解 原式$\xlongequal{\frac{0}{0}}\lim\limits_{x\to1}\dfrac{x\int_x^1 f(u)\mathrm{d}u}{-3(1-x)^2}\xlongequal{\frac{0}{0}}\lim\limits_{x\to1}\dfrac{\int_x^1 f(u)\mathrm{d}u-xf(x)}{6(1-x)}$

$$\xlongequal{\frac{0}{0}}\lim_{x\to1}\frac{-2f(x)-xf'(x)}{-6}=\lim_{x\to1}\frac{1}{6}xf'(x)=\frac{1}{6}.$$

4. 证明推论 1.2 中 ξ 可在开区间(a,b)内取得，即若 $f\in C[a,b]$，则至少存在一点 $\xi\in(a,b)$，使得 $\int_a^b f(x)\mathrm{d}x=f(\xi)(b-a)$.

证 $f\in C[a,b]$，由微积分第一基本定理，$\Phi(x)=\int_a^x f(t)\mathrm{d}t$ 在$[a,b]$上可导. 对 $\Phi(x)$在$[a,b]$上运用 Lagrange 微分中值定理，$\exists\,\xi\in(a,b)$，使

$$\Phi(b)-\Phi(a)=\Phi'(\xi)(b-a).$$

注意到 $\Phi(a)=0$，$\Phi'(\xi)=f(\xi)$. 故 $\exists\,\xi\in(a,b)$，使 $\int_a^b f(t)\mathrm{d}t=f(\xi)(b-a)$.

5. 设函数 f 在$[a,c]$上连续，在(a,c)内可导，且 $\int_a^b f(x)\mathrm{d}x=\int_b^c f(x)\mathrm{d}x=0$，其中 $b\in(a,c)$，证明至少存在一点 $\xi\in(a,c)$，使 $f'(\xi)=0$.

证 因为 $f\in C[a,c]$，由上题知：$\exists\,\xi_1\in(a,b)$，$\xi_2\in(b,c)$，使 $\int_a^b f(x)\mathrm{d}x=f(\xi_1)(b-a)$，$\int_b^c f(x)\mathrm{d}x=f(\xi_2)(c-b)$. 从而 $f(\xi_1)=f(\xi_2)=0$，对 $f(x)$在$[\xi_1,\xi_2]$上应用 Rolle 定理，$\exists\,\xi\in(\xi_1,\xi_2)\subset(a,c)$，使 $f'(\xi)=0$.

6. 设 $f,g\in C[a,b]$，证明至少存在一点 $\xi\in(a,b)$使

$$f(\xi)\int_\xi^b g(x)\mathrm{d}x=g(\xi)\int_a^\xi f(x)\mathrm{d}x.$$

证 令 $F(u)=\int_a^u f(x)\mathrm{d}x\cdot\int_u^b g(x)\mathrm{d}x$，则 $F(u)$在$[a,b]$上可导，且 $F(a)=F(b)=0$，于是 $F(u)$在$[a,b]$上满足 Rolle 定理条件，故 $\exists\,\xi\in(a,b)$，使 $F'(\xi)=0$，即

$$f(\xi)\int_\xi^b g(x)\mathrm{d}x=g(\xi)\int_a^\xi f(x)\mathrm{d}x.$$

习 题 3.3

(A)

1. 利用不定积分换元法则（Ⅰ）计算下列不定积分：

(4) $\int x^2(3+2x^3)^{\frac{1}{5}}\mathrm{d}x=\int\frac{1}{6}(3+2x^3)^{\frac{1}{5}}\mathrm{d}(3+2x^3)$

$$= \frac{1}{7}(3+2x^3)^{\frac{7}{6}} + C$$

(5) $\int \frac{3x^3+x}{1+x^4}\mathrm{d}x = \int \frac{3x^3}{1+x^4}\mathrm{d}x + \int \frac{x\mathrm{d}x}{1+x^4} = \frac{3}{4}\int \frac{\mathrm{d}(x^4+1)}{1+x^4} + \frac{1}{2}\int \frac{\mathrm{d}x^2}{1+(x^2)^2}$

$$= \frac{3}{4}\ln(1+x^4) + \frac{1}{2}\arctan x^2 + C.$$

(6) $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\mathrm{d}x = 2\int \sqrt{1+\sqrt{x}}\,\mathrm{d}(\sqrt{x}+1) = \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C.$

(7) $\int \frac{\cos\ln|x|}{x}\mathrm{d}x = \int \cos\ln|x|\,\mathrm{d}\ln|x| = \sin(\ln|x|) + C.$

(8) $\int \frac{\ln\ln x}{x\ln x}\mathrm{d}x = \int \ln\ln x\,\mathrm{d}\ln\ln x = \frac{1}{2}(\ln\ln x)^2 + C.$

(9) $\int \frac{\cos^3 x}{\sin^2 x}\mathrm{d}x = \int \frac{1-\sin^2 x}{\sin^2 x}\mathrm{d}\sin x = -\frac{1}{\sin x} - \sin x + C.$

(10) $\int \cos^4 x\mathrm{d}x = \int \left(\frac{1+\cos 2x}{2}\right)^2\mathrm{d}x = \frac{1}{4}\int \left(1+2\cos 2x + \frac{\cos 4x+1}{2}\right)\mathrm{d}x$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C.$$

