Lecture 2: Models of Computation

Lecture Overview

- What is an algorithm? What is time?
- Random access machine
- Pointer machine
- Python model
- Document distance: problem & algorithms

History

Al-Khwārizmī "al-kha-raz-mi" (c. 780-850)

- "father of algebra" with his book "The Compendious Book on Calculation by Completion & Balancing"
- linear & quadratic equation solving: some of the first algorithms

What is an Algorithm?

- Mathematical abstraction of computer program
- Computational procedure to solve a problem

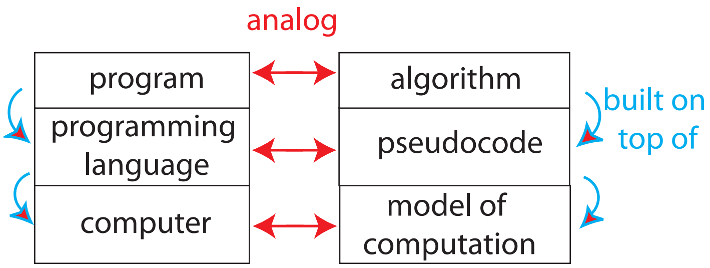
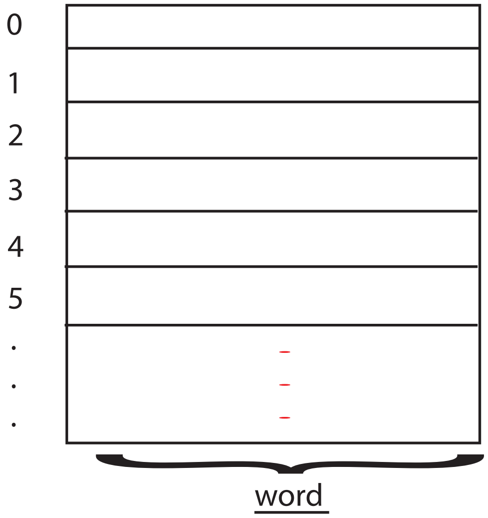


Figure 1: Algorithm

Model of computation specifies

- what operations an algorithm is allowed
- cost (time, space, . . .) of each operation
- cost of algorithm = sum of operation costs

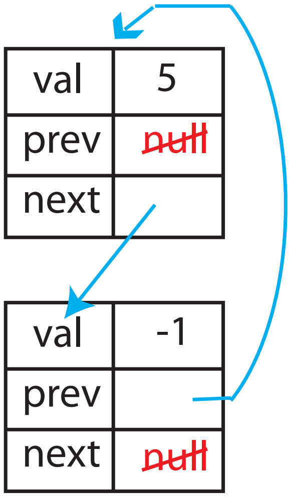
Random Access Machine (RAM)



- Random Access Memory (RAM) modeled by a big array
- $\Theta(1)$ registers (each 1 word)
- In $\Theta(1)$ time, can
  - load word @ r_i into register r_j
  - compute $(+, -, *, /, \&, |, \hat{})$ on registers
  - store register r_j into memory @ r_i
- What's a word? $w \geq \lg$ (memory size) bits
  - assume basic objects (e.g., int) fit in word
  - unit 4 in the course deals with big numbers
- realistic and powerful → implement abstractions

Pointer Machine

- dynamically allocated objects (namedtuple)
- object has $O(1)$ fields
- field = word (e.g., int) or pointer to object/null (a.k.a. reference)
- weaker than (can be implemented on) RAM



Python Model

Python lets you use either mode of thinking

1. "list" is actually an array → RAM

   $L[i] = L[j] + 5 \to \Theta(1)$ time

2. object with $O(1)$ attributes (including references) → pointer machine

   $x = x.next \to \Theta(1)$ time

Python has many other operations. To determine their cost, imagine implementation in terms of (1) or (2):

1. list

   (a) L.append(x) $\to \theta(1)$ time

   obvious if you think of infinite array

   but how would you have > 1 on RAM?
   via *table doubling* [Lecture 9]

   (b) $\underbrace{L = L1 + L2}_{(\theta(1+|L1|+|L2|)\text{ time})} \equiv L = [\,] \to \theta(1)$

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{for } x \text{ in } L1:\\ \quad \text{L.append(x)} \to \theta(1)\end{array}\right\}\theta(|L_1|)\\ \left.\begin{array}{l}\text{for } x \text{ in } L2:\\ \quad \text{L.append(x)} \to \theta(1)\end{array}\right\}\theta(|L_2|)\end{array}\right\}\theta(\,|L1| + |L2| + 1\,)$$

(c) $$\left.\begin{array}{ll} L1.\text{extend}(L2) \equiv \text{for } x \text{ in } L2: & \\ \quad \equiv L1 += L2 \qquad L1.\text{append(x)} & \rightarrow \theta(1) \end{array}\right\} \theta(1+|L_2|) \text{ time}$$

(d) $$\left.\begin{array}{l} L2 = L1[i:j] \equiv L2 = [] \\ \qquad\qquad\qquad \text{for } k \text{ in range}(i,j): \\ \qquad\qquad\qquad\quad L2.\text{append}(L1[i]) \rightarrow \theta(1) \end{array}\right\} \theta(j-i+1) = O(|L|)$$

(e) $$\left.\begin{array}{lll} \begin{array}{l} b = x \text{ in } L \\ \text{\& L.index(x)} \\ \text{\& L.find(x)} \\ \\ \\ \\ \end{array} & \equiv & \begin{array}{l} \text{for } y \text{ in } L: \\ \left.\begin{array}{l} \quad \text{if } x == y: \\ \qquad b = \mathit{True}; \\ \qquad \text{break} \end{array}\right\} \theta(1) \\ \quad \text{else} \\ \qquad b = \mathit{False} \end{array} \end{array}\right\} \theta(\text{index of } x) = \theta(|L|)$$

