Heavy quark evolution in heavy-ion collisions

and Bayesian estimation of its transport coefficients

Yingru Xu

Duke University

yx59@phy.duke.edu

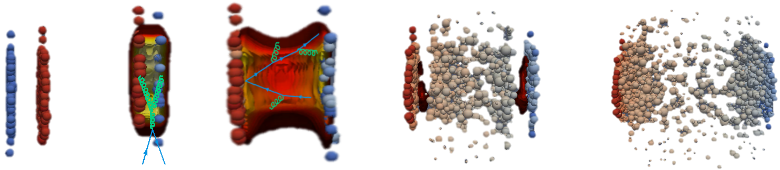
RHIC-AGS user meeting,
Brookhaven National Laboratory, NY
June 12, 2018

In collaboration with :
Weiyao Ke
Steffen A. Bass

This work has been supported by the U.S Department of Energy under grand DE-FG02-05ER41367. Computational resources were provided by the Open Science Grid (OSG) and NERSC.

Introduction

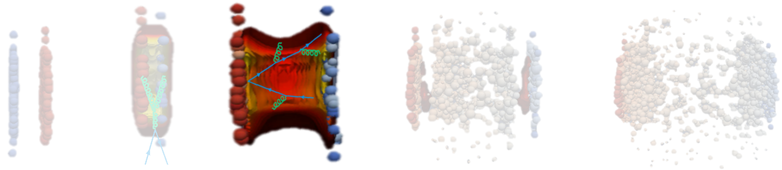
AuAu collision @ 200 GeV, figure credit: Hannah Peterson



Heavy quarks are unique (hard) probes of the Quark-Gluon Plasma:

- Large masses ($M_C \sim 1.3$ GeV, $M_b \sim 4.2$ GeV) $\gg T$
  - Thermal production is negligible $\rightarrow Q\bar{Q}$ primarily produced from initial hard processes
  - Thermal relaxation time $\tau_{c/b} \sim \tau_{\mathrm{QGP}} \rightarrow$ experience full evolution of the QGP
- Probe different transverse momentum region $\rightarrow$ low p_T (collective flow); high p_T (jet quenching)
- Heavy flavor conservation

HQ in-medium transport



Langevin dynamics

- No assumption of medium constituents $\Rightarrow (T, \vec{u})$
- HQ gets frequent kicks from the medium $\Rightarrow$ transport coefficients

Boltzmann dynamics

- Medium constituents: light partons $\Rightarrow f_q, f_g \sim \exp\{-\frac{p \cdot u}{T}\}$
- HQ scatters with medium partons based on pQCD matrix elements

Hybrid models

- **Model A:** Improved Langevin (with radiative energy loss)
- **Model B:** `Lido`: **Li**nearized Boltzmann with **d**iffusion m**o**del

HQ in-medium transport - model A

Improved Langevin transport model

S.Cao, G.Qin, and S.A.Bass, Phys.Rev.C 92, 024907(2015)

$$\frac{d\vec{p}}{dt} = -\eta_D(p)\vec{p} + \vec{\xi} + \vec{f}_g \tag{1}$$

- Drag force: $\eta_D(p) = \hat{q}/(4TE)$
- Thermal random force: $\langle \xi^i(t)\xi^j(t') \rangle = \frac{1}{2}\hat{q}\delta^{ij}\delta(t-t')$
- Recoil force from gluon radiation: $\vec{f}_g(\hat{q}) = -d\vec{p}_g/dt$
  - Gluon emission probability:

$$\frac{dN_g}{dxdk_\perp^2 dt} = \frac{2\alpha_s P(x) C_A/C_F \hat{q}}{\pi k_\perp^4} \sin^2\left(\frac{t-t_i}{2\tau_f}\right)\left(\frac{k_\perp^2}{k_\perp^2 + x^2M^2}\right)^4 \tag{2}$$

HQ in-medium transport - model A

Transport coefficients $\hat{q} \Leftrightarrow D_s 2\pi T$

M. Mannarelli and R. Rapp, Phys. Rev. C 72, 064905 (2005)
A. Francis et al., Phys. Rev. D 92, 116003 (2015)
D. Banerjee et al. Phys. Rev. D 85.014510

- Direct calculation from theory: pQCD, T-matrix, lattice QCD ...

- Data-driven approach: empirical parametrization

$$D_s 2\pi T(T,p) = \frac{1}{1+(\gamma^2 p)^2}(D_s 2\pi T)^{\mathrm{soft}} + \frac{(\gamma^2 p)^2}{1+(\gamma^2 p)^2}(D_s 2\pi T)^{\mathrm{pQCD}}$$

  - Linear-T component: $(D_s 2\pi T)^{\mathrm{soft}} = \alpha \cdot \left[1 + \beta \cdot (\frac{T}{T_c} - 1)\right]$
  - pQCD component: $(D_s 2\pi T)^{\mathrm{pQCD}}(\alpha_s) = 8\pi T^3 / \hat{q}^{\mathrm{pQCD}}(\alpha_s)$
  - $p \ll 1/\gamma^2$: $D_s 2\pi T \rightarrow D_s 2\pi T^{\mathrm{soft}}$
    $p \gg 1/\gamma^2$: $D_s 2\pi T \rightarrow D_s 2\pi T^{\mathrm{pQCD}}$

HQ in-medium transport - model B

Weiyao Ke, Y.Xu, S.A.Bass, paper in preparation

`Lido`: Linearized Boltzmann with diffusion transport model

Scattering component:

$$\frac{\partial}{\partial t} f_Q - \frac{\vec{p}}{E} \cdot \nabla f_Q = \mathcal{C}[f_Q] \tag{3}$$

- Collision integral $\mathcal{C}[f_Q]$ directly related to scattering matrix element: $\mathcal{C} = \mathcal{C}^{2\to3} + \mathcal{C}^{2\leftrightarrow3}$ (detailed balance, LPM)
- Running coupling $\alpha_s(Q) = \alpha_s(\max\{Q, \mu\pi T\})$

Diffusion component:

$$\frac{d\vec{p}}{dt} = -\eta_D(p)\vec{p} + \vec{\xi} \tag{4}$$

- Parametrize diffusion: $\frac{1}{2}\hat{q}_{\mathrm{diffusion}} = T^3 \kappa_D \left(x_D + \frac{1-x_D}{ET} \right)$

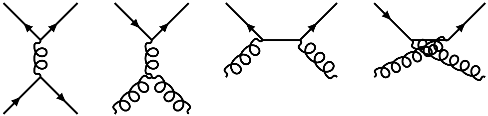
Transport coefficient: $\hat{q} = \hat{q}_{\mathrm{scatter}} + \hat{q}_{\mathrm{diffusion}}, \hat{q} \Leftrightarrow D_s 2\pi T$

HQ in-medium transport - model B

Weiyao Ke, Y.Xu, S.A.Bass, paper in preparation

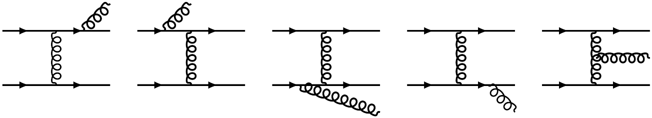
`Lido`: Linearized Boltzmann with diffusion transport model

- Elastic scattering $\mathcal{C}^{2\to2}$: $\frac{1}{t^2} \to \frac{1}{(t-m_D^2)(t-\Lambda_{\mathrm{QCD}})}$



