Simple Auctions and Approximations

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Mechanism Design

A mechanism design instance is denoted as $\Gamma_M = \left(N, \Omega, (v_i)_{i \in N}, (\Theta_i)_{i \in N}, F\right)$ where

- N is the set of agents;
- Ω is the set of outcomes;
- Θ_i is the set of agent i's "types" where $\theta_i \in \Theta_i$ is private information of i;
- $v_i : \Omega \times \Theta_i \to \mathbb{R}$ is agent i's value function;
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Let B_i be the report space of agent i.
A mechanism $M = (x, p)$:

- $x : B \to \Delta(\Omega)$;
- $p_i : B \to \mathbb{R}$, $\forall i$.

Single-item Auctions

Auctions: a single item, n agents.

- each agent i has a private value $v_i \sim F_i \in \Delta(\mathbb{R}_+)$;
- each agent i has linear utility $u_i = v_i x_i - p_i$ where $x_i \in [0,1], p_i \in \mathbb{R}$;
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Welfare: $\mathbf{E}[\sum_i v_i x_i]$.

Revenue: $\mathbf{E}[\sum_i p_i]$.

Welfare Approximations

Inefficiency of Standard Auctions

First-price auctions / all-pay auctions are not efficient in general.

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Price of Anarchy (PoA): given any auction M, letting $M(F)$ be the equilibrium welfare for auction M,

$$\mathrm{PoA}(M) = \max_F \frac{\mathrm{Wel}(F)}{M(F)}.$$

Smooth Auctions

Let $u_i(\mathbf{b};\theta_i) = v_i(x(\mathbf{b}),\theta_i) - p_i(\mathbf{b})$ be the utility of agent i given bid profile b.
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Definition (Smooth Auctions)

For parameters $\lambda \geq 0$ and $\mu \geq 1$, an auction is (λ, μ)-smooth if for every valuation profile $\mathbf{v} \in \mathcal{V}$ there exist bidding distributions $D_1^*(\mathbf{v}), \ldots, D_n^*(\mathbf{v})$ such that, for every bid profile $\mathbf{b}$,

$$\sum_i \mathbb{E}_{b_i^* \sim D_i^*(\mathbf{v})}[u_i(b_i^*, \mathbf{b}_{-i}; \mathbf{v}_i)] \geq \lambda \mathrm{Wel}(\mathbf{v}) - \mu \mathcal{R}(\mathbf{b}).$$

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First-price auction is $(\frac{1}{2}, 1)$-smooth.

- by bidding $\frac{v_i}{2}$, either wins and the utility is high, or loses and the total payment is high.

Smoothness of First-Price Auction

For each agent i, one possible strategy is to bid $b_i^* = \frac{v_i}{2}$ regardless of the opponents' strategy.

$$u_i(b_i^*, \mathbf{b}_{-i}; v_i) \geq \frac{1}{2}v_i - \mathcal{R}(\mathbf{b}).$$

since the bidder either wins and obtains utility $v_i - b_i^* = v_i - \frac{1}{2}v_i = \frac{1}{2}v_i \geq \frac{1}{2}v_i - \mathcal{R}(\mathbf{b})$, or loses and obtains utility $0 \geq \frac{1}{2}v_i - \mathcal{R}(\mathbf{b})$.

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Let x_i^* be the welfare optimal allocation. Since the bid $b_i^* = \frac{v_i}{2}$ guarantees non-negative utility,

$$u_i(b_i^*, \mathbf{b}_{-i}; v_i) \geq \left(\frac{1}{2}v_i - \mathcal{R}(\mathbf{b})\right) \cdot x_i^*(\mathbf{v}).$$

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Summing this inequality over all bidders i, we obtain

$$\sum_{i=1}^{n} u_i(b_i^*, \mathbf{b}_{-i}; v_i) \geq \sum_{i=1}^{n} \left(\frac{1}{2} v_i - \mathcal{R}(\mathbf{b})\right) \cdot x_i^*(\mathbf{v}) = \frac{1}{2} \mathrm{Wel}(\mathbf{v}) - \mathcal{R}(\mathbf{b}),$$

for every valuation profile $\mathbf{v}$ and bid profile $\mathbf{b}$.

