3.3 方向导数与梯度

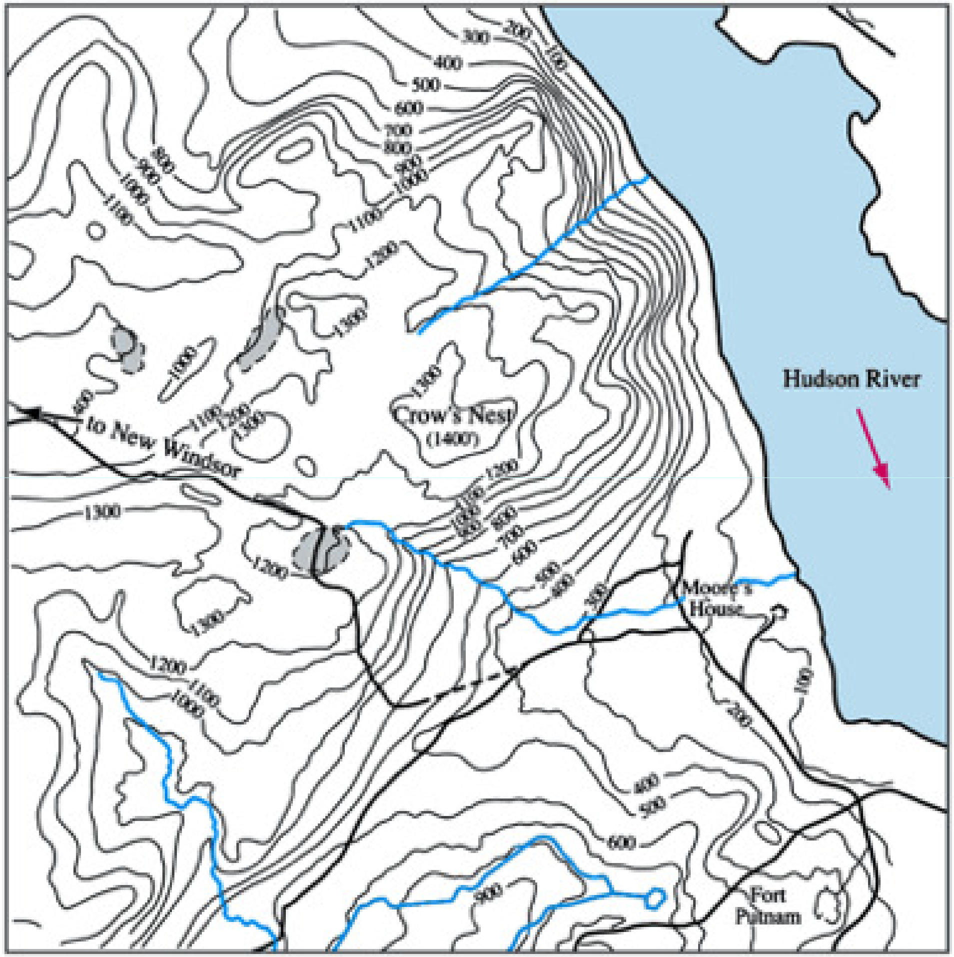


FIGURE 14.25 Contours along the Hudson River in New York show streams, which follow paths of steepest descent, running perpendicular to the contours.

3.3 方向导数与梯度

定义3.3（方向导数）　设点$x_0\in \mathbf{R}^2$,l是平面上一向量，其单位向量为e_l. f:$U(x_0)\subseteq\mathbf{R}^2\to\mathbf{R}$.在$U(x_0)$内让自变量$x$由$x_0$沿与$e_l$平行的直线变到$x_0+te_l$，从而函数值的改变量$f(x_0+te_l)-f(x_0)$.

若

$$\lim_{t\to 0}\frac{f(x_0+te_l)-f(x_0)}{t}$$

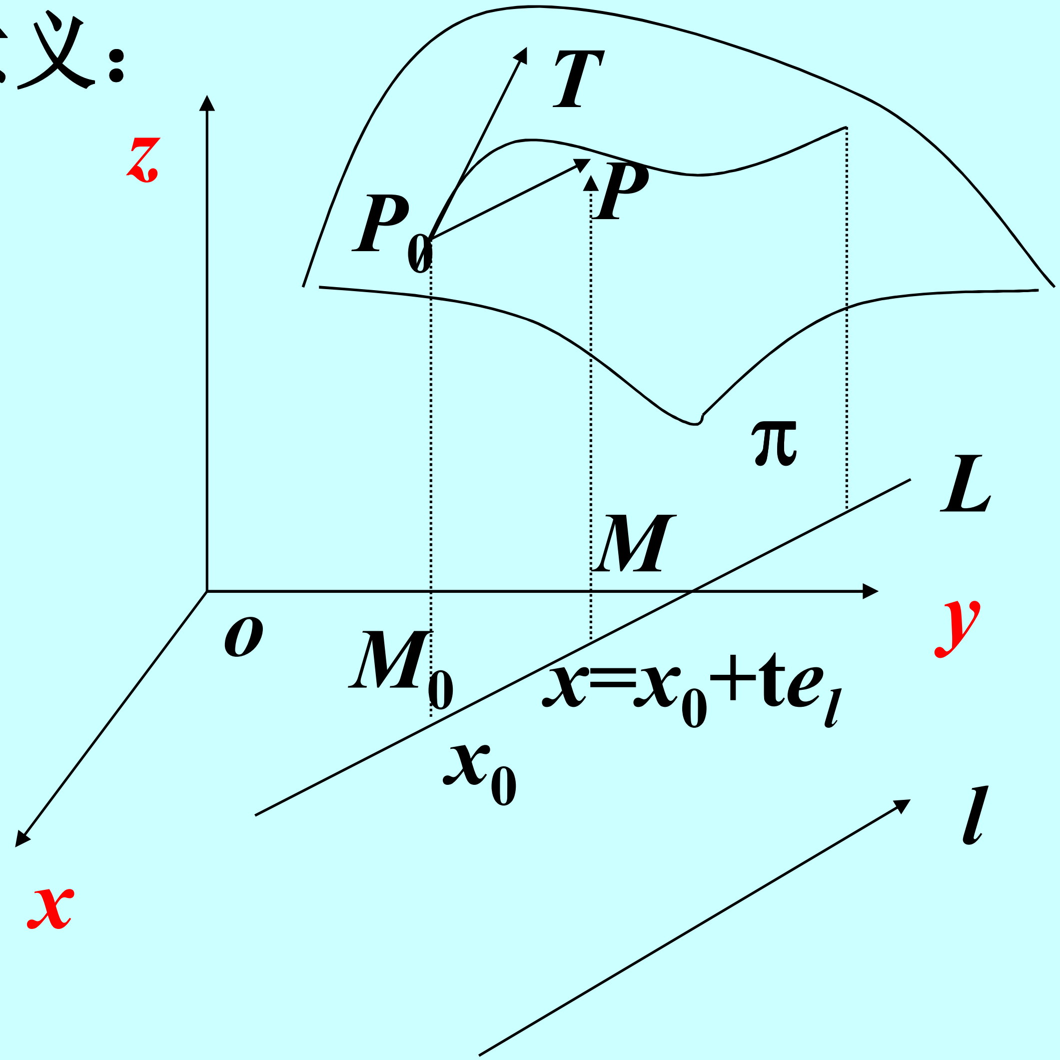
存在，则称此极限为f在点x_0沿l方向的方向导数。

记作：

$$\frac{\partial f(x_0)}{\partial l}=\left.\frac{\partial f}{\partial l}\right|_{x_0}=\lim_{t\to 0}\frac{f(x_0+te_l)-f(x_0)}{t}$$

方向导数的几何意义：

过直线L:$x=x_0+te_l$作平行于z轴的平面π,它与曲面在$z=f(x,y)$所交的曲线C在P_0点唯一切线关于l方向的斜率(与向量l交角的正切值)



例 3.12 设二元函数

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$$

求f在点(0,0)沿方向e_l=(cosθ,sin θ)的方向导数。

解：

$$\frac{\partial f(0,0)}{\partial l}=\begin{cases}\dfrac{\sin^2\theta}{\cos\theta} & \cos\theta\neq 0\\ 0 & \cos\theta=0\end{cases}$$

注1: $$\left.\frac{\partial f}{\partial(-l)}\right|_{x_0} = -\left.\frac{\partial f}{\partial l}\right|_{x_0}$$

注2:在一点的所有的方向导数都存在，也不一定在此点连续。

设e_l是$\mathbf{R}^n$中的一个单位向量，用其方向余弦可表示为

$$e_l=(\cos\theta_1,\cos\theta_2,\cdots,\cos\theta_n),$$

$$\|e_l\|=\sqrt{\cos^2\theta_1+\cos^2\theta_2+\cdots+\cos^2\theta_n}=1.$$

$e_1=(1,0,\cdots,0),\mathrm{e}_2=(0,1,0,\cdots,0),\cdots,$

$e_n=(0,\cdots,0,1)$是$\mathbf{R}^n$的一个标准正交基，

$x_0\in\mathbf{R}^n, f:\mathbf{R}^n\supseteq U(x_0)\to\mathbf{R}$，则

$u=f(x)$在点x_0处沿l方向的方向导数

$$\frac{\partial f(x_0)}{\partial l}=\frac{\partial f}{\partial l}\bigg|_{\mathrm{x}_0}=\lim_{t\to 0}\frac{f(x_0+t\mathrm{e}_l)-f(x_0)}{t}$$

$u = f(x)$在点x_0处对x_i的偏导数

就是它在点 x_0沿方向 $\mathbf{e}_i\ (i = 1,2,\cdots,n)$

的方向导数，即

$$\left.\frac{\partial u}{\partial x_i}\right|_{\mathbf{x}_0} = \frac{\partial f(\mathbf{x}_0)}{\partial x_i} = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)}{t}$$

其中$x_0 = (x_{0,1}, x_{0,2}, \cdots, x_{0,n})$，记$\Delta x_i = t$，则有

$$\frac{\partial f(\mathbf{x}_0)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_0 + \Delta x_i \mathbf{e}_i) - f(x_0)}{\Delta x_i}$$

$$= \lim_{\Delta x_i \to 0} \frac{f(x_{0,1}, \cdots, x_{0,i-1}, x_{0,i} + \Delta x_i, x_{0,i+1}, \cdots, x_{0,n}) - f(x_{0,1}, x_{0,2}, \cdots, x_{0,n})}{\Delta x_i}$$

