Chapter 3
Random Variables and Distributions
Exercises

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多选题 5分

设置

Suppose that the c.d.f. of a continuous random variable X is $F(x)$, its p.d.f. is f(x), where $-\infty<x<\infty$, then $\Pr(0<X<2)=?$

A $F(0)-F(2)$

B $1-F(2)$

C $\int_0^2 F(x)dx$

D $F(2)-F(0)$

E $\int_0^2 f(x)dx$

提交

主观题 10分

设置

Suppose that a random variable X has distribution with the following c.d.f:

$$F(x)=\begin{cases}1 & x\geq\theta\\ \ln x & 1\leq x\leq\theta\\ 0 & x<1\end{cases}$$

1) Determine the constant θ.
2) Find $\Pr(X<2)$, $\Pr(1<X\leq 4)$, $\Pr(X=3/2)$.
3) Find the p.d.f of X.

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作答

Ex2-Solution

- Suppose that a random variable X has distribution with the following c.d.f:

$$F(x)=\begin{cases}1 & x\geq\theta\\ \ln x & 1\leq x\leq\theta\\ 0 & x<1\end{cases}$$

1) Determine the constant θ. **Since** $\ln\theta=1$, **we have** $\theta=e$

2) Find Pr(X<2), Pr(1<X≤4), Pr(X=3/2).

$$P\{X<2\}=P\{X\leq 2\}=F(2)=\ln 2$$

$$P\{1<X\leq 4\}=F(4)-F(1)=1-\ln 1=1$$

$$P\{X=\frac{3}{2}\}=0$$

3) Find the p.d.f of X. If 1<x <e,

$$f(x)=\frac{dF(x)}{dx}=\frac{1}{x},$$

$$\therefore\ f(x)=\begin{cases}\frac{1}{x}, & 1<x<e\\ 0, & \text{otherwise}\end{cases}$$

主观题 10分

设置

Suppose that the $R.V.$ X has the p.d.f.

$$f(x) = \begin{cases} 2x, if\ 0 < x < 1, \\ 0, \qquad otherwise. \end{cases}$$

Determine the quantile function for X.

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作答

Ex3-Solution

Suppose that the $R.V.\ X$ has the p.d.f.

$$f(x) = \begin{cases} 2x, if\ 0 < x < 1, \\ 0, \quad\quad otherwise. \end{cases}$$

Determine the quantile function for X.

Sol:

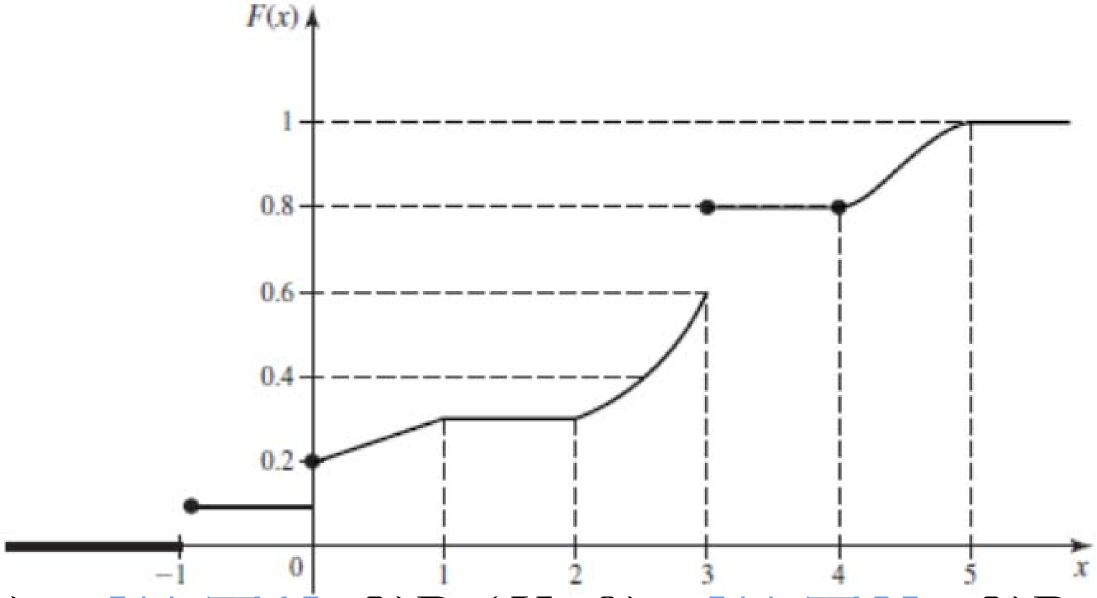
$$F(x) = \begin{cases} 0, x \leq 0 \\ \int_0^x 2tdt = x^2, 0 < x < 1 \\ 1, x \geq 1 \end{cases}$$

For each $0<p<1$, we solve for x in the equation $F(x)=p$.
For $0<x<1$, Because $x^2=p$, we have $x=\sqrt{p}$.
Thus, the quantile function for X is $F^{-1}(p)=\sqrt{p}$ for $0<p<1$.