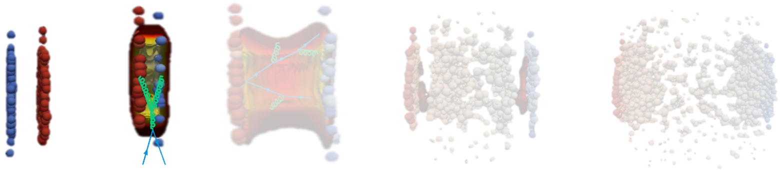
J. Uphoff, O. Fochler, Z. Xu, and C. Greiner, Phys. Rev. C 84, 024908

- Inelastic scattering $\mathcal{C}^{2\to3}$: Gunion Bertsch ($|\mathcal{M}_{2\to3}|^2 \propto |\mathcal{M}|^2_{2\to2} P^g_M$)



- Detailed balance $\mathcal{C}^{3\to2}$
- Landau-Pomeranchuk-Migdal effect: $\int \frac{dk^3}{2k} \to \int \frac{dk^3}{2k} 2 \left[1 - cos\left(\frac{t-t_0}{\tau_f}\right)\right]$, τ_f gluon formation time

Full space-time EbE evolution model



Initial conditions

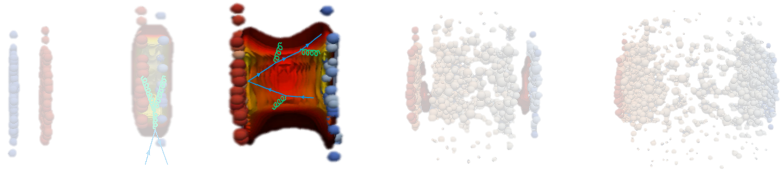
- Soft matter: T_RENTo
  - Entropy deposition proportional to empirical parametrization: $\frac{ds}{dy}\Big|_{\tau=\tau_0} \propto \sqrt{T_A T_B}$

J.S.Moreland,J.Bernhard, and S.A.Bass, Phys.Rev.C 92, 011901(2015)

- Heavy quarks
  - Momentum space: (initial hard scattering) Fixed-Order + Next-to-Leading Log (FONLL)
  - Position space: binary collision density

M.Cacciari,S.Frixione, and P.Nason, arxiv:hep-ph/0102134

Full space-time EbE evolution model



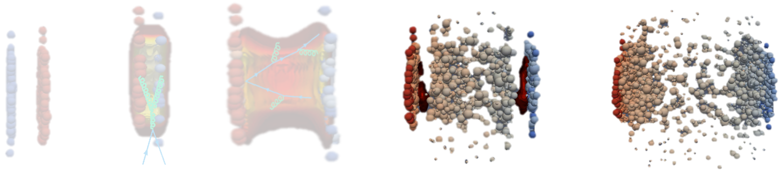
Soft medium evolution

- Event-by-event (2+1)D viscous hydrodynamic model: `iEbE-VISHNU`
  H.Song and U.W.Heinz, Phys.Rev.C 77, 064901(2008)
  - Shear and bulk viscosities: $\eta/s(T), \zeta/s(T)$
  - All the soft medium related parameters are calibrated on soft hadronic observables
    J.Bernhard,J.S.Moreland,S.A.Bass,J.Liu, and U.Heinz Phys.Rev.C 94, 024907(2015)

HQ in-medium transport

- Model A: Improved Langevin model
- Model B: `Lido` - linearized Boltzmann + diffusion model

Full space-time EbE evolution model



Hadronization/particalization

- Soft medium: particlization (hydrodynamic model $\rightarrow$ hadron gas) at $T_{\rm switch}$
- $c \rightarrow D$-meson, charmed baryons at $T_c = 154$ MeV:
  combined model of recombination and fragmentation

Hadronic re-scattering

S. A. Bass et al., Prog. Part. Nucl. Phys. 41 (1998)

M. Bleicher et al. J. Phys. G: Nucl. Part. Phys. 25 (1999)

- UrQMD: solving the Boltzmann equation of hadron scattering
- D-mesons scatter with π, ρ:
  $\pi D \rightarrow \pi D, \pi D^* \rightarrow \pi D^*, \pi D \leftrightarrow \rho D^*$
  $\rho D \rightarrow \rho D, \rho D^* \rightarrow \rho D^*, \rho D \leftrightarrow \pi D^*$

Z.-W. Lin, T. Di, and C. Ko, Nucl. Phys. A689, 965 (2001)

Model-to-data comparison

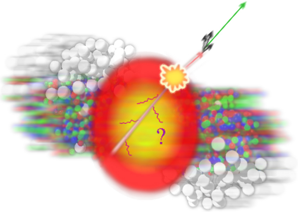


Figure: lbl.gov/Science-Articles

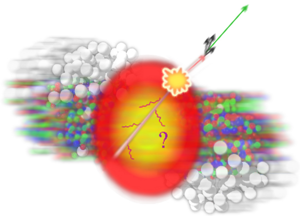
A modeling problem

Parameters
e.g. transport properties
$D_s 2\pi T(\alpha), \hat{q}(\alpha_s), \cdots$

Model
HQ transport model
Langevin, Boltzmann ...

Outputs
e.g. R_{AA}, v_2
model calculation vs.
experimental measurements

Model-to-data comparison



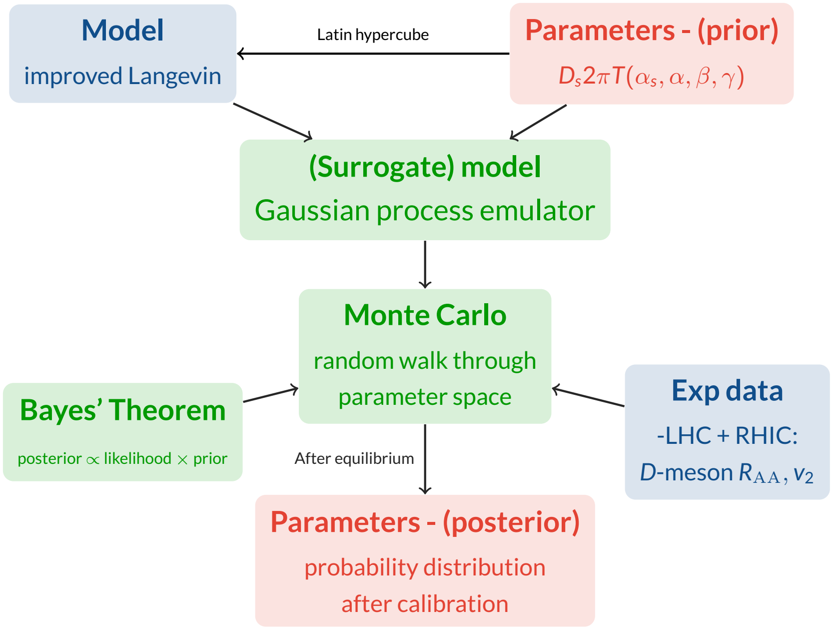
A modeling problem (data-driven)

