英才实验学院线性代数模拟答案

!!! 此答案仅供参考，如有异议，欢迎指出

一、 填空题

1、设$A=\begin{pmatrix} 1 & 2 & -2 \\ 4 & a & 3 \\ 3 & -1 & 1 \end{pmatrix}$，$B$为3阶非零矩阵，且$AB=0$，则$a=$____

-3

2、设矩阵$A=(a_{ij})_{3\times3}$满足$A^*=A^T$,如果a_{11},a_{12},a_{13}是3个相等的正数，则$a_{11}=$____

$$A^*=A^T\Rightarrow\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$AA^*=|A|I=AA^T \qquad \left||A|I\right|=|AA^T|\Rightarrow|A|^n=|A|^2\Rightarrow|A|=0or1$$

$$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=a_{11}^2+a_{12}^2+a_{13}^2=1$$

$$a_{11}=\sqrt{3}/3$$

3、设α,β,γ为3? 1矩阵，已知行列式$|\alpha,\beta,\gamma|$=3，行列式$|\alpha+\beta,\beta+\gamma,\gamma+\alpha|=$____

$$|\alpha,\beta,\gamma|=|\alpha,\beta+\gamma,\gamma|=|\alpha,\beta+\gamma,\gamma+\alpha|$$

$$|\alpha+\beta,\beta+\gamma,\gamma+\alpha|=|\alpha,\beta+\gamma,\gamma+\alpha|+|\beta,\beta+\gamma,\gamma+\alpha|$$

$$|\beta,\beta+\gamma,\gamma+\alpha|=|\beta,\gamma,\gamma+\alpha|=|\beta,\gamma,\alpha|$$

$$|\alpha+\beta,\beta+\gamma,\gamma+\alpha|=|\alpha,\beta+\gamma,\gamma+\alpha|+|\beta,\beta+\gamma,\gamma+\alpha|=|\alpha,\beta,\gamma|+|\beta,\gamma,\alpha|=6$$

4、设$\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$是方程$x^4-6x^2+2x+1=0$的四个实根，则$\begin{vmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_4 & \varepsilon_3 & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_2 & \varepsilon_1 \end{vmatrix}=$____

$$\begin{vmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_4 & \varepsilon_3 & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_4 & \varepsilon_2 & \varepsilon_1 \end{vmatrix} = \begin{vmatrix} \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 & \varepsilon_3 & \varepsilon_1 & \varepsilon_2 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 & \varepsilon_4 & \varepsilon_2 & \varepsilon_1 \end{vmatrix} = (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \begin{vmatrix} 1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ 1 & \varepsilon_3 & \varepsilon_1 & \varepsilon_2 \\ 1 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ 1 & \varepsilon_4 & \varepsilon_2 & \varepsilon_1 \end{vmatrix}$$

$$(x-\varepsilon_1)(x-\varepsilon_2)(x-\varepsilon_3)(x-\varepsilon_4) = x^4 - 6x^2 + 2x + 1 = 0$$

$$x^4 - (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)x^3 + f(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)x^2 + g(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)x + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = x^4 - 6x^2 + 2x + 1 = 0$$

$(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)=0$ $\therefore$ 行列式为0

5、设$D_n = \begin{vmatrix} 1 & 0 & \cdots & 1 \\ \vdots & & \iddots & \\ 1 & 1 & & 0 \\ 1 & 1 & \cdots & 1 \end{vmatrix}$，**计算**$A_{n1} + 2A_{n2} + \cdots + nA_{nn} =$ ____

$$A_{n1} + 2A_{n2} + \cdots + nA_{nn} = \begin{vmatrix} 1 & 0 & \cdots & 1 \\ \vdots & & \iddots & \\ 1 & 1 & & 0 \\ 1 & 2 & \cdots & n \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 1 \\ \vdots & & \iddots & \\ 0 & 1 & & 0 \\ 1\text{-}2\text{-}3\text{-}\cdots\text{-}n & 2 & \cdots & n \end{vmatrix}$$

$$= \left((-1)^{n+1} \cdot 1\right)\left((-1)^{n} \cdot 1\right)\cdots\left((-1)^{2} \cdot (1-2-3-\cdots-n)\right)$$

$$= \left((-1)^{n+1}\right)^{n}(1-2-3-\cdots-n) = (-1)^{\frac{n(n-1)}{2}}\left(2-\frac{n(n+1)}{2}\right)$$