填空题 11分

设置

Suppose the c.d.f. F of a $R.V.$ X is as follows. Find:



1)Pr(X=-1) = [填空1], 2)Pr(X<0)= [填空2] , 3)Pr(X≤0)= [填空3], 4)Pr(X=1)= [填空4],5)Pr(0<X ≤3)= [填空5],6)Pr(0<X<3)= [填空6], 7)Pr(0≤X ≤3)= [填空7],8)Pr(1<X≤2)= [填空8], 9)Pr(3≤X ≤4)= [填空9],10)Pr(X>5)= [填空10],

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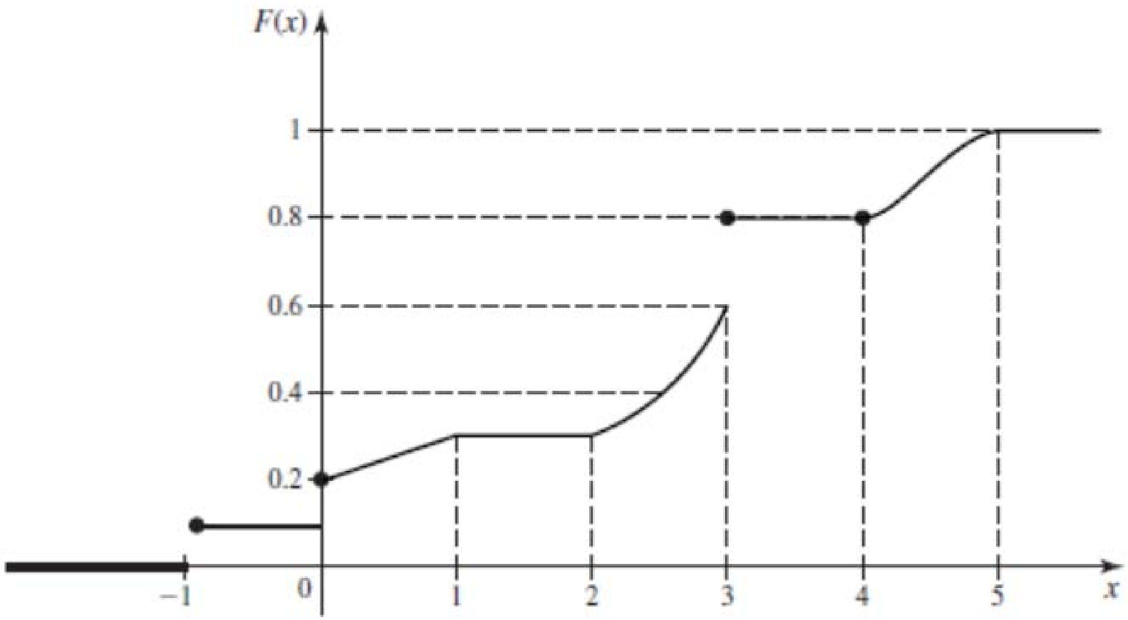
11)Pr(X≥3)= [填空11] .

作答

Ex4-Solution

Suppose the c.d.f. F of a *R.V. X* is



Find each of the following probabilities:

1)Pr(X=-1), 2)Pr(X<0), 3)Pr($X\le$0), 4)Pr(X=1), 5)Pr(0<$X\le$3), 6)Pr(0<X<3),7)Pr(0$\le X\le$3), 8)Pr(1<$X\le$2), 9)Pr(3$\le X\le$4), 10)Pr(X>5),

11)Pr($X\ge$3) $=\Pr(X=3)+1-F(3)=0.2+1-0.8=0.4.$

Sol:

1)Pr(X=-1)=0.1,

2)Pr(X<0)=0.1,

3)Pr($X\le$0)=0.2,

4)Pr(X=1)=0,

5)Pr(0<$X\le$3)=0.6,

6)Pr(0<X<3)=0.4,

7)Pr(0$\le X\le$3)=0.7,

8)Pr(1<X $\le$2)=0,

9)Pr(3$\le X\le$4)=0.2,

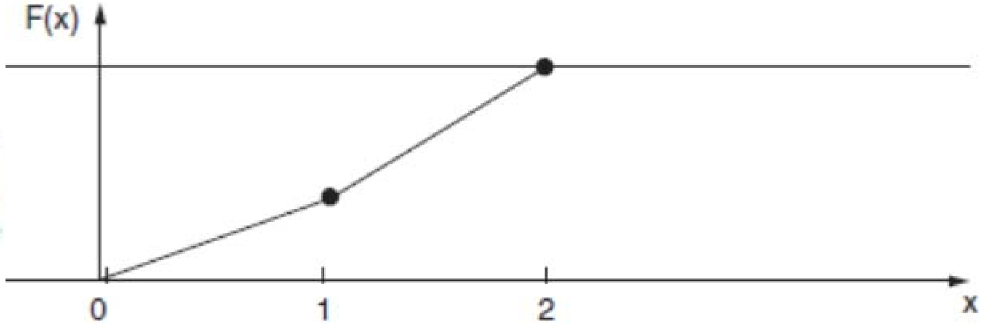
10)Pr(X>5)=0

主观题 5分

设置

Suppose that the $R.V.\ X$ has the following c.d.f.

$$F(x) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{2}{5}x & \text{for } 0 < x \le 1, \\ \frac{3}{5}x - \frac{1}{5} & \text{for } 1 < x \le 2, \\ 1 & \text{for } x > 2. \end{cases}$$



Determine the p.d.f. of X.

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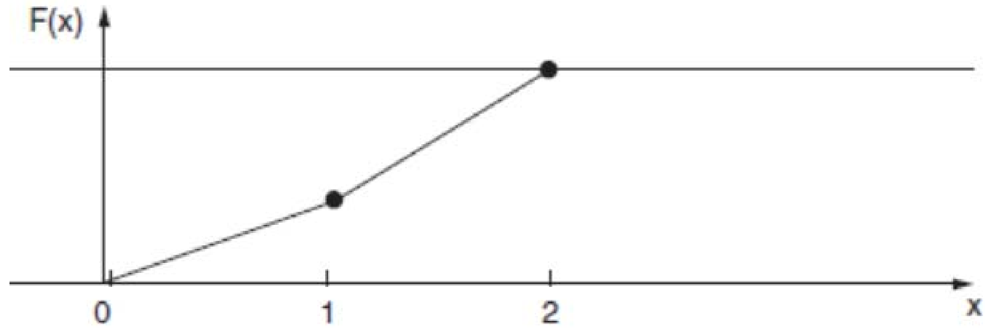
作答

Ex5-Solution

Suppose that the *R.V. X* has the following c.d.f.

$$F(x)=\begin{cases} 0 & \text{for } x \le 0, \\ \frac{2}{5}x & \text{for } 0 < x \le 1, \\ \frac{3}{5}x-\frac{1}{5} & \text{for } 1 < x \le 2, \\ 1 & \text{for } x > 2. \end{cases}$$



Determine the p.d.f. of *X*.

Sol: $F(x)$ is continuous and differentiable everywhere except the points x=0,1,2,

$$f(x)=\frac{dF(x)}{dx}\begin{cases} \frac{2}{5} & \text{for } 0 < x < 1, \\ \frac{3}{5} & \text{for } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

单选题 5分

设置

Suppose that internet users access a particular Web site according to a Poisson process with rate λ per hour, but λ is unknown. The Web site maintainer believes that λ has a continuous distribution with p.d.f.

$$f(\lambda)=\begin{cases}2e^{-2\lambda} & \text{for} \quad \lambda>0,\\ 0 & \text{otherwise.}\end{cases}$$

Let X be the number of users who access the Web site during a one-hour period. If X=1 is observed, find the p.d.f of λ given X=1.

A $g_2(\lambda|x=1)=9\lambda\exp(-3\lambda)$ for $\lambda>0$.

C $g_2(\lambda|x=1)=2\lambda\exp(-3\lambda)$ for $\lambda>0$.

B $g_2(\lambda|x=1)=3\lambda\exp(-9\lambda)$ for $\lambda>0$.

D $g_2(\lambda|x=1)=\lambda\exp(-9\lambda)$ for $\lambda>0$.

提交

Ex6-Solution

◆Suppose that internet users access a particular Web site according to a Poisson process with rate λ per hour, but λ is unknown. The Web site maintainer believes that λ has a continuous distribution with p.d.f.

$$f(\lambda)=\begin{cases}2e^{-2\lambda} & \text{for} \quad \lambda>0,\\ 0 & \text{otherwise.}\end{cases}$$

Let X be the number of users who access the Web site during a one-hour period. If X=1 is observed, find the p.d.f of λ given X=1.