PoA of Smooth Auctions

Theorem

For any $\lambda \leq 1, \mu \geq 1$, if an auction M is (λ, μ)-smooth, then for every product distribution F, every Bayes-Nash equilibrium of the auction has expected welfare at least $\frac{\lambda}{\mu} \cdot \mathrm{Wel}(F)$.

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Corollary

The price of anarchy for first-price auction is at most 2.

- The PoA for first-price auction can be improved to $\frac{e^2}{e^2-1} \approx 1.16$ [Jin and Lu '22].
- Techniques can be applied to other auction formats: all-pay auction is $(\frac{1}{2}, 1)$-smooth.

Proof of the Smooth Auction Theorem (1/2)

Extension Theorem: We construct a valid randomized deviation $D'_i(v_i)$.

1. Agent i samples fictitious valuations $\mathbf{v}'_{-i} \sim F_{-i}$.
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Let $\mathbf{s}$ be a BNE strategy profile, and let $\mathcal{G}$ be the resulting distribution of bids when $\mathbf{v} \sim F$.
By the BNE condition, the expected utility of $s_i(v_i)$ is at least that of the deviation $D'_i(v_i)$.

$$\begin{aligned}\mathbb{E}_{BNE}[u_i|v_i] &\geq \mathbb{E}_{b_i^* \sim D'_i(v_i), \mathbf{b}_{-i} \sim \mathcal{G}_{-i}}[u_i(b_i^*, \mathbf{b}_{-i}; v_i)] \\ &= \mathbb{E}_{\mathbf{v}'_{-i} \sim F_{-i}}\left[\mathbb{E}_{b_i^* \sim D_i^*(v_i, \mathbf{v}'_{-i}), \mathbf{b}_{-i} \sim \mathcal{G}_{-i}}[u_i(b_i^*, \mathbf{b}_{-i}; v_i)]\right].\end{aligned}$$

We take expectation over v_i and sum over i. Let U^{BNE} be the total expected utility.

$$U^{BNE} \geq \sum_i \mathbb{E}_{v_i, \mathbf{v}'_{-i}}\left[\mathbb{E}_{b_i^* \sim D_i^*(v_i, \mathbf{v}'_{-i}), \mathbf{b}_{-i} \sim \mathcal{G}_{-i}}[u_i(b_i^*, \mathbf{b}_{-i}; v_i)]\right].$$

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Let $\mathbf{v} = (v_i, \mathbf{v}'_{-i})$. We rewrite the inequality using linearity of expectation:

$$U^{BNE} \geq \mathbb{E}_{\mathbf{v} \sim F, \mathbf{b} \sim \mathcal{G}}\left[\sum_i \mathbb{E}_{b_i^* \sim D_i^*(\mathbf{v})}[u_i(b_i^*, \mathbf{b}_{-i}; v_i)]\right].$$

Crucially, $\mathbf{v}$ (from sampling) and $\mathbf{b}$ (from equilibrium) are independent here.

Proof of the Smooth Auction Theorem (2/2)

We apply the smoothness definition inside the expectation:

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Substituting this bound:

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Since $M(F) = U^{BNE} + R^{BNE}$, we have $M(F) + (\mu - 1)R^{BNE} \geq \lambda \mathrm{Wel}(F)$. Since $\mu \geq 1$ and $M(F) \geq R^{BNE}$:

$$\mu \cdot M(F) \geq M(F) + (\mu - 1)R^{BNE} \geq \lambda \mathrm{Wel}(F).$$

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In practice, many auctions run in parallel.

