

$$y^*=\cos x+\sin x.$$

方程的通解为

$$y = e^{-2x}(C_1 \cos x + C_2 \sin x) + \cos x + \sin x.$$

要使 $y(x)$在$(-\infty, 0)$内有界，必须$C_1 = C_2 = 0$,

$$\therefore\ y = \cos x + \sin x.$$

9.求 $x+yy'=f(x)g(\sqrt{x^2+y^2})$的通解,并利用此结果求$x+yy'=(\sqrt{x^2+y^2}-1)\cdot\tan x$的通解.

解 原方程可变为 $2x+2yy'=2f(x)g(\sqrt{x^2+y^2})$

即 $(x^2+y^2)'=2f(x)g(\sqrt{x^2+y^2})$ 令$u=x^2+y^2$

方程变为 $\dfrac{du}{dx}=2f(x)g(\sqrt{u})$ 通解为$\displaystyle\int\frac{du}{g(\sqrt{u})}=2\int f(x)dx$

取$f(x)=\tan x, g(\sqrt{u})=\sqrt{u}-1$ 所以$\displaystyle\int\frac{du}{\sqrt{u}-1}=2\int\tan x dx$

通解为:$\sqrt{x^2+y^2}+\ln|\sqrt{x^2+y^2}-1|=-\ln|\cos x|+C.$

10. (2018研) 已知微分方程$y'+y=f(x)$, $f(x)$是R上的连续函数.

(1)若$f(x)=x$时, 求微分方程的通解;

(2)若$f(x)$是周期为T的函数, 证明:方程存在唯一的以T为周期的解.

解 (1)由一阶线性非齐次微分方程的公式可知

$$y=e^{-\int \mathrm{d}x}[\int xe^{\int \mathrm{d}x}\mathrm{d}x+C]=e^{-x}[\int xe^{x}\mathrm{d}x+C]=Ce^{-x}+x-1.$$

(2)由一阶线性非齐次微分方程的公式可知

$$y(x)=e^{-x}[\int_0^x e^t f(t)\mathrm{d}t+C]$$

$$\therefore\ y(x+T)=e^{-(x+T)}[\int_0^{(x+T)} e^t f(t)\mathrm{d}t+C]\quad 令t=u+T,$$

$$\therefore y(x+T)=e^{-(x+T)}[\int_0^{(x+T)} e^t f(t)\mathrm{d}t+C] \quad 令 t=u+T,$$

$$=e^{-(x+T)}[\int_{-T}^{x} e^{u+T} f(u+T)\mathrm{d}u+C]$$

$$y(x)=e^{-x}[\int_0^x e^t f(t)\mathrm{d}t+C]$$

$$=e^{-x}\int_{-T}^{x} e^u f(u)\mathrm{d}u+Ce^{-(x+T)}$$

$$\therefore e^{-x}\int_{-T}^{x} e^u f(u)\mathrm{d}u+Ce^{-(x+T)}=e^{-x}\int_0^x e^t f(t)\mathrm{d}t+Ce^{-x}$$

即：$y(x+T)-y(x)=e^{-x}\int_{-T}^{0} e^t f(t)\mathrm{d}t+Ce^{-(x+T)}-Ce^{-x}=0$

即 $C=\dfrac{1}{1-e^{-T}}\displaystyle\int_{-T}^{0} f(u)e^u\mathrm{d}u$为确定的常数,

所以符合条件的周期解$y(x)$唯一.

11. 设方程$(2x+1)y''+(4x-2)y'-8y=0$有多项式型的特解和形如$y=e^{mx}$(m为常数)之特解,求方程的通解.

