Stochastic Signal Processing

Lesson 8
Memoryless and Linear Systems

Weize Sun

More examples from the last week

- 1: if the power spectrum of the stationary process $X(t)$ is $S_X(\omega) = \frac{1}{[1+\omega^2]^2}$, calculate the autocorrelation.

Tips：

$$S_Y(\omega) = \frac{1}{1+\omega^2} \Leftrightarrow \frac{1}{2}e^{-|\tau|} \qquad S_X(\omega) = S_Y^2(\omega)$$

$$R_X(\tau) = R_Y(\tau) * R_Y(\tau) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|z|}e^{-|\tau-z|}dz \text{ (频域的乘对应时域的卷积)}$$

- 2: let $X(t)\ and\ Y(t)$ be statistically independent stationary processes, with non-zero means m_X and m_Y. Define $Z(t) = X(t) + Y(t)$, given $S_X(\omega)$, calculate $S_{XY}(\omega)$ and $S_{XZ}(\omega)$.

More examples from the last week

- 1: if the power spectrum of the stationary process $X(t)$ is $S_X(\omega) = \frac{1}{[1+\omega^2]^2}$, calculate the autocorrelation.

Tips：

$S_Y(\omega) = \frac{1}{1+\omega^2} \Leftrightarrow \frac{1}{2}e^{-|\tau|}$ $\quad S_X(\omega) = S_Y^2(\omega)$; $\quad R_X(\tau) = R_Y(\tau) * R_Y(\tau) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|z|}e^{-|\tau-z|}dz$ (频域的乘对应时域的卷积)

Solution:

When $\tau \geq 0$:

$$R_X(\tau) = \frac{1}{4}\left(\int_{-\infty}^{0} e^{z}e^{-(\tau-z)}\,dz + \int_{0}^{\tau} e^{-z}e^{-(\tau-z)}\,dz + \int_{\tau}^{\infty} e^{-z}e^{\tau-z}\,dz\right)$$

$$= \frac{1}{4}(e^{-\tau} + \tau e^{-\tau})$$

注：

$$\int_{-\infty}^{0} e^{z}e^{-(\tau-z)}\,dz = \int_{-\infty}^{0} e^{-\tau}e^{2z}\,dz = \frac{e^{-\tau}}{2}\int_{-\infty}^{0} e^{2z}\,d2z = \frac{e^{-\tau}}{2}$$

$$\int_{0}^{\tau} e^{-z}e^{-(\tau-z)}\,dz = \int_{0}^{\tau} e^{-\tau}\,dz = \tau e^{-\tau}$$

$$\int_{\tau}^{\infty} e^{-z}e^{\tau-z}\,dz = \frac{e^{\tau}}{2}\int_{\tau}^{\infty} e^{-2z}\,d2z = -\left(-\frac{e^{-\tau}}{2}\right) = \frac{e^{-\tau}}{2}$$

More examples from the last week

- 1: if the power spectrum of the stationary process $X(t)$ is $S_X(\omega) = \frac{1}{[1+\omega^2]^2}$, calculate the autocorrelation.

Tips：

$$S_Y(\omega) = \frac{1}{1+\omega^2} \Leftrightarrow \frac{1}{2} e^{-|\tau|} \qquad S_X(\omega) = S_Y^2(\omega)$$

$$R_X(\tau) = R_Y(\tau) * R_Y(\tau) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|z|} e^{-|\tau-z|} dz \text{ (频域的乘对应时域的卷积)}$$

Solution:

When $\tau \geq 0$:

$$R_X(\tau) = \frac{1}{4}\left(\int_{-\infty}^{0} e^{z} e^{-(\tau-z)}\, dz + \int_{0}^{\tau} e^{-z} e^{-(\tau-z)}\, dz + \int_{\tau}^{\infty} e^{-z} e^{\tau-z}\, dz\right)$$

$$= \frac{1}{4}(e^{-\tau} + \tau e^{-\tau})$$

When $\tau \leq 0$:

$$R_X(\tau) = \frac{1}{4}(e^{\tau} - \tau e^{\tau})$$

→

$$R_X(\tau) = \frac{1}{4}\left(e^{-|\tau|} + |\tau| e^{-|\tau|}\right)$$

More examples from the last week

- 2: let $X(t)\ and\ Y(t)$ be statistically independent stationary processes, with non-zero means m_X and m_Y. Define $Z(t) = X(t) + Y(t)$, given $S_X(\omega)$, calculate $S_{XY}(\omega)$ and $S_{XZ}(\omega)$.

