Stochastic Signal Processing

Lesson 9
Memoryless and Linear Systems II

Weize Sun

Memoryless and Linear Systems – Outline

- The average power – Time domain
- Differentiators (微分器)
- Power spectrum of the output of a linear system
  - The average power – Frequency domain
  - Differentiators
- Hilbert transforms

The average power – Time domain

We now give the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system <u>if its input is white noise</u>.

- If the input to a linear system $h(t)$ is (complex) stationary white noise with autocorrelation $R_X(\tau) = q\delta(\tau)$, then:

$$E\{|\mathbf{y}(t)|^2\} = R_Y(0) = R_X(0) * h(0) * h^*(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(0+\alpha-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q\delta(\alpha-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$$ (for the $\delta(\alpha-\beta)$, now-zero only when $\alpha=\beta=t$)

$$= \int_{-\infty}^{\infty} qh(t)h^*(t)dt = qE$$ where $E = \int_{-\infty}^{\infty} |h(t)|^2 dt$ is the energy of $h(t)$.

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

Properties when WSS ($\tau = t_1 - t_2$): (complex valued case)

$R_{XY}(t_1,t_2) = R_X(t_1,t_2) * h^*(t_2)$ → $R_{XY}(\tau) = R_X(\tau) * h^*(-\tau)$

$R_{YX}(t_1,t_2) = R_X(t_1,t_2) * h(t_1)$ → $R_{YX}(\tau) = R_X(\tau) * h(\tau)$

$R_Y(t_1,t_2) = R_X(t_1,t_2) * h(t_1) * h^*(t_2)$ → $R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$

$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-\alpha, t_2-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau+\alpha-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$$

Differentiators

- Recall

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

Properties when WSS ($\tau = t_1 - t_2$): (complex valued case)

$$R_{XY}(t_1, t_2) = R_X(t_1, t_2) * h^*(t_2) \quad \rightarrow \quad R_{XY}(\tau) = R_X(\tau) * h^*(-\tau)$$

$$R_{YX}(t_1, t_2) = R_X(t_1, t_2) * h(t_1) \quad \rightarrow \quad R_{YX}(\tau) = R_X(\tau) * h(\tau)$$

$$R_Y(t_1, t_2) = R_X(t_1, t_2) * h(t_1) * h^*(t_2) \quad \rightarrow \quad R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$$

$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta)h(\alpha)h^*(\beta)d\alpha d\beta$$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau + \alpha - \beta)h(\alpha)h^*(\beta)d\alpha d\beta$$

- Now we discuss a widely used system called Differentiators

Differentiators

- A differentiator is a linear system whose output is the derivative of the input

$$L[\mathbf{x}(t)] = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{x}'(t)$$

We can, therefore, use the preceding results to find the mean of $\mathbf{x}'(t)$:

$$m_{\mathrm{X}'}(t) = L[m_X(t)] = \frac{\mathrm{d}m_X(t)}{\mathrm{d}t} = m_X'(t)$$

Which is, the find the mean of $\mathbf{x}'(t)$, we can first take the mean of $\mathbf{x}(t)$ and thus get $m_X(t)$, and then take the derivation $m_X(t)$

- If $\mathbf{x}(t)$ is WSS, then $m_X(t)$ is constant; hence $\frac{\mathrm{d}m_X(t)}{\mathrm{d}t} = 0$, which is:

$$E\{\mathbf{x}'(t)\} = 0$$

Differentiators

- A differentiator is a linear system whose output is the derivative of the input $L[\mathbf{x}(t)] = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{x}'(t)$

We can, therefore, use the preceding results to find the autocorrelation of $\mathbf{x}'(t)$:

$$\begin{cases} R_{XX'}(t_1,t_2) = L_2[R_{XX}(t_1,t_2)] = \dfrac{\partial R_{XX}(t_1,t_2)}{\partial t_2} \\ R_{X'X'}(t_1,t_2) = L_1[R_{XX'}(t_1,t_2)] = \dfrac{\partial R_{xx'}(t_1,t_2)}{\partial t_1} \end{cases} \Rightarrow R_{X'X'}(t_1,t_2) = \frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$$

