Solutions to Random Mathematics Homework #4 Fall 2020

Instructor: Jing Liang
Assigned Date: Oct.15, 2020 Due Date: Oct.22, 2020

H4.1

a. Add up the elements in the last row of the table: 0.15.

b. Add up the elements in the last two columns of the table: 0.23.

c. Add up the elements in the intersection of the first two columns and the first two rows : 0.09.

d. Add up the elements of $(0, 0)$, $(1, 1)$, $(2, 2)$ and $(3, 3)$: 0.3.

e. Add up the elements of $(1, 0)$, $(2, 0)$, $(3, 0)$, $(2, 1)$, $(3, 1)$ and $(3, 2)$: 0.25.

f. The random variable X has a discrete distribution, the marginal p.f. of X is:

$$f(x) = \begin{cases} 0.23 & x = 0, \\ 0.35 & x = 1, \\ 0.27 & x = 2, \\ 0.15 & x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

g. The conditional p.f. of Y given X:

	Y				
X	0	1	2	3	4
0	8/23	7/23	6/23	1/23	1/23
1	6/35	1/35	12/35	1/7	2/35
2	5/27	2/9	1/3	4/27	1/9
3	2/15	1/5	1/5	1/5	4/15

Figure 1: p.f. of Y given X

H4.2

a. Since the joint c.d.f. if continuous and is twice-differentiable in given rectangle, the joint distribution of X and Y is continuous. Therefore,

$$\begin{aligned}\Pr(1 \le X \le 2 \text{ and } 2 \le Y \le 4) &= \Pr(1 < X \le 2 \text{ and } 2 < Y \le 4)\\ &= F(2,4) - F(1,4) - F(2,2) + F(1,2)\\ &= \frac{64}{156} - \frac{20}{156} - \frac{24}{156} + \frac{6}{156} = \frac{13}{78}.\end{aligned}$$

b.

$$\begin{aligned}\Pr(-1 \le X \le 5 \text{ and } 2 \le Y \le 5) &= \Pr(0 < X \le 3 \text{ and } 2 < Y \le 4)\\ &= F(3,4) - F(0,4) - F(3,2) + F(0,2)\\ &= 1 - 0 - \frac{66}{156} + 0 = \frac{15}{26}.\end{aligned}$$

c. Since y must lie in the interval $0 \le y \le 4$, $F_2(y) = 0$ for $y < 0$ and $F_2(y) = 1$ for $y > 4$,

$$F_2(y) = \lim_{x\to\infty} F(x,y) = \lim_{x\to 3} \frac{1}{156}xy(x^2+y) = \frac{1}{52}y(9+y).$$

d. We have $f(x,y) = 0$ unless $0 < x < 3$ and $0 < y < 4$. In this rectangle we have

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{1}{156}(3x^2+2y).$$

e. To calculate $g_1(x|y)$, we need to obtain $f_2(y)$ from (c) first. When $y \le 0$ and $y \ge 4$, $f_2(y) = 0$ and when $0 < y < 4$

$$f_2(y) = \frac{\partial F_2(y)}{\partial y} = \frac{9}{52} + \frac{1}{26}y$$

For $0 < y < 4$

$$g_1(x|y) = \frac{f(x,y)}{f_2(y)} = \begin{cases} \dfrac{3x^2+2y}{27+6y} & \text{for } 0 < x < 3,\\ 0 & \text{otherwise }.\end{cases}$$

f. We first calculate $F_1(x)$. $F_1(x) = 0$ for $x \le 0$ and $x \ge 3$ and for $0 < x < 3$

$$F_1(x) = \lim_{y\to\infty} F(x,y) = \lim_{y\to 4} \frac{1}{156}xy(x^2+y) = \frac{1}{39}(x^3+4x)$$

Then we can have $f_1(x)$. $f_1(x) = 0$ for $x \le 0$ and $x \ge 3$ and for $0 < x < 3$

$$f_1(x) = \frac{\partial F_1(x)}{\partial x} = \frac{3x^2+4}{39}$$

For $0 < x < 3$

$$g_2(y|x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} \dfrac{3x^2+2y}{12x^2+16} & \text{for } 0 < y < 4,\\ 0 & \text{otherwise.}\end{cases}$$

$$g_2(y|0) = \begin{cases} y/8 & \text{for } 0 < y < 4,\\ 0 & \text{otherwise.}\end{cases}$$

g. The region over which to intergrate is shaded in Fig.2.

$$\Pr(2X+Y \le 3) = \int_0^{1.5}\int_0^{3-2x} \frac{1}{156}(3x^2+2y)\,dy\,dx = \frac{75}{1664}.$$

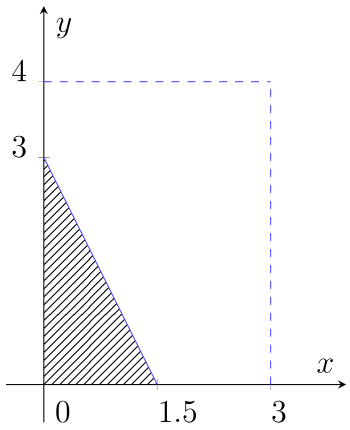

Figure 2: Figure for H4.2.

