Stochastic Signal Processing

Lesson 14: Basic of Random Walk

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Outline

- Start with some examples
- Random walk

Start with some examples

- Example 1: In one lottery, the player picks 6 numbers from a sequence of 1 through 51. At the lottery drawing, 6 balls are drawn at random from a box containing 51 balls numbered 1 through 51 . What is the probability that a player has k matches, $k = 4,5,6$?

Start with some examples

- Example 1: In one lottery, the player picks 6 numbers from a sequence of 1 through 51. At the lottery drawing, 6 balls are drawn at random from a box containing 51 balls numbered 1 through 51 . What is the probability that a player has k matches, $k = 4,5,6$?

Solution:

Let n represent the total number of balls in the box among which for any player there are m 'good ones'(those chosen by the player). The remaining $(n-m)$ balls are 'bad ones'. There are in total $\binom{n}{m}$ samples of size m each with equal probability of occurrence. To determine the probability of the event 'k matches', we need to determine the number of samples containing exactly k 'good' balls (and hence $m-k$ 'bad' ones). Since the k good balls must be chosen from m and the $(m-k)$ bad ones from $n-m$, the total number of such samples is $\binom{m}{k}\binom{n-m}{m-k}$, This gives:

$$P(k \text{ matches}) = \frac{\binom{m}{k}\binom{n-m}{m-k}}{\binom{n}{m}}, \qquad k = 0,1,2,\ldots,m$$

In particular, with $k = m$, we get a perfect match, and a win. Thus

$$P(\text{ winning the lottery}) = \frac{1}{\binom{n}{m}} = \frac{m\cdot(m-1)\cdots 2\cdot 1}{n(n-1)\cdots(n-m+1)}$$

Start with some examples

Take $n = 51, m = 6$, and $k = 4,5,6$, we get

$$P(\text{ winning the lottery, or } k = 6) \quad = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46} = 1{:}\,18{,}009{,}460 \approx 5.5 \times 10^{-8}$$

For $k = 5$ and 4, we get $P(k = 5) = 1{:}\,00.701$ and $P(k = 4) = 1{:}\,1213$, respectively.

- In a typical game suppose the lottery pays \$4 million to the winner and $\$15{,}000$ for 5 matches and $\$200$ for 4 matches. Since the ticket costs \$1, this gives the average gain for the player to be

$$\eta_6 = \frac{4{,}000{,}000}{18{,}009{,}460} - 1 \simeq -0.778,\ \eta_5 = \frac{15{,}000}{66{,}701} - 1 \simeq -0.775,\ \eta_4 = \frac{200}{1213} - 1 \simeq -0.835$$

for winning 6,5,4 matches, respectively.

Notice that the average gain for the player is always negative. On the other hand, the average gain for the lottery institution is always positive, and because of the large number of participants involved in the lottery, the state stands to gain a very large amount in each game.

Start with some examples

- Inference from Bernoulli's theorem: when a large number of games are played under identical conditions between two parties, the one with a positive average gain in a single game wins.
- The assumption of this inference: play long-time game, or play the game a lot of times ($\rightarrow \infty$)
- Interestingly, the stock market situation does allow the possibility of long-time play without the need to settle accounts intermittently(间歇地). Hence if one holds one stocks with positive average gains, in the long run he/she will win.
- In regular gambling, however, payment adjustment is made at the end of each game, and it is quite possible that one may lose all his capital and will have to quit playing long before reaping the advantage that a large number of games would have brought to him.
- Therefore, we will examine a classic problem: the ruin of gamblers (赌徒破产).
- It leads to the knowledge of random walk, and Markov Chain. The related principles are used also by casinos, lotteries, and more respectable institutions such as insurance companies in deciding their operational strategies.

Start with some examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of $\$a$ and B with a capital of $\$b$ and the loser pays \$1 to the winner in each game. Let p represent the probability of winning each game for A and $q = 1 - p$ for player B. Find the probability of ruin for each player if no limit is set for the number of games.

Start with some examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$$a$ and B with a capital of \$$b$ and the loser pays \$1 to the winner in each game. Let p represent the probability of winning each game for A and $q = 1 - p$ for player B. Find the probability of ruin for each player if no limit is set for the number of games.

Solution(*): Let P_n denote the probability of the event $X_n =$ " A's ultimate ruin when his wealth is \$$n$" $(0 \leq n \leq a + b)$. His ruin can occur in only two mutually exclusive ways: either A can win the next game with probability p and his wealth increases to \$$(n + 1)$ so that the probability of being ruined ultimately equals P_{n+1}, or A can lose the next game with probability q and reduce his wealth to \$$(n - 1)$, in which case the probability of being ruined later is P_{n-1}. More explicitly, with $H =$ "A succeeds in the next game", by the theorem of total probability:

$$X_n = X_n(H \cup \bar{H}) = X_n H \cup X_n \bar{H}$$

→ $$P_n = P(X_n) = P(X_n \mid H)P(H) + P(X_n \mid \bar{H})P(\bar{H}) = pP_{n+1} + qP_{n-1}$$

With initial condition $P_0 = 1$ (A is certainly ruined if he has no money left), and $P_{a+b} = 0$ (A's wealth is $a + b$ then B has no money left to play, and the ruin of A is impossible). And:

$$p(P_{n+1} - P_n) = q(P_n - P_{n-1}),\ P_{n+1} - P_n = \left(\frac{q}{p}\right)(P_n - P_{n-1}) = \left(\frac{q}{p}\right)^n (P_1 - P_0)$$
$$= \left(\frac{q}{p}\right)^n (P_1 - 1)$$

Start with some examples

Note that when $q \neq p$, we have:

$$P_{\mathrm{a}+b} - P_n = \sum_{k=n}^{a+b-1} (P_{\mathrm{k}+1} - P_k) = \sum_{k=n}^{a+b-1} \left(\frac{q}{p}\right)^k (P_1 - 1) = (P_1 - 1)[\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}]/(1 - \frac{q}{p})$$

Since $P_{a+b} = 0$, it follows that: $P_n = (1 - P_1)\left[\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}\right]/(1 - \frac{q}{p})$, and since $P_0 = 1$, this expression also gives:

$$P_0 = 1 = (1 - P_1)\left[\left(\frac{q}{p}\right)^0 - \left(\frac{q}{p}\right)^{a+b}\right]/(1 - \frac{q}{p})$$

Eliminating $(1 - P_1)/(1 - \frac{q}{p})$ from the above two equations, we get

$$P_n = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}}{\left(\frac{q}{p}\right)^0 - \left(\frac{q}{p}\right)^{a+b}} = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^{a+b}}{1 - \left(\frac{q}{p}\right)^{a+b}} = \frac{1 - \left(\frac{p}{q}\right)^{a+b-\mathrm{n}}}{1 - \left(\frac{p}{q}\right)^{a+b}} \qquad (14 - 1)$$

Let $n = a$, we obtain the probability of ruin for player A when his wealth is \$$a$ to be (for $p \neq q$)

$$P_a = \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{a+b}}$$

Start with some examples

the probability of ruin for player A when his wealth is \$$a$ (for $p \neq q$): $P_a = \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}}$

the probability of ultimate ruin for player B (when his wealth is \$$b$) for $p \neq q$ is:

$$Q_b = \frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}$$

Note that $$P_a + Q_b = 1$$

- If the players are of equal skill ($p = q = 1/2$), we get $P_a = \mathrm{b}/(\mathrm{a}+\mathrm{b})$; $Q_b = \mathrm{a}/(\mathrm{a}+\mathrm{b})$
- If both players are of equal skill, their probabilities of ruin are inversely proportional to the wealth of the players. It is unwise to play indefinitely even against some one of equal skill whose fortune is very large, since the risk of losing all money is practically certain in the long run ($P_a \to 1$, if $b \gg a$). Needless to say if the adversary is also skillful ($q > p$) and wealthy, then as it shows, A 's ruin is certain in the long run ($P_a \to 1$, as $b \to \infty$).
- All casino games against the house(庄家) amount to this situation, and a sensible strategy in such cases would be to quit while ahead.

Start with some examples

- What if odds are in your favor $\{p > q$, P(win) $>$ P(lose)$\}$, so that $q/p < 1$:

$$\text{Probability of A ruin: } P_a = \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} = \left(\frac{q}{p}\right)^a \frac{1-\left(\frac{q}{p}\right)^b}{1-\left(\frac{q}{p}\right)^{a+b}} < \left(\frac{q}{p}\right)^a$$

P_a converges to $(q/p)^a$ as $b \to \infty$. Thus, while playing a series of advantageous games even against an infinitely rich adversary, the probability of escaping ruin (or gaining wealth) is

$$1 - P_a = 1 - \left(\frac{q}{p}\right)^a$$

If a is large enough, this probability can be made as close to 1 as possible. Thus a skillful player ($p > q$) who also happens to be reasonably rich ($a \to \infty$), will never be ruined, and in fact he will end up even richer in the long run. (Of course, one has to live long enough for all this to happen)

Start with some examples

the probability of ruin and average duration for some typical values of a, b, and p:

p	q	Capital a	Gain b	Prob. Ruin P_a	Prob. Success $1 - P_a$	Average duration
0.50	0.50	9	1	0.100	0.900	9
0.50	0.50	90	10	0.100	0.900	900
0.50	0.50	90	5	0.053	0.947	450
0.50	0.50	500	100	0.167	0.833	50 000
0.45	0.55	9	1	0.210	0.790	11
0.45	0.55	50	10	0.866	0.134	419
0.45	0.55	90	5	0.633	0.367	552
0.45	0.55	90	10	0.866	0.134	765
0.45	0.55	100	5	0.633	0.367	615
0.45	0.55	100	10	0.866	0.134	852

Start with some examples

- Let's see some special cases:

p	q	a	b	$each\ bet\ c$	$Ruin\ P_a$	$Success\ 1 - P_a$	$Average\ duration$
0.5	0.5	10	90	10	0.9	0.1	9
0.5	0.5	10	90	1	0.9	0.1	900
0.55	0.45	10	90	10	0.790	0.210	11
0.55	0.45	10	90	1	0.134	0.866	765
0.55	0.45	10	100	1	0.134	0.866	852

- Note that the case $c = 10, a = 10, b = 90$ equals to the case $c = 1, a = 1, b = 9$

This shows:

- Better skill ($p > q$), higher prob. to win.
- Same skill, larger b, the prob. of success does not change, but requires longer time.

Start with some examples

- Let's see some special cases:

p	q	a	b	$each\ bet\ c$	$Ruin\ P_a$	$Success\ 1-P_a$	$Average\ duration$
0.55	0.45	10	90	10	0.790	0.210	11
0.55	0.45	10	90	1	0.134	0.866	765

- For $p > q$: Smaller c leads to higher prob. to win, higher expectation gain value, but longer time.
- For example, when $c = 10$, expectation gain E_c under $c = 10$ and $c = 1$ are:

$$E_{10} = 0.21 * 90 - 0.79 * 10 = 11$$
$$E_1 = 0.866 * 90 - 0.134 * 10 = 76.6$$

However, when $c = 10$, to earn 11, you should play 11 times, thus, earn 1 dollar each time; for $c = 1$, to earn 76.6, you should play 765 times, approx. 0.1 dollar each time. This means:

- Smaller absolute risk (smaller c/a), you will win less in each gain, might need more time to earn a fixed b (or says, earn less in a fixed time), and safer (less prob. to ruin)

Start with some examples

This leads to a simple solution to this problem:

- Suppose now you are in stock market, and you are going to but one stock. You have 100 dollar, you can estimate the p, and you can bet any amount c each time (less than your max., which is 100 in the initial). your target is:
  - Make the ruin prob. less than α, for example, $\alpha = 0.02$
  - Make it as fast as possible to earn as more as money.

Then, how to determine the value c?

Solution:

p	q	a	b	$each\ bet\ c$	$Ruin\ P_a$	$Success\ 1 - P_a$	$Average\ duration\ N$
X	X	100	100	X	X	X	X

- We can set $a = b = 100$, and do the following programming/calculation:
  - Set $P_a = 0.02$, find c

  or

  - Take a c value, calculate P_a,
  - if $P_a < 0.02$, increase c, otherwise, decrease c, until $P_a \approx 0.02$

Start with some examples

- Another problem is how long one should play to maximize the returns.
- Example 3: A and B plays a series of games where the probability of winning p in a single play for A is unfairly kept at less than $1/2$. However, A gets to choose in advance the total number of plays. To win the whole game one must score more than half the plays (大于1/2). If the total number of plays is to be even, how many plays should A choose?

Start with some examples

Solution:

- On any play A wins with probability p and B wins with probability $q = 1 - p > p > 0.5$. Notice that the expected gain for A on any play is $p - q < 0$.

