# Stochastic Signal Processing

## Lesson 12: Filters II

Weize Sun

# Examples from last week

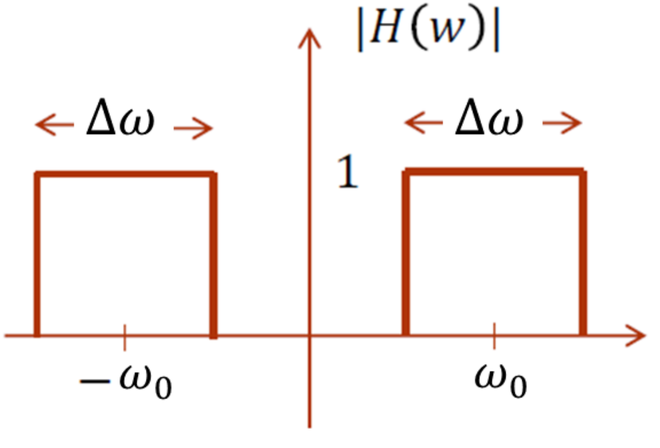
- 1: given input signal $X(t)$, output signal $Y(t)$ and their relationship:

$$\frac{4dY(t)}{dt} + 2Y(t) = X(t)$$

When $X(t) = \delta(t)$, find $R_Y(\tau)$ and $R_{YX}(\tau)$.

It is well known that if feeding an input Gaussian process to a stationary linear system (any deterministic system is stationary), the output is also a Gaussian process.

- 2: given an input Gaussian white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the 1-D pdf of the output.



# Examples from last week

- 1: given input signal $X(t)$, output signal $Y(t)$ and their relationship:

$$\frac{4dY(t)}{dt} + 2Y(t) = X(t)$$

When $X(t) = \delta(t)$, find $R_Y(\tau)$ and $R_{YX}(\tau)$.

Solution:

$$S_X(\omega) = 1 \qquad H(\omega) = \frac{1}{2 + 4j\omega} \qquad |H(\omega)|^2 = \frac{1}{4 + 16\omega^2}$$

$$S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \frac{1}{4 + 16\omega^2} = \frac{1}{16}\frac{1}{\frac{1}{4} + \omega^2} \Rightarrow R_Y(\tau) = \frac{1}{16}e^{-\frac{1}{2}|\tau|}$$

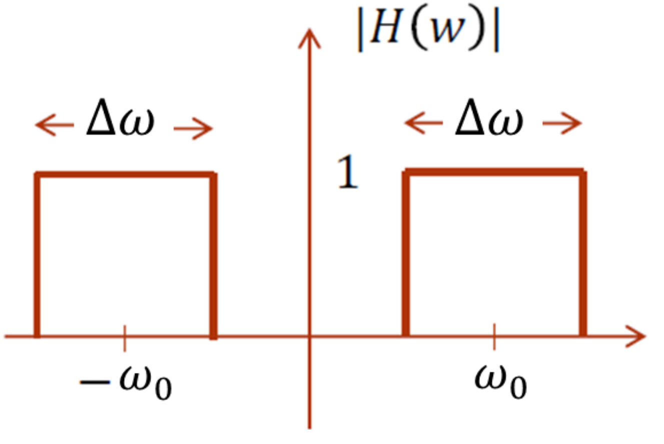
$$S_{YX}(\omega) = S_X(\omega)H(\omega) = \frac{1}{2 + 4j\omega} = \frac{1}{2}\frac{1}{1 + \frac{1}{2}j\omega}$$

$$\Rightarrow R_{YX}(\tau) = \frac{1}{2}\frac{1}{2}e^{-\frac{1}{2}\tau}U(\tau) = \frac{1}{4}e^{-\frac{1}{2}\tau}U(\tau)$$

# Examples from last week

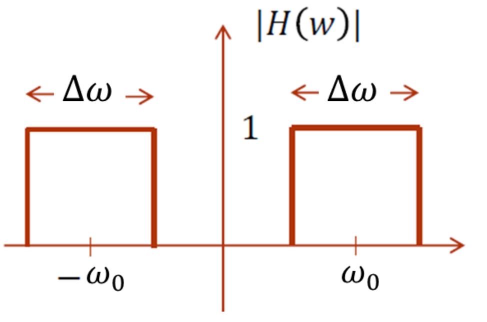
It is well known that if feeding an input Gaussian process to a stationary linear system (any deterministic system is stationary), the output is also a **Gaussian process**.

- 2: given an input Gaussian white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the 1-D pdf of the output.



# Examples from last week

- 2: given an input Gaussian white noise $X(t)$ with power spectrum $S_X(\omega) = N_0/2$, and it is inputted to the ideal band-pass filter as below, calculate the 1-D pdf of the output.




Solution:

$$\sigma_Y^2 = R_Y(0) = \frac{\frac{\Delta\omega}{2} N_0}{\pi} sinc(\frac{\Delta\omega}{2}\ 0\ )\cos(\omega_0\ 0) = \frac{N_0 \Delta\omega}{2\pi}$$

And $E[Y(t)] = 0$

Therefore, the output $Y(t)$ is a Gaussian process with mean 0, and its 1-D variance is $\frac{N_0\Delta\omega}{2\pi}$, thus

$$f_Y(y,t) = f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} = \frac{1}{\sqrt{N_0\Delta\omega}} e^{-\frac{\pi y^2}{N_0\Delta\omega}}$$

# Outline

- Optimal Linear Filter
- Matched filter

# Optimal Linear Filter – introduction

- Many electronic systems are based on echo detection:
  - Radar, sonar, ultrasonic, infrared
- Sometimes, the noise power is much larger then the signal power
  - Signal-to-noise ratio: SNR = signal power / noise power
  - The higher the signal-to-noise ratio, the higher the accuracy of target detection (in radar, sonar, …) and the lower the probability of error (in communication).
- Therefore, it is necessary to design a filter to enhance the power of signal and reduce the power of noise.

# Optimal Linear Filter

- Given the following linear systems:

Deterministic signal

Zero mean stationary stochastic process (noise)

$$z(t) = x(t) + n(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = x_o(t) + n_o(t)$$

$x_o$: 'o' for 'output'

- then:

spectrum

output signal

For signal $X_o(\omega) = X(\omega)H(\omega) \longrightarrow x_o(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}\,d\omega$

Instantaneous output signal power at time $t$: $x_o^2(t)$

For noise

$$S_{n_o}(\omega) = S_n(\omega)|H(\omega)|^2 \longrightarrow E\left(n_o^2(t)\right) = R_{n_o}(\tau = 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2\,d\omega$$

Power spectrum

Average noise power

# Optimal Linear Filter

Given: $z(t) = x(t) + n(t)$

- Now let's define the problem:
  - Suppose we are tackling a detection problem as follows
  - Detection: the goal is to determine whether a signal exists, or to locate the signal. (for example, where is $x(t)$ located within the whole time domain?)
  - For example: there is a strong noise $n(t)$ (time: 0 to 1 second), and we can see that the maximum intensity is 15; however, the signal is weak (red), with an intensity of 1, between 0.2 and 0.4 seconds.

