The main results of this section are proven in [26]. Closely related work includes [17], regarding compactness of the solution space, and [82], [9], and [12], regarding the dependence of solutions on initial conditions.

Basic Assumptions

The Basic Assumptions are the following three conditions on the data (C, F, D, G) of a hybrid system:

A1) C and D are closed sets in $\mathbb{R}^n$.

A2) $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on C, and such that $F(x)$ is nonempty and convex for each $x \in C$.

A3) $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on D, and such that $G(x)$ is nonempty for each $x \in D$.

A set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n,\ y \in F(x)\} \subset \mathbb{R}^{2n}$ is closed. In terms of set convergence (see "Set Convergence"), F is outer semicontinuous if and only if, for each $x \in \mathbb{R}^n$ and each sequence $x_i \to x$, the outer limit $\limsup_{i \to \infty} F(x_i)$ is contained in $F(x)$. The mapping F is *locally bounded* on a set C if, for each compact set $K \subset C$, $F(K)$ is bounded. If F is locally bounded and $F(x)$ is closed for each $x \in \mathbb{R}^n$, then F is outer semicontinuous if and only if, for each x and $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x + \delta\mathbb{B}) \subset F(x) + \varepsilon\mathbb{B}$. A continuous function $f: C \to \mathbb{R}^n$, where C is closed, can be viewed as a set-valued mapping whose values on C consist of one point and are the empty set outside of C. Then, f is locally bounded on C and outer semicontinuous.

The Basic Assumptions combine what is typically assumed, in continuous- and in discrete-time systems, to obtain the continuous- and discrete-time versions of the results we state below for hybrid systems. "Robustness and Generalized Solutions" provides further motivation for introducing these assumptions.

Graphical Convergence and Sequential Compactness of the Space of Solutions

Solutions to differential equations and inclusions are continuous functions parameterized by $t \in \mathbb{R}_{\geq 0}$, and, thus, uniform distance and uniform convergence are adequate tools for analyzing them. To analyze solutions to hybrid systems, the more elaborate yet intuitive concepts of graphical distance and graphical convergence are needed. Before discussing these concepts, we briefly illustrate difficulties in using uniform distance for discontinuous functions.

Example 3: Bouncing Ball and the Uniform Distance Between Solutions

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x : x_1 \geq 0\}, \qquad f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x : x_1 = 0,\ x_2 \leq 0\}, \qquad g(x) := -\rho x \quad \text{for all } x \in D,$$

which is the bouncing ball system considered in Example S4 of "Existence, Uniqueness, and Other Well-Posedness Issues." We use $\gamma = 1$ and $\rho = 1/2$ in the calculations of this example.

Suppose that we abandon hybrid time domains and view the trajectories of the bouncing ball system as piecewise continuous functions parameterized by t only. For example, given $\delta \in [0, 1)$, the velocity resulting from dropping the ball from height $1 + \delta$ with initial velocity zero is given on the interval $t \in [0, 2]$ by

$$x_2^{\delta}(t) = \begin{cases} -t, & t \in [0, \sqrt{2(1+\delta)}), \\ -t + 3\sqrt{(1+\delta)/2}, & t \in [\sqrt{2(1+\delta)}, 2]. \end{cases}$$

See Figure 14. The uniform, that is, L_∞, distance between the velocities of two balls, one dropped from height 1 and the other from $1 + \delta$ with $\delta > 0$, is given by

$$\sup_{t \in [0,2]} |x_2^0(t) - x_2^{\delta}(t)| = 3\sqrt{2}/2$$

and does not decrease to zero as δ decreases to zero. In particular, the velocity of a ball dropped from height $1 + \delta$ does not converge uniformly, as δ decreases to zero, to the velocity of the ball dropped from height one. In other words, the velocity of the ball does not depend continuously on initial conditions, when uniform distance is used. On the intuitive level though, velocities and in fact the whole trajectories of balls dropped from nearby initial conditions appear close to one another. ■

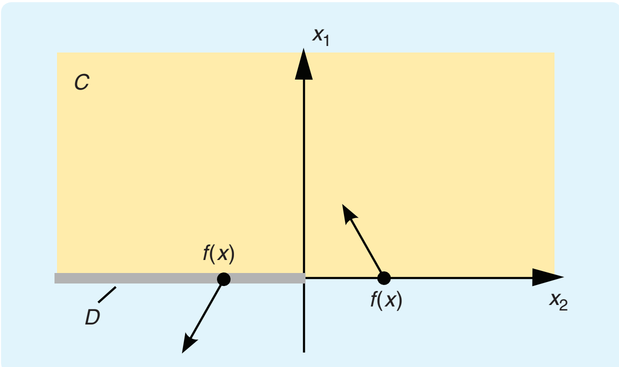

FIGURE 12 Flow and jump sets for the bouncing ball system in Example 3. The state x_1, represented on the vertical axis, is the height of the ball. The state x_2, represented on the horizontal axis, is the velocity of the ball. The flow set C is the closed upper half-plane. The flow map is shown at two points on the x_2-axis. Flow is possible from the point where the flow map is directed into the flow set but is not possible from the point where the flow map is directed out of the flow set. At the latter point, the flow map and the tangent cone to the flow set do not intersect, which makes flow impossible. Since the jump set D contains the latter point, a jump, that is, a bounce of the ball, is possible.

Existence, Uniqueness, and Other Well-Posedness Issues

The term "well-posed" for a mathematical problem usually implies that a solution exists, is unique, and depends continuously on the data of the problem. For hybrid dynamical systems, there are reasons to consider models that do not have solutions from some initial conditions, that do not have unique solutions, and that exhibit only a semicontinuous dependence on the data of the problem. Issues related to semicontinuous dependence on data are discussed in the main text. Here we discuss issues related to existence and uniqueness.

EXISTENCE OF SOLUTIONS AND BEHAVIOR OF MAXIMAL SOLUTIONS

At a point on the boundary of the flow set, if the flow map points out of the flow set then the hybrid system can fail to have a nontrivial solution, that is, a solution x such that dom x contains at least one point different from (0, 0). In the main text, we consider systems where existence of a nontrivial solution can fail at some points since this flexibility can be helpful in stability analysis. Indeed, for system (15), there are points on the boundary of C_i that do not admit a nontrivial solution. Nevertheless, the behavior of the nontrivial solutions to (15) are used to draw stability conclusions about mode-switching control for a disk drive system. For a further discussion of the role of existence in stability analysis, see "Why 'Pre'-Asymptotic Stability?"