H4.3

a. To obtain the joint c.d.f of $f(x, y)$, the whole area shall be divided into several parts by $x = 1$ and $y = 1$. It can be obtained easily that:

(1) When $x > 1$ AND $y > 1$, $F(x, y) = 1$.

(2) When $x < 0$ OR $y < 0$, $F(x, y) = 0$.

(3) When $0 \leq x < 1$ and $0 \leq y < 1$ and $x + y \leq 1$, as shown in Figure 1:

$$F(x, y) = \int_0^y \int_0^x 24uvdudv = 6x^2y^2$$

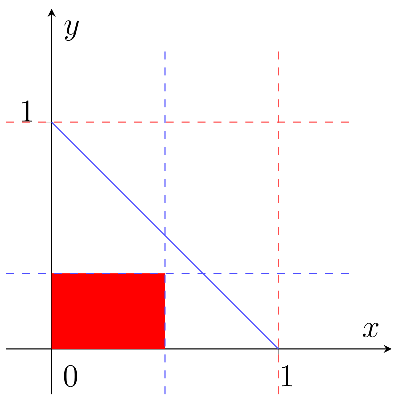

Figure 3: Figure for HW.4.3 (a)

(4) When $0 \leq x < 1$ and $0 \leq y < 1$ and $x + y > 1$, as shown in Figure 2:

$$\begin{aligned} F(x, y) &= \int_0^{1-y} \int_0^y 24uvdvdu + \int_y^x \int_0^{1-x} 24uvdvdu \\ &= \int_0^{1-x} \int_0^x 24uvdudv + \int_{1-x}^y \int_0^{1-y} 24uvdudv \\ &= 6\text{y}^2(1-y)^2 + 6\left[x^2 - (1-y)^2\right] - 8\left[x^3 + (1-y)^3\right] + 3\left[x^4 + (1-y)^4\right] \end{aligned}$$

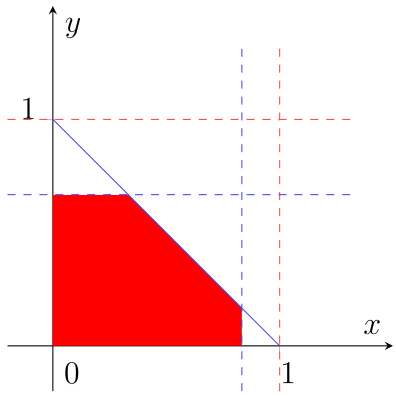

Figure 4: Figure for HW.4.3 (a)

(5) When $x \geq 1$ and $0 \leq y < 1$, as shown in Figure 3:

$$F(x,y) = \int_0^y \int_0^{1-y} 24uvdudv = 6y^2 - 8y^3 + 3y^4$$

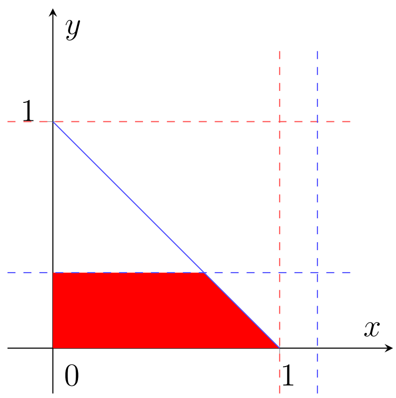

Figure 5: Figure for HW.4.3 (a)

(6) When $y \geq 1$ and $0 \leq x < 1$, as shown in Figure 4:

$$F(x,y) = \int_0^x \int_0^{1-x} 24uvdvdu = 6x^2 - 8x^3 + 3x^4$$

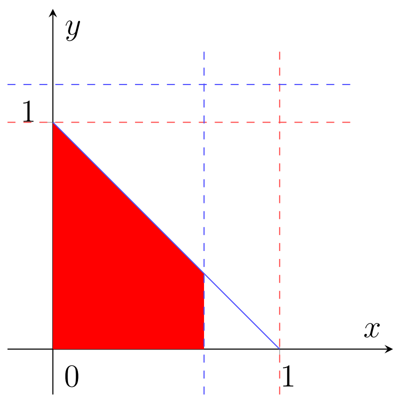

Figure 6: Figure for HW.4.3 (a)

b. For $0 \le x \le 1$

$$\begin{aligned} F_1(x) &= \int_0^x \int_0^{1-x} 24uv dv du \\ &= 6x^2 + 3x^4 - 8x^3 \end{aligned}$$

$$F_1(x) = \begin{cases} 0 & x < 0 \\ 6x^2 + 3x^4 - 8x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c.

