Chapter 1
Introduction to Probability

Instructor : Dr. Jing Liang

School of Information and Communication Engineering

liangjing@uestc.edu.cn

Outlines

- What is Probability
- The History of Probability
- Experiments and Events
- The Sample Space & Finite Sample Spaces
- Set Operations
- Axioms & Properties of Probability
- Counting Methods
- Combinatorial Methods
- Multinomial Coefficients

What is Probability

- Ex1: It probably will be sunny tomorrow.
  (sunny or not)
- Ex2: The chance that I will win the game is 80%.
  (the chance to win)

- Ex3: Toss a coin.
  (head or tail)
- Ex4: Toss a die.
  (frequency of the number)

The **likelihood** that a specific event will occur.

Measure **randomness** and **uncertainty**.

The History of Probability

- 3500B.C. - 2000B.C.:
  Play games of chance with bone objects or cubical dice with markings in Egypt (early development of probability theory)
- 1500 – 1700:
  Solve gambling problems by mathematicians (e.g. Blaise Pascal, Galileo Galilei, etc.)
- 1700 – Current
  Steadily developed and widely applied in diverse fields of study (marketing, finance, earthquake prediction, computer science, telecommunications, etc.)

Experiments and Events

- **Definition 1.3.1 Experiment**: any process, real or hypothetical, in which the possible outcomes can be identified ahead of time.
- **Event**: a well-defined set of possible outcomes of the experiment.

Experiment	Event
Toss a coin 5 times	At least obtain 3 heads
Toss a die 10 times	Obtain number 4 once
52 cards distributed to 4 players	Each player receives one ace

The Sample Space-1

- **Definition 1.4.1 Sample Space**: a set of all possible outcomes, S
- **Outcome**: a point, an element, a member, s
- **Events**: subsets of the sample space, E
- Ex1: Sample space of rolling a six-sided die?
  $S = \{1,2,3,4,5,6\}$ e.g., $E=\{1,3,5\}$
  Name some events
- Ex2: Sample space of rolling 2 dice?
  $S = \{(\mathrm{i,j}), \mathrm{i,j} = 1,2,3,4,5,6\}$
- Ex3: Toss a coin 2 times?
  $S = \{(x,y), x,y = \text{Head,Tail}\}$
  $s \in S$ $s \in E$ $S \supset E$

Ex4 (Book Ex1.4.5) -1

Demands for Utilities. Plan water and electricity demand for an office complex. The demand for electricity will range somewhere between 1 million and 150 million kilowatt-hours per day and water demand will be between 4 and 200 (in thousands of gallons per day). All combinations of electrical and water demand are possible.

