General Equilibrium

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EC5301

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How people should trade for exchange items? What can be sustained in equilibrium in the exchange economy? What are the properties of the equilibria?

- use equilibrium price to exchange the items for efficient allocations.

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- utility function $U^a : \mathbb{R}_+^\ell \to \mathbb{R}$
- endowment $\omega^a = (\omega_1^a, \omega_2^a, ..., \omega_l^a)$ in $\mathbb{R}_+^\ell$.

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Given market prices $p \in \mathbb{R}^\ell$, the income of agent a is $w^a = p \cdot \omega^a$.

- what are the demand of the agents given market prices and their income?

Recap on Demands

An economy with ℓ commodities

- consumption space is $\mathbb{R}^{\ell}_{+}$ (the positive orthant)
- utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$
- endowment/income/budget w
- price vector $p = (p_1, \ldots, p_\ell) \in \mathbb{R}^{\ell}_{++}$

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The budget set of the agent is

$$B(p, w) = \left\{ x \in \mathbb{R}^\ell_+ : p \cdot x \leq w \right\}.$$

Given price-budget pair (p, w), the demand is

$$x^* \in \underset{x \in B(p,w)}{\operatorname{argmax}} \; U(x).$$

Recap on Demands

Suppose that the utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ is
(P1) continuous, (P2) strongly monotone, and (P3) strictly quasi-concave.

Then for any (p, w) in $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$, there exists a *unique* element x^* in $\arg\max_{x \in B(p,w)} U(x)$. Moreover, for any $(p, w) \gg 0$, the demand function $\bar{x}(p, w) = \operatorname{argmax}_{x \in B(p,w)} U(x)$ has the following properties:

(a) it is continuous;

(b) it obeys the budget identity [i.e., $p \cdot \bar{x}(p, w) = w$];

(c) it is zero-homogeneous, [i.e. $\bar{x}(tp, tw) = \bar{x}(p, w)$ for any $t > 0$];

(d) it obeys the boundary condition: if $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and $I = \{i : \bar{p}_i = 0\}$ is nonempty, then

$$\sum_{i \in I} \bar{x}_i(p^n, w^n) \to \infty.$$

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Define $\hat{x}^a : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$.
Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

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The lemma holds since the demand function $\bar{x}^a(p, w)$ is zero-homogeneous and obeys the budget identity for any agent a.

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Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

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Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

- existence of equilibrium price $p^* \gg 0$ such that market clears;
- since Z is zero-homogeneous, if p^* is an equilibrium price so is λp^* for any $\lambda > 0$.

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
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$$\hat{x}^A(p) = \left(\frac{1}{3}, \frac{2p_1}{3p_2}\right).$$

Similarly, $\hat{x}^B(p) = \left(\frac{2p_2}{3p_1}, \frac{1}{3}\right)$. Therefore,

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

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Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Equilibrium price is (λ, λ) for any $\lambda > 0$.

Exchange Economy: Excess Demand

Theorem

The excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ of the economy $\mathcal{E}$ (under assumption (P1), (P2), (P3)) has the following properties:

(1) *it is zero-homogeneous,*

(2) *it obeys Walras' Law,*

(3) *it is continuous,*

(4) *it satisfies the boundary condition,*

(5) *it is bounded below.*

Note: Clear that Z is bounded below since

$$Z(p) = X(p) - \bar{\omega} > -\bar{\omega}.$$

Illustration: Cobb-Douglas utilities.

- demand of agent a for commodity j is $\alpha_j \cdot \frac{w^a}{p_j}$ where $w^a = p \cdot \omega^a$.

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Theorem (Arrow and Debreu '54; McKenzie '59)

Suppose Z satisfies properties (1) to (5). Then there is $p^* \gg 0$ such that $Z(p^*) = 0$.

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Brouwer's fixed point theorem is a (far-reaching) generalization of the intermediate value theorem.

Intermediate Value Theorem

Theorem (Intermediate value theorem)

Let f be a continuous function defined on some interval $[a, b]$. If $f(a)$ and $f(b)$ are of different signs, then there is $c \in [a, b]$ such that $f(c) = 0$.

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- $Z_1(p_1', 1) > 0$ for sufficiently small p_1', and $Z_1(p_1'', 1) < 0$ for sufficiently large p_1'' (by boundary condition & bounded below).
- There exists p_1 such that $Z_1(p_1, 1) = 0$ (by continuity and intermediate value theorem).

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