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Definition

A utility function is complement-free if there exists m additive valuations $f_1, \ldots, f_m$ such that for any set S,
$$f(S) = \max_{k \leq m} f_k(S).$$

Theorem (Composition Theorem)

If players have complement-free utility functions, then the simultaneous composition of (λ, μ)-smooth auctions is again a (λ, μ)-smooth auction.

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Corollary: PoA of the simultaneous composition of (λ, μ)-smooth auctions is at most $\frac{\mu}{\lambda}$.

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Illustration for unit-demand auction and simultaneous first-price auction.

- given valuation profile v, find optimal allocation $x(v)$;
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Reference: Roughgarden, T., Syrgkanis, V., & Tardos, E. (2017). The price of anarchy in auctions. Journal of Artificial Intelligence Research, 59, 59-101.

Reduction from Algorithms to Mechanisms

VCG Mechanisms

VCG mechanism: mechanism that implements efficient allocation in general environments.

- **allocation**: chooses outcome

$$\omega^* = \operatorname*{argmax}_{\omega \in \Omega} \sum_i v_i(\omega, \theta_i).$$

- **payment**: each agent i pays his externality on the welfare

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \geq 0.$$

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VCG mechanism may not be implementable in polynomial time.

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Is this tractable in practice? **NO!**

Example: (Knapsack problem) consider the allocation problem of servicing agents, where $\Omega \subseteq 2^N$.

- each agent has private value θ_i for being serviced;
- servicing each agent i requires a resource of r_i;
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How to find the optimal allocation? Trying all combination requires time exponential in $|N|$. Not practical if $n = |N|$ is large!

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Theorem

The maximum of greedy algorithm and max-value is a 2-approximation to the optimal value in the knapsack problem.

Example: 3D Matching

3D Matching: serving each agent requires two types of resources. N: agents; X: resource type 1; Y: resource type 2.

- $L = \{(i, x, y)\}$: the set of feasible ways to serve the agents;
- find the maximum number of agents that can be served simultaneously.

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Theorem

The greedy algorithm for finding the maximal matching is a 3-approximation to the optimal.

Intuition: in the greedy algorithm, when an agent is served, it will exclude at most two additional agents from the optimal matching.

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Reference: Hartline, J. D., Kleinberg, R., & Malekian, A. (2015). Bayesian incentive compatibility via matchings. Games and Economic Behavior, 92, 401-429.

General Equilibrium

Consider a market with n agents and n items.

- each agent i has unit value v_{ij} for item j;
- each agent i has demand at most f_i;
- each item j has supply at most g_j;
- $\sum_i f_i = \sum_j g_j = 1$.

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Intuition: use tâtonnement rule to adjust the price

- gradually increase the price of the item with excessive demand.

Revenue Approximations

Posted Pricing

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Question: is posted pricing mechanisms also approximately optimal for revenue maximization?

Recap: Prophet Inequality

Online Selection Problem: n items arriving online.

- item i has value $v_i \sim F_i$;
- the agent knows $F_1, \ldots, F_n$ at time 0.
- at time $i \le n$, the agent observes value v_i and decides whether to select item i (if the selection has not been made).

Theorem

There exists a threshold policy that achieves a 2-approximation, i.e., it achieves expected value at least $\frac{1}{2}\mathbf{E}[\max_i v_i]$.

Connection to Revenue Maximization

Prophet inequality: n items

- value distributions $F = F_1 \times \cdots \times F_n$;
- threshold τ for each item;
- arrival order π.

Posted pricing mechanism: n agents

- marginal revenues $F = F_1 \times \cdots \times F_n$;
- threshold τ for each agent i;
- tie breaking rule π.

Any threshold τ in the marginal revenue space corresponds to a price p_i in the value space.

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expected marginal revenue $=$ expected revenue
$\Rightarrow$ posted pricing mechanism has a 2-approximation to the expected revenue.

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Example: n agents. For agent $i \leq n$, $v_i = 2^i$ w.p $\frac{1}{2^{i+1}}$, and $v_i = 0$ w.p $1 - \frac{1}{2^{i+1}}$.