- **Sample space is the set of ordered pairs:**

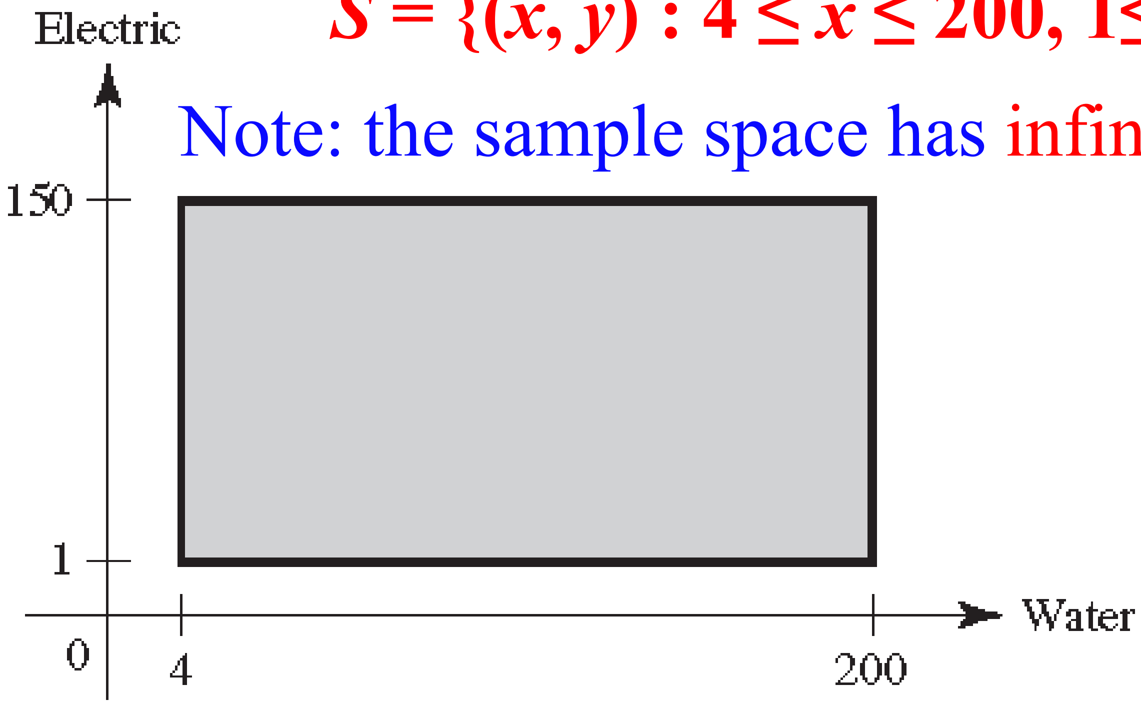
$$S = \{(x, y) : 4 \le x \le 200,\ 1 \le y \le 150\}$$

- x stands for water demand in thousands of gallons per day
  y stands for the electric demand in millions of kilowatt-hours per day.

Ex4 (Book Ex1.4.5) -2

$S = \{(x, y) : 4 \leq x \leq 200, 1 \leq y \leq 150\}$

Note: the sample space has infinitely many points



Some events:

A=\{water demand is at least 100\} $= \{(x, y) : x \geq 100\}$

B=\{electricity no more than 35\}$= \{(x, y) : y \leq 35\}$

The Sample Space-2

Notes:

- The sample space must be exhaustive, i.e. all possible outcomes should be included.

  Rolling a die: $S = \{1,2,3,4,5\}$ is not complete

- The outcomes in sample space must be mutually exclusive

  $S = \{1,2,2,3,4,5,6\}$ is not mutually exclusive

- Is sample space S an event?

  Yes! It must!

Condition 1 must be met by events

Finite Sample Spaces (Book 1.6)

- Experiment has only finitely possible outcomes.

Sample space S contains only a finite number of points $s_1, s_2, \cdots, s_n$.

Requirement of Probabilities: p_i - the prob. of $s_i (i = 1, \cdots, n)$

1) $p_i \geq 0$ for $i = 1, \cdots, n$

2) $\sum_{i=1}^{n} p_i = 1$

Q: Prob. of event E?

Sol: Add p_i of s_i belong to E.

Simple Sample Spaces (classical probability):

Prob. of each outcome s_i is $1/n$. (The assumption)

An event A in this simple sample space has m outcomes

$$\Pr(A) = \frac{m}{n}$$

Ex5 (Book Ex1.4.4)

- **Tossing a Coin.** Toss a coin 3 times. Sample space S?

$S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ s_i - a member of S

$s_i = (xyz),\ \ i = 1, \cdots 8,\ x, y, z = H, T$

Is this a finite sample space? Yes!

Event A: at least 1 head is obtained in the 3 tosses. 7/8

Event B: a head is obtained on the 2nd toss. 1/2

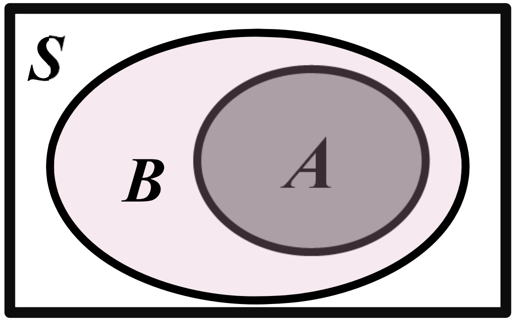
Event C: no heads are obtained. 1/8

Event D: obtaining exactly two heads. 3/8

Q: Prob. of each event?

