Chapter 2
Conditional Probability

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Outlines

- The Definition of Conditional Probability
- The Multiplication Rule for Conditional Prob.
- Conditional Probability and Partitions
- Independent Events
- Bayes' Theorem

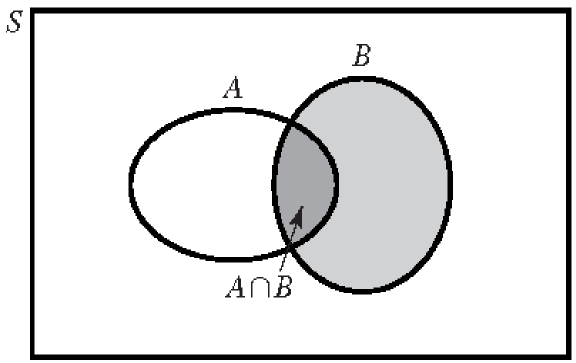
Conditional Probability Definition-1

- **Definition 2.1.1 <u>Conditional Probability</u>**

An event B has occurred. Compute the prob. of another event A taking into account that B has occurred. The new prob. of A after event B is the ***conditional prob. of the event A given that the event B has occurred***.

- Denoted $\Pr(A|B)$. If $\Pr(B) > 0$,

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$



- $\Pr(A|B)$ is not defined if $\Pr(B) = 0$

The sample space is changed from *S* to *B*

Conditional Probability Definition-2

- **Ex1 (Book Ex 2.1.2) Lottery Ticket.**

Consider a lottery game that six numbers are drawn without replacement from a bin containing the numbers 1–30.Suppose that you hold a lottery with the numbers 1, 14,15, 20, 23, and 27. You watch the television, and one number, 15, being drawn. Now what is the prob. that your ticket will be a winner?

Sol: $A=\{1,14,15,20,23,27 \text{are drawn}\}$

$B=\{\text{one of the numbers drawn is 15}\}$

$$\Pr(A\mid B)=\frac{\Pr(A\cap B)}{\Pr(B)}$$

$$=8.4\times 10^{-6}$$

$$\Pr(B)=\frac{\binom{29}{5}}{\binom{30}{6}}=0.2,\ \Pr(A\cap B)=\Pr(A)=\frac{1}{\binom{30}{6}}=1.68\times 10^{-6}$$

Conditional Probability Definition-3

- **Ex2 (Book Ex 2.1.3) Rolling Dice.**

Two dice were rolled and it was observed that the sum T of the two numbers was odd. Determine the prob. that T was less than 8.

Sol: $T = \{\text{the sum of the two numbers was odd}\}$

$A = \{\text{T was less than 8}\}$

Sample Space:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Numbers of $A \cap T$:

12

Numbers of T:

18

$\Pr(A \mid T) = 12/18 = 2/3$

The Multiplication Rule - 1

- **Theorem 2.1.1** Let A and B be events. If $\Pr(B)>0$, then $\Pr(A \cap B) = \Pr(B)\Pr(A \mid B)$, if $\Pr(A)>0$, then $\Pr(A \cap B) = \Pr(A)\Pr(B \mid A)$.
- **Ex 3 (Book Ex 2.1.6) Selecting Two Balls.**

Select two balls at random without replacement from a box containing ***r*** red balls and ***b*** blue balls. What's the prob. that the 1^{st} ball is red and the 2^{nd} ball is blue?

Sol:

Let A={the 1^{st} ball is red}, $\Pr(A) = r/(r+b)$

B={the 2^{nd} ball is blue}, $\Pr(B \mid A) = b/(r+b-1)$

$$\Pr(A \cap B) = \Pr(A)\Pr(B \mid A) = \frac{r}{r+b} \cdot \frac{b}{r+b-1}$$

The Multiplication Rule - 2

- **Theorem 2.1.2** Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\Pr(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$. Then

$\Pr(A_1 \cap A_2 \cap \ldots \cap A_n) =$

$\Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\ldots\Pr(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

Proof

- **Ex4 (Book Ex 2.1.7) Selecting Four Balls.**

Suppose that four balls are selected one at a time without replacement from a box containing ***r*** red balls and ***b*** blue balls ($r \geq 2$, $b \geq 2$). What's the prob. to obtain [R B R B]?