- At first it appears that since the game is unfairly biased toward A, the best strategy for A is to quit the game as early as possible. If A must play an even number, then perhaps quit after two plays? Indeed if p is extremely small that is the correct strategy. However, if $p = 1/2$, then as $2n$, the total number of plays increases, the probability of a tie (the middle binomial term) decreases and the limiting value of A 's chances to win tends to $1/2$.

- In that case, the more plays, the better are the chances for A to succeed. Hence if p is somewhat less that $1/2$, it is reasonable to expect a finite number of plays as the optimum strategy.

Start with some examples

- Let's directly go to the result: (see Example 1-17 of the text book)

$$\frac{1}{1-2p} - 1 \leq 2n \leq \frac{1}{1-2p} + 1$$

- which determines $2n$ uniquely as the even integer that is nearest to $1/(1-2p)$. Thus for example, if $p = 0.47$, then $2n = 16$.

- However, if $1/(1-2p)$ is an odd integer $(such\ as\ p = 0.48)$, both adjacent even integers $2n = 1/(1-2p) - 1(= 24)$ and $2n + 2 = 1/(1-2p) + 1$ $(= 26)$ give the same probability (show this).

- Finally, if $p \simeq 0$, then it gives the optimum number of plays to be 2.

- If p is very close to 0.5, for example, 0.492929 there, which gives $2n = 70$ to be the optimum number of plays. Most people make the mistake of quitting the game long before 70 plays, one reason being the slow progress of the game. (Recall that each play may require many throws because of the do-not-count throws.) However, here is one game where the strategy should be to execute a certain number of plays.

Start with some examples

- Furthermore, we can design an optimum strategy for the game of craps involving the amounts of capital(a), expected return(b), stakes(the amount of each bet c) and probability of success($1 - P_a$).
- The below table lists the probability of success and average duration for some typical values of capital a and gain b ($c = 1$) :

Strategy for a game of craps ($p = 0.492929$)

Capital, a	Gain, b	Probability of Ruin, P_a	Probability of Success, $1 - P_a$	Expected duration, N_a
9	8	0.5306	0.4694	72.14
10	7	0.4707	0.5293	70.80
11	6	0.4090	0.5910	67.40
12	6	0.3913	0.6087	73.84
13	5	0.3307	0.6693	67.30
14	5	0.3173	0.6827	72.78
15	5	0.3054	0.6946	78.32
16	4	0.2477	0.7523	67.47
17	4	0.2390	0.7610	71.98

Start with some examples

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13	5	0.3307	0.6693	67.30
14	5	0.3173	0.6827	72.78
15	5	0.3054	0.6946	78.32
16	4	0.2477	0.7523	67.47
17	4	0.2390	0.7610	71.98

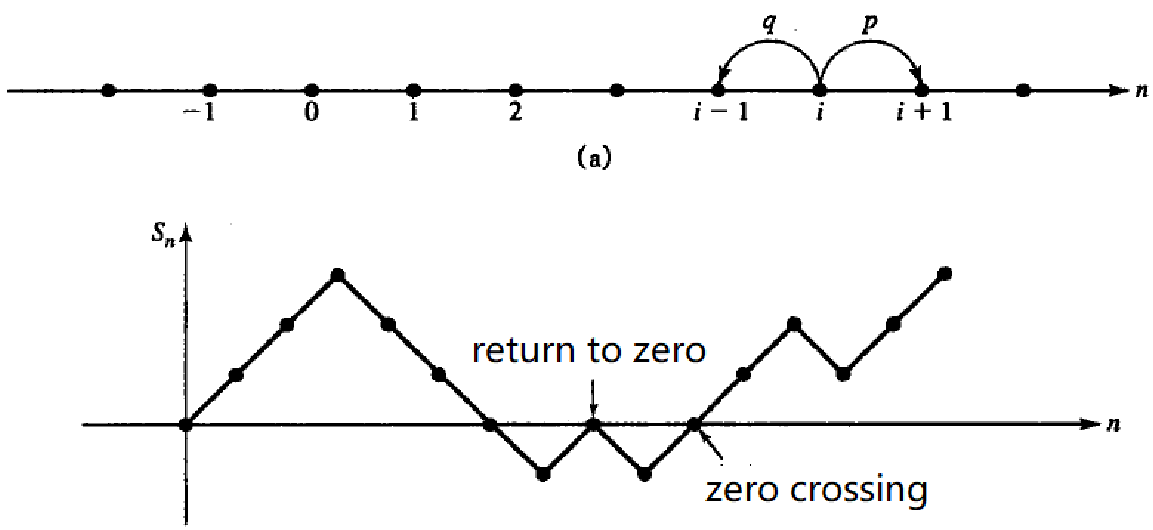
- Here P_a represents the probability of ruin computed with $p = 0.492929$, and N_a represents the corresponding expected number of games. Notice that a and b have been chosen here so that the expected number of games is around its optimum value of 70 .
- Thus starting with \$10, in a \$1 stake game of craps the probability of gaining \$7 is 0.529 in about 70 games. Clearly if the capital is increased to \$100, then to maintain the same number of games and risk level, one should raise the stakes to \$10 for an expected gain of \$70.

Random Walk

- Consider a sequence of independent random variables that assume values $+1$ and -1 with probabilities p and $q = 1 - p$. A natural example is the sequence of Bernoulli trials $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \dots$ with probability of success equal to p in each trial, where $\mathrm{x}_k = +1$ if the k-th trial results in a success and $\mathrm{x}_k = -1$ otherwise. Let $\mathbf{s}_n$ denote the partial sum:

$$\mathbf{s}_n = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n; \qquad \mathbf{s}_0 = 0$$

- that represents the accumulated positive or negative excess at the n-th trial. In a random walk model, the particle takes a unit step up or down at regular intervals, and $\mathbf{s}_n$ represents the position of the particle at the nth step (see Fig. below).



(a)

(b)

Random Walk

- The random walk is said to be symmetric if $p = q = 1/2$ and unsymmetric if $p \neq q$.

- In the gambler's ruin problem discussed in Example 2, s_n represents the accumulated wealth of the player A at the n-th stage. This model will enable us to study the long-time behavior of a prolonged series of individual observations.

- What we will discuss here are: in n successive steps, "return to the origin (or zero)" that represents the return of the random walk to the starting point **(no gain nor loss)**. In particular, the events "the first return (or visit) to the origin" and more generally "the r-th return to the origin", "waiting time for the first gain (first visit to $+1$)," "first passage through $r > 0$ (waiting time for r th gain)" are also of interest.

- In addition, the number of sign changes (zero crossings, means gain → loss or the opposite), the level of maxima(maximum gain) and minima (maximum loss) and their corresponding probabilities are also of great interest. (not discussed in details here)

Random Walk

- To compute the probabilities of these events, let $\{\mathbf{s}_n = r\}$ represent the event "at stage n, the particle is at the point r", and $p_{n,r}$ is the probability:

$$p_{n,r} \triangleq P\{\mathbf{s}_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k is the number of successes in n trials and $n-k$ the number of failures. The net gain is:

$$r = k - (n-k) = 2k - n$$

or $k = (n+r)/2$, so that

$$p_{n,r} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

where the binomial coefficient is zero unless $(n+r)/2$ is an integer between 0 and n, both inclusive. Note that n and r must be therefore odd or even together.

Random Walk

Definition (* know only): the moment generating function(矩生成函数)

- If X is a discrete random variable, the moment generating function is:

$$U_X(z) = \sum_{k=0}^{\infty} p_k z^k$$

Where

- $k = 0,1,2, \ldots$ is the discrete-type values, and p_k is the prob.
- Usually, we use $z = 1$ to calculate the moments of the r.v X.
- If $U_X(z) = \sum_{k=0}^{\infty} p_k = 1$, it means that the $\{p_k,\ k = 0,1,2, \ldots\}$ is a probability distribution, which is, in a random trail, one of the events $\{k = 0,1,2, \ldots\}$ must occur.
- However, if $U_X(z) \neq 1$, it means that the $\{p_k,\ k = 0,1,2, \ldots\ \}$ is not a probability distribution. A simple example is 'the prob. of raining of 365 days', and for this example, usually we have $\sum_{k=0}^{\infty} p_k > 1$.
- $E[X] = U_X'(1)$, which is, the expectation of X is the derivative of $U_X(1)$. This is always true!

Random Walk – Return to the origin

- If the accumulated number of successes and failures are equal at stage n, then $\mathbf{s}_n = 0$, and the random walk has returned to the origin. In that case $r = 0$ or $n = 2k$ so that n is necessarily even, and the probability of return at the $2n$-th trial is given by

$$P\{\mathbf{s}_{2n} = 0\} = \binom{2n}{n} (pq)^n \triangleq u_{2n}$$

- with $u_0 = 1$. Alternatively we have $u_{2n} = (-1)^n \binom{-1/2}{n} (4pq)^n$ (see page 401 of text book), so that the moment generating function of the sequence $\{u_{2n}\}$ is given by

$$U(z) = \sum_{n=0}^{\infty} u_{2n} z^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} (pqz^2)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4pqz^2)^n = \frac{1}{\sqrt{1 - 4pqz^2}}$$

- Note that $U(1) = \sum_{n=0}^{\infty} u_{2n} \neq 1$, therefore the sequence $\{u_{2n}\}$ does not represent a probability distribution. (u_{2n} means the prob. of 'return to zero at stage $2n$', $u_{2n} + u'_{2n} = 1$ where u'_{2n} is the prob. of 'not return to zero at stage $2n$', and $\{u_{2n},\ u'_{2n}\}$ is a probability distribution, but $\{u_{2n}\}$ is not)

- However, if $p = q = 1/2$, $U(1) = \infty$, it means 'return to zero' will occur infinitely. (the prob. of return with infinite stages is infinite)

Random Walk – The first return to origin

- Among the returns to origin or equilibrium point, the first return to the origin commands special attention. A first return to zero occurs at stage $2n$ if the event $B_n = \{\mathbf{s}_1 \neq 0, \mathbf{s}_2 \neq$

Random Walk – The first return to origin

Since $u_0 = 1$ (start from the origin), we get

$$U(z) = 1 + \sum_{n=1}^{\infty} u_{2n} z^{2n} = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} v_{2k} u_{2n-2k} \right) z^{2n}$$

$$= 1 + \sum_{m=0}^{\infty} u_{2m} z^{2m} \cdot \sum_{k=0}^{\infty} v_{2k} z^{2k} = 1 + U(z)V(z)$$

Where $V(z) = \sum_{n=0}^{\infty} v_{2n} z^{2n}$, and thus $U(z) = 1/[1 - V(z)]$

$$V(z) = \sum_{n=0}^{\infty} v_{2n} z^{2n} = 1 - \frac{1}{U(z)} = 1 - \sqrt{1 - 4pqz^2}$$

- What important is that we can use it to compute the probability that the particle will sooner or later return to the origin. In that case, one of the mutually exclusive events B_2 or B_4, ... must happen:

$$P\left\{ \begin{array}{c} \text{particle will ever} \\ \text{return to the origin} \end{array} \right\} = \sum_{n=0}^{\infty} P(B_n) = \sum_{n=0}^{\infty} v_{2n} = V(1) = 1 - \sqrt{1 - 4pq}$$

$$= 1 - |p - q| = \begin{cases} 1 - |p - q| < 1 & p \neq q \\ 1 & p = q \end{cases}$$

Random Walk – The first return to origin

$$P\left\{\begin{array}{c}\text{particle will ever}\\ \text{return to the origin}\end{array}\right\} = \begin{cases} 1-|p-q\,|<1 & p \neq q \\ 1 & p = q \end{cases}$$

- If $p \neq q$, the probability that the particle never return to the origin $\left(P(s_{2k} \neq 0), k \geq 1\right)$ is $|p-q| \neq 0$ (finite). This means that ‘you might never go back to zero (no gain no loss) if your winning prob. is not 0.5’, even you can bet infinite times!

- If $p = q = 1/2$, then with probability 1 the particle will return to the origin. In this case, the return to origin is a certain event(确定性事件) and $\{v_{2n}\}$ represents the probability distribution for the waiting time for the first return to origin. The expected value of this random variable is given by

$$\mu = V'(1) = \begin{cases} \dfrac{4pq}{|p-q|} & p \neq q \\ \infty & p = q \end{cases}$$

- In gambling terminology this means that in a fair game ($p = q$) with infinite resources on both sides, sooner or later one should be able to recover all losses, since return to the break even point is bound to happen.

- How long would it take to realize that break even point? It is ∞! Thus even in a fair game, a player with finite resources may never get to this point, let alone realize a positive net gain.
    - For example, the probability that no break even occurs in 100 trials in a fair game is around 0.08.

Random Walk – Later returns to the origin (* know only)

- Now we discuss the more general event "r-th return to the origin at $2n$-th trial". Let $v_{2n}^{(r)}$ represent the probability of this cumulative event

$$v_{2n}^{(r)} = \sum_{k=0}^{n} v_{2k} v_{2n-2k}^{(r-1)}$$

- We directly go to the result:

$$\eta_c \triangleq \sum_{2n=r\leq cr^2} v_{2n}^{(r)} = \sum_{m=0}^{c^2} v_m^{(r)} = \frac{1}{2}\int_0^{c^2} \sqrt{\frac{2}{\pi}} \frac{r}{m^{3/2}} e^{-r^2/2m} dm$$

where η_c is the "the probability that r returns to origin occur before the instant $t = cr^2$", and this equation is valid only when $m + r$ is an even number. Let $r^2/m = x^2$:

$$\eta_c = \sum_{m=0}^{cr^2} v_m^{(r)} = 2\int_{1/\sqrt{c}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Random Walk – Later returns to the origin (* know only)

$$\eta_c = \sum_{m=0}^{cr^2} v_m^{(r)} = 2\int_{1/\sqrt{c}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

- c is a constant. For example: $c = 10$, $\eta_c = 0.7566$:
  - in a fair game, to observe r returns to zero with 75% confidence, one must be prepared to play $10r^2$ games, or
  - the waiting time to the r-th return to zero in a fair game increases as the square of r (or r increases only as $\sqrt{n}$).
- (*know only) To make $\eta_c \to 1$, $c \to r$
- The time to the r-th return to zero can be interpreted as the sum of r independent waiting times, all of which add up to cr^2. This means, if you want to get $+1$ return to zero, or says from r to $(r+1)$, the time change is $r\{(r+1)^2 - r^2\} = 2r^2 + r$.

→some of the waiting times between successive returns to zero in long runs can be quite large, or saysm

→returns to zero are rare events in a long run.

- Note that the number of zero-crossings does not exceed the number of returns to zero, it is rare, too.

Random Walk

- In fact, the random walk model is used to be used to describe the price of a stock. The conclusion ‘some of the waiting times between successive returns to zero in long runs can be quite large’ or ‘returns to zero are rare events in a long run’ tells us that:
  - If the current stock price is long and far below your buy price, it is almost impossible that you can ‘return to the original’ in your life.

Random Walk

- If the problem of random walk is appeared in the final exam, corresponding equations will be given. All you need is to understand the equations.
- Example 4: supposed that you join a dice game (dice result: 1,2,3,4,5,6) and start with 0 point. Rolling 1,2 you will gain 1 point, and 3,4,5,6 will lose 1 point. Given the prob. of the event 'first return to the origin' v_{2n} and the moment generating function $V(z) = \sum_{n=0}^{\infty} v_{2n} z^{2n} = 1 - \sqrt{1 - 4pqz^2}$. You can quit the game only you first achieve the situation 'no gain no loss'.

(a) What is the prob. that $v_4 = 0$ happens given $v_2 = 0$ happens

(b) What is the prob. that you cannot quit the game?

Random Walk

- Example 4: supposed that you join a dice game (dice result: 1,2,3,4,5,6) and start with 0 point. Rolling 1,2 you will gain 1 point, and 3,4,5,6 will lose 1 point. Given the prob. of the event 'first return to the origin' v_{2n} and the moment generating function $V(z) = \sum_{n=0}^{\infty} v_{2n} z^{2n} = 1 - \sqrt{1 - 4pqz^2}$. You can quit the game only you first achieve the situation 'no gain no loss' .

Solution:

(a) $p = 1/3, \qquad q = 2/3$

'$v_4 = 0$ happens given $v_2 = 0$ happens' is $p \times q + q \times p = 4/9$

(b) $V(1) = \sum_{n=0}^{\infty} v_{2n} = 1 - \sqrt{1 - 4pq}$ is the prob. that 'will sooner or later return to origin/zero'

Thus 'the prob. that you cannot quit the game' is

$$1 - V(1) = \sqrt{1 - 4pq} = \sqrt{1 - 4\frac{1}{3}\frac{2}{3}} = \frac{1}{3}$$

Reading

- This week:
  - Random walk (12.1)

- Next week:
  - 15.1,15.2

- Go on with Experiment 3