Expert Learning

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EC4501/EC4501HM

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Consider an online decision process with T periods and n experts.

- the sequence of payoffs $\{v_{i,t}\}_{i\in[n],t\in[T]}$ are determined by an adversary, where $v_{i,t} \in [0, 1]$.

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At any time $t \leq T$:

- designer selects an expert i_t^*;
- the designer receives a payoff of $v_{i_t^*,t}$;
- the designer observes the realized payoffs for all experts.

Regret Minimization

Optimal-in-hindsight Benchmark:

$$B_T = \max_{i \in [n]} \sum_{t \in T} v_{i,t}.$$

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An algorithm has no-regret if $\mathrm{R}_T = o(T)$.

- Is it possible to design no-regret algorithms without any knowledge about the future reward realizations?

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Follow-the-Leader Algorithm:

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Consider an example with two experts:

- expert 1 has reward sequence $1, 0, 0, 1, 1, 0, 0, \ldots;$
- expert 2 has reward sequence $0, 1, 1, 0, 0, 1, 1, \ldots;$
- each expert gets $\frac{T}{2}$, the algorithm gets $\frac{T}{4}$. Regret is $\frac{T}{4}$.

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Need randomization in algorithms: hedge against adversarial rewards.

- Any deterministic algorithm (e.g., Explore-then-Exploit, UCB) has linear regret.

Hedge Algorithm

Hedge algorithm with learning rate η:
the probability choosing action i at time t is

$$p_t(i) = \frac{\exp(\eta \cdot \hat{\mu}_{i,t})}{\sum_{j=1}^{n} \exp(\eta \cdot \hat{\mu}_{i,t})}.$$

where $\hat{\mu}_{i,t} = \sum_{s<t} v_{i,s}$ is the historical rewards for expert i.

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Theorem

The worst-case regret of Hedge is $O(\sqrt{T \cdot \log n})$.

Hedge Algorithm

Lemma

The worst-case regret of Hedge is $\mathrm{R}_T \leq \frac{\log n}{\eta} + \frac{\eta T}{2}$.

By setting $\eta = \sqrt{\frac{2 \log n}{T}}$, we have $\mathrm{R}_T \leq \sqrt{2T \cdot \log n}$.

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Trade-offs for learning rate:

High η (Aggressive Learner): You react very strongly to daily results.

- Upside: You quickly identify and exploit a winning expert.
- Downside (The "Mistake Cost"): If the best expert has one unlucky bad day, you slash their weight dramatically. This cost of overreactions is proportional to $\eta \cdot T$.

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Low η (Cautious Learner): You react very calmly to daily results.

- Upside: You are stable and don't get thrown off by a single bad day.
- Downside (The "Ignorance Cost"): If one expert is consistently brilliant, it takes you a very long time to give them the majority of your trust. This cost of slow adaption is proportional to $\frac{\log n}{\eta}$.

Potential Function Analysis

Define the potential functions as the exponential of the rewards

$$W_{i,t} \triangleq e^{\eta \hat{\mu}_{i,t}}, \qquad W_t \triangleq \sum_{i=1}^{n} W_{i,t} = \sum_{i=1}^{n} e^{\eta \hat{\mu}_{i,t}}.$$

Initially, $W_{i,t} = 1$ and $W_1 = n$. After one step:

$$W_{t+1} = \sum_{i=1}^{n} e^{\eta \hat{\mu}_{i,t+1}} = \sum_{i=1}^{n} e^{\eta \hat{\mu}_{i,t}} \cdot e^{\eta v_{i,t}} = W_t \cdot \sum_{i=1}^{n} p_t(i) \cdot e^{\eta v_{i,t}}.$$

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Potential of the Hedge algorithm:

$$W_H \triangleq e^{\eta \sum_{t=1}^{T} \sum_{i=1}^{n} p_t(i) v_{i,t}}.$$

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Closely track the total potential function via Hedge:

$$W_H \approx W_T \geq \max_i W_{i,t} \Rightarrow \text{Hedge} \gtrsim B_T = \max_{i \in [n]} \sum_{t \in T} v_{i,t}.$$

