combined test v2

Define X: {not infected, infected} $\rightarrow$ {0, 1} by X(not infected) = 0 and X(infected) = 1.
Define T a sample space that describes the sequence of two tests for which T_1 is the first test (Test 1) and T_2 is the second test (Test 2). Then we have T_1, T_2: {negative, positive}$\rightarrow$ {0, 1} by T_1, T_2(negative) = 0 and T_1, T_2(positive) = 1.
Define sensitivity $p = P(T = 1|X = 1)$
Define specificity $q = P(T = 0|X = 0)$
Let p_1, q_1 denote the sensitivity and specificity of Test 1.
Let p_2, q_2 denote the sensitivity and specificity of Test 2.
Let p_T, q_T denote the sensitivity and specificity of the combined tests.
Let π denotes the prevalence.
Assume that there is no change of infection status throughout the period we consider.

Section 1: Calculating sensitivity and specificity of combined tests

Believe the Positive:

We have $P(T_1 = 1|X = 1) = p_1$, $P(T_1 = 0|X = 0) = q_1$, $P(T_2 = 1|X = 1) = p_2$, $P(T_2 = 0|X = 0) = q_2$ by definition.
Therefore, the overall sensitivity of combined tests is

$$\begin{aligned} p_T &= P(T\ positive|X = 1) \\ &= 1 - P(T\ negative|X = 1) \\ &= 1 - P(T_1 = 0, T_2 = 0|X = 1) \\ &= 1 - P(T_2 = 0|T_1 = 0, X = 1)P(T_1 = 0|X = 1) \end{aligned}$$

We do not know how to simplify $P(T_2 = 0|T_1 = 0, X = 1)$.
However, we can assume that if T_1 and T_2 are independent, then $P(T_2 = 0|T_1 = 0, X = 1) = P(T_2 = 0|X = 1)$.
In the case that the two tests are not independent, we can define a dependent variable "**a**" such that

$$P(T_2 = 0|T_1 = 0, X = 1) = aP(T_2 = 0|X = 1) \in [0, 1]$$

Therefore,

$$\begin{aligned} p_T &= 1 - P(T_2 = 0|T_1 = 0, X = 1)P(T_1 = 0|X = 1) \\ &= 1 - aP(T_2 = 0|X = 1)P(T_1 = 0|X = 1) \\ &= 1 - a(1 - p_1)(1 - p_2) \end{aligned}$$

We also need $p_T = 1 - a(1 - p_1)(1 - p_2) \in [0, 1]$
And the overall specificity of combined test is

$$\begin{aligned} q_T &= P(T\ negative|X = 0) \\ &= P(T_1 = 0, T_2 = 0|X = 0) \\ &= P(T_2 = 0|T_1 = 0, X = 0)P(T_1 = 0|X = 0) \end{aligned}$$

Similar as above, we can define a dependent variable "**b**" such that

$$P(T_2 = 0|T_1 = 0, X = 0) = bP(T_2 = 0|X = 0) \in [0, 1]$$

Therefore,

$$\begin{aligned} q_T &= P(T_2 = 0|T_1 = 0, X = 0)P(T_1 = 0|X = 0) \\ &= bP(T_2 = 0|X = 0)P(T_1 = 0|X = 0) \\ &= bq_1q_2 \end{aligned}$$

We also need $q_T = bq_1q_2 \in [0, 1]$
Note that if Test 1 and Test 2 are independent,
$P(T_2 = 0|T_1 = 0, X = 1) = P(T_2 = 0|X = 1) = 1 - p_2$
$P(T_2 = 0|T_1 = 0, X = 0) = P(T_2 = 0|X = 0) = q_2$
We have the overall sensitivity of combined tests is

$$p_T = 1 - (1 - p_1)(1 - p_2)$$

And the overall specificity of combined test is

$$q_T = q_1q_2$$

Believe the Negative:

By carrying out the similar calculation as for the BP case, we have the following results: The overall sensitivity of combined tests is

$$\begin{aligned} p_T &= P(T\ positive|X = 1) \\ &= ap_1p_2 \end{aligned}$$

We also need $p_T = ap_1p_2 \in [0, 1]$
And the overall specificity of combined test is

$$\begin{aligned} q_T &= P(T\ negative|X = 0) \\ &= 1 - b(1 - q_1)(1 - q_2) \end{aligned}$$

We also need $q_T = 1 - b(1 - q_1)(1 - q_2) \in [0, 1]$
The overall sensitivity of combined tests is

$$p_T = p_1p_2$$

And the overall specificity of combined test is

$$q_T = 1 - (1 - q_1)(1 - q_2)$$

If Test 1 and Test 2 are different tests, we have $P(T_1 = 1|X = 1) = p_1$, $P(T_1 = 0|X = 0) = q_1$
$P(T_2 = 1|X = 1) = p_2$, $P(T_2 = 0|X = 0) = q_2$
The overall sensitivity of combined tests is

$$\begin{aligned} p_T &= P(T\ positive|X = 1) \\ &= P(T_1 = 1, T_2 = 1|X = 1) \\ &= P(T_2 = 1|T_1 = 1, X = 1)P(T_1 = 1|X = 1) \\ &= aP(T_2 = 1|X = 1)P(T_1 = 1|X = 1)\ (using\ dependent\ variable\ a\ defined\ above) \\ &= ap_1p_2 \end{aligned}$$

