Stochastic Signal Processing

Lesson 7

Spectrum Analysis of Stochastic Processes 1: Power Spectrum

Weize Sun

More examples from the last week

- 1: given a stochastic phase signal $X(t) = Acos(\omega t + \varphi)$, where A and ω are constants, and φ is a r.v uniformly distributed in $(0,2\pi)$. Is $X(t)$ Wide-Sense Stationary? Strict-Sense Stationary?

- 2: The autocovariance of a stationary stochastic process $X(t)$ is:

$$C_X(\tau) = \frac{9}{1+\tau^2}$$

If an 'engineering correlation time' is defined as the value t_e where $r_X(t_e) \leq 0.1 * r_X(0)$ with the positive and minimum value t_e (if there are a lot values satisfying this rule), calculate t_e

More examples from the last week

- 1: given a stochastic phase signal $X(t) = Acos(\omega t + \varphi)$, where A and ω are constants, and φ is a r.v uniformly distributed in $(0,2\pi)$. Is $X(t)$ Wide-Sense Stationary? Strict-Sense Stationary?

Solution:

$$E(X(t)) = \int_0^{2\pi} A\cos(\omega t + \varphi)\frac{1}{2\pi}d\varphi = 0$$

$$R_X(t_1, t_2) = A^2\int_0^{2\pi} \cos(\omega t_1 + \varphi)\cos(\omega t_2 + \varphi)\frac{1}{2\pi}d\varphi$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi} \{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\varphi)\}\, d\varphi$$

Integrate a periodic function. If the integration range is an integer number of cycles, the integration result is 0

$$= \frac{A^2}{4\pi}\int_0^{2\pi} \cos(\omega(t_1 - t_2))d\varphi = \frac{A^2}{2}\cos w\tau$$

$The\ mean\ is\ independent$ to $T, and\ the\ autocorrelation$ $is\ only\ a\ function\ of$ the time $interval$ τ→WSS process

- Based on the given information, we cannot determine whether it is SSS or not

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Solution:

$C_X(0) = 9$, therefore $r_X(\tau) = \frac{C_X(\tau)}{C_X(0)} = \frac{1}{1+\tau^2}$, $r_X(0) = 1$, $r_X(3)$

$= 0.1$ satisfies the rule, therefore $t_e = 3$

Power Spectrum – Outline

- Definitions
- Properties
- Cross-power spectrum
- The white noise

Definitions

- In signal theory, spectra are associated with Fourier transforms.
- For a deterministic signal $s(t)$, the spectra is Fourier transforms of $s(t)$:

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t}\, dt$$

- However, we cannot use only one deterministic expression to express a stochastic process.
- At one time or state, a process will give different deterministic expressions.
- Therefore we define the power spectrum.

Definitions

- The power spectrum (or power spectral density) of a WSS process $\mathbf{x}(t)$, real or complex, is the Fourier transform $S(\omega)$ of its autocorrelation $R(\tau) = E\{\mathbf{x}(t + \tau)\mathbf{x}^*(t)\}$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau} d\tau$$

The Inverse Fourier transform:

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau} d\omega$$

Definitions

- Here we list a number of usually used autocorrelations and their corresponding power spectrum, please also see Table 7. 1 in page 373 of text book

Widely used Fourier transform pairs

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,\mathrm{e}^{\mathrm{j}\omega\tau}\mathrm{d}\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$\begin{cases} \alpha e^{-\beta\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases} \leftrightarrow \frac{\alpha}{\beta + j\omega}$$

$$\delta(\tau) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$\mathrm{e}^{\mathrm{i}\beta\tau} \leftrightarrow 2\pi\delta(\omega - \beta)$$

$$\cos\beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$$

$$\mathrm{e}^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\mathrm{e}^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$$

Definitions

Some examples and explanations

- For a signal with DC component a in the autocorrelation:

$$\int_{-\infty}^{\infty} a\, e^{-j\omega\tau} d\tau = a \int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau \triangleq a \cdot 2\pi\delta(\omega$$

Which is a pulse in frequency 0Hz (DC component)