6、设$\alpha_1,\alpha_2,\alpha_3$**均为3维列向量，记矩阵**$A=(\alpha_1,\alpha_2,\alpha_3), B=$
$(\alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+4\alpha_3, \alpha_1+3\alpha_2+9\alpha_3)$,**如果**$|A|=1$,**求**$|B| =$ ____

$$B=\left(\alpha_1,\alpha_2,\alpha_3\right)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}$$

$$|B|=\left|\left(\alpha_1,\alpha_2,\alpha_3\right)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}\right|=|A|\begin{vmatrix}1&1&1\\1&2&3\\1&4&9\end{vmatrix}$$

$$=(2-1)(3-1)(3-2)=2$$

7、已知 $\begin{vmatrix}a_1&a_2&a_3&a_4\\2&2&1&1\\2&3&4&5\\1&1&2&2\end{vmatrix}=9,$**则**$A_{21}+A_{22}=$ _____

$$\begin{vmatrix}a_1&a_2&a_3&a_4\\2&2&1&1\\2&3&4&5\\1&1&2&2\end{vmatrix}=9,\Rightarrow\begin{cases}2A_{21}+2A_{22}+A_{23}+A_{24}=9\\A_{21}+A_{22}+2A_{23}+2A_{24}=0\end{cases}$$

$$A_{21}+A_{22}=6$$

二、设A**的伴随矩阵**$A^*=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&-3&0&8\end{pmatrix}$,**且**$ABA^{-1}=BA^{-1}+3I$,**求**$B$

$$A^{-1}ABA^{-1}A = A^{-1}\left(BA^{-1}+3I\right)A$$

$$B = A^{-1}B + 3I$$

$$A^{-1} = \frac{A^*}{|A|} \qquad \left|A^*\right| = 8 \Rightarrow |A|^3 = 8 \Rightarrow |A| = 2$$

$$2B = A^*B + 6I \Rightarrow B = 6\left(2I - A^*\right)^{-1}$$

$$= \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 6 & 0 & 6 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}$$

三、设n阶矩阵A和B满足$A+B=AB$,(1)证明$A-I$为可逆矩阵;

(2)已知$B=\begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,求$A$.(3)证明$AB=BA$

(1)设$A-I=C \Rightarrow A=C+I$

代入方程得

$$C+I+B=(C+I)B$$

$$C(B-I)=I$$

$$(A-I)(B-I)=I$$

所以$A-I$为可逆矩阵

$$(3)(A-I)(B-I)=I\cdots①$$

$$(B-I)(A-I)=I\cdots②$$

$$AB=A+B=BA$$

四、设$H_1=\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $H_2=\begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix}$,$\cdots$, $H_n=\begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$.

1)确定矩阵H_n的阶，并计算${H_1}^2$和${H_2}^2$;

2)证明${H_n}^{-1}=2^{-n}H_n$.

1）矩阵 H_n 的阶为 2^n。

$$H_1^2=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=2I_2;$$

$$H_2^2=\begin{bmatrix}H_1 & H_1\\ H_1 & -H_1\end{bmatrix}\cdot\begin{bmatrix}H_1 & H_1\\ H_1 & -H_1\end{bmatrix}=\begin{bmatrix}2H_1^2 & 0\\ 0 & 2H_1^2\end{bmatrix}=\begin{bmatrix}2^2 I_2 & 0\\ 0 & 2^2 I_2\end{bmatrix}=2^2 I_{2^2}.$$

2）

$H_1^{-1}=?,H_2^{-1}=?$

$H_1^{-1}=2^{-1}H_1,H_2^{-1}=2^{-2}H_2.$

$H_n^{-1}=2^{-n}H_n \Rightarrow H_n^2=2^n I_{2^n}$?

归纳假设 $H_k^2=2^k I_{2^k}.$

1.当 k=1，2时，结论成立；

2.假设 $H_{k-1}^2=2^{k-1}I_{2^{k-1}}$ 成立.