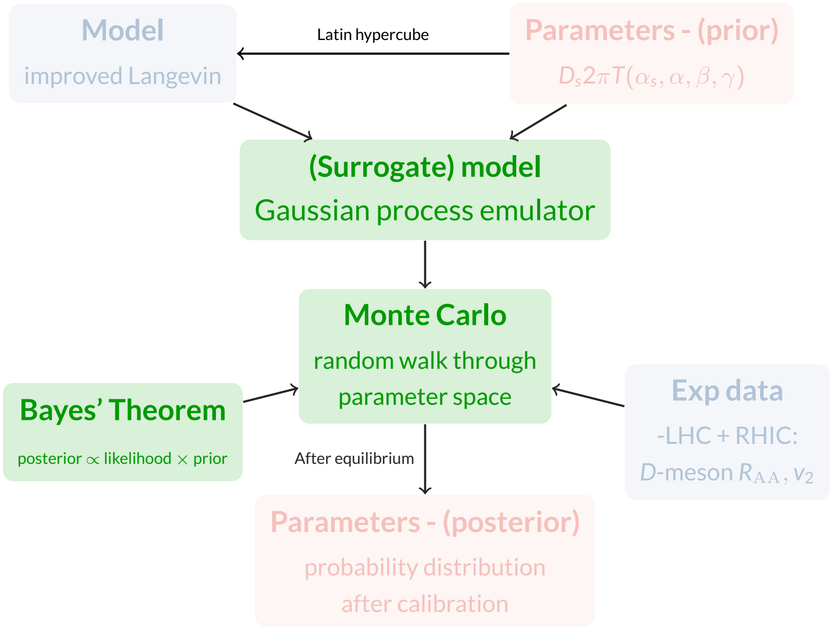
Parameters
e.g. transport properties
$D_s 2\pi T(\alpha), \hat{q}(\alpha_s), \cdots$

Bayes' Theorem

Model
HQ transport model
Langevin, Boltzmann ...

Outputs
e.g. R_{AA}, v_2
model calculation vs.
experimental measurements


Model
improved Langevin
Latin hypercube
Parameters - (prior)
$D_s 2\pi T(\alpha_s, \alpha, \beta, \gamma)$
(Surrogate) model
Gaussian process emulator
Monte Carlo
random walk through
parameter space
Bayes' Theorem
posterior $\propto$ likelihood $\times$ prior
Exp data
-LHC + RHIC:
D-meson R_{AA}, v_2
After equilibrium
Parameters - (posterior)
probability distribution
after calibration


Model
improved Langevin
Latin hypercube
Parameters - (prior)
$D_s 2\pi T(\alpha_s, \alpha, \beta, \gamma)$
(Surrogate) model
Gaussian process emulator
Monte Carlo
random walk through parameter space
Bayes' Theorem
posterior $\propto$ likelihood $\times$ prior
Exp data
-LHC + RHIC:
D-meson R_{AA}, v_2
After equilibrium
Parameters - (posterior)
probability distribution after calibration

Bayes' Theorem

$$p(\theta|y = y_{\rm exp}) \propto \mathcal{L}(y = y_{\rm exp}|\theta) \times p(\theta) \tag{5}$$

- Posterior distribution: probability of θ given observation $y_{\rm exp}$
- Likelihood: $\mathcal{L}(y = y_{\rm exp}|\theta) \propto \exp[-(y(\theta) - y_{\rm exp})\Sigma^{-1}(y(\theta) - y_{\rm exp})^T]$
- Prior distribution $P(\theta)$: prior knowledge of parameters

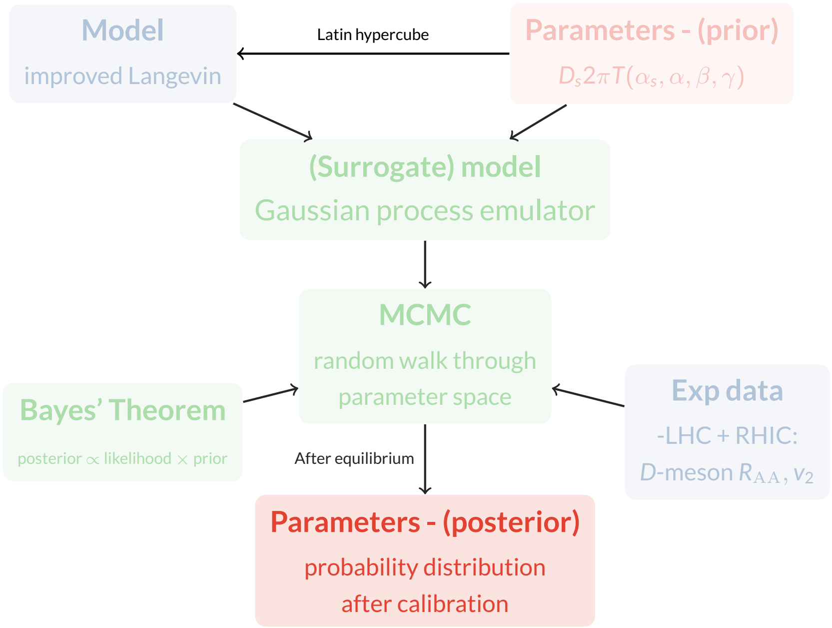
---

Gaussian process emulator

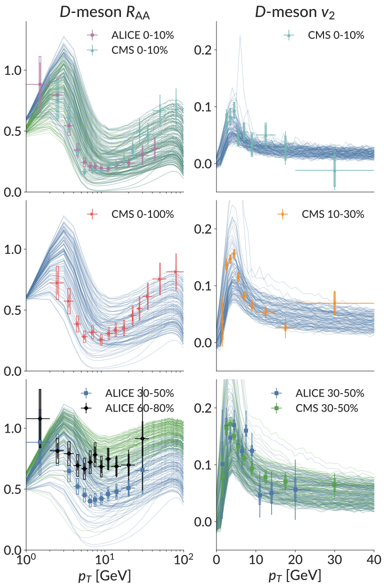
- Non-parametric regression
- Quickly predict model output given input
- Returns not only mean of prediction, but also uncertainty $\sigma_{\rm GP}$

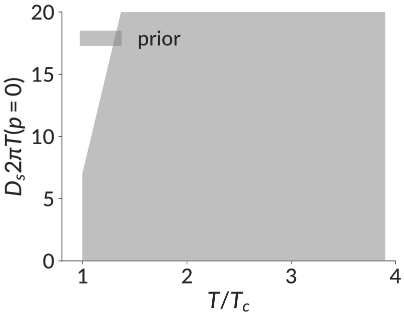
Markov chain Monte Carlo

- Random walk through the parameter space
- Accepted/rejected based on likelihood
- Posterior ensembles achieved after equilibrium


Model
improved Langevin
Latin hypercube
Parameters - (prior)
$D_s 2\pi T(\alpha_s, \alpha, \beta, \gamma)$
(Surrogate) model
Gaussian process emulator
MCMC
random walk through parameter space
Bayes' Theorem
posterior $\propto$ likelihood $\times$ prior
Exp data
-LHC + RHIC:
D-meson R_{AA}, v_2
After equilibrium
Parameters - (posterior)
probability distribution after calibration

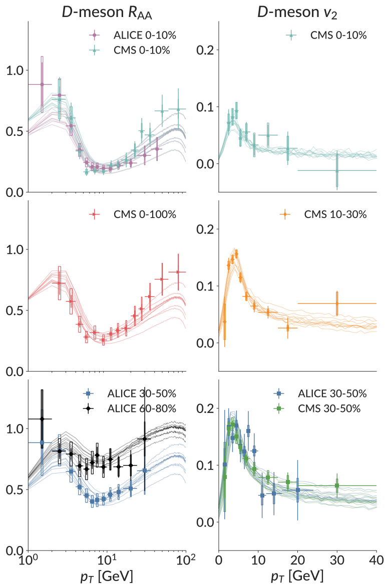
Results (model A): prior and training data