- optimal revenue $\Theta(n)$;
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Third-degree price discrimination is crucial for revenue maximization.

- competition and simultaneous implementation is not.

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Theorem (Yan '11)

Sequential posted pricing mechanism has an $\frac{e}{e-1}$-approximation to the expected revenue.

Correlation Gap

A non-negative real-valued set function f over subsets S of an n element ground set $N = \{1, \ldots, n\}$ and a distribution over subsets given by $\mathcal{D}$.

- $\hat{q}_i$: ex ante probability that element i is in the random set $S \sim \mathcal{D}$
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The correlation gap is the ratio of the expected value of the set function for the (correlated) distribution $\mathcal{D}$ to that with independent distribution $\mathcal{D}^I$, i.e.,

$$\frac{E_{S\sim\mathcal{D}}[f(S)]}{E_{S\sim\mathcal{D}^I}[f(S)]}.$$

Correlation Gap

Definition

A set function $f : 2^S \to \mathbb{R}$ defined on the subsets of a finite set S is called submodular if for all $A \subseteq B \subseteq S$ and $x \notin B$, the following inequality holds:

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B).$$

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Theorem

If the set function f is submodular, the correlation gap for function f is at most $\frac{e}{e-1}$.

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Example: submodular function $f(S) = \mathbf{1}(S \neq \emptyset)$.

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As $n \to \infty$, $\frac{E_{S\sim\mathcal{D}}[f(S)]}{E_{S\sim\mathcal{D}^I}[f(S)]} = \frac{e}{e-1}$.

Connection to Sequential Posted Pricing

Ex ante relaxation: consider the relaxed problem where the sum of ex ante probabilities of receiving an item is at most 1.

$$\mathrm{EAR} = \sum_i R_i(q_i) \quad \text{s.t.} \quad \sum_i q_i \leq 1.$$

EAR is an upper bound on the optimal revenue.

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Correlation gap implies that

$$\frac{E_{S \sim \mathcal{D}}[f(S)]}{E_{S \sim \mathcal{D}^I}[f(S)]} \leq \frac{e}{e-1}.$$

Extension of Approximations Under Non-linear Utilities

Non-linear Utilities

Two options, which one would you choose:

- get \$10M;
- draw a lottery, with probability $\frac{1}{2}$, get \$20M, and get nothing otherwise.

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Risk aversion: $t_i = (v_i, \varphi_i)$ where $v_i \in \mathbb{R}_+$, φ_i is an increasing concave function, and

$$u_i(t_i, x_i, p_i) = \varphi_i(v_i x_i - p_i).$$

Private budgets: $t_i = (v_i, B_i)$ where $v_i, B_i \in \mathbb{R}_+$, and

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Are simple mechanisms approximately optimal for non-linear utilities?

Demands and Ordinary Goods Assumption

In single-agent environments, a mechanism is posting a per-unit price p if the agent can purchase any lottery x with price $x \cdot p$ for any $x \in [0, 1]$.

- agent pays price $x \cdot p$ even if the realized allocation is 0.

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Assumption (Ordinary Goods)

$d^u(t, p)$ is non-increasing in p for all $t \in T$.

Excludes Giffen goods or Veblen goods.

Quantiles for Non-linear Agents

Recall for linear utilities: let $q = \mathbf{Pr}_{t' \sim F}[t' \geq t]$ be the quantile for type t.

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Demand of an agent with a private budget given per-unit pricing:

	$t_1 : (v = 5, B = 1)$	$t_2 : (v = 2, B = 2)$
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There isn't a simple deterministic and consistent way of ordering types for a non-linear agent.