3.梯度

定义3.4(梯度)

设$u=f(\mathrm{x})=f(x_1,x_2,\cdots,x_n)$在点$x_0$处可微,

则称向量$\left(\dfrac{\partial f(x_0)}{\partial x_1},\cdots,\dfrac{\partial f(x_0)}{\partial x_n}\right)$为$f$在点$x_0$处的

梯度向量,简称**梯度**,记为$\mathrm{grad}\ f(x_0)$或$\nabla f(x_0)$.

$$\mathrm{grad}\ f(x_0)=\nabla f(x_0)=\left(\frac{\partial f(x_0)}{\partial x_1},\cdots,\frac{\partial f(x_0)}{\partial x_n}\right)$$

所以方向导数的计算公式：

$$\frac{\partial f(x_0)}{\partial l} = L(e_l) = \sum_{i=1}^{n} \frac{\partial f(x_0)}{\partial x_i} \cos\theta_i$$

$$\frac{\partial f\left(x_0\right)}{\partial l} = \left\langle \mathbf{grad}\ f\left(x_0\right), \mathbf{e}_l \right\rangle = \left\langle \nabla f\left(x_0\right), \mathbf{e}_l \right\rangle$$

$$\mathrm{d}f\left(x\right) = \left\langle \nabla f\left(x\right), \mathrm{d}x \right\rangle, \quad \text{其中} \quad \mathrm{d}x = \left(\mathrm{d}x_1, \cdots, \mathrm{d}x_n\right)$$

梯度的意义

梯度是一个向量，其方向指向函数在该点处增大最快的方向，其模等于这个最大的方向导数的值。沿梯度的反方向，函数减小最快。

例7 求$z = x^2 - xy + y^2$在点$(-1, 1)$沿$\vec{l} = \{2, 1\}$的方向导数，并指出z在该点沿哪个方向的方向导数最大？该最大 的方向导数是多少？z沿哪个方向减小得最快 ？

解
$$\mathrm{grad}z(M_0) = \left\{ \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\}\bigg|_{M_0}$$

$$= \{2x - y, 2y - x\}\big|_{(-1,1)}$$

$$= (-3, 3)$$

$$\frac{\vec{l}}{|\vec{l}|} = \frac{1}{\sqrt{5}}\{2, 1\}$$

$$\left.\frac{\partial z}{\partial l}\right|_{(-1,1)} = \operatorname{grad} f(M_0) \cdot \frac{\vec{l}}{|\vec{l}|}$$

$$= \frac{1}{\sqrt{5}}(-6+3) = \frac{-3}{\sqrt{5}}$$

z在该点沿梯度方向，即 $\{-3, 3\}$的方向导数最大，

这个最大 的方向导数 $= |\operatorname{grad} z| = 3\sqrt{2}$.

z沿负梯度方向，即 $\{3, -3\}$的方向减小得最快。

梯度的运算法则。

设函数u,v及 f 均可微，C_1,C_2为常数

(1) $\operatorname{grad}(C_1u+C_2v)=C_1\operatorname{grad}u+C_2\operatorname{grad}v$,

或 $\nabla(C_1u+C_2v)=C_1\nabla u+C_2\nabla v$;

(2) $\operatorname{grad}(uv)=u\operatorname{grad}v+v\operatorname{grad}u$,

或 $\nabla(uv)=u\nabla v+v\nabla u$;

(3) $\mathrm{grad}\left(\dfrac{u}{v}\right)=\dfrac{1}{v^2}\left[v\,\mathrm{grad}\,u-u\,\mathrm{grad}\,v\right]$,

$$\nabla\left(\frac{u}{v}\right)=\frac{1}{v^2}\left[v\nabla u-u\nabla v\right],\left(v\neq 0\right);$$

(4) $\mathrm{grad}\,f\left(u\right)=f'\left(u\right)\mathrm{grad}\,u$,

$$\nabla f\left(u\right)=f'\left(u\right)\nabla u$$

4. 高阶偏导数

如果n元函数$u=f(x)$的偏导函数$\frac{\partial f(x)}{\partial x_i}$

在点x_0对变量x_j的偏导数存在,则称这个偏导数为f在点x_0先对变量x_i再对变量x_j的**二阶偏导数**,记为

$$\frac{\partial^2 f(\mathrm{x}_0)}{\partial x_j \partial x_i}=\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)_{\mathrm{x}=\mathrm{x}_0},\text{或 } f_{x_i x_j}(\mathrm{x}_0)\text{或 } f_{ij}^{2}(\mathrm{x}_0),$$

其中$1\le i\le n, 1\le j\le n.$

二元函数$z=f(x,y)$的二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx},\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=f_{yx},\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$

并称f_{xy}和f_{yx}为**二阶混合偏导数.**

类似,可由$n-1$阶偏导函数的偏导数来定义n阶偏导数,二阶及二阶以上的偏导数统称为**高阶偏导数.**

例8 设 $z = f(x, y) = x^y$,

求 $\dfrac{\partial^2 z}{\partial x \partial y}, \dfrac{\partial^2 z}{\partial y \partial x}, \dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$.

