Exam
DIT008: Discrete Mathematics

University of Gothenburg
Lecturer: Jan Gerken

28$^{\text{th}}$ October 2024, 14:00 – 18:00

No calculators or other aids are allowed.

You must show your calculations or describe your argument. Unless otherwise stated, if you provide just the final answer, you will not be awarded any points.

The grade boundaries are

- pass (3): at least 20 points (50%)
- pass with credit (4): at least 28 points (70%)
- distinction (5): at least 36 points (90%)

Good luck!

Problem:	1	2	3	4	5	6	7
Points:	2	2	3	4	4	7	4
Score:							

Problem:	8	9	10	11	12		Total
Points:	5	3	2	1	3		40
Score:							

Problem 1 (2 points)
Write negations of the following statements (answers suffice).

(a) (1 point) There exists a real number x such that for all real numbers y, $xy > y$.

(b) (1 point) For all real numbers x, if $x(x-2) > 0$ then $x > 2$ or $x < 0$.

Solution:

(a) For all real numbers x there exists a real number y such that $xy \leq y$.

(b) $\exists$ a real number x such that $x(x-2) > 0$ and $0 \leq x \leq 2$.

Problem 2 (2 points)
Let T be the statement

$$\forall x \in \mathbb{R},\ \text{if } -1 < x \leq 0 \text{ then } x + 1 > 0\,.$$

(a) (1 point) Write the converse of T. (Answer suffices.)

(b) (1 point) Write the contrapositive of T. (Answer suffices.)

Solution:

(a) $\forall$ real numbers x, if $x + 1 > 0$ then $-1 < x \leq 0$.

(b) $\forall$ real numbers x, if $x + 1 \leq 0$ then $-1 \geq x$ or $x > 0$.

Problem 3 (3 points)
Prove the following statement. Use only the definitions, no other theorems/lemmas etc.

$$\forall a, b, c \in \mathbb{Z},\ \text{if } a \mid b \text{ and } a \mid c,\ \text{then } a \mid (5b + 3c).$$

Solution: Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. By definition of divisibility, $b = ar$ and $c = as$ for some integers r and s. Then

$$\begin{aligned} 5b + 3c &= 5(ar) + 3(as) \\ &= a(5r + 3s)\,. \end{aligned}$$

Let $t = 5r + 3s$. Then t is an integer because products and sums of integers are integers, and $5b + 3c = at$. Thus, by definition of divisibility, $a \mid (5b + 3c)$.

Problem 4 (4 points)
Prove that

$$\sum_{i=3}^{n} i(i-1) = \frac{(n-2)(n^2 + 2n + 3)}{3} \qquad \forall n \geq 3\,.$$

Solution: Prove this by induction. Let the property $P(n)$ be the equation

$$\sum_{i=3}^{n} i(i-1) = \frac{(n-2)(n^2+2n+3)}{3}\,.$$

Base case ($n = 3$): $P(3)$ is true because the left-hand side is

$$\sum_{i=3}^{3} (i-1)\cdot i = (3-1)\cdot 3 = 6$$

and the right-hand side is

$$\frac{(3-2)(3^2+2\cdot 3+3)}{3} = \frac{1\cdot(9+6+3)}{3} = 6$$

also.

Inductive step: Let k be any integer with $k \geq 1$, and suppose that $P(k)$ is true, i.e. that

$$\sum_{i=3}^{k} (i-1)\cdot i = \frac{(k-2)(k^2+2k+3)}{3}\,.$$

Then,

$$\begin{aligned}
\sum_{i=3}^{k+1} (i-1)\cdot i &= \sum_{i=3}^{k} (i-1)\cdot i + (k+1-1)(k+1) \\
&= \frac{(k-2)(k^2+2k+3)}{3} + k(k+1) \\
&= \frac{1}{3}\left(k^3+3k^2+2k-6\right)\,.
\end{aligned}$$

On the other hand we have also,

$$\frac{(k-1)((k+1)^2+2(k+1)+3)}{3} = \frac{1}{3}\left(k^3+3k^2+2k-6\right)$$

and hence $P(k+1)$ is true.

Problem 5 (4 points)

Use mathematical induction to prove that $\forall n \in \mathbb{Z}$ with $n \geq 5$, $1 + 4n < 2^n$.

Solution: Let $P(n)$ be the property

$$1 + 4n < 2^n .$$

Base case ($n = 5$): $P(5)$ is true because the left-hand side is

$$1 + 4 \cdot 5 = 21$$

and the right-hand side is

$$2^5 = 32,$$

and $21 < 32$.

Inductive step: Let k be any integer with $k \geq 5$, and suppose $P(k)$, i.e. suppose

$$1 + 4k < 2^k .$$

By algebra and the inductive hypothesis, we have that

$$1 + 4(k+1) = 1 + 4k + 4 < 2^k + 4 .$$

Now, since $k \geq 5$,

$$4 < 2^k = 2^k(2-1) = 2^{k+1} - 2^k ,$$

and thus,

$$2^k + 4 < 2^{k+1} .$$

Hence,

$$1 + 4(k+1) < 2^{k+1} ,$$

which completes the proof.

Problem 6 (7 points)

In a *Double Tower of Hanoi with Adjacency Requirement*, there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially pole A (at one end of the row) contains all the disks, placed on top of each other in *pairs* of decreasing size. Disks may only be transferred one-by-one from one pole to an *adjacent* pole and at no time may a larger disk be placed on top of a smaller one. However, a disk may be placed on top of another one of the same size. Let C be the pole at the other end of the row and let

$$s_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } 2n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}.$$

(a) (2 points) Find s_1 and s_2.