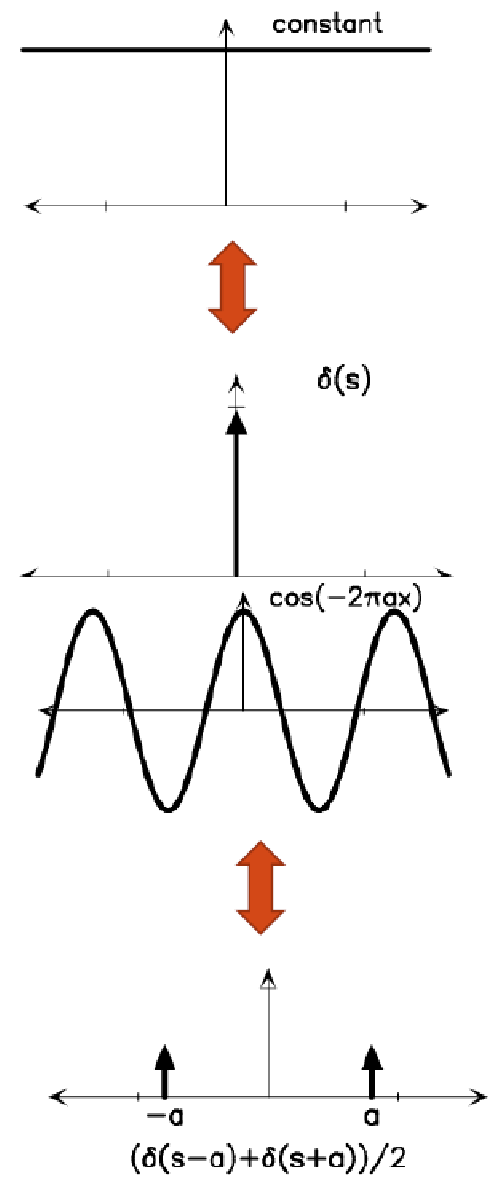
- For a periodic component in the autocorrelation $R_X(\tau)$:
    - For example, $R_X(\tau) = a\cos(\omega_1\tau)$, we have:

$$\cos(\omega_1\tau) = \frac{e^{j\omega_1\tau} + e^{-j\omega_1\tau}}{2}$$

$$\int_{-\infty}^{\infty} e^{j\omega_1\tau} e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_1)\tau}\, d\tau = 2\pi\delta(\omega - \omega_1)$$

$$\int_{-\infty}^{\infty} e^{-j\omega_1\tau} e^{-j\omega\tau}\, d\tau = 2\pi\delta(\omega + \omega_1)$$

$$\rightarrow S(w) = a\pi\delta(\omega - \omega_1) + a\pi\delta(\omega + \omega_1)$$



- **If the mean value of the stochastic process is not zero, the power spectrum will have a δ function at 0**
- **If there is a periodic component, there is a δ function at the corresponding frequency**

Definitions: Line spectra（线谱）

- The process $\mathbf{x}(t) = \sum_i \mathbf{c}_i e^{j\omega_i t}$ is WSS if the random variables $\mathbf{c}_i$ are uncorrelated with zero mean. Then:

$$R(\tau) = \sum_i \sigma_i^2 e^{j\omega_i \tau} \qquad S(\omega) = 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i)$$

where $\sigma_i^2 = E\{\mathbf{c}_i^2\}$. Thus $S(\omega)$ consists of lines (线谱). Such a process is predictable: its present value is uniquely determined in terms of its past (过往取值可以预测未来). This is sometimes called complex line spectra.

- Similarly, the process $\mathbf{y}(t) = \sum_i (\mathbf{a}_i \cos \omega_i t + \mathbf{b}_i \sin \omega_i t)$ is WSS if the random variables $\mathbf{a}_i$ and $\mathbf{b}_i$ are uncorrelated with zero mean and $E\{\mathbf{a}_i^2\} = E\{\mathbf{b}_i^2\} = \sigma_i^2$. In this case,

$$R(\tau) = \sum_i \sigma_i^2 \cos \omega_i \tau \qquad S(\omega) = \pi \sum_i \sigma_i^2 [\delta(\omega - \omega_i) + \delta(\omega + \omega_i)]$$

This is sometimes called real line spectra.