$\Pr(R_1 \cap B_2 \cap R_3 \cap B_4)$

$= \Pr(R_1)\Pr(B_2|R_1)\Pr(R_3|R_1 \cap B_2)\Pr(B_4|R_1 \cap B_2 \cap R_3)$

$$= \frac{r}{r+b} \cdot \frac{b}{r+b-1} \cdot \frac{r-1}{r+b-2} \cdot \frac{b-1}{r+b-3}$$

The Multiplication Rule - 3

- **Theorem 2.1.3** Suppose that $A_1, A_2, \ldots, A_n, B$ are events such that $\Pr(B) > 0$, and $\Pr(A_1 \cap A_2 \cap \ldots \cap A_{n-1} | B) > 0$. Then

$\Pr(A_1 \cap A_2 \cap \ldots \cap A_n | B) =$

$\Pr(A_1 | B) \Pr(A_2 | A_1 \cap B) \ldots \times \Pr(A_n | A_1 \cap A_2 \ldots \cap A_{n-1} \cap B)$

Proof

$$\frac{\Pr(A_1 \cap B)}{\Pr(B)} \cdot \frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(A_1 \cap B)} \cdots \frac{\Pr(A_1 \cap A_2 \cdots \cap A_n \cap B)}{\Pr(A_1 \cap A_2 \cdots \cap A_{n-1} \cap B)}$$

$$= \frac{\Pr(A_1 \cap A_2 \cdots \cap A_n \cap B)}{\Pr(B)}$$

$$= \Pr(A_1 \cap A_2 \cdots \cap A_n | B)$$

Prove $\Pr(A^c | B) = 1 - \Pr(A | B)$

Partitions

◆ **Definition 2.1.2 <u>Partition</u>**

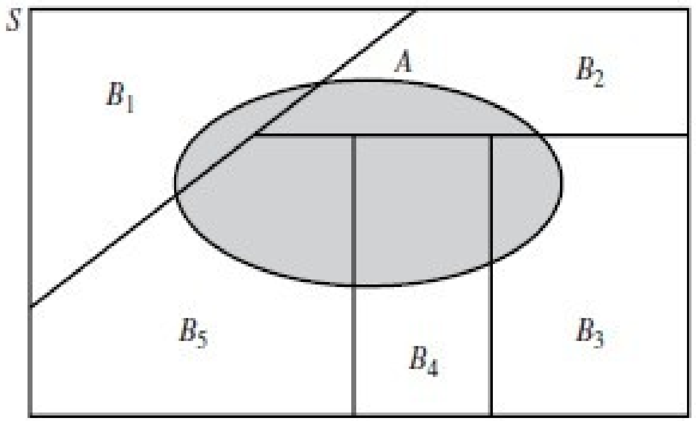
Let S denote the sample space of some experiment, and consider k events $B_1,\ldots,B_k$ are **disjoint** and $\bigcup_{i=1}^{k} B_i = S$. It is said that these events form a partition of S .

◆ **Theorem 2.1.4 <u>Law of total probability</u>**

Suppose that the events $B_1,\ldots,B_k$ form a partition of the space S and $\Pr(B_j)>0$ for $j=1,\ldots,k$. Then, for every event A in S,

$$\Pr(A) = \sum_{j=1}^{k} \Pr(B_j)\Pr(A \mid B_j).$$

Proof



$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_k \cap A)$$

$$\Pr(A) = \sum_{j=1}^{k} \Pr(B_j \cap A)$$

Ex5 (Book Ex 2.1.9)

- **Selecting Bolts.** Two boxes contain long bolts and short bolts. One box contains 60 long bolts and 40 short bolts, and that the other box contains 10 long bolts and 20 short bolts. Suppose that one box is selected at random and a bolt is then selected at random from that box. determine the prob. that this bolt is long.

Sol: let L be the event that a long bolt is selected.