(12) $\int \sec^4 x\mathrm{d}x = \int \sec^2 x \cdot \sec^2 x\mathrm{d}x = \int (\tan^2 x+1)\mathrm{d}\tan x$

$$= \frac{1}{3}\tan^3 x + \tan x + C.$$

(13) $\int \csc^3 x\cot x\mathrm{d}x = -\int \csc^2 x\mathrm{d}\csc x = -\frac{1}{3}\csc^3 x + C.$

(14) **解法一** $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^{-x}\mathrm{d}x}{\mathrm{e}^{-x}(1+\mathrm{e}^x)} = -\int \frac{\mathrm{d}(\mathrm{e}^{-x}+1)}{\mathrm{e}^{-x}+1}$

$$= -\ln(1+\mathrm{e}^{-x}) + C.$$

解法二 $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{(1+\mathrm{e}^x)-\mathrm{e}^x}{1+\mathrm{e}^x}\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(\mathrm{e}^x+1)}{1+\mathrm{e}^x}$

$$= x - \ln(1+\mathrm{e}^x) + C.$$

解法三 $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^x\mathrm{d}x}{\mathrm{e}^x(1+\mathrm{e}^x)} = \int \left(\frac{1}{\mathrm{e}^x} - \frac{1}{1+\mathrm{e}^x}\right)\mathrm{d}\mathrm{e}^x$

$$= x - \ln(1+\mathrm{e}^x) + C.$$

(15) **解法一** $\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\csc^2 x}{\csc^2 x+1}\mathrm{d}x = \int \frac{-\mathrm{d}\cot x}{\cot^2 x+2}$

$$= -\frac{\sqrt{2}}{2}\arctan\left(\frac{\cot x}{\sqrt{2}}\right) + C.$$

解法二 $\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\sec^2 x\,\mathrm{d}x}{\sec^2 x+\tan^2 x} = \int \frac{\mathrm{d}\tan x}{1+2\tan^2 x}$

$$= \frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x) + C.$$

(16) $\int \frac{x}{\sqrt{1+x^2}} e^{-\sqrt{1+x^2}} dx = \int e^{-\sqrt{1+x^2}} d\sqrt{1+x^2} = -e^{-\sqrt{1+x^2}} + C.$

(18) $\int \frac{dx}{\sqrt{4-x^2}\arccos\frac{x}{2}} = \int \frac{-d\arccos\frac{x}{2}}{\arccos\frac{x}{2}} = -\ln\left|\arccos\frac{x}{2}\right| + C.$

(20) $\int \frac{dx}{x^2-2x+3} = \int \frac{dx}{2+(x-1)^2} = \frac{1}{\sqrt{2}}\int \frac{d\frac{x-1}{\sqrt{2}}}{1+\left(\frac{x-1}{\sqrt{2}}\right)^2}$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x-1}{\sqrt{2}} + C.$$

(21) $\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}} = \arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) + C.$

(22) $\int \frac{\sin x\cos x}{1-\sin^4 x}dx = \frac{1}{2}\int \frac{d\sin^2 x}{1-(\sin^2 x)^2} = \frac{1}{4}\ln\frac{1+\sin^2 x}{1-\sin^2 x} + C.$

(23) $\int \frac{\sin x+\cos x}{\sqrt[5]{\sin x-\cos x}}dx = \int (\sin x-\cos x)^{-\frac{1}{5}} d(\sin x-\cos x)$

$$= \frac{5}{4}(\sin x-\cos x)^{\frac{4}{5}} + C.$$

(24) $\int \frac{dx}{e^x+e^{\frac{x}{2}}} = \int \frac{dx}{e^{\frac{x}{2}}(e^{\frac{x}{2}}+1)} = \int e^{-\frac{x}{2}}dx - \int \frac{(1+e^{\frac{x}{2}})-e^{\frac{x}{2}}}{1+e^{\frac{x}{2}}}dx$

$$= -2e^{-\frac{x}{2}} - x + 2\ln(1+e^{\frac{x}{2}}) + C.$$

2. 证明下列各式($m,n\in \mathbf{N}_+$)：

(1) $\int_{-\pi}^{\pi} \sin mx\sin nx\,dx = \begin{cases} 0, & m\neq n, \\ \pi, & m=n; \end{cases}$

(2) $\int_{-\pi}^{\pi} \cos mx\cos nx\,dx = \begin{cases} 0, & m\neq n, \\ \pi, & m=n; \end{cases}$

(3) $\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0$

证　(1) $\int_{-\pi}^{\pi} \sin mx\sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x-\cos(m+n)x]dx$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{n+m}\right]_{-\pi}^{\pi} = 0, & m\neq n, \\ \frac{1}{2}\left[x - \frac{1}{2m}\sin 2mx\right]_{-\pi}^{\pi} = \pi, & m=n. \end{cases}$$

(2) $\int_{-\pi}^{\pi}\cos mx\cos nx\,\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x+\cos(m-n)x]\mathrm{d}x$

$$=\begin{cases}0, & m\neq n,\\ \pi, & m=n.\end{cases}$$

(3) $\int_{-\pi}^{\pi}\sin mx\cos nx\,\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x+\sin(m-n)x]\mathrm{d}x=0.$

3. 利用不定积分换元法则(Ⅱ)计算下列不定积分:

(4) $\int\frac{x^2\mathrm{d}x}{\sqrt{a^2-x^2}}\xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=a\sin t}\int\frac{a^2\sin^2t}{a\cos t}\cdot a\cos t\mathrm{d}t=\frac{a^2}{2}\int(1-\cos 2t)\mathrm{d}t$

$$=\frac{a^2}{2}t-\frac{a^2}{4}\sin 2t+C=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}+C.$$