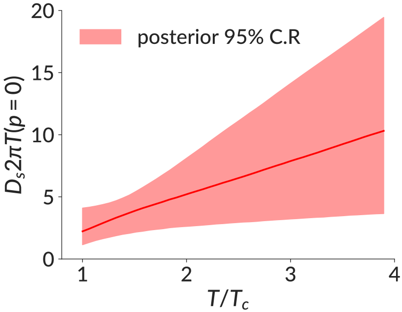




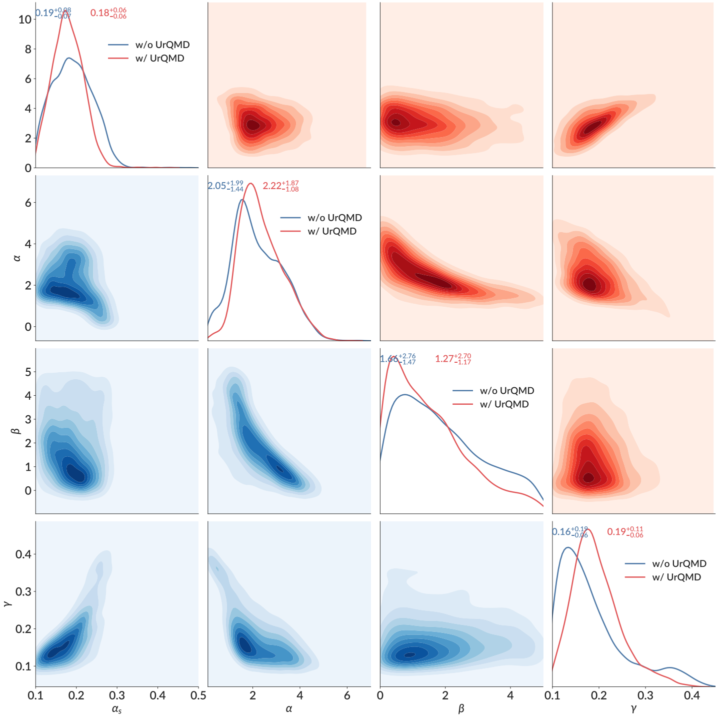
- 100 sets of parameters $\xrightarrow{(\alpha_s,\alpha,\beta,\gamma)}$ Model(θ)
- Before calibration, a wide spread due to the large range of $D_s 2\pi T$

Results (model A): posterior after calibration



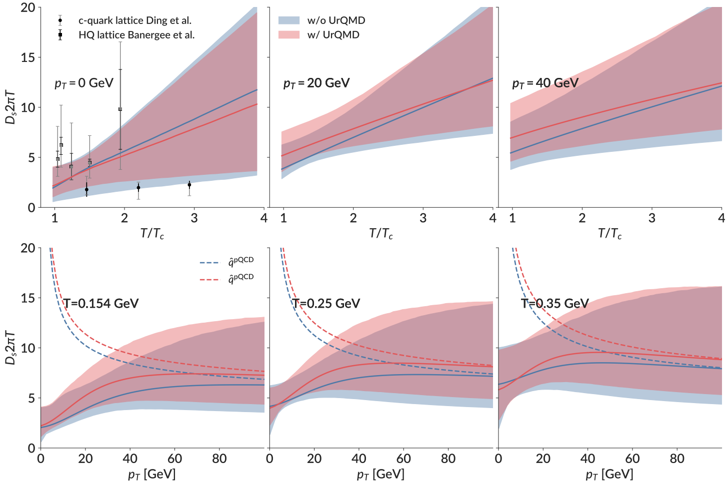


- 10 sets of parameters randomly drawn from posterior distribution
- After calibration, good description of experimental measurements



$0.19^{+0.08}_{-0.07}$
$0.18^{+0.06}_{-0.06}$
$2.05^{+1.99}_{-1.44}$
$2.22^{+1.87}_{-1.08}$
$1.66^{+2.76}_{-1.47}$
$1.27^{+2.70}_{-1.17}$
$0.16^{+0.19}_{-0.06}$
$0.19^{+0.11}_{-0.06}$
w/o UrQMD
w/ UrQMD
α_s
α
β
γ

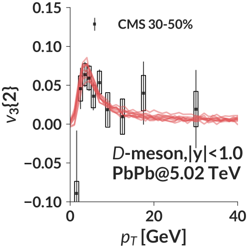
Results (model A): posterior range of $D_s 2\pi T$

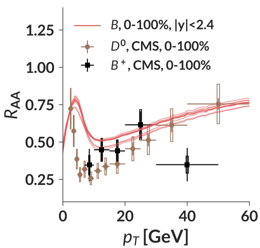


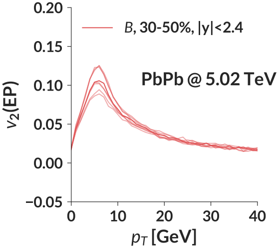
- $D_s 2\pi T$ increases with rising temperature
- Large non-perturbative contribution at low p_T
- Hadronic interaction $\rightarrow$ little impact on QGP phase D_s estimation

Results (model A): prediction beyond calibration

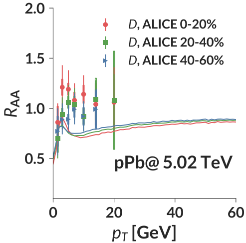
D-meson v_3, B-meson R_{AA}, v_2

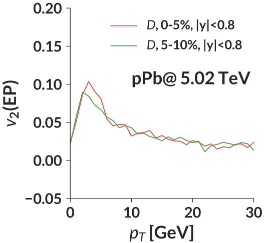






p-Pb collisions

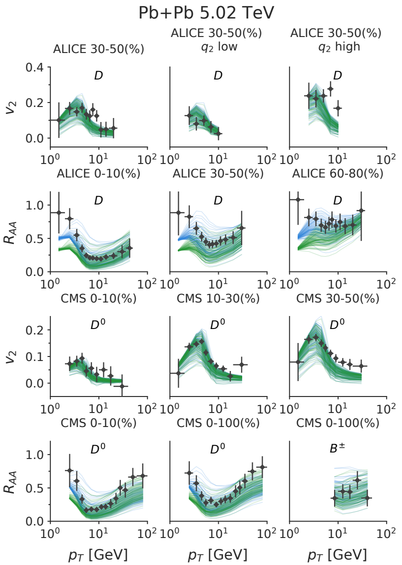




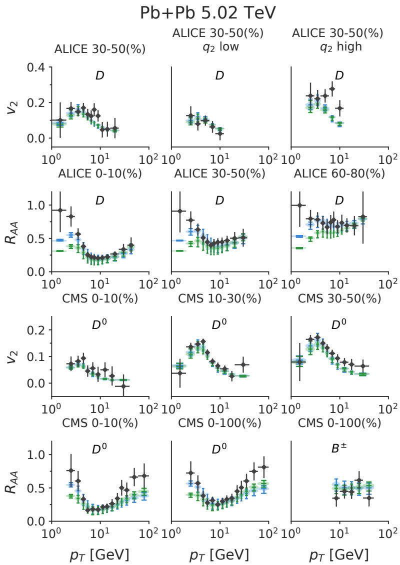
Same framework,
applied to linearized Boltzmann + diffusion model –`Lido`

Lido: before vs. after calibration

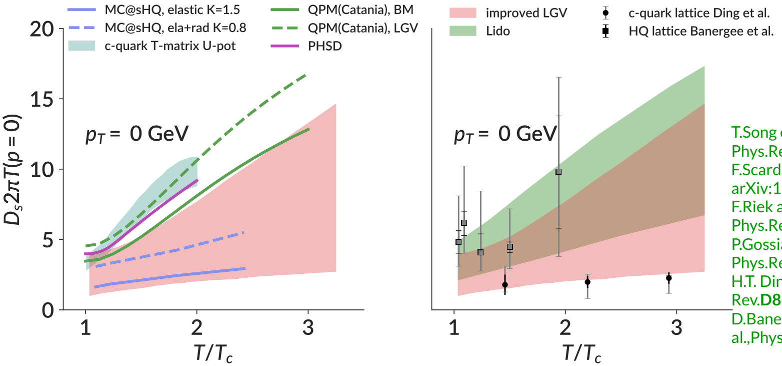
Weiyao Ke, Y.Xu, S.A.Bass, paper in preparation



$\Downarrow$ After calibration



Comparison between models



T.Song et al., Phys.Rev.C**93**,034906
F.Scardina et al. arXiv:1707.05452.
F.Riek and R.Rapp, Phys.Rev.**C82**,035201
P.Gossiaux and J. Aichelin, Phys.Rev.**C78**, 014904
H.T. Ding *et al.* Phys. Rev.**D86**,014509
D.Banerjee et al.,Phys.Rev.**D85**,014510

Different models' estimation qualitatively converging.