Definitions: Doppler effect

- A harmonic oscillator(谐波振荡器) located at point P of the x axis moves in the x direction with velocity $\mathbf{v}$. The emitted signal equals $e^{j\omega_0 t}$ and the signal received by an observer located at point O equals

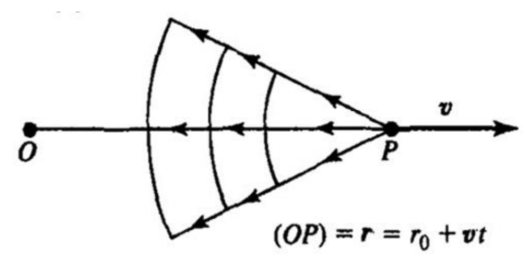
$$\mathbf{s}(t) = ae^{j\omega_0(t-r/c)}$$

where c is the velocity of propagation and $r = r_0 + \boldsymbol{v}t$. We assume that $\boldsymbol{v}$ is a r.v with pdf $f_v(v)$. Clearly,

$$\mathbf{s}(t) = ae^{j(\omega t-\varphi)} \text{ where } \omega = \omega_0\left(1 - \frac{v}{c}\right), \varphi = \frac{r_0\omega_0}{c}$$

- And the spectrum of the received signal is:

$$S(\omega) = 2\pi a^2 f_\omega(\omega) = \frac{2\pi a^2 c}{\omega_0} f_v\left[\left(1 - \frac{\omega}{\omega_0}\right)c\right]$$



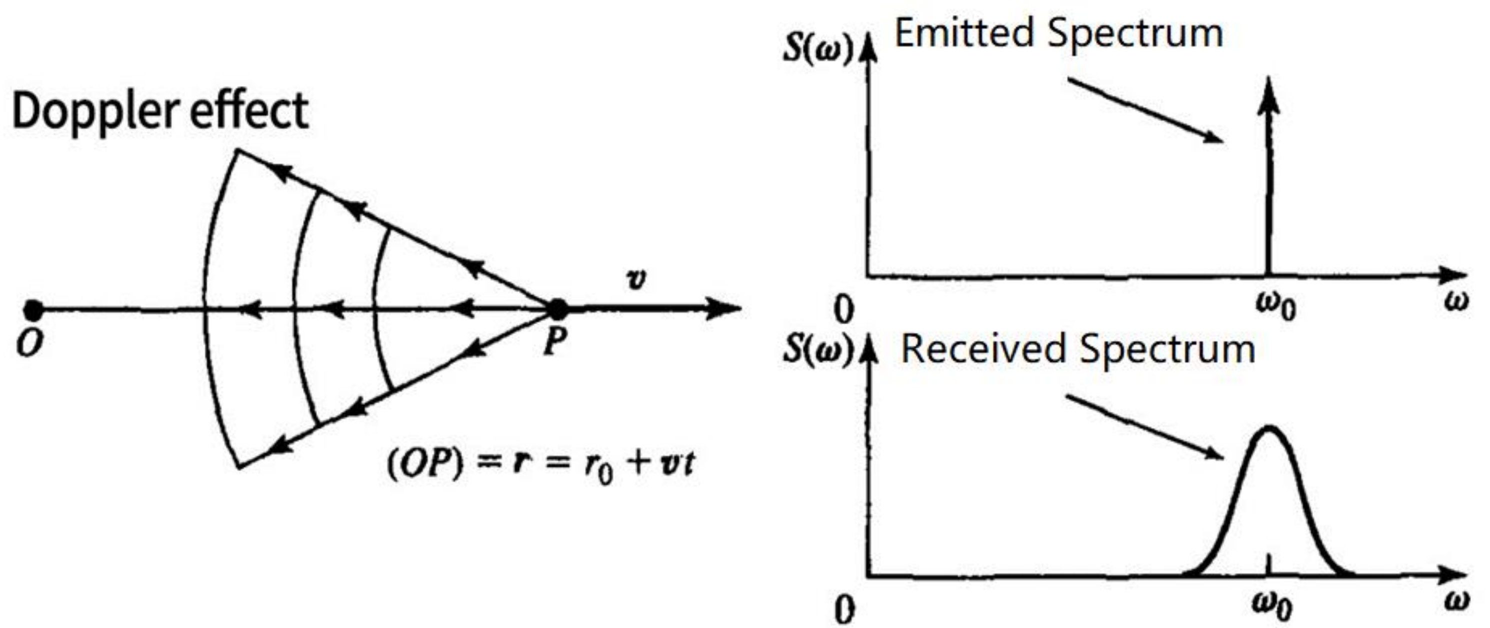
- Note that this development holds also if the motion forms an angle with the x axis provided that $\boldsymbol{v}$ is replaced by its projection $\boldsymbol{v}_x$ on OP.

