# Prior Independent Mechanism Design

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# Leading Example: Auctions

Selling a single item to $n$ buyers to maximize expected revenue.

- each buyer $i$ has private value $v_i$ drawn independently from $F_i$;
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**Optimal mechanism:** virtual value maximization [Myerson '81] or equivalently marginal revenue maximization [Bulow and Robert '89].

# Robustness to Distributional Knowledge

Relaxing the knowledge of the seller:

1. seller only has sample access to the valuation distributions;
2. seller has no information about the distributions except knowing that they are i.i.d.;
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If buyers have common knowledge about the valuation distribution:

- Implementation theory [Caillaud and Robert '05]: implements the Bayesian optimal mechanism when there are multiple agents.

# Robust Mechanism Design Frameworks

Let $\mathbb{M}$ be the set of possible mechanisms (e.g., DSIC mechanisms). For any $M \in \mathbb{M}$

- $M(v)$ is the performance of mechanism $M$ given realized value profile $v$;
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## Prior-free Optimal (for Benchmark $B$)

$$\gamma_B = \min_{M \in \mathbb{M}} \max_{v \in \mathbb{V}} \frac{B(v)}{M(v)}$$

# Mechanism Design with Samples

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A sampling mechanism $\widehat{M} : \mathbb{V}^K \to \Delta(\mathbb{M})$.

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Prior-independent approximation:

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**Question:** what is the minimum $K$ that guarantees a $(1 + \epsilon)$-approximation to the optimal.

- depends on the set $\mathbb{F}$ of possible distributions.

# Necessity of Distributional Assumptions

No mechanism can guarantee any approximation to the optimal without any additional assumption on the set of possible distribution.

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Given any finite sample size $K$, $\exists \epsilon > 0$ s.t. the seller only sees $K$ samples with value $0$ with high probability.
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$\Rightarrow$ given any mechanism, there exists a $v$ such that the revenue loss is large.

The revenue contribution concentrates too heavily on the tail events.

# Sample Complexity

**Definition (Regularity)**

A distribution $F$ is regular if $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is non-decreasing in $v$.

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**Theorem (Guo, Huang and Zhang '19)**

*For any $\epsilon, \delta \in (0,1)$, if the valuation distribution is regular for all agents, there exists a sampling mechanism $\widehat{M}$ such that with $O(n\epsilon^{-3} \cdot \ln^2(\frac{n}{\epsilon\delta}))$ samples, with probability at least $1-\delta$, the multiplicative revenue loss from $\widehat{M}$ is at most $\epsilon$.*

# Sample Complexity

| Setting | Lower Bound (Sec. 4) | Upper Bound (Sec. 3) |
|---|---|---|
| Regular | $\Omega(n\epsilon^{-3})$ | $\tilde{O}(n\epsilon^{-3})$ |
| MHR | $\tilde{\Omega}(n\epsilon^{-2})$ | $\tilde{O}(n\epsilon^{-2})$ |
| $[1, H]$ | $\Omega(nH\epsilon^{-2})$ | $\tilde{O}(nH\epsilon^{-2})$ |
| $[0, 1]$-additive | $\Omega(n\epsilon^{-2})$ | $\tilde{O}(n\epsilon^{-2})$ |

Figure: Various sample complexity bounds in [Guo, Huang and Zhang '19].

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## Lemma

*For any $\epsilon, \delta > 0$, with $K = \frac{1}{\epsilon^2} \cdot \log \frac{1}{\delta\epsilon}$ samples, with probability at least $1 - \delta$, for all prices $p \in \{0, \epsilon, 2\epsilon, \ldots, 1\}$, $|Q(p) - \hat{Q}(p)| \leq \epsilon$.*

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**Proof of Upper Bound:** Let $p^*$ be the optimal price given true distribution and $\hat{p}^*$ be the optimal price given sample.
With $K = \frac{1}{\epsilon^2} \cdot \log \frac{1}{\delta\epsilon}$ samples, with probability at least $1 - \delta$,

$$\hat{p}^* \cdot Q(\hat{p}^*) \ge \hat{p}^* \cdot (\hat{Q}(\hat{p}) - \epsilon) \ge p^* \cdot \hat{Q}(p^*) - \epsilon \ge p^* \cdot (Q(p^*) - \epsilon) - \epsilon = p^* \cdot Q(p^*) - 2\epsilon.$$

# Proof of Sample Complexity: General Case

Given samples $S^K = (s_1, \ldots, s_K)$, let $\hat{F}(S^K)$ be the empirical distribution.