Remaining questions:

- What are the difference and similarity? (intrinsic dynamics: diffusion vs. scattering, radiative energy loss; medium evolution)
- Whether we can distinguish/further constrain? (novel observables, higher statistics)

Summary

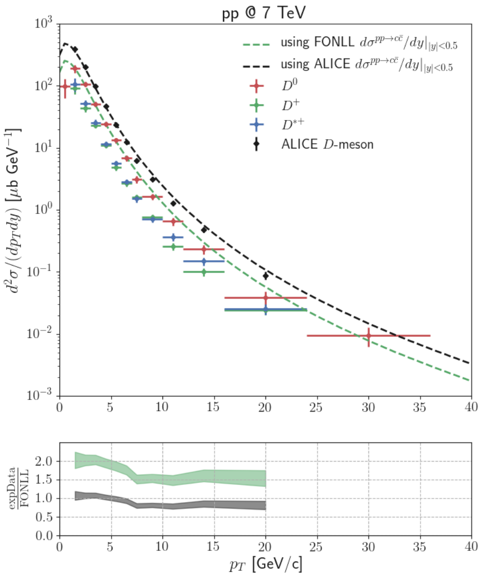
- Simultaneously described D-meson observables: improved Langevin, `Lido`
- Predicted B-meson observables and pPb collisions
- Quantified charm quark diffusion coefficient $D_s 2\pi T$ using systematic Bayesian analysis

Outlook

- Systematic comparison between different models, understanding the discrepancy between estimation
- More heavy flavor hadron observables $\rightarrow$ hadronization process

Back up

1.1 $c \to D$ @ pp 7 TeV



$$\frac{d\sigma^{pp\to D}}{dp_T} = \frac{dN^D}{\langle N_D\rangle\, dp_T} \cdot \sigma^{pp\to D}$$
$$= \frac{N^D}{N^c}\frac{dN^D}{\langle N_D\rangle\, dp_T} \cdot \sigma^{pp\to c\bar{c}}$$

pQCD calculation of diffusion coefficients

$$\langle X(\vec{p})\rangle = \frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\int \frac{d^3p_4}{(2\pi)^3 2E_4}\frac{1}{\gamma_Q}\sum |M|^2 \times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)f_{q,g}(p_2)X(\vec{p}) : \tag{6}$$

$$\lim_{p>>m_Q}\frac{d|\vec{p}|}{dt} = \langle|\vec{p}_1| - |\vec{p}_3|\rangle \approx \frac{dE}{dt} \tag{7}$$

$$\kappa_\perp = \left\langle (p_{1\perp} - p_{2\perp})^2 \right\rangle - \langle (p_{1\perp} - p_{2\perp})\rangle^2 \tag{8}$$

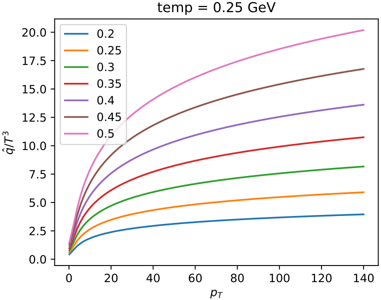
$$\kappa_{//} = \left\langle (p_{1//} - p_{2//})^2 \right\rangle - \langle (p_{1//} - p_{2//})\rangle^2 \tag{9}$$

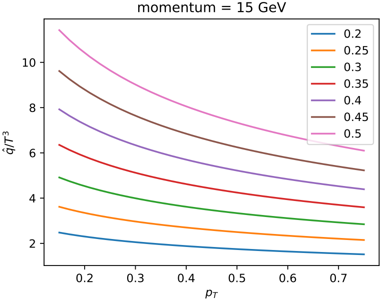
1. Parametrization

$$D_s 2\pi T(T,p) = \frac{1}{1+(\gamma^2 p)^2} D2\pi T^{\text{linear}} + \frac{(\gamma^2 p)^2}{1+(\gamma^2 p)^2} D2\pi T^{\text{pQCD}}(\alpha_s)$$

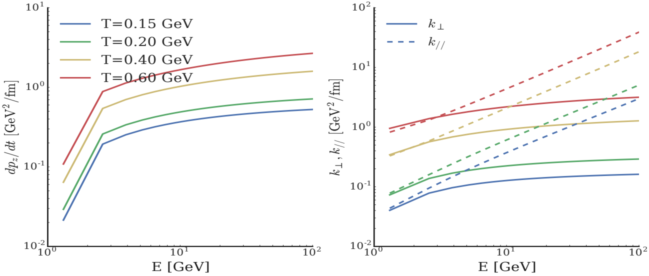
$$D2\pi T^{\text{linear}} = \alpha \cdot (1 + \beta \cdot (\frac{T}{T_c} - 1))$$

pQCD-components $\hat{q}/T^3$:

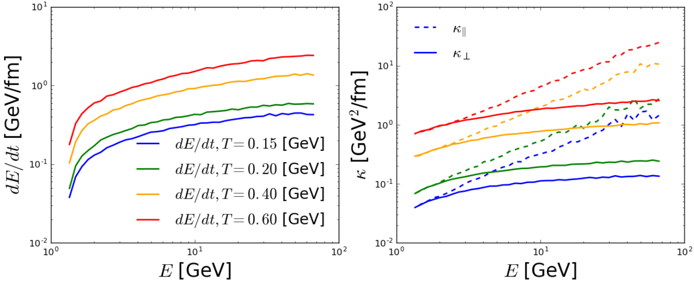




B. analytic results compared with Monte Carlo

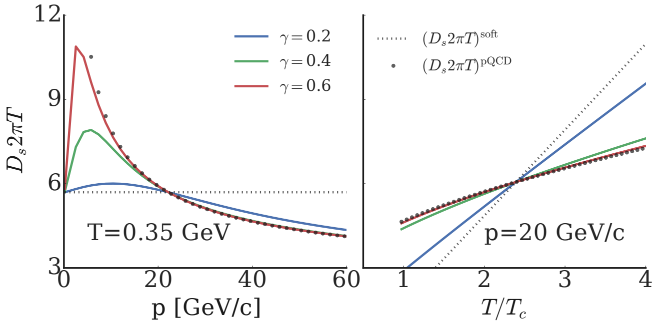


Elastic



Input: prior parameter

- An example of $D_s 2\pi T$ dependence on γ, at fixed $(\alpha, \beta) = (1.8, 1.7)$:
  - Larger γ, quicker conversion to $(D_s 2\pi T)^{\mathrm{pQCD}}$
  - $p = 1/\gamma^2$, $(D_s 2\pi T)^{\mathrm{soft}} = (D_s 2\pi T)^{\mathrm{pQCD}}$



