例 ① $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ vs $\sum \frac{1}{n}$

② $\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$ $\ln n < n^{1/4}$ vs $\sum \frac{1}{n^{5/4}}$

对于 $\sum a_n$, $a_n > 0$, 若 $\lim \frac{a_{n+1}}{a_n} = l$

则 $\lim \sqrt[n]{a_n} = l$.

proof. 由 $\lim \frac{a_{n+1}}{a_n} = l$, 则 $\forall \varepsilon > 0$, $\exists N$, s.t.

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon. \quad (\varepsilon < l)$$

即 $l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon$

于是: $a_{n+1} < (l+\varepsilon) a_n$, $n > N$.

$$\Rightarrow a_n < (l+\varepsilon) a_{n-1} < \cdots < (l+\varepsilon)^{n-N} a_N$$

$$\Rightarrow \sqrt[n]{a_n} < (l+\varepsilon)^{\frac{n-N}{n}} \sqrt[n]{a_N}$$

$\to l+\varepsilon$ $\to 1$

$$\Rightarrow \lim \sqrt[n]{a_n} \le l+\varepsilon.$$

再由ε的任意性得:

$$\lim \sqrt[n]{a_n} \le l.$$

同理, $\lim \sqrt[n]{a_n} \ge l$

从而 $\lim \sqrt[n]{a_n} = l$. #

保号性: 若 $a_n \le b_n$ 则 $\lim a_n \le \lim b_n$

注: 若 $a_n < b_n$, 则 $\lim a_n < \lim b_n$?

$\le$.

积分判别法:

① $\sum \frac{1}{n^p}$, $\sum \frac{1}{n(\ln n)^p}$

② $\sum a_n \Rightarrow a_n \sim f(x)$ 可积

注 $\lim a_n = l \Rightarrow \lim \frac{a_1+\cdots+a_n}{n} = l$ ✓