- Furthermore, for WSS process, $R_X(t_1,t_2) = R_X(\tau)$, $\tau = t_1 - t_2$, then

$$\begin{cases} \dfrac{\partial R_X(t_1-t_2)}{\partial t_2} = -\dfrac{dR_X(\tau)}{d\tau} \\ \dfrac{\partial^2 R_X(t_1-t_2)}{\partial t_1 \partial t_2} = -\dfrac{d^2 R_X(\tau)}{d\tau^2} \end{cases} \Rightarrow \begin{cases} R_{XX'}(\tau) = -R'_{XX}(\tau) \\ R_{X'X'}(\tau) = -R''_{XX}(\tau) \end{cases}$$

Powe spectrum of the output of a linear system

- For the linear system and a stationary input $\mathbf{x}(t)$:

$$\mathbf{y}(t) = \mathbf{x}(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)d\alpha$$

- the autocorrelation/cross-correlation $R_Y(\tau)/R_{XY}(\tau)$ and power spectrum/cross power spectrum $S_Y(\omega)$/ $S_{XY}(\omega)$ of the response are Fourier Transform pairs:

$$R_{XY}(\tau) = R_X(\tau) * h^*(-\tau) \quad \leftrightarrow \quad S_{XY}(\omega) = S_X(\omega)H^*(\omega)$$

  - Note that for real process, $R_{XY}(\tau) = R_X(\tau) * h(-\tau) \quad \leftrightarrow \quad S_{XY}(\omega) = S_X(\omega)H^*(\omega)$, the conjugate in $H^*(\omega)$ refers to the Fourier Transform '$h(-\tau)$', no matter it is in conjugate (complex valued) or not (real valued)

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) \quad \leftrightarrow \quad S_Y(\omega) = S_{XY}(\omega)H(\omega)$$

Where $H(\omega)$ is the Fourier Transform of $h(\mathrm{t})$, also referred to as Transfer function.

Convolution in the time domain is multiplication in the frequency domain, and vice versus

（时域上的卷积等价于频域上的乘，反之亦然）

Powe spectrum of the output of a linear system

$$R_{XY}(\tau) = R_X(\tau) * h^*(-\tau) \quad \leftrightarrow \quad S_{XY}(\omega) = S_X(\omega)H^*(\omega)$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) \quad \leftrightarrow \quad S_Y(\omega) = S_{XY}(\omega)H(\omega)$$

- Combining the above two equations we get:

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau) = R_X(\tau) * \rho(\tau)$$

$$\leftrightarrow \quad S_Y(\omega) = S_X(\omega)H(\omega)H^*(\omega) = S_X(\omega)|H(\omega)|^2$$

where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{\infty} h(t+\tau)h^*(t)dt \leftrightarrow |H(\omega)|^2$$

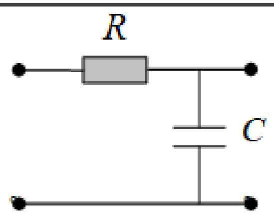
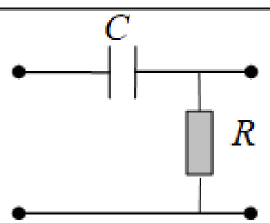
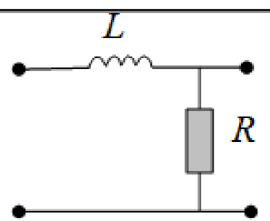
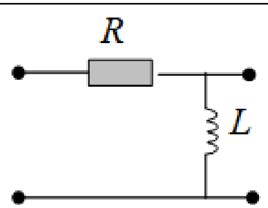
In particular, if $\mathbf{x}(t)$ is white noise with average power q, then

$$R_X(\tau) = q\delta(\tau) \quad \leftrightarrow \quad S_X(\omega) = q$$

$$R_Y(\tau) = q\rho(\tau) \quad \leftrightarrow \quad S_Y(\omega) = q|H(\omega)|^2$$