- Initially: no knowledge/constraint for the parameters $\to$ **uniform prior distribution**
- $\alpha_s \in (0.1, 0.5)$, $\alpha \in (0, 7)$, $\beta \in (0, 5)$, $\gamma \in (0, 0.6)$

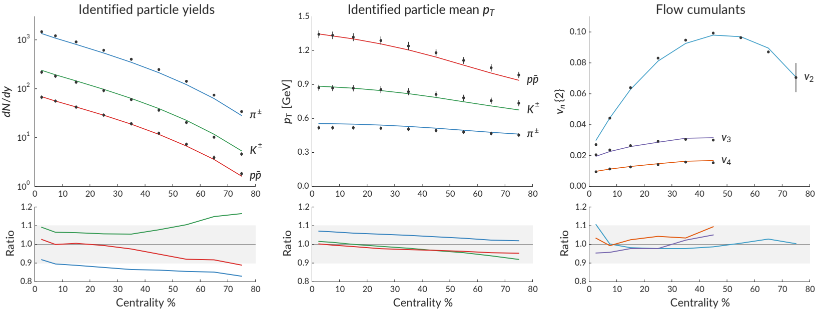
(2+1)D viscous hydro: iEbE-VISHNU

H.Song and U.W.Heinz,
Phys.Rev.C 77, 064901(2008)

- Equation of state from lattice QCD (HotQCD collaboration)
- Temperature-dependent shear + bulk vis correction

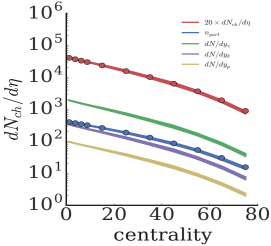
$$(\eta/s)(T) = (\eta/s)_{\mathrm{min}} + (\eta/s)_{\mathrm{slope}}(T - T_c), T_c = 154\mathrm{MeV}$$
$$(\zeta/s)(T) = (\zeta/s)_{\mathrm{norm}} \times f(T)$$

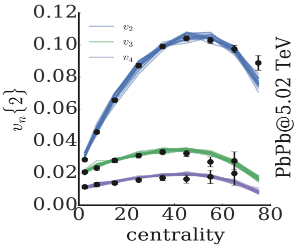
- All the initial/medium related parameters ($\mathrm{norm}, p, \eta/s$ etc.) are calibrated by Bayesian analysis with experimental data

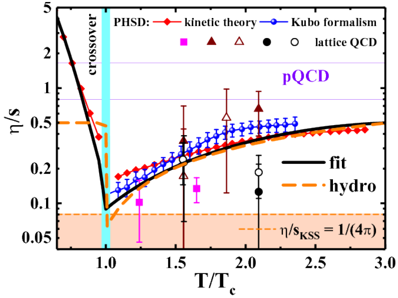


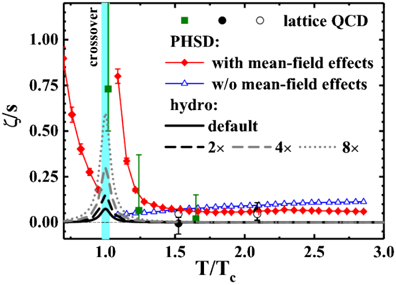
J.Bernhard,J.S.Moreland,S.A.Bass,J.Liu, and U.Heinz
Phys.Rev.C 94, 024907(2015)

- Medium background









Matrix elements

$$
\begin{aligned}
|\overline{\mathcal{M}}_{gQ\to gQ}|^2 = 16\pi^2 \Bigg[& 2\alpha_s^2(t)\frac{(s-M^2)(M^2-u)}{[t-\kappa_t\, m_D^2(\alpha_s(t))]^2} \\
& + \frac{4}{9}\alpha_s^2(s-M^2)\frac{(s-M^2)(M^2-u)+2M^2(s+M^2)}{[s-M^2+m_q^2(\alpha_s(s-M^2))]^2} \\
& + \frac{4}{9}\alpha_s^2(u-M^2)\frac{(s-M^2)(M^2-u)+2M^2(u+M^2)}{[M^2-u+m_q^2(\alpha_s(u-M^2))]^2} \\
& + \frac{1}{9}\alpha_s(s-M^2)\alpha_s(u-M^2) \\
& \quad \times \frac{M^2(4M^2-t)}{[s-M^2+m_q^2(\alpha_s(s-M^2))][M^2-u+m_q^2(\alpha_s(u-M^2))]} \\
& + \alpha_s(t)\alpha_s(s-M^2)\frac{(s-M^2)(M^2-u)+M^2(s-u)}{[t-\kappa_t\, m_D^2(\alpha_s(t))][s-M^2+m_q^2(\alpha_s(s-M^2))]} \\
& - \alpha_s(t)\alpha_s(u-M^2)\frac{(s-M^2)(M^2-u)-M^2(s-u)}{[t-\kappa_t\, m_D^2(\alpha_s(t))][M^2-u+m_q^2(\alpha_s(u-M^2))]}\Bigg] .
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
|\overline{\mathcal{M}}_{qQ\to qQg}|^2 &= 4g^2\,\frac{C(N)^2\,C_2(G)\,(N^2-1)}{N^2}\,\left|\overline{M}_0^{qQ}\right|^2\,(1-x)^2 \\
&\quad \times \left[\frac{\mathbf{k}_\perp}{k_\perp^2+x^2M^2}+\frac{\mathbf{q}_\perp-\mathbf{k}_\perp}{(\mathbf{q}_\perp-\mathbf{k}_\perp)^2+x^2M^2}\right]^2 \\
&= 12g^2\,\left|\overline{\mathcal{M}}_{qQ\to qQ}\right|^2\,(1-x)^2\left[\frac{\mathbf{k}_\perp}{k_\perp^2+x^2M^2}+\frac{\mathbf{q}_\perp-\mathbf{k}_\perp}{(\mathbf{q}_\perp-\mathbf{k}_\perp)^2+x^2M^2}\right]^2
\end{aligned} \tag{3.63}
$$

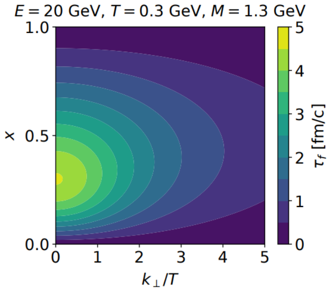
LPM effect

- Multiple collisions during gluon formation time τ_f,

$$\tau_f = \frac{2x(1-x)E}{k_\perp^2 + x^2M^2 + (1-x)m_g^2}$$

Destructive interference $\rightarrow$ suppress gluon radiation.

