（1）设 $M(x_0,y_0)$ 为区域 $D$ 上的一个点，问 $h(x,y)$ 在该点沿平面上什么方向的方向导数最大？若记此方向导数的最大值为 $g(x_0,y_0)$，试写出 $g(x_0,y_0)$ 的表达式；

（2）现欲利用此小山开展攀岩活动，为此需在山脚寻找一上山坡度最大的点作为攀登的起点，试确定攀登起点的位置.

**解** （1）由方向导数及梯度的概念可知 $h(x,y)$ 在 $M(x_0,y_0)$ 处沿 $\operatorname{grad} h(x_0,y_0)=\{-2x_0+y_0,-2y_0+x_0\}$ 的方向导数最大，且此最大的方向导数 $g(x_0,y_0)=\|\operatorname{grad} h(x_0,y_0)\|=\sqrt{5x_0^2+5y_0^2-8x_0y_0}$.

（2）也就是求 $[g(x,y)]^2=5x^2+5y^2-8xy$ 在 $D$ 内满足约束条件 $x^2+y^2-xy=75$ 的最大值点. 构造 Lagrange 函数，令 $L=5x^2+5y^2-8xy+\lambda(x^2+y^2-xy-75)$. 由

$$\begin{cases}L_x=10x-8y+\lambda(2x-y)=0,\\L_y=10y-8x+\lambda(2y-x)=0,\\L_\lambda=x^2+y^2-xy-75=0\end{cases}$$

可得驻点 $M_1(5,-5)$，$M_2(-5,5)$，$M_3(5\sqrt{3},5\sqrt{3})$，$M_4(-5\sqrt{3},-5\sqrt{3})$. 而 $g^2|_{M_1}=g^2|_{M_2}=450$，$g^2|_{M_3}=g^2|_{M_4}=150$. 故攀登起点为 $M_1(5,-5)$ 或 $M_2(-5,5)$.

## 习 题 5.5

### (A)

1. 用导数定义求下列向量值函数的导数.

（1）$f:\mathbf{R}^n\to\mathbf{R}^m$ 是常向量；

（2）$\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{Ax}+\boldsymbol{a}$，其中 $(a_{ij})_{m\times n}$ 为常矩阵，$\boldsymbol{a}\in\mathbf{R}^m$ 为常向量.

**解** （1）$\forall\boldsymbol{x}\in\mathbf{R}^n$，$\boldsymbol{f}(\boldsymbol{x})$ 为常向量，则 $\mathrm{d}\boldsymbol{f}(\boldsymbol{x})=(a_{ij})_{m\times n}\mathrm{d}\boldsymbol{x}$，其中 $a_{ij}=0$，$i=1,2,\cdots,m$，$j=1,2,\cdots,n$. 故 $\boldsymbol{Df}(\boldsymbol{x})=(a_{ij})_{m\times n}$.

（2）由
$$\begin{aligned}\mathrm{d}\boldsymbol{f}(\boldsymbol{x})=\mathrm{d}(\boldsymbol{Ax}+\boldsymbol{a})&=\Big(\mathrm{d}\Big(\sum_{j=1}^n a_{1j}x_j+a_1\Big),\mathrm{d}\Big(\sum_{j=1}^n a_{2j}x_j+a_2\Big),\\&\qquad\cdots,\mathrm{d}\Big(\sum_{j=1}^n a_{mj}x_j+a_m\Big)\Big)^{\mathrm{T}}\\&=\Big(\sum_{j=1}^n a_{1j}\mathrm{d}x_j,\sum_{j=1}^n a_{2j}\mathrm{d}x_j,\cdots,\sum_{j=1}^n a_{mj}\mathrm{d}x_j\Big)^{\mathrm{T}}=(a_{ij})_{m\times n}\mathrm{d}\boldsymbol{x}\\&=\boldsymbol{A}\mathrm{d}\boldsymbol{x},\end{aligned}$$
其中 $a_1,a_2,\cdots,a_m$ 为 $\boldsymbol{a}$ 的 $m$ 个分量，$\mathrm{d}\boldsymbol{x}=(\mathrm{d}x_1,\mathrm{d}x_2,\cdots,\mathrm{d}x_n)^{\mathrm{T}}$. 故 $\boldsymbol{Df}(\boldsymbol{x})=\boldsymbol{A}$.

