Monotone Comparative Statics

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Introduction

Understand the response of different economic agents to changes in the underlying environment or conditions.

- firm's investments in innovations in response to changes in market competitions;
- firm's production in response to changes in market demands or production costs;
- investor's portfolio selections in stock markets in response to income shocks.

Reference:
`https://sites.duke.edu/toddsarver/files/2021/07/Micro-Lecture-Notes.pdf`

Decision Environments

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- T : parameters.

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Example: In monopoly production problem:

- X : set of quantities the monopoly can produce;
- T : set of possible cost of production;
- f : revenue function based on the produced quantity and the production cost.

Decision Environments

Assume that X is compact and $f(x,t)$ is continuous in x given any $t \in T$.

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Denote the set of optimal choice given parameter $t \in T$ as

$$\mathbf{X}(t) = \operatorname*{argmax}_{x \in X} f(x,t).$$

$\mathbf{X}(t)$ is a correspondence, and we denote the optimal choice as $\mathbf{x}(t)$ if the optimal choice set is a singleton.

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Comparative Statics: how does $\mathbf{X}(t)$ changes as a function of t.

Implicit Function Theorem

Assumption 1:

- f is twice continuously differentiable;
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FOC:

$$f_x(\mathbf{x}(t), t) = 0.$$

Implicit Function Theorem

Theorem (Implicit Function Theorem)

Given Assumption 1, we have

$$\mathbf{x}'(t) = -\frac{f_{xt}(\mathbf{x}(t), t)}{f_{xx}(\mathbf{x}(t), t)}$$

Taking total derivative over t given the FOC:

$$f_{xx}(\mathbf{x}(t), t) \cdot \mathbf{x}'(t) + f_{xt}(\mathbf{x}(t), t) = 0.$$

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Corollary

Given Assumption 1, we have $\mathbf{x}'(t) \geq 0$ if and only if $f_{xt}(\mathbf{x}(t), t) \geq 0$.

Proof: by the concavity assumption, $f_{xx}(\mathbf{x}(t), t) < 0$.

Applications: Optimal Production I

Consider a firm who can produce a quantity $x \in \mathbb{R}_+$ to maximize profit

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$$\max_{x \in \mathbb{R}_+^k} \ \pi(x,t) \triangleq x \cdot P(x) - C(x,t).$$

Applications: Optimal Production I

To apply the implicit function theorem, we need $\pi(x,t)$ is concave in x for all t.

$$\pi_{xx}(x,t) = P''(x) \cdot x + 2P'(x) - C_{xx}(x,t) \leq 0.$$

A sufficient condition is that

1. $P''(x) \leq 0$;
2. $P'(x) \leq 0$;
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2. $P'(x) \le 0$;
3. $C_{xx}(x,t) \ge 0$.

Note that $\pi_{xt}(x,t) = -C_{xt}(x,t)$.
$\Rightarrow$ By implicit value theorem, $\mathbf{x}'(t) \ge 0$ if and only if $\pi_{xt}(x,t) \ge 0$, or $C_{xt}(x,t) \le 0$.

- with higher t, the marginal cost for production decreases, e.g., $C(x,t) = \frac{x}{t}$.

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- with higher t, the marginal cost for production decreases, e.g., $C(x,t) = \frac{x}{t}$.

Question: are all the assumptions necessary for the comparative statics analysis? NO!

- especially $P''(x) \leq 0$ may not fit well with practical application.

Limitations

The implicit function theorem approach requires strong assumption on the objective f.

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- twice continuously differentiable;
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Both are not necessary for understanding how $x(t)$ changes in response to t.

The intuition on the requirement that $f_{xt}(x,t) \geq 0$ is roughly correct.

- increasing differences in choices and parameters.

Comparing Choice Sets

Optimal choice $\mathbf{X}(t)$ is a correspondence.