Solution:

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] = m_X\ m_Y = R_{XY}(\tau)$$

$$\Rightarrow S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\, d\tau = m_X m_Y \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau = 2\pi m_X m_Y \delta(\omega)$$

$$R_{XZ}(t_1, t_2) = E[X(t_1)Z(t_2)] = E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)]$$

$$= R_X(\tau) + m_X m_Y$$

$$\Rightarrow S_{XZ}(\omega) = S_X(\omega) + 2\pi m_X m_Y \delta(w)$$

Memoryless and Linear Systems – Outline

- Introduction
- Memoryless Systems
- Linear Systems
- Differentiators (微分器)(next week)

Introduction

- Given a stochastic process $\mathbf{x}(t)$, we assign according to some rule to each of its samples $\mathbf{x}(t, \zeta_i)$ a function $\mathbf{y}(t, \zeta_i)$. We have thus created another process

$$\mathbf{y}(t) = T[\mathbf{x}(t)]$$

whose samples are the functions $\mathbf{y}(t, \zeta_i)$. ζ_i referes to the state i.

- The process $\mathbf{y}(t)$ so formed can be considered as the output of a system (transformation) with input process $\mathbf{x}(t)$.

- The system is completely specified in terms of the operator T, that is, the rule of correspondence between the samples(样本间的对应规则) of the input $\mathbf{x}(t)$ and the output $\mathbf{y}(t)$.

Introduction

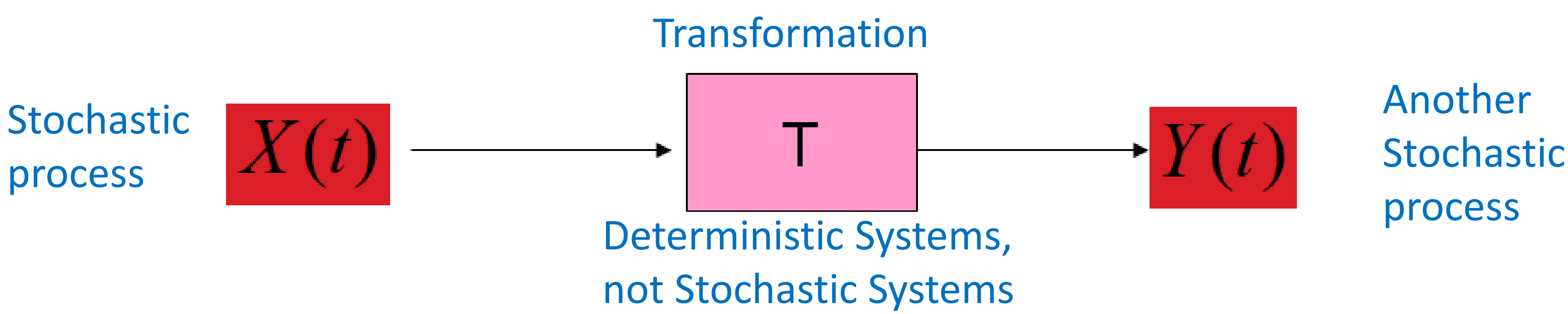
$$\mathbf{y}(t) = T[\mathbf{x}(t)]$$

- The system is **deterministic** if it operates only on the variable t treating ζ as a parameter. This means that if two samples $\mathbf{x}(t,\zeta_1)$ and $\mathbf{x}(t,\zeta_2)$ of the input are identical in t, then the corresponding samples $\mathbf{y}(t,\zeta_1)$ and $\mathbf{y}(t,\zeta_2)$ of the output are also identical in t.

- The system is called **stochastic** if T operates on both variables t and ζ. This means that there exist two outcomes ζ_1 and ζ_2 such that $\mathbf{x}(t,\zeta_1) = \mathbf{x}(t,\zeta_2)$ identically in t but $\mathbf{y}(t,\zeta_1) \neq \mathbf{y}(t,\zeta_2)$. These classifications are based on the terminal properties of the system. If the system is specified in terms of physical elements or by an equation, then it is deterministic (stochastic) if the elements or the coefficients of the defining equations are deterministic (stochastic).

- Throughout this book we shall consider only **deterministic systems (确定性系统)**.

- In principle, the statistics of the output of a system can be expressed in terms of the statistics of the input. However, in general this is a complicated problem. We consider next two important special cases: Memoryless Systems and Linear Systems.



Memoryless Systems

- A system is called memoryless if its output is given by

$$\mathbf{y}(t) = g[\mathbf{x}(t)]$$

- where $g(x)$ is a function of x. Thus, at a given time $t = t_1$, the output $\mathbf{y}(t_1)$ depends only on $\mathbf{x}(t_1)$ and not on any other past or future values of $\mathbf{x}(t)$.

