Solutions to Random Mathematics Homework #1 Fall 2020

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Assigned Date: Sept.10, 2020 Due Date: Sept.17, 2020

H1.1

It can be known that $B = (A \cap B) \cup (A^c \cap B)$, and $A \cap B$ and $A^c \cap B$ are disjoint, so $\Pr(B)$ can be written as $\Pr(B) = \Pr(A \cap B) + \Pr(A^c \cap B)$. In (a), when A and B are disjoint, $\Pr(A \cap B) = 0$, so $\Pr(A^c \cap B) = 1/4 - 0 = 1/4$. In (b), when $A \supset B$, $\Pr(A \cap B) = \Pr(B)$, so $\Pr(A^c \cap B) = 0$. In (c), $\Pr(A^c \cap B) = 1/4 - 1/5 = 1/20$.

H1.2

Let the sample space consist of all three-tuples of dice rolls. There are $6^3 = 216$ possible outcomes. The outcomes with all three rolls different consist of all of the permutations of six items taken three at a time. There are $P_{6,3} = 120$ of these outcomes. So the probability we want is $120/216 = 5/9$.

H1.3

(a) To prove the first result, let $x \in (A \cup B)^c$. This means that x is not in $A \cup B$. In other words, x is neither in A nor in B. Hence $x \in A^c$ and $x \in B^c$. So $x \in A^c \cap B^c$. This proves that $(A \cup B)^c \subset A^c \cap B^c$. Next, suppose that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. So x is neither in A nor in B, so it can't be in $A \cup B$. Hence $x \in (A \cup B)^c$. This shows that $A^c \cap B^c \subset (A \cup B)^c$. The second result follows from the first by applying the first result to A^c and B^c and then taking complements of both sides.

(b) First, show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$. That is, $x \in A$ and either $x \in B$ or $x \in C$ (or both). So either ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) or both. That is, either $x \in A \cap B$ or $x \in A \cap C$. That is what it means to say that $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Basically, running these steps backwards shows that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

H1.4

There are 10^6 possible outcomes in the sample space. If the 6 balls are to be thrown into different boxes, the first ball can be thrown into any one of the 10 boxes, the second ball can then be thrown into any one of the other 9 boxes, etc. Thus there are $10 \cdot 9 \cdot 8 \cdot 7 \cdots 5$ possible outcomes in the event. So the probability is $P_{10,6}/10^6 \approx 0.1512$.

H1.5

From the description of what counts as a collection of customer choices, we see that each collection consists of a tuple $(m_1, ..., m_n)$, where m_i is the number of customers who choose item

i for $i = 1, ..., n$. Each m_i must be between 0 and k and $m_1 + \cdots m_n = k$. Each such tuple is equivalent to a sequence of $n + k - 1$ 0's and 1's as follows. The first m_1 terms are 0 followed by a 1. The next m_2 terms are 0 followed by a 1, and so on up to m_{n-1} 0's followed by a 1 and finally m_n 0's. Since $m_1 + ... + m_n = k$ and since we are putting exactly $n - 1$ 1's into the sequence, each such sequence has exactly $n + k - 1$ terms. Also, it is clear that each such sequence corresponds to exactly one tuple of customer choices. The numbers of 0's between successive 1's give the numbers of customers who choose that item, and the 1's indicate where we switch from one item to the next. So, the number of combinations of choices is the number of such sequences: $\binom{n+k-1}{k}$.

H1.6

The total number of ways of choosing the n seats that will be occupied by the n people is $\binom{2n}{n}$. Offhand, it would seem that there are only two ways of choosing there these seats so that no two adjacent seats are occupied, namely

○●○●···● and ●○●○···○ .

Upon further consideration, however, $n - 1$ more ways can be found, namely:

●○○●○●···○● , ●○●○○●○●···○● ,etc.

Therefore, the total number of ways of choosing the seats so that no two adjacent seats are occupied is $n + 1$. The probability is $(n + 1)/\binom{2n}{n}$.

H1.7

There are $\binom{20}{10} \times \binom{10}{6}$ ways of dividing the books into the three boxes. The number of ways of choosing the 10 books for the first box so as to include both A and B is $\binom{18}{8} \times \binom{10}{6}$. The number of ways of choosing the 6 books for the second box so as to include both A and B is $\binom{18}{4} \times \binom{14}{10}$. And the number of ways of choosing the 4 books for the third box so as to include both A and B is $\binom{18}{2} \times \binom{16}{10}$. Thus, the probability we want is then

$$\frac{\binom{18}{8} \times \binom{10}{6} + \binom{18}{4} \times \binom{14}{10} + \binom{18}{2} \times \binom{16}{10}}{\binom{20}{10} \times \binom{10}{6}} \approx 0.3474.$$

H1.8

Call the four players A, B, C, and D. The number of ways of choosing the positions in the deck that will be occupied by the four aces is $\binom{52}{4}$. Since player A will receive 13 cards, the number of ways of choosing the positions in the deck for the four aces so that all of them will be received by player A is $\binom{13}{4}$. Similarly, since player B will receive 13 other cards, the number of ways of choosing the positions for the four aces so that all of them will be received by player B is $\binom{13}{4}$. A similar result is true for each of the other players. Therefore, the total number of ways of choosing the positions in the deck for the four aces so that all of them will be received by the same player is $4\binom{13}{4}$. Thus, the final probability is $4\binom{13}{4}/\binom{52}{4} \approx 0.01056$.

H1.9

If we do not distinguish among boys with the same last name, then there are $\binom{9}{2,3,4}$ possible arrangements of the nine boys. We are interested in the porbability of a particular one of these arrangements. So, the probability we need is $\frac{1}{\binom{9}{2,3,4}} = 2!3!4!/9! \approx 7.937 \times 10^{-4}$

H1.10

The event that exactly one of the three events will occur can be described as:
$(A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3)$
Since these three events are disjoint, according to Theorem 1.5.2, the probability can be written as:
$\Pr(A_1 \cap A_2^c \cap A_3^c) + \Pr(A_1^c \cap A_2 \cap A_3^c) + \Pr(A_1^c \cap A_2^c \cap A_3)$
Take the first term for example.
According to Theorem 1.4.9 (De Morgan's Laws): $A_1 \cap A_2^c \cap A_3^c = A_1 \cap (A_2 \cup A_3)^c$.
Since $[A_1 \cap (A_2 \cup A_3)^c] \cup [A_1 \cap (A_2 \cup A_3)] = A_1$ and the events are disjoint, we have:
$\Pr[A_1 \cap (A_2 \cup A_3)^c] = \Pr(A_1) - \Pr[A_1 \cap (A_2 \cup A_3)] = \Pr(A_1) - \Pr[(A_1 \cap A_2) \cup (A_1 \cap A_3)] = \Pr(A_1) - [\Pr(A_1 \cap A_2) + \Pr(A_1 \cap A_3) - \Pr(A_1 \cap A_2 \cap A_3)]$
Take all three terms into consideration, add them up, then we can obtain the final result.