

Welfare Theorems

Yingkai Li

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Definition

An allocation $\{z^a\}_{a \in A}$ is a **Pareto improvement** of another allocation $\{y^a\}_{a \in A}$ if $U^a(z^a) \geq U^a(y^a)$ for all $a \in A$ and the inequality is strict for at least one agent.

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Illustration in Edgeworth box.

The First Welfare Theorem

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An allocation $\{x^a\}_{a \in A}$ is a Walrasian allocation if there exists $p \in \mathbb{R}_+^{+\ell}$ such that $Z(p) = 0$ and $x^a = \hat{x}(p)$.

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Intepretation: equilibrium allocation is always efficient.

Remark: we do not assume quasi-concave or continuous utility here.

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Suppose that there exists an allocation $\{z^a\}_{a \in A}$ that is a Pareto improvement of $\{\hat{x}^a\}_{a \in A}$:

$$U^a(z^a) \geq U^a(\hat{x}^a(p^*)), \quad \forall a \in A,$$

and $\exists \tilde{a}$ such that it holds with a strict inequality.

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Lemma

- ① $p \cdot z^a \geq p \cdot \omega^a$ for all agents a .
- ② $p \cdot z^{\tilde{a}} > p \cdot \omega^{\tilde{a}}$.

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Combining the inequalities, we have that

$$p \cdot \left[\sum_{a \in A} z^a \right] > p^* \cdot \left[\sum_{a \in A} \omega^a \right],$$

which implies that $\sum_{a \in A} z^a \neq \sum_{a \in A} \omega^a = \bar{\omega}$, violating the feasibility condition.

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(ii) $x^{\tilde{a}}$ maximizes agent \tilde{a} 's utility in budget set $B(p, p \cdot \omega^{\tilde{a}})$.

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(ii) $x^{\tilde{a}}$ maximizes agent \tilde{a} 's utility in budget set $B(p, p \cdot \omega^{\tilde{a}})$.

(i) and (ii) \Rightarrow bundle $z^{\tilde{a}}$ is not budget feasible for agent \tilde{a} , i.e.,

$$p \cdot z^{\tilde{a}} > p \cdot \omega^{\tilde{a}}.$$

The Second Welfare Theorem

Can Pareto optimal allocation implemented as a Walrasian equilibrium given any endowment?

No!

Illustration of in Edgeworth box with two commodities.

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Definition

x is a **Walrasian allocation with transfers** if there exists a price p and an endowment of monetary transfer t^a for each agent a such that sum of excess demand is zero.

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Theorem

Suppose that U^a is strongly monotone, strictly quasiconcave, and continuous for all a . Then every Pareto optimal allocation is a Walrasian allocation with transfers.

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Motivation for exchange economy with transfers:

- government collects taxes and redistributes them as subsidies to achieve a more efficient allocation in equilibrium.

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Define $t^a = p^* \cdot y^a - p^* \cdot \omega^a$. Then

$$\sum_{a \in A} t^a = p^* \cdot \left(\sum_{a \in A} y^a - \sum_{a \in A} \omega^a \right) = 0.$$