(6) $\int\frac{x^3}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x=\frac{1}{2}\int\frac{x^2\mathrm{d}(1+x^2)}{(1+x^2)^{\frac{3}{2}}}\xlongequal{t=\sqrt{1+x^2}}\frac{1}{2}\int\frac{t^2-1}{t^3}\cdot 2t\mathrm{d}t$

$$=t+\frac{1}{t}+C=\sqrt{1+x^2}+\frac{1}{\sqrt{1+x^2}}+C.$$

(7) $\int\frac{\sqrt{x^2+2x}}{x^2}\mathrm{d}x=\int\frac{1}{|x|}\sqrt{1+2x}\,\mathrm{d}x.$

令 $u=\sqrt{1+\frac{2}{x}}=\frac{\sqrt{x^2+2x}}{|x|}$(当 $x\in(-\infty,-2]$时,取 $u\in[0,1)$;当 $x\in(0,+\infty)$时,取 $u\in(1,+\infty)$),则 $\mathrm{d}x=\frac{-4u\mathrm{d}u}{(u^2-1)^2}$,于是当 $x\in(-\infty,-2]$,即 $u=-\frac{1}{x}\sqrt{x^2+2x}$时,

$$\int\frac{1}{|x|}\sqrt{1+\frac{2}{x}}\mathrm{d}x=\int-\frac{2u^2}{1-u^2}\mathrm{d}u$$

$$=\int\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)\mathrm{d}u$$

$$=2u+\ln\left|\frac{u-1}{u+1}\right|+C$$

$$=-\frac{2\sqrt{x^2+2x}}{x}+\ln\left|\frac{x+\sqrt{x^2+2x}}{x-\sqrt{x^2-2x}}\right|+C.$$

同理可得当 $x\in(0,+\infty)$时,上式也成立.故

$$\int\frac{\sqrt{x^2+2x}}{x^2}\mathrm{d}x=-\frac{2}{x}\sqrt{x^2+2x}+\ln\left|\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}-x}\right|+C.$$

(8) $\int\frac{\mathrm{d}x}{(x+1)\sqrt{x^2+2x+3}}\xlongequal{x+1=u}\int\frac{\frac{1}{2}\mathrm{d}u^2}{u^2\sqrt{u^2+2}}$

$$\xlongequal{t=\sqrt{u^2+2}}\frac{1}{2}\int\frac{2t\mathrm{d}t}{(t^2-2)t}$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{x^2+2x+3}-\sqrt{2}}{\sqrt{x^2+2x+3}+\sqrt{2}}\right|+C.$$

(9) $$\int\frac{\sqrt{1+\ln x}}{x\ln x}\mathrm{d}x=\int\frac{\sqrt{1+\ln x}}{\ln x}\mathrm{d}\ln x\xlongequal{t=\sqrt{1+\ln x}}\int\frac{t}{t^2-1}\cdot 2t\mathrm{d}t$$

$$=2\int\frac{(t^2-1)+1}{t^2-1}\mathrm{d}t=2t+\ln\left|\frac{t-1}{t+1}\right|+C$$

$$=2\sqrt{1+\ln x}+\ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right|+C.$$

(10) $$\int\frac{\mathrm{e}^{2x}}{\sqrt{3\mathrm{e}^x-2}}\mathrm{d}x=\frac{1}{3}\int\frac{\mathrm{e}^x}{\sqrt{3\mathrm{e}^x-2}}\mathrm{d}(3\mathrm{e}^x-2)$$

$$=\frac{1}{9}\int\left(\sqrt{3\mathrm{e}^x-2}+\frac{2}{\sqrt{3\mathrm{e}^x-2}}\right)\mathrm{d}(3\mathrm{e}^x-2)$$

$$=\frac{2}{27}(3\mathrm{e}^x-2)^{\frac{3}{2}}+\frac{4}{9}\sqrt{3\mathrm{e}^x-2}+C$$

$$=\frac{2}{27}(3\mathrm{e}^x+4)\sqrt{3\mathrm{e}^x-2}+C.$$

(11) $$\int\frac{\mathrm{d}x}{1+\sin x+\cos x}=\int\frac{\mathrm{d}x}{2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}}$$

$$=\int\frac{\frac{1}{2}}{\cos^2\frac{x}{2}\left(1+\tan\frac{x}{2}\right)}\mathrm{d}x$$

$$=\int\frac{\mathrm{d}\tan\frac{x}{2}}{1+\tan\frac{x}{2}}=\ln\left|1+\tan\frac{x}{2}\right|+C.$$

(12) $$\int x\sqrt{\frac{1-x}{1+x}}\mathrm{d}x=\int\frac{x(1-x)}{\sqrt{1-x^2}}\mathrm{d}x=\int\frac{-\frac{1}{2}\mathrm{d}(1-x^2)}{\sqrt{1-x^2}}-\int\frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x$$

$$=-\sqrt{1-x^2}+\frac{1}{2}x\sqrt{1-x^2}-\frac{1}{2}\arcsin x+C.$$

其中 $$\int\frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x\xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{x=\sin t}\int\sin^2t\mathrm{d}t=\frac{1}{2}t-\frac{1}{4}\sin 2t+C$$

$$=\frac{1}{2}(\arcsin x-x\sqrt{1-x^2})+C.$$

(13) $$\int\sqrt{e^{2x}+5}\,dx\xlongequal[t\in\left(0,\frac{\pi}{2}\right)]{\frac{1}{\sqrt{5}}e^x=\tan t}\int\sqrt{5}\sec t\,\frac{\sec^2 t}{\tan t}dt$$