Powe spectrum of the output of a linear system

Some widely used Fourier Transform pairs

Circuit System	$H(\omega)$	$h(t)$
	$\dfrac{1}{1+j\omega RC}$	$\dfrac{1}{RC}e^{-t/RC}U(t)$
	$\dfrac{j\omega RC}{1+j\omega RC}$	$\delta(t)-\dfrac{1}{RC}e^{-t/RC}U(t)$
	$\dfrac{R}{R+j\omega L}$	$\dfrac{R}{L}e^{-Rt/L}U(t)$
	$\dfrac{j\omega L}{R+j\omega L}$	$\delta(t)-\dfrac{R}{L}e^{-Rt/L}U(t)$

The average power – Frequency domain

- As shown in page 3 of this ppt, the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system is:

$$E\{|\mathbf{y}(t)|^2\} = R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)|H(\omega)|^2 d\omega \geq 0$$

- This equation describes the filtering properties of a system when the input is a random process. It shows, for example, if $H(\omega) = 0$ for $|\omega| > \omega_0$ and $S_{xx}(\omega) = 0$ for $|\omega| < \omega_0$, then $E\{|\mathbf{y}(t)|^2\} = 0$.

- If $\mathbf{x}(t)$ is white noise with average power q (it means $R_X(\tau) = q\delta(\tau)$), and note that the Fourier Transform of $R_X(\tau) = q\delta(\tau)$ is $S_X(\omega) = q$, we get

$$E\{|\mathbf{y}(t)|^2\} = R_Y(0) = q\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

- And $E = \int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$ is the energy of $h(t)$. (时域和频域求积分都可以得到系统或信号的平均功率)

The average power – Time domain

We now give the average power $E\{|\mathbf{y}(t)|^2\}$ of the output of a system if its input is white noise.

- If the input to a linear system $h(t)$ is (complex) stationary white noise with autocorrelation $R_X(\tau) = q\delta(\tau)$, then:

$$E\{\mathbf{y}^2(t)\} = R_Y(0) = R_X(0) * h(0) * h^*(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(0+\alpha-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q\delta(\alpha-\beta)h(\alpha)h^*(\beta)d\alpha d\beta$$ (for the $\delta(\alpha-\beta)$, now-zero only when $\alpha = \beta = t$)

$$= \int_{-\infty}^{\infty} qh(t)h^*(t)dt = qE$$

where $E = \int_{-\infty}^{\infty} |h(t)|^2 dt$ is the energy of $h(t)$.

Linear Systems – output autocorrelation

- In exam, we consider only the Steady state analysis (稳态分析) of stochastic processes, in this case

If input is stationary, output is stationary, for Steady state analysis (稳态分析) only

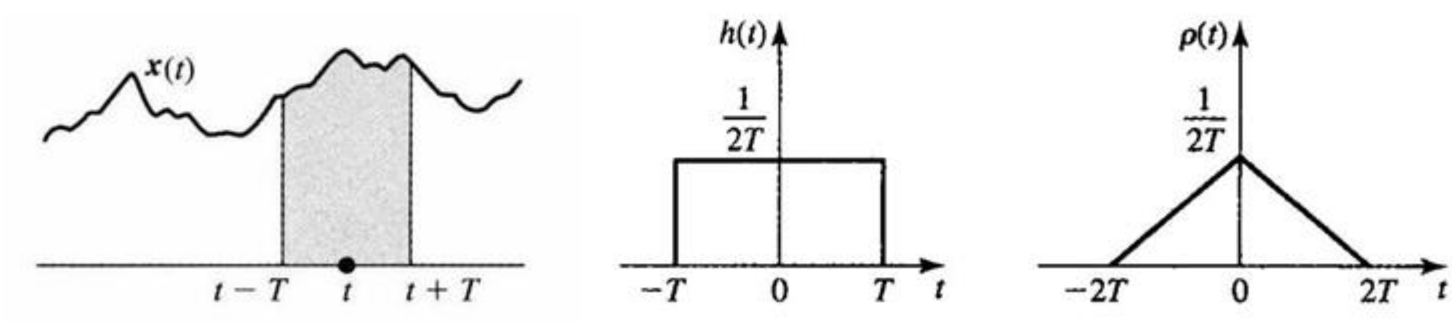
Properties when WSS ($\tau = t_1 - t_2$): (for complex process)