Set Operations -1

- **Definition 1.4.2 Containment**: if A and B are any two sets, A is contained in B (or B contains A) if every element of the set A also belongs to the set B.
- Expressed symbolically by $A \subset B$ or $B \supset A$



e.g., rolling a die

$A = \{1,3,5\}, B = \{1,2,3,4,5\}$

- **Theorem 1.4.1**

If $A \subset B$ and $B \subset A$, then $A = B$

If $A \subset B$ and $B \subset C$, then $A \subset C$

Set Operations -2

- **Definition 1.4.3 <u>Empty Set</u>:** the subset of S that contains no elements. Any event that can't occur, or ***null set***, $\varnothing$ denoted by $\varnothing \subset A$.
- **Theorem 1.4.2** Let A be an event. Then
- **<u>Finite and Infinite Sets</u>**: contain finite/infinite elements
- **Two sizes of infinite sets – Definition 1.4.4**

  <u>Countable</u>: there is a one-to-one correspondence between the elements of A and the set of natural numbers. e.g., integers, non-negative integers, prime numbers

  <u>Uncountable</u>: neither finite nor countable. e.g., real numbers, the numbers in the interval [0,1].

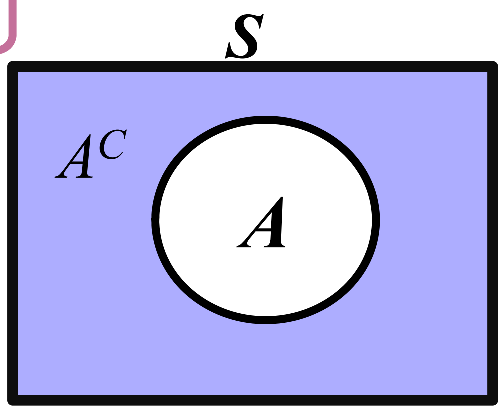
Set Operations - 3

- **Definition 1.4.5 Complement**: the complement of a set A is defined to be the set that contains all outcomes of the sample space S that do not belong to A: $A^c = \{s \in S \mid s \notin A\}$

  The event A does not occur.

- Condition 2 must be met by events.

  If A is an event, then A^c is also an event.



- Ex 6. Rolling a die. Suppose A is the event that an even number is rolled.

$$A^c = \{1, 3, 5\}$$

- **Theorem 1.4.3** $(A^c)^c = A, \quad (\varnothing)^c = S, \quad (S)^c = \varnothing$

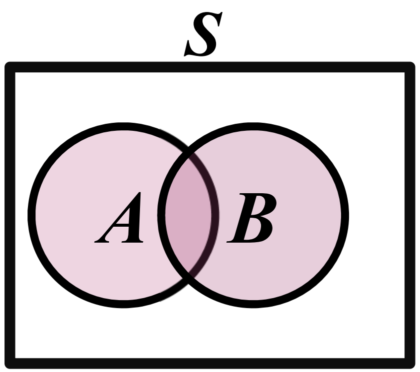
Set Operations - 4

- **Definition 1.4.6 Union of two Sets**: If A and B are any two sets, the union of A and B is defined to be the set containing all outcomes that belong to A alone, to B alone, or to both A and B.

  The event either A, or B, or both occur.

  e.g. Roll a die. $A = \{1,3,5\}$, $B = \{3,4,6\}$

  $$A \cup B = \{1,3,4,5,6\}$$



- **Theorem 1.4.3**

  $A \cup B = B \cup A$, $A \cup A = A$, $A \cup A^c = S$, $A \cup \varnothing = A$, $A \cup S = S$.

