

Bandit Learning

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Multi-arm Bandits

Consider an online decision process with T periods and n arms.

- each arm i has stochastic return $F_i \in \Delta([0, 1])$ with mean μ_i for each time period;
- the designer cannot observe F_i for any i .

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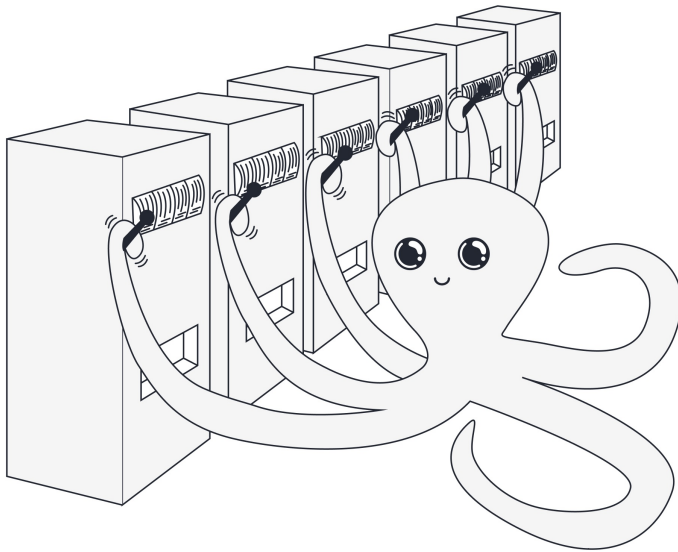
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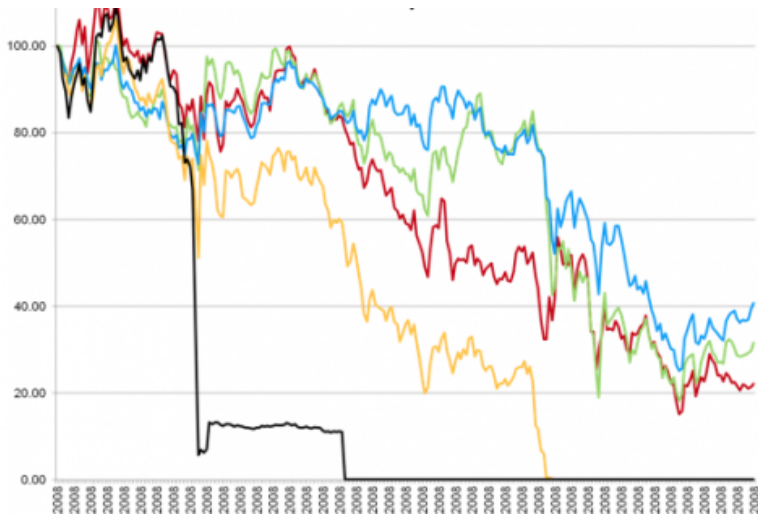
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Question: how to design online algorithms with good online performance even without knowing $\{F_i\}_{i \in [n]}$?

Applications



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An algorithm has **no-regret** if $R_T = o(T)$.

- Is it possible to design no-regret algorithms without any knowledge about the reward distributions?

Simple Question



PollEv.com/quietsalute502

Myopic Exploitation

Myopic Exploitation Algorithm:

- at any time $t \leq T$, select the arm with highest average reward

$$i_t^* = \operatorname{argmax}_{i \in [n]} \hat{\mu}_{i,t} \quad \text{where} \quad \hat{\mu}_{i,t} \triangleq \frac{\sum_{s < t} v_{i,s} \cdot \mathbf{1}(i = i_s^*)}{\sum_{s < t} \mathbf{1}(i = i_s^*)}.$$

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Myopic exploitation has regret $R_T = \Theta(T)$.

- two arms, arm 1 has fixed reward $\frac{1}{3}$, arm 2 has reward uniform in $\{0, 1\}$;
- myopic exploitation will always choose the inferior arm 1 if in the first time arm 2 only provides a reward of 0; with **expected regret at least $\frac{T}{12}$** .

Explore then Exploit

Learn the distributions first and choose the best one later.

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Given parameter $K \leq \frac{T}{n}$:

- choose each arm one by one for each period $t \leq nK$;
- for any period $t \in [nK + 1, T]$, choose arm

$$i_t^* = \operatorname{argmax}_{i \in [n]} \hat{\mu}_{i, nK}.$$

Concentration Inequalities

Estimation error with large samples.

Lemma (Hoeffding's Inequality)

Let X_1, X_2, \dots, X_n be independent random variables such that $X_i \in [a_i, b_i]$ almost surely. Then, for the sum of these variables, we have the following concentration bound:

$$\mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n X_i \right] \right| \geq \epsilon \right) \leq 2 \exp \left(- \frac{2n^2 \epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2} \right).$$

In the special case where $X_i \in [0, 1]$ for all i :

$$\mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n X_i \right] \right| \geq \epsilon \right) \leq 2 \exp(-2n\epsilon^2).$$

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Bound on sample size for an ϵ -estimation with error probability at most δ .

Lemma

Fixing any arm i , for any $\epsilon, \delta > 0$, if $K \geq \frac{1}{2\epsilon^2} \cdot \log \frac{2}{\delta}$, we have $|\hat{\mu}_{i,nK} - \mu_i| \leq \epsilon$ with probability at least $1 - \delta$.