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Dominated Empirical Distributions: Let $\tilde{F}(S^K)$ be the distribution that shifts the quantiles of $\hat{F}(S^K)$ down by $\tilde{O}(\frac{1}{\sqrt{K}})$.

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Run Myerson's optimal auction based on $\tilde{F}(S^K)$.

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*With probability at least* $1-\delta$, $F_i$ *first order stochastically dominates* $\tilde{F}_i(S^K)$ *for all* $i$.

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*If* $K = O(n\epsilon^{-3} \cdot \ln^2(\frac{n}{\epsilon\delta}))$, *w.p. at least* $1-\delta$, $\mathrm{OPT}_{\tilde{F}_i(S^K)}(\tilde{F}_i(S^K)) \geq \frac{1}{1+\epsilon} \cdot \mathrm{OPT}_F(F)$.

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Application of concentration inequalities: Bernstein's inequality [Bernstein '24].

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Consider $\tilde{F}$ that is obtained by shifting the quantiles of $F$ down by $\tilde{O}(\frac{1}{\sqrt{K}})$.

- $\tilde{F}$ is close to $F$ in KL-distance;
- optimal revenue of $\tilde{F}$ and $F$ must be close [Pinsker '60].

# Strong Revenue Monotonicity

**Theorem** (Devanur, Huang and Psomas '16)

*For any distributions $F$ and $F'$ such that $F'_i$ first order stochastically dominates $F_i$ for all $i$, letting $\mathrm{OPT}_F$ be the optimal mechanism for $F$, we have $\mathrm{OPT}_F(F') \geq \mathrm{OPT}_F(F)$.*

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**Idea:** Coupling argument.

- See illustration for the weaker version on board.

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Even characterizing the robust performance of an arbitrary mechanism is challenging with small sample sizes.

- consider mechanisms with specific forms that are "easy" to analyze.

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Prove by graphical illustration.

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Can we do better?

- Not for deterministic mechanisms;
- Yes for using lotteries.

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- still no closed-form characterization for worst case distribution given the mechanism;
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Intuition on why random mechanisms improve the worst case performance

- see graphical illustration.
- markdown the sample by a small factor significantly improves the worst case performance for large $q^*$;
- markup the sample improves the worst case performance for small $q^*$.

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- intuitive reason: samples may overestimates the distributions.

Alternative ideas: shift the empirical distributions.

- difficult to analyze its worst case performance for this class of mechanism.

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- the analysis turn out to be pretty tight.

# A Few Samples

**Allouah, Bahamou, and Besbes:** *Pricing with Samples*
Operations Research, 2022, vol. 70, no. 2, pp. 1088–1104, © 2022 INFORMS 1103

**Table 4.** Lower Bounds on the Maximin Ratio $\mathcal{R}(\mathcal{P}_N, \mathcal{F}_\alpha)$

| | | | Performance of mechanism | | Distribution parameters | |
|---|---|---|---|---|---|---|
| Class | $N$ | Pricing mechanism $(i, \psi)$ | Lower bound | Upper bound | $q_0$ | $q_1$ |
| mhr | 1 | $(1, \delta_{0.76})$ | 64.4% | 64.8% | 0.448 | 0.079 |
| | 2 | $(2, \delta_{0.73})$ | 71.6% | 72.3% | 0.999 | 0.154 |
| | 5 | $(3, \delta_{0.85})$ | 79.1% | 79.9% | 0.41 | 0.07 |
| | 10 | $(6, \delta_{0.81})$ | 80.4% | 81.0% | 1 | 0 |
| Regular | 1 | $(1, 0.9483\,\delta_{0.98} + 0.0517\,\delta_{2.1})$ | 50.2% | 50.4% | 0.016 | 0.002 |
| | 2 | $(2, \delta_{0.75})$ | 61.5% | 61.9% | 0.003 | 0.001 |
| | 5 | $(4, \delta_{0.80})$ | 62.4% | 62.5% | 0.002 | 0.001 |
| | 10 | $(8, \delta_{0.70})$ | 65.3% | 66.0% | 0.002 | 0.001 |

*Note.* The table reports mechanisms and an interval in which its performance is guaranteed to belong for various number of the number of samples $N$. The table also reports near-worst-case distributions for the mechanisms proposed (for all of these, we fixed $\overline{v} = +\infty$).

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In general, in robust environments, the principal cannot simulate arbitrary mechanisms using revelation mechanisms due to the lack of information.

# Revelation Principle

Revelation principle: in Bayesian environments with common knowledge, it is without loss to focus on revelation mechanisms [Myerson '81].

In general, in robust environments, the principal cannot simulate arbitrary mechanisms using revelation mechanisms due to the lack of information.

**Definition (Revelation Gap (Informal))**

The revelation gap of an robust mechanism design environments is defined as the multiplicative gap between the worst case approximation ratio of non-revelation mechanisms and revelation mechanisms.

**Question:** is there a setting where the revelation gap is strictly larger than $1$.

# Revelation Gap for Pricing from Samples

**Pricing from samples:** a single buyer

- private value $v \sim F$;
- seller has access to a sample $s \sim F$;
- buyer knows both $v$ and $F$.

**Remark:** Requires distributional knowledge of the buyer.

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In this setting, revelation mechanism is equivalent to sample-based pricing:

- post a price $p \sim G(s)$ to the agent as take-or-leave-it offer.

# Non-revelation Mechanisms

### Definition (Sample-bid Mechanism)

For any $\alpha > 0$, sample-bid mechanism $\mathrm{SB}_\alpha$

1. solicits a non-negative bid $b \geq 0$;
2. charges the agent $\alpha \cdot \min\{b, s\}$;
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**Remark:** $\mathrm{SB}_\alpha$ is individual rational: the utility of bidding $0$ is $0$.

# Performance of Sample-bid Mechanisms

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$$u(b; v, F) = v \cdot \underbrace{F(b)}_{\mathbf{Pr}_{s \sim F}[s \leq b]} - \underbrace{\alpha b \cdot (1 - F(b))}_{\text{payment when } s \geq b} - \underbrace{\alpha \int_0^b t dF(t)}_{\text{payment when } s \leq b}$$

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Setting $\alpha = 0.824$ achieves an approximation of $1.296$.

- the probability of sale is at least $\frac{1}{e}$ when selling at price $w$ if $F$ is MHR [Barlow and Marshall '65].

# Approximation Guarantees

| | Class of revelation mechanisms [Allouah, Bahamou and Besbes '22] | | Class of all mechanisms [Feng, Hartline and Li '21] | |
|---|---|---|---|---|
| | Regular dists. | MHR dists. | Regular dists. | MHR dists. |
| Upper bound | 1.996 | 1.575 | 1.835 | 1.296 |
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**Theorem (Revelation Gap [Feng, Hartline and Li '21])**

*In single-item single-agent auction with single sample access, for the revenue maximization problem, the revelation gap is*

- *For MHR distributions* $\mathbb{F}_M$, $\Gamma(\mathbb{F}_M) \in [1.190, 1.467]$;
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**Take away:** importance of analyzing non-revelation mechanisms in robust settings.

# Non-truthful Samples

In practical applications, non-truthful auctions are widely adopted [Hartline and Taggart '19]

- the seller may be restricted to only adopt mechanisms with all-pay format or winner-pays-bid format;
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Informal Statement of Result: polynomial number of samples are sufficient to guarantee an $(1+\epsilon)$-approximation [Hartline and Taggart '19] .

# Prior Independent Mechanism Design

## Prior-independent Optimal

$$\beta = \min_{M \in \mathbb{M}} \max_{F \in \mathbb{F}} \frac{\mathrm{OPT}_F(F)}{M(F)}$$

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- robust but not optimal for the prior-independent analysis.

Better idea: second-price auction [Bulow and Klemperer '96; Allouah and Besbes '20]
random markup mechanisms [Fu, Immorlica, Lucier and Strack '15; Hartline, Johnson and Li '20]

- avoid revenue loss by just using part of agents as samples.

# Auctions vs Negotiations

**Theorem (Bulow and Klemperer '96)**

*For any $n \geq 1$, assuming that all agents have i.i.d. regular value distributions, the expected revenue from second-price auction with $n+1$ agents is at least the optimal revenue with $n$ agents.*

**Proof:** [Hartline '20] Given i.i.d. regular value distributions, the optimal mechanism allocates the item to the agent with highest non-negative (virtual) value.

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- $M_S$ is the second-price auction.

Revenue from $\mathrm{OPT}$ with $n$ agents is at most the revenue from $M_S$ with $n+1$ agents.

$$\mathrm{SPA}(F^{n+1}) = M_S(F^{n+1}) \geq \mathrm{OPT}(F^n).$$

One mechanism that always sell with $n+1$ agents: run optimal with $n$ agents, give the item to the additional agent for free if no sale.

# Prior Independent Approximations

## Corollary

*When there are $n \geq 2$ agents, assuming that all agents have i.i.d. regular value distributions, the expected revenue from second-price auction (SPA) is at least $\frac{n-1}{n}$ fraction of the optimal revenue.*

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Optimal revenue is submodular.

- Given any $n' < n$ agents, simulate the values for $n - n'$ agents and run the optimal mechanism for $n$ agents on $n'$ real agents and $n - n'$ simulated agents.

$$\mathrm{SPA}(F^n) \geq \mathrm{OPT}(F^{n-1}) \geq \frac{n-1}{n} \cdot \mathrm{OPT}(F^n).$$

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SPA is asymptotically optimal as $n \to \infty$.

- 2-approximation to the optimal when $n = 2$.

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**Theorem** (Allouah and Besbes '20)

*For $n = 2$, if $\mathbb{F}$ is the set of i.i.d. MHR distributions, second-price auction is prior-independent optimal with approximation ratio $1.398$.*

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- SPA and truncated exponential distributions are mutual best response.

If $\mathbb{F}$ is the set of i.i.d. regular distributions, SPA is not prior-independent optimal.

Randomization helps for improving the prior-independent approximation when $n = 2$ [Fu, Immorlica, Lucier and Strack '15].

# Prior Independent Optimal Auction

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- Draw $\alpha \sim G$
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**Theorem (Hartline, Johnson and Li '20)**

*Random markup mechanism with scale distribution $G^*$ is prior-independent optimal with $\beta \approx 1.91.^*$*

*lower bound holds under a technical restriction on the family of mechanisms.

# Prior Independent Optimal Auction

A mechanism is scale-invariant if for any $\alpha > 0$, the outcome distributions given $(v_1, v_2)$ is the same as $(\alpha v_1, \alpha v_2)$.

**Lemma**

*It is without loss to focus on scale-invariant mechanism.*

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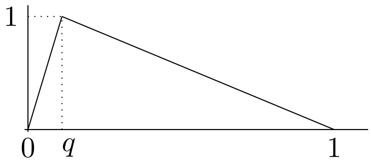
**Lemma**

*It is without loss to focus on scale-invariant mechanism.*

**Triangle distributions:** $\mathrm{Tri}_q$ with cumulative distribution function

$$\mathrm{Tri}_q(v) = \begin{cases} 1 - \frac{1}{1+v(1-q)} & v \leq {}^1/_q, \\ 1 & \text{otherwise.} \end{cases}$$

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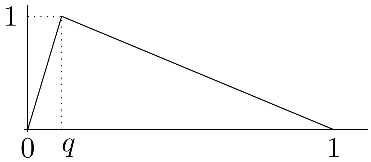


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**Lemma** (Hartline, Johnson and Li '20)

*It is without loss to focus on worst case distributions that are triangle distributions.*

# Prior Independent Optimal Auction

Scale invariance mechanisms are essentially random markup mechanism [Hartline, Johnson and Li '20].

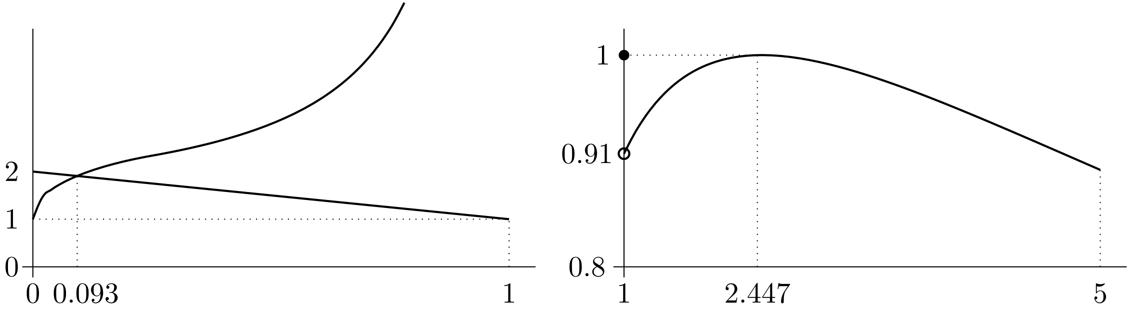


Figure: Left: approximation of second-price and non-trivial markup mechanism for distribution $\mathrm{Tri}_q$.

Right: revenue of the markup mechanisms for triangle distribution $\mathrm{Tri}_{q^*}$ with $q^* \approx 0.093$.