解：$\dfrac{\partial z}{\partial x} = yx^{y-1}$, $\quad \dfrac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-2}$

$\dfrac{\partial z}{\partial y} = x^y \ln x$, $\quad \dfrac{\partial^2 z}{\partial y^2} = x^y (\ln x)^2$

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1} \ln x + x^{y-1} = \frac{\partial^2 z}{\partial y \partial x}$$

例9 证明 $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ 满足拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

证明: $\dfrac{\partial u}{\partial x}=-\dfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x=-\dfrac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}-x[-\frac{3}{2}(x^2+y^2+z^2)^{\frac{-5}{2}}\cdot 2x]$$

$$=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

同样可得：

$$\frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - z^2 - x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 u}{\partial z^2} = \frac{2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

例10 设二元函数

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases},$$

求$f_{xy}(0,0)$, $f_{yx}(0,0)$.

解：$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x+0,0)-f(0,0)}{\Delta x}=0$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y+0)-f(0,0)}{\Delta y}=0$$

当 $x^2+y^2\neq 0$ 时，

$$f_x(x,y)=\frac{x^4y+3x^2y^3}{(x^2+y^2)^2},$$

$$f_y(x,y)=\frac{x^5-2x^3y^2}{(x^2+y^2)^2}$$

$$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{0}{\Delta y}=0$$

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x,0) - f_y(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$$

例 7 中 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$， 而例 9 中 $\frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y \partial x}$，

, 混合偏导数相等需要什么条件？

定理3

若$f_{xy}(x,y), f_{yx}(x,y)$在点(x,y)的某邻域内连续,则有 $f_{yx}(x,y)=f_{xy}(x,y)$, 即与求偏导数的次序无关.

证明：设 $F = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)$

$$- f(x+\Delta x, y) + f(x, y)$$

$$\Phi(x,y) = f(x+\Delta x, y) - f(x,y)$$

则 $F = \Phi(x, y+\Delta y) - \Phi(x, y)$

$$= \Phi_y(x, y+\theta_1\Delta y)\Delta y \qquad 0 < \theta_1 < 1$$

$$= [f_y(x+\Delta x, y+\theta_1\Delta y) - f_y(x, y+\theta_1\Delta y)]\Delta y$$

$$= f_{yx}(x+\theta_2\Delta x, y+\theta_1\Delta y)\Delta x\Delta y \qquad 0 < \theta_2 < 1$$

同样可得 ：

$$F = f_{xy}(x+\theta_3\Delta x, y+\theta_4\Delta y)\Delta x\Delta y$$

$$0 < \theta_3, \theta_4 < 1$$

$$\therefore f_{yx}(x+\theta_2\Delta x, y+\theta_4\Delta y) = f_{xy}(x+\theta_3\Delta x, y+\theta_4\Delta y)$$

由于 f_{xy}, f_{yx} 连续，令 $\Delta x \to 0, \Delta y \to 0$ 得：

$$f_{xy}(x, y) = f_{yx}(x, y)$$

5. 多元复合函数的偏导数与全微分

一元函数求导法中，复合函数的链式求导法则推广到多元上来：

定理3.3

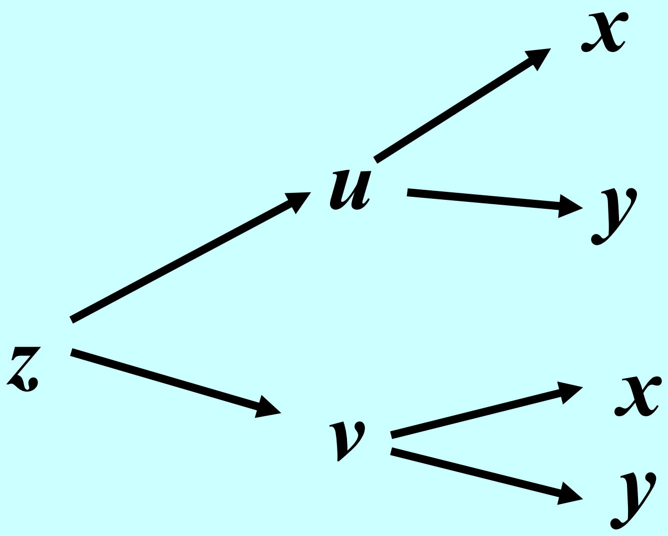
设$u=\varphi(x,y),v=\psi(x,y)$,均在点$(x,y)$处可微,而$z=f(u,v)$在对应的点$(u,v)$处可微,则复合函数$z=f[\varphi(x,y),\psi(x,y)]$在点$(x,y)$处也必可微,且其全微分为

$$\mathrm{d}z=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

故多元函数有如下**链式求导法则**：

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\left\langle \nabla f(u,v),\left(\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\right)\right\rangle$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\left\langle \nabla f(u,v),\left(\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}\right)\right\rangle$$



按线相乘，分线相加

几种特殊的情形：

(1) 设$z=f(u,v),u=\varphi(x),v=\psi(x)$均分别可微,则复合以后是$x$的一元函数$z=f[\varphi(x),\psi(x)]$,于是有

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}$$

它称为复合函数z对x的**全导数**

(2) 设$w=f(u),u=\varphi(x,y,z)$均可微,则有

$$\frac{\partial w}{\partial x}=\frac{\mathrm{d}w}{\mathrm{d}u}\frac{\partial u}{\partial x},\quad \frac{\partial w}{\partial y}=\frac{\mathrm{d}w}{\mathrm{d}u}\frac{\partial u}{\partial y},\quad \frac{\partial w}{\partial z}=\frac{\mathrm{d}w}{\mathrm{d}u}\frac{\partial u}{\partial z}$$