# Optimal Linear Filter

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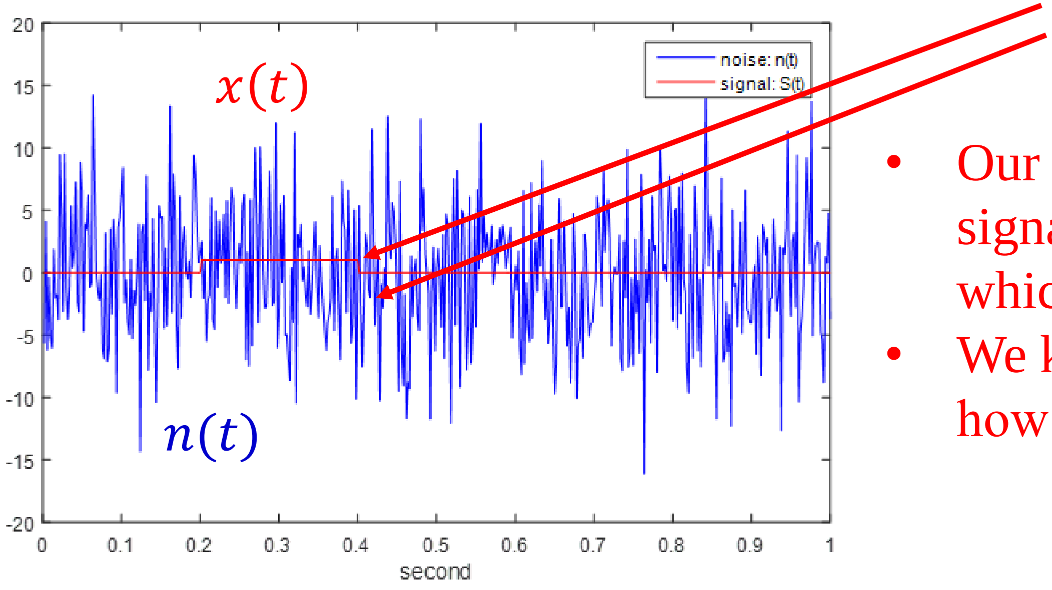


end time of signal: 0.4s

- Our goal is to find out the end time of the signal correctly in the time period of 0-1s, which is,
- We know that the signal ends at 0.4s, but how to ‘detect’ it?

# Optimal Linear Filter

Given: $z(t) = x(t) + n(t)$

- Now let's define the problem:
  - Suppose we are tackling a detection problem as follows
  - Detection: the goal is to determine whether a signal exists, or to locate the signal. (for example, where is $x(t)$ located within the whole time domain?)
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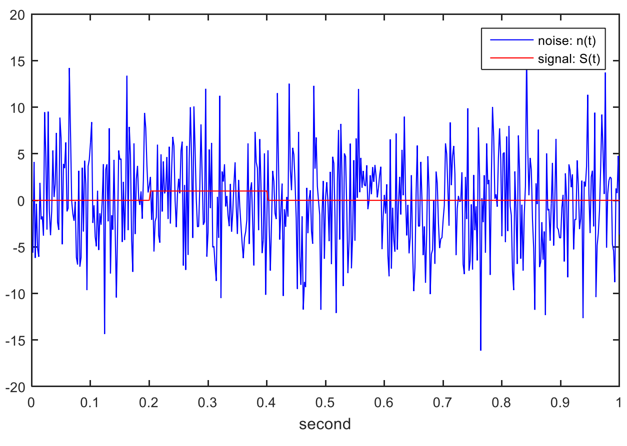
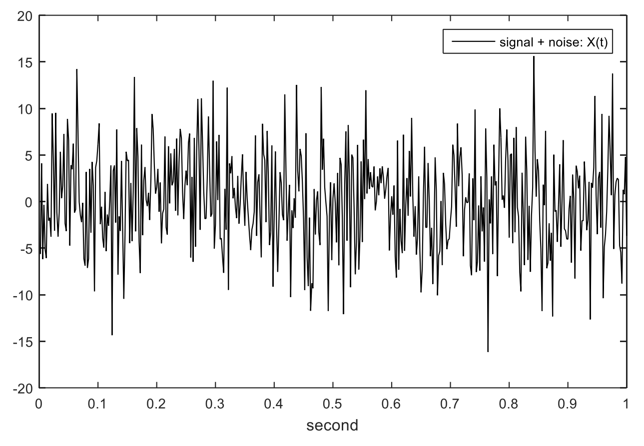


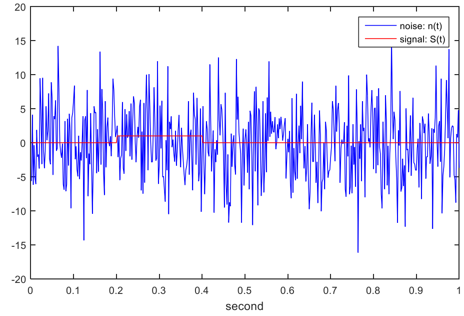

Note that what we receive is the right image, **<u>we cannot 'see' the signal dircectly</u>**!

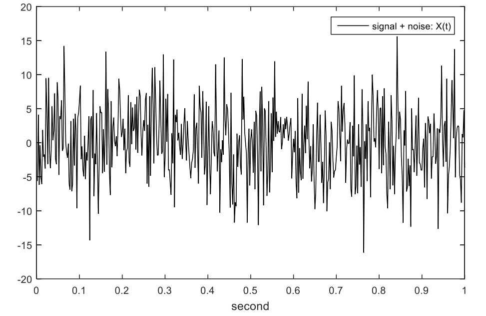
# Optimal Linear Filter

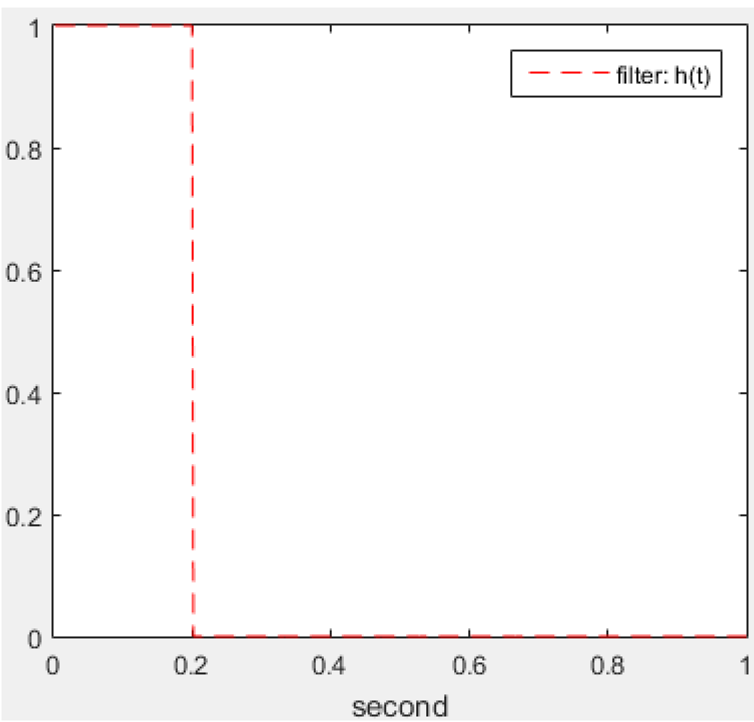
Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

- Now let's solve the problem:
  - Define a filter $h(t)$ as follows:



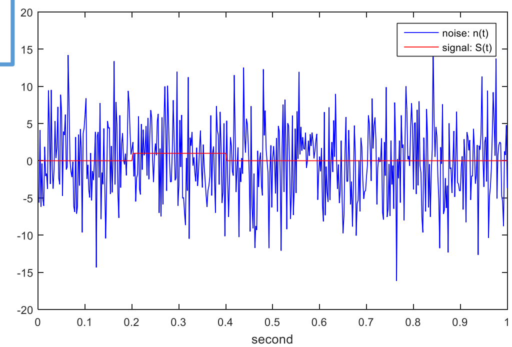



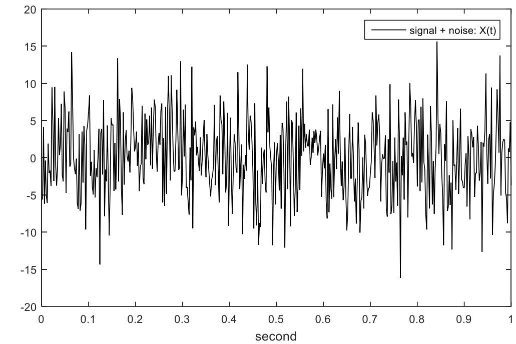

- The duration of the filter is 0.2 seconds, which is consistent with the duration of the signal.
- The filter is a square wave (look similar as the signal)
- Now we feed the input $z(t)$ to $h(t)$ and get the output $y(t)$