- The 1-D pdf $f_y(y;t)$ of $\mathbf{y}(t)$ can be expressed in terms of the corresponding density $f_x(x;t)$ of $\mathbf{x}(t)$ (see lesson 3), and

$$E\{\mathbf{y}(t)\} = \int_{-\infty}^{\infty} g(x) f_x(x;t)dx$$

- Similarly, since $\mathbf{y}(t_1) = g[\mathbf{x}(t_1)]$ and $\mathbf{y}(t_2) = g[\mathbf{x}(t_2)]$, the 2-D pdf $f_y(y_1, y_2; t_1, t_2)$ of $\mathbf{y}(t)$ can be determined in terms of the $f_x(x_1, x_2; t_1, t_2)$ of $\mathbf{x}(t)$ (see lesson 3), and

$$E\{\mathbf{y}(t_1)\mathbf{y}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)f_x(x_1, x_2; t_1, t_2)dx_1 dx_2$$

Memoryless Systems – Stationary

- Suppose that the input to a memoryless system is an SSS (严格平稳) process $\mathbf{x}(t)$, the resulting output $\mathbf{y}(t)$ is also SSS.
- However, if $\mathbf{x}(t)$ is WSS (广义平稳), $\mathbf{y}(t)$ might not be stationary
  - $\mathbf{y}(t)$ might be WSS
  - $\mathbf{y}(t)$ can be not WSS

Memoryless Systems – Square-law detector

- A square-law detector is a memoryless system whose output equals

$$\mathbf{y}(t) = \mathbf{x}^2(t)$$

- Its 1-D and 2-D pdf can then be determined:
  - For 1-D, if $y > 0$, then the system $y = x^2$ has the two solutions $\pm\sqrt{y}$, and $x'(\mathrm{y}) = \pm\frac{1}{2\sqrt{y}}$
  - →

$$f_y(y;t) = \frac{1}{2\sqrt{y}}[f_x(\sqrt{y};t) + f_x(-\sqrt{y};t)]$$

  - For 2-D, if $y_1 > 0$ and $y_2 > 0$, then the system

$$y_1 = x_1^2; \qquad y_2 = x_2^2$$

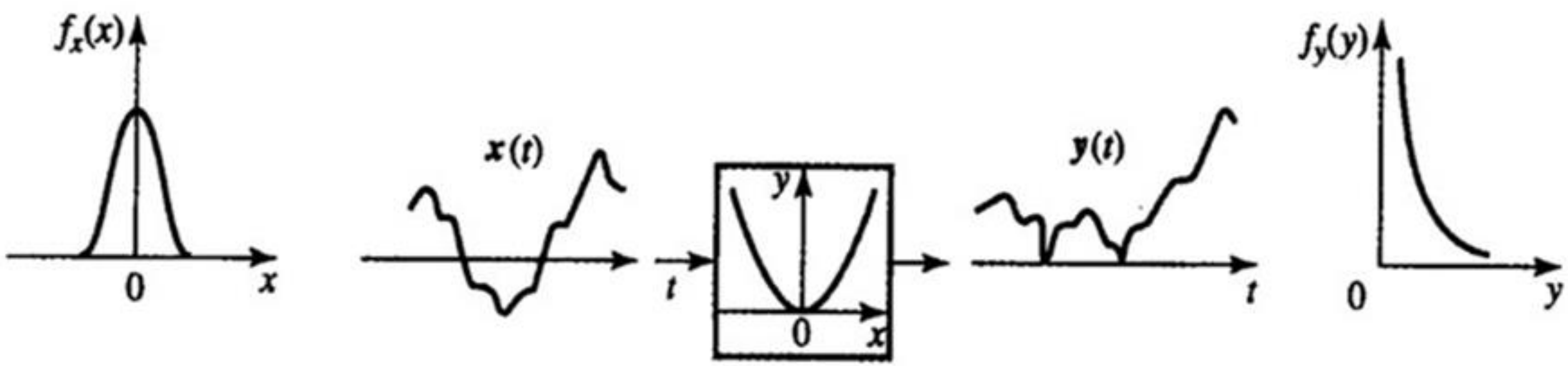
has the four solutions $\left(\pm\sqrt{y_1}, \pm\sqrt{y_2}\right)$. Its Jacobian equals $\pm\frac{1}{4\sqrt{y_1 y_2}}$ →

$$f_y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_x(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2)$$

where the summation has four terms.

Memoryless Systems – Square-law detector

- If $\mathbf{x}(t)$ is SSS, then $f_x(x;t) = f_x(x)$ is independent of t and $f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; \tau)$ depends only on $\tau = t_1 - t_2$. Hence $f_y(y)$ is independent of t and $f_y(y_1, y_2; \tau)$ depends only on $\tau = t_1 - t_2$.
  - Note: $\mathbf{x}(t)$ WSS $\neq \boldsymbol{y}(t)$ WSS
  - Note that memoryless systems and linear systems are two different things
- Suppose that $\mathbf{x}(t)$ is a normal stationary process with zero mean and autocorrelation $R_x(\tau)$. (For normal process, WSS = SSS), If $\mathbf{y}(t) = \mathbf{x}^2(t)$, then:



Memoryless Systems – Square-law detector

- If $\mathbf{y}(t) = \mathbf{x}^2(t)$, and $\mathbf{x}(t)$ is a normal stationary process:
  - $f_x(x)$ is normal with variance $R_x(0)$
  - $E\{\mathbf{y}(t)\} = R_x(0)$ and

$$f_y(y) = \frac{1}{\sqrt{2\pi R_x(0) y}} e^{-y/2R_x(0)} U(y)$$ (understanding is OK)

  - The random variables $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ are jointly normal with zero mean, and

$$E\{\mathbf{y}(t+\tau)\mathbf{y}(t)\} = E\{\mathbf{x}^2(t+\tau)\mathbf{x}^2(t)\}$$
$$= E\{\mathbf{x}^2(t+\tau)\}E\{\mathbf{x}^2(t)\} + 2E^2\{\mathbf{x}(t+\tau)\mathbf{x}(t)\}$$

(understanding is OK, see page 183 of text book)

$$\rightarrow \quad R_y(\tau) = R_x^2(0) + 2R_x^2(\tau)$$

- Note:

$$E\{\mathbf{y}^2(t)\} = R_y(0) = 3R_x^2(0)$$

$$\sigma_y^2 = 2R_x^2(0)$$

Memoryless Systems – Square-law detector

- Still in some researches / engineering applications

Linear Systems

- If for any $\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1(t), \mathbf{x}_2(t)$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are random variables, $\mathbf{y}(t)$ is the output of a system with input $\mathbf{x}(t)$, we have
$$L[\mathbf{a}_1\mathbf{x}_1(t) + \mathbf{a}_2\mathbf{x}_2(t)] = \mathbf{a}_1 L[\mathbf{x}_1(t)] + \mathbf{a}_2 L[\mathbf{x}_2(t)]$$
- The $\mathbf{y}(t) = L[\mathbf{x}(t)]$ is a Linear System. Usually, we call it Linear System *L*. Note that the system is deterministic and it operates only on the variable t.
- Here we do not consider any input with initial conditions. (我们这里不讨论带初始响应的系统)
- A system *L* is called time-invariant if its response to $\mathbf{x}(t + c)$ equals $\mathbf{y}(t + c)$.
- In this course we mainly discuss linear time-invariant systems.

Linear Systems

- It is well known that **the output of a linear system is a convolution**

$$\mathbf{y}(t) = \mathbf{x}(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)d\alpha \qquad \text{(8-1)}$$

Where $h(t) = L[\delta(t)]$ is its impulse response.

- In the following, when writing $\mathbf{y}(t) = L[\mathbf{x}(t)]$, we refer to the convolution (8-1) in most cases.
- The following observations are immediate consequences of the linearity and time invariance of the system:
  - If $\mathbf{x}(t)$ is a normal process, then $\mathbf{y}(t)$ is also a normal process. This is an extension of the familiar property of linear transformations of normal random variables and can be justified if we approximate the integral in (8-1) by a sum:

$$\mathbf{y}(t_i) \approx \sum_k \mathbf{x}(t_i - \alpha_k)h(\alpha_k)\Delta(\alpha)$$

  - If $\mathbf{x}(t)$ is SSS, then $\mathbf{y}(t)$ is also SSS.
    - Indeed, since $\mathbf{y}(t+c) = L[\mathbf{x}(t+c)]$ for every c, we conclude that if the processes $\mathbf{x}(t)$ and $\mathbf{x}(t+c)$ have the same statistical properties, so do the processes $\mathbf{y}(t)$ and $\mathbf{y}(t+c)$.
  - **If $x(t)$ is WSS, the processes $x(t)$ and $y(t)$ are jointly WSS. （本课程一般只讨论稳态情况）**

Linear Systems – property

- For any linear system:

$$E\{L[\mathbf{x}(t)]\} = L[E\{\mathbf{x}(t)\}]$$

Also written as $$\boldsymbol{m}_y(t) = L[\boldsymbol{m}_x(t)]$$

- In other words, the expectation of the output $\mathbf{y}(t) = L[\mathbf{x}(t)]$ equals the response of the system to the mean $E\{\mathbf{x}(t)\}$ of the input, as

$$E\{\mathbf{y}(t)\} = \int_{-\infty}^{\infty} E\{\mathbf{x}(t-\alpha)\}h(\alpha)d\alpha = \boldsymbol{m}_x(t) * h(t)$$

Linear Systems – output autocorrelation

- We wish to express the autocorrelation $R_{yy}(t_1, t_2)$ of the output $\mathbf{y}(t)$ of a linear system in terms of the autocorrelation $R_{xx}(t_1, t_2)$ of the input $\mathbf{x}(t)$.

- In fact, it is easier to find first the cross-correlation $R_{xy}(t_1, t_2)$ between $\mathbf{x}(t)$ and $\mathbf{y}(t)$:

$$R_{xy}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] \qquad (8\text{-}2)$$

Where L_2 means that the system operates on the variable t_2 when treating t_1 as a parameter, this means:

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - \alpha)h(\alpha)d\alpha$$

- Then

$$R_{yy}(t_1, t_2) = L_1\left[R_{xy}(t_1, t_2)\right] \qquad (8\text{-}3)$$

Where the system operates on t_1:

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xy}(t_1 - \alpha, t_2)h(\alpha)d\alpha$$

- 注意，对$Y(t)$的autocorrelation，可以写作R_{yy}或者R_y， 也可以写作R_Y，也有写作R_{YY}的

Linear Systems – output autocorrelation

Proof:

- If for any $\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1(t), \mathbf{x}_2(t)$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are random variables, $\mathbf{y}(t)$ is the output of a linear system with input $\mathbf{x}(t)$, we have
$$L[\mathbf{a}_1\mathbf{x}_1(t) + \mathbf{a}_2\mathbf{x}_2(t)] = \mathbf{a}_1 L[\mathbf{x}_1(t)] + \mathbf{a}_2 L[\mathbf{x}_2(t)]$$

- Multiplying $\mathbf{y}(t) = L_t[\mathbf{x}(t)]$ by $\mathbf{x}(t_1)$ gives
$$\mathbf{x}(t_1)\mathbf{y}(t) = \mathbf{x}(t_1)L_t[\mathbf{x}(t)]$$

Note that, the $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are the input & output process, but $\mathbf{x}(t_1)$, the sample of the process $\mathbf{x}(t)$ at time t_1, as it already be sampled, is a constant! →
$$\mathbf{x}(t_1)\mathbf{y}(t) = L_t[\mathbf{x}(t_1)\mathbf{x}(t)]$$

- where L_t means that the system operates on t. Hence
$$E\{\mathbf{x}(t_1)\mathbf{y}(t)\} = L_t[E\{\mathbf{x}(t_1)\mathbf{x}(t)\}]$$

- Setting $t = t_2$ gives (8-2). The proof of (8-3) is similar: We multiply $\mathbf{y}(t) = L_t[\mathbf{x}(t)]$ by $\mathbf{y}(t_2)$ and yields
$$E\{\mathbf{y}(t)\mathbf{y}(t_2)\} = L_t[E\{\mathbf{x}(t)\mathbf{y}(t_2)\}]$$

$$E\{L[\mathbf{x}(t)]\} = L[E\{\mathbf{x}(t)\}]$$

- and (8-3) follows with $t = t_1$

Linear Systems – output autocorrelation

Conclusion:

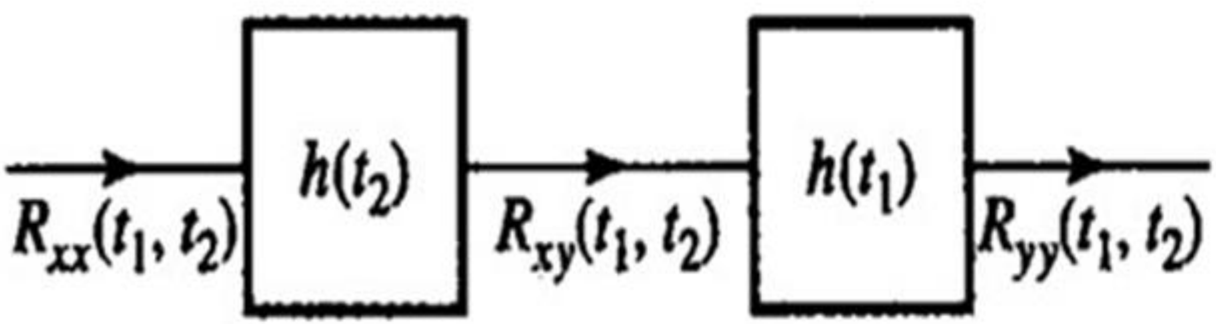
$$R_{XY}(t_1,t_2) = L_{t_2}[R_{XX}(t_1,t_2)] = L_{t_2}[R_X(t_1,t_2)],\ \text{or}\ R_{XY}(t_1,t_2) = R_X(t_1,t_2) * h(\mathrm{t}_2)$$

$$R_{YX}(t_1,t_2) = L_{t_1}[R_{XX}(t_1,t_2)] = L_{t_1}[R_X(t_1,t_2)],\ \text{or}\ R_{YX}(t_1,t_2) = R_X(t_1,t_2) * h(\mathrm{t}_1)$$

$$R_Y(t_1,t_2) = R_{YY}(t_1,t_2) = L_{t_1}[R_{XY}(t_1,t_2)] = L_{t_1}L_{t_2}[R_X(t_1,t_2)]$$

Or says: $$R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1-\alpha,t_2-\beta)h(\alpha)h(\beta)d\alpha d\beta$$

Which is: $$R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h(\mathrm{t}_1) * h(t_2)$$