(3) 设$u = f(x, y, z), z = \varphi(x, y)$均可微，则有

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

左端 $\frac{\partial u}{\partial x}$ 表示复合后对x的偏导数,

右端 $\frac{\partial f}{\partial x}$ 表示复合前对x的偏导数,

例11 设 $z = e^u \sin 2v, u = xy, v = x + y$,求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解:$\frac{\partial z}{\partial x} = e^u y \sin 2v + 2e^u \cos 2v$

$$= e^{xy} y \sin 2(x+y) + 2e^{xy} \cos 2(x+y)$$

$$\frac{\partial z}{\partial y} = e^u \sin 2v \cdot x + 2e^u \cos 2v$$

$$= e^{xy} y \sin 2(x+y)x + 2e^{xy} \cos 2(x+y)$$

例12 设 $z=f(x,y)$可微，$x=r\cos\theta$，$y=r\sin\theta$，求 $\dfrac{\partial z}{\partial r},\dfrac{\partial z}{\partial\theta}$.

$$\begin{tikzcd} & & r \\ & x \arrow[ur] \arrow[r] & \theta \\ z \arrow[ur] \arrow[dr] & & \\ & y \arrow[r] \arrow[dr] & r \\ & & \theta \end{tikzcd}$$

解：
$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta}=r\left[\frac{\partial z}{\partial y}\cos\theta-\frac{\partial z}{\partial x}\sin\theta\right]$$

$$\Rightarrow\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2$$

$$+\frac{1}{r^2}r^2\left[\frac{\partial z}{\partial y}\cos\theta-\frac{\partial z}{\partial x}\sin\theta\right]^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

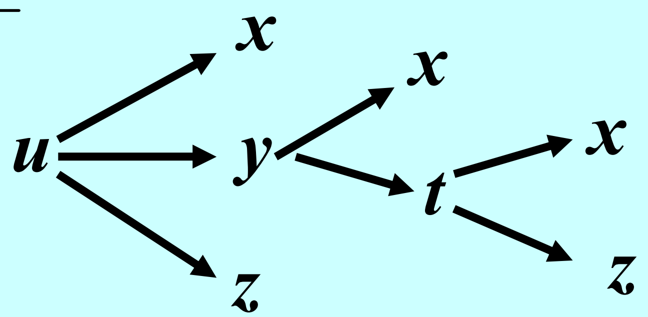
例13 设 $u = f(x, y, z),\ y = \varphi(x, t),\ t = \psi(x, z)$均可微，

求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial z}$.

解：

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x}$$

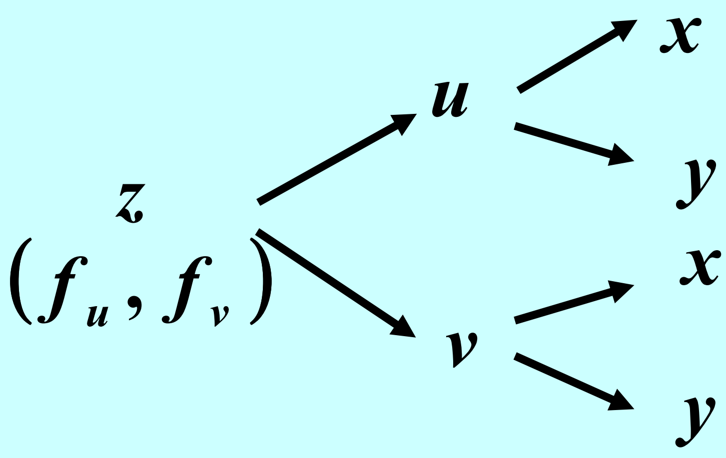
$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial z}$$



注意题中$\dfrac{\partial u}{\partial x}$与$\dfrac{\partial f}{\partial x}$,$\dfrac{\partial u}{\partial z}$与$\dfrac{\partial f}{\partial z}$是不同的

例14 设 $z = f(x-y, xy^2)$, f 有二阶连续偏导数，

求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial y \partial x}, \dfrac{\partial^2 z}{\partial x^2}$.



解：设 $u = x - y, \quad v = xy^2$

$$\frac{\partial z}{\partial x} = f_u + f_v \cdot y^2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{uu} + f_{uv} y^2 + y^2 (f_{vu} + f_{vv} y^2)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= f_{uu}(-1) + f_{uv} \cdot 2xy + 2y f_v + y^2 \left[f_{vu}(-1) + f_{vv}\, 2xy \right]$$

$$= -f_{uu} + 2xy^3 f_{vv} + \left(2xy - y^2\right) f_{uv} + 2y f_v$$

为了书写简单起见，可不引入符号 u, v，而把 $x-y, xy^2$ 分别简记为1, 2, 则有：

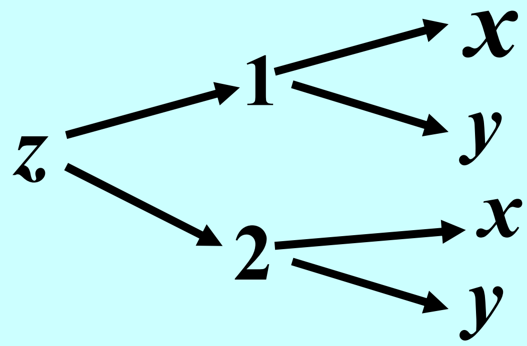
$$\frac{\partial z}{\partial x}=f_1+f_2y^2, \qquad \frac{\partial^2 z}{\partial x^2}=f_{11}+2f_{12}y^2+f_{22}y^4$$

$$\frac{\partial^2 z}{\partial y\partial x}=-f_{11}+2xy^3f_{22}+(2x-y)y\,f_{12}+2y\,f_2$$

在求二阶偏导数时，一定要注意 f_1, f_2 仍是原变量的复合函数.