Definition (Strong Set Order)

For any $Y, Z \subseteq X$, we say Y dominates Z in strong set order if for any $y \in Y$ and $z \in Z$, $\min\{y, z\} \in Z$ and $\max\{y, z\} \in Y$. We denote this as $Y \geq_s Z$.

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In the special case where Y and Z are singletons, i.e., $Y = \{y\}$ and $Z = \{z\}$, strong set order is equivalent to $y \geq z$.

- see general graphic illustration on board for examples of Y, Z that satisfy/violate the strong set order.

Monotone Comparative Statics

Definition (Increasing Differences)

A function $f : X \times T \to \mathbb{R}$ has increasing differences in (x, t) if for any $x' > x$ and $t' > t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

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Increasing difference coincides with the definition that $f_{xt}(x, t) \geq 0$ when f is twice continuously differentiable.

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Monotone Comparative Statics

Theorem (Topkis '78)

If function $f : X \times T \to \mathbb{R}$ has increasing differences in (x,t), the optimal choice set $\mathbf{X}(t) = \mathrm{argmax}_{x \in X} f(x,t)$ is monotone non-decreasing in t in strong set order. That is, for any $t' \geq t$,

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Proof.

For any $t' \geq t$, and any $x' \in \mathbf{X}(t'), x \in \mathbf{X}(t)$, if $x' \geq x$, the theorem holds.

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Proof.

For any $t' \geq t$, and any $x' \in \mathbf{X}(t'), x \in \mathbf{X}(t)$, if $x' \geq x$, the theorem holds.
If $x' < x$,

$$\begin{aligned} 0 &\leq f(x, t) - f(x', t) && (x \in \mathbf{X}(t)) \\ &\leq f(x, t') - f(x', t') && \text{(increasing differences)} \\ &\leq 0 && (x' \in \mathbf{X}(t')) \end{aligned}$$

All equalities must hold with equality. □

Monotone Comparative Statics

Theorem (Topkis '78)

If function $f : X \times T \to \mathbb{R}$ has decreasing differences in (x, t), the optimal choice set $\mathbf{X}(t) = \operatorname{argmax}_{x \in X} f(x, t)$ is monotone non-increasing in t in strong set order. That is, for any $t' \geq t$,

$$\mathbf{X}(t') \geq_s \mathbf{X}(t).$$

Similar argument as increasing differences.

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Consider a firm who can produce a quantity $x \in \mathbb{R}_+$ to maximize profit

$$\max_{x \in \mathbb{R}_+^k} \ \pi(x,t) \triangleq x \cdot P(x) - C(x,t).$$

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Let $\mathbf{X}(t) = \mathrm{argmax}_{x \in \mathbb{R}^k_+} \ \pi(x,t)$.

By [Topkis '78], if $-C_{xt}(x,t) \leq 0, \forall x, t$, $\pi(x,t)$ has increasing differences in (x,t),

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If the marginal cost of production decreases, the optimal quantity produced weakly increases.

- Indeed, we don't need π to be concave in x.

Applications: Optimal Production II

Consider a firm who can produce a quantity $x \in \mathbb{R}_+$ of products using $z \in \mathbb{R}^k_+$ as inputs.

- $F : z \in \mathbb{R}^k_+ \to \mathbb{R}_+$: production function;
- $p > 0$: price for the product;
- $w \in \mathbb{R}^k_{++}$: price vector for the inputs.

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The firm's problem is

$$\max_{z \in \mathbb{R}^k_+} \; p \cdot F(z) - w \cdot z.$$

Applications: Optimal Production II

The firm's problem is equivalent to

$$\pi(x,p) = \max_{x\in\mathbb{R}_+} \; p\cdot x - C(x).$$

$$\text{where } C(x) = \min\left\{ w\cdot z : F(z) \geq x \right\}.$$

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$\pi(x,p)$ satisfies increasing differences in $(x,p) \Rightarrow$ optimal production satisfies

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Applications: Monopoly Auction

A single item, a single agent.

- the agent has value $v \sim F$;
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Lemma

Given any direct revelation mechanism, the interim allocation of the agent is non-decreasing in his value.

Recall that this can be proved by the Envelope Theorem [Milgrom and Segal '02].

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Taxation principle: the agent is offered a menu of allocation-payment pairs $\{m_i = (x_i, p_i)\}$.

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By [Topkis '78], the optimal menu choice $m(v)$ is non-decreasing in v.

- the allocation is non-decreasing in v since higher menu choice represents higher allocations.

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Extension: non-linear utility: $u = f(v, x) - p$

- f is increasing in v and x;
- f has increasing differences in (v, x), e.g, $f(v, x) = e^{v \cdot x^2}$.

Applications: Signaling Games

Workers with type $\theta \in \Theta = \{\theta_0, \ldots, \theta_n\}$.

- $\theta_0 < \cdots < \theta_n$;
- q_θ: prior probability of type θ;
- $C(e, \theta)$: cost of education/signal $e \geq 0$ given type θ; increasing in e and decreasing in θ.

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Theorem

If the cost of signaling $-C(e, \theta)$ has increasing differences in (e, θ), the separating equilibrium exists.

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Construction by induction: $e(\theta_0) = 0$,

$$\theta_{i-1} - C(e(\theta_{i-1}), \theta_{i-1}) = \theta_i - C(e(\theta_i), \theta_{i-1}), \quad \forall i \in \{1, \dots, n\}.$$

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Local IC implies global IC under increasing differences.

Monotone Transformation

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Idea: apply monotone transformation on f to create increasing difference while preserving the optimal solution.

- see graphical illustrations for adjusting f without affecting choice sets.

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Idea: apply monotone transformation on f to create increasing difference while preserving the optimal solution.

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Let $g : \mathbb{R} \times T \to \mathbb{R}$ be a function that is strictly increasing in its first argument for all $t \in T$.

$$\mathbf{X}(t) \triangleq \underset{x \in X}{\operatorname{argmax}}\ f(x,t) = \underset{x \in X}{\operatorname{argmax}}\ g(f(x,t),t), \quad \forall t \in T.$$

If $g(f(x,t),t)$ has increasing differences in (x,t), $\mathbf{X}(t)$ is non-decreasing in t in strong set order.

Applications: Market Size

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Letting $N > 0$ be the market size, the firm's problem is

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where $C(Nq)$ is the cost of producing quantity Nq.

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where $C(Nq)$ is the cost of producing quantity Nq.

Question: how optimal q or $P(q)$ changes with respect to N.

- $\pi(q, N)$ does not have increasing differences in (q, N) in general (without strong assumptions on $P(q)$).

Applications: Market Size

Let $g(\pi(q, N), N) = \frac{\pi(q,N)}{N}$.

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Assuming C is twice continuously differentiable,

$$g_{qN}(\pi(q,N),N) = -q\cdot C''(Nq).$$

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Larger market ($\Rightarrow$ higher marginal cost of production) $\Rightarrow$ smaller quantity produced $\Rightarrow$ higher market price.

Necessity of Assumptions

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- recall the trick of monotone transformation.

The cardinal values of f is not always important for comparative statics.

Single Crossing

A ordinal version for comparative statics:

Definition

A function $f : X \times T \to \mathbb{R}$ has the single crossing property in (x, t) if for any $x' > x$ and $t' > t$,

$$f(x', t) - f(x, t) \geq 0 \Rightarrow f(x', t') - f(x, t') \geq 0;$$
$$f(x', t) - f(x, t) > 0 \Rightarrow f(x', t') - f(x, t') > 0.$$

- The single crossing property is a property based on the ordinal preference.
- $\forall x' > x,\ g(t) \triangleq f(x', t) - f(x, t)$ cross 0 from below by at most once.

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Remark: f has increasing differences in $(x, t) \Rightarrow f$ has the single crossing property in (x, t).