Let B_i be the event that the *ith* Box is selected, i=1 or 2.

$$\Pr(L) = \Pr(B_1)\Pr(L \mid B_1) + \Pr(B_2)\Pr(L \mid B_2)$$
$$= \frac{1}{2}\cdot\frac{60}{100} + \frac{1}{2}\cdot\frac{10}{30}$$
$$= \frac{7}{15}$$

Ex6

- A box contains 3 cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. One card is selected from the box at random, and the color on one side is observed. If this side is green, what is the probability that the other side of the card is also green?
- Sol: Let B be the event that the observed side is green. Let A be the event that the select card is green on both sides.

$$\Pr(B) = \frac{1}{2}; \quad \Pr(A \cap B) = \Pr(A) = \frac{1}{3};$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{2}{3}.$$

Independent Events - 1

- **Ex7 (Book Ex2.2.1) Tossing Coins.** A fair coin is tossed twice. Calculate the prob. of event A={H on second toss} and the prob. of event B={T on first toss}.

Sol: $S=\{HH, HT, TH, TT\}$, $\Pr(A)=1/2$, $\Pr(B)=1/2$,

$\Pr(A\cap B)=1/4$,

$\Pr(A|B) = 1/2 = \Pr(A)$.

Learning B has occurred does not changed the prob. of A, we say that ***A and B are independent***.

Ex7 again We toss a coin and then roll a die. Tossing the coin is in isolation of the rolling a die. Let A be the event that that the die shows 3 and let B be the event that the coin lands with heads up.

$\Pr(A|B) = \Pr(A) = 1/6.$

Independent Events - 2

- **Definition 2.2.1 Independent Events.** Suppose $\Pr(A)>0$ and $\Pr(B)>0$. Two events A and B are ***independent*** if

$$\Pr(A\cap B) = \Pr(A)\Pr(B)$$

Necessary and sufficient condition

A and B are ***independent*** if and only if $\Pr(A|B)=\Pr(A)$ and $\Pr(B|A)=\Pr(B)$.

Ex8(Book Ex 2.2.2) Machine Operation. Two machines m1 & m2 operated independently. A is the event that m1 will become inoperative during a given 8-hour period. B is the event that m2 will become inoperative during the same period. $\Pr(A)=1/3$, $\Pr(B)=1/4$. Determine the prob. that at least one m will become inoperative given 8 hours.

Sol: $\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$

$$=1/3+1/4-(1/3)\times(1/4)=1/2$$

Independent Events - 3

- **Ex9(Book Ex 2.2.3) Rolling a Die.** A balanced die is rolled. A is the event that an even number is obtained, and B is the event that one of the numbers 1,2,3,or 4 is obtained. Show that events A and B are independent.

Sol: $\Pr(A) = 1/2$;

$\Pr(B) = 2/3$;

$\Pr(A \cap B) = 1/3$;

Hence, $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

A and B are independent.

Independence of Complements

◆ **Theorem 2.2.1** If two events A and B are independent, then the events A and B^C are also independent.

Proof $\quad A = (A \cap B^C) \cup (A \cap B)$

$$\Pr(A \cap B^C) = \Pr(A) - \Pr(A \cap B)$$
$$= \Pr(A) - \Pr(A)\Pr(B)$$
$$= \Pr(A)\Pr(B^C)$$

Therefore, the events A and B^C are independent.

Note: B and A^C, A^C and B^C are also independent.

Proof $\Pr(A^C)\Pr(B^C) = [1-\Pr(A)][1-\Pr(B)]$

$$= 1-\Pr(A \cup B)$$
$$= \Pr[(A \cup B)^C]$$
$$= \Pr(A^C \cap B^C)$$

Independence of Several Events

- **Definition 2.2.2 (Mutually) Independent Events**

The k events $A_1, \ldots, A_k$ are ***independent (or mutually independent)*** if, for every subset $A_{i_1}, \ldots, A_{i_j}$ of j of these events (j=2,3, ...,k)

$$\Pr(A_{i_1} \cap \ldots \cap A_{i_j}) = \Pr(A_{i_1}) \ldots \Pr(A_{i_j})$$

e.g., $S=\{A,B,C\}$

$$\left.\begin{aligned} \Pr(A \cap B) &= \Pr(A)\Pr(B) \\ \Pr(A \cap C) &= \Pr(A)\Pr(C) \\ \Pr(B \cap C) &= \Pr(B)\Pr(C) \end{aligned}\right\} \text{pair-wise independent}$$

$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$$

Pairwise Independence

- **Ex10 (Book Ex 2.2.4)** A fair coin is tossed twice. S=?