  If $A \subset B$, then $A \cup B = B$.

- **Definition 1.4.7 Union of Many Sets** $\bigcup_{i=1}^{n} A_i$

Set Operations - 5

- **Theorem 1.4.5**

  The union of a finite number of events $A_1, \cdots, A_n$ is an event.

- If $A_1, A_2, \cdots$ is a countable collection of events, then $\bigcup_{i=1}^{\infty} A_i$ is also an event.

Condition 3 must be met by events.

We do not require that the union of an arbitrary collection of events be an event.

We require the above 3 simple conditions in order to do be able to do all the probability calculation.

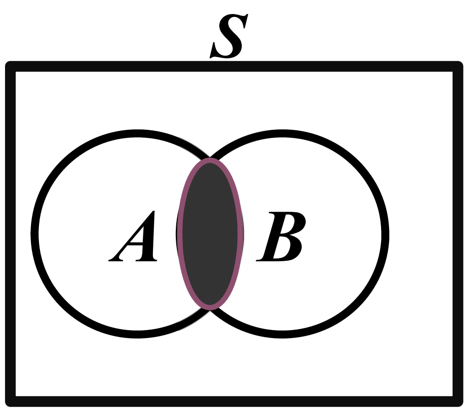
Set Operations - 6

- **Definition 1.4.8 <u>Intersection of two Sets</u>**: If A and B are any two sets, the intersection of A and B is defined to be the set that contains all outcomes that belong both to A and to B.

  The event that both A and B occur.

  e.g. Roll a die. $A = \{1,3,5\}$, $B = \{3,4,6\}$

  $$A \cap B = \{3\}$$



- **Theorem 1.4.7**

  $A \cap B = B \cap A$, $A \cap A = A$, $A \cap A^c = \varnothing$, $A \cap \varnothing = \varnothing$, $A \cap S = A$.

  If $A \subset B$, then $A \cap B = A$.

- **Definition 1.4.9 <u>Intersection of Many Sets</u>**

  $$\bigcap_{i=1}^{n} A_i$$

Set Operations - 7

- **Properties of Set Operations**
- **Theorem 1.4.6 & 1.4.8 Associative Property**

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$$

$$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$$

- **Theorem 1.4.9 De Morgan's Laws**

$$(A \cup B)^C = A^C \cap B^C \qquad (A \cap B)^C = A^C \cup B^C$$

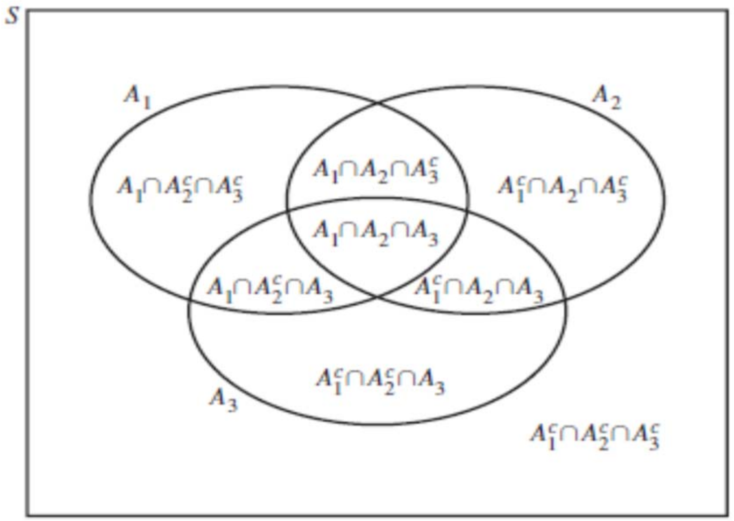
- **Theorem 1.4.10 Distributive Property**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Operations - 8

- **Definition 1.4.10 Disjoint/Mutually Exclusive**: It is said that two sets A and B are ***disjoint, or mutually exclusive***, if A and B have no outcomes in common, that is, if $A \cap B = \varnothing$.
- The event that A and B cannot both occur.
- The sets $A_1, \ldots, A_n$ ($n \geq 2$) are disjoint if for every $i \neq j$, we have that A_i and A_j are disjoint, that is, $A_i \cap A_j = \varnothing$ for all $i \neq j$.



