Revenue Maximization

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Single-item Auctions

Auctions: a single item, n agents.

- each agent i has value $v_i \sim F_i$;
- each agent i has utility $u_i = v_i x_i - p_i$.

Revenue maximization: maximize $\sum_i p_i$.

Incentives

Given any $v > v'$:

$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$
$$v' \cdot x(v') - p(v') \geq v' \cdot x(v) - p(v)$$

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Combining inequalities:

$$v' \cdot (x(v) - x(v')) \leq p(v) - p(v') \leq v \cdot (x(v) - x(v')) \Rightarrow x(v) - x(v') \geq 0.$$

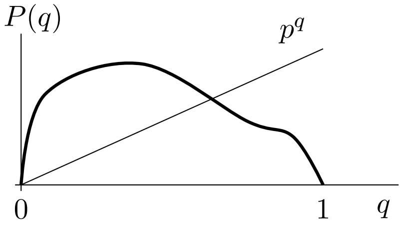
In any incentive compatible mechanism, allocation must be weakly increasing in values.

Revenue Optimal Mechanisms

Revenue Curves: Single-agent Analysis

Price posting revenue curve $P(q)$: expected revenue from selling the item using market clearing price p^q.

- p^q: per-unit price that sells the item with total demand q;



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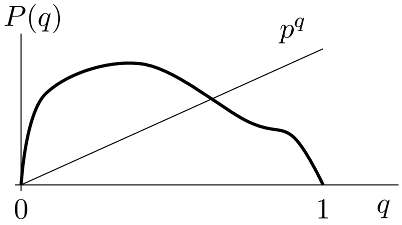
- $\bar{P}$: concave hull of P.



Pricing-based Mechanisms

Quantile space: let $q = \mathbf{Pr}_{t' \sim F}[t' \geq t]$ be the quantile for type t.

- $q \sim U[0, 1]$ (assuming continuous type distribution)
- lower quantile $\Leftrightarrow$ higher willingness to pay.

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- selling to quantiles lower than $\hat{q}_i$ $\Leftrightarrow$ posting market clearing price $p^{\hat{q}_i}$ to agent i.

For each agent i, given Q_i, the distribution over thresholds $\hat{q}_i$ does not depend on the type distribution of other agents.

Marginal Revenue Maximization

Expected revenue from pricing-based mechanisms:

$$\sum_{i \in N} \mathbf{E}_{\forall j \neq i, q_j \sim U[0,1]} [P_i(Q_i(q_{-i}))] = \sum_{i \in N} \mathbf{E}_{\forall j, q_j \sim U[0,1]} \left[P_i'(q_i) x_i(q_i, q_{-i}) \right]$$

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Marginal revenue maximization: allocate to agents with m highest $P_i'(q_i)$ [Bulow and Roberts '89].

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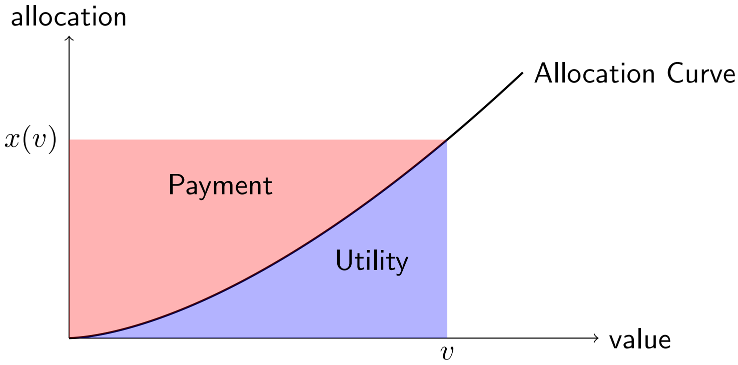
Marginal revenue maximization is optimal among all possible mechanisms.

Alternative Geometric Proof



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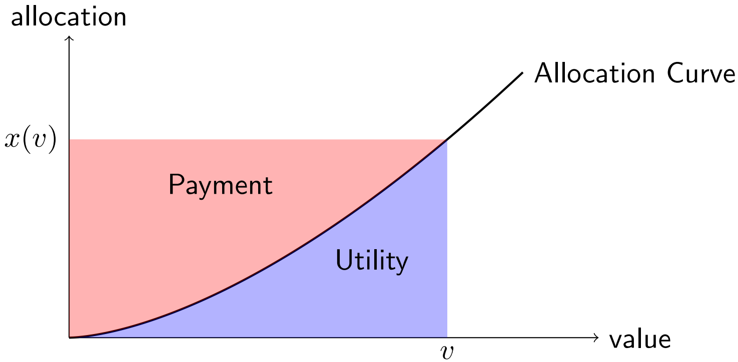

$$p(v) = v \cdot x(v) - \int_0^v x(z)\,\mathrm{d}z$$

$$\Rightarrow \mathbf{E}[p(v)] = \mathbf{E}\left[v \cdot x(v) - \int_0^v x(z)\right]$$

$$= \mathbf{E}\left[\left(v - \frac{1-F(v)}{f(v)}\right) \cdot x(v)\right] = \mathbf{E}\left[P'(q) \cdot x(q)\right]$$

Revenue Equivalence

The marginal revenue $P_i'(q_i)$ for value $v_i(q_i)$ sometimes is also referred to as the virtual value for $v_i(q_i)$ [Myerson'81].

Lemma

Given any mechanism M with allocation rule x, the expected revenue of the mechanism equals the expected marginal revenue / virtual value. That is,

$$\mathrm{Rev}(M) = \mathbf{E}_{\forall j, q_j \sim U[0,1]}\left[\sum_{i \in N} P_i'(q_i) x_i(q_i, q_{-i})\right].$$

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Expected ironed marginal revenue is an upper bound for expected marginal revenue, and they have the same maximizer.

- ironed marginal revenue is always weakly decreasing.

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Remark: the optimal reserve price v^* does not depend on the number of agents.

- it is also the optimal price in the single agent problem.

Approximation Under Linear Utilities

Posted Pricing

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Question: is posted pricing mechanisms also approximately optimal for revenue maximization?

Recap: Prophet Inequality

Online Selection Problem: n items arriving online.

- item i has value $v_i \sim F_i$;
- the agent knows $F_1, \ldots, F_n$ at time 0.
- at time $i \leq n$, the agent observes value v_i and decides whether to select item i (if the selection has not been made).

Theorem

There exists a threshold policy that achieves a 2-approximation, i.e., it achieves expected value at least $\frac{1}{2}\mathbf{E}[\max_i v_i]$.

Connection to Revenue Maximization

Prophet inequality: n items

- value distributions $F = F_1 \times \cdots \times F_n$;
- threshold τ for each item;
- arrival order π.

Posted pricing mechanism: n agents

- marginal revenues $F = F_1 \times \cdots \times F_n$;
- threshold τ for each agent i;
- tie breaking rule π.

Any threshold τ in the marginal revenue space corresponds to a price p_i in the value space.

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Given any valuation profile $v = (v_1, \ldots, v_n)$, the selected value and the optimal value in both problems are the same.

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expected marginal revenue $=$ expected revenue
$\Rightarrow$ posted pricing mechanism has a 2-approximation to the expected revenue.

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Example: n agents. For agent $i \leq n$, $v_i = 2^i$ w.p $\frac{1}{2^{i+1}}$, and $v_i = 0$ w.p $1 - \frac{1}{2^{i+1}}$.

- optimal revenue $\Theta(n)$;
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Third-degree price discrimination is crucial for revenue maximization.

- competition and simultaneous implementation is not.

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Theorem (Yan '11)

Sequential posted pricing mechanism has an $\frac{e}{e-1}$-approximation to the expected revenue.

Correlation Gap

A non-negative real-valued set function f over subsets S of an n element ground set $N = \{1, \ldots, n\}$ and a distribution over subsets given by $\mathcal{D}$.

- $\hat{q}_i$: ex ante probability that element i is in the random set $S \sim \mathcal{D}$
- $\mathcal{D}^I$: distribution over subsets induced by independently adding each element i to the set with probability equal to its ex ante probability $\hat{q}_i$.

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The correlation gap is the ratio of the expected value of the set function for the (correlated) distribution $\mathcal{D}$ to that with independent distribution $\mathcal{D}^I$, i.e.,

$$\frac{E_{S\sim\mathcal{D}}[f(S)]}{E_{S\sim\mathcal{D}^I}[f(S)]}.$$

Correlation Gap

Definition

A set function $f : 2^S \to \mathbb{R}$ defined on the subsets of a finite set S is called submodular if for all $A \subseteq B \subseteq S$ and $x \notin B$, the following inequality holds:

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B).$$

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Theorem

If the set function f is submodular, the correlation gap for function f is at most $\frac{e}{e-1}$.

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As $n \to \infty$, $\frac{E_{S \sim \mathcal{D}}[f(S)]}{E_{S \sim \mathcal{D}^I}[f(S)]} = \frac{e}{e-1}$.

Connection to Sequential Posted Pricing

Ex ante relaxation: consider the relaxed problem where the sum of ex ante probabilities of receiving an item is at most 1.

$$\mathrm{EAR} = \sum_i R_i(q_i) \quad \text{s.t.} \quad \sum_i q_i \le 1.$$

EAR is an upper bound on the optimal revenue.

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Correlation gap implies that

$$\frac{E_{S \sim \mathcal{D}}[f(S)]}{E_{S \sim \mathcal{D}^I}[f(S)]} \leq \frac{e}{e-1}.$$

Extension of Approximations Under Non-linear Utilities

Non-linear Utilities

Two options, which one would you choose:

- get \$10M;
- draw a lottery, with probability $\frac{1}{2}$, get \$20M, and get nothing otherwise.

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Risk aversion: $t_i = (v_i, \varphi_i)$ where $v_i \in \mathbb{R}_+$, φ_i is an increasing concave function, and

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Private budgets: $t_i = (v_i, B_i)$ where $v_i, B_i \in \mathbb{R}_+$, and

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Are simple mechanisms approximately optimal for non-linear utilities?

Demands and Ordinary Goods Assumption

In single-agent environments, a mechanism is posting a per-unit price p if the agent can purchase any lottery x with price $x \cdot p$ for any $x \in [0, 1]$.

- agent pays price $x \cdot p$ even if the realized allocation is 0.

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Assumption (Ordinary Goods)

$d^u(t, p)$ *is non-increasing in* p *for all* $t \in T$.

Excludes Giffen goods or Veblen goods.

Quantiles for Non-linear Agents

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Demand of an agent with a private budget given per-unit pricing:

	$t_1 : (v = 5, B = 1)$	$t_2 : (v = 2, B = 2)$
$p_1 = 4$	0.25	0
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There isn't a simple deterministic and consistent way of ordering types for a non-linear agent.

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Definition (Quantiles for Non-linear Agents)

The randomized quantile q for type $t \in T$ is drawn from distribution with CDF $d(t, p^q)$.

- $d(t, p^0) = 0$ and $d(t, p^1) = 1$;
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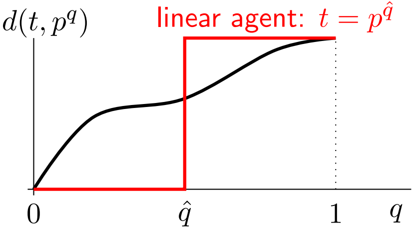

Quantiles for Non-linear Agents

Solution: a random mapping from types to quantiles based on demand functions.

Intuition: given any threshold $\hat{q}$ and any type t, the following two quantities should coincide

1. probability the quantile of t is below $\hat{q}$;
2. the demand of t given market clearing price $p^{\hat{q}}$.