$$=-\sqrt{5}\int\left[\frac{1}{(1-\cos^2 t)}-\frac{1}{\cos^2 t}\right]d\cos t$$

$$=-\frac{\sqrt{5}}{2}\ln\left|\frac{1+\cos t}{1-\cos t}\right|+\sqrt{5}\,\frac{1}{\cos t}+C$$

$$=-\frac{\sqrt{5}}{2}\ln\left|(\sqrt{e^{2x}+5}+\sqrt{5})/(\sqrt{5}-\sqrt{e^{2x}+5})\right|$$

$$+\sqrt{e^{2x}+5}+C.$$

4. 求下列定积分的值：

(2) $$\int_0^1\frac{dx}{e^x+e^{-x}}=\int_0^1\frac{de^x}{e^{2x}+1}=\arctan e^x\Big|_0^1=\arctan e-\frac{\pi}{4}.$$

(4) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x}\,|\sin x|\,dx$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\sin x\,dx-\int_{-\frac{\pi}{2}}^{0}\sqrt{\cos x}\sin x\,dx=\frac{4}{3}.$$

(6) $$\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\sqrt{1-x^2}}{x^2}dx\xlongequal{x=\sin t}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot^2 t\,dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\csc^2 t-1)dt=1-\frac{\pi}{4}.$$

(8) $$\int_0^{\pi}\sqrt{1+\cos 2x}\,dx=\int_0^{\pi}\sqrt{2}\,|\cos x|\,dx$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\cos x\,dx+\sqrt{2}\int_{\frac{\pi}{2}}^{\pi}(-\cos x)dx$$

$$=2\sqrt{2}.$$

5. 设 f 在 $[-a,a]$ 上连续，利用定积分的换元法证明：

(1) 如果 $f(x)$ 为奇函数，那么 $\int_{-a}^{a}f(x)dx=0$；

(2) 如果 $f(x)$ 为偶函数，那么 $\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx$；

(3) 计算 $\int_{-1}^{1}|x|\left(x^2+\frac{\sin^3 x}{1+\cos x}\right)dx$.

证 (1) 由 $f(x)$ 为奇函数知：$\int_{-a}^{a}f(x)dx\xlongequal{t=-x}\int_{a}^{-a}-f(-t)dt=-\int_{-a}^{a}f(t)dt$，

故 $$\int_{-a}^{a}f(t)dt=0.$$

(2) 由 f 为偶函数知：$\int_{-a}^{0}f(x)dx\xlongequal{x=-t}\int_{a}^{0}f(-t)d(-t)=\int_0^a f(t)dt$，

从而
$$\int_{-a}^{a} f(x)\mathrm{d}x = \int_0^a f(x)\mathrm{d}x + \int_{-a}^{0} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x.$$

(3) 利用(1)、(2)结论，则
$$\int_{-1}^{1} |x| \left(x^2 + \frac{\sin^3 x}{1+\cos x}\right)\mathrm{d}x = 2\int_0^1 |x| x^2 \mathrm{d}x = 2\int_0^1 x^3 \mathrm{d}x = \frac{1}{2}.$$

6. 设 $f(x)$ 为连续的周期函数，其周期为 T，利用定积分换元法证明
$$\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x \quad (a \text{ 为常数}).$$

证　由于
$$\int_T^{a+T} f(x)\mathrm{d}x \xlongequal{x=T+t} \int_0^a f(T+t)\mathrm{d}t = \int_0^a f(t)\mathrm{d}t,$$
因而
$$\begin{aligned}\int_a^{a+T} f(x)\mathrm{d}x &= \int_a^0 f(x)\mathrm{d}x + \int_0^T f(x)\mathrm{d}x + \int_T^{a+T} f(x)\mathrm{d}x \\ &= \int_0^T f(x)\mathrm{d}x.\end{aligned}$$

7. 利用分部积分法计算下列积分：

(2) $$\begin{aligned}\int x^3 \operatorname{ch} x\mathrm{d}x &= \int x^3 \mathrm{d}\operatorname{sh} x = x^3 \operatorname{sh} x - 3\int x^2 \mathrm{d}\operatorname{ch} x \\ &= x^3\operatorname{sh} x - 3x^2 \operatorname{ch} x + 6\int x\mathrm{d}\operatorname{sh} x \\ &= x^3\operatorname{sh} x - 3x^2\operatorname{ch} x + 6x\operatorname{sh} x - 6\operatorname{ch} x + C.\end{aligned}$$

(5) $$\begin{aligned}\int \frac{xe^x}{(1+e^x)^2}\mathrm{d}x &= \int -x\mathrm{d}\frac{1}{1+e^x} = -\frac{x}{1+e^x} + \int \frac{\mathrm{d}x}{1+e^x} \\ &= -\frac{x}{1+e^x} + \int \frac{1+e^x-e^x}{e^x+1}\mathrm{d}x \\ &= -\frac{x}{1+e^x} + x - \ln(1+e^x) + C.\end{aligned}$$

(6) $$\begin{aligned}\int \frac{\arcsin x}{\sqrt{1-x}}\mathrm{d}x &= -2\int \arcsin x\mathrm{d}\sqrt{1-x} \\ &= -2\sqrt{1-x}\arcsin x + 2\int \frac{\sqrt{1-x}}{\sqrt{1-x^2}}\mathrm{d}x \\ &= -2\sqrt{1-x}\arcsin x + 4\sqrt{1+x} + C.\end{aligned}$$