S={HH, HT, TH, TT} is a simple sample space. Define

A = {H on the first toss} ={HH, HT}

B = {H on the second toss} ={HH, TH}

C = {Both tosses the same} ={HH, TT}

$\Pr(A) = \Pr(B) = \Pr(C) = 1/2$

$A \cap B = B \cap C = A \cap C = A \cap B \cap C = \{HH\}$

$\Pr(A \cap B) = \Pr(B \cap C) = \Pr(A \cap C) = \Pr(A \cap B \cap C) = 1/4$

$\Pr(A \cap B \cap C) \neq \Pr(A)\Pr(B)\Pr(C)$

A, B, and C are ***pairwise independent***, but **all three events are not independent**.

Ex10 (Ex book 2.2.5&2.2.6&2.2.8)

- **Inspecting Items.** A machine produces a defective item with prob. p $(0 < p < 1)$ and produces a nondefective item with prob. $1 - p$. Six items produced by the machine are selected at random and inspected, and that the results (defective or nondefective) for these six items are independent. 1) Determine the prob. that exactly two of the six items are defective.

Sol:
$$\binom{6}{2} p^2 (1-p)^4$$

2) Determine the prob. that at least 1 item defective.

Sol:
$$1-(1-p)^6$$

3) Items are inspected one at a time until exactly 5 defective items obtained. The prob. that exactly n items $(n \geq 5)$ must be selected to obtain the five detectives?

$$C_{n-1,4} p^4 (1-p)^{n-5} \cdot p = C_{n-1,4} p^5 (1-p)^{n-5}$$

Ex11 (Ex book 2.2.7)

- **Tossing a Coin Until a Head Appears.** a fair coin is tossed until a head appears for the first time, and assume that the outcomes of the tosses are independent. determine the probability p_n that exactly n tosses will be required.

Sol: $$p_n = \left(\frac{1}{2}\right)^n, n = 1, \cdots, \infty$$

$$\sum_{n=1}^{\infty} p_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Prob. of obtaining an infinite sequence of tails without ever obtaining a head must be 0.

Independence and Conditional Prob.

- **Theorem 2.2.2** Let $A_1,\ldots, A_k$ be events such that $\Pr(A_1\cap\ldots\cap A_k) > 0$. Then $A_1,\ldots, A_k$ are independent if and only if, for every two disjoint subsets $\{i_1,\ldots, i_m\}$ and $\{j_1,\ldots, j_l\}$ of $\{1,\ldots,k\}$, we have

$$\Pr(A_{i_1}\cap\ldots\cap A_{i_m} \mid A_{j_1}\cap\ldots\cap A_{j_l}) = \Pr(A_{i_1}\cap\ldots\cap A_{i_m})$$

- k events are independent if and only if learning that some of the events occur does not change the prob. that any combination of the other events occurs.
- Two events A and B with positive prob. are independent if and only if $\Pr(A|B) = \Pr(A)$. Similar results hold for larger collections of independent events.

Conditionally Independent Events

Definition 2.2.3 Conditional Independence

We say that events $A_1, \ldots, A_k$ are conditionally independent given B if, for every subcollection $A_{i_1} \ldots A_{i_j}$ of j of these events (j=2,3, ...,k)

$$\Pr(A_{i_1} \cap \ldots \cap A_{i_j} \mid B) = \Pr(A_{i_1} \mid B) \ldots \Pr(A_{i_j} \mid B)$$

It is not necessary that $A_1, \ldots, A_k$ are conditionally independent given B^C.