则 $H_k^2=\begin{bmatrix}H_{k-1} & H_{k-1}\\ H_{k-1} & -H_{k-1}\end{bmatrix}\cdot\begin{bmatrix}H_{k-1} & H_{k-1}\\ H_{k-1} & -H_{k-1}\end{bmatrix}=\begin{bmatrix}2H_{k-1}^2 & 0\\ 0 & 2H_{k-1}^2\end{bmatrix}=2^k I_{2^k}.$

因此有 $H_n^2=2^n I_{2^n}$，从而 $H_n^{-1}=2^{-n}H_n.$

五、 $A_n=\begin{vmatrix}a & a-1 & 0 & 0 & \cdots & 0 & 0\\ 1 & a & a-1 & 0 & \cdots & 0 & 0\\ 0 & 1 & a & a-1 & \cdots & 0 & 0\\ 0 & 0 & 1 & a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & a & a-1\\ 0 & 0 & 0 & 0 & \cdots & 1 & a\end{vmatrix}$

$$例8. A_n=\begin{vmatrix} a & a-1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a & a-1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a & a-1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & a-1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a \end{vmatrix} = a\begin{vmatrix} a & a-1 & 0 & \cdots & 0 & 0 \\ 1 & a & a-1 & \cdots & 0 & 0 \\ 0 & 1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & a-1 \\ 0 & 0 & 0 & \cdots & 1 & a \end{vmatrix} -(a-1)\begin{vmatrix} 1 & a-1 & 0 & \cdots & 0 & 0 \\ 0 & a & a-1 & \cdots & 0 & 0 \\ 0 & 1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & a-1 \\ 0 & 0 & 0 & \cdots & 1 & a \end{vmatrix}$$

$$A_n=aA_{n-1}-(a-1)A_{n-2}$$

$$A_n\text{-}A_{n-1}=(a-1)\left(A_{n-1}-A_{n-2}\right)=(a-1)^2\left(A_{n-2}-A_{n-3}\right)=\cdots=(a-1)^{n-2}(A_2-A_1)=(a-1)^n$$

$$A_n\text{-}A_{n-1}=(a-1)^n \Rightarrow \frac{A_n}{(a-1)^n}-\frac{1}{a-1}\frac{A_{n-1}}{(a-1)^{n-1}}=1 \quad 记\frac{A_n}{(a-1)^n}=B_n \quad \Rightarrow B_n=\frac{1}{a-1}B_{n-1}+1$$

$$B_n+\frac{a-1}{2-a}=\frac{1}{a-1}\left(B_{n-1}+\frac{a-1}{2-a}\right) \quad B_1+\frac{a-1}{2-a}=\frac{1}{(a-1)(2-a)} \quad B_n+\frac{a-1}{2-a}=\left(\frac{1}{a-1}\right)^{n-1}\left(\frac{1}{(2-a)(a-1)}\right)$$

$$A_n=\frac{1-(a-1)^{n+1}}{2-a}$$

三线对角矩阵 $\longrightarrow$ 构造数列求解

六、 $\begin{vmatrix} 0 & b & b & \cdots & b & b \\ c & 0 & a & \cdots & a & a \\ c & a & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & a & a & \cdots & 0 & a \\ c & a & a & \cdots & a & 0 \end{vmatrix}=$

$$\begin{vmatrix} 0 & b & b & \cdots & b & b \\ c & 0 & a & \cdots & a & a \\ c & a & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & a & a & \cdots & 0 & a \\ c & a & a & \cdots & a & 0 \end{vmatrix}=\begin{vmatrix} 1 & c & a & a & \cdots & a & a \\ 0 & 0 & b & b & \cdots & b & b \\ 0 & c & 0 & a & \cdots & a & a \\ 0 & c & a & 0 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c & a & a & \cdots & 0 & a \\ 0 & c & a & a & \cdots & a & 0 \end{vmatrix}=\begin{vmatrix} 1 & c & a & a & \cdots & a & a \\ -1 & -c & b-a & b-a & \cdots & b-a & b-a \\ -1 & 0 & -a & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & -a & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -a \end{vmatrix}$$

$$=\begin{vmatrix} 1+(n\text{-}1)(\text{-}1) & c & a & a & \cdots & a & a \\ -1+(n-1)\left(\frac{a-b}{a}\right) & -c & b-a & b-a & \cdots & b-a & b-a \\ 0 & 0 & -a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a \end{vmatrix}=\begin{vmatrix} 1+(n\text{-}1)(\text{-}1)\text{-}1+(n\text{-}1)\left(\frac{a-b}{a}\right) & c & a & a & \cdots & a & a \\ 0 & -c & b-a & b-a & \cdots & b-a & b-a \\ 0 & 0 & -a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a \end{vmatrix}$$

$$=(n-1)(-b)ca^{n-2}(-1)^n=(-1)^{n+1}(n-1)a^{n-2}bc$$