Stochastic Signal Processing

Lesson 15: Basic of Markov Chain

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Outline

- Markov Process and Markov Chain: introduction
- Transition probabilities
- Transition matrix
- Higher transition probability and Chapman-Kolmogorov equation

Markov Process and Markov Chain

- Markov processes: the outcome at any instant depends only on the outcome that preceeds it and none before that. In a Markov process $\mathbf{x}(t)$, the only the present influences the future:

For $t_{n-1} < t_n$: $\quad P[\mathbf{x}(t_n) \leq x_n \mid \mathbf{x}(t), t \leq t_{n-1}] = P[\mathbf{x}(t_n) \leq x_n \mid \mathbf{x}(t_{n-1})]$

For $t_1 < t_2 < \cdots < t_n$: $P[\mathbf{x}(t_n) \leq x_n \mid \mathbf{x}(t_{n-1}), \dots, \mathbf{x}(t_1)] = P[\mathbf{x}(t_n) \leq x_n \mid \mathbf{x}(t_{n-1})]$

- A special kind of Markov process is a Markov chain where the system can occupy a finite or countably infinite number of states $e_1, e_2, \dots, e_j, \dots$ such that the future evolution of the process, once it is in a given state, depends only on the present state and not on how it arrived at that state.
  - We only discuss discrete-time cases in this course.
- We first use some examples to illustrate the abundance of Markov processes in real world problems.

Markov Chain Examples

Random walk: The 1-D random walk model considered in last lecture is a special case of a Markov chain.

$$\mathbf{s}_n = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n; \qquad \mathbf{s}_0 = 0$$

- The sequence of Bernoulli trials $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \dots$, at each stage are independent, and the accumulated partial sum $\mathbf{s}_n$ that represents the relative position of the particle satisfies the recursion $\mathbf{s}_{n+1} = \mathbf{s}_n + \mathbf{x}_{n+1}$. Given $\mathbf{s}_n = j$, for $j = 0, \pm 1, \pm 2, \dots, \pm n, \dots$, the random variable $\mathbf{s}_{n+1}$ can assume only two values: $\mathbf{s}_{n+1} = j + 1$ with probability p, and $\mathbf{s}_{n+1} = j - 1$ with probability q:

$$P(\mathbf{s}_{n+1} = j + 1 \mid \mathbf{s}_n = j) = p$$
$$P(\mathbf{s}_{n+1} = j - 1 \mid \mathbf{s}_n = j) = q$$

- These conditional probabilities for $\mathbf{s}_{n+1}$ depend only on the values of $\mathbf{s}_n$ and are not affected by the values of $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{n-1}$.

Markov Chain Examples

Social mobility (社会流动)

- Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. Then the change of the social class is a Markov chain:
  - If $X_n = 1$, the $P(X_{n+1} = 1|X_n = 1) = p_{11}$, $P(X_{n+1} = 2|X_n = 1) = p_{12}$, $P(X_{n+1} = 3|X_n = 1) = p_{13}$, note that $p_{11} + p_{12} + p_{13} = 1$.
  - If $X_n = 2$, the $P(X_{n+1} = 1|X_n = 2) = p_{21}$, $P(X_{n+1} = 2|X_n = 2) = p_{22}$, $P(X_{n+1} = 3|X_n = 2) = p_{23}$
  - If $X_n = 3$, the $P(X_{n+1} = 1|X_n = 3) = p_{31}$, $P(X_{n+1} = 2|X_n = 3) = p_{32}$, $P(X_{n+1} = 3|X_n = 3) = p_{33}$
- For a completely fair society, $p_{ik} = p_{jk}$ for any i, j, k.

Transition probabilities

- In a discrete-time Markov chain $\{\mathbf{x}_n\}$ with a finite or infinite set of states $e_1, e_2, \dots, e_i, \dots$, let $\mathbf{x}_n = \mathbf{x}(t_n)$ represent the state of the system at $t = t_n$. If $t_n = nT$, then for $n \geq m \geq 0$, the sequence $\mathbf{x}_m \to \mathbf{x}_{m+1} \to \cdots \ \mathbf{x}_n, \dots$ represents the evolution of the system. Let

$$p_i(m) = P\{\mathbf{x}_m = e_i\}$$

represents the probability that at time $t = t_m$ the system occupies the state e_i, and

$$p_{ik}(m, r) \triangleq P\{\boldsymbol{x}_r = e_k \mid \boldsymbol{x}_m = e_i\}$$

represents the probability that the system goes into state e_k at $t = t_r$ given that it was in state e_i at $t = t_m$ (regardless of its behavior prior to t_m). Futhermore, assume at time t_n, $n > r$, the state is e_j, and let the $p_{kj}(r, n)$ represent the transition probabilities of the Markov chain from state e_k at t_r to state e_j at t_n. Then:

$$\boxed{P\{\mathbf{x}_n = e_j, \mathbf{x}_r = e_k, \mathbf{x}_m = e_i\}}$$

probability of three states at three times, which is, the whole system if what we care is the states at these three times

$$= P\{\mathbf{x}_n = e_j \mid \mathbf{x}_r = e_k\} P\{\mathbf{x}_r = e_k \mid \mathbf{x}_m = e_i\} P\{\mathbf{x}_m = e_i\} = p_{kj}(r, n) p_{ik}(m, \mathrm{r}) p_i(m)$$

Homogeneous chain

- A Markov chain is said to be homogeneous (齐次) if $p_{ij}(m,n)$ depends only on the difference $n-m$. In that case, the transition probabilities are said to be stationary and

$$P\{\mathbf{x}_{m+n}=e_j \mid \mathbf{x}_m=e_i\} \triangleq p_{ij}(n)=p_{ij}^{(n)}$$

represents the conditional probability that a homogeneous Markov chain will move from state e_i to state e_j in n steps. The one-step transition probabilities are usually denoted simply as p_{ij}. Thus

$$p_{ij}=P\{\mathbf{x}_{n+1}=e_j \mid \mathbf{x}_n=e_i\}$$

- The time duration y that a homogeneous Markov process spends in a given state (interarrival time) must be memoryless, since the present state is sufficient to determine the future. Thus in the discrete case if the time instants t_n are uniformly placed at $t_n = nT$, then y satisfies the relation

$$P(\mathbf{y}>m+n \mid \mathbf{y}>m)=P(\mathbf{y}>n)$$

- In this course, we will mainly discuss the Homogeneous Chain.

Transition matrix (转移矩阵)

- It is convenient to arrange the transition probabilities $p_{ij}(m,n)$ in a matrix form $P(m,n)$ as

$$P(m,n) = \begin{pmatrix} p_{11}(m,n) & p_{12}(m,n) & \cdots & p_{1j}(m,n) & \cdots \\ p_{21}(m,n) & p_{22}(m,n) & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1}(m,n) & \cdots & \cdots & p_{ij}(m,n) & \cdots \\ \ldots & \ldots & \cdots & \ldots & \ldots \end{pmatrix}$$

$P(m,n)$ is a matrix whose entries are all nonnegative, and $\sum_j p_{ij}(m,n) = \sum_j P\{\mathbf{x}_n = e_j \mid \mathbf{x}_m = e_i\} = 1$

- This is the transition probability matrix (转移概率矩阵) that completely define the Markov chain.

- In the special case of a homogeneous Markov chain, we have the one step transition matrix P

$$p_k(0) \triangleq P\{\mathbf{x}_0 = e_k\}$$

That can completely defines the process.

Transition matrix: examples

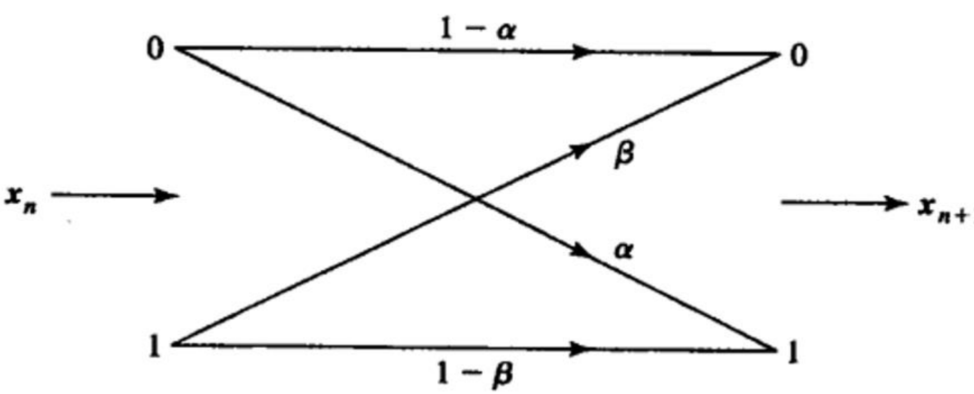
Binary communication channel:

- The figure below represents a time-invariant binary communication channel: $\mathbf{x}_n$ denotes the input and $\mathbf{x}_{n+1}$ the output. The input and the output each possess two states e_0 and e_1 that represent the two binary symbols "0" and "1" respectively. The channel delivers the input symbol to the output with a certain error probability that may depend on the symbol being transmitted. In a time invariant channel, these error probabilities remain constant over various transmitted symbols so that

$$P\{\mathbf{x}_{n+1} = 1 \mid \mathbf{x}_n = 0\} = p_{01} = \alpha \qquad P\{\mathbf{x}_{n+1} = 0 \mid \mathbf{x}_n = 1\} = p_{10} = \beta$$

and the corresponding Markov chain is homogeneous. The 2×2 homogeneous one step transition matrix P is

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$



Transition matrix: examples

Social mobility (社会流动)

- Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. Then the change of the social class is a Markov chain:
  - If $X_n = 1$, the $P(X_{n+1} = 1|X_n = 1) = p_{11}$, $P(X_{n+1} = 2|X_n = 1) = p_{12}$, $P(X_{n+1} = 3|X_n = 1) = p_{13}$, note that $p_{11} + p_{12} + p_{13} = 1$.
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  - A possible one step transition matrix might be:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Now we have a question: in the long run, the percentage of the three classes will approach a certain number?

Let's solve it!

Transition matrix: examples

- Example 1: Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. The one step transition matrix is:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Calculate the stationary distributions of the three classes, which is defined as

$$[a \quad b \quad c]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [a \quad b \quad c]$$

Which means, at step n, the distribution is $[a \quad b \quad c]$ (note that the summation is 1), and at step $n+1$, the distribution is still $[a \quad b \quad c]$, this is defined as stationary distributions.

Try to calculate it yourself!

Transition matrix: examples

- Example 1: Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. The one step transition matrix is:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Calculate the stationary distributions of the three classes.

Solution:

$$[a \quad b \quad c]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [a \quad b \quad c] \quad \rightarrow$$

$$\begin{aligned} 0.7a + 0.3b + 0.2c &= a \\ 0.2a + 0.5b + 0.4c &= b \\ 0.1a + 0.2b + 0.4c &= c \end{aligned}$$

And according to the definition: $a + b + c = 1$

$\rightarrow$

$$a = 22/47, b = 16/47, c = 9/47$$

Transition matrix: examples

Random walk:

- Consider a general one-dimensional random walk on the possible states $e_0, e_1, e_2, \ldots$. Let $\boldsymbol{s}_n$ represent the location of the particle at time n on a straight line such that at each interior state e_j, the particle either moves to the right to e_{j+1} with probability p_j, or to the left to e_{j-1} with probability q_j or remains where it is at e_j with probability r_j. Obviously when at state e_0, it can either stay there with probability r_0 or move to the right to e_1 with probability p_1. This gives the corresponding transition matrix P to be

$$P = \begin{pmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{15-1}$$

Transition matrix: examples

Random walk with absorbing barriers (吸收壁): let the number of states in a random walk be finite $(e_0, e_1, e_2, \ldots, e_N)$ and consider the special case of (14-1) as

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ q & 0 & p & 0 & 0 & \cdot & \cdot & 0 \\ 0 & q & 0 & p & 0 & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & . & \cdot & \cdot & q & 0 & p \\ 0 & 0 & . & \cdot & \cdot & 0 & 0 & 1 \end{pmatrix}$$

- Thus from the interior states $e_1, e_2, \ldots, e_{N-1}$, transitions to the left and right neighbors are possible with probabilities q and p, respectively, while no transition is possible from e_0 and e_N to any other state. The system may move from one interior state to the other, but once it reaches a boundary it stays there forever (the particle gets absorbed).
  - It is easy to see that the gambler's ruin problem discussed in last lecture, where both players have finite wealth, corresponds to this case with $N = a + b$. In that case the game starts from the fixed point a (state e_a) of the interval $(0, a + b)$.

Now we have a question: can we calculate the gambler's ruin probability from this Markov chain model?

Transition matrix: examples

Random walk with reflecting barriers (反射壁):

- Suppose the two boundaries in previous example reflect the particle back to the adjacent state instead of absorbing it. With $e_1, e_2, \dots, e_N$ representing the N states, the end reflection probabilities to the right and left are given by

$$p_{1,2} = p \text{ and } p_{N,N-1} = q$$

and this gives the $N \times N$ transition matrix to be

$$P = \begin{pmatrix} q & p & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ q & 0 & p & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & q & 0 & p & 0 & \cdot & \cdot & \cdot \\ \vdots & & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & q & 0 & p \\ 0 & 0 & 0 & \cdot & \cdot & 0 & q & p \end{pmatrix}$$

- In gambling, this corresponds to a fun game where every time a player loses the game, his counterparty returns just the stake amount so that the game is kept alive and it continues forever.

Higher transition probability and Chapman-Kolmogorov equation

- Recall

$$\boxed{P\{\mathbf{x}_n = e_j, \mathbf{x}_r = e_k, \mathbf{x}_m = e_i\}}$$

probability of three states at three times, which is, the whole system if what we care is the states at these three times

$$= P\{\mathbf{x}_n = e_j | \mathbf{x}_r = e_k\} P\{\mathbf{x}_r = e_k | \mathbf{x}_m = e_i\} P\{\mathbf{x}_m = e_i\} = p_{kj}(r, n) p_{ik}(m, \mathrm{r}) p_i(m)$$

- The transition probability function of any Markov chain $\{\mathbf{x}_n\}$ satisfies the Chapman-Kolmogorov equation, and for $n > r > m$, we have

$$P\{\mathbf{x}_n = e_j, \mathbf{x}_m = e_i\} = \sum_k P\{\mathbf{x}_n = e_j, \mathbf{x}_r = e_k, \mathbf{x}_m = e_i\}$$

$$= \sum_k P\{\mathbf{x}_n = e_j | \mathbf{x}_r = e_k, \mathbf{x}_m = e_i\} P\{\mathbf{x}_r = e_k, \mathbf{x}_m = e_i\}$$

$$= \sum_k P\{\mathbf{x}_n = e_j | \mathbf{x}_r = e_k\} P\{\mathbf{x}_r = e_k, \mathbf{x}_\mathrm{m} = e_i\}$$

Higher transition probability and Chapman-Kolmogorov equation

$$p_{ij}(m,n) = P\{\boldsymbol{x}_n = e_j \mid \boldsymbol{x}_m = e_i\} = \sum_k P\{\boldsymbol{x}_n = e_j \mid \boldsymbol{x}_r = e_k\} P\{\boldsymbol{x}_r = e_k \mid \boldsymbol{x}_m = e_i\}$$

that is $p_{ij}(m,n) = \sum_k p_{ik}(m,r) p_{kj}(r,n)$

- In matrix form, it is $\boldsymbol{P}(m,n) = \boldsymbol{P}(m,r)\boldsymbol{P}(r,n)$ where $m < r < n$
- by letting $r = m+1, m+2, \ldots$ we get $\boldsymbol{P}(m,n) = \boldsymbol{P}(m,m+1)\boldsymbol{P}(m+1,m+2)\ldots\boldsymbol{P}(n-1,n)$
- It means: to obtain $\boldsymbol{P}(m,n)$ for all $n \geq m$, it is sufficient to know the one-step transition probability matrices $P(0,1), P(1,2), P(2,3), \ldots, P(n,n+1), \ldots$

Higher transition probability and Chapman-Kolmogorov equation

$$P(m,n) = P(m,m+1)P(m+1,m+2)\dots P(n-1,n) \qquad \text{(15-2)}$$

- For a homogeneous Markov chain, all transition probability matrices in (15-2) are equal to P:

$$P(m,n) = P^{n-m}$$

Or

$$P(n) = P^{n}$$

This is called n step transition matrix

More examples

- Example 1-1: Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. The one step transition matrix is:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

If the initial distribution is $[1 \quad 0 \quad 0]$, what is the distribution after 1 generation? 2 generations?

More examples

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If the initial distribution is $[1 \quad 0 \quad 0]$, what is the distribution after 1 generation? 2 generations?

Solution:

the distribution after 1 generation is

$$[1 \quad 0 \quad 0] \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [0.7 \quad 0.2 \quad 0.1]$$

More examples

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If the initial distribution is $[1 \quad 0 \quad 0]$, what is the distribution after 1 generation? 2 generations?

Solution:

the distribution after 1 generation is

$$[1 \quad 0 \quad 0]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [0.7 \quad 0.2 \quad 0.1]$$

the distribution after 2 generation is

$$[0.7 \quad 0.2 \quad 0.1]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [1 \quad 0 \quad 0]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}^2 = [0.57 \quad 0.28 \quad 0.15]$$

n step transition matrix and distributions

Note that

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.57 & 0.28 & 0.15 \end{bmatrix}$$

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.57 & 0.28 & 0.15 \end{bmatrix}$$

We get the following for a homogeneous Markov chain:

- Given the n step transition matrix P^n
- If the initial distribution is $p(0)$, then the distribution after n steps is

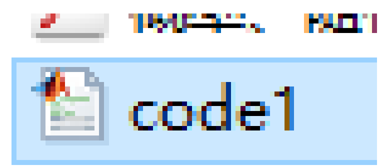
$$p(n) = p(0)P^n$$

More examples

- Example 1-2: Let X_n represents the social class (社会阶层) of a family at the n-th generation, and assume that there are three classes: 1-lower, 2-middle, 3-upper. The one step transition matrix is:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

If the initial distribution is $[1 \quad 0 \quad 0]$, what is the distribution for 30 generations?

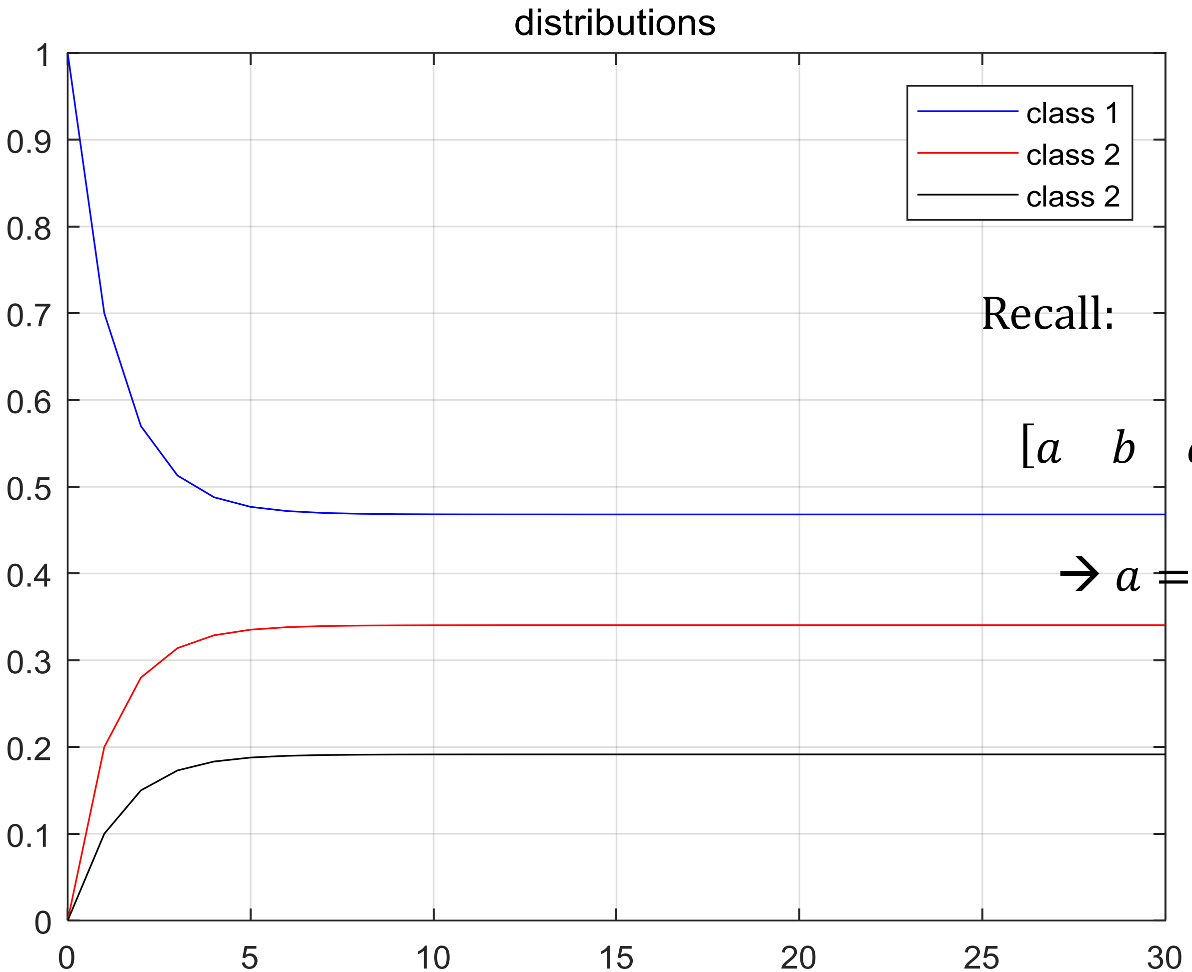


More examples

```
clear
clc

P = [0.7,0.2,0.1;0.3,0.5,0.2;0.2,0.4,0.4];

Total_Gen_No = 30;
Vec_ini = [1,0,0];
% Vec_ini = [0,0,1];

gens = 1:Total_Gen_No;
Vec_current = Vec_ini;
Vec_all = Vec_ini;
for i = gens
    Vec_current = Vec_current*P;
    Vec_all = [Vec_all; Vec_current];
end
```

```
gens = [0, gens];

figure(1)
plot(gens, Vec_all(:, 1), 'b'), grid on, hold on
plot(gens, Vec_all(:, 2), 'r'), grid on, hold on
plot(gens, Vec_all(:, 3), 'k'), grid on, hold on
xlim([0,Total_Gen_No])
ylim([0,1])
legend('class 1', 'class 2', 'class 2')
title('distributions')
```

More examples



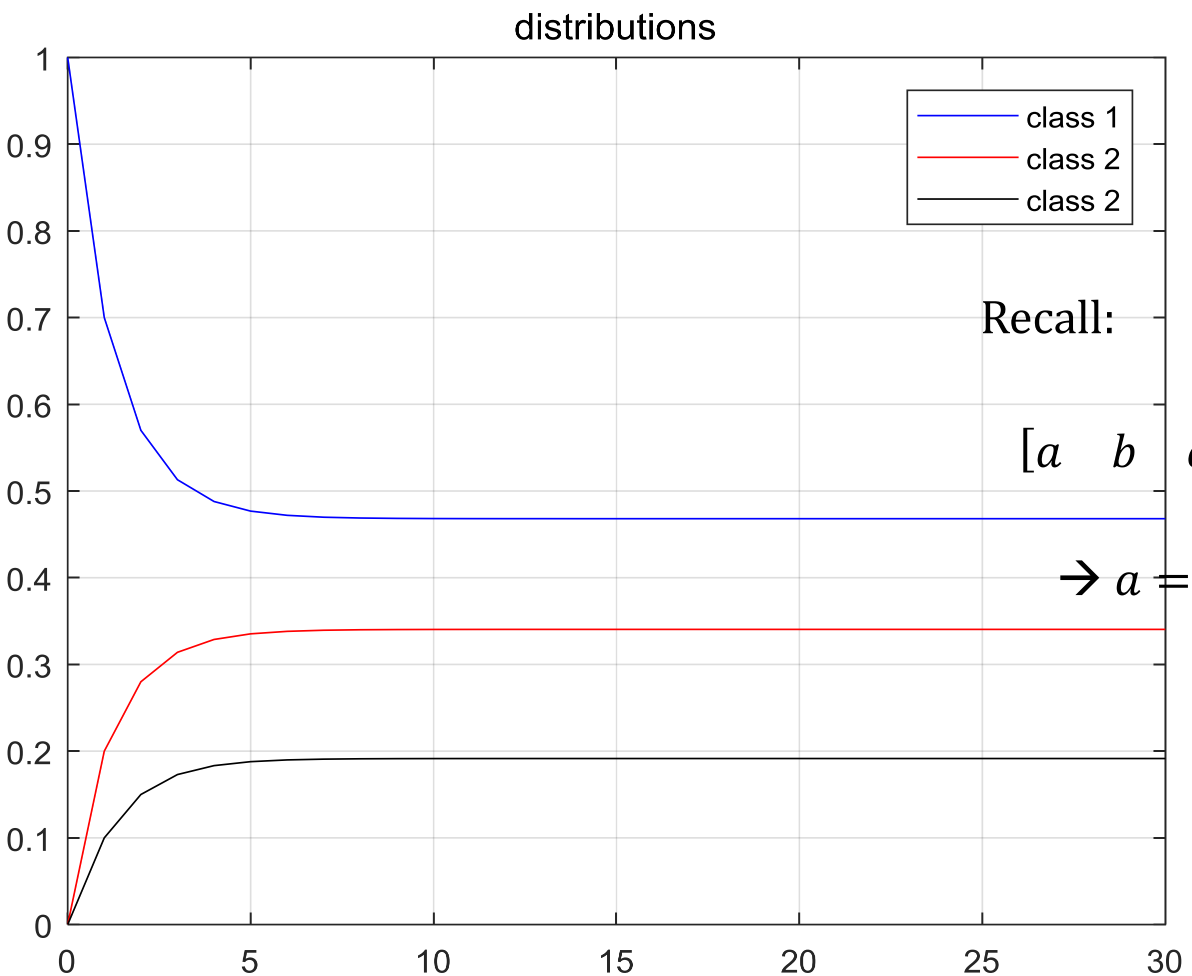
Recall:

$$[a \quad b \quad c]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = [a \quad b \quad c]$$

$\rightarrow a = 22/47, b = 16/47, c = 9/47$

More examples

If the initial distribution is $[0 \quad 0 \quad 1]$, what is the distribution for 30 generations?



Recall:

$$[a \quad b \quad c]\begin{bmatrix}0.7 & 0.2 & 0.1\\0.3 & 0.5 & 0.2\\0.2 & 0.4 & 0.4\end{bmatrix} = [a \quad b \quad c]$$

$$\rightarrow a = 22/47, b = 16/47, c = 9/47$$

The resulted stationary distributions do not change according to the initial distribution!

More examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$2 and B with a capital of \$2 and the loser pays \$1 to the winner in each game. Assume that there are three outcomes of one game: A wins with probability 0.4, draw (平局) with probability 0.3, and A loses with probability 0.3. What is the probability of A ruined in 1,2,3,…,30 games?
- The one step transition matrix is:

Note that the maximum is 2+2=4, thus it is 5*5 matrix (start from 0)

$$
\begin{array}{lc}
 & \text{Next state: A \$0} \\
\begin{array}{l}
\text{Current state: A \$0} \\
\text{Current state: A \$1} \\
\text{Current state: A \$2} \\
\text{Current state: A \$3} \\
\text{Current state: A \$4}
\end{array}
& P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0.3 & 0.3 & 0.4 & 0 & 0 \\
0 & 0.3 & 0.3 & 0.4 & 0 \\
0 & 0 & 0.3 & 0.3 & 0.4 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
 & \text{Next state: A \$1} \qquad \text{Next state: A \$4}
\end{array}
$$

More examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$2 and B with a capital of \$2 and the loser pays \$1 to the winner in each game. Assume that there are three outcomes of one game: A wins with probability 0.4, draw (平局) with probability 0.3, and A loses with probability 0.3. What is the probability of A ruined in 1,2,3,…,30 games?
- The one step transition matrix is:

Next state: A \$0 (A ruined)

State 0: A \$2

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In 1 game, the prob. Of A ruined is 0

More examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$2 and B with a capital of \$2 and the loser pays \$1 to the winner in each game. Assume that there are three outcomes of one game: A wins with probability 0.4, draw (平局) with probability 0.3, and A loses with probability 0.3. What is the probability of A ruined in 1,2,3,…,30 games?
- The one step transition matrix is:

```
P2 = P*P;
```

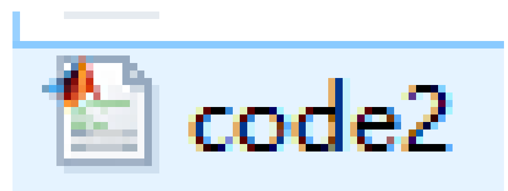
Next state: A \$0 (A ruined)

State 0: A \$2

P2

5x5 double

	1	2	3	4	5
1	1	0	0	0	0
2	0.3900	0.2100	0.2400	0.1600	0
3	0.0900	0.1800	0.3300	0.2400	0.1600
4	0	0.0900	0.1800	0.2100	0.5200
5	0	0	0	0	1



In 2 games, the prob. Of A ruined is 0.09

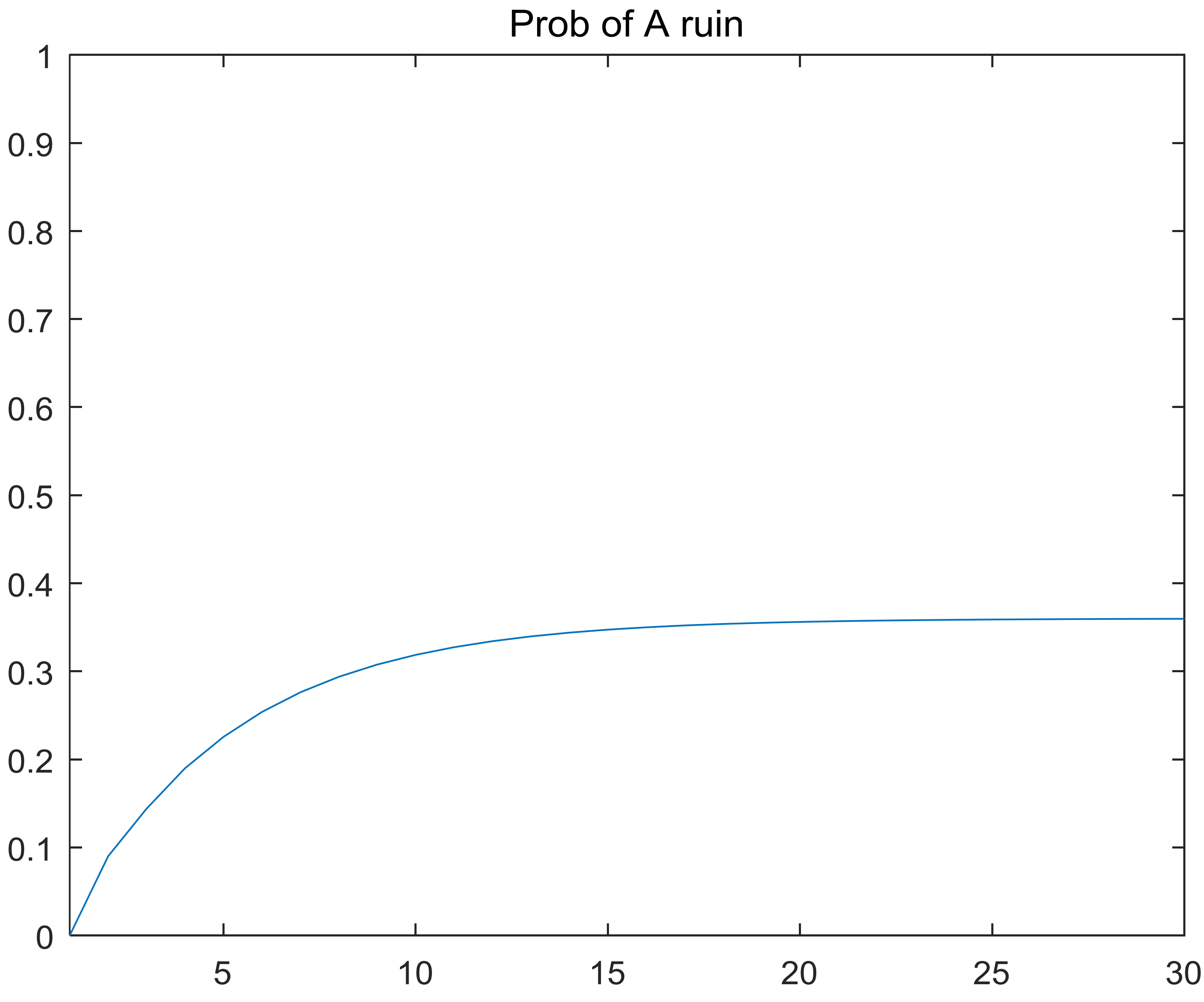
More examples

```
clear
clc

P = [1,0,0,0,0;0.3,0.3,0.4,0,0;0,0.3,0.3,0.4,0;0,0,0.3,0.3,0.4;0,0,0,0,1];
% P2 = P*P;

Prob_of_A_ruin = [];
Prob_of_B_ruin = [];
Prob_of_no_ruin = [];

Total_Game_No = 30;
A_start = 2;
B_start = 2;

P_current = eye(A_start+B_start+1);
games = 1:Total_Game_No;
for i = games
    P_current = P_current*P;
    Prob_of_A_ruin = [Prob_of_A_ruin, P_current(A_start+1, 1)];
    Prob_of_B_ruin = [Prob_of_B_ruin, P_current(A_start+1, end)];
    Prob_of_no_ruin = [Prob_of_no_ruin, 1-P_current(A_start+1, 1)-P_current(A_start+1, end)];
end
```

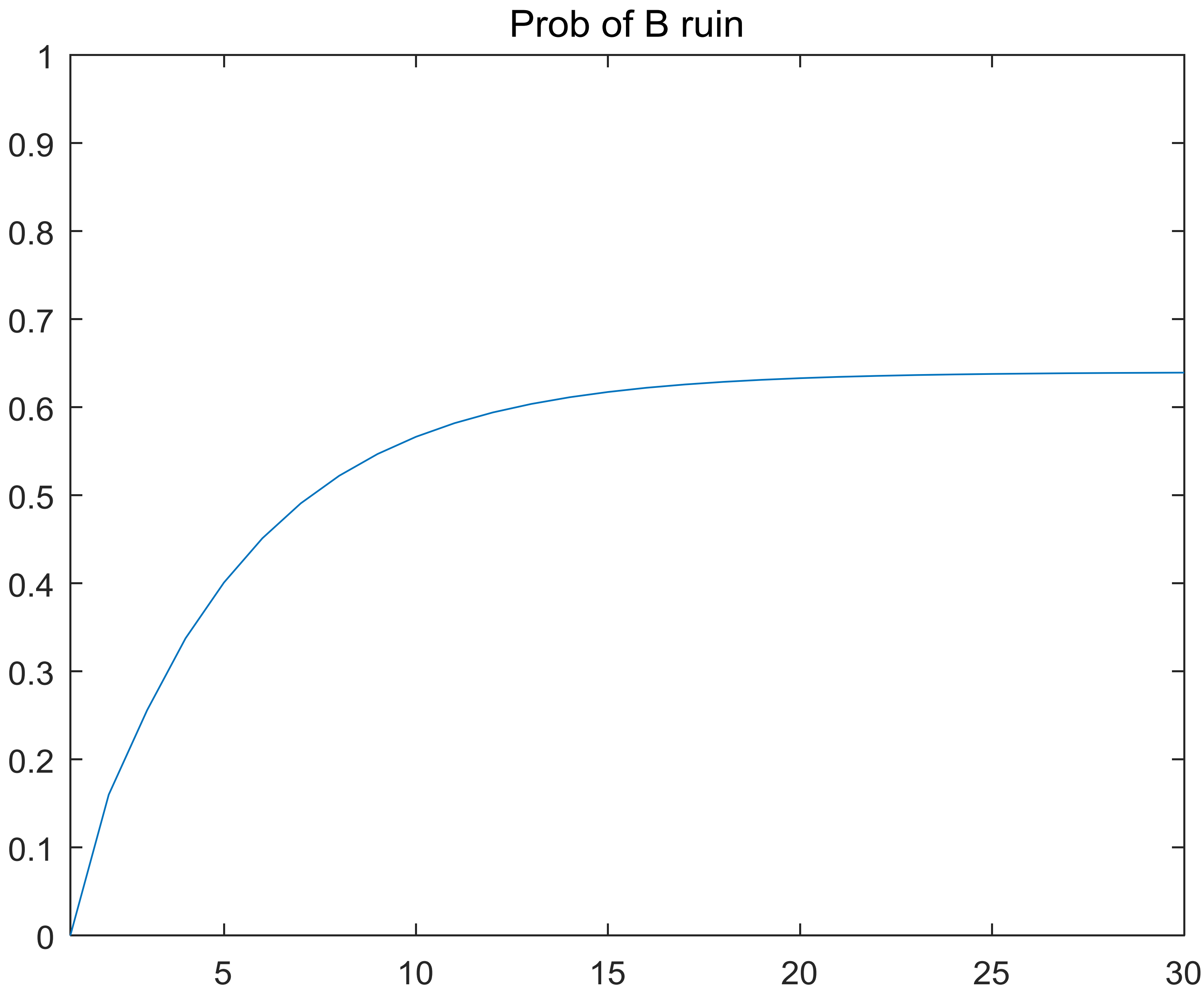
More examples

```
figure(1)
plot(games, Prob_of_A_ruin)
xlim([1,Total_Game_No])
ylim([0,1])
title('Prob of A ruin')

figure(2)
plot(games, Prob_of_B_ruin)
xlim([1,Total_Game_No])
ylim([0,1])
title('Prob of B ruin')

figure(3)
plot(games, Prob_of_no_ruin)
xlim([1,Total_Game_No])
ylim([0,1])
title('Prob of no ruin')
```

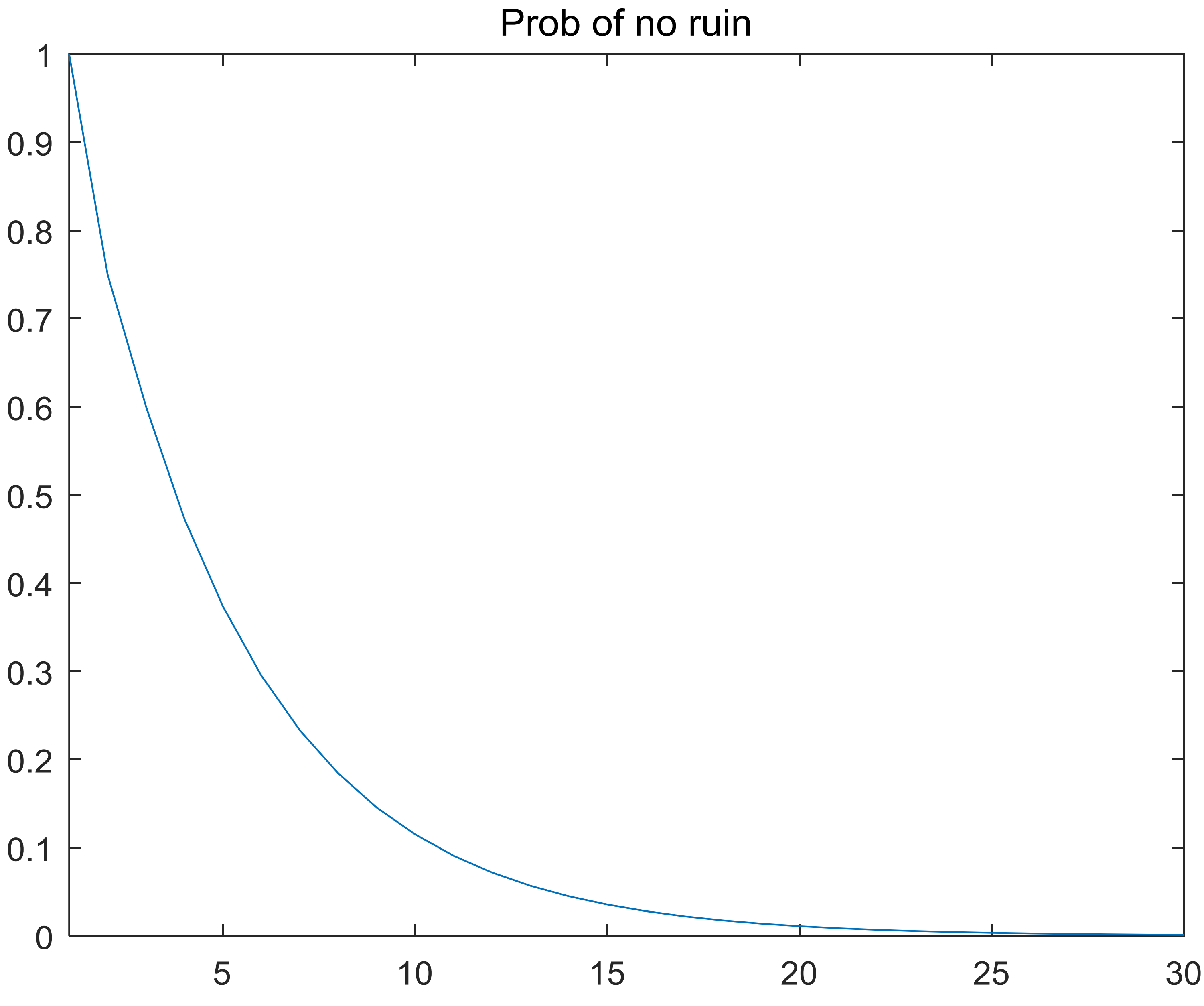
More examples



More examples



More examples



More examples

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of $2 and B with a capital of $2 and the loser pays $1 to the winner in each game. Assume that there are three outcomes of one game: A wins with probability 0.4, draw (平局) with probability 0.3, and A loses with probability 0.3. What is the probability of A ruined in the 5-th game?

Let

```
Total_Game_No = 5;
A_start = 2;
B_start = 2;
```

令行窗口

```
>> Prob_of_A_ruin

Prob_of_A_ruin =

         0    0.0900    0.1080    0.1215    0.1247
```

More examples

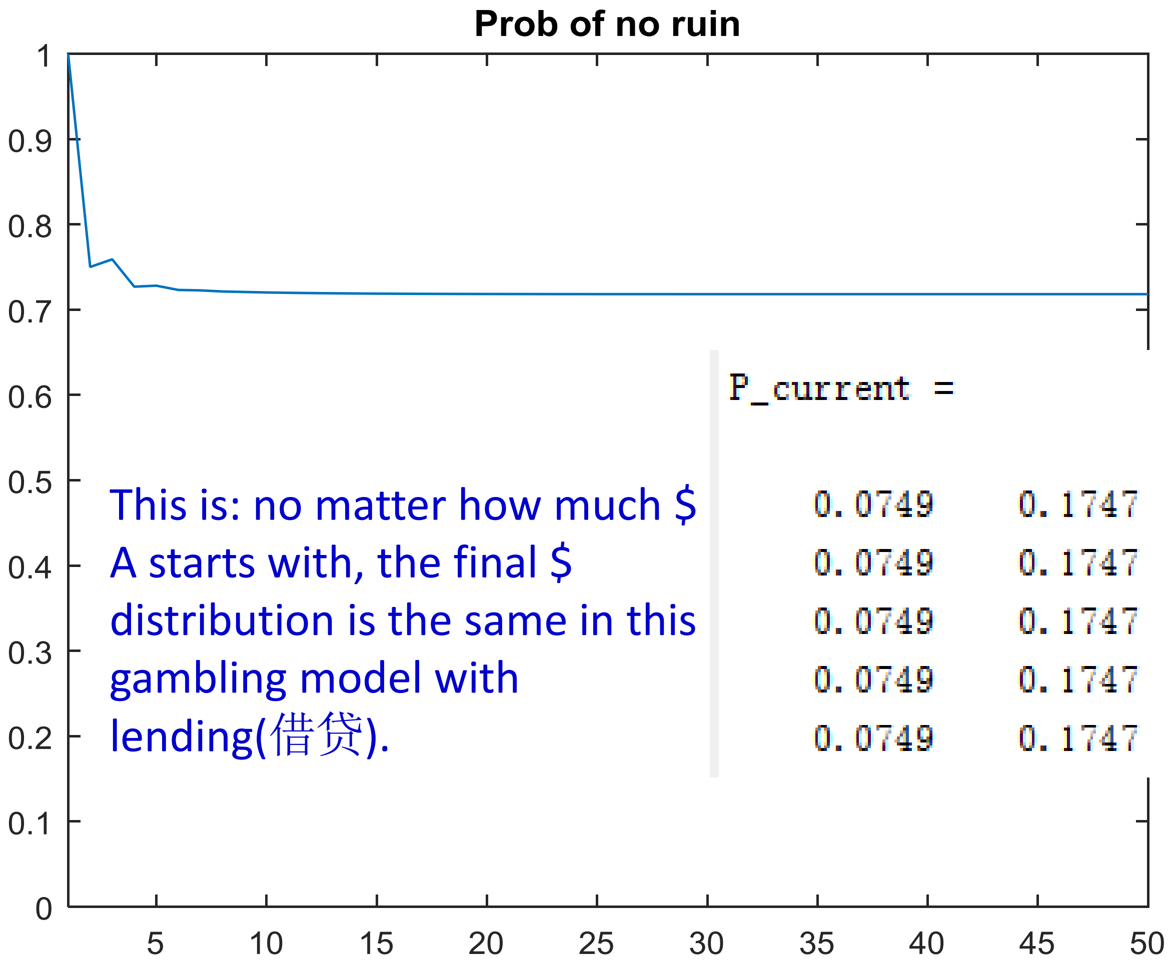
Now we consider the reflecting barriers.

- Example 2: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of \$2 and B with a capital of \$2 and the loser pays \$1 to the winner in each game. Assume that there are three outcomes of one game: A wins with probability 0.4, draw (平局) with probability 0.3, and A loses with probability 0.3. Once A or B ruins (becomes \$0), the non-ruined person will lend the ruined person \$1 and thus the game goes on. What is the probability of the money of A and B are not 0 in the 100-th game?

The one step transition matrix is:

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

More examples



This is: no matter how much $ A starts with, the final $ distribution is the same in this gambling model with lending(借贷).

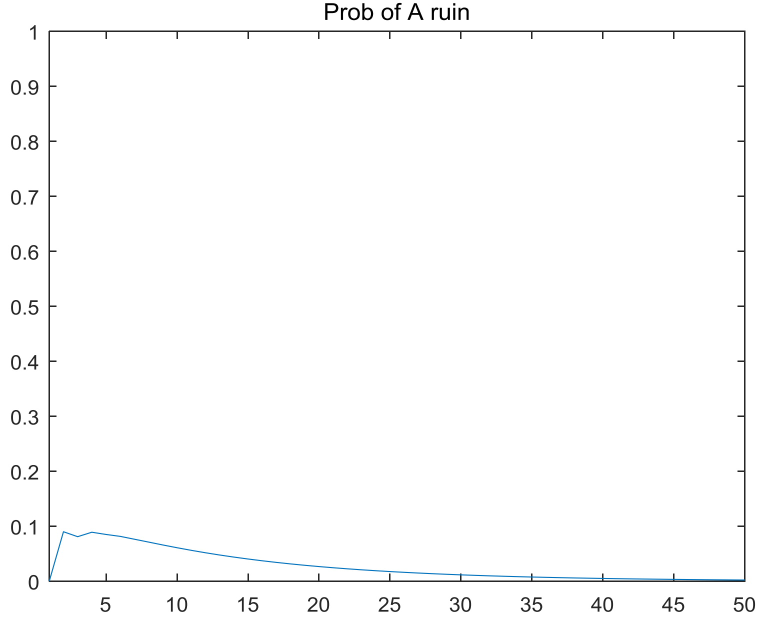
P_current =

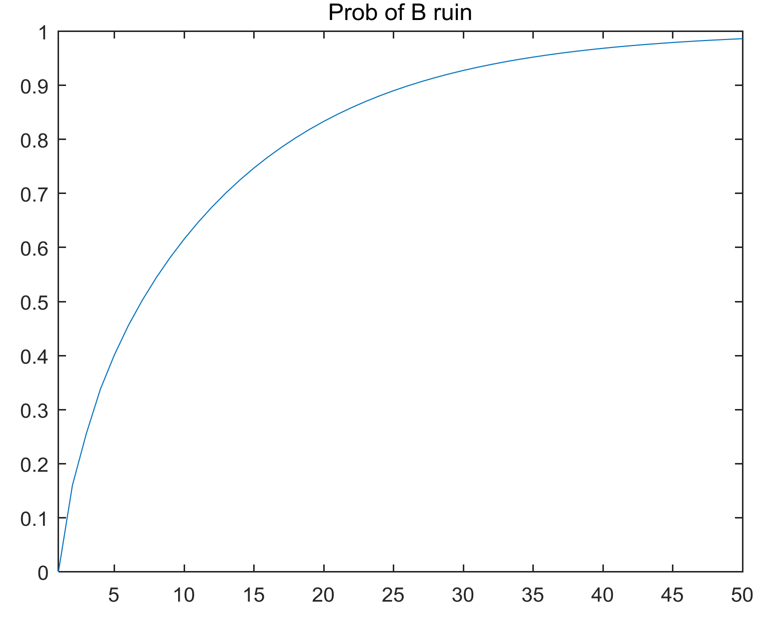
0.0749	0.1747	0.2329	0.3105	0.2070
0.0749	0.1747	0.2329	0.3105	0.2070
0.0749	0.1747	0.2329	0.3105	0.2070
0.0749	0.1747	0.2329	0.3105	0.2070
0.0749	0.1747	0.2329	0.3105	0.2070

More examples

Now we consider the reflecting barriers.

- How about $P = \begin{pmatrix} \color{blue}{0.3} & \color{blue}{0.7} & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & \color{blue}{0} & \color{blue}{1} \end{pmatrix}$?





Arrangement

- Reading: 15.1,15.2
- The final week:
  - Review
  - Q&A
    - All courseworks
    - Exercise questions
  - Experiment 4

Others

- We start experiment 4 this week

- Submit your coursework 3 today

- Submit your experimental report 3 before 2024.06.19,23:59:59