例15 设 $z = f(e^x \sin y, x^2 + y^2)$, f 有二阶连续偏导数，

求 $\dfrac{\partial^2 z}{\partial y \partial x}$.



解： $\dfrac{\partial z}{\partial x} = f_1 e^x \sin y + f_2 2x$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left[f_{11} e^x \cos y + f_{12} \cdot 2y\right] e^x \sin y + f_1 e^x \cos y$$

$$+ 2x\left[f_{21} e^x \cos y + f_{22} \cdot 2y\right]$$

$$= e^x \cos y f_1 + \frac{1}{2} e^{2x} \sin 2y f_{11} + 4xy f_{22}$$

$$+ 2e^x (y \sin y + x \cos y) f_{12}$$

例16 设f有二阶导数，g有二阶连续偏导数，

$$z = f(2x - y) + g(x, xy).\text{ 求 }\frac{\partial^2 z}{\partial y \partial x}.$$

解：$\dfrac{\partial z}{\partial x} = 2f' + g_1 + yg_2$

$$\frac{\partial^2 z}{\partial y \partial x} = -2f'' + xg_{12} + yg_{22} \cdot x + g_2$$

例17 设f, g二阶连续可微，$u = y f\left(\frac{x}{y}\right) + x g\left(\frac{y}{x}\right)$，求

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x}.$$

解：$$\frac{\partial u}{\partial x} = yf' \cdot \frac{1}{y} + g\left(\frac{y}{x}\right) + xg' \cdot \left(-\frac{y}{x^2}\right) = f' + g - \frac{y}{x}g'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{y}f'' + g' \cdot \left(-\frac{y}{x^2}\right) + \frac{y}{x^2}g' - \frac{y}{x}g''\left(-\frac{y}{x^2}\right) = \frac{1}{y}f'' + \frac{y^2}{x^3}g''$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f''\left(-\frac{x}{y^2}\right) + g' \cdot \frac{1}{x} - \frac{1}{x} \cdot g' - \frac{y}{x}g'' \cdot \frac{1}{x}$$

$$= -\frac{x}{y^2}f'' - \frac{y}{x^2}g'' \qquad \therefore x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y \partial x} = 0$$

例18 设$u = u(\xi, \eta), \xi = x + ay, \eta = x + by (a \neq b)$，问$a, b$为何值时，可使 $u_{xx} + 4u_{xy} + 3u_{yy} = 0$ 变换为 $u_{\xi\eta} = 0$。

解
$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$
$$u_{yy} = a^2 u_{\xi\xi} + 2ab u_{\xi\eta} + b^2 u_{\eta\eta},$$
$$u_{xy} = a u_{\xi\xi} + (a + b) u_{\xi\eta} + b u_{\eta\eta}$$

$$u_{xx}+4u_{xy}+3u_{yy}=(3a^2+4a+1)u_{\xi\xi}$$

$$+(6ab+4a+4b+2)u_{\xi\eta}$$

$$+(3b^2+4b+1)u_{\eta\eta}=0$$

变换为 $u_{\xi\eta}=0 \Rightarrow 3a^2+4a+1=0,$

$$3b^2+4b+1=0,$$

$$6ab+4a+4b+2\neq 0$$

$$\Rightarrow a=-\frac{1}{3}, b=-1 \text{或} a=-1, b=-\frac{1}{3}$$

推广到n元函数

设$x=(x_1,\cdots,x_n)\in \mathbf{R}^n$, n元数量值函数$u_i=\varphi_i(\mathbf{x})$在$x$处可微$(i=1,\cdots,m)$,而数量值函数$y=f(u)$在对应的$u=u(x)=(\varphi_1(x),\cdots,\varphi_m(x))$处可微,则复合函数$y=F(x)=f[u(x)]$在$x$处也必可微,从而$F(x)$关于各个变量$x_1,\cdots,x_n$的偏导数均存在,且有

$$\frac{\partial F}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f}{\partial u_j}\frac{\partial u_j}{\partial x_i}=\left\langle \nabla f(u),\left(\frac{\partial u_1}{\partial x_i},\cdots,\frac{\partial u_m}{\partial x_i}\right)\right\rangle,$$

$$i=1,\cdots,n$$

$$\mathbf{d}\boldsymbol{y} = \left(\sum_{j=1}^{m} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_j} \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_1} \right) \mathbf{d}\boldsymbol{x}_1 + \cdots + \left(\sum_{j=1}^{m} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_j} \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_n} \right) \mathbf{d}\boldsymbol{x}_n$$

$$= \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_1}, \cdots, \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_m} \right) \begin{pmatrix} \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_1} & \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_2} & \cdots & \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \boldsymbol{u}_m}{\partial \boldsymbol{x}_1} & \frac{\partial \boldsymbol{u}_m}{\partial \boldsymbol{x}_2} & \cdots & \frac{\partial \boldsymbol{u}_m}{\partial \boldsymbol{x}_n} \end{pmatrix} \begin{pmatrix} \mathbf{d}\boldsymbol{x}_1 \\ \vdots \\ \mathbf{d}\boldsymbol{x}_n \end{pmatrix}$$

一阶微分形式不变性

设有m元函数$y=f(u)=f(u_1,\cdots,u_m)$与m个n元函数$u_i=u_i(\mathrm{x})=u_i(x_1,\cdots,x_n),i=1,\cdots,m$复合,若$f$在u可微,且u在x也可微,则复合函数的全微分可写成

$$\mathrm{d}y=\left(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_m}\right)\begin{pmatrix}\sum_{i=1}^{n}\frac{\partial u_1}{\partial x_i}\mathrm{d}x_i\\ \vdots\\ \sum_{i=1}^{n}\frac{\partial u_m}{\partial x_i}\mathrm{d}x_i\end{pmatrix}=\left(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_m}\right)\begin{pmatrix}\mathrm{d}u_1\\ \vdots\\ \mathrm{d}u_m\end{pmatrix}$$

$$=\frac{\partial f}{\partial u_1}\mathrm{d}u_1+\cdots+\frac{\partial f}{\partial u_m}\mathrm{d}u_m$$

这一全微分的形式与把$y=f(u_1,\cdots,u_m)$中的中间变量$u_i\ (i=1,\cdots,m)$看作是自变量时的全微分形式完全一样,这一性质称为**一阶全微分形式不变性**

由一阶微分形式不变性得:

$$(1)\ \mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$$

$$(2)\ \mathrm{d}(uv)=v\mathrm{d}u+u\mathrm{d}v$$

$$(3)\ \mathrm{d}\left(\frac{u}{v}\right)=\frac{1}{v^2}(v\mathrm{d}u-u\mathrm{d}v),v\neq 0$$

高阶微分不具有形式不变性。

例13′ 设 $u = f(x,y,z),\ y = \varphi(x,t),\ t = \psi(x,z)$ 均可微，求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial z}$.

解：
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\left(\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial t}dt\right) + \frac{\partial f}{\partial z}dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial x}dx + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial t}\left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dz\right) + \frac{\partial f}{\partial z}dz$$

$$= \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial t}\cdot\frac{\partial \psi}{\partial x}\right)}_{\frac{\partial u}{\partial x}}dx + \underbrace{\left(\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial t}\cdot\frac{\partial \psi}{\partial y} + \frac{\partial f}{\partial z}\right)}_{\frac{\partial u}{\partial z}}dz$$

6.一个方程确定的隐函数微分法

定理4(隐函数存在定理)

如果二元函数$F(x,y)$满足

(1) $F(x_0,y_0)=0$;

(2) 在点(x_0,y_0)的某邻域中有连续的偏导数;

(3) $F_y(x_0,y_0)\neq 0$.

则方程$F(x,y)=0$在的某一邻域内唯一确定了一个具有连续导数的函数$y=f(x)$,它满足

$y_0=f(x_0)$及$F[x,f(x)]\equiv 0$,并且 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{F_x}{F_y}$

可推广到多元函数：

定理4′

设 (1) $n+1$元函数$F(x_1,x_{2,}\cdots,x_n,u)$在点

$(x_{0,1},\cdots,x_{0,n},u_0)$的某邻域内具有连

续的偏导数，

(2) $F(x_{0,1},\cdots x_{0,n},u_0)=0, F_u(x_{0,1},\cdots,x_{0,n},u_0)\neq 0,$

则方程$F(x_1,\cdots,x_n,u)=0$在点$x_0=(x_{0,1},\cdots,x_{0,n})$

的邻域内能唯一确定一个连续且有一阶连续

偏导数的函数 $u=f(x_1,\cdots x_n)$,满足 $u_0=f(x_0)$,且

$$\frac{\partial u}{\partial x_i}=-\frac{F_{x_i}}{F_u}\quad(i=1,2,\cdots,n).$$

例19 设方程 $z^3 - 3xyz = a^2$ 确定 z 是 x, y 的函数，

求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解：法一： **公式法**

令 $F(x, y, z) = z^3 - 3xyz - a^2$,

$$F_x = -3yz, F_y = -3xz, F_z = 3z^2 - 3xy,$$

则：
$$\begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = \dfrac{yz}{z^2 - xy} \\[2ex] \dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = \dfrac{xz}{z^2 - xy} \end{cases}$$

法二： **直接法**

在 $z^3-3xyz=a^2$ 两边分别对x,y求导,得

$$3z^2\frac{\partial z}{\partial x}-3yz-3xy\frac{\partial z}{\partial x}=0,$$

$$3z^2\frac{\partial z}{\partial y}-3xz-3xy\frac{\partial z}{\partial y}=0$$

$$\therefore \frac{\partial z}{\partial x}=\frac{yz}{y^2-xy}\ ,\qquad \frac{\partial z}{\partial y}=\frac{xz}{y^2-xy}$$

