Correlated Equilibrium and Coarse Correlated Equilibrium

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EC4501/EC4501HM

Complete Information Game

A static game with complete information is denoted as $\Gamma = \left(N, (A_i)_{i \in N}, (u_i)_{i \in N}\right)$

- N is the set of players;
- A_i is the set of player i's actions;
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In game Γ, a (mixed) strategy of player i is denoted by $\sigma_i \in \Delta(A_i)$.

Correlated Equilibrium

Definition

A strategy profile $\sigma \in \Delta(A)$ is a correlated equilibrium if for any player i and any action a_i in the support of σ, we have

$$\mathbf{E}_\sigma[u_i(a_i, a_{-i}) \mid a_i] \geq \mathbf{E}_\sigma\left[u_i(a_i', a_{-i}) \mid a_i\right], \quad \forall a_i' \in A_i.$$

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Example: coordination game

	movie (M)	concert (C)
movie (M)	2,1	0,0
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Examples of correlated equilibria:

- (MM) or (CC);
- with probability $\frac{1}{2}$ (MM) and with probability $\frac{1}{2}$ (CC).
- with probability $\frac{1}{3}$ (MM), with probability $\frac{1}{3}$ (MC), and with probability $\frac{1}{3}$ (CC).

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If $\sigma \in \Delta(A)$ is a Nash equilibrium, then σ must also be a correlated equilibrium.

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In Nash equilibrium, agents' strategies are independent.

- the requirement that

$$\mathbf{E}_\sigma [u_i(a_i, a_{-i}) \mid a_i] \geq \mathbf{E}_\sigma \left[u_i(a_i', a_{-i}) \mid a_i\right], \quad \forall a_i' \in A_i$$

  is the same as the condition for Nash equilibrium as the distribution over a_{-i} is the same conditional on any a_i.

Coarse Correlated Equilibrium

Definition

A strategy profile $\sigma \in \Delta(A)$ is a coarse correlated equilibrium if for any player i, we have

$$\mathbf{E}_\sigma[u_i(a_i, a_{-i})] \geq \mathbf{E}_\sigma\left[u_i(a_i', a_{-i})\right], \quad \forall a_i' \in A_i.$$

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Taking expectation over a_i for

$$\mathbf{E}_\sigma[u_i(a_i, a_{-i}) \,|\, a_i] \geq \mathbf{E}_\sigma\left[u_i(a_i', a_{-i}) \,|\, a_i\right], \quad \forall a_i' \in A_i.$$

Learning in Games

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Model as an adversarial bandit / expert learning from the perspective of each agent i':

- set of arms A_i;
- realized payoff $u_i(a_i^t, a_{-i}^t)$ for any period $t \leq T$.

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Theorem

Given a complete information game Γ, if all agents adopt no-regret learning algorithms, the empirical action profile converges to the coarse correlated equilibrium when the time horizon $T \to \infty$.

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Given a complete information game Γ, if all agents adopt no-regret learning algorithms, the empirical action profile converges to the coarse correlated equilibrium when the time horizon $T \to \infty$.

From the perspective of each agent i, treating the action profile of other agents as given, no-regret requires that

$$\frac{1}{T} \sum_{t \in [T]} u_i(a_{i,t}, a_{-i,t}) \geq \frac{1}{T} \sum_{t \in [T]} u_i(a_i, a_{-i,t}) - o(1), \quad \forall a_i \in A_i,$$

coinciding with the requirements for coarse correlated equilibrium.

Convergence Under No-Swap-Regret

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Given a complete information game Γ, if all agents adopt no-swap-regret learning algorithms, the empirical action profile converges to the correlated equilibrium when the time horizon $T \to \infty$.

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From the perspective of each agent i, treating the action profile of other agents as given, no-swap-regret requires that

$$\frac{1}{T} \sum_{t \in [T]} u_i(a_{i,t}, a_{-i,t}) \geq \frac{1}{T} \sum_{t \in [T]} u_i(\pi_i(a_i), a_{-i,t}) - o(1), \quad \forall \pi_i : A_i \to A_i,$$

coinciding with the requirements for correlated equilibrium.

Convergence

Not all correlated equilibria or coarse correlated equilibria are reachable under natural learning algorithms (You may attain those equilibria under bizarre algorithms).

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Example: second-price auction, two agents with value $v_1 > v_2$, each using Hedge algorithms [Kolumbus and Nisan '22]

- the bid distribution of the high value agent converges to uniform distribution over $[v_2, v_1]$;
- the bid distribution of the low value agent converges to uniform distribution over $[0, v_2]$.