Economics and Computation

Yingkai Li

EC4501/EC4501HM Semester 2, AY2024/25

Logistics

Instructor: Yingkai Li

Office: AS2 05-21

Office hour: by appointment.

Reading Lists

1. Aleksandrs Slivkins. *Introduction to Multi-Armed Bandits.* `https://arxiv.org/abs/1904.07272`
2. Jason Hartline. *Mechanism Design and Approximation.* `https://jasonhartline.com/MDnA/`
3. Tim Roughgarden. *Twenty Lectures on Algorithmic Game Theory.* `https://timroughgarden.org/notes.html`

Additional readings:

- Noam Nisan, Tim Roughgarden, Éva Tardos, Vijay V. Vazirani. *Algorithmic Game Theory.* Cambridge University Press.
- Federico Echenique, Nicole Immorlica, Vijay V. Vazirani. *Online and Matching-Based Market Design.* Cambridge University Press.

Prerequisite

Required: Basics in probabilities, calculus, and how to prove formal theorems.

Not required: solid background knowledge about algorithm design (CS), mechanism design (Econ), or game theory (Econ). Coding is also not required.

Evaluations

- Two assignments (40%); due on Sep 29th, Nov 7th.
- Course project (30%); due on Oct 31th, mid-term review on Oct 6th.
- Final exam (30%); scheduled on Nov 21th, 5pm.
- Survey paper (25%); due on Nov 10th; only for HM students.

Syllabus

Week 1:	Preview of the course
Week 2/3/4:	Learning: bandits, experts, calibration
Week 5/6:	Learning in games
Week 7/8:	Mechanism design: welfare, revenue
Week 9/10:	Robust mechanism design
Week 11/12:	Topic courses: fairness, contracts, etc.
Week 13:	Project presentation by students

Course Philosophy

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Computer science approach: provide simple and practical solutions that are “good enough” for complex problems.

This Course: focus on the computer science approaches in various economic problems.

A Simple Example: Complexity of Optimal Solutions

Knapsack problems:
n tasks, each task $i \in [n]$ requires a resource of c_i, and generates a value of v_i.

Objective: find a set of tasks to maximize the total value subject to a budget B on resources.

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- naïvely, the optimal solution can be found by enumerating all possible subsets, taking time $\exp(n)$, not practical.
- NP-hard means that "no algorithm" can find the optimal solution much faster than that.

A Simple Example: Simple Algorithms

Let us give up the optimal solution and only look for a “good enough” one.

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$$\text{Upper-Bound} \leq \text{Greedy} + \text{Max-Val} \Rightarrow \max\{\text{Greedy}, \text{Max-Val}\} \geq \frac{1}{2} \cdot \text{Upper-Bound}.$$

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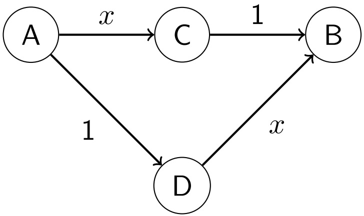
- A $\to$ C, D $\to$ B: travel time x, fraction of travelers.
- A $\to$ D, C $\to$ B: travel time 1.

Network Before Adding Shortcut



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- New road in network: open a portal from C to D with zero travel time.

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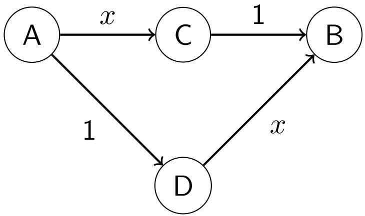

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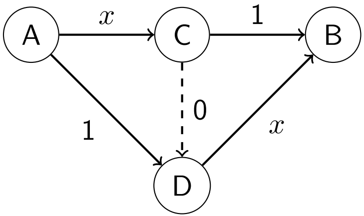

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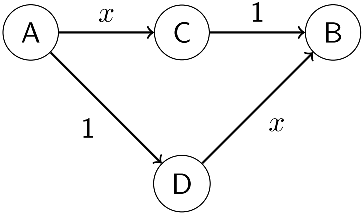

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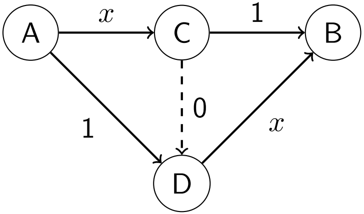

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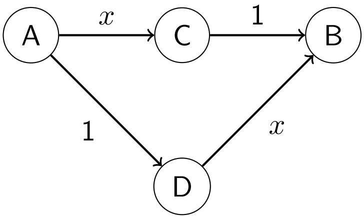

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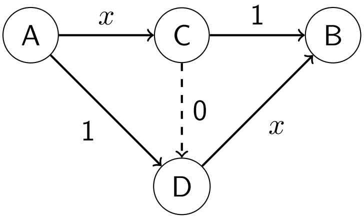

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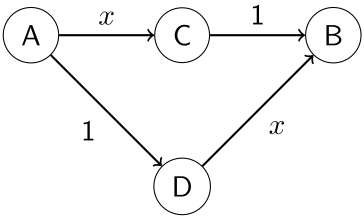