解　设$y_1=e^{mx}$，代入原方程，得

$$(2m^2+4m)x+(m^2-2m-8)=0,$$

$$\Rightarrow \begin{cases} 2m^2+4m=0, \\ m^2-2m-8=0, \end{cases} \Rightarrow m=-2, \quad \Rightarrow y_1=e^{-2x},$$

设$y_2=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x+a_0$,

代入原方程，得 $(4n-8)a_n x^n+\cdots=0$,

$$\Rightarrow (4n-8)a_n=0,$$

代入原方程，得 $(4n-8)a_n x^n + \cdots = 0,$

$\Rightarrow (4n-8)a_n = 0,$

$\therefore n = 2,$ 即 $y_2 = a_2 x^2 + a_1 x + a_0,$

代入原方程，得 $-4a_1 x + (2a_2 - 2a_1 - 8a_0) = 0,$

$$\Rightarrow a_1 = 0,\ \ a_2 = 4a_0,$$

令 $a_0 = 1,$ 则 $a_2 = 4,$ $y_2 = 4x^2 + 1.$

$$\because \frac{y_1}{y_2} = \frac{e^{-2x}}{4x^2+1} \neq \text{常数},$$

故所求通解为 $y = C_1 e^{-2x} + C_2(4x^2+1).$

12.设方程 $y''-\frac{1}{x}y'+q(x)y=0$ 有两个特解 $y_1(x)$ 和 $y_2(x)$，

且 $y_1\cdot y_2=1$，求 $q(x)$，并求方程的通解.

解 (1) $y_1=a$，由 $y_1 \bullet y_2=1$ 知 $a\neq 0$，并且由 $y_1(x)=a$

是所给的微分方程 $y''-\frac{1}{x}y'+q(x)y=0,$

的特解知 $aq(x)=0,$ 由此推得 $q(x)=0,$

因此 $y''-\frac{1}{x}y'=0,$ 得特解 $x^2,$

它与 $y_1(x)=a$ 无关. 故通解：$y=C_1a+C_2x^2.$

$$y'' - \frac{1}{x} y' + q(x) y = 0$$

(2) 由 $y_2 = \dfrac{1}{y_1}, \Rightarrow y_2' = -\dfrac{y_1'}{y_1^2}, \quad \Rightarrow y_2'' = -\dfrac{y_1 y_1'' - 2 y_1'^2}{y_1^3},$

$$又 \because q(x) = -\frac{y_1'' - \frac{1}{x} y_1'}{y_1}, \quad q(x) = -\frac{y_2'' - \frac{1}{x} y_2'}{y_2},$$

$$\therefore \frac{y_1'' - \frac{1}{x} y_1'}{y_1} = y_1 \left(-\frac{y_1 y_1'' - 2 y_1'^2}{y_1^3} + \frac{1}{x} \frac{y_1'}{y_1^2} \right),$$

$$\Rightarrow y_1'' - \frac{1}{x}y_1' - \frac{1}{y_1}y_1'^2 = 0. \quad \Rightarrow y_1'' - - \frac{1}{y_1}y_1'^2 = \frac{1}{x}y_1'.$$

$$\Rightarrow \frac{y_1 y_1'' - y_1 y_1'^2}{y_1^2} = \frac{1}{x}\frac{y_1'}{y_1}. \qquad \Rightarrow \left(\frac{y_1'}{y_1}\right)' = \frac{1}{x}\frac{y_1'}{y_1}.$$

令$z = \dfrac{y_1{}'}{y_1}$，可得 $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{x}\cdot z$, $\Rightarrow z = 2x$,

$$\Rightarrow y_1(x) = e^{x^2}, y_2(x) = e^{-x^2}. \qquad q(x) = -4x^2.$$

故原方程的通解：$y = C_1 e^{x^2} + C_2 e^{-x^2}$.

13.设 $y_1(x)$ 和 $y_2(x)$ 是方程 $y''+p(x)y'+q(x)y=0$ 的解，试证：$y_1y_2'-y_2y_1'=Ce^{-\int p(x)\mathrm{d}x}$.

证　设 $u=y_1y_2'-y_2y_1'$，则

$$u'=y_1y_2''-y_2y_1''$$

$$=-y_1[p(x)y_2'+q(x)y_2]+y_2[p(x)y_1'+q(x)y_1]$$

$$=-(y_1y_2'-y_2y_1')p(x)=-up(x),$$

即 $u'+p(x)u=0$, $\Rightarrow u=Ce^{-\int p(x)\mathrm{d}x}$,

即 $y_1y_2'-y_2y_1'=Ce^{-\int p(x)\mathrm{d}x}$.