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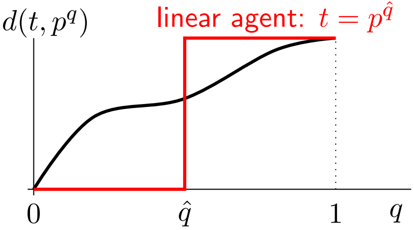

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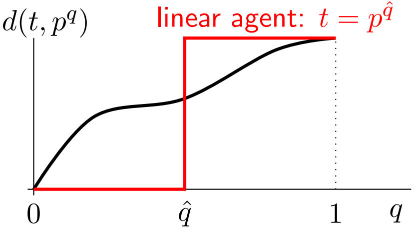
Remark: $q \sim U[0, 1]$: **Pr**$[z \leq q] = \mathbf{E}_{t \sim F}[d(t, p^q)] = q$.



Pricing-based Mechanisms in Quantile Space

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1. Map type t_i to quantile q_i according to $d(t, p^q)$, and calculate threshold as $\hat{q}_i = Q_i(q_{-i})$.
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Price-posting Equivalence Interpretation: Fixing any $\hat{q}_i$, from perspective of agent i

- wins the item with probability $d(t_i, p^{\hat{q}_i})$;
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Expected revenue: from any pricing-based mechanisms M for non-linear agents,

$$M(P) = \sum_i \mathbf{E}_{\forall j\neq i, q_j\sim U[0,1]}\left[P_i(Q_i(q_{-i}))\right].$$

Suboptimality of Pricing for Non-linear Agents

For linear agents: $R = \bar{P}$ [Bulow and Robert '89].

For non-linear agents: pricing-based mechanisms in general are not optimal, i.e., $R \neq \bar{P}$.

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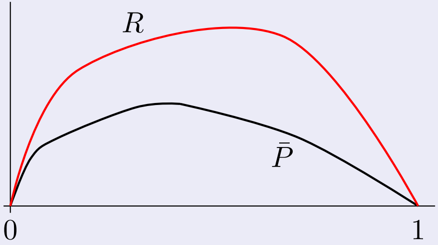
Lottery mechanism:

- offer menu of lotteries $(x_1 = \frac{1}{2}, p_1 = 1)$ and $(x_2 = 1, p_2 = 3)$;
- expected revenue equals 2.

Resemblance: Approximations in Single-agent Settings

Definition (ζ-resemblance)

A non-linear agent is ζ-resemblant to a linear agent if given any supply constraint $q \in [0,1]$, there exists a posted pricing mechanism with expected demand $q^\dagger \leq q$ such that $\bar{P}(q^\dagger) \geq \frac{1}{\zeta} R(q)$.



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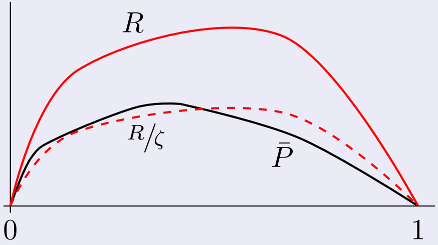

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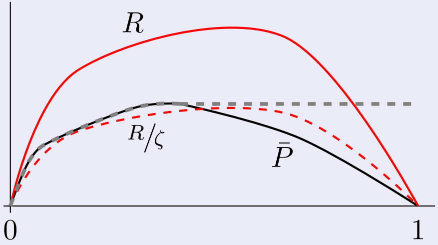

Reduction from Non-linear to Linear Agents

Theorem

For non-linear agents that are ζ-resemblant to linear agents,
pricing-based mechanism M is a γ-approximation to ex ante relaxation for linear agents
$\Rightarrow$ *M is a $\zeta\gamma$-approximation to ex ante relaxation for non-linear agents.*

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Non-linearities are often details that can be dispensed from the model without affecting main economic conclusions.

Economic conclusions for linear agents $\Rightarrow$ economic conclusions for non-linear agents.

ζ-resemblance for Non-linear Agents

	independent private budget	risk averse
revenue	3	e
welfare	2	1

Table: Summary of results for ζ-resemblance, assuming regularity for budgeted utility and MHR for risk averse utility under revenue objective.

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Corollary

For risk averse agents, sequential posted pricing is an ${}^{e}/_{(e-1)}$*-approximation to the optimal welfare.*