3. 求下列向量值函数在给定点的导数.

(2) $f(x,y)=(\arctan x,\mathrm{e}^{xy})^{\mathrm{T}}$，在$(1,0)^{\mathrm{T}}$处；

(4) $f(x,y,z)=\left(\sin(x^2-y^2),\ln(x^2+z^2),\dfrac{1}{\sqrt{y^2+z^2}}\right)^{\mathrm{T}}$，在$(1,1,1)^{\mathrm{T}}$处.

**解**　(2) $\boldsymbol{D}f(1,0)=\begin{pmatrix}\dfrac{1}{1+x^2} & 0\\ y\mathrm{e}^{xy} & x\mathrm{e}^{xy}\end{pmatrix}\Bigg|_{\substack{x=1\\y=0}}=\begin{pmatrix}\dfrac{1}{2} & 0\\ 0 & 1\end{pmatrix}.$

$$(4)\ \boldsymbol{D}f(1,1,1)=\begin{pmatrix}2x\cos(x^2-y^2) & -2y\cos(x^2-y^2) & 0\\ \dfrac{2x}{x^2+z^2} & 0 & \dfrac{2z}{x^2+z^2}\\ 0 & -y(y^2+z^2)^{-\frac{3}{2}} & -z(y^2+z^2)^{-\frac{3}{2}}\end{pmatrix}_{\substack{x=1\\y=1\\z=1}}$$

$$=\begin{pmatrix}2 & -2 & 0\\ 1 & 0 & 1\\ 0 & -\dfrac{1}{2\sqrt{2}} & -\dfrac{1}{2\sqrt{2}}\end{pmatrix}.$$

4. 设$\boldsymbol{r}=\boldsymbol{r}(t)$为空间$\mathbf{R}^3$中动点$(x(t),y(t),z(t))^{\mathrm{T}}$的向径，证明$\|\boldsymbol{r}(t)\|=C$($C$为常数)$\Leftrightarrow\langle\boldsymbol{r}'(t),\boldsymbol{r}(t)\rangle=0$.

**证明**　$\|\boldsymbol{r}(t)\|=C\Leftrightarrow\|\boldsymbol{r}(t)\|^2=C^2\Leftrightarrow\dfrac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}(t)\|^2=0\Leftrightarrow\dfrac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{r}(t),\boldsymbol{r}(t)\rangle=$

$0\xLeftrightarrow{\text{定理5.2(2)}}2\langle\boldsymbol{r}(t),\boldsymbol{r}'(t)\rangle=0\Leftrightarrow\langle\boldsymbol{r}(t),\boldsymbol{r}'(t)\rangle=0.$

或　由$\boldsymbol{r}(t)=\{x(t),y(t),z(t)\}$知$\boldsymbol{r}'(t)=\{x'(t),y'(t),z'(t)\}$，于是$\langle\boldsymbol{r}(t),\boldsymbol{r}'(t)\rangle=x(t)x'(t)+y(t)y'(t)+z(t)z'(t)$.

故$\|\boldsymbol{r}(t)\|=C\Leftrightarrow x^2(t)+y^2(t)+z^2(t)=C^2$

$\Leftrightarrow 2x(t)x'(t)+2y(t)y'(t)+2z(t)z'(t)=0$

$\Leftrightarrow\langle\boldsymbol{r}(t),\boldsymbol{r}'(t)\rangle=0.$

5. 求由下列方程组所确定的隐函数的导数.