$$f_1(x) = \int_0^{1-x} 24xy dx = 12x(1-x)^2, \text{ for } 0 < x < 1$$

$$f_2(y) = \int_0^{1-y} 24xy dx = 12y(1-y)^2, \text{ for } 0 < y < 1$$

Since $f(x,y) \neq f_1(x) f_2(y)$, X and Y are not independent.

d.

$$\Pr(Y \ge X) = \int_0^{0.5} \int_0^y 24xy dx dy + \int_{0.5}^1 \int_0^{1-y} 24xy dx dy = 0.5,$$

or

$$\Pr(Y \ge X) = \int_0^{0.5} \int_x^{1-x} 24xy dy dx = 0.5.$$

H4.4

a. For $0 \le x \le 2$ we have

$$f_1(x) = \int_0^1 f(x,y) dy = \frac{1}{2}.$$

Also, $f_1(x) = 0$ for x outside the interval $0 \le x \le 2$. Similarly, for $0 \le y \le 1$,

$$f_2(y) = \int_0^2 f(x,y) dx = 3y^2.$$

Also, $f_2(y) = 0$ for y outside the interval $0 \le y \le 1$.

b. X and Y are independent because $f(x,y) = f_1(x) f_2(y)$ for $-\infty < x < \infty$ and $-\infty < y < \infty$.

c. We have

$$\begin{aligned} \Pr(X < 1 \text{ and } Y \ge 1/2) &= \int_0^1 \int_{1/2}^1 f(x,y) dx dy \\ &= \int_0^1 \int_{1/2}^1 f_1(x) f_2(y) dx dy \\ &= \int_0^1 f_1(x) dx \int_{1/2}^1 f_2(y) dy \\ &= \Pr(X < 1) \Pr(Y \ge 1/2). \end{aligned}$$

Therefore, by the definition of the independence of two event (Definition 2.2.1), the two given events are independent.

We can also reach this answer, without carrying out the above calculation, by reasoning as follows: Since the random variables X and Y are independent. and since the occurrence or nonoccurence of the event $\{X < 1\}$ depends on the value of X only while the occurrence or nonoccurence of the event $\{Y \geq 1/2\}$ depends on the value of Y only, it follows that these two events must be independent.

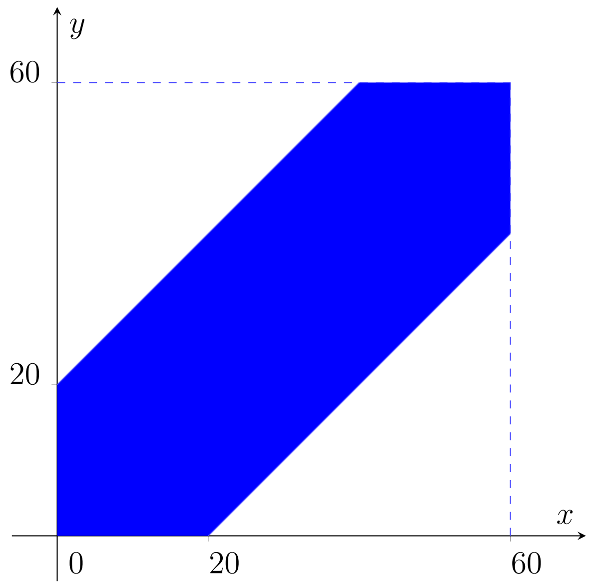

Figure 7: Figure for H4.5.

H4.5

Let X and Y denote the arrival times of the two persons, measured in terms of the number of after 5 P.M. Then X and Y each have the uniform distribution on the interval $(0, 60)$ and they are independent. Therefore, the joint. p.d.f. of X and Y is

$$f(x,y) = \begin{cases} \dfrac{1}{3600} & \text{for } 0 < x < 60, 0 < y < 60, \\ 0 & \text{otherwise.} \end{cases}$$

We must calculate Pr$(|X - Y| < 20)$, which is equal to the probability that the point (X, Y) lies in shaded region in Fig. 7. Since the joint p.d.f. of X and Y is constant over the entire square, this probability is equal to (area of shaded region) $/3600$. The area of the shaded region is 2000. Therefore, the required probability is $2000/3600 = 5/9$.

H4.6

a. Since $f(x, y)$ is constant over the rectangle S and the area of S is 15 units, it follows that $f(x, y) = 1/15$ inside S and $f(x, y) = 0$ outside S.

   Next, for $0 \leq x \leq 5$,

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_1^4 \frac{1}{15}dy = \frac{1}{5}.$$

   Also, $f_1(x) = 0$ otherwise.

Similarly, for $1 \leq y \leq 4$,

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^5 \frac{1}{15}dx = \frac{1}{3}.$$

Also, $f_2(y) = 0$ otherwise.

Thus, the marginal distribution of both X and Y are uniform distributions.

b. Since $f(x,y) = f_1(x)f_2(y)$ for all values of x and y, it follows that X and Y are independent.

H4.7

a. We have Pr(Junior) $= 0.04 + 0.20 + 0.09 = 0.33$. Therefore,

$$\text{Pr(Never} \mid \text{Junior)} = \frac{\text{Pr(Junior and Never)}}{\text{Pr(Junior)}} = \frac{0.04}{0.33} = \frac{4}{33}.$$

b. The only way we can use the fact that a student visited the museum three times is to classify the student as having visited more than once. We have

$$\text{Pr(More than once)} = 0.04 + 0.04 + 0.09 + 0.10 = 0.27.$$

Therefore,

$$\text{Pr(Senior} \mid \text{More than once)} = \frac{\text{Pr(Senior and More than once)}}{\text{Pr(More than once)}} = \frac{0.10}{0.27} = \frac{10}{27}.$$

H4.8

The joint p.d.f. of X and Y is positive inside the triangle S shown in Fig. 8. It is seen from Fig. 8 that the possible values of X lie between 0 and 2. Hence, for $0 < x < 2$,

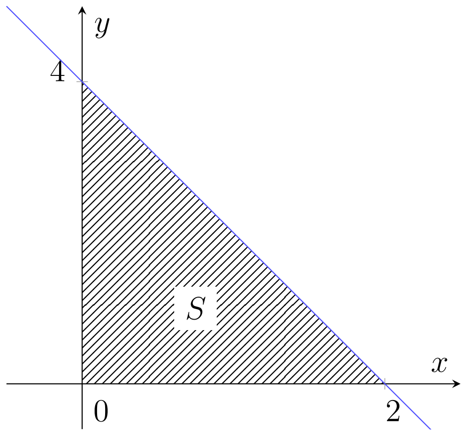

Figure 8: Figure for H4.8.

$$f_1(x) = \int_0^{4-2x} f(x,y)dy = \frac{3}{8}(x-2)^2.$$

a. It follows that for $0 < x < 2$ and $0 < y < 4 - 2x$,

$$g_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{4-2x-y}{2(x-2)^2}.$$

b. When $X = 1$, it follows from part (a) that

$$g_2(y|x=1) = \begin{cases} \dfrac{2-y}{2} & \text{for } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\Pr(Y \le 1 | X = 1) = \int_0^1 g_2(y|x=1)\,dy = \frac{3}{4}.$$

H4.9

a. The joint p.d.f./p.f. of X and Y is the product $f_2(y)g_1(x \mid y)$.

$$f(x,y) = \begin{cases} (2y)^x \exp(-3y)/x! & \text{if } y > 0 \text{ and } x = 0, 1, ..., \\ 0 & \text{otherwise.} \end{cases}$$

The marginal p.f. of X is obtained by integrating over y.

$$f_1(x) = \int_0^\infty \frac{(2y)^x}{x!} \exp(-3y)dy = \frac{1}{3}(\frac{2}{3})^x,$$

for $x = 0, 1, ...$

b. The conditional p.d.f. of Y given $X = 0$ is the ratio of the joint p.f./p.d.f. to $f_1(0)$.

$$g_2(y \mid 0) = \frac{(2y)^0 \exp(-3y)/0!}{(1/3)(2/3)^0} = 3\exp(-3y),$$

for $y > 0$.

c. The conditional p.d.f. of Y given $X = 1$ is the ratio of the joint p.f./p.d.f. to $f_1(1)$.

$$g_2(y \mid 1) = \frac{(2y)^1 \exp(-3y)/1!}{(1/3)(2/3)^1} = 9y\exp(-3y),$$

for $y > 0$.

d. The ratio of the two conditional p.d.f. 's is

$$\frac{g_2(y \mid 1)}{g_2(y \mid 0)} = \frac{9y\exp(-3y)}{3\exp(-3y)} = 3y.$$

The ratio is smaller than 1 if $y < \frac{1}{3}$. This corresponds to the intuition that if we observe more calls, then we should think the rate is higher.

