第二次作业

一、 Fill in the blanks.（10 points）

1. The autocorrelation of a real stationary stochastic process is $R_X(\tau)=\frac{40+72\tau^2}{1+2\tau^2}$, then for this process, the mean is __________, the variance is ____________________.（3 points each）
2. Mean value of white noise is________________, The autocorrelation function is an impulse function (or δ Function). （4 points）

Sol：

1. ±6， 4
2. 0（zero）

二、 Calculation. （90 points）

1. (20 points) If $Y(t)=X(t-\alpha)$, and the system input $X(t)$ is a stationary stochastic process with autocorrelation $R_X(\tau)$ and power spectrum is $S_X(\omega)$, find the autocorrelation $R_Y(\tau)$ and power spectrum $S_Y(\omega)$. (written as $R_X(\tau)$ and $S_X(\omega)$)

Sol:

As $Y(t)=X(t-\alpha)$, we have

$$R_Y(\tau)=E[Y(t)Y(t-\tau)]=E[\{X(t-\alpha)\}\{X(t-\alpha-\tau)\}]=R_x(\tau)$$

Thus $S_Y(\omega)=S_X(\omega)$。

Try: how about $Y(t)=X(t)+X(t-\alpha)$?

2. (20 points) The power spectrum of a stationary stochastic process is $S_x(\omega)=\frac{\omega^2+17}{\omega^4+34\omega^2+225}$: Calculate the autocorrelation, mean, variance and correlation coefficient of the stochastic process.

Sol:

$$S_x(\omega)=\frac{\omega^2+17}{\omega^4+34\omega^2+225}=\frac{1}{2}\frac{(\omega^2+25)+(\omega^2+9)}{(\omega^2+9)(\omega^2+25)}$$
$$=\frac{1}{2}\left(\frac{1}{6}\frac{2\cdot 3}{\omega^2+3^2}+\frac{1}{10}\frac{2\cdot 5}{\omega^2+5^2}\right)$$

Given Fourier Transform $e^{-\alpha|\tau|}\leftrightarrow\frac{2\alpha}{\omega^2+\alpha^2}$

The autocorrelation is: $R_x(\tau)=\frac{1}{12}e^{-|\tau|}+\frac{1}{20}e^{-5|\tau|}$

The mean $m_x^2=R_x(\infty)=0, \therefore m_x=0$

The variance $\sigma_x^2=R_x(0)-m_x^2=\frac{2}{15}-0=\frac{2}{15}$

The correlation coefficient $r_{x(\tau)}=\frac{R_x(\tau)-m_x}{\sigma_x^2}=\frac{5}{8}e^{-3|\tau|}+\frac{3}{8}e^{-5|\tau|}$

3. (20 points) The stochastic process $X(t) = A\cos(\omega_0 t + \Phi)$, where ω_0 is a constant, and A and Φ are independent random variables. Φ is uniformly distributed in $(-3.5\pi, 2.5\pi)$, and A is a zero mean Gaussian random variable with variance 1.

(1) Is $X(t)$ Wide Sense Stationary (WSS)? Prove it. (10 points)

(2) Calculate the power spectrum of this process $X(t)$. (10 points)

Solution:

(1) Prove the WSS:

Note that $E(A^2) = Var(A) + m_A^2 = 1$

$E[X(t)] = E[A\cos(\omega_0 t + \Phi)] = E[A]E[\cos(\omega_0 t + \Phi)] = 0$

$E[X(t_1)X(t_2)] = E[A^2\cos(\omega_0 t_1 + \Phi)\cos(\omega_0 t_2 + \Phi)]$

$= E[A^2]E[\cos(\omega_0 t_1 + \Phi)\cos(\omega_0 t_2 + \Phi)]$

$$= 1 \times \frac{1}{2}E[\cos(\omega_0 t_1 + \omega_0 t_2 + 2\Phi) + \cos(\omega_0 t_1 - \omega_0 t_2)]$$

$$= \frac{1}{2}E[\cos\omega_0(t_1 - t_2)]$$

$= \frac{1}{2}\cos(\omega_0\tau) = R_X(\tau)$, where $\tau = t_1 - t_2$

Thus $X(t)$ is WSS

(2) power spectrum is $S_X(\omega) = \frac{\pi}{2}\delta(\omega - \omega_0) + \frac{\pi}{2}\delta(\omega + \omega_0)$

4. (30 points) Given real joint stationary processes $X(t)$ and $Y(t)$: $\alpha Y(t) + \frac{d^3Y(t)}{d^3t} = X(t) - \beta\frac{d^2X(t)}{d^2t}$, and the power spectrum of $X(t)$ is $S_X(\omega)$,

(1) Calculate the transfer function $H_Y(\omega)$ and the cross-power spectrum of $S_{XY}(\omega)$ and $S_{YX}(\omega)$ (represented by α, β and $S_X(\omega)$). (15 points)

(2) If the input $X(t)$ is a white noise with power spectrum q, and $S_Y(\omega) = \frac{2\beta\omega^2+2}{\omega^4-\omega^2+1}$, calculate α, β, q (value only, do not consider the units). (15 points)

Solution:

(1) we have: $\alpha Y(\omega) + (j\omega)^3Y(\omega) = X(\omega) - \beta(j\omega)^2X(\omega)$

Thus the transfer function is $H_Y(\omega) = \frac{1+\beta\omega^2}{\alpha-j\omega^3}$

Define $\tau = t_1 - t_2$:

$$S_{YX}(\omega) = S_X(\omega)H_Y(\omega) = \frac{1+\beta\omega^2}{\alpha-j\omega^3}S_X(\omega)$$

$$S_{XY}(\omega) = S_X(\omega)H_Y^*(\omega) = \frac{1+\beta\omega^2}{\alpha+j\omega^3}S_X(\omega)$$

(2) $S_Y(\omega) = |H_Y(\omega)|^2 S_X(\omega) = \frac{(1+\beta\omega^2)^2}{\alpha^2+\beta^2\omega^6}q = \frac{2\beta\omega^2+2}{\omega^4-\omega^2+1}$

Then: $2(\alpha^2+\beta^2\omega^6)=q(\beta\omega^2+1)(\omega^4-\omega^2+1)=q[\beta\omega^6+(1-\beta)\omega^4+(\beta-1)\omega^2+1]$

➔ $q\beta=2\beta^2$, $\beta-1=0$, $2\alpha^2=q$

Thus

$$q=2$$
$$\beta=1$$
$$\alpha=\pm 1$$