(1) $\begin{cases}xu+yv=0,\\ yu+xv=0,\end{cases}$ 求$\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial y}$；

(2) $\begin{cases}u+v+w=x,\\ uv+vw+wu=y,\\ uvw=z,\end{cases}$ 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}$.

**解**　(1) 将$u,v$看作$x,y$的隐函数，两端分别对$x$求导，得

$$\begin{cases}u+xu_x+yv_x=0,\\ yu_x+v+xv_x=0,\end{cases}\text{解之得 } u_x=\frac{\partial u}{\partial x}=\frac{-ux+yv}{x^2-y^2}.$$

原方程组两端对 $y$ 求偏导得$\begin{cases} xu_y + yv_y + v = 0, \\ yu_y + xv_y + u = 0, \end{cases}$解之得 $v_y = \dfrac{\partial v}{\partial y} = \dfrac{-ux + yv}{x^2 - y^2}$.

（2）方程两端求全微分，

得
$$\begin{cases} du + dv + dw = dx, \\ (u+w)dv + (v+w)du + (v+u)dw = dy, \\ vwdu + uwdv + uvdw = dz. \end{cases}$$

解之得
$$\begin{cases} du = \dfrac{v-w}{J}[u^2dx - udy + dz], \\ dv = \dfrac{u-w}{J}[-v^2dx + vdy - dz], \\ dw = \dfrac{u-v}{J}(w^2dx - wdy + dz). \end{cases}$$

故 $\dfrac{\partial u}{\partial x} = \dfrac{u^2}{(u-v)(u-w)}, \dfrac{\partial u}{\partial y} = \dfrac{-u}{(u-v)(u-w)}$,

$\dfrac{\partial u}{\partial z} = \dfrac{1}{(u-v)(u-w)}$.

其中 $J = (u-v)(u-w)(v-w)$.

6. 设函数 $u = u(x)$ 由方程组 $u = f(x,y,z), \varphi(x^2, e^y, z) = 0, y = \sin x$ 确定，其中 $f,\varphi$ 都具有连续的一阶偏导数，且$\dfrac{\partial \varphi}{\partial z} \neq 0$，求$\dfrac{du}{dx}$.

**解** 由一阶全微分形式不变性可得

$$\begin{cases} du = f_x dx + f_y dy + f_z dz, & ① \\ \varphi_1 dx^2 + \varphi_2 de^y + \varphi_3 dz = 2x\varphi_1 dx + e^y\varphi_2 dy + \varphi_3 dz = 0, & ② \\ dy = d\sin x. & ③ \end{cases}$$

将第③式代入第②式可得 dz. 再将 dy 与 dz 的表达式代入第①式可得

$$du = \left[f_1 + \cos x \cdot f_2 - \frac{f_3}{\varphi_3}(2x\varphi_1 + e^y\varphi_2\cos x)\right]dx$$

所以
$$\frac{du}{dx} = f_1 + f_2\cos x - \frac{f_3}{\varphi_3}(2x\varphi_1 + e^y\varphi_2\cos x).$$

8. 设方程组 $F(y-x, y-z) = 0, G\left(xy, \dfrac{z}{y}\right) = 0$ 确定隐函数 $x = x(y), z = z(y)$，其中 $F,G$ 都具有一阶连续偏导数，求$\dfrac{dx}{dy}$.

**解**　利用一阶全微分形式不变性得

$$\begin{cases} F_1\mathrm{d}(y-x)+F_2\mathrm{d}(y-z)=0, \\ G_1\mathrm{d}xy+G_2\mathrm{d}\dfrac{z}{y}=0. \end{cases}$$

即
$$\begin{cases} (F_1+F_2)\mathrm{d}y-F_1\mathrm{d}x-F_2\mathrm{d}z=0, \\ \left(xG_1-\dfrac{zG_2}{y^2}\right)\mathrm{d}y+yG_1\mathrm{d}x+\dfrac{1}{y}G_2\mathrm{d}z=0. \end{cases}$$