H4.10

Let Y denote the instrument that is chosen. Then $\Pr(Y = 1) = \Pr(Y = 2) = \frac{1}{2}$. In this exercise the distribution of X is continuous and the distribution of Y is discrete. Hence, the joint distribution of X and Y is a mixed distribution, as described in Sec.3.4. In this case, the joint p.f./p.d.f. of X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{1}{2} \cdot 2x = x & \text{for } y = 1 \text{ and } 0 < x < 1, \\ \frac{1}{2} \cdot 3x^2 = \frac{3}{2}x^2 & \text{for } y = 2 \text{ and } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. It follows that for $0 < x < 1$,

$$f_1(x) = \sum_{y=1}^{2} f(x,y) = x + \frac{3}{2}x^2,$$

and $f_1(x) = 0$ otherwise.

b. For $y = 1, 2$ and $0 < x < 1$, we have

$$\Pr(Y = y \mid X = x) = g_2(y \mid x) = \frac{f(x,y)}{f_1(x)}.$$

Hence,

$$\Pr(Y = 1 \mid X = \frac{1}{5}) = \frac{f(\frac{1}{5}, 1)}{f_1(\frac{1}{5})} = \frac{\frac{1}{5}}{\frac{1}{5} + \frac{3}{2} \cdot \frac{1}{25}} = \frac{10}{13}.$$

H4.11

a. The answer is

$$\int_0^1 \int_{0.8}^1 f(x,y)dxdy = 0.264$$

b. For $0 < y < 1$, the marginal p.d.f. of Y is

$$f_2(y) = \int_0^1 f(x,y)dx = \frac{2}{5}(1 + 3y)$$

Hence, for $0 < x < 1$ and $0 < y < 1$

$$g_1(x \mid y) = \frac{2x + 3y}{1 + 3y}$$

When $Y = 0.3$, it follows that

$$g_1(x \mid y = 0.3) = \frac{2x + 0.9}{1.9} \qquad \text{for } 0 < x < 1$$

Hence,

$$\Pr(X > 0.8 \mid Y = 0.3) = \int_{0.8}^1 g_1(x \mid y = 0.3)dx = 0.284$$

H4.12

a. We have

$$\int_0^1 \int_0^1 \int_0^1 f(x_1, x_2, x_3)dx_1 dx_2 dx_3 = 3c.$$

Since the value of this integral must be equal to 1, it follows that $c = 1/3$.

b. For $0 \le x_1 \le 1$ and $0 \le x_3 \le 1$,

$$f_{13}(x_1, x_3) = \int_0^1 f(x_1, x_2, x_3)dx_2 = \frac{1}{3}(x_1 + 1 + 3x_3).$$

c. The conditional p.d.f. of x_3 given that $x_1 = 1/4$ and $x_2 = 3/4$ is, for $0 \le x_3 \le 1$,

$$g_3(x_3|x_1 = \frac{1}{4}, x_2 = \frac{3}{4}) = \frac{f(\frac{1}{4}, \frac{3}{4}, x_3)}{f_{12}(\frac{1}{4}, \frac{3}{4})} = \frac{7}{13} + \frac{12}{13}x_3.$$

Therefore,

$$\Pr(X_3 < \frac{1}{2}|X_1 = \frac{1}{4}, X_2 = \frac{3}{4}) = \int_0^{\frac{1}{2}} (\frac{7}{13} + \frac{12}{13}x_3)dx_3 = \frac{5}{13}.$$

d. For $0 \le x_3 \le 1$,

$$f_3\left(x_3\right) = \int_0^1 \int_0^1 f\left(x_1, x_2, x_3\right) dx_1 dx_2 = 0.5 + x_3$$

Otherwise, $f_3(x_3) = 0$. Thus, the c.d.f is

$$F_3(x) = \int_0^{x_3} f_3(t)dt = \begin{cases} 0 & \text{for} \quad x_3 < 0 \\ 0.5\left(x_3 + x_3^2\right) & \text{for} \quad 0 \le x_3 \le 1 \\ 1 & \text{for} \quad x_3 > 1 \end{cases}$$

H4.13

As x varies over all positive values, y also varies over all positive values. Also, $x = y^2$ and $dx/dy = 2y$. Therefore, for $y > 0$,

$$g(y) = f(y^2)(2y) = 2y \exp(-y^2).$$

H4.14

The inverse transformation is $z = 1/t$ with derivative $-1/t^2$. The p.d.f. of T is

$$g(t) = f(1/t)/t^2 = 2\exp(-2/t)/t^2,$$

for $t > 0$.