Linear Systems – output autocorrelation & autocovariance

- The autocovariance $C_{yy}(t_1, t_2)$ of $\mathbf{y}(t)$ is the autocorrelation of the process

$$\tilde{\mathbf{y}}(t) = L[\tilde{\mathbf{x}}(t)] = L[\mathbf{x}(t) - \boldsymbol{m}_x(t)] = \mathbf{y}(t) - \boldsymbol{m}_y(t)$$

- Therefore:

$$C_{xy}(t_1, t_2) = C_{xx}(t_1, t_2) * h(t_2); \qquad C_{yx}(t_1, t_2) = C_{xx}(t_1, t_2) * h(t_1)$$

$$C_{yy}(t_1, t_2) = C_{xy}(t_1, t_2) * h(t_1)$$

- For complex processes, following the same approach we get:

$$R_{xy}(t_1, t_2) = R_{xx}(t_1, t_2) * h^*(t_2); \qquad R_{yx}(t_1, t_2) = R_{xx}(t_1, t_2) * h(t_1)$$

$$R_{yy}(t_1, t_2) = R_{xx}(t_1, t_2) * h(t_1) * h^*(t_2)$$

$$C_{xy}(t_1, t_2) = C_{xx}(t_1, t_2) * h^*(t_2); \qquad C_{yx}(t_1, t_2) = C_{xx}(t_1, t_2) * h(t_1)$$

$$C_{yy}(t_1, t_2) = C_{xx}(t_1, t_2) * h(t_1) * h^*(t_2)$$

Linear Systems – output autocorrelation*

An example requires understanding only (not in exam): A real world stationary process $\boldsymbol{v}(t)$ with autocorrelation $R_{vv}(\tau) = q\delta(\tau)$ (white noise) is applied at $\boldsymbol{t} = \mathbf{0}$ to a linear system with

$$h(t) = e^{-ct}U(t)$$

Then, for $0 < t_1 < t_2$, the full autocorrelation of the resulting output $\mathbf{y}(t)$ equals

$$R_{yy}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1})e^{-c(t_2 - t_1)}$$

Proof: we assume that the input to the system is the real world process

$$\mathbf{x}(t) = \boldsymbol{v}(t)U(t)$$

- Physical meaning of $h(t) = e^{-ct}U(t)$: at time 0, the system starts, for example, you power on the receiver at time 0 (在时间为0的时候打开接收机), then the receiver starts working. In this case, for any time before time 0, which is when the receiver is not turned on, there are no response of the system. (在系统开机之前，系统不会有任何响应)

- Physical meaning of $\mathbf{x}(t) = \boldsymbol{v}(t)U(t)$: at time 0, as the system starts, the system begin to receive input $\boldsymbol{v}(t)$. Although the process $\boldsymbol{v}(t)$ exists before time 0, for example, you speak (there is signal before time 0) before you turn on a microphone, the system cannot only receive the signal after it is turned on. (在系统开机之前，系统不可能可以接收到任何信号，即使该信号存在)

注：这里讨论的是一个实际系统对应的例子，但是，在考试中，我们不会涉及如此复杂的问题

*：本页ppt理解即可

Linear Systems – output autocorrelation*

An example requires understanding only (not in exam): A real world stationary process $\boldsymbol{v}(t)$ with autocorrelation $R_{\nu\nu}(\tau) = q\delta(\tau)$ (white noise) is applied at $t = 0$ to a linear system with

$$h(t) = e^{-ct}U(t)$$

Then, for $0 < t_1 < t_2$, the full autocorrelation of the resulting output $\mathbf{y}(t)$ equals

$$R_{yy}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1})e^{-c(t_2 - t_1)}$$

Proof: we assume that the input to the system is the real world process

$$\mathbf{x}(t) = \boldsymbol{v}(t)U(t)$$

- With this assumption, all correlations are 0 if $t_1 < 0$ or $t_2 < 0$.

- For $t_1 > 0$ and $t_2 > 0$,

$$R_{xx}(t_1, t_2) = E\{\boldsymbol{v}(t_1)\boldsymbol{v}(t_2)\} = q\delta(t_1 - t_2)$$

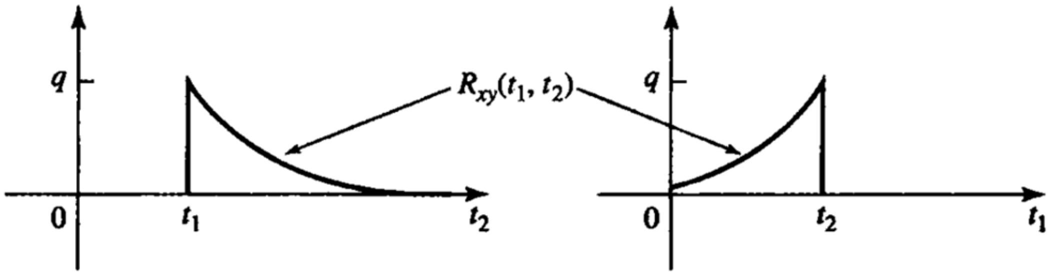
- Then $R_{xy}(t_1, t_2)$ equals the response of the system to $q\delta(t_1 - t_2)$ considered as a function of t_2. Since $\delta(t_1 - t_2) = \delta(t_2 - t_1)$ and $L[\delta(t_2 - t_1)] = h(t_2 - t_1)$ (time invariance), we conclude that

$$R_{xy}(t_1, t_2) = qh(t_2 - t_1) = qe^{-c(t_2 - t_1)}U(t_2 - t_1)U(t_1)^{[1]}$$