Information Intuition

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\begin{aligned}
W_{t+1} &= W_t \cdot \sum_{i=1}^{n} p_t(i) \cdot e^{\eta v_{i,t}} \\
&= W_t \cdot \sum_{i=1}^{n} p_t(i)(1 + \eta v_{i,t}) && (e^x = 1 + x) \\
&= W_t \cdot (1 + \eta \sum_{i=1}^{n} p_t(i) v_{i,t}) && (\textstyle\sum_{i=1}^{n} p_t(i) = 1) \\
&= W_t \cdot e^{\eta \sum_{i=1}^{n} p_t(i) v_{i,t}} && (e^x = 1 + x)
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Thus,

$$
W_T = W_1 \cdot e^{\eta \sum_{t=1}^{T} \sum_{i=1}^{n} p_t(i) v_{i,t}} = n \cdot W_H.
$$

Proof of Hedge Algorithm

Bounding individual potentials:

Rewriting $\eta v_{i,t}$ as $\eta + \eta(v_{i,t} - 1)$, we have

$$e^{\eta v_{i,t}} \leq e^{\eta} \cdot \left(1 + \eta(v_{i,t} - 1) + \frac{1}{2}\eta^2(v_{i,t} - 1)^2 \right) \qquad (\exp(x) \leq 1 + x + \tfrac{1}{2}x^2 \text{ for } x \leq 0)$$

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Bounding updates of aggregated potentials:

$$\begin{aligned}
\sum_{i=1}^{n} p_t(i) \cdot e^{\eta v_{i,t}} &\leq \sum_{i=1}^{n} p_t(i) \cdot e^{\eta} \cdot \left(1 + \eta(v_{i,t} - 1) + \frac{1}{2}\eta^2(v_{i,t} - 1)^2 \right) \\
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&\leq e^{\eta \sum_{i=1}^{n} p_t(i) v_{i,t} + \frac{\eta^2}{2}}. && (v_{i,t} \in [0,1] \Rightarrow (v_{i,t} - 1)^2 \in [0,1])
\end{aligned}$$

Proof of Hedge Algorithm

Unrolling the recursion of potential function with $W_1 = n$:

$$W_{T+1} \leq n \cdot e^{\eta \sum_{t=1}^{T} \sum_{i=1}^{n} p_t(i) v_{i,t} + \frac{\eta^2 T}{2}}.$$

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Thus, for any expert i, we have $e^{\eta \hat{\mu}_{i,T+1}} \leq W_{T+1}$ and hence:

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Taking logs and rearranging:

$$\mathrm{R}_T = \hat{\mu}_{i,T+1} - \sum_{t=1}^{T} \sum_{i=1}^{n} p_t(i) v_{i,t} \leq \frac{\log n}{\eta} + \frac{\eta T}{2}.$$

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Example Regularization:

- *L2 regularization:* $l(p) = \frac{\lambda}{2}\|p\|^2$.
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Hedge is FTRL with entropy regularization.

Calibration

We want the prediction of the forecast to be credible and trustworthy:

- If a weather forecaster predicts the probability of raining, we want the frequency of raining to match the prediction; e.g., if the forecaster predicts the probability of raining is 50% for some days, the prediction is calibrated if half of those days are raining.
- If a financial manager/LLM/AI predicts the probability of a positive return for an investment option, we want the frequency of positive return to match the prediction.

Calibration

prediction	50%	50%	33.3%	50%	33.3%	33.3%	50%
outcome	rain	sunny	sunny	rain	rain	sunny	sunny

Table: Calibrated Forecast

prediction	42.9%	42.9%	42.9%	42.9%	42.9%	42.9%	42.9%
outcome	rain	sunny	sunny	rain	rain	sunny	sunny

Table: Calibrated Forecast

prediction	50%	25%	25%	50%	25%	25%	50%
outcome	rain	sunny	sunny	rain	rain	sunny	sunny

Table: Non-calibrated Forecast

Swap Regret

Swap Regret (Internal Regret):

$$\mathrm{SR}_T = \max_{\pi:A\to A} \sum_{t\in T} v_{\pi(i_t^*),t} - \sum_{t\in T} v_{i_t^*,t}.$$

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Lemma

For any bandit instance and any learning algorithm, $\mathrm{SR}_T \geq \mathrm{R}_T$.

Intuitive Connections

Calibration: probabilistic forecasts; no improvement by changing any forecast.

No-swap-regret: utility maximization; no improvement by switching actions.