Set Operations - 9

- **Theorem 1.4.11: <u>Partitioning a Set</u>**

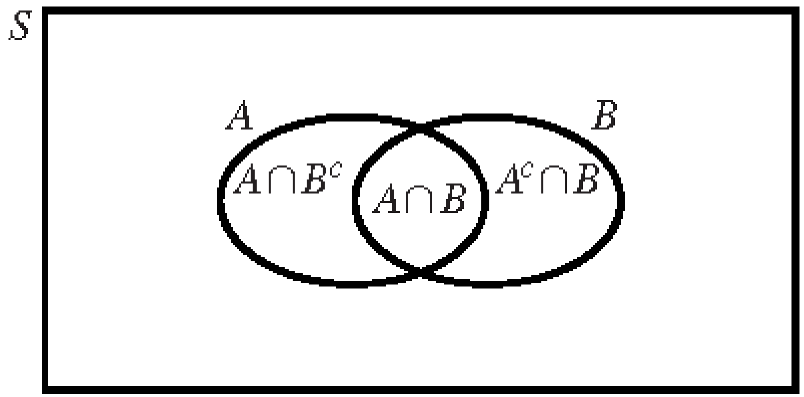
For every two sets A and B, $A \bigcap B$ and $A \bigcap B^C$ are disjoint,

$$A = (A \bigcap B) \bigcup (A \bigcap B^C)$$

$$B = (A \bigcap B) \bigcup (A^C \bigcap B)$$

- In addition, B and $A \bigcap B^C$ are disjoint,

$$A \bigcup B = B \bigcup (A \bigcap B^C) \qquad A \bigcup B = A \bigcup (A^C \bigcap B)$$



Axioms of Probability

- **Axiom 1** for every event A, $\Pr(A) \geq 0$.
- **Axiom 2** $\Pr(S)=1$.
- **Axiom 3** For every infinite sequence of disjoint events

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

- **Definition 1.5.1 Probability** A ***probability measure***, or simply a ***probability***, on a sample space S is a specification of numbers $\Pr(A)$ for all events A that satisfy Axiom 1-3.
- **Theorem 1.5.1** $\Pr(\varnothing)=0$.
- **Theorem 1.5.2** For every finite sequence of n disjoint events $A_1, A_2, \ldots,$ $\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$

Properties of Probability

- **Theorem 1.5.3** $\Pr(A^c) = 1 - \Pr(A)$

Proof Hint $A \cup A^c = S$, $\Pr(A) + \Pr(A^c) = \Pr(S)$

- **Theorem 1.5.4** If $A \subset B$, then $\Pr(A) \le \Pr(B)$

Proof Hint $B = A \cup (A^C \cap B)$, $\Pr(B) = \Pr(A) + \Pr(A^C \cap B)$

- **Theorem 1.5.5** For every event A, $0 \le \Pr(A) \le 1$
- **Theorem 1.5.6** $\Pr(A \cap B^C) = \Pr(A) - \Pr(A \cap B)$

Proof Hint $A = (A \cap B) \cup (A \cap B^C)$

- **Theorem 1.5.7** $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Proof Hint $A \cup B = B \cup (A \cap B^C)$

$$\Pr(A \cup B) = \Pr(B) + \Pr(A) - \Pr(A \cap B)$$

Ex6 (Book 1.5.3)