Monotone Comparative Statics

Theorem (Milgrom and Shannon '94)

If function $f : X \times T \to \mathbb{R}$ has the single crossing property in (x, t), the optimal choice set $\mathbf{X}(t) = \operatorname{argmax}_{x \in X} f(x, t)$ is monotone non-decreasing in t in strong set order. That is, for any $t' \geq t$,

$$\mathbf{X}(t') \geq_s \mathbf{X}(t).$$

Same argument as in [Topkis '78].

Necessity of Single Crossing

Theorem

Suppose $X, T \subseteq \mathbb{R}$ and $f : X \times T \to \mathbb{R}$. The optimal choice set $\mathbf{X}_S(t) = \operatorname{argmax}_{x \in S} f(x,t)$ is monotone non-decreasing in t in strong set order for any $S \subseteq X$ if and only if f has the single crossing property in (x,t).

If direction: [Milgrom and Shannon '94].

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If direction: [Milgrom and Shannon '94].

Only if direction: prove by contradiction.
(partial proof): there exists $x' > x, t' > t$ such that

$$f(x',t) - f(x,t) \geq 0 \text{ and } f(x',t') - f(x,t') < 0.$$

Restrict attention to $S = \{x, x'\}$.

- $x' \in \mathbf{X}(t)$;
- $x' \notin \mathbf{X}(t')$.

Necessity of Increasing Differences

Optimal production: A firm can generate a revenue of $f(x,t)$ by acquiring an input of $x \in \mathbb{R}_+$. If the price of the input is p, the profit of the firm given input x is

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Theorem

Suppose $X, T \subseteq \mathbb{R}$ and $f : X \times T \to \mathbb{R}$. Then $f(x,t) - px$ has the single crossing property in (x,t) for all $p \in \mathbb{R}$ if and only if f has increasing differences in (x,t).

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$f(x,t)$ has increasing differences in (x,t)
$\Rightarrow f(x,t) - px$ has increasing differences in (x,t)
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There exists $x' > x, t' > t$ such that

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Let $p > 0$ be a real number such that

$$f(x', t') - f(x, t') < p(x' - x) < f(x', t) - f(x, t).$$

By rearranging the terms, the single crossing property is violated.

Multivariate Comparative Statics

The agent maximizes a parametric objective function $f : X \times T \to \mathbb{R}$.

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For any $x, x' \in \mathbb{R}^n$:

- the meet of x and x' is $x \wedge x' \triangleq (\min\{x_1, x'_1\}, \ldots, \min\{x_n, x'_n\})$;
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Definition (Lattice)

$X \subseteq \mathbb{R}^n$ is a lattice if for any $x, x' \in X$, both $x \wedge x'$ and $x \vee x'$ are in X.

Comparing Choice Sets

Comparing choices: for any $x, x' \in \mathbb{R}^n$

- $x \geq x'$: $x_i \geq x'_i$ for all i;
- $x < x'$: $x_i \geq x'_i$ for all i and $x \neq x'$.

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Definition (Strong Set Order)

For any $Y, Z \subseteq X$, we say Y dominates Z in strong set order if for any $y \in Y$ and $z \in Z$, $y \wedge z \in Z$ and $y \vee z \in Y$. We denote this as $Y \geq_s Z$.

In the special case where $n = 1$, the reduces to the previous definition of strong set order.

Single Crossing

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Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $T \subseteq \mathbb{R}$. A function $f : X \times T \to \mathbb{R}$ has the single crossing property in (x, t) if for any $x' > x$ and $t' > t$,

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Quasisupermodularity

In multivariate comparative statics, the single crossing property is not sufficient for guaranteeing the strong set order in optimal choices.

- Intuition: When different coordinates of choices are substitutes, increasing the parameter may cause the choice variable to increase in one coordinate while decreasing in the other.
- Counterexample: exercise.

