3.1 偏导数

偏导数的定义及其计算

高阶偏导数

一、偏导数的定义及其计算法

定义 3.1 设函数$z=f(x,y)$在点(x_0,y_0)的某一邻域内有定义，当y固定在y_0而x在x_0处有增量Δx时，相应地函数有增量

$$f(x_0+\Delta x,y_0)-f(x_0,y_0),$$

如果$\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$存在，则称此极限为函数$z=f(x,y)$在点$(x_0,y_0)$处对$x$的偏导数，记为

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\quad \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\quad z_x\Big|_{\substack{x=x_0\\y=y_0}} \text{或} f_x(x_0,y_0).$$

同理可定义函数$z=f(x,y)$在点(x_0,y_0)处对y的偏导数， 为

$$\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

记为$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$，$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$，$z_y\Big|_{\substack{x=x_0\\y=y_0}}$或$f_y(x_0,y_0)$.

如果函数$z=f(x,y)$在区域D内任一点(x,y)处对x的偏导数都存在，那么这个偏导数就是x、y的函数，它就称为函数$z=f(x,y)$对自变量x的偏导数，

记作$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial f}{\partial x}$，$z_x$或$f_x(x,y)$.

同理可以定义函数$z=f(x,y)$对自变量y的偏导数，记作$\dfrac{\partial z}{\partial y}$，$\dfrac{\partial f}{\partial y}$，$z_y$或$f_y(x,y)$.

偏导数的概念可以推广到二元以上函数

如 $u=f(x,y,z)$ 在 (x,y,z) 处

$$f_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x},$$

$$f_y(x,y,z)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y},$$

$$f_z(x,y,z)=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}.$$

例 1 求 $z = x^2 + 3xy + y^2$ 在点 $(1,2)$ 处的偏导数.

解 $\dfrac{\partial z}{\partial x} = 2x + 3y$; $\dfrac{\partial z}{\partial y} = 3x + 2y$.

$$\therefore \quad \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\ y=2}} = 2\times 1 + 3\times 2 = 8,$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\ y=2}} = 3\times 1 + 2\times 2 = 7.$$

例 2 设 $z = x^y\ (x > 0, x \neq 1)$，

求证 $$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z.$$

证 $$\frac{\partial z}{\partial x} = yx^{y-1}, \qquad \frac{\partial z}{\partial y} = x^y \ln x,$$

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = \frac{x}{y} yx^{y-1} + \frac{1}{\ln x} x^y \ln x$$

$$= x^y + x^y = 2z.$$ 原结论成立.

例 3 设$z=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_x$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}\qquad(\sqrt{y^2}=|y|)$$

$$=\frac{|y|}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_y$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn}\frac{1}{y}\qquad(y\neq 0)$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x\neq 0\\ y=0}}$$ 不存在.

例 4 已知理想气体的状态方程 $pV = RT$（R为常数），求证：$\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -1.$

证 $p = \dfrac{RT}{V} \Rightarrow \dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2};$

$V = \dfrac{RT}{p} \Rightarrow \dfrac{\partial V}{\partial T} = \dfrac{R}{p};$ $\quad T = \dfrac{pV}{R} \Rightarrow \dfrac{\partial T}{\partial p} = \dfrac{V}{R};$

$$\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{RT}{V^2}\cdot\frac{R}{p}\cdot\frac{V}{R} = -\frac{RT}{pV} = -1.$$

有关偏导数的几点说明：

1、 偏导数$\dfrac{\partial u}{\partial x}$是一个整体记号，不能拆分;

2、 求分界点、不连续点处的偏导数要用定义求;

例如, 设$z = f(x, y) = \sqrt{|xy|}$, 求$f_x(0, 0)$, $f_y(0, 0)$.

解 $$f_x(0,0) = \lim_{x \to 0} \frac{\sqrt{|x \cdot 0|} - 0}{x} = 0 = f_y(0,0).$$

3、偏导数存在与连续的关系

一元函数中在某点可导 $\longrightarrow$ 连续，

多元函数中在某点偏导数存在 $\xrightarrow{?}$ 连续，

例如,函数$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$,

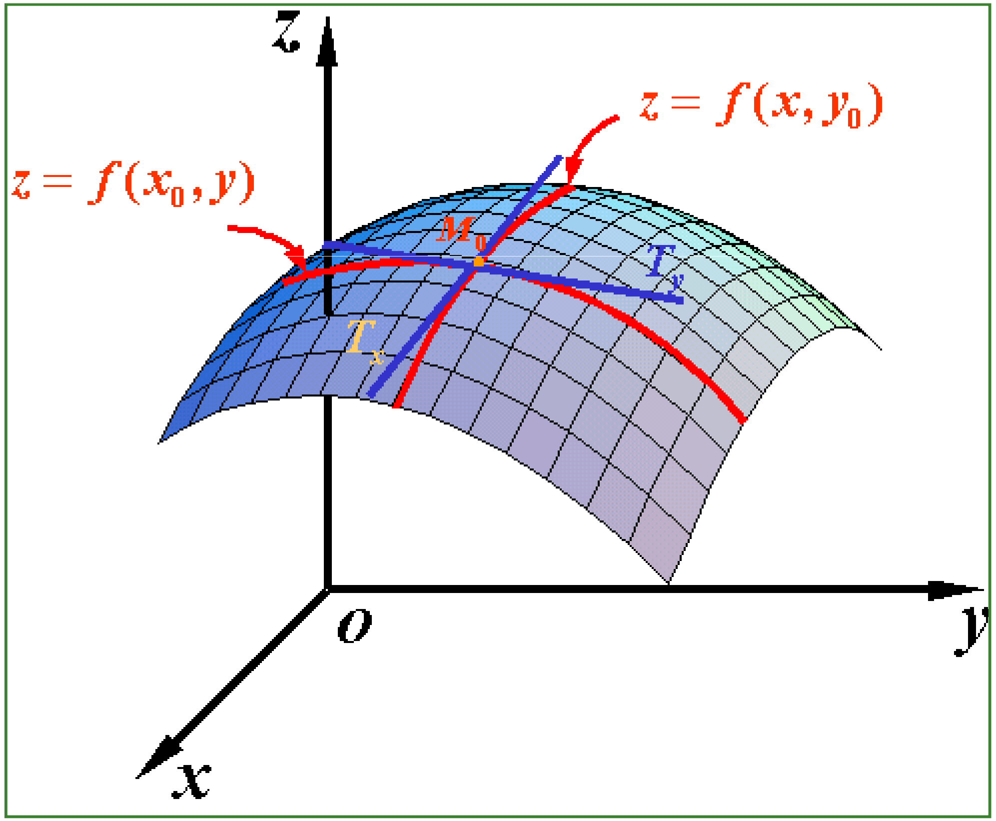
依定义知在(0,0)处，$f_x(0,0)=f_y(0,0)=0.$

但函数在该点处并不连续. 偏导数存在 $\not\longrightarrow$ 连续.

4、偏导数的几何意义

设 $M_0(x_0, y_0, f(x_0, y_0))$ 为曲面 $z = f(x, y)$ 上一点，

如图



几何意义：

偏导数$f_x(x_0,y_0)$就是曲面被平面$y=y_0$所截得的曲线在点M_0处的切线M_0T_x对x轴的斜率.

偏导数$f_y(x_0,y_0)$就是曲面被平面$x=x_0$所截得的曲线在点M_0处的切线M_0T_y对y轴的斜率.

二、高阶偏导数

函数$z = f(x, y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y), \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y)$$

混合偏导

定义：二阶及二阶以上的偏导数统称为高阶偏导数.

例 5 设 $z = x^3y^2 - 3xy^3 - xy + 1$，

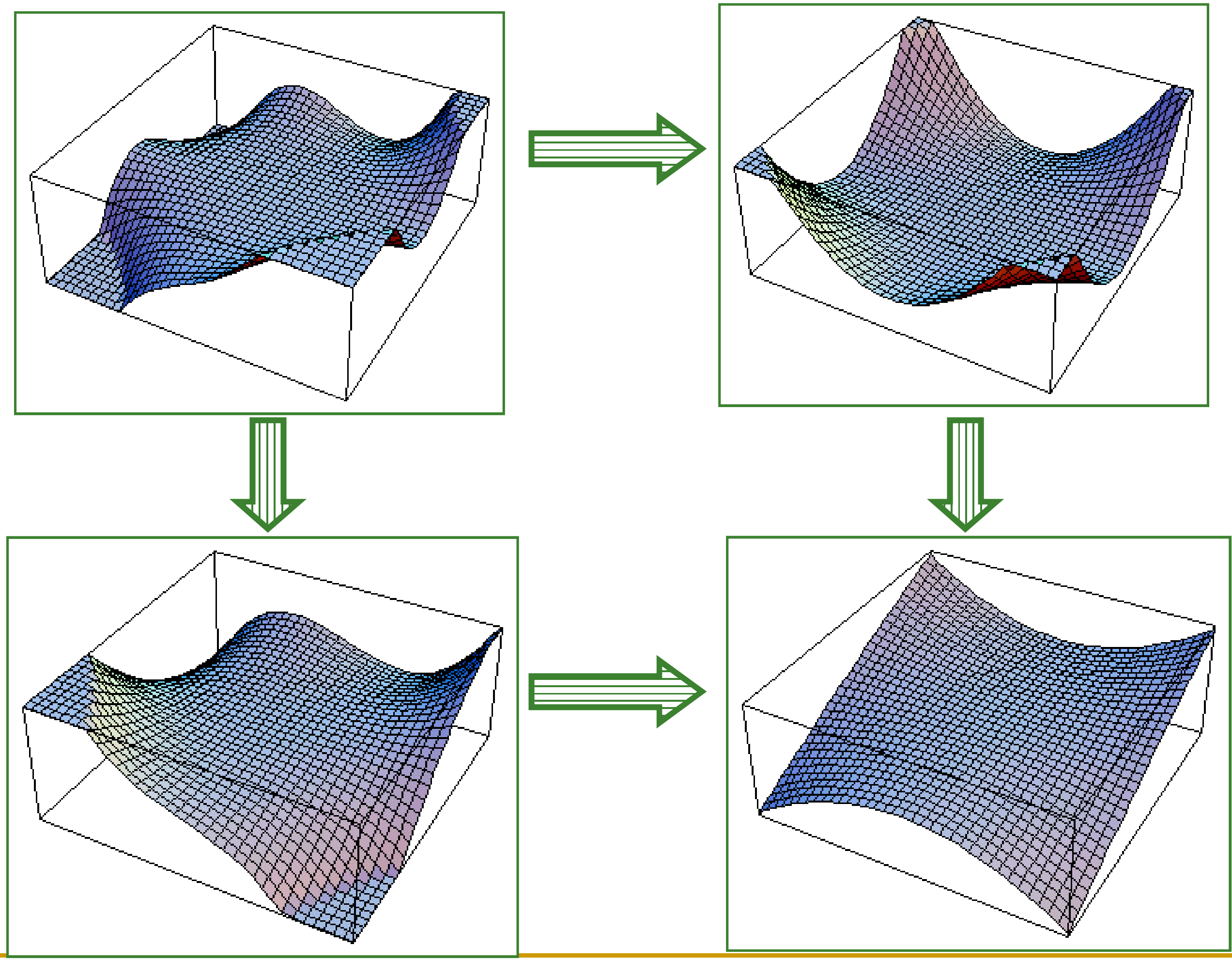
求 $\dfrac{\partial^2 z}{\partial x^2}$、$\dfrac{\partial^2 z}{\partial y \partial x}$、$\dfrac{\partial^2 z}{\partial x \partial y}$、$\dfrac{\partial^2 z}{\partial y^2}$ 及 $\dfrac{\partial^3 z}{\partial x^3}$.

解 $\dfrac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\dfrac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$;

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \qquad \frac{\partial^3 z}{\partial x^3} = 6y^2, \qquad \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy;$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1, \qquad \frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1.$$

观察上例中原函数、偏导函数与二阶混合偏导函数图象间的关系：



原函数图形

偏导函数图形

偏导函数图形

二阶混合偏导函数图形

例 6　设$u = e^{ax}\cos by$，求二阶偏导数.

解　$\dfrac{\partial u}{\partial x} = ae^{ax}\cos by$,　　$\dfrac{\partial u}{\partial y} = -be^{ax}\sin by$;

$$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax}\cos by, \qquad \frac{\partial^2 u}{\partial y^2} = -b^2 e^{ax}\cos by,$$

$$\frac{\partial^2 u}{\partial x \partial y} = -abe^{ax}\sin by, \qquad \frac{\partial^2 u}{\partial y \partial x} = -abe^{ax}\sin by.$$

问题：混合偏导数都相等吗？具备怎样的条件才相等？

定理 如果函数$z=f(x,y)$的两个二阶混合偏导数$\frac{\partial^2 z}{\partial y\partial x}$及$\frac{\partial^2 z}{\partial x\partial y}$在区域 D 内连续，那末在该区域内这两个二阶混合偏导数必相等.

例 6 验证函数$u(x,y)=\ln\sqrt{x^2+y^2}$满足拉普拉斯方程$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

解 $\because \ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2),$

$$\therefore \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0.$$

三、小结

偏导数的定义（偏增量比的极限）

偏导数的计算、偏导数的几何意义

高阶偏导数 $\begin{cases} \text{纯偏导} \\ \text{混合偏导（相等的条件）} \end{cases}$

思考题

若函数 $f(x,y)$ 在点 $P_0(x_0,y_0)$ 连续，能否断定 $f(x,y)$ 在点 $P_0(x_0,y_0)$ 的偏导数必定存在？

思考题解答

不能.

例如，$f(x,y)=\sqrt{x^2+y^2}$,

在(0,0)处连续,

但 $f_x(0,0)=f_y(0,0)$不存在.

练 习 题

一、填空题:

1、设$z=\ln\tan\frac{x}{y}$，则$\frac{\partial z}{\partial x}=$__________；$\frac{\partial z}{\partial y}=$__________.

2、设$z=e^{xy}(x+y)$，则$\frac{\partial z}{\partial x}=$__________；$\frac{\partial z}{\partial y}=$__________.

3、设$u=x^{\frac{y}{z}}$，则$\frac{\partial u}{\partial x}=$__________；$\frac{\partial u}{\partial y}=$__________；

$\frac{\partial u}{\partial z}=$__________.

4、设$z=\arctan\frac{y}{x}$，则$\frac{\partial^2 z}{\partial x^2}=$__________；$\frac{\partial^2 z}{\partial y^2}=$__________；

$\frac{\partial^2 z}{\partial x\partial y}=$__________.

5、设$u=(\frac{x}{y})^z$，则$\frac{\partial^2 u}{\partial z\partial y}=$____________.

二、求下列函数的偏导数:

1、$z=(1+xy)^y$；

2、$u=\arctan(x-y)^z$.

三、曲线$\begin{cases} z=\frac{x^2+y^2}{4} \\ y=4 \end{cases}$，在点(2, 4, 5)处的切线与正向$x$轴所成的倾角是多少?

四、设$z=y^x$，求$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial y^2}$和$\frac{\partial^2 z}{\partial x\partial y}$.

五、设$z=x\ln(xy)$，求$\frac{\partial^3 z}{\partial x^2\partial y}$和$\frac{\partial^3 z}{\partial x\partial y^2}$.

六、验证:

1、$z = e^{-(\frac{1}{x}+\frac{1}{y})}$, 满足 $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 2z$;

2、$r = \sqrt{x^2 + y^2 + z^2}$ 满足

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{z}{r}.$$

七、设

$$f(x, y) = \begin{cases} x^2 \arctan\frac{y}{x} - y^2 \arctan\frac{x}{y}, xy \neq 0 \\ 0, xy = 0 \end{cases}$$

求 f_x, f_{xy}.

练习题答案

一、1、$\frac{2}{y}\csc\frac{2x}{y}, -\frac{2x}{y^2}\csc\frac{2x}{y}$；

2、$e^{xy}(xy+y^2+1), e^{xy}(xy+x^2+1)$；

3、$\frac{y}{z}x^{\frac{y}{z}-1}, \frac{1}{z}x^{\frac{y}{z}}\ln x, \quad -\frac{y}{z^2}x^{\frac{y}{z}}\ln x$；

4、$\frac{2xy}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2}$；

5、$-(\frac{x}{y})^z(\frac{1}{y}+\frac{z}{y}\ln\frac{x}{y})$.

二、1、

$$\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}, \frac{\partial z}{\partial y}=(1+xy)^y\left[\ln(1+xy)+\frac{xy}{1+xy}\right];$$

2、$\dfrac{\partial u}{\partial x}=\dfrac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$ ，$\dfrac{\partial u}{\partial y}=\dfrac{-z(x-y)^{z-1}}{1+(x-y)^{2z}}$,

$\dfrac{\partial u}{\partial z}=\dfrac{(x-y)\ln(x-y)}{1+(x-y)^{2z}}$.

三、$\dfrac{\pi}{4}$.

四、$\dfrac{\partial^2 z}{\partial x^2}=y^x\ln^2 y,\dfrac{\partial^2 z}{\partial y^2}=x(x-1)y^{x-2}$,

$\dfrac{\partial^2 z}{\partial x\partial y}=y^{x-1}(x\ln y+1)$.

五、$\dfrac{\partial^3 z}{\partial x^2\partial y}=0,\dfrac{\partial^3 z}{\partial x\partial y^2}=-\dfrac{1}{y^2}$.

七、$f_x = \begin{cases} 2x\arctan\dfrac{y}{x} - y, xy \neq 0 \\ -y, x = 0, y \neq 0 \\ 0, x = y = 0; x \neq 0, y = 0 \end{cases}$,

$$f_{xy} = \begin{cases} -1, x = 0 \\ \dfrac{x^2 - y^2}{x^2 + y^2}, xy \neq 0. \\ 1, x \neq 0, y = 0 \end{cases}$$