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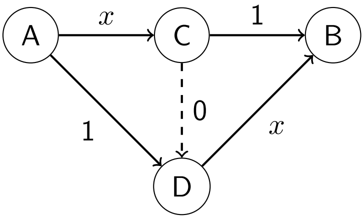

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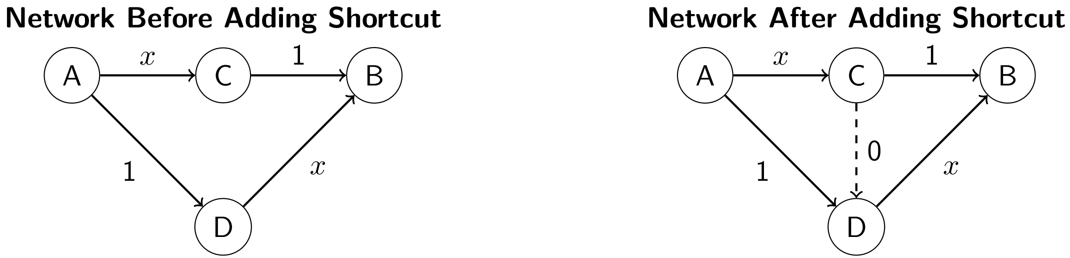

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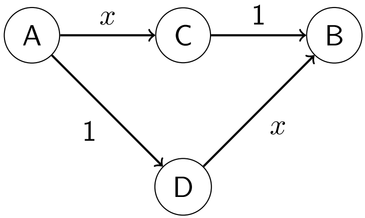

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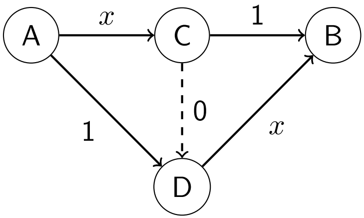

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A sample question: how to design "good" mechanisms in complex strategic environments.

Methodologies

Methodology Overview

Economic analysis using algorithmic tools.

- approximation analysis: design and analysis of simple mechanisms in complex environments where finding the optimal is infeasible or undesirable.
- robust analysis: design robust mechanisms in the absence of detailed knowledge about the environment.
- data analysis: how to design good mechanisms with access to historical data.

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Goal: understand the design of good mechanisms in practical applications.

- online platforms (Google/Meta);
- resource allocations (FCC Spectrum/Land Resource/Cloud Computing);
- blockchains and cryptocurrencies (Bitcoin);
- recommendation system (Yelp/Netflix);
- etc.

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In future lectures, we will see an algorithm for making good forecasts without any information about the future.

Worst-case Approximations

- Algorithm / Mechanism M;
- Benchmark B;
- Set of possible inputs $\mathcal{F}$.

Worst-case approximation:

$$\mathrm{APX}(M) \triangleq \max_{F \in \mathcal{F}} \frac{B(F)}{M(F)}$$

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How do we evaluate this approximation? Is 2 "good enough"?

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Asymptotic analysis: understand how the worst-case approximation guarantee scales with the instance size.

- $\mathrm{APX}(M; n)$: worst-case approximation of M when input size is n.

Asymptotic Analysis

- $f(n) = O(g(n)) : \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$;
- $f(n) = \Omega(g(n)) : \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$.
- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$;
- $f(n) = o(g(n))$ if $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$;
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Example:

- $2n^2 + 8n + 100 = O(n^2)$;
- $16n^3 = o(2^n)$.
- $4n - 32 = \Theta(n)$.
- $\log(n) = o(n^\epsilon)$ for any constant $\epsilon > 0$.

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2-approximation to the optimal: Great! Same rate as the optimal!

Online Selection Problems

Problem: n items arriving online.

- item i has value $v_i \sim F_i$;
- the agent knows $F_1, \ldots, F_n$ at time 0.
- at time $i \leq n$, the agent observes value v_i and decides whether to select item i (if the selection has not been made).

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How to make good decision without knowing the future?

Application: Job Hiring Process

A firm wants to hire for a vacant position.

- optimal policy: interview all the candidates, and selects the best one after the interviews.
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Question: how to design good online hiring policies? What is the loss of adhering to online policies?

- designing the online policy is the same as the previous online selection problem.

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- can we do better?

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The designer cannot foresee the future values. How would she know whether to select the current value or not?

Threshold Policies

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Simple policy in practice: threshold policies

- set threshold τ;
- at time i, selects item i if and only if $v_i \geq \tau$.

τ is an approximation of what the designer can gain in the future.

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The expected performance of the algorithm is

$$\begin{aligned}
\mathrm{ALG}_\tau &= p_\tau \cdot \tau + \sum_{i \le n} \Pr[v_j < \tau, \forall j < i] \cdot \mathbf{E}\left[(v_i - \tau)^+\right] \\
&\ge p_\tau \cdot \tau + (1 - p_\tau) \cdot \sum_{i \le n} \mathbf{E}\left[(v_i - \tau)^+\right] \\
&\ge p_\tau \cdot \tau + (1 - p_\tau) \cdot \left(\mathbf{E}\left[\max_i v_i\right] - \tau\right)
\end{aligned}$$

Last inequality holds since $\max_i v_i \le \tau + \max_i (v_i - \tau)^+ \le \tau + \sum_i (v_i - \tau)^+$.