- **Diagnosing Diseases**. A doctor decides that a patient has either a bacterial infection or a viral infection or both. The doctor decides that there is a probability of 0.7 that the patient has a bacterial infection and a probability of 0.4 that the person has a viral infection. What is the probability that the patient has both infections?
- Hint: B-the event that a patient has a bacterial infection
  V-the event that a patient has a viral infection

  $$\Pr(V \cup B) = \Pr(B) + \Pr(V) - \Pr(V \cap B)$$

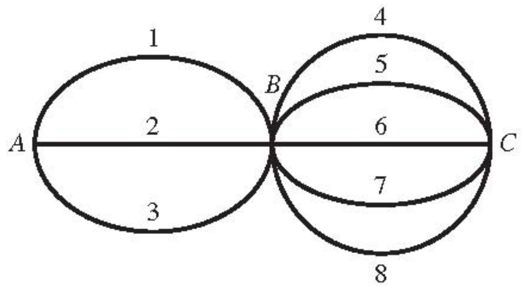
  $$\Pr(V \cap B) = 0.7 + 0.4 - 1$$
- Prove that real numbers are uncountable P13-14.

Counting Methods - 1

- Count the number of outcomes in a set.
- **Theorem 1.7.2 <u>Multiplication Rule</u>**

Suppose that an experiment has k parts (k>1), the ith part have n_i possible outcomes (i=1,…,k). Each part occurs regardless of other parts. The total number of outcomes in S will be $\prod_{i=1}^{k} n_i$.

Ex7 (Book Example 1.7.1) Routes between Cities.



The number of total routes:

$3\times 5 = 15$

Ex8 (Book 1.7.4) Toss six coins. The prob. to obtain H on all six coins? 1/64

The prob. to obtain exactly one H? 6/64=3/32

Counting Methods - 3

- **Definition 1.7.1 <u>Permutations</u>**

Suppose that a set has n elements. An experiment to select k elements **one at a time without replacement**. Each outcome consists of the k elements **in the order selected.** Each such outcome is ***a permutation of n elements taken k at a time***. Denote by the symbol $P_{n,k}$. ***Notice: here n≥k!***

- **Theorem 1.7.3 Number of Permutations.**

The number of permutations of n elements taken k at a time is $P_{n,k} = n(n-1)\cdots(n-k+1) = n!/(n-k)!$

Ex 9 (Book Ex 1.7.8) choose a p. and a s. from 25

$$P_{25,2} = (25)(24) = 600$$

Ex10(Book Ex1.7.9) Arrange 6 different books to a shelf $P_{6,6} = 6! = 720$

Counting Methods - 4

- **Sampling with replacement**

Ex11 (Book 1.7.10&1.7.11) Obtaining Different No.

Consider a box containing n balls numbered 1,…,n. First, one ball is selected at random and its number is noted. This ball is then put back in the box and anther ball is selected. This process is ***sampling with replacement***. Assume that each ball is equally likely to be selected at each stage and each selection is independent.

Can $k > n$? Yes! Why not?

Q1: What's the sample space of this experiment?

S contains all all vectors of the form $(x_1, \cdots, x_k)$, where x_i is the outcome of the ith selection $(i = 1, \cdots, k)$.

Q2: What's the total number of vectors in S? n^k

Counting Methods - 4

- **Sampling with replacement**

Ex11 (Book 1.7.10&1.7.11) continues. What's the prob. of the event E that selected k balls will have a different number (assume $k \le n$)?

$$\Pr(E) = \frac{P_{n,k}}{n^k} = \frac{n!}{(n-k)!\,n^k}.$$

Combinatorial Methods-1

- Count how many subsets are contained in a fixed set.

Ex12 (Book Ex1.8.1) Choosing Subsets. Consider the set {a,b,c,d}. Count the number of distinct subsets of size two. Note: {a,b} and {b,a} are the same subsets.

