Discussion problem assignment:

问题一：

2. For a periodic signal $x(t) \xleftrightarrow{FS} a_k$, find the FS for the following signal:

$$y(t) = x^*(-t-1)$$

答案：

信号 y(t)可以看成是对原信号 x(t)做时域的多次变换得到的。这里，也可以信号变换的方法，一步一步执行，每一步根据 FS 性质得到对应变换后的 FS。

$$y(t) = x^*(-t-1)$$

$$x(t) \xleftrightarrow{FS} a_k$$

$$g(t) = x(t-1) \xleftrightarrow{FS} b_k = a_k e^{-jk\omega_0}$$

$$h(t) = g(-t) = x(-t-1) \xleftrightarrow{FS} c_k = b_{-k} = a_{-k} e^{jk\omega_0}$$

$$x^*(-t-1) = h^*(t) \xleftrightarrow{FS} d_k = c^*_{-k} = a^*_k e^{jk\omega_0}$$

$$c_{-k} = a_k e^{-jk\omega_0}$$

答案二：使用合成公式

$$y(t) = x^*(-t-1)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(-t-1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t-1)}$$

$$x^*(-t-1) = \sum_{k=-\infty}^{\infty} a^*_k e^{-jk\omega_0(-t-1)} = \sum_{k=-\infty}^{\infty} a^*_k e^{jk\omega_0(t+1)}$$

$$= \sum_{k=-\infty}^{\infty} a^*_k e^{jk\omega_0} e^{jk\omega_0 t}$$

$$x^*(-t-1) \xleftrightarrow{FS} a^*_k e^{jk\omega_0}$$

第二题：

Prove the following two FS properties:

multiplication property $x(t)y(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

periodic convolution property $\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{FS} Ta_k b_k$

答案：

First, confirm that the new signal has the same period and same fundamental frequency as $x(t)$ and $y(t)$.

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad y(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t}$$

$$x(t)y(t)=\left(\sum_{l=-\infty}^{\infty}a_l e^{jl\omega_0 t}\right)\times\left(\sum_{m=-\infty}^{\infty}b_m e^{jm\omega_0 t}\right)=\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_l b_m e^{j(l+m)\omega_0 t}$$

$$=\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}a_l b_{k-l} e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}a_l b_{k-l} e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$$

First, confirm that the new signal has the same period and same fundamental frequency as $x(t)$ and $y(t)$.

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad y(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t}$$

$$\begin{aligned}
\frac{1}{T}\int_T z(t)e^{-jk\omega_0 t}dt &= \frac{1}{T}\int_0^T\int_0^T x(\tau)y(t-\tau)d\tau e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_0^T\int_0^T y(t-\tau)e^{-jk\omega_0(t-\tau)}dt\,x(\tau)e^{-jk\omega_0\tau}d\tau \\
&= \frac{1}{T}\int_0^T Tb_k x(\tau)e^{-jk\omega_0\tau}d\tau = Tb_k\frac{1}{T}\int_0^T x(\tau)e^{-jk\omega_0\tau}d\tau \\
&= Ta_k b_k
\end{aligned}$$

First, confirm that the new signal has the same period and same fundamental frequency as $x(t)$ and $y(t)$.

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad y(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t}$$

$$\begin{aligned}
\int_0^T x(\tau)y(t-\tau)d\tau &= \int_0^T\sum_{m=-\infty}^{\infty}a_m e^{jm\omega_0\tau}\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0(t-\tau)}d\tau \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_m b_k e^{jk\omega_0 t}\int_0^T e^{j(m-k)\omega_0\tau}d\tau \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_m b_k e^{jk\omega_0 t}T\delta_{m,k} \\
&= \sum_{k=-\infty}^{\infty}Ta_m b_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}
\end{aligned}$$

Orthogonal property of harmonics