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$

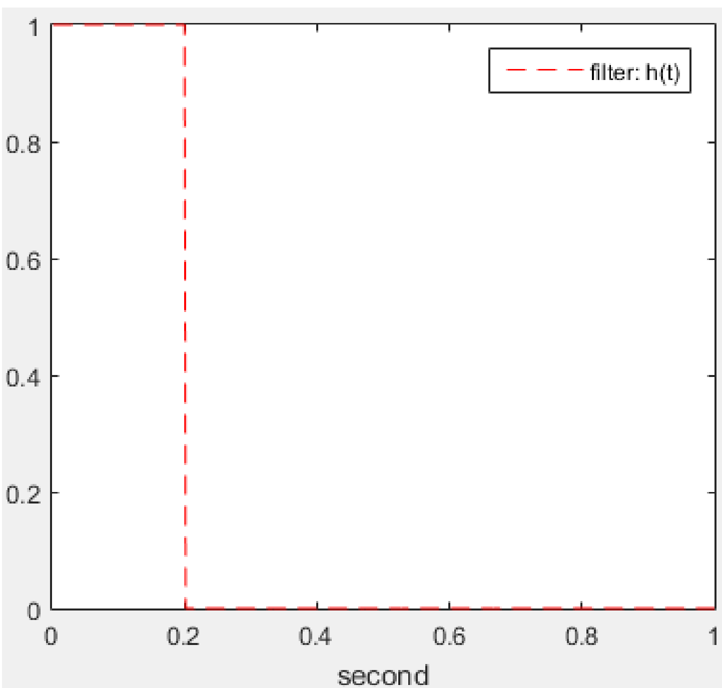
Output: $y(t) = x_o(t) + n_o(t)$

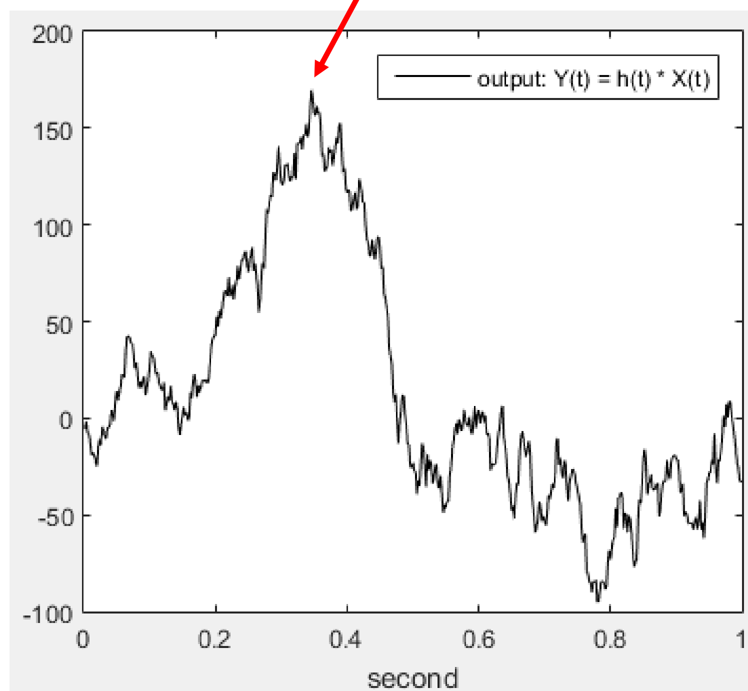



- Now let's solve the problem:
  - Define a filter $h(t)$, and feed the input $z(t)$ to $h(t)$ and get the output $y(t)$

Obviously, the signal seems located here. Although this result is not exactly 0.4 second (i.e. the correct end time of the signal), it is close. Moreover, the peak characteristic (峰值特征) is obvious.



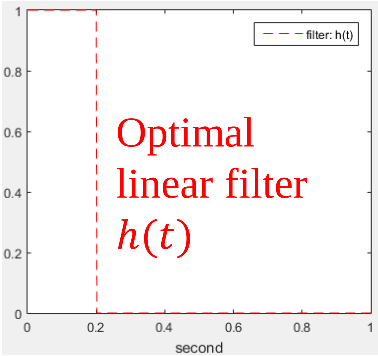



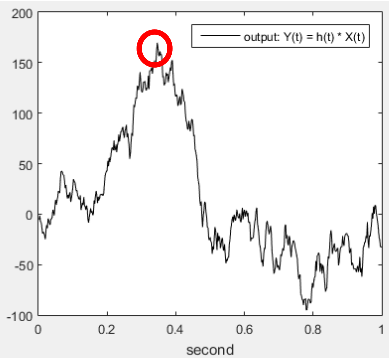
# Optimal Linear Filter

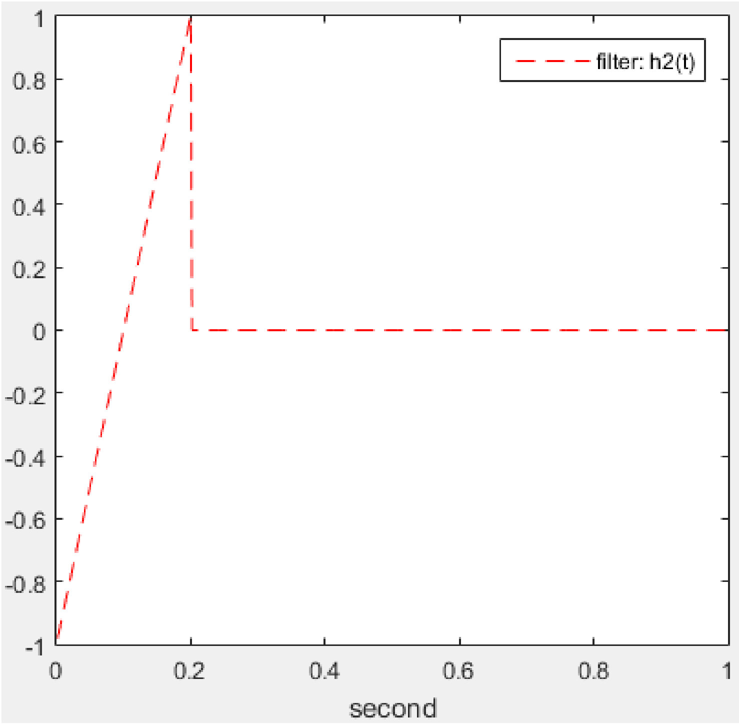
Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

- Now let's solve the problem:
  - Define another filter $h_2(t)$ :



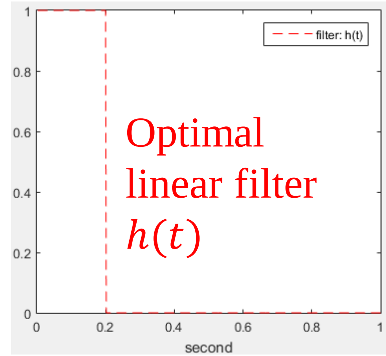

- The duration of the filter is 0.2 seconds, which is consistent with the duration of the signal.
- The filter is a triangular wave.
- Now we feed the input $z(t)$ to $h_2(t)$ and get the output $y_2(t)$

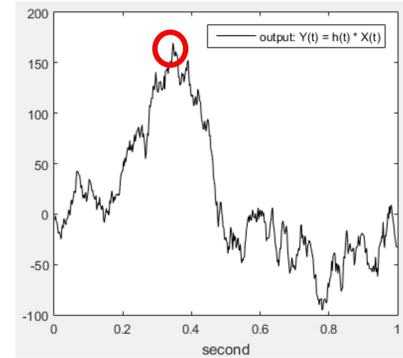
# Optimal Linear Filter

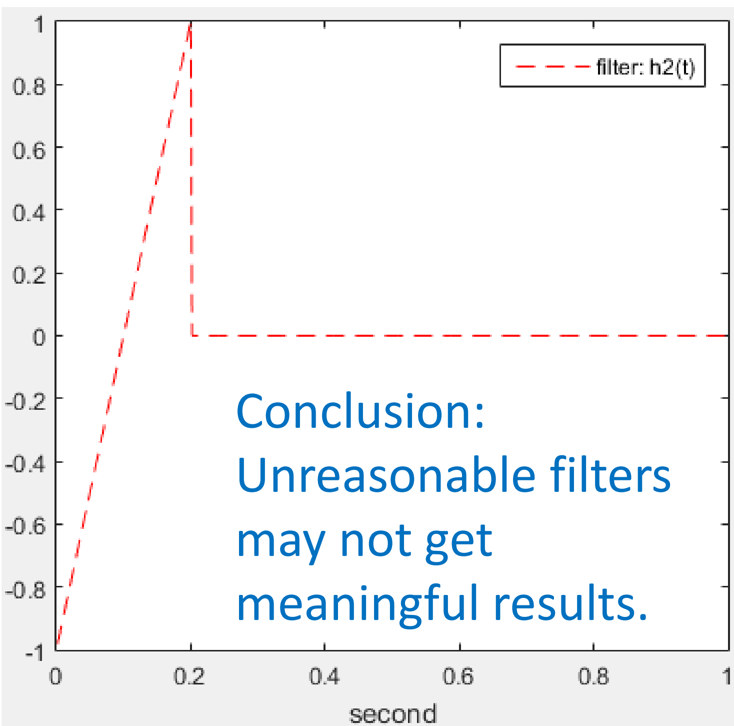
Input: $z(t) = x(t) + n(t)$

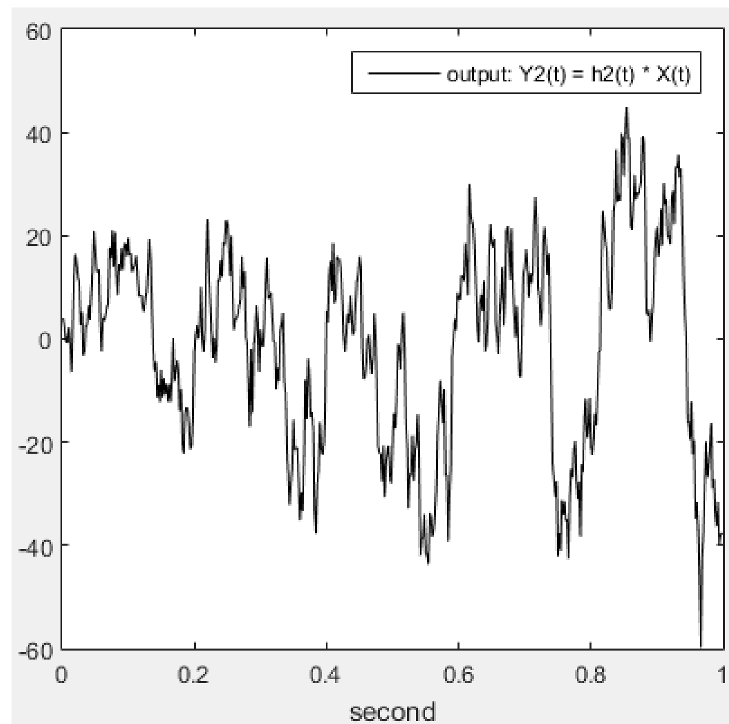
Output: $y(t) = x_o(t) + n_o(t)$

- Now let's solve the problem:
  - Define another filter $h_2(t)$ :



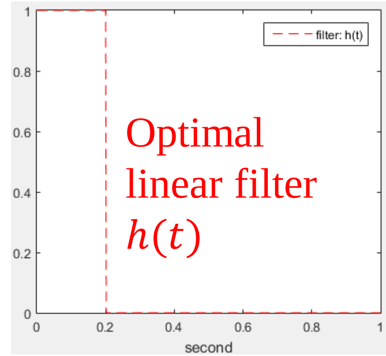



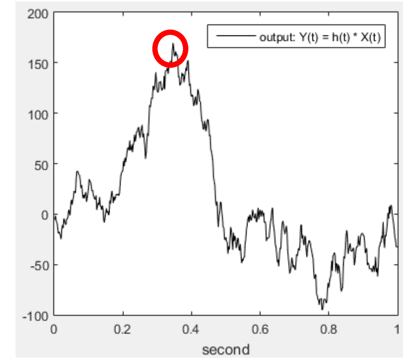



# Optimal Linear Filter

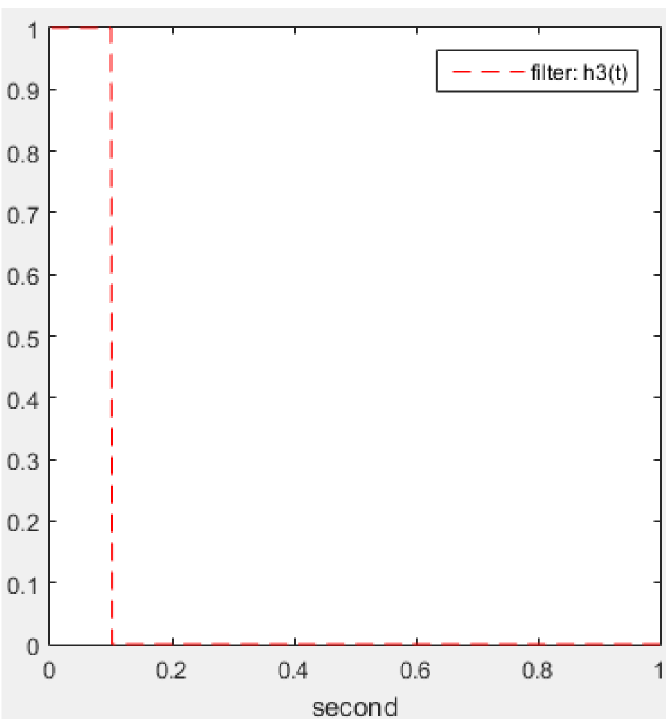
Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$




- Now let's solve the problem:
  - assume that the square wave filter is reasonable, how about a filter $h_3(t)$ with shorter time:




- The filter is a square wave.
- However, the duration of the filter is 0.1 seconds, which is shorter than the duration of the signal.
- Now we feed the input $z(t)$ to $h_3(t)$ and get the output $y_3(t)$

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

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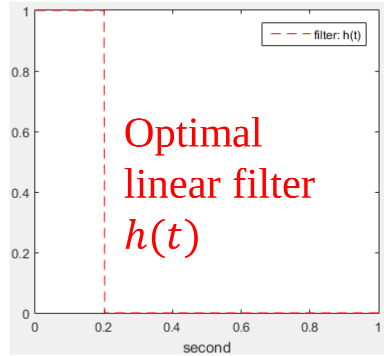
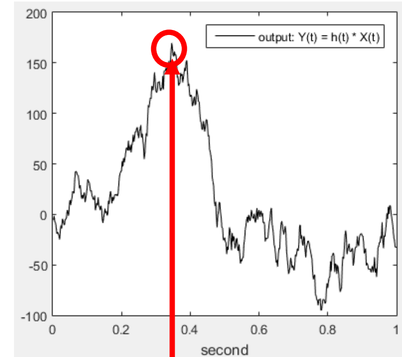
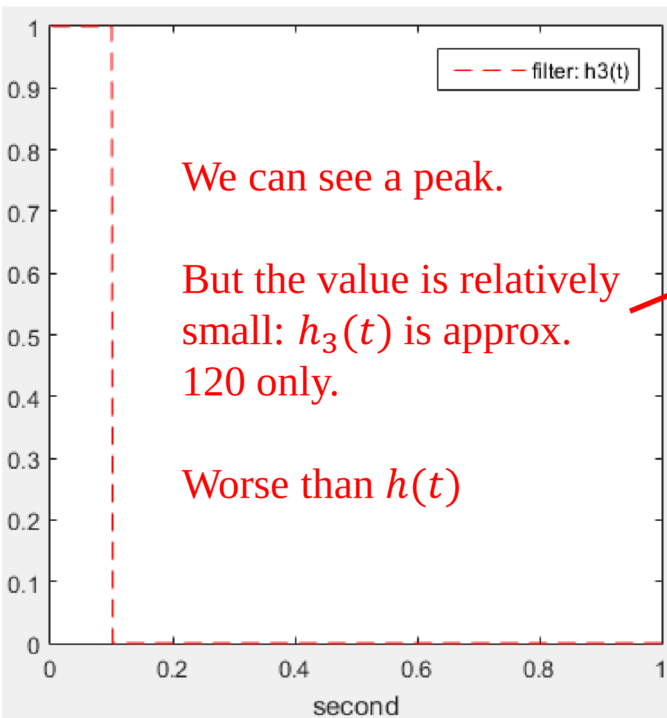
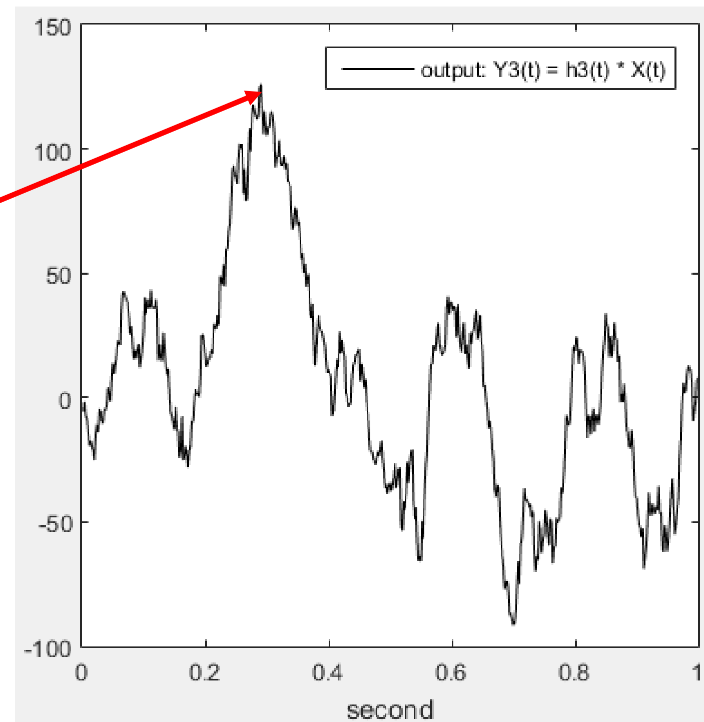


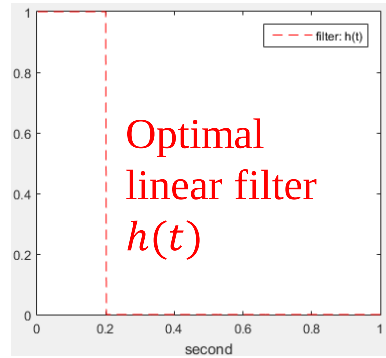



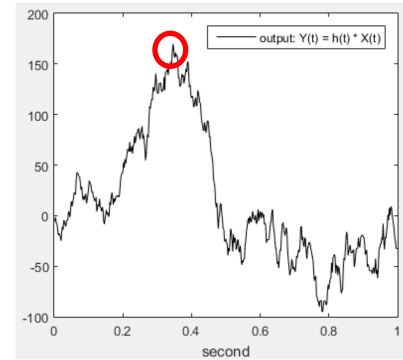

*The peak value of filter* $h(t)$ *is approx.* 160;



# Optimal Linear Filter

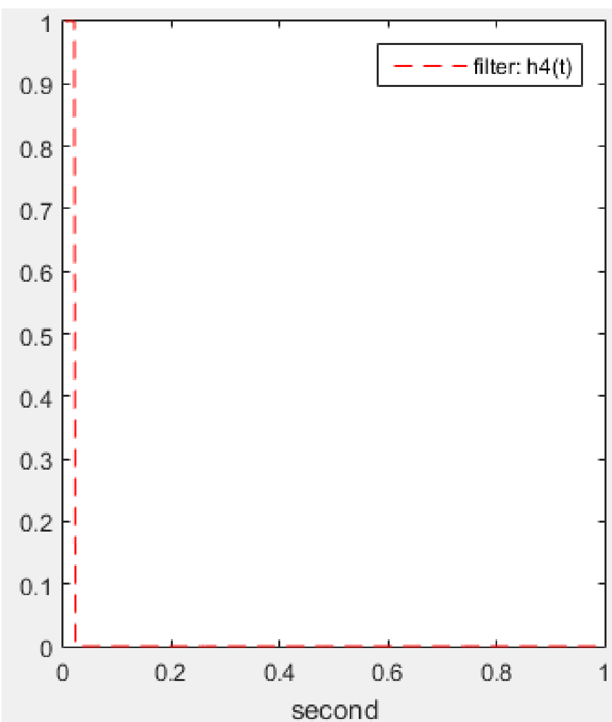


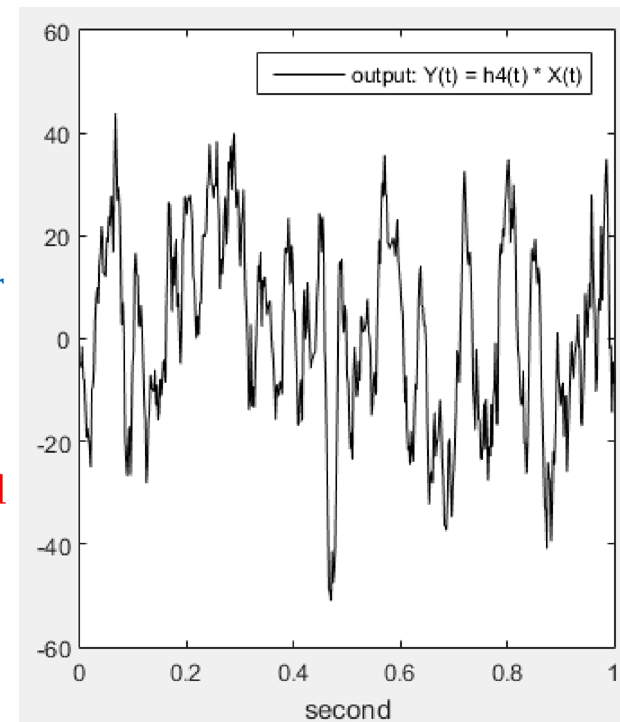

Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

- Now let's solve the problem:
  - How about a filter $h_4(t)$ with only 0.02 seconds. (1 / 10 of the signal time)




- the result looks like noise. Unable to determine the location of the signal.
- Conclusion: Unreasonable filter time length does not work well.
- In fact, the filter duration time should be the same as the signal duration time



# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

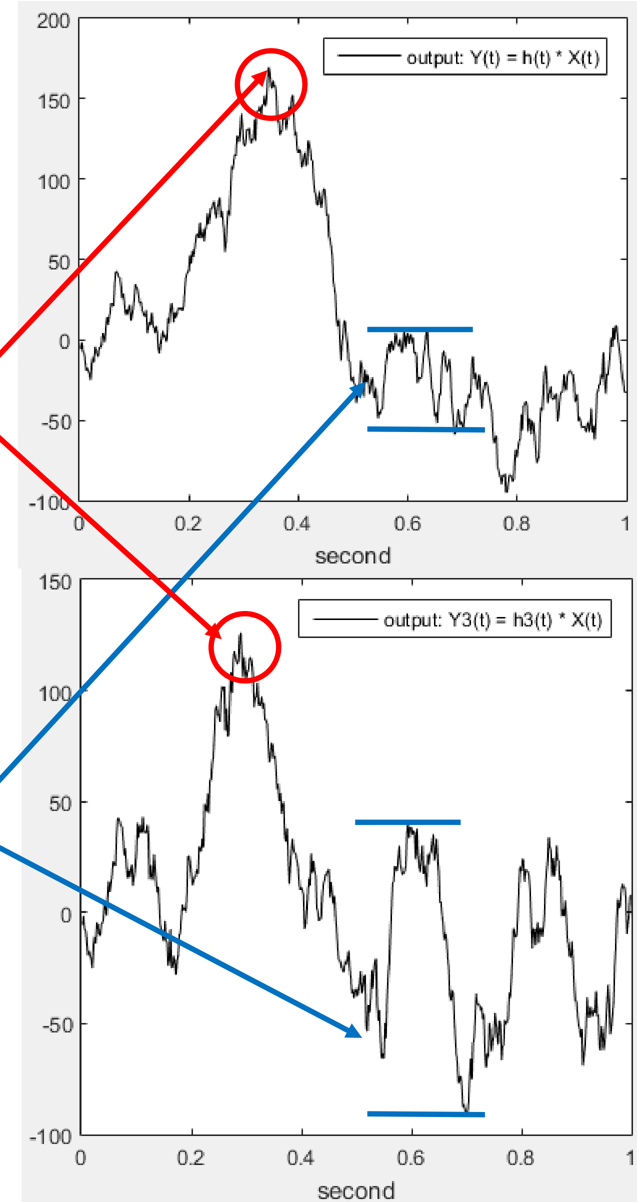
- From the above experiments,

Optimal linear filter output

Peak value, that is, the maximum signal power of output $y(t)$, the greater the better