14.设$x''(t)+2mx'(t)+n^2x(t)=0$, $x(0)=x_1, x'(0)=x_2$, 其中$m>n>0$, 试求$\int_0^{+\infty}x(t)\mathrm{d}t$.

解　方程的通解为

$$x(t)=c_1e^{(-m+\sqrt{m^2-n^2})t}+c_2e^{(-m-\sqrt{m^2-n^2})t},$$

$\because m>n>0,$

$\therefore -m+\sqrt{m^2-n^2}<0,\ -m-\sqrt{m^2-n^2}<0,$

$\Rightarrow \lim\limits_{t\to+\infty}x(t)=0,\ \lim\limits_{t\to+\infty}x'(t)=0,$

$$\int_0^{+\infty}x'(t)\mathrm{d}t=\lim_{b\to+\infty}[x(t)]_0^b=\lim_{b\to+\infty}x(b)-x(0)$$

$=-x(0)=-x_1,$

$$\int_0^{+\infty} x''(t)\mathrm{d}t = \lim_{b\to+\infty}[x'(t)]_0^b = \lim_{b\to+\infty} x'(b) - x'(0)$$

$$= -x'(0) = -x_2,$$

由原方程可得：$n^2 x(t) = -[x''(t) + 2mx'(t)],$

$$\Rightarrow \int_0^{+\infty} n^2 x(t)\mathrm{d}t = -\int_0^{+\infty}[x''(t) + 2mx'(t)]\mathrm{d}t$$

$$= -\int_0^{+\infty} x''(t)\mathrm{d}t - 2m\int_0^{+\infty} x'(t)\mathrm{d}t$$

$$= x_2 + 2mx_1,$$

故$\int_0^{+\infty} x(t)\mathrm{d}t = \dfrac{1}{n^2}(x_2 + 2mx_1).$

15 设 $f(x)$可微,且满足$x=\int_0^x f(t)\mathrm{d}t+\int_0^x tf(t-x)\mathrm{d}t$,求

(1) $f(x)$的表达式.(2) $\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} |f(t)|^n \,\mathrm{d}x$ (其中 $n=2,3,\cdots$).

解 $\int_0^x tf(t-x)\mathrm{d}t = \int_{-x}^0 (u+x)f(u)\mathrm{d}u$

$$= \int_{-x}^0 tf(t)\mathrm{d}t + x\int_{-x}^0 f(t)\mathrm{d}t$$

故 $x=\int_0^x f(t)\mathrm{d}t+\int_{-x}^0 tf(t)\mathrm{d}t+x\int_{-x}^0 f(t)\mathrm{d}t$

两边求导得

$$1 = f(x) \boxed{- xf(-x)} + \int_{-x}^0 f(t)\mathrm{d}t \boxed{+xf(-x)}$$

$$i.e.\quad 1 = f(x) + \int_{-x}^{0} f(t)\mathrm{d}t \quad i.e.\quad 0 = f'(x) + f(-x) \quad (1)$$

$$\Rightarrow f''(x) - f'(-x) = 0 \quad (2)$$

$$\Rightarrow f'(-x) + f(x) = 0 \quad (3) \quad \Rightarrow \boxed{f''(x) + f(x) = 0}$$

$$\Rightarrow f(x) = C_1 \cos x + C_2 \sin x,$$

又 $f'(x) = -C_1 \sin x + C_2 \cos x.$ 又 $\because \quad f(0) = 1,$

$f'(0) = -1 \quad \Rightarrow C_1 = 1, C_2 = -1.$

$$\Rightarrow f(x) = \cos x - \sin x = \sqrt{2}\cos(x + \frac{\pi}{4}).$$

$$(2)\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} |f(t)|^n \, \mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sqrt{2})^n \left|\cos(x + \frac{\pi}{4})\right|^n \mathrm{d}x$$

$$= (\sqrt{2})^n \int_0^{\pi} |\cos t|^n \, \mathrm{d}t$$

$$= 2(\sqrt{2})^n \int_0^{\frac{\pi}{2}} |\cos t|^n \, \mathrm{d}t$$

$$= 2(\sqrt{2})^n \int_0^{\frac{\pi}{2}} \cos^n t \mathrm{d}t = \cdots$$

利用教材P251　15T

16 求满足条件$f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$的可微实函数$f(x)$.