(b) (5 points) Find a recurrence relation expressing s_k in terms of s_{k-1} for all integers $k \geq 2$. Justify your answer carefully.

Solution:

(a) $s_1 = 4$ (two moves to transfer the two disks to pole B and two more moves to transfer them to pole C)

$s_2 = 16$ (by the reasoning above, 4 moves are needed to transfer the top two disks from pole A to pole C, then 2 moves are needed to transfer the two bottom disks from pole A to pole B, another 4 moves to transfer the top two disks back to pole A, then 2 more to transfer the two bottom disks from pole B to pole C, and finally 4 moves to transfer the two top disks from pole A to pole C)

(b)

$$\begin{aligned} s_k = s_{k-1} \quad & \text{(moves to transfer the top } 2(k-1) \text{ disks from pole } A \text{ to pole } C) \\ + 2 \quad & \text{(moves to transfer the two bottom disks from pole } A \text{ to pole } B) \\ + s_{k-1} \quad & \text{(moves to transfer the top } 2(k-1) \text{ disks from pole } C \text{ to pole } A) \\ + 2 \quad & \text{(move to transfer the two bottom disks from pole } B \text{ to pole } C) \\ + s_{k-1} \quad & \text{(moves to transfer the top } 2(k-1) \text{ disks from pole } A \text{ to pole } C). \end{aligned}$$

So,

$$s_k = 3s_{k-1} + 4 \quad \text{for all integers } k \geq 2\,.$$

Problem 7 (4 points)
A club has seven members. Three are to be chosen to go as a group to a national meeting.

(a) (1 point) How many distinct groups of three can be chosen?

(b) (1 point) If the club contains four men and three women, how many distinct groups of three contain two men and one woman?

(c) (2 points) If two members of the club refuse to travel together as part of the group (but each is willing to go if the other does not), how many distinct groups of three can be chosen?

Solution:

(a) There are $\binom{7}{3} = 35$ distinct groups of three that can be chosen from the seven members of the club.

(b) There are $\binom{4}{2} \cdot \binom{3}{1} = 6 \cdot 3 = 18$ distinct groups of three that contain two men and one woman.

(c) There are $\binom{5}{2} + \binom{5}{2} + \binom{5}{3} = 10 + 10 + 10 = 30$ distinct groups of three that can be chosen assuming that two members of the club refuse to travel together as part of the group but each is willing to go if the other does not.

Problem 8 (5 points)
Let $T = \{3, 4, 5, 6, 7, 8, 9, 10\}$ and suppose five integers are chosen from T. Must two of these integers have the property that the difference of the larger minus the smaller equals 2? Why or why not?

Solution: Yes. Partition the set T into the following four subsets:

$$\{3, 5\}, \{4, 6\}, \{7, 9\}, \{8, 10\}.$$

These subsets have three important properties: (1) within each subset, the larger number minus the smaller number equals 2, (2) every number in T is in one of the subsets, and (3) no number in T is in more than one subset. Define a function f from the set of the five chosen numbers to the four subsets of the partition by the rule: to each number x associate the subset containing that number. By the pigeonhole principle, f is not one-to-one, so two numbers belong to the same subset. But within each subset, the difference of the larger number minus the smaller one is 2. So at least two numbers out of the five have the property that the larger minus the smaller is 2.

Problem 9 (3 points)
Either draw a graph with the given specification or explain why no such graph exists.

(a) (1 point) A full binary tree with 16 vertices of which 6 are internal vertices.

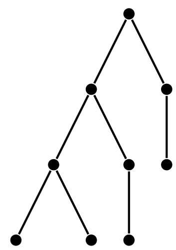
(b) (1 point) A binary tree, height 3, 9 vertices.

(c) (1 point) A binary tree, height 4, 18 terminal vertices.

Solution:

(a) No such graph exists because a full binary tree with 6 internal vertices has 7 leaves and a total of 13 vertices.

(b)



(c) No such graph exists because the number of leaves in a binary tree of height 4 is less than or equal to $2^4 = 16$ and $18 > 16$.

Problem 10 (2 points)

Explain why the following statement is true.

$$3 + 6 + 9 + \cdots + 3n \quad \text{is} \quad \Theta(n^2)$$

Solution:

$$3 + 6 + 9 + \cdots + 3n = 3\sum_{i=1}^{n} i = \frac{3n(n+1)}{2}$$

This is $\Theta(n^2)$ by the theorem on polynomial orders.

Problem 11 (1 point)

Use O-notation to express the following statement.

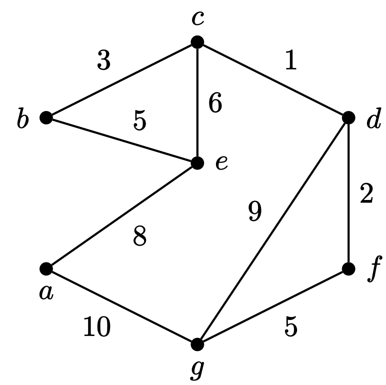
$$5x + x\log_2 x \leq 6x\log_2 x \qquad \forall x > 2$$

Solution:

$$5x + x\log_2 x \quad \text{is} \quad O(x\log_2 x)\,.$$

Problem 12 (3 points)

For the following weighted graph, use Prim's algorithm starting with vertex a to find a minimum spanning tree. Indicate the order in which the edges are added to build the tree.



Solution: Order of adding the edges: $\{a, e\}, \{b, e\}, \{b, c\}, \{c, d\}, \{d, f\}, \{f, g\}$