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Let X_j be the random variable for pulling arm i for the j th time.

$$\begin{aligned}\mathbb{P}(|\hat{\mu}_{i,nK} - \mu_i| \geq \epsilon) &= \mathbb{P}\left(\left|\frac{1}{K} \sum_{j=1}^K X_j - \mathbb{E}\left[\frac{1}{K} \sum_{j=1}^K X_j\right]\right| \geq \epsilon\right) \\ &\leq 2 \exp(-2K\epsilon^2) \leq \delta.\end{aligned}$$

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Lemma (Union Bound)

For any probability events X, Y , we have

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Therefore, letting $\delta = \frac{1}{T}$ and $\epsilon = (\frac{n}{T})^{\frac{1}{3}}$, we have $K = \frac{1}{2} \cdot (\frac{T}{n})^{\frac{2}{3}} \cdot \log 2nT$ and the regret of Explore-then-Exploit is

$$R_T \leq \underbrace{nK}_{\text{Exploration Regret}} + \underbrace{T((1 - \delta) \cdot 2\epsilon + \delta)}_{\text{Exploitation Regret}} \leq nK + T \cdot 2\epsilon + 1 = O(n^{\frac{1}{3}} \cdot T^{\frac{2}{3}} \cdot \log 2nT).$$

Better Algorithms

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- Active arm elimination;
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- Active arm elimination;
- Upper confidence bound;
- Thompson sampling.

The worst-case regrets for these three algorithms are $O(\sqrt{nT \cdot \log nT})$

Active Arm Elimination

- Maintain an active set S , which is initialized as $[n]$;
- Choose an arm in S in a sequential order;
- Update the active set S : eliminate arm $i \in S$ if there exists $j \in S$ such that

$$\hat{\mu}_{j,t} \geq \hat{\mu}_{i,t} + 2C_t$$

where $C_t = \sqrt{\frac{\log nT}{K_t}}$ and K_t is the number of times arms in S has been chosen.

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Intuition: if the history of rewards indicates that an arm is not the best arm with high probability, the algorithm never chooses that arm again in the future.

- in contrast, Explore-then-Exploit keeps exploring bad arms until after $\tilde{O}(n^{\frac{1}{3}} \cdot T^{\frac{2}{3}})$.

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With probability $1 - \frac{2}{nT}$, arm i^ is never eliminated, and arm $i \neq i^*$ is removed before time*

$$T_i \triangleq \frac{16 \log nT}{\Delta_i^2}.$$

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Again by applying Hoeffding's inequality, at any time $t \in [T]$, for any arm $i \in [n]$,

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$$\hat{\mu}_{i^*,t} - \hat{\mu}_{i,t} > (\mu_{i^*} - C_t) - (\mu_i + C_t) = \Delta_i - 2C_t.$$

To guarantee elimination of i , we require $\Delta_i - 2C_t \geq 2C_t$, or $\Delta_i \geq 4C_t = 4\sqrt{\frac{\log(nT)}{K_t}}$.

Solving for K_t : $K_t \geq \frac{16 \log(nT)}{\Delta_i^2}$.

Instance-dependent Bound:

$$\begin{aligned} R_T(\mathcal{E}) &\leq \sum_{i \neq i^*} \Delta_i \cdot T_i \\ &= \sum_{i \neq i^*} \Delta_i \cdot \frac{16 \log(nT)}{\Delta_i^2} \\ &= 16 \log(nT) \cdot \sum_{i \neq i^*} \frac{1}{\Delta_i}. \end{aligned}$$

Active Arm Elimination

Lemma (Cauchy-Schwarz inequality)

For two vectors $\mathbf{u} = (u_1, \dots, u_k)$ and $\mathbf{v} = (v_1, \dots, v_k)$,

$$\left(\sum_{i=1}^k u_i v_i \right)^2 \leq \left(\sum_{i=1}^k u_i^2 \right) \cdot \left(\sum_{i=1}^k v_i^2 \right).$$

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Worst-case Bound: Let $L = 16 \log(nT)$. Worst case occurs when $\sum_{i \neq i^*} T_i = T$, i.e., $\sum_{i \neq i^*} \frac{L}{\Delta_i^2} = T$.

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The regret of active-arm-elimination is

$$\begin{aligned} R_T &\leq L \cdot \sum_{i \neq i^*} \frac{1}{\Delta_i} \\ &\leq L \sqrt{n \sum_{i \neq i^*} \frac{1}{\Delta_i^2}} && \text{(Cauchy-Schwarz)} \\ &= O(\sqrt{nT \cdot \log nT}). \end{aligned}$$

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Intuition: somewhat similar to active-arm-elimination (AAE), UCB never chooses suboptimal arms for too many periods.

- AAE rules out all overly pessimistic arms;
- UCB chooses the most optimistic arm.

Upper Confidence Bound (UCB)

Similar to the analysis of AAE: with probability at least $1 - \frac{2}{nT}$, each arm $i \neq i^*$ is pulled by at most $\frac{8 \log nT}{\Delta_i^2}$ times.

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Thompson Sampling

- At any time t , estimate a reward distribution $\hat{F}_{i,t}$ for each arm i ;
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Intuition: better arms are exploited with higher probability, and bad arms are still explored with small probability.

- empirically, Thompson sampling usually have better performance than UCB or active-arm-elimination, despite the fact that they have the same worst-case regret.

Unknown Time Horizon

Previous algorithms such as UCB set confident intervals based on the time horizon.

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Doubling trick: consider time horizons $T = 2, 4, 8, 16 \dots$

- for time $t \in [2^i, 2^{i+1})$, apply bandit algorithms as if the time horizon is 2^{i+1} ;
- same regret bound with only constant additional loss.