解之得

$$\mathrm{d}x=\frac{\dfrac{1}{y}F_1G_2+xF_2G_1+\dfrac{1}{y}\left(1-\dfrac{z}{y}\right)F_2G_2}{\dfrac{1}{y}F_1G_2-yG_1F_2}\mathrm{d}y$$

即
$$\frac{\mathrm{d}x}{\mathrm{d}y}=\left[\frac{1}{y}F_1G_2+xF_2G_1+\frac{1}{y}\left(1-\frac{z}{y}\right)F_2G_2\right]\Big/\left[\frac{1}{y}F_1G_2-yG_1F_2\right].$$

9. 设$\boldsymbol{f}=(f_1,f_2)^{\mathrm{T}}$,$f_1(x_1,x_2,x_3,y_1,y_2)=2e^{y_1}+x_1y_2-4x_2+3$,$f_2(x_1,x_2,x_3,y_1,y_2)=y_2\cos y_1-6y_1+2x_1-x_3$,$\boldsymbol{x}_0=(3,2,7)^{\mathrm{T}}$,$\boldsymbol{y}_0=(0,1)^{\mathrm{T}}$. 求由向量方程$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=0$所确定的隐函数$\boldsymbol{y}=\boldsymbol{g}(\boldsymbol{x}_0)$在$\boldsymbol{x}_0$处的导数,其中$\boldsymbol{x}=(x_1,x_2,x_3)^{\mathrm{T}}$,$\boldsymbol{y}=(y_1,y_2)^{\mathrm{T}}$.

**解**　向量方程两边求微分得

$$\begin{pmatrix} y_2 & -4 & 0 & 2e^{y_1} & x_1 \\ 2 & 0 & -1 & -6-y_2\sin y_1 & \cos y_1 \end{pmatrix}\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \\ \mathrm{d}y_1 \\ \mathrm{d}y_2 \end{pmatrix}=0.$$

在$\boldsymbol{x}_0=(3,2,7)^{\mathrm{T}}$,$\boldsymbol{y}_0=(0,1)^{\mathrm{T}}$有

$$\begin{pmatrix} 1 & -4 & 0 & 2 & 3 \\ 2 & 0 & -1 & -6 & 1 \end{pmatrix}(\mathrm{d}x_1\quad \mathrm{d}x_2\quad \mathrm{d}x_3\quad \mathrm{d}y_1\quad \mathrm{d}y_2)^{\mathrm{T}}=0.$$

解之得

$$\mathrm{d}\boldsymbol{y}=\begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \end{pmatrix}=\begin{pmatrix} \dfrac{1}{4} & \dfrac{1}{5} & -\dfrac{3}{20} \\ -\dfrac{1}{2} & \dfrac{6}{5} & \dfrac{1}{10} \end{pmatrix}\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{pmatrix}$$

故
$$\mathrm{D}\boldsymbol{g}(\boldsymbol{x}_0)=\begin{pmatrix}\frac{1}{4} & \frac{1}{5} & -\frac{3}{20}\\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10}\end{pmatrix}.$$

**(B)**

1. 设$\boldsymbol{f}$: $U(\boldsymbol{x}_0)\subseteq\mathbf{R}^n\to\mathbf{R}^m$,其中$\boldsymbol{f}=(f_1,\cdots,f_m)^{\mathrm{T}}$,$\boldsymbol{x}_0\in\mathbf{R}^n$,$\boldsymbol{x}=(x_1,\cdots,x_n)^{\mathrm{T}}\in\mathbf{R}^n$,若$\frac{\partial f_i(\boldsymbol{x}_0)}{\partial x_j}(i=1,2,\cdots,m,j=1,\cdots,n)$在$\boldsymbol{x}_0$的某邻域内存在,且在$\boldsymbol{x}_0$处连续,证明$\boldsymbol{f}$在$x_0$处可微.

**证明** 由于$\frac{\partial f_i(\boldsymbol{x}_0)}{\partial x_j}i=1,2,\cdots,m,j=1,2,\cdots,n$在$\boldsymbol{x}_0$的某邻域内存在,且在$\boldsymbol{x}_0$处连续,由定理3.2知数量值函数$f_i(\boldsymbol{x})$在$\boldsymbol{x}_0$处可微,$i=1,2,\cdots,m$. 即向量值函数的每一个分量$f_i(\boldsymbol{x})(i=1,\cdots,m)$在$\boldsymbol{x}_0$处可微,由向量值函数可微的定义知$\boldsymbol{f}(\boldsymbol{x})$在$\boldsymbol{x}_0$处可微.

3. 设$\boldsymbol{x}_0,\boldsymbol{y}_0\in\mathbf{R}^n$,$S$是联结$\boldsymbol{x}_0,\boldsymbol{y}_0$的线段,$\Omega$是包含$S$的区域,$\boldsymbol{f}=(f_1,\cdots,f_m)^{\mathrm{T}}:\Omega\to\mathbf{R}^m$连续,$\boldsymbol{f}$在$S$上($\boldsymbol{x}_0,\boldsymbol{y}_0$可以除外)可微,则存在$\boldsymbol{\xi}_1\cdots,\boldsymbol{\xi}_m\in S$使

$$\boldsymbol{f}(\boldsymbol{y}_0)-\boldsymbol{f}(\boldsymbol{x}_0)=\left(\frac{\partial f_i(\boldsymbol{\xi}_i)}{\partial x_j}\right)_{m\times n}(\boldsymbol{y}_0-\boldsymbol{x}_0)$$

**证明** 令$\Delta\boldsymbol{x}=\boldsymbol{y}_0-\boldsymbol{x}_0$. 首先证明对$\boldsymbol{f}$的每个分量($n$元数量值函数)$f_i(\boldsymbol{x})$,存在$\boldsymbol{\xi}_i\in S,i=1,2,\cdots,m$,使

$$f_i(\boldsymbol{y}_0)-f_i(\boldsymbol{x}_0)=f_i(\boldsymbol{x}_0+\Delta\boldsymbol{x})-f_i(\boldsymbol{x}_0)=\langle\nabla f_i(\boldsymbol{\xi}_i),\Delta\boldsymbol{x}\rangle.$$

为此考虑一元函数$\varphi_i(t)=f_i(\boldsymbol{x}_0+t\Delta\boldsymbol{x}),0\leqslant t\leqslant1$.

则$\varphi_i(1)=f_i(\boldsymbol{x}_0+\Delta\boldsymbol{x})=f_i(\boldsymbol{y}_0),\varphi_i(0)=f_i(\boldsymbol{x}_0)$.

由于$\boldsymbol{f}$在$S$上可微,从而复合函数$\varphi_i(t)=f_i(\boldsymbol{x}_0+t\Delta\boldsymbol{x})$在$[0,1]$对$t$可导,由一元函数的Lagrange公式知,存在$\eta_i\in(0,1)$,使$\varphi_i(1)-\varphi_i(0)=\left.\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right|_{t=\eta_i}$.

$$f_i(\boldsymbol{x}_0+\Delta\boldsymbol{x})-f_i(\boldsymbol{x}_0)=\sum_{j=1}^{n}\frac{\partial f_i(\boldsymbol{x}_0+\eta_i\Delta\boldsymbol{x})}{\partial x_j}\Delta x_j=\langle\nabla f_i(\boldsymbol{x}_0+\eta_i\Delta\boldsymbol{x}),\Delta\boldsymbol{x}\rangle.$$

令$\boldsymbol{\xi}_i=\boldsymbol{x}_0+\eta_i\Delta\boldsymbol{x}$,由于$\eta_i\in(0,1)$,则$\boldsymbol{\xi}_i\in S$,其中

$$(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)^{\mathrm{T}}=\Delta\boldsymbol{x}=\boldsymbol{y}_0-\boldsymbol{x}_0,i=1,2,\cdots,m.$$

故$\exists \boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\cdots,\boldsymbol{\xi}_n\in S$，使

$$\boldsymbol{f}(\boldsymbol{x}_0+\Delta\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{x}_0)=\begin{pmatrix} f_1(\boldsymbol{x}_0+\Delta\boldsymbol{x})-f_1(\boldsymbol{x}_0)\\ \vdots \\ f_m(\boldsymbol{x}_0+\Delta\boldsymbol{x})-f_m(\boldsymbol{x}_0)\end{pmatrix}=\begin{pmatrix} \sum\limits_{j=1}^{n}\dfrac{\partial f_1(\boldsymbol{\xi}_1)}{\partial x_j}\Delta x_j\\ \vdots \\ \sum\limits_{j=1}^{n}\dfrac{\partial f_m(\boldsymbol{\xi}_m)}{\partial x_j}\Delta x_j\end{pmatrix}$$

$$=\left(\frac{\partial f_i(\boldsymbol{\xi}_i)}{\partial x_j}\right)_{m\times n}\Delta\boldsymbol{x}.$$

注意到$\boldsymbol{y}_0=\boldsymbol{x}_0+\Delta\boldsymbol{x}$，$\Delta\boldsymbol{x}=\boldsymbol{y}_0-\boldsymbol{x}_0$，本题得证.

## 习　题　5.6

### (A)

2. 求曲线$\boldsymbol{r}=(t,-t^2,t^3)$上的与平面$x+2y+z=4$平行的切线方程.

**解**　设曲线上$\boldsymbol{r}_0=(t_0,-t_0^2,t_0^3)$的切线与平面$x+2y+z=4$平行，则$\boldsymbol{r}_0$处的切向量$\dot{\boldsymbol{r}}_0=(1,-2t_0,3t_0^2)$与平面的法向量$(1,2,1)$垂直，即$1-4t_0+3t_0^2=0$. 解之得$t_0=1$或$t_0=\dfrac{1}{3}$. 则所求切线过$\boldsymbol{r}_0=(1,-1,1)$或$\boldsymbol{r}_0=\left(\dfrac{1}{3},-\dfrac{1}{9},\dfrac{1}{27}\right)$. 所求切线为

$$\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z-1}{3}\text{和}\frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{9}}{-\frac{2}{3}}=\frac{z-\frac{1}{27}}{\frac{1}{3}}.$$

3. 证明螺线$\boldsymbol{r}=(a\cos\theta,a\sin\theta,k\theta)$上任一点的切线与$Oz$轴交成定角.

**证明**　$P_0(x_0,y_0,z_0)$为螺线上任一点，则$x_0=a\cos\theta_0,y_0=a\sin\theta_0,z_0=k\theta_0$. $P_0$处的切向量为$\boldsymbol{r}'=(-a\sin\theta,a\cos\theta,k)$与$z$轴夹角为$\alpha$，则$\cos\alpha=\dfrac{\langle\boldsymbol{r}',\boldsymbol{k}\rangle}{\|\boldsymbol{r}'\|\cdot\|\boldsymbol{k}\|}=\dfrac{k}{\sqrt{a^2+k^2}}$，其中$\boldsymbol{k}=(0,0,1)$为$z$轴正向的单位向量. 即$\alpha$与$P_0$无关的常数. 本题得证.

4. 求下列平面曲线的弧长.

(2) $x=x(t)=\displaystyle\int_0^{t^2}\sqrt{1+u}\,\mathrm{d}u,\ y=y(t)=\int_0^{t^2}\sqrt{1-u}\,\mathrm{d}u$，$0\leqslant t\leqslant 1$；

**解**　$S=\displaystyle\int_0^1\sqrt{[\dot{x}(t)]^2+[\dot{y}(t)]^2}\,\mathrm{d}t$