(f) len(L) $\rightarrow \theta(1)$ time - list stores its length in a field

(g) L.sort() $\rightarrow \theta(|L| \log |L|)$ - via *comparison sort* [Lecture 3, 4 & 7)]

2. tuple, str: similar, (think of as immutable lists)

3. dict: via *hashing* [Unit 3 = Lectures 8-10]

   $$\left.\begin{array}{l} \text{D[key] = val} \\ \text{key in D} \end{array}\right\} \theta(1) \text{ time w.h.p.}$$

4. set: similar (think of as dict without vals)

5. heapq: heappush & heappop - via *heaps* [Lecture 4] $\rightarrow \theta(\log(n))$ time

6. long: via *Karatsuba algorithm* [Lecture 11]

   $x + y \rightarrow O(|x| + |y|)$ time where $|y|$ reflects # words

   $x * y \rightarrow O((|x| + |y|)^{\log(3)}) \approx O((|x| + |y|)^{1.58})$ time

Document Distance Problem — compute $d(D_1, D_2)$

The document distance problem has applications in finding similar documents, detecting duplicates (Wikipedia mirrors and Google) and plagiarism, and also in web search (D_2 = query).

Some Definitions:

- Word = sequence of alphanumeric characters
- Document = sequence of words (ignore space, punctuation, etc.)

The idea is to define distance in terms of shared words. Think of document D as a vector: $D[w]$ = # occurrences of word W. For example:

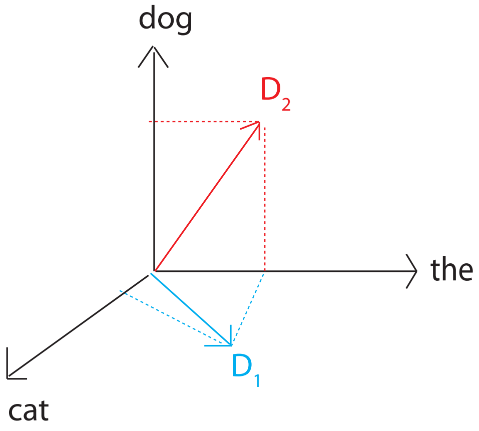


Figure 2: D_1 = "the cat", D_2 = "the dog"

As a first attempt, define document distance as

$$d'(D_1, D_2) = D_1 \cdot D_2 = \sum_W D_1[W] \cdot D_2[W]$$

The problem is that this is not scale invariant. This means that long documents with 99% same words seem farther than short documents with 10% same words.
This can be fixed by normalizing by the number of words:

$$d''(D_1, D_2) = \frac{D_1 \cdot D_2}{|D_1| \cdot |D_2|}$$

where $|D_i|$ is the number of words in document i. The geometric (rescaling) interpretation of this would be that:

$$d(D_1, D_2) = \arccos(d''(D_1, D_2))$$

or the document distance is the angle between the vectors. An angle of 0° means the two documents are identical whereas an angle of 90° means there are no common words. This approach was introduced by [Salton, Wong, Yang 1975].

Document Distance Algorithm

1. split each document into words
2. count word frequencies (document vectors)
3. compute dot product (& divide)

```
(1) re.findall (r“ w+”, doc) → what cost?
    in general re can be exponential time
    → for char in doc:                                   } $\Theta(|doc|)$
        if not alphanumeric
          add previous word                  } $\Theta(1)$
              (if any) to list
          start new word

(2) sort word list   ← $O(k \log k \cdot |word|)$ where $k$ is #words
    for word in list:                                         } $O(\sum |word|) = O(|doc|)$
        if same as last word:   ← $O(|word|)$
          increment counter                        } $\Theta(1)$
        else:
          add last word and count to list
          reset counter to 0

(3)     for word, count1 in doc1:   ← $\Theta(k_1)$          } $O(k_1 \cdot k_2)$
            if word, count2 in doc2:   ← $\Theta(k_2)$
                total += count1 * count2    $\Theta(1)$

(3)'    start at first word of each list                    } $O(\sum |word|) = O(|doc|)$
        if words equal:   ← $O(|word|)$
            total += count1 * count2
        if word1 $\leq$ word2:  ← $O(|word|)$
            advance list1
        else:
            advance list2
        repeat either until list done
```

Dictionary Approach

```
(2)'      count = {}

              for word in doc:                                                   } $O(|doc|)$ w.h.p.
                  if word in count:   ← $\Theta(|word|) + \Theta(1)$ w.h.p
                      count[word] += 1   } $\Theta(1)$
                  else
                      count[word] = 1

(3)'   as above → $O(|doc_1|)$ w.h.p.
```

Code (lecture2_code.zip & _data.zip on website)

t2.bobsey.txt 268,778 chars/49,785 words/3354 uniq
t3.lewis.txt 1,031,470 chars/182,355 words/8534 uniq
seconds on Pentium 4, 2.8 GHz, C-Python 2.62, Linux 2.6.26

- docdist1: 228.1 — (1), (2), (3) (with extra sorting)
  words = words + words_on_line
- docdist2: 164.7 — words += words_on_line
- docdist3: 123.1 — (3)' . . . with insertion sort
- docdist4: 71.7 — (2)' but still sort to use (3)'
- docdist5: 18.3 — split words via string.translate
- docdist6: 11.5 — merge sort (vs. insertion)
- docdist7: 1.8 — (3) (full dictionary)
- docdist8: 0.2 — whole doc, not line by line

6.006 Introduction to Algorithms
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