To establish the existence of solutions, the following theorem points out that the existence of nontrivial solutions from a point $\xi \in C \cup D$ amounts to the existence of a discrete-time nontrivial solution or a continuous-time nontrivial solution.

Proposition S2

Suppose that $\mathcal{H}$ satisfies the Basic Assumptions and that $\xi \in \mathbb{R}^n$ is such that either $\xi \in D$ or there exists a nontrivial solution z to $\dot{z} \in F(z)$, that is, an absolutely continuous function $z:[0,\varepsilon] \to \mathbb{R}^n$ with $\varepsilon > 0$ satisfying $\dot{z}(t) \in F(z(t))$ for almost all $t \in [0,\varepsilon]$, such that $z(0) = \xi$ and $z(t) \in C$ for all $t \in (0,\varepsilon]$. Then there exists a nontrivial solution x to $\mathcal{H}$, with $x(0,0) = \xi$.

Indeed, if $\xi \in D$, then $G(\xi)$ is nonempty and $x(0,0) = \xi$, $x(0,1) \in G(\xi)$ provides a nontrivial solution to $\mathcal{H}$ with dom $x = (0,0) \cup (0,1)$. If there exists a nontrivial solution z to $\dot{z} \in F(z)$, as described in the assumption of Proposition S2, then $x(t,0) = z(t)$ provides a nontrivial solution to $\mathcal{H}$ with dom $x = [0,\varepsilon] \times \{0\}$.

Viability theory for differential inclusions provides sufficient conditions on F and C for flowing solutions to exist. One simple condition involves tangent cones to the set C at points near ξ.If C is closed, $\xi \in C$, and there exists a neighborhood U of ξ such that, for each $\eta \in U \cap C$, $F(\eta) \cap T_C(\eta) \neq \emptyset$, then there exist $\varepsilon > 0$ and $z:[0,\varepsilon] \to \mathbb{R}^n$ such that $z(0) = \xi$, $\dot{z}(t) \in F(z(t))$, and $z(t) \in C$ for almost all $t \in (0,\varepsilon]$. For details, see [S16, Prop. 3.4.2]. Figure S5 depicts a flow map F and the tangent cone to a given set C at several points $\xi \in C$. Additional discussion about existence of solutions to hybrid systems appears in [S17] and [51].

Existence of nontrivial solutions from each initial condition in $C \cup D$ has bearing on the structure of maximal solutions.

Theorem S3

Suppose that $\mathcal{H}$ satisfies the Basic Assumptions and, for every $\xi \in C \cup D$, there exists a nontrivial solution to $\mathcal{H}$ starting from ξ. Let x be a maximal solution to $\mathcal{H}$. Then exactly one of of the following three cases holds:

a) x is complete.

b) x blows up in finite (hybrid) time. In other words, $J = \max\{j : \text{there exists } t \text{ such that } (t,j) \in \operatorname{dom} x\}$ and $T = \sup\{t : (t,J) \in \operatorname{dom} x\}$ are both finite, the interval $\{t : (t,J) \in \operatorname{dom} x\}$ has nonempty interior, is open to the right so that $(T,J) \notin \operatorname{dom} x$, and $|x(t,J)| \to \infty$ when $t \to T$.

c) x eventually jumps out of $C \cup D$. In other words, $(T,J) \in \operatorname{dom} x$ and $x(T,J) \notin C \cup D$, where T and J defined in (b) are finite.

Note the lack of symmetry between continuous time and discrete time in b) and c) above. Finite-time blowup of a solution to a hybrid system cannot result from jumping, while a solution ending up outside $C \cup D$ cannot result from flowing. Consequently, no solutions can leave $C \cup D$ when $G(D) \subset C \cup D$. Finite-time blowup is excluded when, for example, C is bounded

We now consider the concept of graphical convergence of hybrid arcs along with the related concept of distance, which focus not just on the values of the hybrid arcs but on their graphs. One benefit of this approach is that different hybrid time domains can be handled easily. Note that bouncing balls dropped from different initial heights lead to different hybrid time domains of the hybrid arcs representing their heights and velocities. For example, the hybrid time domain of the trajectory of the bouncing ball dropped from height $1+\delta$, where $\delta \in [0,1)$, depends on δ since the time at which the jump occurs depends on the initial height. In fact, for $t \le 2$, the hybrid time domain is given by $[0, \sqrt{2(1+\delta)}] \times \{0\} \cup [\sqrt{2(1+\delta)}, 2] \times \{1\}$.

The *graph* of a hybrid arc x is the set

$$\operatorname{gph} x = \{(t,j,\xi) : (t,j) \in \operatorname{dom} x, \xi = x(t,j)\}.$$

The sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs *converges graphically* if the sequence $\{\operatorname{gph} x_i\}_{i=1}^{\infty}$ of graphs converges in the sense of set convergence; see "Set Convergence." The *graphical limit* of a graphically convergent sequence or there exists $c \in \mathbb{R}_{\geq 0}$ such that, for each $x \in C$ and each $f \in F(x)$, $|f| \leq c(|x| + 1)$.

Example S4: Bouncing Ball and Existence of Solutions

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x : x_1 \geq 0\}, \quad f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \text{for all } x \in C,$$
$$D := \{x : x_1 = 0,\ x_2 \leq 0\}, \quad g(x) := -\rho x \quad \text{for all } x \in D,$$

where $\rho \in [0, 1)$ is the restitution coefficient and $\gamma > 0$ is the gravity constant. This data models a ball bouncing on a floor. The state x_1 corresponds to height above the floor and x_2 corresponds to vertical velocity. Figure 12 depicts the flow and jump sets as well as the flow map at two points. The Basic Assumptions are satisfied. To verify sufficient conditions for the existence of nontrivial solutions from each point in $C \cup D$, it is enough to show that $f(\xi) \in T_C(\xi)$ for each $\xi \in C \setminus D$. For all $\xi \in C$ such that $\xi_1 > 0$, $T_C(\xi) = \mathbb{R}^2$. Consequently, for $\xi \in C \setminus D$ with $\xi_1 > 0$, $f(\xi) \in T_C(\xi)$ trivially holds. For all $\xi \in C$ with $\xi_1 = 0$, $T_C(\xi) = \mathbb{R}_{\geq 0} \times \mathbb{R}$, that is, the tangent cone is the right-half plane. For $\xi \in C \setminus D$ with $\xi_1 = 0$ we also have $\xi_2 > 0$, and consequently $f(\xi) \in T_C(\xi)$ holds. In summary, the assumption of Proposition S2 holds for every point $\xi \in C \cup D$, and nontrivial solutions to the hybrid system exist from each such point. Note though that $f(\xi) \notin T_C(\xi)$ for $\xi \in C \cap D$.