Theorem 2.2.4 Suppose that A_1, A_2 and B are events such that $\Pr(A_1 \cap B) > 0$. Then A_1 and A_2 are conditionally independent given B if and only if

$$\Pr(A_2 \mid A_1 \cap B) = \Pr(A_2 \mid B)$$

Theorem 2.2.4 Proof

- For the "only if" direction, A_1 and A_2 are conditionally independent implies that

$$\Pr(A_1 \cap A_2 \mid B) = \Pr(A_2 \mid B)\Pr(A_1 \mid B)$$

$$\Pr(A_2 \mid B) = \frac{\Pr(A_1 \cap A_2 \mid B)}{\Pr(A_1 \mid B)}$$

$$= \frac{\Pr(A_1 \cap A_2 \cap B)/\Pr(B)}{\Pr(A_1 \cap B)/\Pr(B)}$$

$$= \Pr(A_2 \mid A_1 \cap B)$$

$$\therefore \Pr(A_2 \mid A_1 \cap B) = \Pr(A_2 \mid B)$$

Theorem 2.2.4 Proof

- For the "if" direction because $\Pr(A_2 \mid A_1 \cap B) = \Pr(A_2 \mid B)$

We have $$\frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(A_1 \cap B)} = \frac{\Pr(A_2 \cap B)}{\Pr(B)}$$

$$\therefore \frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(B)} = \frac{\Pr(A_1 \cap B)\Pr(A_2 \cap B)}{\Pr(B)\Pr(B)}$$

$$\therefore \Pr(A_1 \cap A_2 \mid B) = \Pr(A_2 \mid B)\Pr(A_1 \mid B)$$

Thus, A_1 and A_2 are conditionally independent given B.

Exclusive ≠ Independent - 1

- Definition 1.4.10 defines ***mutually exclusive events***.
- Definition 2.2.2 defines ***mutually independent events***.
- **Theorem 2.2.3** Let $n>1$ and $A_1, \ldots, A_n$ be events that are mutually exclusive. The events are also mutually independent if and only if all the events except possibly one of them has probability 0.

Proof

- For the "if" direction, assume at most 1 of the events has positive prob. The prob. product of every 2 collection or more is 0. $\because$ mutually exclusive, the intersection of every collection of size 2 or more has prob. 0.

$\therefore$ mutually independent.

Exclusive ≠ Independent - 2

◆For the “only if ” direction, assume that the events are mutually independent.

∵Mutually exclusive, the intersection of every collection of size 2 or more is empty and must have prob. 0.

∴The product of the probabilities of every collection of size 2 or more must be 0.

This means that at least one factor from every product of at least 2 probabilities must be 0.

∴ There can be no more than one prob. greater than 0, otherwise the product of the two nonzero probabilities would be nonzero.

Bayes' Theorem

◆ **Theorem 2.3.1 Bayes' thereom**. Let the events $B_1, \ldots, B_k$ **form a partition** of the space S such that $\Pr(B_j) > 0$ for $j = 1, \ldots, k$, and let A be an event such that $\Pr(A) > 0$. Then, for $i = 1, \ldots, k$,

$$\Pr(B_i \mid A) = \frac{\Pr(B_i)\Pr(A \mid B_i)}{\sum_{j=1}^{k} \Pr(B_j)\Pr(A \mid B_j)}.$$

Proof

Multiplication Rule

$$\frac{\Pr(B_i)\Pr(A \mid B_i)}{\sum_{j=1}^{k} \Pr(B_j)\Pr(A \mid B_j)} = \frac{\Pr(A \cap B_i)}{\Pr(A)}$$

Law of total probability

Ex12 (Book Ex 2.3.2)

Selecting Bolts. One box contains 60 long bolts and 40 short bolts, and that the other box contains 10 long bolts and 20 short bolts. let A be the event that a long bolt is selected. Let B_i be the event that the *ith* Box is selected, i=1 or 2. If we learn A has occurred, compute the conditional probabilities of the two boxes given A.

Sol:

$$\Pr(B_i \mid A) = \frac{\Pr(A \cap B_i)}{\Pr(A)} = \frac{\Pr(B_i)\Pr(A \mid B_i)}{\sum_{j=1}^{k} \Pr(B_j)\Pr(A \mid B_j)}$$

$$\Pr(B_1 \mid A) = \frac{(1/2)(3/5)}{(1/2)(3/5) + (1/2)(1/3)} = \frac{9}{14}$$

$$\Pr(B_2 \mid A) = 1 - \frac{9}{14} = \frac{5}{14}$$

events B_i form a partition of S

Ex13 (Book Ex 2.3.1&2.3.3)

- **Test for a Disease.** If a person has the disease, there is a prob. of 0.9 that the test will give a positive response; whereas, if a person does not have the disease, there is a prob. of 0.1 that the test will give a positive response. The chances of having the disease are 1 in 10,000. You take the test and learn that you had a positive response to the test. what is the prob. that you have the disease?