Definitions: Doppler effect

- Note that if $\boldsymbol{v} = 0$, then

$$\mathbf{s}(t) = a e^{j(\omega_0 t - \varphi)} \; ; \quad R(\tau) = a^2 e^{j\omega_0 \tau} \; ; \quad S(\omega) = 2\pi a^2 \delta(\omega - \omega_0)$$

This is the spectrum of the emitted signal. Thus the motion causes broadening of the spectrum.



Definitions

- Example 1: Given a spectrum $S_X(\omega) = \frac{\omega^2+4}{\omega^4+10\omega^2+9}$, calculate the autocorrelation.

  Tip: Fourier transform pairs $e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$

Power Spectrum – Outline

- Definitions
- Properties
- Cross-power spectrum
- The white noise

Properties

$$S(\omega)=\int_{-\infty}^{\infty}R(\tau)\,e^{-j\omega\tau}d\tau \;\leftrightarrow\; R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\,e^{j\omega\tau}d\omega$$

- Since $R(-\tau)=R^*(\tau)$ it follows that $S(\omega)$ is a real function of ω, no matter the process is complex or real, because:

$$S(\omega)=\int_0^{\infty}R(\tau)\,e^{-j\omega\tau}d\tau+\int_{-\infty}^{0}R(\tau)\,e^{-j\omega\tau}d\tau$$

$$\int_{-\infty}^{0}R(\tau)\,e^{-j\omega\tau}d\tau=\int_0^{\infty}R(-\tau)\,e^{-j\omega(-\tau)}d\tau=\int_0^{\infty}R^*(\tau)\,e^{-j\omega(-\tau)}d\tau$$

$$=\int_0^{\infty}\left(R(\tau)e^{-j\omega\tau}\right)^*d\tau$$

$\rightarrow$ $$S(\omega)=\int_0^{\infty}\{R(\tau)e^{-j\omega\tau}+\left(R(\tau)e^{-j\omega\tau}\right)^*\}\,d\tau \qquad (7\text{-}1)$$

Note that for any complex value a, $a+a^*$ is real.

- If process is real, any additional properties?

Properties

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau} d\tau \;\leftrightarrow\; R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau} d\omega$$

- If $\mathbf{x}(t)$ is a real process, its power spectrum is a real, nonnegative and even function:

$$S(-\omega) = \int_{-\infty}^{\infty} R(\tau)e^{j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} R\,(-\tau)e^{j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} R(a)e^{-j\omega a}\, da = S(\omega) \geq 0$$

From (7-1):

$$S(\omega) = \int_0^{\infty} R(\tau)(e^{-j\omega\tau} + \left(e^{-j\omega\tau}\right)^*)d\tau = \int_0^{\infty} R(\tau)(e^{-j\omega\tau} + e^{j\omega\tau})d\tau$$

$$= 2\int_0^{\infty} R(\tau)\cos{(\omega\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} R(\tau)\cos{(\omega\tau)}\, d\tau \qquad (*)$$

(*) is due to: $R(\tau)$ and $\cos{(\omega\tau)}$ are even functions, thus $R(\tau)\cos{(\omega\tau)}$ is even function.

& For any even functions f, $\int_{-\infty}^{0} f = \int_0^{\infty} f$

- Also: $R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos{\omega\tau} d\omega = \frac{1}{\pi}\int_0^{\infty} S(\omega)\cos{\omega\tau} d\omega$

Properties

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau} d\tau \;\leftrightarrow\; R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau} d\omega$$

- Note that at $\tau = 0$ for the process $X(t)$:

$$R_X(0) = m_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) d\omega$$

is the total average power (DC + AC power) of $X(t)$, which is the integral of the power spectrum over 2π.

- The relationship between autocorrelation and power spectrum is:
  - The weaker the autocorrelation, the wider the power spectrum; The stronger the autocorrelation, the steeper and narrower the power spectrum.

Definitions

- Example 2: Calculate the power spectrum $S(\omega)$ of the following stationary processes:

$$u_1(t) = \mathrm{A}\cos(\omega_0 t + \theta)$$
$$u_2(t) = [\mathrm{A}\cos(\omega_0 t + \theta)]^2$$

Where θ is uniformly distributed in (0, 2π). And calculate the power of two processes.