Sol: $\because f(x,\lambda)=g_1(x|\lambda)f_2(\lambda)$, we have the joint p.f./p.d.f is

$$f(x,\lambda)=\exp(-\lambda)\frac{\lambda^x}{x!}2\exp(-2\lambda)=2\exp(-3\lambda)\frac{\lambda^x}{x!}.$$

The marginal p.f. of X at x=1 is

$$f_1(1)=\int_0^\infty 2\lambda\exp(-3\lambda)d\lambda=\frac{2}{9}.$$

We want $g_2(\lambda|x=1)=\dfrac{f(x,\lambda)}{f_1(x=1)}$

$$g_2(\lambda|x=1)=9\lambda\exp(-3\lambda) \text{ for } \lambda>0.$$

主观题 14分

设置

Suppose that X_1 and X_2 are i.i.d. $R.V.$s and each of them has the uniform distribution on the interval (0,1). Find the p.d.f. of:

1) $Y = X_1 + X_2$.

2) $Z = (X_1 + X_2)/2$.

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作答

Ex7-1-Solution

Suppose that X_1 and X_2 are i.i.d. $R.V.$s and each of them has the uniform distribution on the interval (0,1). Find the p.d.f. of:

1) $Y=X_1+X_2$.

2) $Z=(X_1+X_2)/2$.

Sol: The joint p.d.f. of X_1 and X_2 is

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Based on convolution, we have

$$g(y) = \int_{-\infty}^{\infty} f(y-z, z)dz.$$

The integrand is positive only for $0<y-z<1$ and $0<z<1$.

Thus, for $0<y\leq 1$, the integrand is positive only for $0<z<y$. We have

$$g(y) = \int_0^y 1 \cdot dz = y.$$

For $1<y<2$, the integrand is positive only for $y-1<z<1$.

$$g(y) = \int_{y-1}^1 1 \cdot dz = 2-y.$$ $g(y)=0$, otherwise.

Ex7-2-Solution

Suppose that X_1 and X_2 are i.i.d. $R.V.$s and each of them has the uniform distribution on the interval [0,1]. Find the p.d.f. of:

1) $Y=X_1+X_2$.

2) $Z=(X_1+X_2)/2$.

Sol: We've obtained that in 1) for $0<y\le 1$, $g(y)=\int_0^y 1\cdot dz = y.$

For $1<y<2$, $g(y)=\int_{y-1}^1 1\cdot dz = 2-y.$

Otherwise, $g(y)=0$.

$Z=Y/2$. $f(z)=\dfrac{f(y)}{|\frac{dz}{dy}|}=2f(2z)=\begin{cases} 4z & \text{for } 0<z\le 1/2, \\ 4(1-z) & \text{for } 1/2<z<1, \\ 0 & \text{otherwise.} \end{cases}$

主观题 10分

设置

Suppose that X_1 and X_2 are i.i.d. *R.V.*s and each of them has the p.d.f. as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f. of $Y = X_1 - X_2$.

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作答

Ex8-Solution

Suppose that X_1 and X_2 are i.i.d. *R.V.*s and each of them has the p.d.f. as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f. of $Y=X_1-X_2$.

Sol: Let $Z=-X_2$. Then the p.d.f. of Z is $f_2(z) = \begin{cases} \exp(z) & \text{for } z < 0, \\ 0 & \text{for } z \geq 0. \end{cases}$

Since X_1 and Z are independent, their joint p.d.f. is

$$f(x_1, z) = \begin{cases} \exp(-(x - z)) & \text{for } x > 0,\ z < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Based on convolution, the p.d.f. of $Y=X_1+Z$: $g(y) = \int_{-\infty}^{\infty} f(y - z, z)\, dz.$

The integrand is positive only for $y-z>0$ and $z<0$.

For $y \leq 0$, $\quad g(y) = \int_{-\infty}^{y} \exp(-(y - 2z))\, dz = \frac{1}{2}\exp(y).$

For $y>0$, $\quad g(y) = \int_{-\infty}^{0} \exp(-(y - 2z))\, dz = \frac{1}{2}\exp(-y).$

Ex9-1

Preliminaries-Direct Transformation of a multivariate p.d.f.

Suppose X, Y, V, W are four $R.V.$s. They have the relations such that: $X=h1(V,W)$, $Y=h2(V,W)$, where $h1$ and $h2$ represent invertible functions. The joint p.d.f. of V and W can be expressed by

$$f_{V,W}(v,w)=\frac{f_{X,Y}(h_1(v,w),h_2(v,w))}{|J(x,y)|},\quad \text{where } J(x,y)=\det\begin{bmatrix}\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{bmatrix}.$$

or

$$f_{V,W}(v,w)=f_{X,Y}(h_1(v,w),h_2(v,w))\,|J(v,w)|,\quad J(v,w)=\det\begin{bmatrix}\frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\end{bmatrix}.$$

The determinant $J(x,y)$ is called the **Jacobian** of the transformation. $J(v,w)$ is called the Jacobian of the inverse transformation.