Additional arguments are needed to show that all maximal solutions are complete. Since $g(D) \subset C \cup D$, solutions do not jump out of $C \cup D$. Additional arguments, carried out when Example S4 is revisited in the section "Asymptotic Stability," show that all solutions are bounded and, hence, they do not blow up in finite time. Consequently, all maximal solutions are complete.

Solutions to the bouncing ball model exhibit Zeno behavior, as discussed in "Zeno Solutions." Simulations for the model are given in "Simulation in Matlab/Simulink." ■

UNIQUENESS OF SOLUTIONS

In dynamical systems, nonunique solutions can arise. One physically-motivated model that exhibits nonunique solutions is the differential equation corresponding the reverse-time evolution of a leaky bucket [S21, Example 4.2.1].

Nonuniqueness can occur in mathematical models that are designed to generate all possible solutions that meet certain conditions. For example, all Lipschitz continuous functions with Lipschitz constant equal to 1 are generated by the differential inclusion $\dot{x} \in [-1, 1]$ where $x \in \mathbb{R}$.

Additionally, consider a nonlinear control system $\dot{x} = f(x, u)$, where f is Lipschitz continuous and $u = \kappa(x)$ is a possibly discontinuous feedback. The family of locally absolutely continuous functions that arise as the limit of a sequence of solutions x_i to $\dot{x}_i = f(x_i, \kappa(x_i + e_i))$, $x_i(0) = x_0$ with e_i measurable and $\lim_{i \to \infty} \sup_{t \geq 0} |e_i(t)| = 0$ is equivalent to the family of solutions to the differential inclusion $\dot{x} \in F(x)$, $x(0) = x_0$ where $F(x) = \bigcap_{\delta > 0} \overline{\text{con}} f(x, \kappa(x + \delta\mathbb{B}))$ [S20]. The signals e_i can be associated with arbitrarily small measurement noise in the control system. A version of this result for hybrid systems is

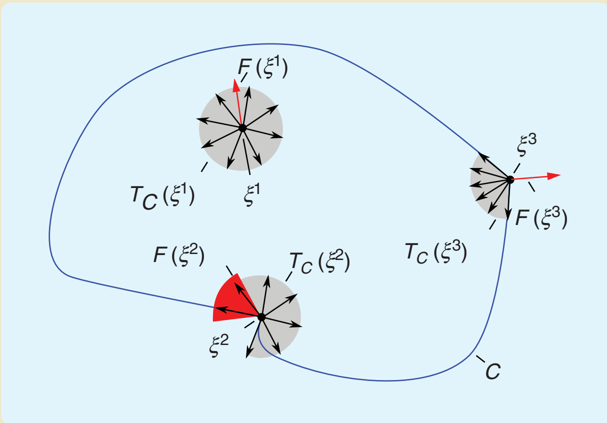

FIGURE S5 A flow map F and the tangent cone to a set C represented at several points $\xi \in C$. Directions in the flow map, single-valued at ξ^1, ξ^3 and set-valued at ξ^2, are shown in red. Tangent cones are represented in gray, sample directions in the tangent cones are in black. At points in the interior of C, such as ξ^1, the tangent cone is the entire space. At ξ^1 and ξ^2, the intersection between the flow map and the tangent cone is nonempty, whereas at ξ^3 this intersection is empty.

of hybrid arcs $\{x_i\}_{i=1}^{\infty}$, defined as the set-valued mapping whose graph is the limit of graphs of arcs x_i, need not be a hybrid arc. However, if the graphically convergent sequence is locally bounded and consists of solutions to a hybrid system that satisfies the Basic Assumptions, then the graphical limit is always a hybrid arc, and in fact, a solution to the hybrid system. The following result, given in [69, Thm. 5.36] and [26, Lem. 4.3], states the sequential compactness of the space of solutions to a hybrid system that satisfies the Basic Assumptions.

Theorem 4

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of solutions to a hybrid system $\mathcal{H}$ meeting the Basic Assumptions. Suppose that the sequence $\{x_i\}_{i=1}^{\infty}$ is locally uniformly bounded in the sense that, for each $\tau > 0$, there exists a compact set $K_\tau \subset \mathbb{R}^n$ such that, for each $i = 1, 2, \ldots$ and each $(t, j) \in \text{dom}\, x_i$ with $t + j \leq \tau$, it follows that $x_i(t, j) \in K_\tau$. Then the sequence $\{x_i\}_{i=1}^{\infty}$ has a graphically convergent subsequence; moreover, if the sequence $\{x_i\}_{i=1}^{\infty}$ is graphically convergent, then its graphical limit is a hybrid arc that is a solution to the hybrid system $\mathcal{H}$.

discussed in "Robustness and Generalized Solutions." The solutions to $\dot{x} \in F(x)$, $x(0) = x_0$ can be nonunique. For example, for a system with $x \in \mathbb{R}$, $u \in \mathbb{R}$, $f(x, u) = u$, $\kappa(x) = 1$ for $x \geq 0$, and $\kappa(x) = -1$ for $x < 0$, we get $F(0) = [-1, 1]$, $F(x) = 1$ for $x > 0$, and $F(x) = -1$ for $x < 0$. Thus, for $x_0 = 0$, there are multiple solutions. One solution is $x(t) = 0$ for all $t \geq 0$. Additional solutions are $x(t) = 0$ for $t \in [0, \bar{t}]$, $x(t) = (t - \bar{t})$ for $t > \bar{t} \geq 0$ or $x(t) = -(t - \bar{t})$ for $t > \bar{t} \geq 0$.

Nonunique solutions arise when developing a model for switched systems that generates all solutions under arbitrary switching among a finite set of locally Lipschitz vector fields. This set of solutions is equivalent to the set of solutions to a differential inclusion where the set-valued right-hand side at a point is equal to the union of the vector fields at that point. See [S18, Chapter 4]. For this differential inclusion, solutions are not unique. Each vector field produces a solution. Additional solutions are produced by following one vector field for some amount of time, switching to a different vector field for some amount of time, and so on.

A similar situation arises when modeling switched systems under a restricted class of switching signals. In this case, the switched system can be modeled by an autonomous hybrid system that produces all of the possible solutions produced by switching signals that belong to the class. In this setup, for each initial condition of the hybrid system, there are many solutions, each generated by a particular switching signal. See "Switching Systems."

Similarly, when addressing networked control systems, an autonomous hybrid model is used that generates all solutions that can occur for a class of transmission time sequences. See Example 27.

