

Prior Independent Mechanism Design

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Leading Example: Auctions

Selling a single item to n buyers to maximize expected revenue.

- each buyer i has private value v_i drawn independently from F_i ;
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Optimal mechanism: virtual value maximization [Myerson '81] or equivalently marginal revenue maximization [Bulow and Robert '89].

Robustness to Distributional Knowledge

Relaxing the **knowledge of the seller**:

- ① seller only has sample access to the valuation distributions;
- ② seller has no information about the distributions except knowing that they are i.i.d.;
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If buyers have common knowledge about the valuation distribution:

- Implementation theory [Caillaud and Robert '05]: implements the Bayesian optimal mechanism when there are multiple agents.

Robust Mechanism Design Frameworks

Let \mathbb{M} be the set of possible mechanisms (e.g., DSIC mechanisms). For any $M \in \mathbb{M}$

- $M(v)$ is the performance of mechanism M given realized value profile v ;
- $M(F) \triangleq \mathbf{E}_{v \sim F}[M(v)]$.

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Prior-free Optimal (for Benchmark B)

$$\gamma_B = \min_{M \in \mathbb{M}} \max_{v \in \mathbb{V}} \frac{B(v)}{M(v)}$$

Mechanism Design with Samples

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A **sampling mechanism** $\widehat{M} : \mathbb{V}^K \rightarrow \Delta(\mathbb{M})$.

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Prior-independent approximation:

$$\beta = \min_{\widehat{M} \in \widehat{\mathbb{M}}} \max_{F \in \mathbb{F}} \frac{\text{OPT}_F(F)}{\widehat{M}(F)}$$

Sample Complexity

With $K = \infty$ samples, the seller can perfectly learn the valuation distributions, and design the Bayesian optimal mechanisms.

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Question: what is the minimum K that guarantees a $(1 + \epsilon)$ -approximation to the optimal.

- depends on the set \mathbb{F} of possible distributions.

Necessity of Distributional Assumptions

No mechanism can guarantee any approximation to the optimal without any additional assumption on the set of possible distribution.

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Given any finite sample size K , $\exists \epsilon > 0$ s.t. the seller only sees K samples with value 0 with high probability.

\Rightarrow seller cannot infer the value v from the sample with high probability.

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The revenue contribution concentrates too heavily on the tail events.

Sample Complexity

Definition (Regularity)

A distribution F is regular if $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is non-decreasing in v .

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Theorem (Guo, Huang and Zhang '19)

For any $\epsilon, \delta \in (0, 1)$, if the valuation distribution is regular for all agents, there exists a sampling mechanism \widehat{M} such that with $O(n\epsilon^{-3} \cdot \ln^2(\frac{n}{\epsilon\delta}))$ samples, with probability at least $1 - \delta$, the multiplicative revenue loss from \widehat{M} is at most ϵ .

Sample Complexity

Setting	Lower Bound (Sec. 4)	Upper Bound (Sec. 3)
Regular	$\Omega(n\epsilon^{-3})$	$\tilde{O}(n\epsilon^{-3})$
MHR	$\tilde{\Omega}(n\epsilon^{-2})$	$\tilde{O}(n\epsilon^{-2})$
$[1, H]$	$\Omega(nH\epsilon^{-2})$	$\tilde{O}(nH\epsilon^{-2})$
$[0, 1]$ -additive	$\Omega(n\epsilon^{-2})$	$\tilde{O}(n\epsilon^{-2})$

Figure: Various sample complexity bounds in [\[Guo, Huang and Zhang '19\]](#).

Proof of Sample Complexity

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Dominated Empirical Distributions: Let $\tilde{F}(S^K)$ be the distribution that shifts the quantiles of $\hat{F}(S^K)$ down by $\tilde{O}(\frac{1}{\sqrt{K}})$.

- overestimation given sample realizations: potentially no sale; avoid
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Run Myerson's optimal auction based on $\tilde{F}(S^K)$.

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Lemma

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With probability at least $1 - \delta$, $\text{OPT}_{\tilde{F}_i(S^K)}(F) \geq \text{OPT}_{\tilde{F}_i(S^K)}(\tilde{F}_i(S^K))$.

Lemma

If $K = O(n\epsilon^{-3} \cdot \ln^2(\frac{n}{\epsilon\delta}))$, w.p. at least $1 - \delta$, $\text{OPT}_{\tilde{F}_i(S^K)}(\tilde{F}_i(S^K)) \geq \frac{1}{1+\epsilon} \cdot \text{OPT}_F(F)$.

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Application of concentration inequalities: Bernstein's inequality [Bernstein '24].

- with high probability, the estimation error of the quantiles based on the empirical distribution is at most $\tilde{O}(\frac{1}{\sqrt{K}})$.

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Consider \tilde{F} that is obtained by shifting the quantiles of F down by $\tilde{O}(\frac{1}{\sqrt{K}})$.

- \tilde{F} is close to F in KL-distance;
- optimal revenue of \tilde{F} and F must be close [Pinsker '60].

Strong Revenue Monotonicity

Theorem (Devanur, Huang and Psomas '16)

For any distributions F and F' such that F'_i first order stochastically dominates F_i for all i , letting OPT_F be the optimal mechanism for F , we have $\text{OPT}_F(F') \geq \text{OPT}_F(F)$.

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Idea: Coupling argument.

- See illustration for the weaker version on board.

A Few Samples

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Even characterizing the robust performance of an arbitrary mechanism is challenging with small sample sizes.

- consider mechanisms with specific forms that are “easy” to analyze.

A Simple Example: A Single Sample

Theorem (Dhangwatnotai, Roughgarden and Yan '15)

For regular valuation distribution, posting a price equal to the sample guarantees a 2-approximation.

Prove by graphical illustration.

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Can we do better?

- Not for deterministic mechanisms;
- Yes for using lotteries.

Randomized Sample-based Pricing

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Further simplification: focus on ψ with binary support $\{\underline{r}, \bar{r}\}$ with $\underline{r} < 1 < \bar{r}$.

- still no closed-form characterization for worst case distribution given the mechanism;
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Intuition on why random mechanisms improve the worst case performance

- see graphical illustration.
- markdown the sample by a small factor significantly improves the worst case performance for large q^* ;
- markup the sample improves the worst case performance for small q^* .

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- intuitive reason: samples may overestimates the distributions.

Alternative ideas: shift the empirical distributions.

- difficult to analyze its worst case performance for this class of mechanism.

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- the analysis turn out to be pretty tight.

Table 4. Lower Bounds on the Maximin Ratio $\mathcal{R}(\mathcal{P}_N, \mathcal{F}_\alpha)$

Class	N	Pricing mechanism (i, ψ)	Performance of mechanism		Distribution parameters	
			Lower bound	Upper bound	q_0	q_1
mhr	1	$(1, \delta_{0.76})$	64.4%	64.8%	0.448	0.079
	2	$(2, \delta_{0.73})$	71.6%	72.3%	0.999	0.154
	5	$(3, \delta_{0.85})$	79.1%	79.9%	0.41	0.07
	10	$(6, \delta_{0.81})$	80.4%	81.0%	1	0
Regular	1	$(1, 0.9483 \delta_{0.98} + 0.0517 \delta_{2.1})$	50.2%	50.4%	0.016	0.002
	2	$(2, \delta_{0.75})$	61.5%	61.9%	0.003	0.001
	5	$(4, \delta_{0.80})$	62.4%	62.5%	0.002	0.001
	10	$(8, \delta_{0.70})$	65.3%	66.0%	0.002	0.001

Note. The table reports mechanisms and an interval in which its performance is guaranteed to belong for various number of the number of samples N . The table also reports near-worst-case distributions for the mechanisms proposed (for all of these, we fixed $\bar{v} = +\infty$).

Revelation Principle

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Definition (Revelation Gap (Informal))

The revelation gap of an robust mechanism design environments is defined as the multiplicative gap between the worst case approximation ratio of non-revelation mechanisms and revelation mechanisms.

Question: is there a setting where the revelation gap is strictly larger than 1.

Revelation Gap for Pricing from Samples

Pricing from samples: a single buyer

- private value $v \sim F$;
- seller has access to a sample $s \sim F$;
- buyer knows both v and F .

Remark: Requires distributional knowledge of the buyer.

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In this setting, revelation mechanism is equivalent to sample-based pricing:

- post a price $p \sim G(s)$ to the agent as take-or-leave-it offer.

Non-revelation Mechanisms

Definition (Sample-bid Mechanism)

For any $\alpha > 0$, sample-bid mechanism SB_α

- ① solicits a non-negative bid $b \geq 0$;
- ② charges the agent $\alpha \cdot \min\{b, s\}$;
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Remark: SB_α is individual rational: the utility of bidding 0 is 0.

Performance of Sample-bid Mechanisms

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Agent's utility for bidding $b \geq 0$ in SB_α :

$$u(b; v, F) = v \cdot \underbrace{F(b)}_{\Pr_{s \sim F}[s \leq b]} - \underbrace{\alpha b \cdot (1 - F(b))}_{\text{payment when } s \geq b} - \underbrace{\alpha \int_0^b t dF(t)}_{\text{payment when } s \leq b}$$

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$u(b; v, F)$ is convex in b given any v and F that is MHR.

- agent maximizes his utility by bidding 0 or ∞ .

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$u(b; v, F)$ is convex in b given any v and F that is MHR.

- agent maximizes his utility by bidding 0 or ∞ .

Let $w = \mathbf{E}_{v \sim F}[v]$ be the expected welfare.

SB_α is equivalent to posting $\alpha \cdot w$ when F is MHR.

- perfectly infer the welfare using a single sample!

Performance of Sample-bid Mechanisms

Consider the simpler case where F is MHR.

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Setting $\alpha = 0.824$ achieves an approximation of 1.296.

- the probability of sale is at least $\frac{1}{e}$ when selling at price w if F is MHR [Barlow and Marshall '65].

Approximation Guarantees

	Class of revelation mechanisms [Allouah, Bahamou and Besbes '22]		Class of all mechanisms [Feng, Hartline and Li '21]	
	Regular dists.	MHR dists.	Regular dists.	MHR dists.
Upper bound	1.996	1.575	1.835	1.296
Lower bound	1.957	1.543	1.073	

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Theorem (Revelation Gap [Feng, Hartline and Li '21])

In single-item single-agent auction with single sample access, for the revenue maximization problem, the revelation gap is

- For MHR distributions \mathbb{F}_M , $\Gamma(\mathbb{F}_M) \in [1.190, 1.467]$;
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Take away: importance of analyzing non-revelation mechanisms in robust settings.

Non-truthful Samples

In practical applications, non-truthful auctions are widely adopted [[Hartline and Taggart '19](#)]

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Informal Statement of Result: polynomial number of samples are sufficient to guarantee an $(1 + \epsilon)$ -approximation [Hartline and Taggart '19] .

Prior Independent Mechanism Design

Prior-independent Optimal

$$\beta = \min_{M \in \mathbb{M}} \max_{F \in \mathbb{F}} \frac{\text{OPT}_F(F)}{M(F)}$$

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Better idea: second-price auction [Bulow and Klemperer '96; Allouah and Besbes '20]

random markup mechanisms [Fu, Immorlica, Lucier and Strack '15; Hartline, Johnson and Li '20]

- avoid revenue loss by just using part of agents as samples.

Auctions vs Negotiations

Theorem (Bulow and Klemperer '96)

For any $n \geq 1$, assuming that all agents have i.i.d. regular value distributions, the expected revenue from second-price auction with $n + 1$ agents is at least the optimal revenue with n agents.

Proof: [Hartline '20] Given i.i.d. regular value distributions, the optimal mechanism allocates the item to the agent with highest non-negative (virtual) value.

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Revenue from OPT with n agents is at most the revenue from M_S with $n + 1$ agents.

$$\text{SPA}(F^{n+1}) = M_S(F^{n+1}) \geq \text{OPT}(F^n).$$

One mechanism that always sell with $n + 1$ agents: run optimal with n agents, give the item to the additional agent for free if no sale.

Prior Independent Approximations

Corollary

When there are $n \geq 2$ agents, assuming that all agents have i.i.d. regular value distributions, the expected revenue from second-price auction (SPA) is at least $\frac{n-1}{n}$ fraction of the optimal revenue.

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Optimal revenue is submodular.

- Given any $n' < n$ agents, simulate the values for $n - n'$ agents and run the optimal mechanism for n agents on n' real agents and $n - n'$ simulated agents.

$$\text{SPA}(F^n) \geq \text{OPT}(F^{n-1}) \geq \frac{n-1}{n} \cdot \text{OPT}(F^n).$$

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SPA is asymptotically optimal as $n \rightarrow \infty$.

- 2-approximation to the optimal when $n = 2$.

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If \mathbb{F} is the set of i.i.d. **regular** distributions, SPA is **not** prior-independent optimal.

Randomization helps for improving the prior-independent approximation when $n = 2$ [Fu, Immorlica, Lucier and Strack '15].

Prior Independent Optimal Auction

Definition (Random Markup Mechanism with Scale Distribution G)

- Draw $\alpha \sim G$
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Theorem (Hartline, Johnson and Li '20)

Random markup mechanism with scale distribution G^ is prior-independent optimal with $\beta \approx 1.91$.**

*lower bound holds under a technical restriction on the family of mechanisms.

Prior Independent Optimal Auction

A mechanism is **scale-invariant** if for any $\alpha > 0$, the outcome distributions given (v_1, v_2) is the same as $(\alpha v_1, \alpha v_2)$.

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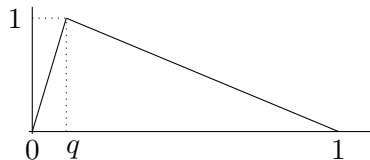
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Triangle distributions: Tri_q with cumulative distribution function

$$\text{Tri}_q(v) = \begin{cases} 1 - \frac{1}{1+v(1-q)} & v \leq 1/q, \\ 1 & \text{otherwise.} \end{cases}$$

Triangle distributions are first order stochastically dominated by other regular distributions.



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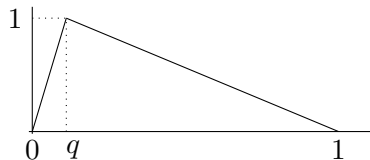
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Lemma (Hartline, Johnson and Li '20)

It is without loss to focus on worst case distributions that are triangle distributions.

Prior Independent Optimal Auction

Scale invariance mechanisms are essentially random markup mechanism [Hartline, Johnson and Li '20].

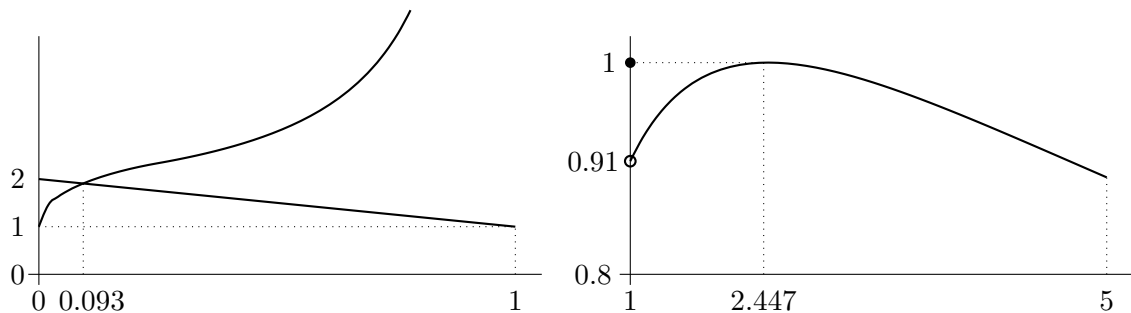


Figure: Left: approximation of second-price and non-trivial markup mechanism for distribution Tri_q .

Right: revenue of the markup mechanisms for triangle distribution Tri_{q^*} with $q^* \approx 0.093$.