主观题 10分

设置

Let X and Y be independent standard normal $R.V.$s. Find the joint p.d.f. of V and W that satisfy following equations: $X=V\cos W$, $Y=V\sin W$, for $v\geq 0$ and $0\leq w<2\pi$. Are V and W independent?

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作答

Ex9-2-Solution

Let X and Y be independent standard normal $R.V.$s. Find the joint p.d.f. of V and W that satisfy following equations: $X=V\cos W$, $Y=V\sin W$, for $v\geq 0$ and $0\leq w<2\pi$. Are V and W independent?

Sol: $\because f(x)=\dfrac{1}{\sqrt{2\pi}}\exp(-\dfrac{1}{2}x^2)$, for $-\infty<x<\infty$, X and Y are i.i.d.,

We have that $f(x,y)=\dfrac{1}{2\pi}\exp[-\dfrac{1}{2}(x^2+y^2)]$, for $-\infty<x,y<\infty$

$$\because f_{V,W}(v,w)=f_{X,Y}(h_1(v,w),h_2(v,w))\,|J(v,w)|,\quad J(v,w)=\det\begin{bmatrix}\frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\end{bmatrix}.$$

$$J(v,w)=\det\begin{bmatrix}\cos w & -v\sin w\\ \sin w & v\cos w\end{bmatrix}=v.$$

$$\therefore f_{V,W}(v,w)=\frac{1}{2\pi}\exp[-\frac{1}{2}(v^2\cos^2 w+v^2\sin^2 w)]v=\frac{v}{2\pi}\exp(-\frac{v^2}{2})$$

for $v\geq 0$ and $0\leq w<2\pi$. Otherwise, $f_{V,W}(v,w)=0$.

The p.d.f. of a Rayleigh $R.V.$ X is $f(x)=\dfrac{x}{\sigma^2}\exp\{-\dfrac{x^2}{2\sigma^2}\},x\geq 0$

We can conclude that V and W are independent.

V is a Rayleigh $R.V.$ and W is a uniform $R.V.$ over $(0,2\pi)$.

主观题 10分

设置

Suppose that X_1 and X_2 have a continuous joint p.d.f. as follows:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f. of $Y = X_1 X_2$.

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作答

Ex10-Solution

Suppose that X_1 and X_2 have a continuous joint p.d.f. as follows:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f. of $Y=X_1X_2$.

Sol: Let $Z=X_1$.

$Y/Z=X_2$. $0<y<z<1$.

$$\because f_{V,W}(v,w) = f_{X,Y}(h_1(v,w), h_2(v,w))\,|J(v,w)|,\ J(v,w) = \det\begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix}.$$

$$J = \det\begin{bmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{bmatrix} = \det\begin{bmatrix} 0 & 1 \\ \frac{1}{z} & -\frac{y}{z^2} \end{bmatrix} = -\frac{1}{z}.$$

For $0<y<z<1$, the joint p.d.f. of Y and Z is:

$$g(y,z) = f\left(z, \frac{y}{z}\right)|J| = \left(z + \frac{y}{z}\right)\left(\frac{1}{z}\right).$$

It follows that for $0<y<1$, the marginal p.d.f. of Y is

$$g_1(y) = \int_y^1 g(y,z)\,dz = 2(1-y).$$

主观题 10分

设置

Suppose that the *R.V.s* X, Y and Z have the following joint p.d.f.

$$f(x,y,z)=\begin{cases}2 & \text{for } 0<x<y<1 \text{ and } 0<z<1,\\ 0 & \text{otherwise.}\end{cases}$$

What is $\Pr(3X>Y|1<4Z<2)$?

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作答

Ex11-Solution

Suppose that the *R.V.s X, Y* and *Z* have the following joint p.d.f.

$$f(x, y, z) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \text{ and } 0 < z < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\Pr(3X>Y|1<4Z<2)$?

Sol: Since $f(x,y,z)$ can be factored in form $g(x,y)h(z)$, it follows that Z is independent of X and Y.

Thus, $\Pr(3X>Y|1<4Z<2)=\Pr(3X>Y)$.

We can obtain that

$g(x,y) = \int_0^1 2dz=2$, for $0<x<y<1$.

Therefore,

$$\Pr(3X > Y) = \int_0^1 \int_{y/3}^{y} 2dxdy = \frac{2}{3}.$$