Solutions to Random Mathematics Homework #2 Fall 2020

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H2.1

Let E_1 be the event that A is in class, let E_2 be the event that B is in class, and let E_3 be the event that C is in class. Let T be the event that at least one of students is in class. That is, $T = E_1 \cup E_2 \cup E_3$. Let K be the event that exactly one of them is in class.

(a) We want $\Pr(T)$. Using Theorem 1.10.1, we have $\Pr(T) = \Pr(E_1) + \Pr(E_2) + \Pr(E_3) - \Pr(E_1 \cap E_2) - \Pr(E_2 \cap E_3) - \Pr(E_1 \cap E_3) + \Pr(E_1 \cap E_2 \cap E_3)$. Since E_1 and E_2 are independent, we have $\Pr(E_1 \cap E_2) = \Pr(E_1)\Pr(E_2)$. Similarly, $\Pr(E_2 \cap E_3) = \Pr(E_2)\Pr(E_3)$, $\Pr(E_1 \cap E_3) = \Pr(E_1)\Pr(E_3)$ and $\Pr(E_1 \cap E_2 \cap E_3) = \Pr(E_1)\Pr(E_2)$ $\Pr(E_3)$. Hence,

$$\Pr(T) = 0.4 + 0.8 + 0.9 - 0.4 \times 0.8 - 0.8 \times 0.9 - 0.4 \times 0.9 + 0.4 \times 0.8 \times 0.9 = 0.9880$$

(b) Firstly we calculate there is only A in class. We have $\Pr(\text{only A in class}) = \Pr(E_1 \cap E_2^c \cap E_3^c)$. Using independence to calculate probabilities of intersections, $\Pr(E_1 \cap E_2^c \cap E_3^c) = \Pr(E_1)$ $\Pr(E_2^c)\Pr(E_3^c)$. Similarly, $\Pr(\text{only B in class}) = \Pr(E_1^c)\Pr(E_2)\Pr(E_3^c)$ and $\Pr(\text{only C in class}) = \Pr(E_1^c)\Pr(E_2^c)\Pr(E_3)$. Hence,

$$\Pr(K) = 0.4 \times 0.2 \times 0.1 + 0.6 \times 0.8 \times 0.1 + 0.6 \times 0.2 \times 0.9 = 0.1640$$

H2.2

(1) Since $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ and $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$, we have

$$\frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{3}{5}$$

Hence $\Pr(A \cap B) = \dfrac{1}{10}$. Using De Morgan's Laws.

$$\Pr(A^c \cup B^c) = 1 - \Pr(A \cap B) = \frac{9}{10}$$

(2) Because A and B are independent,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) = 0.25 + 0.5 - 0.25 \times 0.5 = 0.625$$

(3) Because A and B are disjoint,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0}{0.5} = 0$$

H2.3

$\Pr(A \cap B) = \dfrac{\Pr(A \cap B \cap C)}{\Pr(C|A \cap B)} = \dfrac{0.02}{0.4} = \dfrac{1}{20}$. Since A and B are independent, $\Pr(A \cap B) = \Pr(A)\Pr(B) = 5[\Pr(B)]^2$. Hence, $5[\Pr(B)]^2 = \dfrac{1}{20}$, so $\Pr(B) = \dfrac{1}{10}$. It now follows that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 5 \times \frac{1}{10} + \frac{1}{10} - \frac{1}{20} = \frac{11}{20}$$

H2.4

In order for the target to be hit for the first time on the fourth throw of boy B, all eight of the following independent events must occur: (1) A misses on his first throw, (2) B misses on his first throw, (3) A misses on his second throw, (4) B misses on his second throw, (5) A misses on his third throw, (6) B misses on his third throw, (7) A misses on his fourth throw, (8) B hits on his fourth throw. The probability of all five events occurring is $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{48}$

H2.5

Let A_1 denote the event that no white balls are selected, let A_2 denote the event that no blue balls are selected, and let A_3 denote the event that no red balls are selected. We must determine the value of $\Pr(A_1 \cup A_2 \cup A_3)$. We shall apply Theorem 1.10.1. The event A_1 will occur if and only if all ten selected balls are blue or red. Since there is probability 0.8 that any given selected ball will be blue or red, we have $\Pr(A_1) = (0.8)^{10}$. Similarly, $\Pr(A_2) = (0.7)^{10}$ and $\Pr(A_3) = (0.5)^{10}$. The event $A_1 \cap A_2$ will occur if and only if all ten selected balls are red. Therefore $\Pr(A_1 \cap A_2) = (0.5)^{10}$. Similarly, $\Pr(A_2 \cap A_3) = (0.2)^{10}$ and $\Pr(A_1 \cap A_3) = (0.3)^{10}$. Finally, the event $A_1 \cap A_2 \cap A_3$ cannot possibly occur, so $\Pr(A_1 \cap A_2 \cap A_3) = 0$. So, the desired probability is

$$(0.8)^{10} + (0.7)^{10} + (0.5)^{10} - (0.5)^{10} - (0.2)^{10} - (0.3)^{10}$$

H2.6

Let D denote the event that the bottles are defective, let R denote the event that the bottles are removed.