(8) $$\begin{aligned}\int \sqrt{x}\sin\sqrt{x}\mathrm{d}x &\xlongequal{t=\sqrt{x}} \int (t\sin t)(2t\mathrm{d}t) = -2\int t^2 \mathrm{d}\cos t \\ &= -2t^2\cos t + 4\int t\mathrm{d}\sin t \\ &= -2t^2\cos t + 4t\sin t + 4\cos t + C. \\ &= (4-2x)\cos\sqrt{x} + 4\sqrt{x}\sin\sqrt{x} + C.\end{aligned}$$

(9) $\int_0^{e-1}\ln(1+x)\mathrm{d}x=(x+1)\ln(1+x)\Big|_0^{e-1}-\int_0^{e-1}(1+x)\mathrm{d}\ln(1+x)=1.$

(11) $\int x\sin x\cos x\mathrm{d}x=\frac{1}{4}\int -x\mathrm{d}\cos 2x=-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C.$

(12) $$\int\sin(\ln x)\mathrm{d}x=x\sin(\ln x)-\int\cos(\ln x)\mathrm{d}x$$

$$=x\sin(\ln x)-x\cos(\ln x)-\int\sin(\ln x)\mathrm{d}x,$$

故 $$\int\sin(\ln x)\mathrm{d}x=\frac{x}{2}[\sin(\ln x)-\cos(\ln x)]+C.$$

(13) $\int\left(\ln x+\frac{1}{x}\right)e^x\mathrm{d}x=e^x\ln x+C.$

其中：$\int\frac{1}{x}e^x\mathrm{d}x=\int e^x\mathrm{d}\ln x=e^x\ln x-\int\ln x\mathrm{d}e^x=e^x\ln x-\int e^x\ln x\mathrm{d}x.$

8. 证明下列递推公式($n=2,3,\cdots$)：

(1) 设 $I_n=\int\tan^n x\mathrm{d}x$，则 $I_n=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}$；

(2) 设 $I_n=\int\frac{\mathrm{d}x}{\sin^n x}$，则 $I_n=\frac{1}{1-n}\cdot\frac{\cos x}{\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}$.

证 (1) $$I_n=\int\tan^{n-2}x(\sec^2 x-1)\mathrm{d}x=\int\tan^{n-2}x\mathrm{d}\tan x-I_{n-2}$$

$$=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}.$$

(2) $$I_n=-\int\frac{\mathrm{d}\cot x}{\sin^{n-2}x}=-\frac{\cot x}{\sin^{n-2}x}-\int\frac{(\cot x)(n-2)\cos x}{\sin^{n-1}x}\mathrm{d}x$$

$$=-\frac{\cos x}{\sin^{n-1}x}-(n-2)\int\frac{1-\sin^2 x}{\sin^n x}\mathrm{d}x$$

$$=-\frac{\cos x}{\sin^{n-1}x}-(n-2)I_n+(n-2)I_{n-2},$$

从而 $$I_n=\frac{-1}{n-1}\frac{\cos x}{\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}.$$

9. 计算下列积分：

(1) $\int\frac{\mathrm{d}x}{x^4+3x^2}=\frac{1}{3}\int\left(\frac{1}{x^2}-\frac{1}{x^2+3}\right)\mathrm{d}x=-\frac{1}{3x}-\frac{1}{3\sqrt{3}}\arctan\frac{x}{\sqrt{3}}+C.$

(2) $$\int\frac{t}{t^4+10t^2+9}\mathrm{d}t=\frac{1}{8}\int\left(\frac{t}{t^2+1}-\frac{t}{t^2+9}\right)\mathrm{d}t$$

$$=\frac{1}{16}[\ln(1+t^2)-\ln(t^2+9)]+C.$$

(3) $\int\frac{x^2}{(x-1)^{100}}\mathrm{d}x=\int\frac{(x^2-1)+1}{(x-1)^{100}}\mathrm{d}x=\int\frac{(x-1)+2}{(x-1)^{99}}\mathrm{d}x+\int\frac{\mathrm{d}x}{(x-1)^{100}}$

$$= \int \frac{dx}{(x-1)^{98}} + \int \frac{2dx}{(x-1)^{99}} + \int \frac{dx}{(x-1)^{100}}$$

$$= -\frac{1}{97}\frac{1}{(x-1)^{97}} - \frac{1}{49(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + C.$$

(4) $\int \frac{1-x^7}{x(1+x^7)}dx = \int\left(\frac{1}{x} - \frac{2x^6}{1+x^7}\right)dx = \ln|x| - \frac{2}{7}\ln(1+x^7) + C.$

(5) $\int \frac{dx}{3+2\cos x} = \int \frac{dx}{1+2(1+\cos x)} = \int \frac{dx}{1+4\cos^2\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}+4}dx$

$$= 2\int \frac{d\tan\frac{x}{2}}{\tan^2\frac{x}{2}+5} = \frac{2}{\sqrt{5}}\arctan\left(\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right) + C.$$

(6) $\int \frac{\cos x - \sin x}{\cos x + \sin x}dx = \ln|\sin x + \cos x| + C.$

(7) $\int \frac{x^2}{a^2 - x^6}dx = \frac{1}{3}\int \frac{dx^3}{a^2-(x^3)^2} = \frac{1}{6a}\ln\left|\frac{a+x^3}{a-x^3}\right| + C.$

(8) $\int \frac{x^{11}dx}{x^8+4x^4+5} = \frac{1}{4}\int\left(1 - \frac{4x^4+5}{(x^4+2)^2+1}\right)\cdot dx^4$

$$= \frac{1}{4}x^4 - \frac{1}{4}\int \frac{4(x^4+2)-3}{(x^4+2)^2+1}dx^4$$

$$= \frac{1}{4}x^4 - \frac{1}{2}\ln|1+(x^4+2)^2| + \frac{3}{4}\arctan(x^4+2) + C.$$

(9) $\int \frac{\ln\tan x}{\sin x\cos x} = \int \frac{\ln\tan x}{\tan x}\cdot\frac{1}{\cos^2 x}dx = \int \frac{\ln\tan x}{\tan x}d\tan x$

$$= \int \ln(\tan x)d\ln(\tan x)$$

$$= \frac{1}{2}\ln^2(\tan x) + C.$$

(10) $\int \frac{\cos 2x}{1+\sin x\cos x}dx = \int \frac{\cos 2x}{1+\frac{1}{2}\sin 2x}dx = \int \frac{d\,\frac{1}{2}\sin 2x}{1+\frac{1}{2}\sin 2x}$

$$= \ln\left|1+\frac{1}{2}\sin 2x\right| + C.$$

(11) $\int \frac{x+\sin x}{1+\cos x}dx = \int \frac{x}{2\cos^2\frac{x}{2}}dx - \int \frac{-\sin x}{1+\cos x}dx$

$$
\begin{aligned}
&= \int x \mathrm{d} \tan \frac{x}{2} - \ln(1+\cos x) \\
&= x \tan \frac{x}{2} - \int \tan \frac{x}{2} \mathrm{d}x - \ln(1+\cos x) \\
&= x \tan \frac{x}{2} + 2\ln \left| \cos \frac{x}{2} \right| - \ln(1+\cos x) + C \\
&= x \tan \frac{x}{2} + C' \quad (C' = C - \ln 2).
\end{aligned}
$$

(12) $\int \frac{x^2+1}{x^4+1} \mathrm{d}x$.

解法一 $\frac{x^2+1}{x^4+1} = \frac{x^2+1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$

$$= \frac{1}{(\sqrt{2}x+1)^2+1} + \frac{1}{(\sqrt{2}x-1)^2+1},$$

原式 $= \frac{1}{\sqrt{2}}[\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)] + C$.

解法二 在任何不包含 0 的区间内

$$
\begin{aligned}
f(x) &= \frac{x^2+1}{x^4+1} = \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} = \frac{\left(x-\frac{1}{x}\right)'}{\left(x-\frac{1}{x}\right)^2+2} \\
&= \left(\frac{1}{\sqrt{2}} \arctan \frac{x^2-1}{\sqrt{2}x}\right)' \stackrel{\text{def}}{=} (F(x))',
\end{aligned}
$$

故
$$\int f(x) \mathrm{d}x = \begin{cases} F(x) + C_1, & x \in (0, +\infty), \\ F(x) + C_2, & x \in (-\infty, 0). \end{cases}$$

又由于被积函数 $f(x)$在$(-\infty,+\infty)$内连续,故应在$(-\infty,+\infty)$内求其原函数,因此应在上式补充定义原函数在 $x=0$ 点的值,使原函数在 $x=0$ 处连续. 又因为 $\lim\limits_{x\to0^+}(F(x)+C_1) = -\frac{\pi}{2\sqrt{2}}+C_1$, $\lim\limits_{x\to0^-}(F(x)+C_2) = \frac{\pi}{2\sqrt{2}}+C_2$, 令 $-\frac{\pi}{2\sqrt{2}}+C_1 = \frac{\pi}{2\sqrt{2}}+C_2 = C$, 于是 $\int f(x)\mathrm{d}x = \begin{cases} \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} + \frac{\pi}{2\sqrt{2}} + C, & x>0, \\ C, & x=0, \\ \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{\pi}{2\sqrt{2}} + C, & x<0. \end{cases}$

(14) $\int \frac{\sin x}{\sin x + \cos x} \mathrm{d}x$.

解法一 原式 $= I = \int \frac{(\sin x - \cos x) + \cos x}{\sin x + \cos x} \mathrm{d}x$

$$=-\int\frac{\mathrm{d}(\sin x+\cos x)}{\sin x+\cos x}+\int\frac{(\cos x+\sin x)-\sin x}{\sin x+\cos x}\mathrm{d}x$$

$$=-\ln|\sin x+\cos x|+x-I.$$

故 $$I=\int\frac{\sin x}{\sin x+\cos x}\mathrm{d}x=-\frac{1}{2}\ln|\sin x+\cos x|+\frac{1}{2}x+C.$$

解法二 $$I=\int\frac{\sin(\cos x-\sin x)}{\cos^2 x-\sin^2 x}\mathrm{d}x$$

$$=\int\left(\frac{\sin x\cos x}{2\cos^2 x-1}-\frac{1-\cos 2x}{2\cos 2x}\right)\mathrm{d}x$$

$$=-\frac{1}{4}\int\frac{\mathrm{d}(2\cos^2 x-1)}{2\cos^2 x-1}-\int\frac{1}{2}(\sec 2x-1)\mathrm{d}x$$

$$=-\frac{1}{4}\ln|2\cos^2 x-1|-\frac{1}{4}\ln|\sec 2x+\tan 2x|+\frac{1}{2}x+C.$$

解法三 $$I=\int\frac{\sin x(\cos x-\sin x)}{\cos^2 x-\sin^2 x}\mathrm{d}x=\int\left(\frac{\sin 2x}{2\cos 2x}-\frac{1-\cos 2x}{2\cos 2x}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int(\tan 2x-\sec 2x+1)\mathrm{d}x$$

$$=-\frac{1}{4}\ln|\cos 2x|-\frac{1}{4}\ln|\sec 2x+\tan 2x|+\frac{x}{2}+C.$$

解法四 令 $\tan x=t$ 得

$$I=\int\frac{\tan x}{\tan x+1}\mathrm{d}x=\int\frac{t\mathrm{d}t}{(t+1)(t^2+1)}=\frac{1}{2}\int\left(\frac{t+1}{t^2+1}-\frac{1}{t+1}\right)\mathrm{d}t,$$ 可以积出.