The fluctuation represents the noise: smaller the better (note that the actual noise power should be calculated by mathematical formula)

Output of a non-optimal filter



# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

- From the above experiments,

it is shown that in order to correctly detect the signal:

- We need to design an "optimal filter", in order to simplify the system design, preferably "optimal linear filter".
- Given signal $x(t)$ and noise $n(t)$, we can determined the shape of this "optimal linear filter".
- Wrong shapes leads to wrong results.
- The duration of this "optimal linear filter" is also important.
- Wrong duration leads to wrong results.
- **How to solve it mathematically?**

Optimal linear filter output

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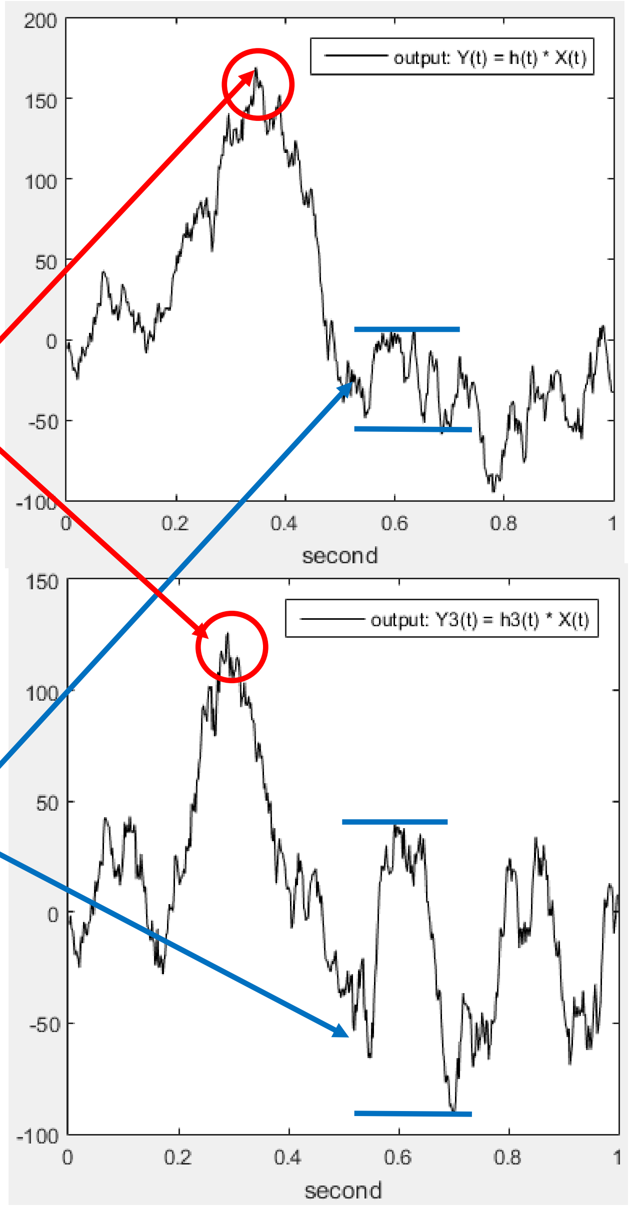
The fluctuation represents the noise: smaller the better (note that the actual noise power should be calculated by mathematical formula)

Output of a non-optimal filter



# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$

Output: $y(t) = x_o(t) + n_o(t)$

- From the above experiments,

it is shown that in order to correctly detect the signal:

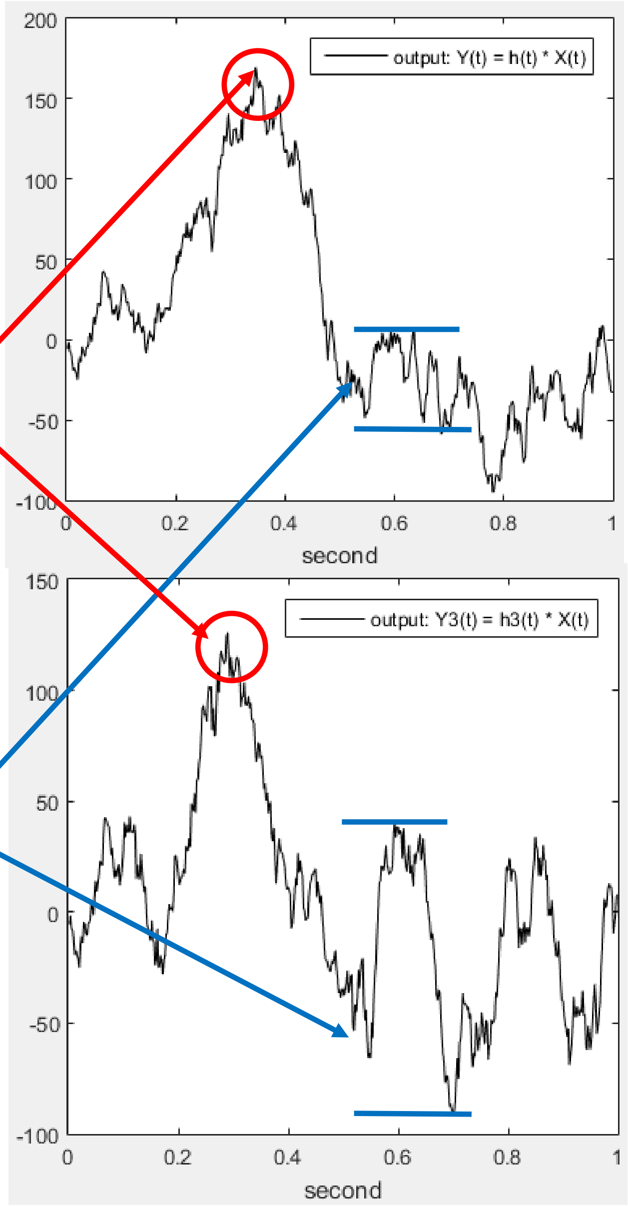
- We need to design an "optimal filter", in order to simplify the system design, preferably "optimal linear filter".
- This "optimal linear filter" aims at making the output **peak signal-to-noise ratio (the ratio of the maximum signal to noise)** as large as possible.
- Therefore, we should first define "signal-to-noise ratio"

Optimal linear filter output

Peak value, that is, the maximum signal power of output $y(t)$, the greater the better

The fluctuation represents the noise: smaller the better (note that the actual noise power should be calculated by mathematical formula)

Output of a non-optimal filter



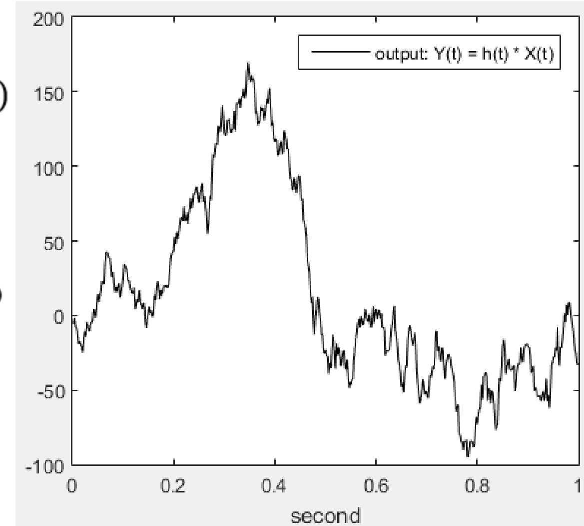
# Optimal Linear Filter (recall page 8 of this ppt)

- Given the following linear systems:

Deterministic signal

Zero mean stationary stochastic process (noise)

$$z(t) = x(t) + n(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = x_o(t) + n_o(t)$$




- then:

spectrum

output signal

For signal $X_o(\omega) = X(\omega)H(\omega) \longrightarrow x_o(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}\, d\omega$

Instantaneous output signal power at time $t$: $x_o^2(t)$

Power spectrum

Average noise power

For noise $S_{n_o}(\omega) = S_n(\omega)|H(\omega)|^2 \longrightarrow E\left(n_o^2(t)\right) = R_{n_o}(\tau = 0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2\, d\omega$

- We can define the **SNR at time *t*** as $x_o^2(t)/E\left(n_o^2(t)\right)$
- Note that this **SNR change with time** as signal change with time
- Thus, when this SNR reaches its maximum, it is where the signal located!

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Signal-to-noise ratio (SNR):
  - $d_0$: the ratio of the 'instantaneous power of the output signal' of the filter to the 'average power of the noise' at a certain time $t = t_0$

$$d_0 = \frac{x_0^2(t_0)}{E\{n_0^2(t)\}}$$

$$x_o(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t_0}\, d\omega$$

$$E\{n_0^2(t)\} = R_{n_0}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2 e^{j\omega\tau}\, d\omega \bigg|_{\tau=0}$$

$$\Longrightarrow \quad d_0 = \frac{1}{2\pi}\frac{\left|\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t_0}\, d\omega\right|^2}{\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2\, d\omega}$$

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Signal-to-noise ratio (SNR):
  - $d_0$: the ratio of the 'instantaneous power of the output signal' of the filter to the 'average power of the noise' at a certain time $t = t_0$

$$x_o(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t_0}\, d\omega$$

$$d_0 = \frac{x_0^2(t_0)}{E\{n_0^2(t)\}} \longrightarrow d_0 = \frac{1}{2\pi}\frac{\left|\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t_0}\, d\omega\right|^2}{\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2\, d\omega}$$

$$E\{n_0^2(t)\} = R_{n_0}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2 e^{j\omega\tau}\, d\omega\,\Big|_{\tau=0}$$

- Target:
  - Design a linear system $h(t)$ to maximize the output signal-to-noise ratio

- Result: when $H(\omega) = cX^*(\omega)e^{-j\omega t_0}/S_n(\omega)$, the signal-to-noise ratio is maximized.
  - The $c$ is a constant and can be any value

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

Signal-to-noise ratio: $d_0 = \frac{1}{2\pi} \frac{\left| \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t_0} d\omega \right|^2}{\int_{-\infty}^{\infty} S_n(\omega) |H(\omega)|^2 d\omega}$

- Result: when $H(\omega) = cX^*(\omega)e^{-j\omega t_0}/S_n(\omega)$, the signal-to-noise ratio is maximized, and we call the $d_0$ at this $H(\omega)$ as $d_m$ ($m$ stands for maximum)

$$d_m = \frac{1}{2\pi} \frac{\left| \int_{-\infty}^{\infty} \frac{c|X(\omega)|^2}{S_n(\omega)} d\omega \right|^2}{\int_{-\infty}^{\infty} \frac{S_n(\omega)|cX(\omega)|^2}{S_n^2(\omega)} d\omega}$$

$$= \frac{1}{2\pi} \frac{c^2 \left( \int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{S_n(\omega)} d\omega \right)^2}{c^2 \int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{S_n(\omega)} d\omega}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{S_n(\omega)} d\omega$$

As all parameters are positive, the absolute is removed.

# Optimal Linear Filter

Input: $z(t) = x(t) + n(t)$
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$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
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- Result: when $H(\omega) = cX^*(\omega)e^{-j\omega t_0}/S_n(\omega)$, the signal-to-noise ratio is maximized, and we call the $d_0$ at this $H(\omega)$ as $d_m$ ($m$ stands for maximum)

$$d_m = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X(\omega)|^2}{S_n(\omega)} d\omega$$

- Although we have not solved the filter $h(t)$, the maximum signal-to-noise ratio is obtained theoretically. It has many characteristics and applications:
  - The maximum signal-to-noise ratio is defined by the signal and noise only.
  - Given the signal and noise, it is necessary to determined whether a signal can be detected without designing the filter $h(t)$ or solving the output $y(t)$
    - For example, for a system, if the signal can only be detected under a signal-to-noise ratio larger than 1 (or says, 0dB). And according to the signal and noise, we calculate that $d_m = 0.5$, which is, the maximum Signal-to-noise ratio is 0.5, or says, SNR=1 is impossible, then we reach to the conclusion that we cannot detect the signal.

# Outline

- Optimal Linear Filter
- Matched filter (匹配滤波器)

# Matched filter

- The previous optimal linear filter is for general stationary noise
- For white noise:
  - The power spectrum of white noise is a constant $q$:
    $H(\omega) = cX^*(\omega)e^{-j\omega t_0}/q = c_n X^*(\omega)e^{-j\omega t_0} = cX^*(\omega)e^{-j\omega t_0}$
- Impulse response:

Fourier transformation

$$h(t) = cx^*(t_0 - t) \xrightarrow{\text{If it's a real signal}} h(t) = cx(t_0 - t)$$

- This is called matched filter:
  - The impulse response of the matched filter is the conjugate image (共轭镜像) of the input signal
  - When $c = 1$, $h(t)$ and $x(t)$ are even symmetric(偶对称) centered at $t_0/2$.

$c$ is a constant, which can be assigned a new constant $c_n = c/q$, and it can be simplified written as $c$

# Matched filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Properties of matched filter
  - Let the power spectrum of white noise be $S_n(\omega) = q$, then the maximum SNR

$$d_m = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X(\omega)|^2}{S_n(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X(\omega)|^2}{q}d\omega$$

$$= \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\, d\omega}{q} = \frac{E_X}{q}$$

Where $E_X = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\, d\omega$ is the energy of the signal; $q$ is noise power

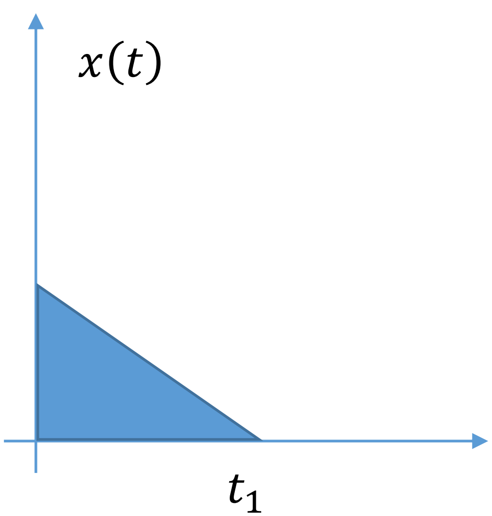
  - Property 1: the maximum signal-to-noise ratio is only related to the energy of the signal and the noise power, and not related to the waveform shape of signal.

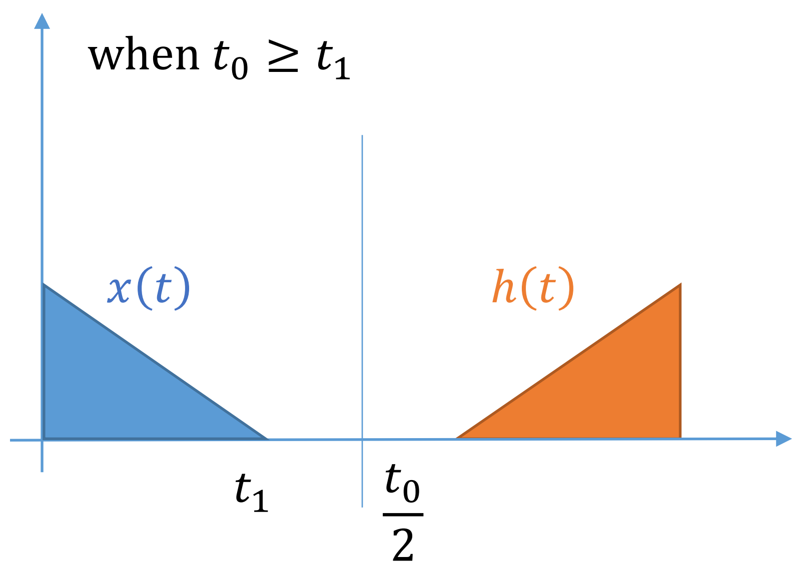
# Matched filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Properties of matched filter
  - Note that for matched filter, the filter is $h(t) = cx^*(t_0 - t)$
  - Property 2: $t_0$ should be larger then the duration time of signal $x(t)$; the $h(t)$ and $x(t)$ are even symmetric(偶对称) centered at $t_0/2$.
  - From $h(t) = cx^*(t_0 - t)$:
    - For a physically realizable system $h(t)$, we always have $h(t) = 0$ for $t < 0$.
    - Take the following real signal as an example:




It works!

# Matched filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Properties of matched filter
  - Note that for matched filter, the filter is $h(t) = cx^*(t_0 - t)$
  - Property 2: $t_0$ should be larger then the duration time of signal $x(t)$; the $h(t)$ and $x(t)$ are even symmetric(偶对称) centered at $t_0/2$.
  - From $h(t) = cx^*(t_0 - t)$:
    - For a physically realizable system $h(t)$, we always have $h(t) = 0$ for $t < 0$.
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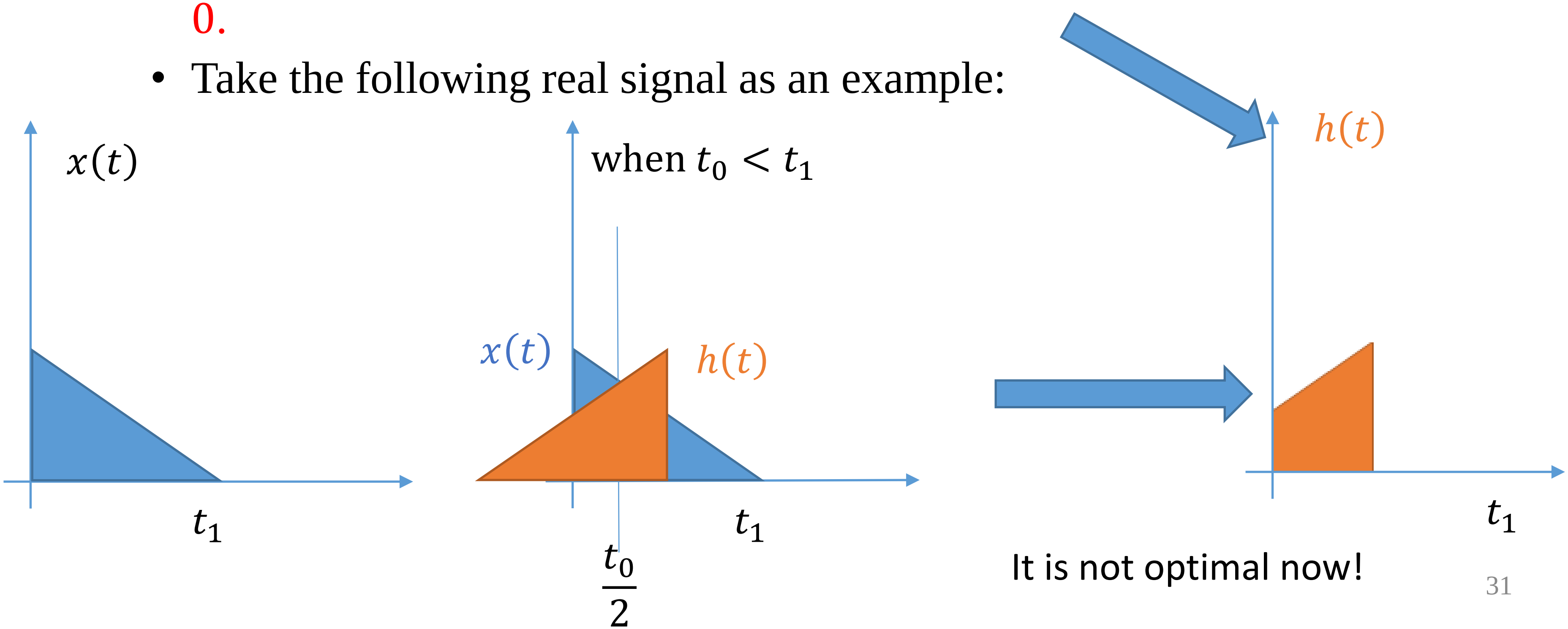



It is not optimal now!

# Matched filter

Input: $z(t) = x(t) + n(t)$
Output: $y(t) = x_o(t) + n_o(t)$

$X(\omega)/H(\omega)$ : spectrum of $x(t)/h(t)$
$S_n(\omega)$: power spectrum of $n(t)$

- Properties of matched filter：
  - Property 3: matched filter is adaptive to signal attenuation and time delay
    - It is assumed that the received signal has certain attenuation and time delay comparing to the transmitted signal

$$x_1(t) = ax(t-\tau) \longleftrightarrow X_1(\omega) = aX(\omega)e^{-j\omega\tau}$$

    - Then the new matched filter is：

$$H_1(\omega) = cX_1^*(\omega)e^{-j\omega t_1} = caX^*(\omega)e^{-j\omega(t_1-\tau)}$$
$$= caX^*(\omega)e^{-j\omega t_0}e^{-j\omega(t_1-\tau-t_0)}$$
$$= aH(\omega)e^{-j\omega(t_1-\tau-t_0)}$$

Matched filter of $x(t)$:
$H(\omega) = cX^*(\omega)e^{-j\omega t_0}$

    - Where $H(\omega)$ is the matched filter of signal $x(t)$. Define $t_1 = t_0 + \tau$ we have:

$$H_1(\omega) = aH(\omega)$$

**Physical meaning: when designing a matched filter, if there is attenuation and time delay in the received signal, the filter can still works well!**

## Matched filter

- Example 1: given an input $z(t) = x(t) + n(t)$, where $x(t)$ is a deterministic signal with spectrum $X(\omega)$ as follows, and $n(t)$ is a stationary Gaussian white noise with power spectrum $A$. Calculate the maximum signal-to-noise ratio of the output of $z(t)$ feeding to its matched filter.

$$X(\omega) = \begin{cases} 1 + j\omega, & (0 < \omega \leq 1) \\ 0, & (other) \end{cases}$$

Solution:

# More examples

- Example 2: given input $X(t)$, system $H(\omega) = \frac{\alpha}{\alpha + j\omega}$ and output $Y(t)$, and the autocorrelation $R_X(\tau) = 3\delta(\tau)$, calculate: (1) the autocorrelation of output (2) if the average power of the output is 3, find $\alpha$. (3) Cross-correlation $R_{YX}(\tau)$
- Example 3: given that the input signal $X(t)$ and output signal $Y(t)$ of a linear system satisfy

$$Y(t) = X(t) + X(t - \tau)$$

Find the power spectrum of $Y(t)$ (expressed by $S_X(\omega)$)

Hint: time delay $\tau$ corresponds to $e^{-j\tau\omega}$ in frequency domain, which is, $X(t-\tau) \leftarrow\rightarrow X(\omega)e^{-j\tau\omega}$

- Example 4: feed a white noise to an RC circuit as figure 1, calculate: (1) the noise equivalent pass-band $\Delta f_e$; (2) the correlation time of the output.

Hint: $\int_a^b \frac{1}{1+x^2} da = \arctan(x)\,|_a^b$

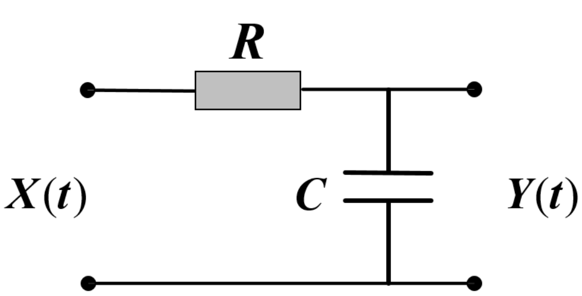
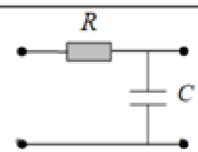



figure 1

Some more widely used Fourier Transform pairs

Hint

| Circuit System | $H(\omega)$ | $h(t)$ |
|---|---|---|
|  | $\frac{1}{1+j\omega RC}$ | $\frac{1}{RC}e^{-t/RC}U(t)$ |

## Reading

- This week:
  - Red book: 3.5 (最佳线性滤波器) 注：最佳线性滤波器的讲解将以此书为主导
- Next week:
  - Text book: 10.1: Ergodicity

# Experiment

- Go on with Experiment 3