第三次作业 答案

1. For a zero mean Gaussian stationary process $X(t)$, whose power spectrum is:

$$S_X(\omega) = \begin{cases} A, & ||\omega| - \omega_0| < \dfrac{\Delta\omega}{2} \\ 0, & \text{others} \end{cases}$$

Where $\omega_0 > \Delta\omega$. Calculate the 1-D pdf of it. (10 points)

Solution:

The power of $X(t)$ is :

$$P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} S_X(\omega)\, d\omega = \frac{1}{\pi} A\Delta\omega$$

Therefore:

$$f_X(x) = \frac{1}{\sqrt{2A\Delta\omega}} e^{-\frac{\pi x^2}{2A\Delta\omega}}$$

2. Assuming that a stationary Gaussian white noise with power spectrum A is fed into a filter $H(\omega) = \frac{2}{j\omega+1}$, calculate the one-dimensional pdf of the output. (20 points)

Sol: The Gaussian process inputted to the system, and output is also Gaussian.

The $|H(\omega)|^2 = \frac{4}{\omega^2+1}$

Then the output $S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \frac{4A}{\omega^2+1}$

And for the output $Y(t)$, mean is $m_Y^2 = 0$ thus $m_Y = 0$.

And $\sigma_Y^2 = R_Y(0) = 2A$

Then the 1-D pdf is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-m_Y)^2}{2\sigma_Y^2}} = \frac{1}{2\sqrt{\pi A}} e^{-\frac{y^2}{2A}}$$

3. A stationary stochastic process $X(t)$ is fed into a low-pass filter $h(t) = \alpha e^{-\alpha t}U(t)$. The autocorrelation of $X(t)$ is $\delta(\tau)$. Calculate the output autocorrelation $R_Y(\tau)$. (20 points)

Sol:

The $h(t) = \alpha e^{-\alpha t}U(t)$, thus $H(\omega) = \int_0^{+\infty} e^{-\alpha t} e^{-j\omega t} dt = \frac{\alpha}{\alpha + j\omega}$

Then

$$|H(\omega)|^2 = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

The autocorrelation of $X(t)$ is $\delta(\tau)$, thus the power spectrum is $S_X(\omega) = 1$.

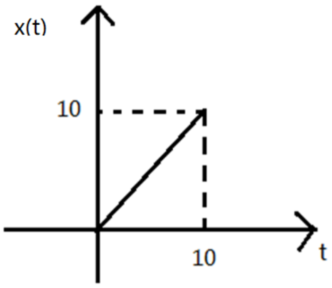
And

$$S_Y(w) = S_X(\omega)|H(\omega)|^2 = \frac{\alpha^2}{\alpha^2+\omega^2},$$

Therefore

$$R_Y(\tau) = \frac{1}{\pi}\int_0^{\infty} \frac{\alpha^2}{\alpha^2+\omega^2}\cos(\omega\tau)\,d\omega = \frac{\alpha}{2}\frac{1}{\pi}\int_0^{\infty} \frac{2\alpha}{\alpha^2+\omega^2}\cos(\omega\tau)\,d\omega = \frac{\alpha}{2}e^{-\alpha|\tau|}$$

4. Given an input $z(t) = x(t) + n(t)$, where $x(t)$ is a deterministic signal as follows (triangular wave), and $n(t)$ is stationary Gaussian white noise with power spectrum of q. (30 points)



a) Calculate the maximum signal-to-noise ratio of the output if this $z(t)$ is fed into its matched filter. (10 points)

b) There is another input $z_1(t) = \frac{1}{3}x(t) + n(t)$, calculate the maximum signal-to-noise ratio of the output of this input if it is fed into its matched filter. (10 points)

c) Calculate the spectrum of the matched filter $H(\omega)$ of the signal. (10 points)

Sol:

a) The signal is

$$s(t) = \begin{cases} t, & 0 \le t \le 10 \\ 0, & \text{others} \end{cases}$$

The maximum signal-to-noise ratio is $d_m = \frac{E}{q}$, where E is the signal power, which is the integral of the signal in time domain

$$E = \int_0^{10} t^2 dt = \frac{1000}{3}$$

Thus $d_m = \frac{1000}{3q}$

b) For time delay and attenuation, the match filter still works. The amplitude of the new signal is $\frac{1}{2}$ of the original one, then the new power is $E_{new} = \int_0^{10}\left(\frac{1}{3}t\right)^2 dt = \frac{1000}{9}$, the noise power is the same, therefore

$$d_{mnew} = \frac{1000}{27q}$$

c) The spectrum of the signal $s(t) = \begin{cases} t, & 0 \le t \le 10 \\ 0, & \text{others} \end{cases}$ is

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt = \int_0^{10} te^{-j\omega t}dt = -\frac{1}{j\omega}\int_0^{10} tde^{-j\omega t}$$

$$= -\frac{1}{j\omega}\left[te^{-j\omega t}|_0^{10} - \int_0^{10} e^{-j\omega t}dt\right]$$

$$= -\frac{1}{j\omega}\left[te^{-j\omega t} - \left(-\frac{1}{j\omega}\right)e^{-j\omega t}\right]_0^{10}$$

$$= \left[-\frac{1}{j\omega}te^{-j\omega t} - \left(\frac{1}{j\omega}\right)^2 e^{-j\omega t}\right]_0^{10}$$

$$= -\frac{1}{j\omega}10e^{-j\omega 10} + \frac{1}{\omega^2}e^{-j\omega 10} - \frac{1}{\omega^2}$$

Therefore the spectrum of the filter is

$$cS^*(\omega)e^{-j\omega t_0} = c[\frac{1}{j\omega}10e^{j\omega 10} + \frac{1}{\omega^2}e^{j\omega 10} - \frac{1}{\omega^2}]e^{-j\omega t_0}$$

5. The stochastic process $Y(t) = Xcos(\omega_0 t + \theta)$, where ω_0 is a constant, X and θ are independent random variables, X is zero mean white Gaussian variable with variance c^2 where c is a constant, and θ follows uniformly distributed in $(-\pi, \pi)$. (20 points)

a) Calculate the power spectrum of $Y(t)$.

b) Is $Y(t)$ an ergodicity process? Prove it.

solution:

a) The mean of $Y(t)$ is $E[Y(t)] = E[X]E[cos(\omega_0 t + \theta)] = 0$

The autocorrelation of $Y(t)$ is

$$R_Y(t_1, t_2) = E[X^2 cos(\omega_0 t_1 + \theta) cos(\omega_0 t_2 + \theta)]$$

$$= \frac{1}{2}E(X^2)E\{cos(\omega_0 t_1 + \omega_0 t_2 + 2\theta) + cos[\omega_0(t_1 - t_2)]\}$$

$$= \frac{1}{2}E(X^2)\,cos(\omega_0(t_1 - t_2)) = \frac{c^2}{2}cos(\omega_0 \tau),\ \ \tau = t_1 - t_2$$

The power spectrum is

$$S_Y(\omega) = FT\left(\frac{c^2}{2}\cos(\omega_0 \tau)\right) = \frac{c^2}{2}\delta(\omega - \omega_0) + \frac{c^2}{2}\delta(\omega + \omega_0)$$

b) Note that according to a), it is a WSS process, and $R_Y(\tau) = \frac{2\pi^2}{3}cos(\omega_0 \tau)$, $\tau = t_1 - t_2$。

The **time autocorrelation** of $Y(t)$ is

$$\overline{R_Y(\tau)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[Xcos(\omega_0 t + \tau + \theta)Xcos(\omega_0 t + \theta)]\,dt$$

$$= \frac{X^2}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[cos(2\omega_0 t + \omega_0 \tau + 2\theta) + cos(\omega_0 \tau)]\,dt = \frac{X^2}{2}cos(\omega_0 \tau)$$

Which is related to the X, $\overline{R_Y(\tau)}$ might not equals to $R_Y(\tau)$ for some X, therefore it is not an ergodicity process