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Definition (Quasisupermodularity)

Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $T \subseteq \mathbb{R}$. A function $f : X \times T \to \mathbb{R}$ is quasisupermodular in x if for any $x, x' \in X$ and $t \in T$,

$$f(x,t) \geq f(x \wedge x', t) \Rightarrow f(x \vee x', t) \geq f(x', t)$$
$$f(x,t) > f(x \wedge x', t) \Rightarrow f(x \vee x', t) > f(x', t)$$

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This definition is vacuous when $n = 1$.

- intuitively, it implies that the choices in different coordinates are complements;
- implied by supermodularity: $f(x \wedge x', t) + f(x \vee x', t) \geq f(x, t) + f(x', t)$.

Quasisupermodularity

Proposition

Suppose $X, Y \in \mathbb{R}$ and $f : X \times Y \to \mathbb{R}$. If f is quasisupermodular in (x, y), f has the single crossing property in (x, y).

For any $x' > x$ and $y' > y$, we have

$$f(x', y) \geq f(x, y) \Rightarrow f(x', y') \geq f(x, y'). \qquad \text{(quasisupermodularity)}$$

Remark: $(x', y) \wedge (x, y') = (x, y)$ and $(x', y) \vee (x, y') = (x', y')$.

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This is identical to the condition in single crossing properties.

Same argument applies if the inequalities are strict.

Monotone Comparative Statics

Theorem (Milgrom and Shannon '94)

Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $T \subseteq \mathbb{R}$. If function $f : X \times T \to \mathbb{R}$ is quasisupermodular in x and has the single crossing property in (x, t), the optimal choice set $\mathbf{X}(t) = \operatorname{argmax}_{x \in X} f(x, t)$ is monotone non-decreasing in t in strong set order.

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Given any $t' \geq t, x \in \mathbf{X}(t)$ and $x' \in \mathbf{X}(t')$,

$$
\begin{aligned}
& f(x,t) \geq f(x \wedge x', t) && (x \in \mathbf{X}(t)) \\
\Rightarrow\ & f(x \vee x', t) \geq f(x', t) && \text{(Quasisupermodularity)} \\
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Suppose by contradiction that $x \wedge x' \notin \mathbf{X}(t)$

$$\begin{aligned} & f(x,t) > f(x \wedge x', t) && (x \wedge x' \notin \mathbf{X}(t)) \\ \Rightarrow & f(x \vee x', t) > f(x', t) && \text{(Quasisupermodularity)} \\ \Rightarrow & f(x \vee x', t') > f(x', t') && \text{(Single crossing)} \\ \Rightarrow & x' \notin \mathbf{X}(t'). \end{aligned}$$

Applications: Le Chatelier Principle

Long run choices are often more elastic than short run ones.

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The agent maximizes a function $f : X \times Y \times T \to \mathbb{R}$.

- choice variables: $x \in X \subseteq \mathbb{R}$ and $y \in Y \subseteq \mathbb{R}$;
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Let $\mathbf{x}^s(y, t)$ be the short-run optimal choice of x fixing y and t.
Let $(\mathbf{x}(t), \mathbf{y}(t))$ be the long-run optimal choice of x, y fixing t.

- break tie by taking the maximum.

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Application: x is labor, y is capital, t is the parameter relates to the price of labor.

- in the short run, when the price of labor is changed, the adjustments in capital may not take in effect immediately, and the choice of labor is optimized given the previously optimal capital.

Applications: Le Chatelier Principle

Theorem (Milgrom and Roberts '96)

Suppose $X, Y, T \subseteq \mathbb{R}$ are compact and $f : X \times Y \times T \to \mathbb{R}$. If f is continuous and quasisupermodular in (x, y) and has the single crossing property in (x, y, t), for any $t' \geq t$,

$$\mathbf{x}(t) \leq \mathbf{x}^s(\mathbf{y}(t), t') \leq \mathbf{x}(t'), \qquad (t \to t')$$

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The optimal choice of labor respond more to the change of parameter t in the long run.