- τ_f can be $\gg$ mean free path.
- A non-local task for transport model.
- In principal, we should go beyond point-like interaction picture. (to be improved)



Hadronization: recombination probability

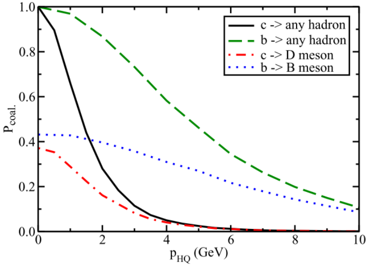
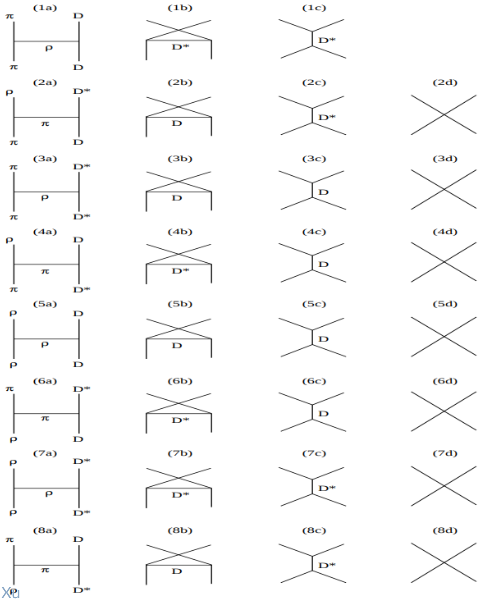
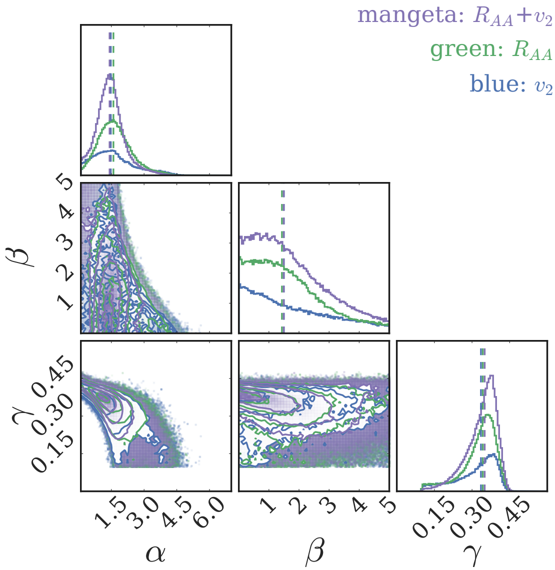


FIGURE 5.1: The coalescence probabilities for heavy-light quarks as a functions of the heavy quark momentum.

Hadronic stage scattering



R_{AA} vs. v_2

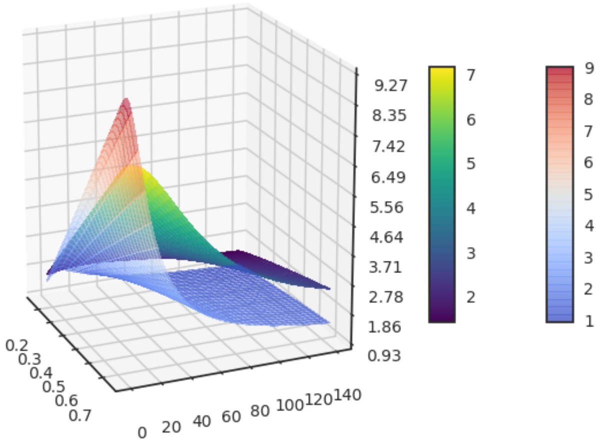


The ability to constrain parameters:

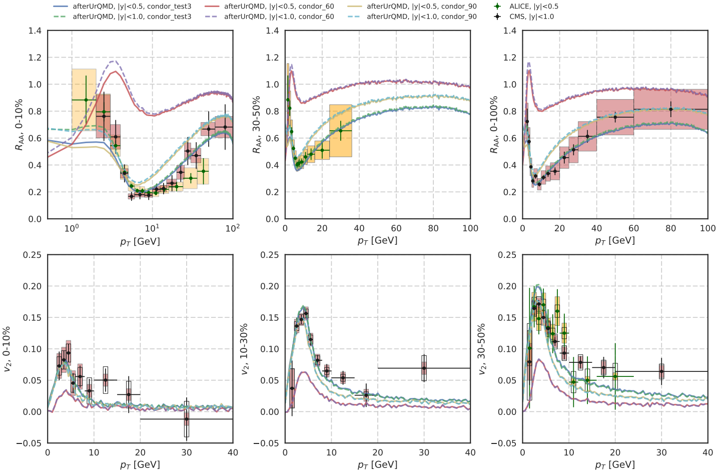
- its uncertainty
- the information it constrains

Maybe slightly better for R_{AA} in terms of (α, β, γ) posterior distribution, though not much of the difference from the posterior range of $D_s 2\pi T$; but **more is better**!

4. $D_s 2\pi T$ posterior distribution



1. Rapidity dependence?

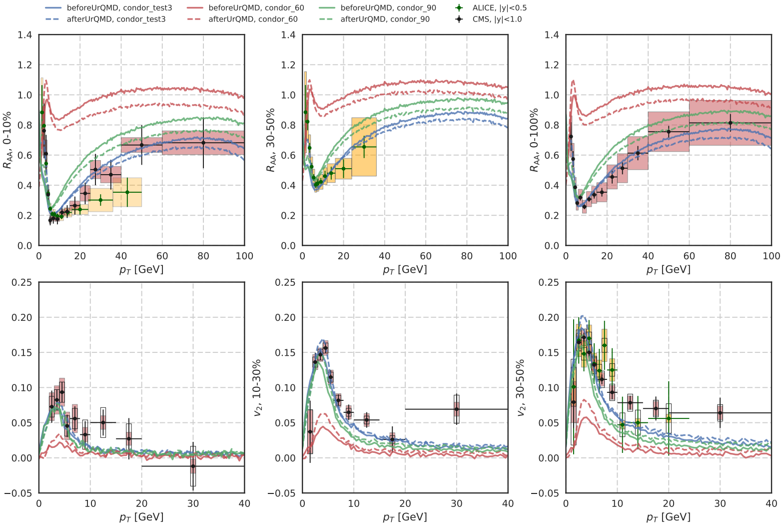


Difference between $|y| < 0.5$ vs. $|y| < 1$, at low $p_T < 5$ GeV/c

2. Hadronic interaction/decay effects

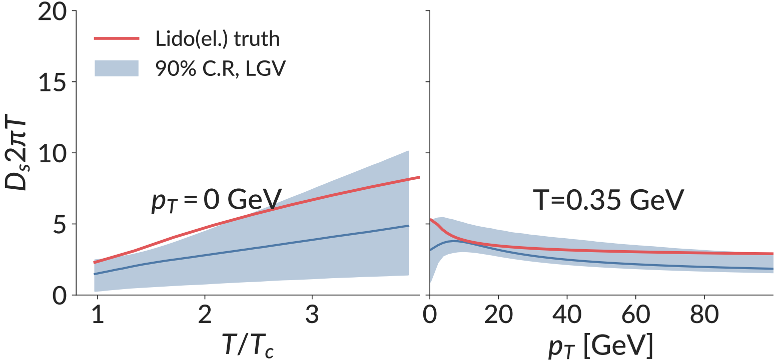
Hadronic interaction (including UrQMD) slightly increases anisotropy, especially at low p_T.