(a)

$$\Pr(D|R) = \frac{\Pr(D \cap R)}{\Pr(R)} = \frac{0.9 \times 0.3}{0.9 \times 0.3 + 0.2 \times 0.7} = 0.659$$

(b)

$$\Pr(D|R^c) = \frac{\Pr(D \cap R^c)}{\Pr(R^c)} = \frac{0.1 \times 0.3}{0.1 \times 0.3 + 0.8 \times 0.7} = 0.051$$

H2.7

Let B_i denote the event that the ith child is a boy, let G_i denote the event that the ith child is a girl. Then what we want is

$$\Pr\left[(G_1 \cap G_2 \cap \cdots G_n)^c \mid (B_1 \cap B_2 \cap \cdots B_n)^c)\right] = \frac{\Pr\left[(G_1 \cap G_2 \cap \cdots G_n)^c \cap (B_1 \cap B_2 \cap \cdots B_n)^c\right]}{\Pr\left[(B_1 \cap B_2 \cap \cdots B_n)^c\right]}$$

$$= \frac{\Pr\left[(G_1 \cap G_2 \cap \cdots G_n) \cup (B_1 \cap B_2 \cap \cdots B_n)\right]^{\mathrm{c}}}{1 - \Pr\left(B_1 \cap B_2 \cap \cdots B_n\right)} = \frac{1 - \Pr\left[(G_1 \cap G_2 \cap \cdots G_n) \cup (B_1 \cap B_2 \cap \cdots B_n)\right]}{1 - (1/2)^n}$$

$$= \frac{1 - \Pr\left[(G_1 \cap G_2 \cap \cdots G_n)\right] - \Pr\left[(B_1 \cap B_2 \cap \cdots B_n)\right]}{1 - (1/2)^n} = \frac{1 - (1/2)^n - (1/2)^n}{1 - (1/2)^n} = \frac{1 - (1/2)^{n-1}}{1 - (1/2)^n}$$

H2.8

Let A denote the event that an item is taken from basket A, let B denote the event that an item is taken from basket B, let X be the event that the fruit is apple and Y be the event that the fruit is pear.

(a) What we want is $\Pr(X)$. By using total probability formula, we have:

$$\Pr(X) = \Pr(X|A)\Pr(A) + \Pr(X|B)\Pr(B) = \frac{5}{10} \times \frac{1}{4} + \frac{7}{10} \times \frac{3}{4} = \frac{13}{20}$$

(b) What we want is $\Pr(\text{Head}|Y)$. By using Bayesian formula, we have:

$$\Pr(\text{Head}|Y) = \frac{\Pr(Y|\text{Head})\Pr(\text{Head})}{\Pr(Y)} = \frac{\frac{5}{10} \times \frac{1}{4}}{1 - \frac{13}{20}} = \frac{5}{14}$$

H2.9

Let A_1, B_1 and C_1 denote the event that the input string is $AAAA$, $BBBB$ and $CCCC$ respectively, and let D denote the event that the output of the system is $ABCA$. Since A_1, B_1 and C_1 are three independent events , and there is $\Pr\left(A_1 \cup B_1 \cup C_1\right) = \Pr\left(A_1\right) + \Pr\left(B_1\right) + \Pr\left(C_1\right) = p_1 + p_2 + p_3 = 1$, we can use the total probability's formula and Bayes' formula. Hence,

$$\Pr\left(A_1|D\right) = \frac{\Pr\left(A_1 \cap D\right)}{\Pr\left(D\right)} = \frac{\Pr\left(D|A_1\right)p_1}{\Pr\left(D|A_1\right)p_1 + \Pr\left(D|B_1\right)p_2 + \Pr\left(D|C_1\right)p_3}$$

When the input string is $AAAA$, and the output of the system is $ABCA$, two letters are oringal and the other two are different letters. Hence,

$$\Pr\left(D|A_1\right) = \alpha^2\left(\frac{1-\alpha}{2}\right)^2$$

Similarly, $\Pr\left(D|B_1\right) = \Pr\left(D|C_1\right) = \alpha\left(\frac{1-\alpha}{2}\right)^3$. So the probability is

$$\Pr\left(A_1|D\right) = \frac{2\alpha p_1}{(3\alpha - 1)\,p_1 + 1 - \alpha}$$

H2.10

Let A, B, C and D denote the event in (1), (2), (3) and (4) respectively.

(1)

$$\Pr(A) = \frac{\binom{8}{2}}{\binom{10}{2}} = \frac{28}{45}$$

(2)

$$\Pr(B) = \frac{\begin{pmatrix} 8 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 10 \\ 2 \end{pmatrix}} = \frac{16}{45}$$

(3)

$$\Pr(C) = \frac{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 10 \\ 1 \end{pmatrix}} \times \frac{\begin{pmatrix} 8 \\ 1 \end{pmatrix}}{\begin{pmatrix} 9 \\ 1 \end{pmatrix}} = \frac{8}{45}$$

(4)

$$\Pr(D) = \frac{2}{10} = \frac{1}{5}$$