And the overall specificity of combined test is

$$
\begin{aligned}
q_T &= P(T\ negative|X=0) \\
&= 1 - P(T\ positive|X=0) \\
&= 1 - P(T_1 = 1, T_2 = 1|X=0) \\
&= 1 - P(T_2 = 1|T_1 = 1, X=0)P(T_1 = 1|X=0) \\
&= 1 - bP(T_2 = 1|X=0)P(T_1 = 1|X=0)\ (using\ dependent\ variable\ b\ defined\ above) \\
&= 1 - b(1-q_1)(1-q_2)
\end{aligned}
$$

Note that if Test 1 and Test 2 are independent,
$P(T_2 = 1|T_1 = 1, X = 1) = P(T_2 = 1|X = 1) = p_2$
$P(T_2 = 1|T_1 = 1, X = 0) = P(T_2 = 1|X = 0) = 1 - q_2$
We have the overall sensitivity of combined tests is

$$p_T = p_1 p_2$$

And the overall specificity of combined test is

$$q_T = 1 - (1-q_1)(1-q_2)$$

Section 2: Calculating positive and negative predictive values of both individual and combined test

Individual tests:

By definition, positive predictive values (PPV) is

$$
\begin{aligned}
PPV &= P(X=1|T=1) \\
&= \frac{P(X=1)P(T=1|X=1)}{P(T=1)} \\
&= \frac{P(X=1)P(T=1|X=1)}{P(X=0)P(T=1|X=0) + P(X=1)P(T=1|X=1)} (By\ Law\ of\ Total\ Probability) \\
&= \frac{\pi p}{\pi p + (1-\pi)(1-q)}
\end{aligned}
$$

By carrying similar calculations above, negative predictive values (NPV) is

$$
\begin{aligned}
NPV &= P(X=0|T=0) \\
&= \frac{(1-\pi)q}{(1-\pi)q + \pi(1-p)}
\end{aligned}
$$

Combined tests:

Calculating the combined positive and negative predictive by inserting the corresponding combined sensitivity and specificity into the formula of positive and negative predictive value in the individual case.

Believe the Positive:

The positive predictive values for combined test is

$$PPV_T = \frac{\pi(1 - a(1-p_1)(1-p_2))}{\pi(1 - a(1-p_1)(1-p_2)) + (1-\pi)(1 - bq_1q_2)}$$

And the negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)bq_1q_2}{(1-\pi)bq_1q_2 + \pi a(1-p_1)(1-p_2)}$$

Note that if Test 1 and Test 2 are independent,
The positive predictive values for combined test is

$$PPV_T = \frac{\pi(p_1 + p_2(1-p_1))}{\pi(p_1 + p_2(1-p_1)) + (1-\pi)(1-q_1q_2)}$$

And the negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)q_1q_2}{(1-\pi)q_1q_2 + \pi(1-p_1)(1-p_2)}$$

Note that if Test 1 and Test 2 are the same kind of tests,simply replacing p_2, q_2 with p_1, q_1 respectively, we have the positive predictive values for combined test is

$$PPV_T = \frac{\pi(2p_1 - p_1^2)}{\pi(2p_1 - p_1^2) + (1-\pi)(1-q_1^2)}$$

The negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)q_1^2}{(1-\pi)q_1^2 + \pi(1-2p_1+p_1^2)}$$

Believe the Negative:

The positive predictive values for combined test is

$$PPV_T = \frac{\pi a p_1 p_2}{\pi a p_1 p_2 + b(1-\pi)(1-q_1)(1-q_2)}$$

And the negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)(1-b(1-q_1)(1-q_2))}{(1-\pi)(1-b(1-q_1)(1-q_2)) + \pi(1-ap_1p_2)}$$

Note that if Test 1 and Test 2 are independent,
The positive predictive values for combined test is

$$PPV_T = \frac{\pi p_1 p_2}{\pi p_1 p_2 + (1-\pi)(1-q_1)(1-q_2)}$$

And the negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)(q_1 + q_2(1-q_1))}{(1-\pi)(q_1 + q_2(1-q_1)) + \pi(1-p_1p_2)}$$

Note that if Test 1 and Test 2 are the same kind of tests,simply replacing p_2, q_2 with p_1, q_1 respectively, we have the positive predictive values for combined test is

$$PPV_T = \frac{\pi p_1^2}{\pi p_1^2 + (1-\pi)(1-2q_1+q_1^2)}$$

The negative predictive values for combined test is

$$NPV_T = \frac{(1-\pi)(2q_1 - q_1^2)}{(1-\pi)(2q_1 - q_1^2) + \pi(1-p_1^2)}$$