And, larger suppression for high p_T D-meson. D^* decay



Calibrate model A to model B

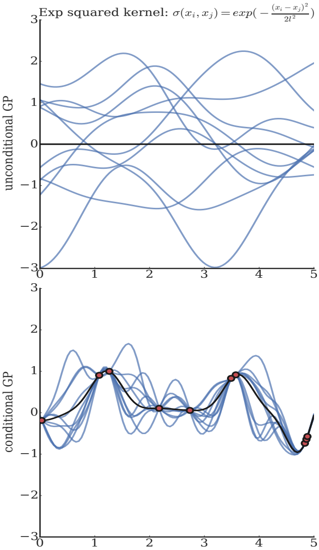
Scenario 2: collisional energy loss only



III: Gaussian process emulator

Gaussian process

- A collection of random variables, which have a joint Gaussian distribution
- Map inputs to normally-distributed outputs
$$\vec{y} \sim \mathcal{N}(\vec{\mu}, \sigma(X)) \tag{10}$$
- Only need to be specified by mean and covariance functions, for example:
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mu(\vec{x}_1) \\ \mu(\vec{x}_2) \end{pmatrix}, \begin{pmatrix} \sigma(\vec{x}_1, \vec{x}_1), & \sigma(\vec{x}_1, \vec{x}_2) \\ \sigma(\vec{x}_2, \vec{x}_1), & \sigma(\vec{x}_2, \vec{x}_2) \end{pmatrix}\right]$$
mean $\vec{\mu} = \vec{0}, \sigma(\vec{x}_1, \vec{x}_2) = \delta^2 \exp\left(-\frac{(\vec{x}_1 - \vec{x}_2)^2}{2\vec{\beta}^2}\right)$
- Given$(\vec{y}, X)$GP parameters (hyper-parameters) $\delta^2, \vec{\beta}$ can be estimated



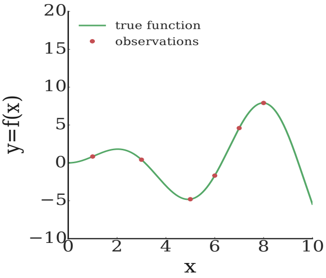
III: Gaussian process emulator

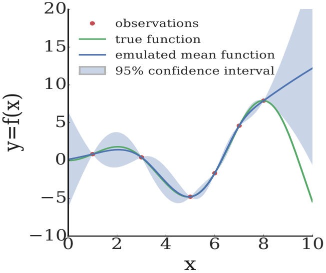
GP as emulator

- Physics model: simulator $y = f(x) + \epsilon$

$$\begin{pmatrix} x_{11} & \dots & x_{1m} \\ \dots & & \dots \\ x_{n1} & \dots & x_{nm} \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} \tag{11}$$

- GP emulator: given a dataset $(X, \vec{y})$, approximation of the simulator + probabilistic prediction
- This work: covariance(include a noise term)
$\sigma(\vec{x}, \vec{x}') = \sigma^2_{\rm GP} \exp \left[-\sum_{k=1}^{m} \frac{(x_k - x_k')^2}{2 l_k^2} \right] + \sigma_n^2 \delta_{\vec{x}, \vec{x}'}$
- Maximize the evidence $\log P(y_* | X, Y, \vec{x}_*) = -\frac{1}{2} Y^T \Sigma^{-1}(X, \vec{x}_*) Y - \frac{1}{2} \log |\Sigma(X, \vec{x}_*)| - N/2 \log(2\pi)$





III: Principal component analysis

Multiple observables, correlated?

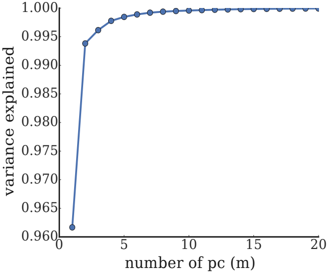
$$\begin{pmatrix} x_{11} & \dots & x_{1m} \\ \dots & & \dots \\ x_{n1} & \dots & x_{nm} \end{pmatrix} \Rightarrow \begin{pmatrix} y_{11} & \dots & y_{1k} \\ \dots & & \dots \\ y_{n1} & \dots & y_{nk} \end{pmatrix} \quad (12)$$

- Decompose into orthogonal linear principal components
- Singular value decomposition: n sets of k-dimension observables $\Rightarrow$ n sets of l-dimensional PCs Z

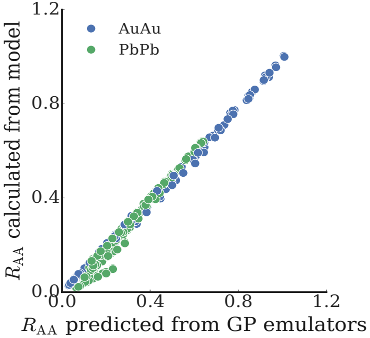
$$Y_{kn} = U_{kl} S_{ll} V_{ln}^T \quad (13)$$

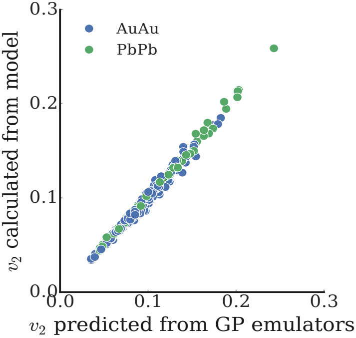
$$Z = \sqrt{n} Y V \quad (14)$$

- Eigenvalue λ_i of Y represents the variance explained by PC $\vec{z}_i$



Another 10 sets of validation inputs $\vec{x}$
Compare between physics model calculation $\vec{y}$ and GP emulator predicted $\vec{y}_{\mathrm{pred}}$

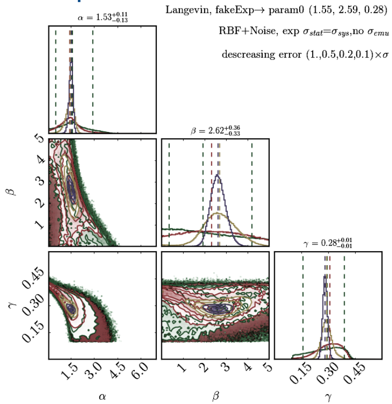


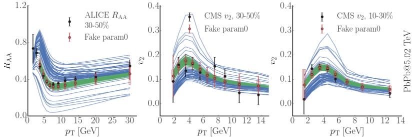


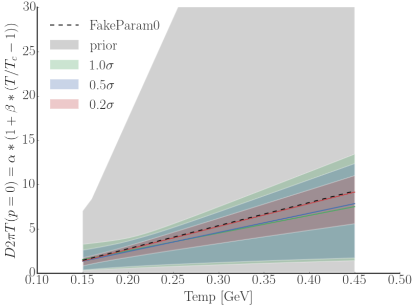
1. Higher precision experimental data?

Question: If there is such "the parameters" exists, how precise the data need to be to recover it? (Exam on **PbPb 5020 GeV**)

test parameter 0:



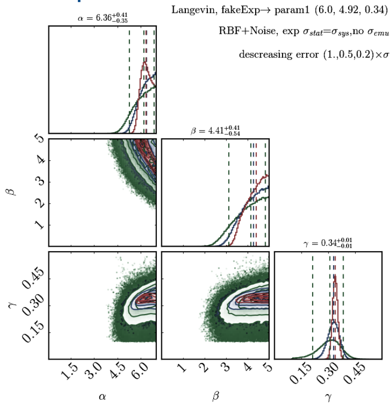


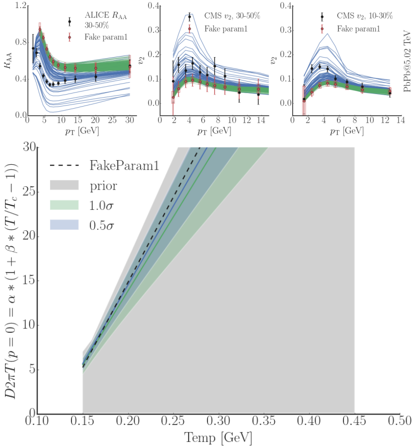


We need about 0.2 of current σ to recover "The fake parameter"

Question: If there is such "the parameters" exists, how precise the data need to be to recover it?(Exam on **PbPb 5020 GeV**)

test parameter 1:





Question: If there is such "the parameters" exists, how precise the data need to be to recover it?(Exam on **PbPb 5020 GeV**)

test parameter 2: