Economics and Computation

Yingkai Li

EC4501/EC4501HM Semester 2, AY2024/25

Logistics

Instructor: Yingkai Li

Office: AS2 05-21

Office hour: by appointment.

Reading Lists

1. Aleksandrs Slivkins. *Introduction to Multi-Armed Bandits.* `https://arxiv.org/abs/1904.07272`
2. Jason Hartline. *Mechanism Design and Approximation.* `https://jasonhartline.com/MDnA/`
3. Tim Roughgarden. *Twenty Lectures on Algorithmic Game Theory.* `https://timroughgarden.org/notes.html`

Additional readings:

- Noam Nisan, Tim Roughgarden, Éva Tardos, Vijay V. Vazirani. *Algorithmic Game Theory.* Cambridge University Press.
- Federico Echenique, Nicole Immorlica, Vijay V. Vazirani. *Online and Matching-Based Market Design.* Cambridge University Press.

Prerequisite

Required: Basics in probabilities, calculus, and how to prove formal theorems.

Not required: solid background knowledge about algorithm design (CS), mechanism design (Econ), or game theory (Econ). Coding is also not required.

Evaluations

- Two assignments (40%); due on Sep 29th, Nov 7th.
- Course project (30%); due on Oct 31th, mid-term review on Oct 6th.
- Final exam (30%); scheduled on Nov 21th, 5pm.
- Survey paper (25%); due on Nov 10th; only for HM students.

Syllabus

Week 1:	Preview of the course
Week 2/3/4:	Learning: bandits, experts, calibration
Week 5/6:	Learning in games
Week 7/8:	Mechanism design: welfare, revenue
Week 9/10:	Robust mechanism design
Week 11/12:	Topic courses: fairness, contracts, etc.
Week 13:	Project presentation by students

Course Philosophy

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This Course: focus on the computer science approaches in various economic problems.

A Simple Example: Complexity of Optimal Solutions

Knapsack problems:
n tasks, each task $i \in [n]$ requires a resource of c_i, and generates a value of v_i.

Objective: find a set of tasks to maximize the total value subject to a budget B on resources.

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- naïvely, the optimal solution can be found by enumerating all possible subsets, taking time $\exp(n)$, not practical.
- NP-hard means that "no algorithm" can find the optimal solution much faster than that.

A Simple Example: Simple Algorithms

Let us give up the optimal solution and only look for a "good enough" one.

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$$\text{Upper-Bound} \leq \text{Greedy} + \text{Max-Val} \Rightarrow \max\{\text{Greedy}, \text{Max-Val}\} \geq \frac{1}{2} \cdot \text{Upper-Bound}.$$

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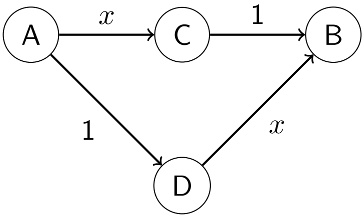
- A $\rightarrow$ C, D $\rightarrow$ B: travel time x, fraction of travelers.
- A $\rightarrow$ D, C $\rightarrow$ B: travel time 1.

Network Before Adding Shortcut



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- New road in network: open a portal from C to D with zero travel time.

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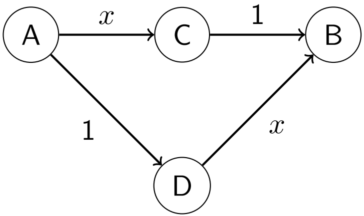

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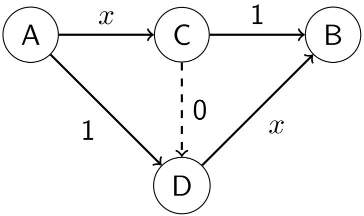

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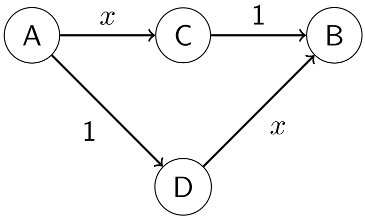

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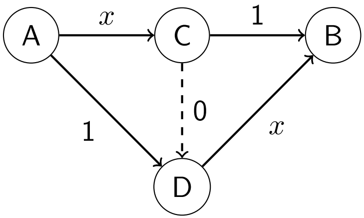

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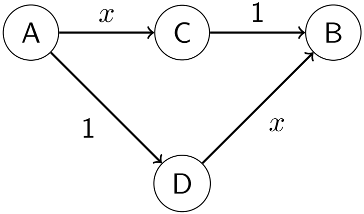

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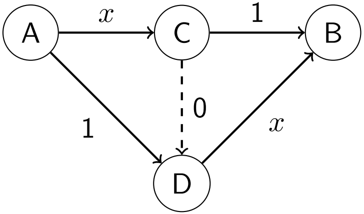