解法五 令 $\cot x=t$,

$$I=\int\frac{\mathrm{d}x}{1+\cot x}=-\int\frac{\mathrm{d}t}{(1+t)(1+t^2)}=\frac{1}{2}\int\left(\frac{t-1}{1+t^2}-\frac{1}{1+t}\right)\mathrm{d}t,$$ 可以积出.

解法六 令 $\tan\frac{x}{2}=t$ 得.

$$I=\int\frac{\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t=\int\left(\frac{1+t}{1+t^2}-\frac{1-t}{1+2t-t^2}\right)\mathrm{d}t,$$ 可以积出.

解法七

$$I=\frac{1}{\sqrt{2}}\int\frac{\sin x}{\cos\left(x-\frac{\pi}{4}\right)}\mathrm{d}x\xlongequal{t=x-\frac{\pi}{4}}\frac{1}{\sqrt{2}}\int\frac{\sin\left(t+\frac{\pi}{4}\right)}{\cos t}\mathrm{d}t$$

$$=\frac{1}{2}\int\frac{\sin t+\cos t}{\cos t}\mathrm{d}t,$$ 可以积出.

解法八

$$I=\int\frac{\frac{1}{\sqrt{2}}\sin\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)}{\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}}\mathrm{d}x=\int\frac{\sin\left(x+\frac{\pi}{4}\right)-\cos\left(x+\frac{\pi}{4}\right)}{2\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x,$$

$$=\frac{1}{2}\int\left(1-\cot\left(x+\frac{\pi}{4}\right)\right)\mathrm{d}x=\frac{1}{2}x-\frac{1}{2}\ln\left|\sin\left(x+\frac{\pi}{4}\right)\right|+C.$$

(15) $\displaystyle\int\frac{x^2-1+3}{(x-1)^4}\mathrm{d}x=\int\left(\frac{x-1+2}{(x-1)^3}+\frac{3}{(x-1)^4}\right)\mathrm{d}x$

$$=\int\left(\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{3}{(x-1)^4}\right)\mathrm{d}x$$

$$=-\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{1}{(x-1)^3}+C.$$

(16) $\displaystyle\int\frac{1}{x}\sqrt{\frac{1+x}{x}}\mathrm{d}x\xlongequal{t=\frac{1}{x}}\int t\sqrt{t+1}\left(-\frac{1}{t^2}\right)\mathrm{d}t$

$$=-\int\frac{1}{t}\sqrt{t+1}\mathrm{d}t\xlongequal{u=\sqrt{t+1}}-\int\frac{u}{u^2-1}2u\mathrm{d}u$$

$$=-\int\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)\mathrm{d}u$$

$$=-2u+\ln\left|\frac{u+1}{u-1}\right|+C$$

$$=-2\sqrt{\frac{1}{x}+1}+\ln\left|\frac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x}}-1}\right|+C.$$

10. 证明下列积分等式(其中 f 为连续函数):

(1) $\displaystyle\int_0^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x=\int_0^{\frac{\pi}{2}}f(\cos x)\mathrm{d}x.$

证 $\displaystyle\int_0^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x\xlongequal{t=\frac{\pi}{2}-x}\int_{\frac{\pi}{2}}^0 f(\cos t)(-\mathrm{d}t)=\int_0^{\frac{\pi}{2}}f(\cos x)\mathrm{d}x.$

(2) $\displaystyle\int_a^b f(x)\mathrm{d}x=(b-a)\int_0^1 f[a+(b-a)x]\mathrm{d}x.$

证 $\displaystyle\int_a^b f(x)\mathrm{d}x\xlongequal{x=a+(b-a)t}\int_0^1 f[a+(b-a)t][(b-a)\mathrm{d}t]$,得证.

(3) $\displaystyle\int_0^1 x^m(1-x)^n\mathrm{d}x=\int_0^1 x^n(1-x)^m\mathrm{d}x$

证 左 $\displaystyle\xlongequal{t=1-x}\int_1^0(1-t)^m t^n(-\mathrm{d}t)=\int_0^1 t^n(1-t)^m\mathrm{d}t=$右.

(4) $\displaystyle\int_0^a x^3f(x^2)\mathrm{d}x=\frac{1}{2}\int_0^{a^2}xf(x)\mathrm{d}x.$

证　$\int_0^a x^3 f(x^2)\mathrm{d}x = \int_0^a \frac{1}{2}x^2 f(x^2)\mathrm{d}x^2 \xlongequal{t=x^2} \int_0^{a^2} \frac{1}{2}tf(t)\mathrm{d}t$

$$= \frac{1}{2}\int_0^{a^2} xf(x)\mathrm{d}x.$$

(B)

1. 证明：$\int_0^{\frac{\pi}{2}} \sin^m x\cos^m x\mathrm{d}x = \frac{1}{2^m}\int_0^{\frac{\pi}{2}}\cos^m x\mathrm{d}x(m=0,1,2,\cdots)$.

