Random Mathematics Homework #4 Fall 2020

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Assigned Date: Oct.15, 2020   Due Date: Oct.22, 2020

1. X and Y are discrete random variables. Suppose that the joint p.f. of X and Y is as specified in the following table:

	Y				
X	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Determine each of the following probabilities:

(a) $\Pr(X = 3)$

(b) $\Pr(Y \geq 3)$

(c) $\Pr(X \leq 1 \text{ and } Y \geq 3)$

(d) $\Pr(X = Y)$

(e) $\Pr(X > Y)$

(f) What's the marginal p.f. of X?

(g) What's the conditional p.f. of Y given X? Please use a Table to show.

2. Suppose that X and Y are random variables such that (X, Y) must belong to the rectangle in the xy-plane containing all points (x, y) for which $0 \leq x \leq 3$ and $0 \leq y \leq 4$. Suppose also that the joint c.d.f. of X and Y at every point (x, y) in this rectangle is specified as follows:

$$F(x, y) = \frac{1}{156} xy(x^2 + y).$$

Determine:

(a) $\Pr(1 \leq X \leq 2 \text{ and } 2 \leq Y \leq 4)$;

(b) $\Pr(-1 < X \leq 5 \text{ and } 2 \leq Y < 5)$;

(c) the marginal c.d.f. of Y;

(d) the joint p.d.f. of X and Y;

(e) $g_1(x \mid y)$;

(f) $g_2(y \mid 0)$;

(g) $\Pr(2X + Y \leq 3)$.

3. Suppose that the joint p.d.f of X and Y is as follows:

$$f(x, y) = \begin{cases} 24xy, & \text{for } x \geq 0, y \geq 0, \text{and } x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What's the joint c.d.f.?

(b) What's the marginal c.d.f. of X?

(c) Are X and Y independent?

(d) What's $\Pr(Y \geq X)$?

4. Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is defined as follows:

$$f(x, y) = \begin{cases} \frac{3}{2}y^2 & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the marginal p.d.f.'s of X and Y.
(b) Are X and Y independent?
(c) Are the event $\{X<1\}$ and the event $\{Y \geq 1/2\}$ independent?

5. Suppose that two persons make an appointment to meet between 5 p.m. and 6 p.m. at a certain location, and they agree that neither person will wait more than 20 minutes for the other person. If they arrive independently at random times between 5 p.m. and 6 p.m., what is the probability that they will meet?

6. Suppose that a point (X,Y) is chosen at random from the rectangle S defined as follows:
$S = \{(x, y): 0 \leq x \leq 5 \text{ and } 1 \leq y \leq 4\}$.
(a) Determine the joint p.d.f. of X and Y, the marginal p.d.f. of X and the marginal p.d.f. of Y.
(b) Are X and Y independent?

7. Each student in a certain high school was classified according to her year in school (freshman, sophomore, junior, or senior) and according to the number of times that she had visited a certain museum (never, once, or more than once). The proportions of students in the various classifications are given in the following table:

	Never	Once	More than once
Freshmen	0.08	0.10	0.04
Sophomores	0.04	0.10	0.04
Juniors	0.04	0.20	0.09
Seniors	0.02	0.15	0.10

(a) If a student selected at random from the high school is a junior, what is the probability that she has never visited the museum?
(b) If a student selected at random from the high school has visited the museum twice, what is the probability that she is a senior?

8. Suppose that the joint p.d.f. of two random variables X and Y is as follows:

$$f(x, y) = \begin{cases} \frac{3}{16}(4 - 2x - y) & \text{for } x > 0,\ y > 0, \\ & \quad \text{and } 2x + y < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Determine:
(a) the conditional p.d.f. of Y for every given value of X, and
(b) $\Pr(Y \le 1 | X = 1)$.

9. Let Y be the rate (calls per hour) at which calls arrive at a switchboard. Let X be the number of calls during a two-hour period. Suppose that the marginal p.d.f. of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that the conditional p.f. of X given $Y = y$ is

$$g_1(x|y) = \begin{cases} \dfrac{(2y)^x}{x!} e^{-2y} & \text{if } x = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal p.f. of X. (You may use the formula $\int_0^\infty y^k e^{-y}\, dy = k!$.)
(b) Find the conditional p.d.f. $g_2(y\,|\,0)$ of Y given $X = 0$.
(c) Find the conditional p.d.f. $g_2(y\,|\,1)$ of Y given $X = 1$.
(d) For what values of y is $g_2(y\,|\,0) > g_2(y\,|\,1)$? Does this agree with the intuition that the more calls you see, the higher you should think the rate is?

10. Suppose that either of two instruments might be used for making a certain measurement. Instrument 1 yields a measurement whose p.d.f. $h1$ is

$$h_1(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Instrument 2 yields a measurement whose p.d.f. $h2$ is

$$h_2(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that one of the two instruments is chosen at random and a measurement X is made with it.
(a) Determine the marginal p.d.f. of X.
(b) If the value of the measurement is $X = 1/5$, what is the probability that instrument 1 was used?

11. Suppose that a person's score X on Random Maths test is a number between 0 and 1, and that his score Y on C Language test is also a number between 0 and 1. Suppose that in the population of all UESTC students, the scores X and Y are distributed according to the following joint p.d.f.:

$$f(x, y) = \begin{cases} \dfrac{2}{5}(2x + 3y) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) What proportion of UESTC students obtain a score greater than 0.8 on the Random Maths test?
(b) If a student's score on the C Language test is 0.3, what's the probability that his score on the Random Maths test will be greater than 0.8?

12. Suppose that three random variables X_1 , X_2 and X_3 have a continuous joint distribution with the following joint p.d.f.:

$$f\ (x_1,\ x_2,\ x_3) = \begin{cases} c(x_1 +\ 2x_2 + 3x_3) & \text{for } 0 \le x_i \le 1 (\mathrm{i} = 1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Determine:
(a) the value of the constant c;
(b) the marginal joint p.d.f. of $X1$ and $X3$;
(c) $\Pr(X_3 < \frac{1}{2} \mid X_1 = \frac{1}{4}, X_2 = \frac{3}{4})$;
(d) the marginal c.d.f. of $X3$.

13. Suppose that the p.d.f. of X is as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Determine the p.d.f. of $Y = X^{1/2}$.

14. Let Z be the rate at which customers are served in a queue. Assume that Z has the p.d.f.

$$f(z) = \begin{cases} 2e^{-2z} & \text{for } z > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.d.f. of the average waiting time $T = 1/Z$.