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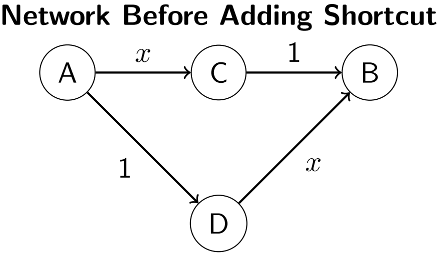

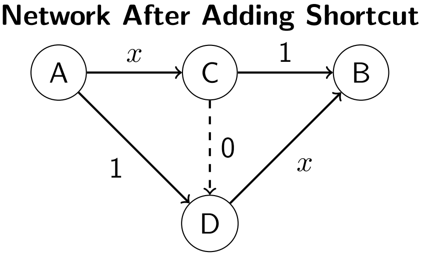

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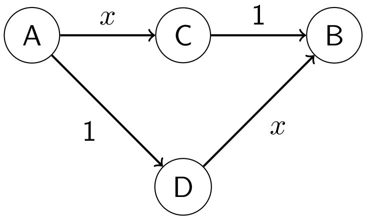

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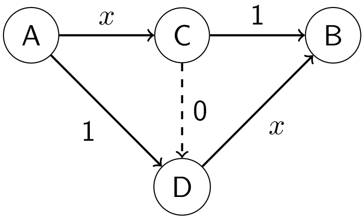

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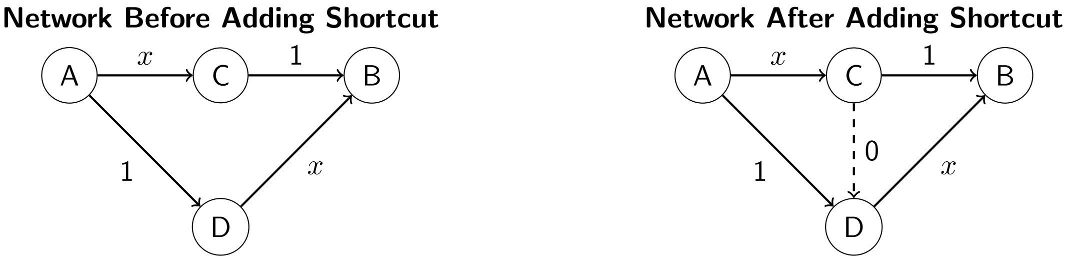

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A sample question: how to design "good" mechanisms in complex strategic environments.

Methodologies

Methodology Overview

Economic analysis using algorithmic tools.

- approximation analysis: design and analysis of simple mechanisms in complex environments where finding the optimal is infeasible or undesirable.
- robust analysis: design robust mechanisms in the absence of detailed knowledge about the environment.
- data analysis: how to design good mechanisms with access to historical data.

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Goal: understand the design of good mechanisms in practical applications.

- online platforms (Google/Meta);
- resource allocations (FCC Spectrum/Land Resource/Cloud Computing);
- blockchains and cryptocurrencies (Bitcoin);
- recommendation system (Yelp/Netflix);
- etc.

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In future lectures, we will see an algorithm for making good forecasts without any information about the future.

Worst-case Approximations

- Algorithm / Mechanism M;
- Benchmark B;
- Set of possible inputs $\mathcal{F}$.

Worst-case approximation:

$$\mathrm{APX}(M) \triangleq \max_{F \in \mathcal{F}} \frac{B(F)}{M(F)}$$

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How do we evaluate this approximation? Is 2 "good enough"?

Parametrized Instances

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Asymptotic analysis: understand how the worst-case approximation guarantee scales with the instance size.

- $\mathrm{APX}(M; n)$: worst-case approximation of M when input size is n.

Asymptotic Analysis

- $f(n) = O(g(n)) : \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$;
- $f(n) = \Omega(g(n)) : \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$.
- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$;
- $f(n) = o(g(n))$ if $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$;
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Example:

- $2n^2 + 8n + 100 = O(n^2)$;
- $16n^3 = o(2^n)$.
- $4n - 32 = \Theta(n)$.
- $\log(n) = o(n^\epsilon)$ for any constant $\epsilon > 0$.

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2-approximation to the optimal: Great! Same rate as the optimal!

Online Selection Problems

Problem: n items arriving online.

- item i has value $v_i \sim F_i$;
- the agent knows $F_1, \ldots, F_n$ at time 0.
- at time $i \leq n$, the agent observes value v_i and decides whether to select item i (if the selection has not been made).

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How to make good decision without knowing the future?

Application: Job Hiring Process

A firm wants to hire for a vacant position.

- optimal policy: interview all the candidates, and selects the best one after the interviews.
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Question: how to design good online hiring policies? What is the loss of adhering to online policies?

- designing the online policy is the same as the previous online selection problem.

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- the probability of choosing the highest value is $\frac{1}{n} \Rightarrow \mathrm{APX}(\mathrm{RS}) = n$.
- can we do better?

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The designer cannot foresee the future values. How would she know whether to select the current value or not?