证　$\int_0^{\frac{\pi}{2}} \sin^m x\cos^m x\mathrm{d}x = \frac{1}{2^m}\int_0^{\frac{\pi}{2}}\sin^m 2x\mathrm{d}x = \frac{1}{2^{m+1}}\int_0^{\pi}\sin^m t\mathrm{d}t(t=2x)$

$$= \frac{1}{2^{m+1}}\left(\int_0^{\frac{\pi}{2}}\sin^m t\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi}\sin^m t\mathrm{d}t\right) = \frac{1}{2^m}\int_0^{\frac{\pi}{2}}\cos^m t\mathrm{d}t.$$

$\Bigg($由本习题(A)第 10 题(1)知 $\int_0^{\frac{\pi}{2}}\sin^m t\mathrm{d}t = \int_0^{\frac{\pi}{2}}\cos^m t\mathrm{d}t$，而

$$\int_{\frac{\pi}{2}}^{\pi}\sin^m t\mathrm{d}t \xlongequal{u=t-\frac{\pi}{2}} \int_0^{\frac{\pi}{2}}\left[\sin\left(u+\frac{\pi}{2}\right)\right]^m\mathrm{d}u = \int_0^{\frac{\pi}{2}}\cos^m u\mathrm{d}u.\Bigg)$$

2. 计算 $\int_0^{n\pi}\sqrt{1-\sin 2x}\mathrm{d}x(n\in\mathbf{N}_+)$.

解　因为$\sqrt{1-\sin 2x}$是周期函数，且最小正周期为 π，所以由习题 3.3(A)第 6 题

$$\int_0^{n\pi}\sqrt{1-\sin 2x}\mathrm{d}x = n\int_0^{\pi}\sqrt{1-\sin 2x}\mathrm{d}x = n\int_0^{\pi}|\sin x-\cos x|\mathrm{d}x$$

$$= n\left[\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\mathrm{d}x + \int_{\frac{\pi}{4}}^{\pi}(\sin x-\cos x)\mathrm{d}x\right]$$

$$= 2\sqrt{2}n.$$

3. 计算 $\int_0^{10\pi}\frac{\sin^3 x+\cos^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x$.

解　被积函数是周期为 $T=2\pi$ 的周期函数，且 $\sin^3 x$ 为奇函数，所以

$$\text{原式} = 5\int_{-\pi}^{\pi}\frac{\sin^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x + 5\int_{-\pi}^{\pi}\frac{\cos^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x$$

$$= 10\int_0^{\pi}\frac{\cos^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x$$

$$= 10\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x + 10\int_{\frac{\pi}{2}}^{\pi}\frac{\cos^3 x\mathrm{d}x}{2\sin^2 x+\cos^4 x}.$$

又因为　$\int_{\frac{\pi}{2}}^{\pi}\frac{\cos^3 x}{2\sin^2 x+\cos^4 x}\mathrm{d}x \xlongequal{-u=\frac{\pi}{2}-x} \int_0^{\frac{\pi}{2}} -\frac{\sin^3 u}{2\cos^2 u+\sin^4 u}\mathrm{d}u$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{2\sin^2 x+\cos^4 x}dx \xlongequal{t=\frac{\pi}{2}-x} \int_{\frac{\pi}{2}}^0 \frac{\sin^3 t}{2\cos^2 t+\sin^4 t}(-dt),$$

故原式=0.

4. 计算$\int_0^{n\pi} x\,|\sin x|\,dx \quad (n\in \mathbf{N}_+)$.

解 $I=\int_0^{n\pi} x\,|\sin x|\,dx \xlongequal{u=n\pi-x} \int_0^{n\pi}(n\pi-u)\,|\sin u|\,du$,

即 $I=n\pi\int_0^{n\pi}|\sin u|-I$,故 $I=\frac{1}{2}n\pi\int_0^{n\pi}|\sin u|\,du$.

又因为$|\sin u|$是周期为π的周期函数,故

$$I=\frac{1}{2}n^2\pi\left(\int_0^{\pi}\sin u du\right)=n^2\pi.$$

5. 计算$\int_{\frac{1}{2}}^2\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$.

解
$$\begin{aligned}\int_{\frac{1}{2}}^2 e^{x+\frac{1}{x}}dx &= xe^{x+\frac{1}{x}}\Big|_{\frac{1}{2}}^2-\int_{\frac{1}{2}}^2 xe^{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)dx\\ &=\frac{3}{2}e^{\frac{5}{2}}-\int_{\frac{1}{2}}^2\left(x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx,\end{aligned}$$

故原式 $=\frac{3}{2}e^{\frac{5}{2}}$.

6. 计算$\int\frac{xe^x}{(1+x)^2}dx$.

解
$$\begin{aligned}\text{原式}&=\int\frac{(x+1-1)e^x}{(1+x)^2}dx=\int\frac{e^x}{1+x}dx+\int e^x\frac{-1}{(1+x)^2}dx\\ &=\int\frac{e^x}{1+x}dx+\frac{e^x}{1+x}-\int\frac{1}{1+x}de^x=\frac{e^x}{1+x}+C.\end{aligned}$$

习 题 3.4

(A)

1. 求由下列各曲线围成平面图形的面积:

(1) 曲线 $y=9-x^2$, $y=x^2$ 与直线 $x=0$, $x=1$.

解 如图所示,面积元为 $dA=[(9-x^2)-x^2]dx=(9-2x^2)dx$,从而所求面积为 $A=\int_0^1(9-2x^2)dx=\frac{25}{3}$.