解 令$y=0$,则 $f(x)=\dfrac{f(x)+f(0)}{1-f(x)f(0)}$,

$$f(0)[1+f^2(x)]=0, \qquad \therefore\ f(0)=0.$$

$$\text{又}\because \frac{f(x+y)-f(x)}{y}=\frac{f(y)-f(0)}{y}\cdot\frac{1+f^2(x)}{1-f(x)f(y)}$$

令$y\to 0$,两边同时取极限,得 $f'(x)=f'(0)[1+f^2(x)]$,

$$\frac{\mathrm{d}f(x)}{1+f^2(x)}=f'(0)\mathrm{d}x, \qquad \cdots\cdots$$

17 (1)求微分方程 $\frac{\mathrm{d}y}{\mathrm{d}x}-xy=xe^{x^2}$, $y(0)=1$的解.

(2)如果$y=f(x)$为上述方程的解,证明:

$$\lim_{n\to\infty}\int_0^1\frac{n}{1+n^2x^2}f(x)\mathrm{d}x=\frac{\pi}{2}.$$

(1)解　　由$P(x)=-x, Q(x)=xe^{x^2}$知,

$$y=e^{\int x\mathrm{d}x}[\int xe^{x^2}e^{-\int x\mathrm{d}x}\mathrm{d}x+C]=e^{\frac{x^2}{2}}[e^{\frac{x^2}{2}}+C]$$

由$y(0)=1$知,$C=0$,　　$\therefore\ y=e^{x^2}.$

(2)如果 $y=f(x)$为上述方程的解,证明:$\lim\limits_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2} f(x)\mathrm{d}x=\frac{\pi}{2}$.

证 $\int_0^1 \frac{n}{1+n^2x^2} e^{x^2}\mathrm{d}x = \int_0^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x + \int_0^1 \frac{n}{1+n^2x^2}\mathrm{d}x$

$\because \int_0^1 \frac{n}{1+n^2x^2}\mathrm{d}x = \int_0^1 \frac{\mathrm{d}(nx)}{1+n^2x^2} = \arctan(nx)\big|_0^1 = \arctan n.$

$\therefore \lim\limits_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2}\mathrm{d}x = \frac{\pi}{2}.$

$\because \forall\varepsilon>0$,由$\lim\limits_{x\to0}(e^{x^2}-1)=0$,$\exists\delta>0$,$\forall 0<x<\delta$时,$|e^{x^2}-1|<\frac{\varepsilon}{\pi}$.

$\therefore \int_0^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x = \int_0^\delta \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x$

$+\int_\delta^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x$

$$\therefore \int_0^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x = \int_0^\delta \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x$$

$$+\int_\delta^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x$$

$$\le \frac{\varepsilon}{\pi}\int_0^\delta \frac{n}{1+n^2x^2}\mathrm{d}x + (e-1)\int_\delta^1 \frac{n}{1+n^2x^2}\mathrm{d}x$$

$$\le \frac{\varepsilon}{\pi}\frac{\pi}{2} + \frac{(e-1)}{n}\int_\delta^1 \frac{1}{x^2}\mathrm{d}x = \frac{\varepsilon}{2} + \frac{1}{n}(e-1)(\frac{1}{\delta}-1) = \frac{\varepsilon}{2} + \frac{1}{n}M_0$$

$$\therefore 当 n > \frac{2M_0}{\varepsilon} 时, \ \left| \int_\delta^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x \right| < \frac{\varepsilon}{2}.$$

$$\therefore \left| \int_0^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x \right| < \varepsilon, \ \therefore \lim_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2}e^{x^2}\mathrm{d}x = \frac{\pi}{2}.$$

$$\int_0^1 \frac{n}{1+n^2x^2}e^{x^2}\mathrm{d}x = \int_0^1 \frac{n}{1+n^2x^2}(e^{x^2}-1)\mathrm{d}x + \int_0^1 \frac{n}{1+n^2x^2}\mathrm{d}x$$