[1]：这里加的$U(t_1)$是因为，只有在$t_1 > 0$时，$R_{xy}(t_1, t_2)$才会有取值，反之是0！对此的刻画就是加一个在t_1时刻的阶跃函数$U(t_1)$

*：本页ppt理解即可

Linear Systems – output autocorrelation*



注： 此图说明的是， 当$0 < t_1 < t_2$时，$R_{xy}(t_1, t_2)$如左图所示； 为了求$R_{yy}(t_1, t_2)$，需要对$R_{xy}(t_1, t_2)$和$h(t)$做卷积， 固定$h(t)$，将$R_{xy}(t_1, t_2)$左右翻转， 因为$0 < t_1 < t_2$，其尾部的一部分去到了负数域， 被截断了（在真实应用中， 不可能能读取到开机前的任何信息）， 变成了右图， 然后以右图去和$h(t)$做卷积

- The above figure shows $R_{xy}(t_1, t_2)$ as a function of t_1 and t_2. As $R_{yy}(t_1, t_2) = R_{xy}(t_1, t_2) * h(t_1)$, we obtain

$$R_{yy}(t_1, t_2) = q \int_0^{t_1} e^{c(t_1 - \alpha - t_2)} e^{-c\alpha} d\alpha = q e^{c(t_1 - t_2)} \int_0^{t_1} e^{-2c\alpha} d\alpha$$

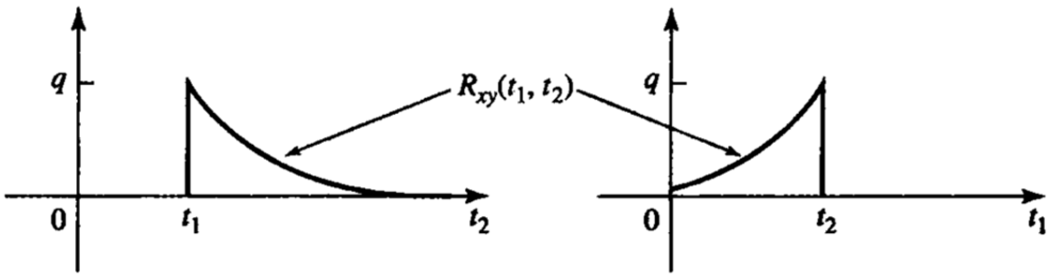
Which is $R_{yy}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1}) e^{-c(t_2 - t_1)}$ (for $0 < t_1 < t_2$).

- Note that

$$E\{\mathbf{y}^2(t)\} = R_{yy}(t, t) = \frac{q}{2c}(1 - e^{-2ct}) = q \int_0^t h^2(\alpha) d\alpha$$

*：本页ppt理解即可

Linear Systems – output autocorrelation



注：此图说明的是，当$0 < t_1 < t_2$时，$R_{xy}(t_1, t_2)$如左图所示；为了求$R_{yy}(t_1, t_2)$，需要对$R_{xy}(t_1, t_2)$和$h(t)$做卷积，固定$h(t)$，将$R_{xy}(t_1, t_2)$左右翻转，因为$0 < t_1 < t_2$，其尾部的一部分去到了负数域，被截断了（在真实应用中，不可能能读取到开机前的任何信息），变成了右图，然后以右图去和$h(t)$做卷积

- The above figure shows $R_{xy}(t_1, t_2)$ as a function of t_1 and t_2. As $R_{yy}(t_1, t_2) = R_{xy}(t_1, t_2) * h(t_1)$, we obtain

$$R_{yy}(t_1, t_2) = q \int_0^{t_1} e^{c(t_1-\alpha-t_2)} e^{-c\alpha} d\alpha = q e^{c(t_1-t_2)} \int_0^{t_1} e^{-2c\alpha} d\alpha$$

Which is $R_{yy}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1}) e^{-c(t_2-t_1)}$ (for $0 < t_1 < t_2$).

- It is not stationary! Related to time t_1! The input is **stationary but the output is not!**
- This is called Transient analysis (瞬态分析) of stochastic processes, is not required in exam.
- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

- 此后，除非特别说明，否则问的都是稳态分析

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

  If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

Properties when WSS ($\tau = t_1 - t_2$): (real valued case)

$$R_{XY}(t_1, t_2) = R_X(t_1, t_2) * h(t_2) \rightarrow R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

$$R_{YX}(t_1, t_2) = R_X(t_1, t_2) * h(t_1) \rightarrow R_{YX}(\tau) = R_X(\tau) * h(\tau)$$

$$R_Y(t_1, t_2) = R_X(t_1, t_2) * h(t_1) * h(t_2) \rightarrow R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau + \alpha - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

- 此后，除非特别说明，否则问的都是稳态分析
- 注意，对$Y(t)$的autocorrelation，可以写作R_{yy}或者R_y， 也可以写作R_Y，也有写作R_{YY}的

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

Properties when WSS ($\tau = t_1 - t_2$): (complex valued case)

$$R_{XY}(t_1, t_2) = R_X(t_1, t_2) * h^*(\mathrm{t}_2) \quad \rightarrow \quad R_{XY}(\tau) = R_X(\tau) * h^*(-\tau)$$

$$R_{YX}(t_1, t_2) = R_X(t_1, t_2) * h(\mathrm{t}_1) \quad \rightarrow \quad R_{YX}(\tau) = R_X(\tau) * h(\tau)$$

$$R_Y(t_1, t_2) = R_X(t_1, t_2) * h(\mathrm{t}_1) * h^*(t_2) \quad \rightarrow \quad R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$$

$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau + \alpha - \beta) h(\alpha) h^*(\beta) d\alpha d\beta$$

- 此后，除非特别说明，否则问的都是稳态分析
- 注意，对$Y(t)$的autocorrelation，可以写作R_{yy}或者R_y，也可以写作R_Y，也有写作R_{YY}的

Linear Systems – output autocorrelation

- Example 1: A stationary process $\boldsymbol{x}(t)$ with autocorrelation $R_X(\tau) = q\delta(\tau)$ (white noise) is applied ~~at $t = 0$~~ to a linear system with

$$h(t) = e^{-ct}U(t)$$

Please calculate the autocorrelation of the output, or says, calculate $R_{yy}(\tau)$ where $\tau = t_1 - t_2$.

Hint: in this case, the input is $\boldsymbol{x}(t)$! Not $\boldsymbol{x}(t)U(t)$!

Linear Systems – output autocorrelation — Previous example

An example requires understanding only (not in exam): A real world stationary process $v(t)$ with autocorrelation $R_{vv}(\tau) = q\delta(\tau)$ (white noise) is applied at $t = 0$ to a linear system with

$$h(t) = e^{-ct}U(t)$$

Then, for $0 < t_1 < t_2$, the full autocorrelation of the resulting output $\mathbf{y}(t)$ equals

$$R_{yy}(t_1, t_2) = \frac{q}{2c}\left(1 - e^{-2ct_1}\right)e^{-c|t_2 - t_1|}$$

Proof: we assume that the input to the system is the real world process

$$\mathbf{x}(t) = \boldsymbol{v}(t)U(t)$$

- With this assumption, all correlations are 0 if $t_1 < 0$ or $t_2 < 0$.

Linear Systems – output autocorrelation

- Example 2: A stationary process $\boldsymbol{x}(t)$ with autocorrelation $R_X(\tau) = \sigma_X^2 e^{-\beta|\tau|}$ (white noise) is applied to a linear system with

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Please calculate the autocorrelation of the output, or says, calculate $R_Y(\tau)$ where $\tau = t_1 - t_2$.

Hint:

$$R_Y(\tau) = \int_0^\infty \int_0^\infty R_X(\tau + u - v)h(u)h(v)\,dudv = \alpha^2\sigma_X^2 \int_0^\infty \int_0^\infty e^{-\beta|\tau+u-v|} e^{-\alpha(u+v)}\,dudv$$

$$= \alpha^2\sigma_X^2 \int_0^\infty \int_0^{\tau+u} e^{-\beta(\tau+u-v)} e^{-\alpha(u+v)}\,dvdu + \alpha^2\sigma_X^2 \int_0^\infty \int_{\tau+u}^\infty e^{-\beta(v-\tau-u)} e^{-\alpha(u+v)}\,dvdu$$

$$\int_0^\infty \int_0^{\tau+u} e^{-\beta(\tau+u-v)} e^{-\alpha(u+v)}\,dvdu = \frac{e^{-\alpha\tau}}{2\alpha(\beta-\alpha)} - \frac{e^{-\beta\tau}}{\beta^2-\alpha^2}$$

$$\int_0^\infty \int_{\tau+u}^\infty e^{-\beta(v-\tau-u)} e^{-\alpha(u+v)}\,dvdu = \frac{e^{-\alpha\tau}}{2\alpha(\alpha+\beta)}$$

注意，这里因为题干里面有α和β，所以在积分的时候，就用u和v代替了α和β

Reading

- This week:
  - Text book: 7.2
  - Red book: 3.1, 3.2.1
- Next week:
  - Text book: 7.3 (part of ‘power spectrum together with systems’)
  - Red book: 3.2.2, 3.2.3

Experiment

- Experiment 2