Thm 3.3 ($\mathbb{R}^2$)

证明：由 f 在 (x_0,y_0) 处可微，则

$$f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0) = a_1\Delta x + a_2\Delta y + o(\rho)$$

其中 $a_1 = f_x(x_0,y_0)$，$a_2 = f_y(x_0,y_0)$，$\rho = \sqrt{\Delta x^2 + \Delta y^2}$.

取 $\Delta x = t\cos\theta_1$，$\Delta y = t\cos\theta_2$ 则 $\rho = |t|$，且

$$f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0) = f(\vec{x}_0 + t e_l) - f(\vec{x}_0) \qquad \boxed{\vec{x}_0 = (x_0,y_0)}$$

由方向导数的定义：

$$\frac{\partial f}{\partial l}\Big|_{x_0} = \lim_{t\to 0}\frac{f(\vec{x}_0 + t e_l) - f(\vec{x}_0)}{t}$$

$$= \lim_{t\to 0}\left(a_1\cos\theta_1 + a_2\cos\theta_2 + \cancel{\frac{o(\rho)}{t}}\right)$$

$$= a_1\cos\theta_1 + a_2\cos\theta_2.$$

$z = f(x,y)$：令 $\vec{r}(t) = g(t)i + h(t)j$ 为 z 的一条等值线。记 $f(g(t),h(t)) = c$（常数）

从而 $0 = \dfrac{dc}{dt} = \dfrac{\partial f}{\partial x}\cdot\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dh}{dt}$ §3.5

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$= \nabla f \cdot \frac{d\vec{r}}{dt}$$

grad　　曲线的切向量

即 ∇f 垂直于等值线 $\vec{r}(t)$.

The chain Rule ① for $z = f(u(x,y), v(x,y))$, then

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} \end{cases}$$

② For $z = f\big(u(x,y), v(x,y), w(x,y)\big)$

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} = \cdots \end{cases}$$

③ For $z = f(u(x), v(x), w(x))$ (一元)

$$\Rightarrow \frac{dz}{dx} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx} + \frac{\partial f}{\partial w}\frac{dw}{dx}$$

一元 $z=f(u)$ $u=\underline{g(x)}$

If $\Delta u \neq 0$ 时,

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

则 $$\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x} = \lim_{\substack{\Delta x\to 0\\ \Rightarrow \Delta u\to 0}}\frac{\Delta z}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

$$\Rightarrow \frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx}$$

Remark: