Welfare Theorems

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Pareto Optimality

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Definition

An allocation $\{z^a\}_{a\in A}$ is a Pareto improvement of another allocation $\{y^a\}_{a\in A}$ if $U^a(z^a) \geq U^a(y^a)$ for all $a \in A$ and the inequality is strict for at least one agent.

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Definition

An allocation $\{z^a\}_{a\in A}$ is a Pareto improvement of another allocation $\{y^a\}_{a\in A}$ if $U^a(z^a) \geq U^a(y^a)$ for all $a \in A$ and the inequality is strict for at least one agent. Moreover, an allocation $\{y^a\}_{a\in A}$ is Pareto optimal if it cannot be Pareto-improved by another feasible allocation.

Illustration in Edgeworth box.

The First Welfare Theorem

Definition

An allocation $\{x^a\}_{a \in A}$ is a Walrasian allocation if there exists $p \in \mathbb{R}_+ +^{\ell}$ such that $Z(p) = 0$ and $x^a = \hat{x}(p)$.

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Intepretation: equilibrium allocation is always efficient.

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Intepretation: equilibrium allocation is always efficient.

Remark: we do not assume quasi-concave or continuous utility here.

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Suppose that there exists an allocation $\{z^a\}_{a \in A}$ that is a Pareto improvement of $\{\hat{x}^a\}_{a \in A}$:

$$U^a(z^a) \geq U^a(\hat{x}^a(p^*)), \quad \forall a \in A,$$

and $\exists \tilde{a}$ such that it holds with a strict inequality.

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Lemma

1. $p \cdot z^a \geq p \cdot \omega^a$ *for all agents* a.
2. $p \cdot z^{\tilde{a}} > p \cdot \omega^{\tilde{a}}$.

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Combining the inequalities, we have that

$$p \cdot \left[\sum_{a \in A} z^a \right] > p^* \cdot \left[\sum_{a \in A} \omega^a \right],$$

which implies that $\sum_{a \in A} z^a \neq \sum_{a \in A} \omega^a = \bar{\omega}$, violating the feasibility condition.

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Suppose by contradiction that $\exists a, p \cdot z^a < p \cdot \omega^a$.
$\Rightarrow \exists \epsilon > 0$ such that

$$p \cdot (z^a + (\epsilon, \epsilon, ..., \epsilon)) < p \cdot \omega^a.$$

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(i) $U^{\tilde{a}}(z^{\tilde{a}}) > U^{\tilde{a}}(x^{\tilde{a}})$.

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Proof of (2) $p \cdot z^{\tilde{a}} > p \cdot \omega^{\tilde{a}}$.
(i) $U^{\tilde{a}}(z^{\tilde{a}}) > U^{\tilde{a}}(x^{\tilde{a}})$.
(ii) $x^{\tilde{a}}$ maximizes agent $\tilde{a}$'s utility in budget set $B(p, p \cdot \omega^{\tilde{a}})$.

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(i) $U^{\tilde{a}}(z^{\tilde{a}}) > U^{\tilde{a}}(x^{\tilde{a}})$.
(ii) $x^{\tilde{a}}$ maximizes agent $\tilde{a}$'s utility in budget set $B(p, p \cdot \omega^{\tilde{a}})$.
(i) and (ii) $\Rightarrow$ bundle $z^{\tilde{a}}$ is not budget feasible for agent $\tilde{a}$, i.e.,

$$p \cdot z^{\tilde{a}} > p \cdot \omega^{\tilde{a}}.$$

The Second Welfare Theorem

Can Pareto optimal allocation implemented as a Walrasian equilibrium given any endowment?
No!
Illustration of in Edgeworth box with two commodities.

Exchange Economy with Transfers

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Definition

x is a Walrasian allocation with transfers if there exists a price p and an endowment of monetary transfer t^a for each agent a such that sum of excess demand is zero.

The Second Welfare Theorem

Theorem

Suppose that U^a is strongly monotone, strictly quasiconcave, and continuous for all a. Then every Pareto optimal allocation is a Walrasian allocation with transfers.

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Motivation for exchange economy with transfers:

- government collects taxes and redistributes them as subsidies to achieve a more efficient allocation in equilibrium.

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$$\sum_{a\in A} \bar{x}^a(p^*, p^* \cdot y^a) = \sum_{a\in A} y^a = \bar{\omega}.$$

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Equilibrium condition
$\Rightarrow u^a(\bar{x}^a(p^*, p^* \cdot y^a)) \geq u^a(y^a)$ for all a since y^a is budget feasible.

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$\Rightarrow y^a = \bar{x}^a(p^*, p^* \cdot y^a)$ since demand is unique (because U^a is strictly quasiconcave).
Define $t^a = p^* \cdot y^a - p^* \cdot \omega^a$. Then

$$\sum_{a \in A} t^a = p^* \cdot \left(\sum_{a \in A} y^a - \sum_{a \in A} \omega^a \right) = 0.$$