Powe spectrum of the output of a linear system

- Quick example – Moving average: The integral $\mathbf{y}(t) = \frac{1}{2T}\int_{t-T}^{t+T} \mathbf{x}(\alpha)d\alpha$ is the average of the process $\mathbf{x}(t)$ in the interval $(t-T, t+T)$. Clearly, $\mathbf{y}(t)$ is the output of a system with input $\mathbf{x}(t)$ and impulse response a rectangular pulse as in the below Fig. The corresponding $\rho(\tau)$ is a triangle. In this case,

$$H(\omega) = \frac{1}{2T}\int_{-T}^{T} 1 \times e^{-j\omega\tau}d\tau = \frac{\sin T\omega}{T\omega} \qquad S_Y(\omega) = S_X(\omega)\frac{\sin^2 T\omega}{T^2\omega^2}$$

- Thus $H(\omega)$ takes significant values only in the interval [-T, T] centered at the origin. Hence the moving average suppresses the high-frequency components of the input
- It is a simple low-pass filter.



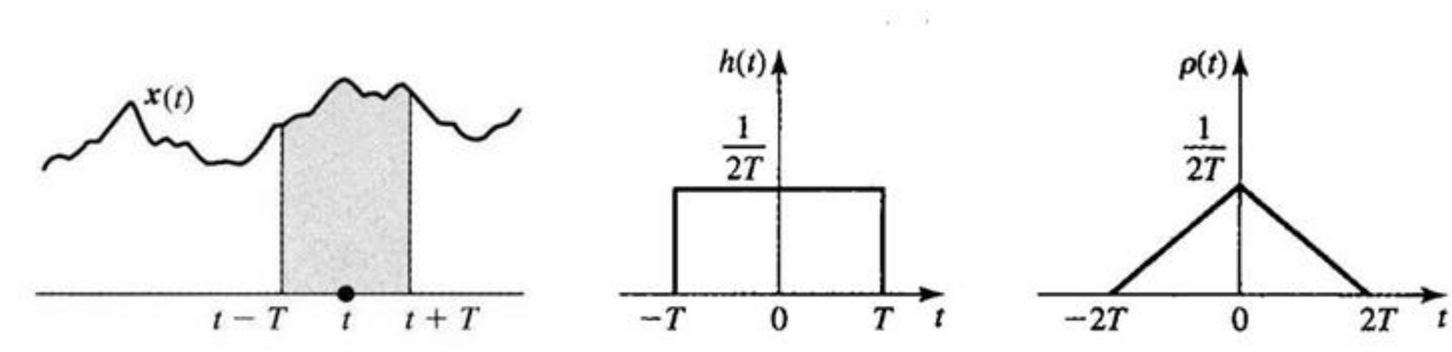
Powe spectrum of the output of a linear system

- Since $\rho(\tau) = h(\tau) * h^*(-\tau)$ is a triangle, it follows from $R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau) = R_X(\tau) * \rho(\tau)$that

$$R_Y(\tau) = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\alpha|}{2T}\right) R_X(\tau - \alpha)d\alpha$$