List all the subsets: {a,b} {a,c} {a,d} {b,c} {b,d} {c,d}

- **Definition 1.8.1 Combinations**

Consider a set with n elements. Each subset of size k chosen from this set is a ***combination of n elements taken k at a time***. Denote by the symbol $C_{n,k}$

- **Theorem 1.8.1**

$$\because P_{n,k} = C_{n,k} k! \qquad \therefore C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{k!(n-k)!}$$

Combinatorial Methods-2

Ex13 (Book Ex1.8.2) Selecting a Committee. Suppose that a committee composed of 8 people will be selected from a group of 20 people. The number of different groups of people that might be on the committee is

$$C_{20,8} = \frac{P_{20,8}}{8!} = \frac{20!}{8!12!} = 125,970$$

Ex14 (Book Ex1.8.3) Selecting a Committee. Suppose that a committed composed of 8 people (each people assign a different job) will be selected from a group of 20 people. The number of ways to choose 8 people with the different job is:

$$P_{20,8} = C_{20,8} \times 8! = 125,970 \times 8! = 5,078,110,400$$

Combinatorial Methods-3

- **Definition 1.8.2 Binomial Coefficients** $C_{n,k} = C_{n,n\text{-}k}$

$C_{n,k}$ is also denoted by the symbol $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

- **Theorem 1.8.2 Binomial Theorem**

For all numbers x and y and each positive integer n

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- **Definition 1.9.1 Multinomial Coefficients**

$$\binom{n}{n_1, n_2, \cdots n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Ex15 (Book Ex1.9.1)

- **Choosing Committees**. 20 people are to be divided into 3 committees A, B and C. A, B and C have 8,8,4 members, respectively. Each person gets assigned to one and only one committee. Determine the number of different ways of assignment.

Sol: $$\binom{20}{8}\binom{12}{8} = \frac{20!}{8!12!}\frac{12!}{8!4!} = 62,355,150$$

n distinct elements to be divided into *k* different groups ($k \geq 2$), for $j = 1, \ldots, k$, the *j*th group contains exactly n_j elements, where $n_1 + n_2 + \ldots + n_k = n$. Determine the number of different ways of division .

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-2}}{n_{k-1}} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

Ex16 (Book Ex1.8.4)

Blood Type. There are 3 alleles O,A,B. The gene for human blood type consists of a pair of alleles chose from these 3 alleles. E.g., AO and BB (called genotypes). AO and OA are the same. How many genotypes for blood type?

Sol:

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

unordered sampling with replacement

The general formula for the number of unordered samples of size k with replacement from n elements is

$$\binom{n+k-1}{k}$$

Prove this in Hw#1

***$k \geq n$* is possible!**

Ex17 (Book Ex1.8.7)

- **Tossing a Coin**. Toss a fair coin 10 times. Determine: (a) the prob. of obtaining exactly 3 heads; (b) the prob. of obtaining 3 or fewer heads.

Sol: (a)

$$p = \frac{C_{10,3}}{2^{10}} = \frac{10!}{3!7!2^{10}} = 0.1172$$

(b)

$$p' = \frac{C_{10,0} + C_{10,1} + C_{10,2} + C_{10,3}}{2^{10}} = \frac{176}{2^{10}} = 0.1719$$

Ex18 (Book Ex1.8.8)

- **Sampling without Replacement**. A class contains 15 boys and 30 girls. 10 students are to be selected randomly. Determine the prob. that exactly 3 boys will be selected.

Sol:

$$p = \frac{\binom{15}{3}\binom{30}{7}}{\binom{45}{10}} = 0.2904$$

Ex19 (Book Ex1.9.3)

- **Rolling Dice**. Suppose that 12 dice are to be rolled. We shall determine the probability p that each of the six different numbers will appear twice.

Sol: The number of the outcomes such that each of the six different numbers will appear twice is

$$\binom{12}{2,2,2,2,2,2} = \frac{12!}{(2!)^6}$$

$$p = \frac{12!}{2^6 6^{12}} = 0.0034$$