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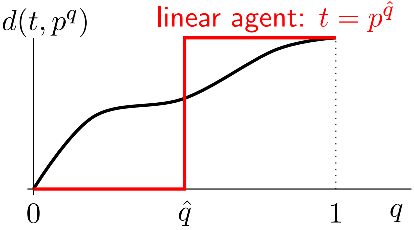
Remark: $q \sim U[0, 1]$: $\mathbf{Pr}[z \leq q] = \mathbf{E}_{t \sim F}[d(t, p^q)] = q$.



Pricing-based Mechanisms in Quantile Space

Definition (Pricing-based Mechanisms in Quantile Space)

Given any profile of feasible thresholds $\{Q_i\}_{i\in[n]}$,

1. Map type t_i to quantile q_i according to $d(t, p^q)$, and calculate threshold as $\hat{q}_i = Q_i(q_{-i})$.
2. The allocation of agent i is $x_i = 1$ if and only if $q_i \leq \hat{q}_i$.
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Price-posting Equivalence Interpretation: Fixing any $\hat{q}_i$, from perspective of agent i

- wins the item with probability $d(t_i, p^{\hat{q}_i})$;
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Expected revenue: from any pricing-based mechanisms M for non-linear agents,

$$M(P) = \sum_i \mathbf{E}_{\forall j\neq i, q_j\sim U[0,1]}\left[P_i(Q_i(q_{-i}))\right].$$

Suboptimality of Pricing for Non-linear Agents

For linear agents: $R = \bar{P}$ [Bulow and Robert '89].

For non-linear agents: pricing-based mechanisms in general are not optimal, i.e., $R \neq \bar{P}$.

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Revenue from posted pricing:

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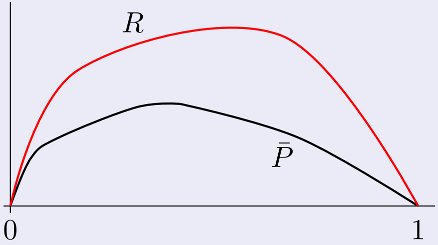
Lottery mechanism:

- offer menu of lotteries $(x_1 = \frac{1}{2}, p_1 = 1)$ and $(x_2 = 1, p_2 = 3)$;
- expected revenue equals 2.

Resemblance: Approximations in Single-agent Settings

Definition (ζ-resemblance)

A non-linear agent is ζ-resemblant to a linear agent if given any supply constraint $q \in [0, 1]$, there exists a posted pricing mechanism with expected demand $q^\dagger \leq q$ such that $\bar{P}(q^\dagger) \geq \frac{1}{\zeta} R(q)$.



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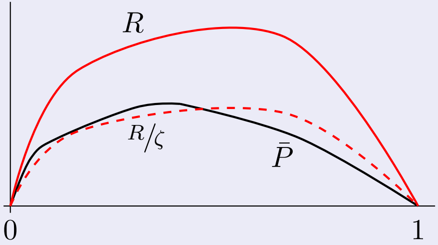

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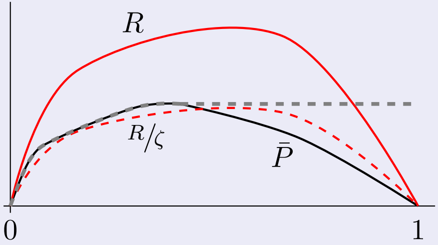

Reduction from Non-linear to Linear Agents

Theorem

For non-linear agents that are ζ-resemblant to linear agents,
pricing-based mechanism M is a γ-approximation to ex ante relaxation for linear agents
$\Rightarrow$ M is a $\zeta\gamma$-approximation to ex ante relaxation for non-linear agents.

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Non-linearities are often details that can be dispensed from the model without affecting main economic conclusions.

Economic conclusions for linear agents $\Rightarrow$ economic conclusions for non-linear agents.

ζ-resemblance for Non-linear Agents

	independent private budget*	risk averse*
revenue	3	e
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Table: Summary of results for ζ-resemblance.

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Corollary

For risk averse agents, sequential posted pricing is an $e/(e-1)$*-approximation to the optimal welfare.*