

Production Economy

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Production: Firms

What is a firm?

- How is it managed/organized?
- What can it do?

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Neoclassical approach:

- A firm is a “black box” that transforms inputs into outputs.
- Firms are profit maximizing.

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$$y = (-z_1, \dots, -z_{\ell-1}, q).$$

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Production set: $Y = \{(z_1, \dots, z_\ell)\}$.

This set characterises the “input-output possibilities.”

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Throughout, we assume that Y :

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- Optimal supply $y(p)$ is unique (due to strict convexity).
- The properties of the solution, and the corresponding profit, mirror those of utility maximization in consumer theory (see graphical illustration).

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- Agent $a \in A$ has the utility function $U^a : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ and endowment $\omega^a \in \mathbb{R}_+^\ell$.

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Interpretation of goods and utility: some of the goods could be thought of as different types of labor.

- An agent can be endowed with his labor, which could be consumed by the agent as **leisure** (and it gives him utility), or it could be supplied as labor to firms, in order to earn money for other forms of consumption.

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Profit maximizing firm: At price $p = (p_1, \dots, p_l) \gg 0$, firm f chooses a bundle $\hat{y}^f(p) \in Y^f$ to maximize its profit $p \cdot y^f(p)$. We denote the maximum profit at price p by $\pi^f(p)$.

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Utility maximizing agent: At price p , the agent a has an income of $p \cdot \omega^a + \sum_{f \in F} s^{fa} \pi^f(p)$. He maximizes $U^a(x)$ subject to

$$x \in B^a(p) = \{x \in \mathbb{R}_+^m : p \cdot x \leq p \cdot \omega^a + \sum_{f \in F} s^{fa} \pi^f(p)\}.$$

We denote agent a 's utility-maximizing choice by $\hat{x}^a(p)$.

Excess Demand

Excess demand: Given any market price p , the (aggregate) excess demand is

$$Z(p) = \sum_{a \in A} \hat{x}^a(p) - \left\{ \sum_{a \in A} \omega^a + \sum_{f \in F} \hat{y}^f(p) \right\}.$$

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Definition

The price vector $p^* \gg 0$ is a **Walrasian equilibrium price** if there is $\hat{y}^f(p^*)$ for each firm f and $\hat{x}^a(p^*)$ for each agent a such that $Z(p^*) = 0$.

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In the case where a good j is a **pure factor**, i.e., it does not give utility to anyone in the economy (e.g., raw materials like coal or timber), then $\sum_{a \in A} \hat{x}_j^a(p^*) = 0$ and so

$$\sum_{a \in A} \omega_j^a + \sum_{f \in F} \hat{y}_j^f(p^*) = 0.$$

So it can be part of the endowment of some agent, who will sell all of his endowment of j at price p_j^* , and it is then used by firms as an input (so $y_j^f < 0$ for some firm f).

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For an **intermediate good** k , which is a good created and traded as part of the production process but is not endowed to any agent and never consumed by consumers (e.g., steel, textiles), we have

$$\sum_{f \in F} \hat{y}_k^f(p^*) = 0.$$

Production Economy: Example

n agents, 2 goods.

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Utilities and Endowments: Every agent in the economy is endowed with one unit of labor/leisure and they all have the following utility function:

$$U(c, L) = \ln c + \ln L,$$

where L is interpreted as the agent's consumption of his own labor, i.e., his leisure, and c is the level of consumption of the consumer good, of which there is just one.

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Firm shares: Agent a in the economy has a share s^a of this firm, so $\sum_{a \in A} s^a = 1$.

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\Rightarrow optimal production of consumer good is $F(L^*) = 2p$;

\Rightarrow optimal profit of the firm is $2p \cdot F(L^*) - L^* = 2p^2 - p^2 = p^2$.

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Optimal demand: The income of agent a 's is $1 + s^a \cdot p^2$. Thus, his budget set is

$$\{x \in \mathbb{R}_+^2 : p \cdot x \leq 1 + s^a \cdot p^2\}.$$

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Excess demand: The aggregate demand for the consumer good is

$$\sum_{a \in A} \frac{1 + s^a \cdot p^2}{2p} = \frac{n + p^2}{2p},$$

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Walrasian equilibrium price: Solving this equation we obtain $p = \sqrt{\frac{n}{3}}$.

Agent a 's supply of labor is

$$1 - \frac{1}{2} \left(1 + \frac{s^a \cdot n}{3} \right).$$

Equilibrium Existence

Theorem (Arrow and Debreu '54; McKenzie '59)

Suppose the excess demand function $Z : \mathbb{R}_{++}^{\ell} \rightarrow \mathbb{R}^{\ell}$ of the economy \mathcal{E} satisfies the following properties, then there is $p^ \gg 0$ such that $Z(p^*) = 0$.*

- (1) *it is zero-homogenous,*
- (2) *it obeys Walras' Law,*
- (3) *it is continuous,*
- (4) *it satisfies the boundary condition,*
- (5) *it is bounded below.*

Remark: applies beyond exchange economy.

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Lemma

If the agents' utility functions are continuous, strictly monotone and strictly quasi-concave, and the firms' production sets are closed, strictly convex, bounded above, and a strictly positive aggregate consumption bundle is producible from the initial endowments, the excess demand function satisfies the above properties.

Productive Efficiency

Theorem

Suppose $p^* \gg 0$ is the Walrasian equilibrium price of a production economy and suppose that at this price firm f is producing $\hat{y}^f(p^*)$. Then there does not exist y^f (for each $f \in F$) such that

$$\sum_{f \in F} \hat{y}^f(p^*) < \sum_{f \in F} y^f. \quad (1)$$

In other words, there is **productive efficiency** at the Walrasian equilibrium (there is no other production that requires less inputs and produces more outputs).

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Proof: By way of contradiction, suppose (1) holds. Taking the inner product of both sides by $p^* \gg 0$, we obtain

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Thus there is some firm \tilde{f} such that

$$p^* \cdot \hat{y}^{\tilde{f}}(p^*) < p^* \cdot y^{\tilde{f}},$$

which means firm \tilde{f} is not maximizing profit – a contradiction.

QED

Pareto Efficiency with Production

An allocation $\{z^a\}_{a \in A}$ is said to be **feasible** if there is $y^f \in Y^f$ such that

$$\sum_{a \in A} \omega^a + \sum_{f \in F} y^f = \sum_{a \in A} z^a. \quad (2)$$

Definition

An allocation $\{x^a\}_{a \in A}$ is **Pareto optimal** if it is not Pareto dominated by another feasible allocation, i.e., there is no feasible allocation $\{z^a\}_{a \in A}$ such that $u^a(z^a) \geq u^a(x^a)$ for all agents a and at least one agent has *strictly* higher utility.

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Proof: Suppose $\{z^a\}_{a \in A}$ is feasible (obeying (2)) and a Pareto improvement of $\{\hat{x}^a(p^*)\}_{a \in A}$. Therefore, $U^a(z^a) \geq U^a(\hat{x}^a(p^*))$ for all $a \in A$ with a strict inequality for some agent \tilde{a} .

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Therefore, $U^a(z^a) \geq U^a(\hat{x}^a(p^*))$ for all $a \in A$ with a strict inequality for some agent \tilde{a} .

By definition, $\hat{x}^{\tilde{a}}(p^*)$ maximizes agent \tilde{a} 's utility in \tilde{a} 's budget set. So the bundle $z^{\tilde{a}}$ cannot be affordable to agent \tilde{a} , i.e.,

$$p^* \cdot z^{\tilde{a}} > p^* \cdot \omega^{\tilde{a}} + \sum_{f \in F} s^{f\tilde{a}} p^* \cdot \hat{y}^f(p^*).$$

Furthermore, for all agents in A ,

$$p^* \cdot z^a \geq p^* \cdot \omega^a + \sum_{f \in F} s^{fa} p^* \cdot \hat{y}^f(p^*).$$

(This claim uses monotonicity; see proof of this theorem in exchange economies.)

Pareto efficiency with Production

Proof, continued: Summing across all agents and using the fact that $p^* \cdot \hat{y}^f(p^*) \geq p^* \cdot y^f$ for all f , we obtain

$$\begin{aligned} p^* \cdot \left[\sum_{a \in A} z^a \right] &> p^* \cdot \left[\sum_{a \in A} \omega^a \right] + p^* \cdot \left[\sum_{f \in F} \hat{y}^f(p^*) \right] \\ &\geq p^* \cdot \left[\sum_{a \in A} \omega^a \right] + p^* \cdot \left[\sum_{f \in F} y^f \right], \end{aligned}$$

which means that feasibility constraint is violated.

QED

The Second Welfare Theorem

Production economy with transfers: Endowment of each agent $a \in A$:

- commodities ω^a ;
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Budget balance constraint: $\sum_{a \in A} t^a = 0$.

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Theorem

Suppose that U^a is strongly monotone, strictly quasiconcave, and continuous for all a , and the production set Y^f is closed and strictly convex for all f . Then every Pareto optimal allocation is a Walrasian allocation with transfers.