法三： $\boxed{\text{全微分法}}$

在等式两边求全微分得：

$$3z^2dz - 3xydz - 3xzdy - 3yzdx = 0$$

$$\therefore\ dz = \frac{yz}{z^2 - xy}dx + \frac{xz}{z^2 - xy}dy$$

即得 $\dfrac{\partial z}{\partial x} = \dfrac{yz}{y^2 - xy}$ $\quad \dfrac{\partial z}{\partial y} = \dfrac{xz}{y^2 - xy}$

例20 设 $z = z(x, y)$ 由方程 $F(x - az, y - bz) = 0$ 所确定，a, b 为常数，求证

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$$

证明：$F_x = F_1 \quad F_y = F_2, \quad F_z = -aF_1 - bF_2$

$$\therefore \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{F_1}{aF_1 + bF_2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{F_2}{aF_1 + bF_2} \quad \Rightarrow a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$$

例21 设 $u=f(x,y,z),\ y=g(\sin x),\ z=z(x)$ 由方程 $\varphi(x^2,e^y,z)=0$ 确定，其中 f,φ 具有一阶连续偏导数， g 可导，且 $\dfrac{\partial\varphi}{\partial z}\neq 0$，求 $\dfrac{du}{dx}$.

解：法一：$\dfrac{du}{dx}=f_x+f_y\dfrac{dy}{dx}+f_z\dfrac{dz}{dx}$，而

$$\frac{dy}{dx}=g'\cos x,$$

$$\frac{dz}{dx}=-\frac{\varphi_x}{\varphi_z}=-\frac{2x\varphi_1+e^y g'\cos x\varphi_2}{\varphi_z},$$

代入即可

法二：求微分得 ：

$$\varphi_1 d(x^2) + \varphi_2 de^y + \varphi_z dz$$

$$= 2x\varphi_1 dx + \varphi_2 e^y g' \cos x dx + \varphi_z dz = 0$$

$$\Rightarrow dz = -\frac{2x\varphi_1 + \varphi_2 e^y g' \cos x}{\varphi_z} dx$$

而 $du = f_x dx + f_y dy + f_z dz$

$$= (f_x + f_y g' \cos x - f_z \frac{2x\varphi_1 + \varphi_2 e^y g' \cos x}{\varphi_z}) dx$$

$$\Rightarrow \frac{du}{dx} = f_x + f_y g' \cos x - f_z \frac{2x\varphi_1 + \varphi_2 e^y g' \cos x}{\varphi_z}$$

7.由方程组确定的隐函数微分法

以$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$为例

设 (1) $F(x,y,u,v), G(x,y,u,v)$在点 $P(x_0,y_0,u_0,v_0)$的某一邻域内有一阶连续偏导数，

(2) $F(x_0,y_0,u_0,v_0)=0, G(x_0,y_0,u_0,v_0)=0,$

Jacobi 行列式 $J|_P=\dfrac{\partial(F,G)}{\partial(u,v)}\Big|_P=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\Big|_P \neq 0,$

则由方程组$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$在$P$的某邻域内能

唯一确定一组连续且具　一阶连续偏导数的函数

$u = u(x, y),\ v = v(x, y),$ 满足$u_0 = u(x_0, y_0), v_0 = v(x_0, y_0),$
且

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(x, v)}, \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(y, v)},$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, x)}, \frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, y)}$$

(由于$\begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0 \\ G(x, y, u(x, y), v(x, y)) \equiv 0 \end{cases}$, 两边对$x$求偏导 :

$$\begin{cases} F_x + F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = 0 \\ G_x + G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = 0 \end{cases},$$

由此解得$\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}. \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$同理可得)

例22 $u=u(x,y), v=v(x,y)$由方程组$\begin{cases} u^2-v+x=0 \\ u+v^2-y=0 \end{cases}$确定，

求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$.

解：法一：两边对x求偏导，得
$$\begin{cases} 2u\dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial x}+1=0 \\ \dfrac{\partial u}{\partial x}+2v\dfrac{\partial v}{\partial x}=0 \end{cases}$$

解得 $\dfrac{\partial u}{\partial x}=\dfrac{-2v}{1+4uv}$, $\dfrac{\partial v}{\partial x}=\dfrac{1}{1+4uv}$

两边对 y 求偏导,得

$$\begin{cases} 2u\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} - 1 = 0 \end{cases}$$

解得 $\dfrac{\partial u}{\partial y} = \dfrac{1}{1+4uv}, \quad \dfrac{\partial v}{\partial y} = \dfrac{2u}{1+4uv}$

法二 ：求微分得 ：

$$\begin{cases} 2udu - dv + dx = 0 \\ du + 2vdv - dy = 0 \end{cases}$$

解得 ：$du = \dfrac{-2vdx + dy}{1+4uv}$， $dv = \dfrac{dx + 2udy}{1+4uv}$

由此得 $\dfrac{\partial u}{\partial x} = \dfrac{-2v}{1+4uv}$, $\dfrac{\partial u}{\partial y} = \dfrac{1}{1+4uv}$

$$\frac{\partial v}{\partial x} = \frac{1}{1+4uv}, \quad \frac{\partial v}{\partial y} = \frac{2u}{1+4uv}$$