Threshold Policies

The designer knows the distribution of values and can predict the expected gain from the future if the current value is not selected.

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Simple policy in practice: threshold policies

- set threshold τ;
- at time i, selects item i if and only if $v_i \geq \tau$.

τ is an approximation of what the designer can gain in the future.

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The expected performance of the algorithm is

$$
\begin{aligned}
\mathrm{ALG}_\tau &= p_\tau \cdot \tau + \sum_{i \le n} \Pr[v_j < \tau, \forall j < i] \cdot \mathbf{E}\left[(v_i - \tau)^+\right] \\
&\ge p_\tau \cdot \tau + (1 - p_\tau) \cdot \sum_{i \le n} \mathbf{E}\left[(v_i - \tau)^+\right] \\
&\ge p_\tau \cdot \tau + (1 - p_\tau) \cdot \left(\mathbf{E}\left[\max_i v_i\right] - \tau \right)
\end{aligned}
$$

Last inequality holds since $\max_i v_i \le \tau + \max_i (v_i - \tau)^+ \le \tau + \sum_i (v_i - \tau)^+$.

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$$\mathrm{ALG}_\tau \geq p_\tau \cdot \tau + (1 - p_\tau) \cdot \left(\mathbf{E}\left[\max_i v_i\right] - \tau \right).$$

- Mean Rule: Let $\tau = \frac{1}{2}\mathbf{E}[\max_i v_i]$. We have

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Prophet Inequalities

Theorem

There exists a threshold policy that achieves a 2-approximation, i.e., it achieves expected value at least $\frac{1}{2}\mathbf{E}[\max_i v_i]$.

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- Median Rule: Let τ such that $p_\tau = \frac{1}{2}$. We have

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The gap is 2 when $z \to \infty$.

Application: Auctions

Auctions: a single item, n agents.

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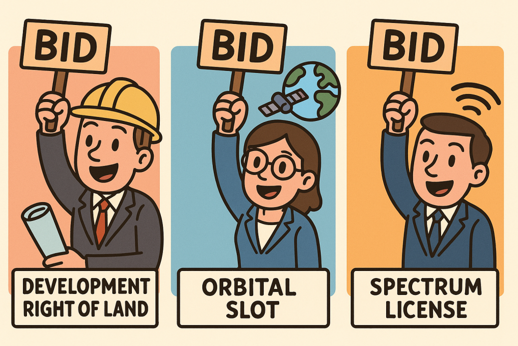
- distributing scarce resource: spectrum license; display of Ad slots; development rights on lands; orbital slots for satellites; pollution permits; ...



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Posted pricing mechanism: given prices $\{p_i\}_{i\in[n]}$

- the item is sold to an agent with value $v_i \geq p_i$.
- tie-breaking π when there are multiple agents with high values.

Connection to Auctions

Prophet inequality: n items

- value distributions $F = F_1 \times \cdots \times F_n$;
- threshold τ;
- arrival order π.

Posted pricing mechanism: n agents

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Given any valuation profile $v = (v_1, \ldots, v_n)$, the selected value and the optimal value in both problems are the same.

- Posted pricing mechanism has a 2-approximation to the optimal welfare.

Basics on Game Theory

Incomplete Information Games

A static game with incomplete information is denoted as
$\Gamma_I = \left(N, (A_i)_{i\in N}, (u_i)_{i\in N}, (\Theta_i)_{i\in N}, \mu\right)$ where

- N is the set of players;
- A_i is the set of player i's actions; (what the agents can do)
- Θ_i is the set of player i's "types" where $\theta_i \in \Theta_i$ is private information of i; (what the agents know)
- $u_i : A \times \Theta \to \mathbb{R}$ is player i's payoff function (where $A = \times_{i\in N} A_i$, and $\Theta = \times_{i\in N} \Theta_i$).
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μ is called a common prior.

- Let μ_i denote the marginal distribution of μ on Θ_i, i.e., $\mu_i(\theta_i) \equiv \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_i, \theta_{-i})$.
- Let $\mu(\theta_{-i}|\theta_i)$ be the belief of agent i over θ_{-i} conditional on his type being θ_i.

Strategies and Bayesian Nash Equilibrium

A strategy of player i in Γ_I is a mapping $s_i : \Theta_i \to \Delta(A_i)$.

- s_i is a pure strategy if the mapping is deterministic, i.e., $s_i : \Theta_i \to A_i$. Let S_i be the set of pure strategies for i.

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Definition (BNE)

A strategy profile s is a Bayesian Nash Equilibrium if for any agent i and any type θ_i (such that $\mu_i(\theta_i) > 0$), for any action a_i^* in the support of $s_i(\theta_i)$, we have

$$a_i^* \in \operatorname*{argmax}_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \cdot \mathbf{E}_{a_{-i} \sim s_{-i}(\theta_{-i})}[u_i(a_i, a_{-i}, \theta)] \,.$$

Informal definition of BNE: all agents are doing the best they can given what they think others are doing.