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- Mean Rule: Let $\tau = \frac{1}{2}\mathbf{E}[\max_i v_i]$. We have

$$\mathrm{ALG}_\tau \geq p_\tau \cdot \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] + (1 - p_\tau) \cdot \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] = \frac{1}{2}\mathbf{E}\left[\max_i v_i\right].$$

Prophet Inequalities

Theorem

There exists a threshold policy that achieves a 2-approximation, i.e., it achieves expected value at least $\frac{1}{2}\mathbf{E}[\max_i v_i]$.

$$\mathrm{ALG}_\tau \geq p_\tau \cdot \tau + (1 - p_\tau) \cdot \left(\mathbf{E}\left[\max_i v_i\right] - \tau \right).$$

- Mean Rule: Let $\tau = \frac{1}{2}\mathbf{E}[\max_i v_i]$. We have

$$\mathrm{ALG}_\tau \geq p_\tau \cdot \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] + (1 - p_\tau) \cdot \frac{1}{2}\mathbf{E}\left[\max_i v_i\right] = \frac{1}{2}\mathbf{E}\left[\max_i v_i\right].$$

- Median Rule: Let τ such that $p_\tau = \frac{1}{2}$. We have

$$\mathrm{ALG}_\tau \geq \frac{1}{2}\tau + \frac{1}{2}\left(\mathbf{E}\left[\max_i v_i\right] - \tau \right) = \frac{1}{2}\mathbf{E}\left[\max_i v_i\right].$$

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The gap is 2 when $z \to \infty$.

Application: Auctions

Auctions: a single item, n agents.

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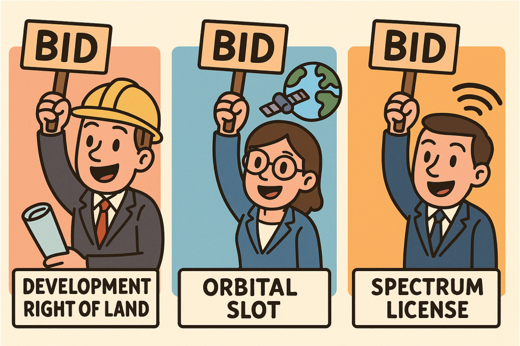
- distributing scarce resource: spectrum license; display of Ad slots; development rights on lands; orbital slots for satellites; pollution permits; ...



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Posted pricing mechanism: given prices $\{p_i\}_{i\in[n]}$

- the item is sold to an agent with value $v_i \geq p_i$.
- tie-breaking π when there are multiple agents with high values.

Connection to Auctions

Prophet inequality: n items

- value distributions $F = F_1 \times \cdots \times F_n$;
- threshold τ;
- arrival order π.

Posted pricing mechanism: n agents

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Given any valuation profile $v = (v_1, \ldots, v_n)$, the selected value and the optimal value in both problems are the same.

- Posted pricing mechanism has a 2-approximation to the optimal welfare.

Basics on Game Theory

Incomplete Information Games

A static game with incomplete information is denoted as
$\Gamma_I = \left(N, (A_i)_{i \in N}, (u_i)_{i \in N}, (\Theta_i)_{i \in N}, \mu\right)$ where

- N is the set of players;
- A_i is the set of player i's actions; (what the agents can do)
- Θ_i is the set of player i's "types" where $\theta_i \in \Theta_i$ is private information of i; (what the agents know)
- $u_i : A \times \Theta \to \mathbb{R}$ is player i's payoff function (where $A = \times_{i \in N} A_i$, and $\Theta = \times_{i \in N} \Theta_i$).
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μ is called a common prior.

- Let μ_i denote the marginal distribution of μ on Θ_i, i.e., $\mu_i(\theta_i) \equiv \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_i, \theta_{-i})$.
- Let $\mu(\theta_{-i}|\theta_i)$ be the belief of agent i over θ_{-i} conditional on his type being θ_i.

Strategies and Bayesian Nash Equilibrium

A strategy of player i in Γ_I is a mapping $s_i : \Theta_i \to \Delta(A_i)$.

- s_i is a pure strategy if the mapping is deterministic, i.e., $s_i : \Theta_i \to A_i$. Let S_i be the set of pure strategies for i.

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Definition (BNE)

A strategy profile s is a Bayesian Nash Equilibrium if for any agent i and any type θ_i (such that $\mu_i(\theta_i) > 0$), for any action a_i^* in the support of $s_i(\theta_i)$, we have

$$a_i^* \in \operatorname*{argmax}_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \cdot \mathbf{E}_{a_{-i} \sim s_{-i}(\theta_{-i})}[u_i(a_i, a_{-i}, \theta)] .$$

Informal definition of BNE: all agents are doing the best they can given what they think others are doing.