- Note that we can define the time average of the process $\mathbf{x}(t)$ as:

$$\boldsymbol{m}_T = \frac{1}{2T}\int_{-T}^{T} \mathbf{x}(t)dt$$



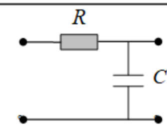
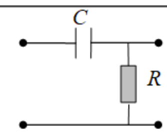
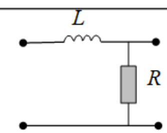
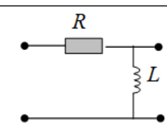
Powe spectrum of the output of a linear system

- Example 1: A stationary process $\boldsymbol{x}(t)$ with autocorrelation $R_X(\tau) = \sigma_X^2 e^{-\beta|\tau|}$ is applied to a linear system with

$$h(t) = \begin{cases} \alpha e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Please calculate the autocorrelation of the output, or says, calculate $R_Y(\tau)$ where $\tau = t_1 - t_2$.

(This is the example 2 from last lesson, this time, we use spectrum domain kind of method to solve this problem)

Circuit System	$H(\omega)$	$h(t)$
	$\frac{1}{1+j\omega RC}$	$\frac{1}{RC}e^{-t/RC}U(t)$
	$\frac{j\omega RC}{1+j\omega RC}$	$\delta(t)-\frac{1}{RC}e^{-t/RC}U(t)$
	$\frac{R}{R+j\omega L}$	$\frac{R}{L}e^{-Rt/L}U(t)$
	$\frac{j\omega L}{R+j\omega L}$	$\delta(t)-\frac{R}{L}e^{-Rt/L}U(t)$

Powe spectrum of the output of a linear system

- Example 2: For the two linear system $h_Y(t)$ and $h_z(t)$ whose Transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha + j\omega}$ and $H_Z(\omega) = \frac{\alpha + j\omega}{2\alpha + j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{ZY}(\omega)$

Solution:

Powe spectrum of the output of a linear system

- Example 2: For the two linear system $h_Y(t)$ and $h_z(t)$ whose Transfer functions are $H_Y(\omega) = \frac{\alpha}{2\alpha+j\omega}$ and $H_Z(\omega) = \frac{\alpha+j\omega}{2\alpha+j\omega}$, and the power spectrum of the stationary input $X(t)$ is $S_X(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}$, calculate $S_Y(\omega)$, $S_Z(\omega)$ and $S_{ZY}(\omega)$

Quick quesion:

Then $S_{YZ}(\omega) =?$

Powe spectrum of the output of a linear system - Differentiators

- The derivative $\mathbf{x}'(t)$ of a process $\mathbf{x}(t)$ can be considered as the output of a linear system with input $\mathbf{x}(t)$ and system function $j\omega$. Which is:

$$R_{XX'}(\tau) = -\frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{XX'}(\omega) = -j\omega S_X(\omega)$$

$$R_{X'X}(\tau) = \frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{X'X}(\omega) = j\omega S_X(\omega)$$

$$R_{X'X'}(\tau) = -\frac{d^2 R_{xx}(\tau)}{d\tau^2} \longleftrightarrow S_{X'X'}(\omega) = \omega^2 S_X(\omega)$$

- The n-th derivative $\boldsymbol{y}(t) = \mathbf{x}^{(n)}(t)$ of $\mathbf{x}(t)$ is the output of a system with input $x(t)$ and system function $(j\omega)^n$, thus:

$$S_Y(\omega) = |j\omega|^{2n} S_X(\omega); \quad R_Y(\tau) = (-1)^{\mathrm{n}} R_X^{(2n)}(\tau)$$

Powe spectrum of the output of a linear system - Differentiators

- Quick example – The differential equation

$$\mathbf{y}'(t) + \mathrm{c}\mathbf{y}(t) = \mathbf{x}(t)$$

specifies a linear system with input $\mathbf{x}(t)$, output $\mathbf{y}(t)$, and system function $1/(j\omega + c)$.