- Tips:
  - The power is the integral of power spectrum $S(\omega)$:

$$P_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_i(\omega)\, d\omega$$

  - The autocorrelation:

$$R_1(t_1, t_2) = E[A^2\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta)] = \frac{A^2}{2}\cos(\omega_0 \tau)$$

$$R_2(t_1, t_2) = E[A^4\cos(\omega_0 t_1 + \theta)^2\cos(\omega_0 t_2 + \theta)^2]$$

Periodic component

$$= A^4 E(\frac{1}{4}(\cos(2\omega_0 t_1 + 2\theta) + 1)(\cos(2\omega_0 t_2 + 2\theta) + 1))$$

$$= \frac{A^4}{4}E([\cos(2\omega_0 t_1 + 2\theta)\cos(2\omega_0 t_2 + 2\theta) + \cos(2\omega_0 t_1 + 2\theta) + \cos(2\omega_0 t_2 + 2\theta) + 1])$$

$$= \frac{A^4}{4}(\cos(2\omega_0(t_1 - t_2)) + 1) = \frac{A^4}{4}(\frac{1}{2}\cos(2\omega_0 \tau) + 1)$$

Periodic component + DC component

Properties

- Example 3: which of the following functions can be a correct expression of one power spectrum of a real stationary process:

$$A.\, S(\omega) = \frac{\omega^2+9}{(\omega^2+4)(\omega+1)^2}$$

$$B.\, S(\omega) = \frac{\omega^2+1}{\omega^4+5\omega^2+6}$$

$$C.\, S(\omega) = \frac{\omega^2+4}{\omega^4-4\omega^2+3}$$

$$D.\, S(\omega) = \frac{e^{-j\omega^2}}{\omega^2+2}$$

Power Spectrum – Outline

- Definitions
- Properties
- Cross-power spectrum
- The white noise

Cross-power spectrum

- The cross-power spectrum of two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is the Fourier transform $S_{XY}(\omega)$ of their cross-correlation $R_{XY}(\tau) = E\{\mathbf{x}(t + \tau)\mathbf{y}^*(t)\}$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega \qquad (7-2)$$

- The function $S_{XY}(\omega)$ is, in general, complex even when both processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real. In all cases,

$$S_{XY}(\omega) = S_{XY}^*(\omega)$$

because $R_{XY}(-\tau) = E\{\mathbf{x}(t-\tau)\mathbf{y}^*(t)\} = E\{\left(\mathbf{y}(t)\mathbf{x}^*(t-\tau)\right)^*\} = R_{YX}^*(\tau)$

Cross-power spectrum: properties

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau \leftrightarrow R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega$$

- Cross-power spectrum is neither even nor odd function, but it satisfies:

$$|S_{XY}(\omega)|^2 \leq S_X(\omega)S_Y(\omega)$$

- If $X(t)$ and $Y(t)$ orthogonal:

$$R_{XY}(\tau) = 0 \leftrightarrow S_{XY}(\omega) = S_{YX}(\omega) = 0$$

  for all τ and ω

- If $X(t)$ and $Y(t)$ uncorrelated:

$$R_{XY}(\tau) = m_X m_Y \leftrightarrow S_{XY}(\omega) = S_{YX}^*(\omega) = 2\pi m_X m_Y \delta(\omega)$$

  for all τ and ω

Cross-power spectrum

- Example 4: if processes $X(t)$ and $Y(t)$ jointly WSS, and $R_{XY}(\tau)$ $= \begin{cases} 9e^{-3\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$, calculate the cross-power spectrum.

Solution:

Power Spectrum – Outline

- Definitions
- Properties
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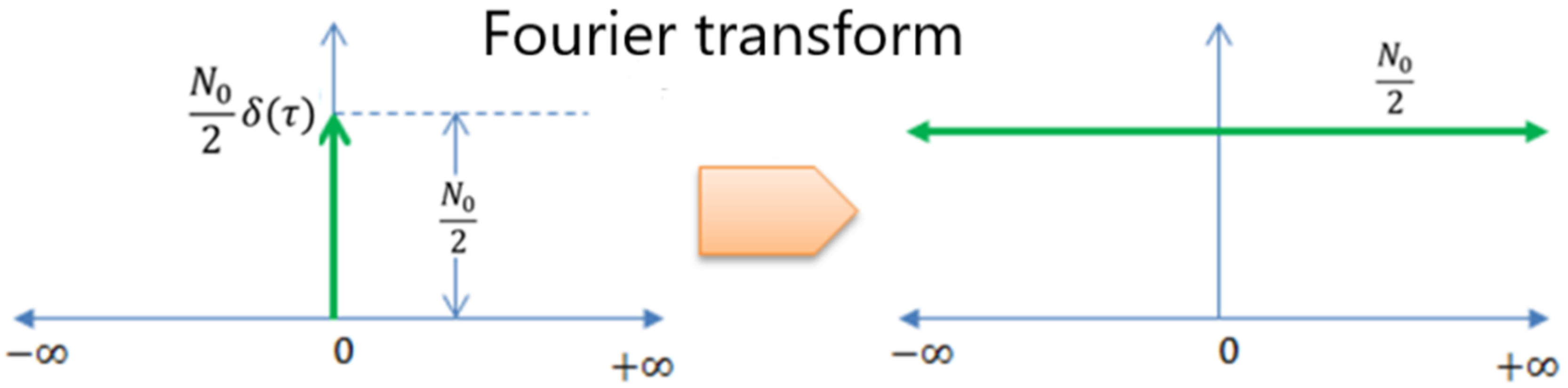
The white noise

- If the mean of one stationary process is 0, and the power pectrum is a non-zero constant ($N_0/2$) in the whole frequency domain ($-\infty \rightarrow \infty$), it is called White noise:

$$S_X(\omega) = \frac{N_0}{2}, \qquad -\infty < \omega < \infty$$



- All Non white noise is called colored noise

The white noise – properties

- Autocorrelation:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} e^{j\omega\tau} d\omega = \frac{1}{2\pi}\frac{N_0}{2} \cdot 2\pi\delta(\tau) = \frac{N_0}{2}\delta(\tau)$$

- Mean and variance:

$$m^2 = R(\infty) = \frac{N_0}{2}\delta(\infty) = 0 \qquad\qquad C(\tau) = R(\tau) - m^2 = \frac{N_0}{2}\delta(\tau)$$

$$\sigma^2 = R(0) - m^2 = \frac{N_0}{2}\delta(0)$$

- Correlation coefficient $r(\tau)$ and correlation time τ_0:

$$r(\tau) = \frac{C(\tau)}{C(0)} = \begin{cases} 1, \tau = 0 \\ 0, \tau \neq 0 \end{cases} \qquad\qquad \tau_0 = \int_0^{\infty} r(\tau)\, d\tau = 0$$

The white noise – properties

- Correlation and independence: the white noise at any two different times (any $t_1 \neq t_2$) is uncorrelated; If the noise $X(t)$ normal distributed, uncorrelation means independence.

- Power of band limited noise: the power within a certain bandwidth $[-W, W]$ is

$$P(W) = \frac{1}{2\pi}\int_{-W}^{W} S(\omega)\, d\omega = \frac{1}{2\pi} * \frac{N_0}{2} * 2W = \frac{W N_0}{2\pi}$$

- Obviously, for infinite bandwidth system, the power (or the variance) of noise is infinite:

$$P(\infty) = R(0) = \sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \infty$$

- (in engineering) when the bandwidth of noise is much larger than the signal bandwidth (about 2-3 times), it can be approximated as white noise

Reading

- This week:
  - Text book: 7.3 (power spectrum)
  - Red book: 2.5
- Next week:
  - Text book: 7.2
  - Red book: 3.1, 3.2

Experiment

- We start Experiment 2 today

More examples

- 1: if the power spectrum of the stationary process $X(t)$ is $S_X(\omega) = \frac{1}{[1+\omega^2]^2}$, calculate the autocorrelation.

Tips：

$$S_Y(\omega) = \frac{1}{1+\omega^2} \Leftrightarrow \frac{1}{2}\mathrm{e}^{-|\tau|} \qquad\qquad S_X(\omega) = S_Y^2(\omega)$$

$$R_X(\tau) = R_Y(\tau) * R_Y(\tau) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|z|}e^{-|\tau-z|}dz$$ (频域的乘对应时域的卷积)

- 2: let $X(t)$ *and* $Y(t)$ be statistically independent stationary processes, with non-zero means m_X and m_Y. Define $Z(t) = X(t) + Y(t)$, given $S_X(\omega)$, calculate $S_{XY}(\omega)$ and $S_{XZ}(\omega)$.