Exam
DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

28th October 2024, 14:00 – 18:00

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

In case of questions, call Jan Gerken at 031-772-14-37.

Good luck!

Problem:	1	2	3	4	5	6	7
Points:	2	2	3	4	4	7	4
Problem:	8	9	10	11	12		Total
Points:	5	3	2	1	3		40

Problem 1 (2 points)
Write negations of the following statements (answers suffice).

(a) (1 point) There exists a real number x such that for all real numbers y, $xy > y$.

(b) (1 point) For all real numbers x, if $x(x-2) > 0$ then $x > 2$ or $x < 0$.

Problem 2 (2 points)
Let T be the statement

$$\forall x \in \mathbb{R},\ \text{if } -1 < x \leq 0 \text{ then } x + 1 > 0\,.$$

(a) (1 point) Write the converse of T. (Answer suffices.)

(b) (1 point) Write the contrapositive of T. (Answer suffices.)

Problem 3 (3 points)
Prove the following statement. Use only the definitions, no other theorems/lemmas etc.

$$\forall a, b, c \in \mathbb{Z},\ \text{if } a \mid b \text{ and } a \mid c,\ \text{then } a \mid (5b + 3c).$$

Problem 4 (4 points)
Prove that

$$\sum_{i=3}^{n} i(i-1) = \frac{(n-2)(n^2+2n+3)}{3} \qquad \forall n \geq 3\,.$$

Problem 5 (4 points)
Use mathematical induction to prove that $\forall n \in \mathbb{Z}$ with $n \geq 5$, $1 + 4n < 2^n$.

Problem 6 (7 points)

In a *Double Tower of Hanoi with Adjacency Requirement*, there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially pole A (at one end of the row) contains all the disks, placed on top of each other in *pairs* of decreasing size. Disks may only be transferred one-by-one from one pole to an *adjacent* pole and at no time may a larger disk be placed on top of a smaller one. However, a disk may be placed on top of another one of the same size. Let C be the pole at the other end of the row and let

$$s_n = \left[\begin{array}{c}\text{the minimum number of moves}\\ \text{needed to transfer a tower of } 2n\\ \text{disks from pole } A \text{ to pole } C\end{array}\right].$$

(a) (2 points) Find s_1 and s_2.

(b) (5 points) Find a recurrence relation expressing s_k in terms of s_{k-1} for all integers $k \geq 2$. Justify your answer carefully.

Problem 7 (4 points)

A club has seven members. Three are to be chosen to go as a group to a national meeting.

(a) (1 point) How many distinct groups of three can be chosen?

(b) (1 point) If the club contains four men and three women, how many distinct groups of three contain two men and one woman?

(c) (2 points) If two members of the club refuse to travel together as part of the group (but each is willing to go if the other does not), how many distinct groups of three can be chosen?

Problem 8 (5 points)

Let $T = \{3, 4, 5, 6, 7, 8, 9, 10\}$ and suppose five integers are chosen from T. Must two of these integers have the property that the difference of the larger minus the smaller equals 2? Why or why not?

Problem 9 (3 points)

Either draw a graph with the given specification or explain why no such graph exists.

(a) (1 point) A full binary tree with 16 vertices of which 6 are internal vertices.

(b) (1 point) A binary tree, height 3, 9 vertices.

(c) (1 point) A binary tree, height 4, 18 terminal vertices.

Problem 10 (2 points)
Explain why the following statement is true.

$$3 + 6 + 9 + \cdots + 3n \quad \text{is} \quad \Theta(n^2)$$

Problem 11 (1 point)
Use O-notation to express the following statement.

$$5x + x \log_2 x \leq 6x \log_2 x \qquad \forall x > 2$$

Problem 12 (3 points)
For the following weighted graph, use Prim's algorithm starting with vertex a to find a minimum spanning tree. Indicate the order in which the edges are added to build the tree.