This is because: $\mathbf{y}'(t) + \mathrm{c}\mathbf{y}(t) = \mathbf{x}(t) \rightarrow j\omega Y(\omega) + cY(\omega) = X(\omega)$, then

$$H(\omega) = Y(\omega)/X(\omega) = 1/(j\omega + c)$$

We assume that $\mathbf{x}$(t) is white noise with $R_X(\tau) = q\delta(\tau)$. Then we obtain

$$S_Y(\omega) = S_X(\omega)\frac{1}{\omega^2+c^2} = \frac{q}{\omega^2+c^2}; \qquad R_Y(\tau) = \frac{q}{2c}e^{-c|\tau|}$$

- Note that $E\{\mathbf{y}^2(t)\} = R_{yy}(0) = q/2c$.
- Similarly, if $\mathbf{y}''(t) + b\mathbf{y}'(t) + c\mathbf{y}(t) = \mathbf{x}(t)$, and $S_X(\omega) = q$, then we have:

$$H(\omega) = \frac{1}{-\omega^2 + jb\omega + c} \qquad S_Y(\omega) = \frac{q}{(c-\omega^2)^2 + b^2\omega^2}$$

$$R_{XX'}(\tau) = -\frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{XX'}(\omega) = -j\omega S_X(\omega)$$

$$R_{X'X}(\tau) = \frac{dR_X(\tau)}{d\tau} \longleftrightarrow S_{X'X}(\omega) = j\omega S_X(\omega)$$

$$R_{X'X'}(\tau) = -\frac{d^2R_{xx}(\tau)}{d\tau^2} \longleftrightarrow S_{X'X'}(\omega) = \omega^2 S_X(\omega)$$

Powe spectrum of the output of a linear system - Differentiators

- Example 3: given a linear time-invariant system with input $x(t)$ and output $y(t)$ satisfying

$$\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$$

Find $H(\omega)$ and $|H(\omega)|^2$

Solution:

Hilbert transforms

- A system with system function

$$H(\omega) = -j\text{sgn}\omega = \begin{cases} -j & \omega > 0 \\ j & \omega < 0 \end{cases}, \text{ where } \text{sgn}\omega = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$$

is called a **quadrature filter** (正交滤波器). The corresponding impulse response is $1/\pi t$.

- $H(\omega)$ is all-pass with $-90°$ phase shift; hence its response to $\cos\omega t$ is $\cos(\omega t - 90°) = \sin\omega t$ and its response to $\sin\omega t$ is $\sin(\omega t - 90°) = -\cos\omega t$.

- The response of a quadrature filter to a real process $\mathbf{x}(t)$ is denoted by $\hat{\mathbf{x}}(t)$ and it is called the Hilbert transform of $\mathbf{x}(t)$. Thus

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbf{x}(\alpha)}{t - \alpha} d\alpha$$

where $*$ is the convolution operator

Hilbert transforms

- The response of a quadrature filter to a real process $\mathbf{x}(t)$ is denoted by $\hat{\mathbf{x}}(t)$ and it is called the Hilbert transform of $\mathbf{x}(t)$. Thus

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathbf{x}(\alpha)}{t - \alpha} d\alpha$$

$$\Rightarrow \left\{ \begin{aligned} S_{X\hat{X}}(\omega) &= jS_{XX}(\omega)\operatorname{sgn}\omega = -S_{\hat{X}X}(\omega) \\ S_{\hat{X}\hat{X}}(\omega) &= S_{XX}(\omega) \end{aligned} \right.$$

Hilbert transforms

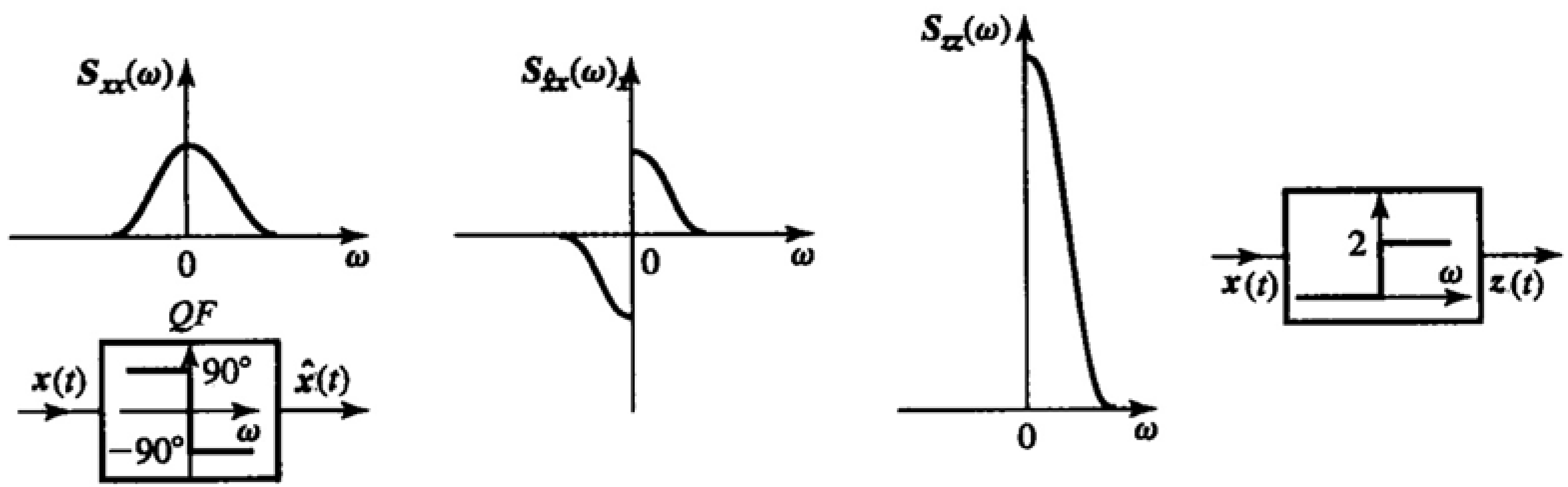
- The complex process $\mathbf{z}(t) = \mathbf{x}(t) + j\hat{\mathbf{x}}(t)$ is calked the analytic signal associated with $\mathbf{x}(t)$. Clearly, $\mathbf{z}(t)$ is the response of the system

$$H(\omega) = 1 + j(-j\operatorname{sgn}\omega) = 2U(\omega)$$

- with input $\mathbf{x}(t)$. we have:

$$S_{ZZ}(\omega) = 4S_{XX}(\omega)U(\omega) = 2S_{XX}(\omega) + 2jS_{\hat{X}X}(\omega)$$

$$R_{ZZ}(\tau) = 2R_{XX}(\tau) + 2jR_{\hat{X}X}(\tau)$$



More examples

1: find the mean and autocorrelation of $Y(t)$ if it is an Poisson impact process (泊松冲击过程) inputted to the Differentiator.

Hint: for Poisson impact process $X(t)$:

$$E[X(t)] = \lambda t \qquad\qquad R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

2: given a process $X(t) = Acos(\omega_0 t + \varphi)$ where A and ω_0 are constants, φ uniformly distributed in $(0,2\pi)$. Input the $X(t)$ to an differentiator and get $X'(t)$, calculate the mean, autocorrelation and power spectrum of $X'(t)$.

Reading

- This week:
  - Text book: 7.3 (part of ‘power spectrum together with systems’)
  - Red book: 3.2.2, 3.2.3
- Next week:
  - Text book: 7.4
  - Red book: 3.3 (限带过程) 注：限带过程的讲解将以此书为主导

Experiment

- Go on with Experiment 2