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Connecting probabilistic forecasts with utility maximization: proper scoring rule $S(p, \omega)$

$$\mathbf{E}_{\omega \sim p}[S(p, \omega)] \geq \mathbf{E}_{\omega \sim p}\left[S(p', \omega)\right], \forall p, p'.$$

- Quadratic scoring rule: $S(p, \omega) = 1 - (p - \omega)^2$.
- Log scoring rule: $S(p, \omega) = \log p(\omega)$.

Reduction

A calibrated forecast based on any no-swap-regret algorithm $\mathcal{A}$:

- construct a proper scoring rule for converting probabilistic forecasts to realized payoffs;
- apply no-swap-regret algorithm $\mathcal{A}$, with actions being probabilistic forecasts, for payoffs given by scoring rules.

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By the definition of proper scoring rules, the following are equivalent:

- the forecast is calibrated, i.e., for any forecast p, the empirical distribution in periods predicting p is also p;
- the algorithm has no swap regret, i.e., for any action i (forecast p_i), the utility of swapping i to another action i' (forecast $p_{i'}$) is lower.

No Swap Regret

Theorem (Blum and Mansour '07)

When there are n actions and T periods, there is an algorithm that achieves swap regret at most $O(n\sqrt{T \log n})$.

Intuition:

- build a no (external) regret algorithm for each expert to ensure the regret of swapping that expert with others is small;
- find a smart way of aggregating the recommendations of different algorithm to ensure no swap regret.

No Swap Regret

1. Initialize an algorithm $\mathcal{A}_i$ for each expert i;
2. Let $q_{i,t}$ be the recommended distribution over experts from algorithm $\mathcal{A}_i$ at time t. Aggregate them into a distribution p_t.
3. Select an expert according to p_t. The designer observes rewards $v_{i,t}$ for all i.
4. For each algorithm $\mathcal{A}_i$, scale the rewards by $p_t(i)$ as feedback. I.e, $\mathcal{A}_i$ sees reward vector $p_t(i) \cdot v_t$.

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In step 2, the aggregate distribution p_t satisfies

$$p_t(i) = \sum_{j \in [n]} p_t(j) \cdot q_{j,t}(i), \forall i \in [n].$$

That is, $p_t = p_t \times q_t$.

No Swap Regret

For algorithm $\mathcal{A}_i$ and any expert $\pi(i) \in [n]$ its regret is

$$\mathrm{R}_{i,T} \geq \sum_{t \leq T} p_t(i) \cdot v_{\pi(i),t} - \sum_{t \leq T} p_t(i) \cdot \sum_{i \in [n]} q_{i,t} v_t.$$

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Summing over $i \in [n]$, we have

$$\begin{aligned}
\sum_i \mathrm{R}_{i,T} &\geq \sum_{i \in [n]} \sum_{t \leq T} p_t(i) \cdot v_{\pi(i),t} - \sum_{i \in [n]} \sum_{t \leq T} p_t(i) \cdot \sum_{i \in [n]} q_{i,t} v_t \\
&= \mathbf{E}\left[\sum_{t \leq T} v_{\pi(i_t^*),t}\right] - \sum_{t \leq T} p_t v_t = \mathrm{SR}_T.
\end{aligned}$$

No Swap Regret

For algorithm $\mathcal{A}_i$ and any expert $\pi(i) \in [n]$ its regret is

$$\mathrm{R}_{i,T} \geq \sum_{t \leq T} p_t(i) \cdot v_{\pi(i),t} - \sum_{t \leq T} p_t(i) \cdot \sum_{i \in [n]} q_{i,t} v_t.$$

Summing over $i \in [n]$, we have

$$\begin{aligned} \sum_i \mathrm{R}_{i,T} &\geq \sum_{i \in [n]} \sum_{t \leq T} p_t(i) \cdot v_{\pi(i),t} - \sum_{i \in [n]} \sum_{t \leq T} p_t(i) \cdot \sum_{i \in [n]} q_{i,t} v_t \\ &= \mathbf{E}\left[\sum_{t \leq T} v_{\pi(i_t^*),t}\right] - \sum_{t \leq T} p_t v_t = \mathrm{SR}_T. \end{aligned}$$

Since we have algorithms such that $\mathrm{R}_{i,T} \leq \sqrt{2T \log n}$ for all $i \in [n]$, we have $\mathrm{SR}_T \leq n\sqrt{2T \log n}$.