Sol: Let A be the event that the test response is positive.
Let B_1 be the event that you have the disease.
Let B_2 be the event that you do not have the disease.

$$\Pr(B_1 \mid A) = \frac{\Pr(B_1)\Pr(A \mid B_1)}{\sum_{j=1}^{2}\Pr(B_j)\Pr(A \mid B_j)} = \frac{0.0001\times 0.9}{0.0001\times 0.9 + 0.9999\times 0.1}$$

$$= 0.0009$$

Ex14 (Book Ex 2.3.4)

- **Indentifying the Source of a Defective Item.** 3 Different machines M_1, M_2, and M_3 were producing a large batch of similar manufactured items. 20,30 and 50 percent of the items were produced by M_1, M2, and M3, respectively. 1, 2, and 3 percent of the items produced by M_1, M_2 and M_3, respectively, are defective. One item is selected at random from the entire batch and it is found to be defective. What's the prob. that this item was by M2?

Sol: Let A describe the event that the item is defective. Let M_i describe the event that the item was given by Mi.

$$\Pr(M_2 \mid A) = \frac{\Pr(M_2)\Pr(A \mid M_2)}{\sum_{j=1}^{3} \Pr(M_j)\Pr(A \mid M_j)}$$

$$= \frac{0.3 \times 0.02}{0.2 \times 0.01 + 0.3 \times 0.02 + 0.5 \times 0.03} = 0.26$$

Prior and Posterior Probabilities - 1

- Back in Ex. 14,

prior prob.

$$\boxed{\Pr(M_2 \mid A)} = \frac{\boxed{\Pr(M_2)}\Pr(A \mid M_2)}{\sum_{j=1}^{3}\Pr(M_j)\Pr(A \mid M_j)}$$

posterior prob.

$\Pr(M_2)$ is the probability of this event before the item is selected and before it is known whether the selected item is defective or nondefective.

$\Pr(M_2|A)$ is the probability of this event after it is known that the selected item is defective.

Q: in **Ex14** for which values of i (i=1,2,3), the posterior prob. is larger than its corresponding prior prob.?

Prior and Posterior Probabilities - 2

$$\Pr(M_1) = 0.2,\ \ \Pr(M_2) = 0.3,\ \ \Pr(M_3) = 0.5$$

$$\Pr(M_1 \mid A) = \frac{0.2 \times 0.01}{0.2 \times 0.01 + 0.3 \times 0.02 + 0.5 \times 0.03} = 0.09$$

$$\Pr(M_2 \mid A) = \frac{0.3 \times 0.02}{0.2 \times 0.01 + 0.3 \times 0.02 + 0.5 \times 0.03} = 0.26$$

$$\Pr(M_3 \mid A) = \frac{0.5 \times 0.03}{0.2 \times 0.01 + 0.3 \times 0.02 + 0.5 \times 0.03} = 0.65$$

$$For\ i = 3,\ \Pr(M_3 \mid A) > \Pr(M_3)$$

Prior and Posterior Probabilities - 3

- **Ex 15.** Let the events $B_1, \ldots, B_k$ form a partition of the space S. For $i = 1, \ldots, k$, let $\Pr(B_i)$ denote the prior prob. of B_i. For each event A such that $\Pr(A) > 0$, let $\Pr(B_i|A)$ denote the posterior prob. of B_i given that event A has occurred. Prove that if $\Pr(B_1|A) < \Pr(B_1)$, then $\Pr(B_i|A) > \Pr(B_i)$ for at least one value of i ($i=2,\ldots,k$).

Proof It is true that

$$\boxed{\sum_{i=1}^{k} \Pr(B_i)=1}, \text{ and } \boxed{\sum_{i=1}^{k} \Pr(B_i \mid A) = 1}$$

If $\Pr(B1 \mid A) < \Pr(B1), \Pr(B_i \mid A) \le \Pr(B_i)$ for $i = 2, \cdots, k,$ then

$$\sum_{i=1}^{k} \Pr(B_i \mid A) < \sum_{i=1}^{k} \Pr(B_i)$$

This is a contradiction!