Sufficient conditions for uniqueness can be invoked, if desired. Several sufficient conditions for uniqueness of solutions to ordinary differential equations are given in the literature. The simplest condition is that the differential equation's vector field is locally Lipschitz continuous. Relaxed conditions also exist. See [S19] for further discussion.

The following result characterizes uniqueness of solutions to hybrid systems. Formally, *uniqueness of solutions* holds

The first conclusion of Theorem 4 is a property of set convergence. The second conclusion is specific to the hybrid system setting. In summary, Theorem 4 states that, from each locally uniformly bounded sequence of solutions to a hybrid system that satisfies the Basic Assumptions, we can extract a graphically convergent subsequence whose graphical limit is a solution to the hybrid system. Consequences of Theorem 4 for asymptotic stability in hybrid systems are discussed in the "Asymptotic Stability" section.

Graphical convergence of hybrid arcs to a hybrid arc has an equivalent pointwise description. Consider a sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid arcs and a hybrid arc x. Then $\{x_i\}_{i=1}^{\infty}$ converges graphically to x if and only if the following two conditions hold:

i) For every sequence of points $(t_i, j) \in \operatorname{dom} x_i$ such that the sequences $\{t_i\}_{i=1}^{\infty}$ and $\{x_i(t_i, j)\}_{i=1}^{\infty}$ are convergent, it follows that $(t, j) \in \operatorname{dom} x$, where $t = \lim_{i\to\infty} t_i$ and $x(t, j) = \lim_{i\to\infty} x_i(t_i, j)$.

ii) For every $(t, j) \in \operatorname{dom} x$ there exists a sequence $\{(t_i, j)\}_{i=1}^{\infty}$, where $(t_i, j) \in \operatorname{dom} x_i$, such that $t = \lim_{i\to\infty} t_i$ and $x(t, j) = \lim_{i\to\infty} x_i(t_i, j)$.

Figure 13 depicts several solutions from a graphically convergent sequence of solutions to the bouncing ball system in Example 3.

Dependence of Solutions on Initial Conditions

One of the consequences of Theorem 4 is the semicontinuous dependence of solutions to a hybrid system on initial conditions. To state this result rigorously, in Theorem 5, we define a concept of distance between hybrid arcs that is closely related to graphical convergence.

Given $\tau \geq 0$ and $\varepsilon > 0$, the hybrid arcs x and y are (τ, ε)-*close* if the following conditions are satisfied:

a) For each $(t, j) \in \operatorname{dom} x$ with $t + j \leq \tau$ there exists $s \in \mathbb{R}_{\geq 0}$ such that $(s, j) \in \operatorname{dom} y$, $|t - s| < \varepsilon$, and

$$|x(t, j) - y(s, j)| < \varepsilon.$$

b) For each $(t, j) \in \operatorname{dom} y$ with $t + j \leq \tau$ there exists $s \in \mathbb{R}_{\geq 0}$ such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \varepsilon$, and

$$|y(t, j) - x(s, j)| < \varepsilon.$$

The concept of (τ, ε)-closeness provides an equivalent characterization of graphical convergence of hybrid arcs. Consider a locally uniformly bounded sequence of hybrid arcs $\{x_i\}_{i=1}^{\infty}$ and a hybrid arc x. Then the sequence $\{x_i\}_{i=1}^{\infty}$ converges graphically to x if and only if, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists i_0 such that, for all $i > i_0$, the hybrid arcs x_i and x are (τ, ε)-close.

Equipped with the concept of (τ, ε)-closeness, we again revisit Example 3.

Example 3 Revisited: Bouncing Ball and (τ, ε)-Closeness

Consider the bouncing ball model with $\gamma = 1$ and $\rho = 1/2$. Given $\delta \in [0, 1)$, the hybrid arc representing the velocity of the ball dropped from height $1 + \delta$ with velocity zero, for times $t \leq 2$, is given by

$$x_2^{\delta}(t, j) = \begin{cases} -t, & t \in [0, \sqrt{2(1+\delta)}],\ j = 0, \\ -t + 3\sqrt{(1+\delta)/2}, & t \in [\sqrt{2(1+\delta)}, 2],\ j = 1. \end{cases}$$

Consider the hybrid arcs, x_2^0 and x_2^{δ}, where $\delta \in (0, 1)$. These arcs are (τ, ε)-close, with any $\tau \geq 0$ and $\varepsilon = 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. To show this, it is sufficient to for the hybrid system $\mathcal{H}$ if, for any two maximal solutions x_1 and x_2 to $\mathcal{H}$, if $x_1(0,0) = x_2(0,0)$, then $\operatorname{dom} x_1 = \operatorname{dom} x_2$ and $x_1(t,j) = x_2(t,j)$ for all $(t,j) \in \operatorname{dom} x_1$.

Proposition S5

Uniqueness of solutions holds for a hybrid system with data (C, F, D, G) if and only if the following conditions hold:

1) For each initial point $\xi \in C$ there exists a unique maximal solution to the differential inclusion $\dot{z}(t) \in F(z(t))$ satisfying $z(0) = \xi$ and $z(t) \in C$.
2) For each initial point $\xi \in D$, $G(\xi)$ is a singleton.
3) For each initial point $\xi \in C \cap D$, there are no nontrivial solutions to $\dot{z}(t) \in F(z(t))$ satisfying $z(0) = \xi$ and $z(t) \in C$.

The first condition is ensured when F is a locally Lipschitz continuous function but can hold when F is set valued. In contrast, set-valuedness of G at a point in D immediately leads to non-unique solutions. Hence, the second condition cannot be weakened. The third condition is ensured when for each $\xi \in C \cap D$, $T_C(\xi) \cap F(\xi) = \emptyset$, where $T_C(\xi)$ denotes the tangent cone to C at ξ. This condition indicates that, roughly speaking, the vector field given by F should point to the outside of C at points in $C \cap D$.

The bouncing ball model in Example S4 satisfies the three conditions of Proposition S5 and thus generates unique solutions from all initial conditions.

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consider $\tau = 3$. Indeed, since we are considering arcs with domains restricted to (t, j) such that $t + j \leq 3$, $(3, \varepsilon)$-closeness implies (τ, ε)-closeness for all $\tau \geq 0$.

To verify condition a) of (τ, ε)-closeness, note that, for each $(t, 0) \in \operatorname{dom} x_2^0$ with $t \leq 3$, we have $(t, 0) \in \operatorname{dom} x_2^\delta$ and $x_2^0(t, 0) = x_2^\delta(t, 0)$. In practical terms, the velocities of the balls are the same until the ball dropped from the lower height bounces. For each $(t, 1) \in \operatorname{dom} x_2^0$ with $t \leq 2$, that is, with $t + 1 \leq 3$, there exists $(s, 1) \in \operatorname{dom} x_2^\delta$ with $|t - s| \leq \sqrt{2(1+\delta)} - \sqrt{2}$ and $|x_2^0(t, j) - x_2^\delta(s, j)| \leq 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. In fact, for $t \leq \sqrt{2(1+\delta)}$, we can take $s = \sqrt{2(1+\delta)}$, while, for the remaining t's, we can take $s = t$. Consequently, for each $(t, j) \in \operatorname{dom} x_2^0$ with $t + j \leq \tau = 3$ there exists $s \geq 0$ such that $(s, j) \in \operatorname{dom} x_2^\delta$, $|t - s| < \varepsilon$ and $|x(t, j) - y(s, j)| < \varepsilon$, where $\varepsilon = 3\sqrt{2}(\sqrt{1+\delta} - 1)/2$. A similar calculation can be carried out for condition b) of (τ, ε)-closeness.

Thanks to the equivalent characterization of graphical convergence in terms of (τ, ε)-closeness, the $(\tau, 3\sqrt{2}(\sqrt{1+\delta} - 1)/2)$-closeness of x_2^0 and x_2^δ, for each $\tau \geq 0$, implies that x_2^δ converge graphically, as $\delta \to 0$, to x_2^0. ■

For the bouncing ball, whose maximal solution is unique for each initial condition, we can establish that the arc x_2^δ depends continuously on δ at zero, for an appropriately defined concept of continuous dependence. In the absence of uniqueness of solutions, continuous dependence cannot be expected. Continuous dependence fails in simpler settings, too. For the differential equation $\dot{x} = 2\sqrt{|x|}$, solutions $(t + \sqrt{\delta})^2$ from initial points $x(0) = \delta > 0$ converge uniformly on compact time intervals to t^2, as $\delta \to 0$. But from the initial point zero, there also exists a solution, identically equal to zero, which is not a limit of any sequence of solutions from positive initial points. This simple setting already illustrates what outer semicontinuous dependence on initial conditions is, namely,

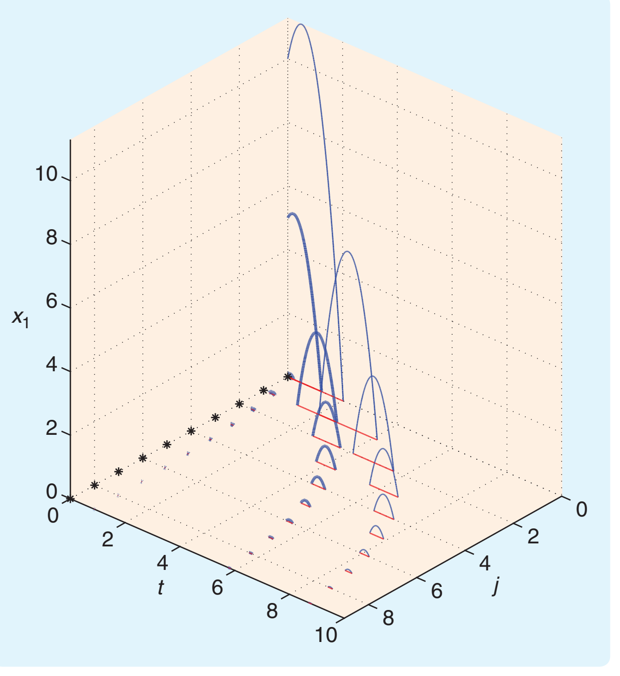

FIGURE 13 The height x_1 of solutions to the bouncing ball system in Example 3. The height of three solutions with nonzero initial height and velocity given by (10, 5), (5, 2), and (0.1, 0.5), are shown in blue. The solution from (0, 0), denoted in black with $*$ marks, is a solution that jumps and does not flow. Initial conditions closer to the origin result in the blue graphs that more closely resemble the black graph. In other words, solutions with initial conditions close to (0, 0) are graphically close to the solution from (0, 0).

that each solution from the initial point $\delta > 0$ close to zero is close to some solution from zero. In the language of sequences, the limit of a sequence of solutions from initial points close to zero is a solution from zero, even though some other solutions from zero are not limits of any solutions from nearby initial points. We stress here, again, that continuous dependence on initial points is not needed to develop fundamental stability theory results, such as converse Lyapunov theorems and invariance principles.

In a hybrid system, even the domains of solutions with close initial points can differ significantly. An example illustrating this phenomenon is the system on $\mathbb{R}$ with $C = \mathbb{R}_{\geq 0}$, $f(x) = -x$, $D = (-\infty, 0]$, and $g(x) = x/2$. For all $\tau > \varepsilon > 0$, the solution $x(t, 0) = 0$ for all $t \geq 0$, from the initial point 0, is not (τ, ε)-close to any of the solutions $y(0, j) = -\delta/2^j$, $j = 1, 2, \ldots$, independently of how small $\delta > 0$ is. Note, however, that the solutions $y(0, j) = -\delta/2^j$, $j = 1, 2, \ldots$ converge to another solution from zero, namely $z(0, j) = 0$, $j = 1, 2, \ldots$.

The following result, proven in [26, Cor. 4.8], concerns the outer semicontinuous dependence of solutions to hybrid systems on initial conditions.

Theorem 5

Suppose that the hybrid system $\mathcal{H}$ meets the Basic Assumptions and $\xi \in \mathbb{R}^n$ is such that each maximal solution to $\mathcal{H}$ from ξ is either complete or bounded. Then, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for each solution x^δ to $\mathcal{H}$ with $|x^\delta(0, 0) - \xi| \leq \delta$, there exists a solution x to $\mathcal{H}$ with $x(0, 0) = \xi$ such that x^δ and x are (τ, ε)-close.

Additional Consequences of Sequential Compactness

Sequential compactness of the space of solutions of a hybrid system, stated in Theorem 4, results in uniformity of various properties. We illustrate this fact with a property related to the lack of complete solutions that jump but do not flow. The result below, and the phrase "uniformly non-Zeno," are taken from [17]. See "Zeno Solutions" for a further discussion of the Zeno phenomenon.

Proposition 6

Consider the hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, and a compact set $K \subset \mathbb{R}^n$ that is forward invariant, in other words, such that all solutions x to $\mathcal{H}$ with $x(0, 0) \in K$ satisfy $x(t, j) \in K$ for all $(t, j) \in \operatorname{dom} x$. Then, exactly one of the following conditions is satisfied:

a) There exists a complete solution x to $\mathcal{H}$ with $x(0, 0) \in K$ and $\operatorname{dom} x = \{0\} \times \mathbb{N}$.

b) The set of all solutions with initial points in K is uniformly non-Zeno, that is, there exist $T > 0$ and $J \in \mathbb{N}$ such that, for each solution x to $\mathcal{H}$ with $x(0, 0) \in K$, each $(t, j), (t', j') \in \operatorname{dom} x$ with $|t - t'| \leq T$ satisfies $|j - j'| \leq J$.

For illustration purposes, we outline a proof. The conditions a) and b) are mutually exclusive. Negating b) yields a sequence of solutions $\{x_i\}_{i=1}^{\infty}$, with $x_i(0, 0) \in K$, and, for each $i = 1, 2, \ldots$, $(t_i, j_i), (t_i', j_i') \in \operatorname{dom} x_i$ with $|t_i - t_i'| \leq 1/i$ and $|j_i - j_i'| > i$. Without loss of generality, we can assume that $t_i \leq t_i'$ and $j_i < j_i'$, for all $i = 1, 2, \ldots$. Define a sequence of hybrid arcs y_i by $y_i(t, j) = x_i(t + t_i, j + j_i)$, which implicitly defines $\operatorname{dom} y_i$ to be the tail of $\operatorname{dom} x_i$. Forward invariance of K implies that, for every i, $y_i(t, j) \in K$ for all $(t, j) \in \operatorname{dom} y_i$. Thus, the sequence $\{y_i\}_{i=1}^{\infty}$ is locally uniformly bounded in the sense of Theorem 4. Part a) of that theorem implies that there exists a graphically convergent subsequence of the sequence y_i. Part b) implies that the graphical limit of the subsequence, denoted y, is a solution to $\mathcal{H}$. Clearly, $y(0, 0) \in K$. It remains to conclude that y is complete and never flows. This conclusion comes out from the fact that the point $(t_i' - t_i, j_i' - j_i)$ is an element of $\operatorname{dom} y_i$, which says that y_i jumps at least i times in at most $1/i$ time units, and from the definition of graphical convergence.

Similar, yet even simpler arguments, are used in the following result to establish the compactness of reachable sets.

Proposition 7 [26, Cor. 4.7]

Suppose that the hybrid system $\mathcal{H}$ satisfies the Basic Assumptions. Consider a compact set K such that every solution to $\mathcal{H}$ starting in K is either complete or bounded, and $m > 0$. Then, the reachable set

$$\mathcal{R}_{\leq m}(K) := \{x(t, j) \mid x(0, 0) \in K, \\ (t, j) \in \operatorname{dom} x, t + j \leq m\}$$

is compact.

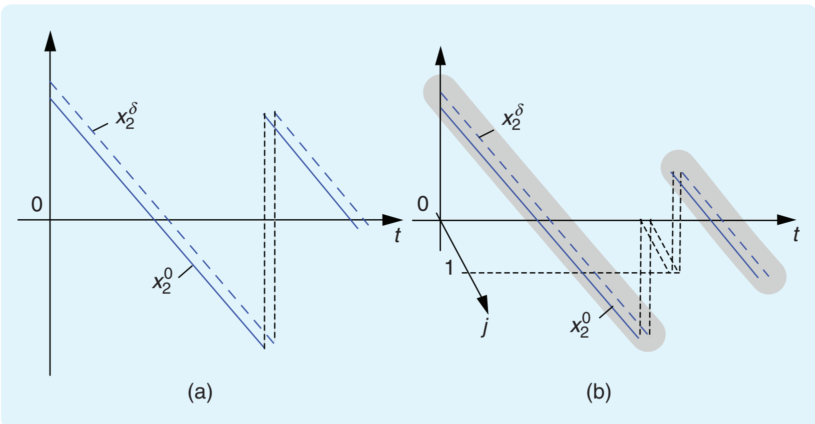

FIGURE 14 Components x_2^0, x_2^δ of solutions from "Example 3 Revisited," representing the velocities of bouncing balls dropped from heights 1 and $1 + \delta$, respectively. (a) Velocities parameterized by t. The velocities are not close in the uniform distance since their difference, at times after the first ball bounces and before the second ball bounces, is large. (b) Velocities on hybrid time domains. The shaded neighborhoods of their graphs indicate that the velocities are graphically close.

This article is a tutorial on modeling the dynamics of hybrid systems, on the elements of stability theory for hybrid systems, and on the basics of hybrid control.

Outer semicontinuous dependence of solutions to a hybrid system on initial conditions, stated in Theorem 5, can be generalized to allow state perturbations of the hybrid system. Given a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$, a continuous function $\sigma: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $\delta \geq 0$, consider a hybrid system $\mathcal{H}_{\delta\sigma}$ with the state $x \in \mathbb{R}^n$ and data

$$C_{\delta\sigma} := \{x: (x + \delta\sigma(x)\mathbb{B}) \cap C \neq \emptyset\}, \tag{8}$$

$$F_{\delta\sigma}(x) := \overline{\text{con}}\, F((x + \delta\sigma(x)\mathbb{B}) \cap C) + \delta\sigma(x)\mathbb{B} \quad \text{for all } x \in C_{\delta\sigma}, \tag{9}$$

$$D_{\delta\sigma} := \{x: (x + \delta\sigma(x)\mathbb{B}) \cap D \neq \emptyset\}, \tag{10}$$

$$G_{\delta\sigma}(x) := \{v: v \in g + \delta\sigma(g)\mathbb{B},\ g \in G((x + \delta\sigma(x)\mathbb{B}) \cap D)\} \quad \text{for all } x \in D_{\delta\sigma}. \tag{11}$$

Perturbations, of the system $\mathcal{H}$, of this kind are used in the analysis of robustness of asymptotic stability. Figure 15 illustrates the idea behind perturbations of the sets C and D.

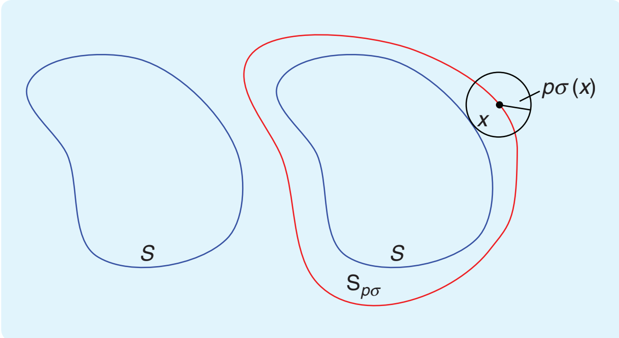

FIGURE 15 Enlargement of a set S due to a state-dependent perturbation of size $p\sigma$. The perturbed set $S_{p\sigma}$ contains all points x with Euclidean distance from the unperturbed set S no larger than $p\sigma(x)$.

Theorem 8 [26, Corollary 5.5]

Suppose that the hybrid system $\mathcal{H}$ satisfies the Basic Assumptions and $\xi \in \mathbb{R}^n$ is such that each maximal solution to $\mathcal{H}$ from ξ is either complete or bounded. Let $\sigma: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function. Then, for every $\tau \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for each solution x^δ to $\mathcal{H}_{\delta\sigma}$ with $|x^\delta(0,0) - \xi| \leq \delta$, there exists a solution x to $\mathcal{H}$ with $x(0,0) = \xi$ such that x^δ and x are (τ, ε)-close.

ASYMPTOTIC STABILITY

This section addresses asymptotic stability in hybrid dynamical systems, including basic definitions and equivalent characterizations of asymptotic stability. Examples are provided to illustrate the main concepts. Some of the definitions and results have a formulation that is slightly different from classical stability theory for differential equations. This difference is mainly due to the fact that existence of solutions is a more subtle issue for hybrid systems than it is for classical systems. See "Existence, Uniqueness, and Other Well-Posedness Issues." Otherwise, the stability theory results that are available for hybrid systems typically parallel the results that are available for classical systems.

Definition and Examples

As discussed in "Motivating Stability of Sets," the solutions of a dynamical system sometimes converge to a set rather than to an equilibrium point. Thus, we study asymptotic stability of sets. The scope is limited to compact sets for simplicity.

Roughly speaking, a compact set $\mathcal{A}$ is asymptotically stable if solutions that start close to $\mathcal{A}$ stay close to $\mathcal{A}$, and complete solutions that start close to $\mathcal{A}$ converge to $\mathcal{A}$. We now make this concept precise. A compact set $\mathcal{A}$ is *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0,0)|_{\mathcal{A}} \leq \delta$ implies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all solutions x to $\mathcal{H}$ and all $(t,j) \in \operatorname{dom} x$. The notation $|x|_{\mathcal{A}} = \min\{|x-y| : y \in \mathcal{A}\}$ indicates the distance of the vector x to the set $\mathcal{A}$. If $\mathcal{A}$ is the origin then $|x|_{\mathcal{A}} = |x|$. A compact set $\mathcal{A}$ is *pre-attractive* if there exists a neighborhood of $\mathcal{A}$ from which each solution is bounded and the complete solutions converge to $\mathcal{A}$, that is, $|x(t,j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, where $(t,j) \in \operatorname{dom} x$. The prefix "pre-" is used since it is not a requirement that maximal solutions starting near $\mathcal{A}$ be complete. See "Why 'Pre'-Asymptotic Stability?" for additional reasons to consider asymptotic stability without insisting on completeness of solutions. A compact set $\mathcal{A}$ is *pre-asymptotically stable* if it is stable and pre-attractive.

For a pre-asymptotically stable compact set $\mathcal{A} \subset \mathbb{R}^n$, its *basin of pre-attraction* is the set of points in $\mathbb{R}^n$ from which each solution is bounded and the complete solutions converge to $\mathcal{A}$. By definition, the basin of pre-attraction contains a neighborhood of $\mathcal{A}$. In addition, each point in $\mathbb{R}^n \backslash (C \cup D)$ belongs to the basin of pre-attraction since no solution starts at a point in $\mathbb{R}^n \backslash (C \cup D)$. If the basin of pre-attraction is $\mathbb{R}^n$ then the set $\mathcal{A}$ is *globally pre-asymptotically stable*. We drop "pre" when all solutions starting in the basin of pre-attraction are complete.

Set Convergence

The concept of convergence of sets, as well as many other elements of set-valued analysis, are important ingredients of modern analysis. For example, in optimization and optimal control, set convergence helps in the study of how sets of optimal solutions or optimal controls, when these solutions or controls are not unique, depend on parameters or initial conditions. The brief exposition of set convergence given below follows the terminology and definitions of [69].

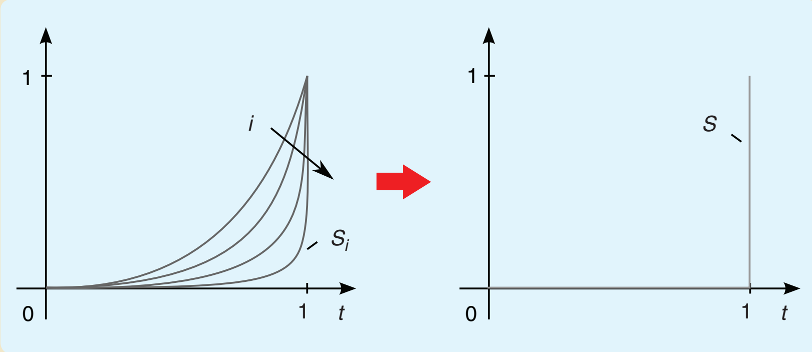

FIGURE S6 A sequence of sets S_i converging to the reflected L-shaped set S. Equivalently, the sequence of functions with graphs given by S_is converges graphically to a set-valued mapping with graph given by S.

Let $\{S_i\}_{i=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^n$. The *outer limit* of this sequence, denoted $\limsup_{i\to\infty} S_i$, is the set of all accumulation points of sequences of points $x_i \in S_i$; more precisely, it is the set of all points $x \in \mathbb{R}^n$ or which there exists a sequence of points x_k, $k = 1, 2, \ldots$, and a subsequence $\{S_{i_k}\}_{k=1}^{\infty}$ of the sequence $\{S_i\}_{i=1}^{\infty}$ such that $x_k \in S_{i_k}$ and $x_k \to x$ as $k \to \infty$. The *inner limit* of this sequence, denoted $\liminf_{i\to\infty} S_i$, is the set of all points $x \in \mathbb{R}^n$ for which there exists a sequence of points $x_i \in S_i$, $i = 1, 2, \ldots$, such that $x_i \to x$ as $i \to \infty$. The *limit* of the sequence $\{S_i\}_{i=1}^{\infty}$, denoted $\lim_{i\to\infty} S_i$, exists if the inner and outer limits are equal, in which case $\lim_{i\to\infty} S_i = \limsup_{i\to\infty} S_i = \liminf_{i\to\infty} S_i$.

The outer and inner limit always exist, but these limits may be empty. Furthermore, the outer limit is nonempty when the sequence S_i does not escapes to infinity, in the sense that, for each $r > 0$, there exists $i_r \in \mathbb{N}$ such that, for all $i > i_r$, the intersection of S_i with a ball of radius r is empty. Finally, the inner and outer limits—and thus the limit, if it exists—are closed, independently of whether or not each of the S_is is closed.

Some basic examples of set convergence are the following:

a) If each S_i is a singleton, that is, $S_i = \{s_i\}$, where $s_i \in \mathbb{R}^n$, then the outer limit of the sequence $\{S_i\}_{i=1}^{\infty}$ is the set of accumulation points of the sequence s_i; the inner limit is nonempty if and only if the sequence s_i is convergent, in which case $\liminf_{i\to\infty} S_i = \lim_{i\to\infty} S_i = \lim_{i\to\infty} s_i$.

b) Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of nonnegative numbers. For the sequence of closed balls of radius r_i, $S_i = r_i\mathbb{B}$, it follows that $\limsup_{i\to\infty} S_i = \bar{r}\mathbb{B}$, where $\bar{r}$ is the upper limit of the sequence $\{r_i\}_{i=1}^{\infty}$, and $\liminf_{i\to\infty} S_i = \underline{r}\mathbb{B}$, where $\underline{r}$ is the lower limit of the sequence $\{r_i\}_{i=1}^{\infty}$. When $\bar{r} = \infty$, $\bar{r}\mathbb{B}$ corresponds to all of $\mathbb{R}^n$.

c) Let $S_i = \{(t, t/i) : t \in \mathbb{R}\}$, in other words, let S_i be the graph of the linear function $t \mapsto t/i$. Then the sequence $\{S_i\}_{i=1}^{\infty}$ is convergent, and the limit is the graph of the function $t \mapsto 0$ defined on $\mathbb{R}$.

d) Let $S_i = \{(t, t^i) : t \in [0, 1]\}$, in other words, let S_i be the graph of the function $t \mapsto t^i$ on $[0, 1]$. Then the sequence $\{S_i\}_{i=1}^{\infty}$ is convergent, and the limit S has the reflected L shape, that is, $S = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$. Figure S6 shows S_i and S.

Example c) suggests that a sequence $\{S_i\}_{i=1}^{\infty}$ of sets can converge to S even though the Hausdorff distance between S_i and S is infinite for all $i = 1, 2, \ldots$. The Hausdorff distance can be used to characterize set convergence of bounded

Example 9: A Planar System

Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$C := \{x: x_1 \geq 0\}, \quad f(x) := \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} x \quad \text{for all } x \in C,$$
$$D := \{x: x_1 = 0, x_2 \leq 0\}, \quad g(x) := -\gamma x \quad \text{for all } x \in D,$$

where $\gamma > 0$, $\omega > 0$, and $\alpha \in \mathbb{R}$. During flows, a solution rotates in the clockwise direction through the set C until reaching the negative x_2-axis. The maximum amount of time spent in C before a jump occurs is π/ω units of time. The sign of α determines whether the norm of a solution increases or decreases during flows. At points on the negative x_2-axis, a solution jumps to the positive x_2-axis. The sign of $\gamma - 1$ determines whether the norm of a solution increases or decreases during jumps. Figure 16 illustrates different possibilities. If $\exp(\alpha\pi/\omega)\gamma < 1$ then the norm of a solution decreases over one cycle from the positive x_2-axis and back again. In this case, each solution x that starts close to the origin remains close to the origin and tends toward the origin as $t + j \to \infty$, where $(t, j) \in \operatorname{dom} x$. Thus, the origin is globally asymptotically stable. ■

Example 1 Revisited: Dual-Mode Control for Disk Drives

We start with a preliminary observation about a hybrid system that has no discrete-time dynamics and has the continuous-time dynamics sequences of sets. More precisely, a uniformly bounded sequence $\{S_i\}_{i=1}^{\infty}$ of closed sets converges to a closed set S if and only if the Hausdorff distance between S and S_i converges to zero.

Examples c) and d) also illustrate the concept of graphical convergence of functions or set-valued mappings. A sequence $\{F_i\}_{i=1}^{\infty}$ of set-valued mappings $F:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *converges graphically* to $F:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ if the sequence of graphs of F_is, which are subsets of $\mathbb{R}^{m+n}$, converges to the graph of F, in the sense of set convergence. In c), the sequence of functions $t \mapsto t/i$ converges graphically, and pointwise, to the function $t \mapsto 0$. In d), the sequence of functions $t \mapsto t^i$ on $[0, 1]$ converges graphically not to a function but to a set-valued mapping. More precisely, the graphical limit of functions $t \mapsto t^i$ on $[0, 1]$ is equal to zero for $t \in [0, 1)$ and $[0, 1]$ for $t = 1$, as can be seen in Figure S6. Note that the sequence of functions $t \mapsto t^i$ on $[0, 1]$ converges pointwise to the function that is equal to zero for $t \in [0, 1)$ and equal to one at $t = 1$; however, this convergence is not uniform on $[0, 1]$.

A natural example of the outer limit of a sequence of sets is provided by omega limits of solutions to dynamical systems. To illustrate omega limits in a continuous time setting, let $x:\mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a function. Although x may be a solution to a differential equation, continuity properties of x are irrelevant for the following discussion. The omega limit of x, denoted $\omega(x)$, is the set of all points $\xi \in \mathbb{R}^n$ for which there exists a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_i \to \infty$ such that $x(t_i) \to \xi$. Then

$$\omega(x) = \limsup_{i\to\infty} S_i$$

where $S_i = \{x(t) : t \geq i\}$. See Figure S7. Since the sequence S_i is nonincreasing, the outer limit is in fact the limit. A property of set limits, as noted above, implies that $\omega(x)$ is closed. Another property implies that if $|x(t)|$ does not diverge to infinity then $\omega(x)$ is nonempty. In fact, properties of set convergence imply that if x is bounded, then $\omega(x)$ is nonempty and compact and x converges to $\omega(x)$. The convergence of x to $\omega(x)$ follows from the following property [69, Thm. 4.10] of set convergence: given a sequence of sets $\{S_i\}_{i=1}^{\infty}$ and a closed set S, $\lim_{i\to\infty} S_i = S$ if and only if, for all $\varepsilon > 0$ and $\rho > 0$, there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$,

$$S \cap \rho\mathbb{B} \subset S_i + \varepsilon\mathbb{B}, \qquad S_i \cap \rho\mathbb{B} \subset S + \varepsilon\mathbb{B}. \tag{S5}$$

The second inclusion (S5), with ρ such that $x(t) \in \rho\mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$, implies that x converges to $\omega(x)$. The characterization of set convergence in (5) is behind the relationship between graphical convergence of hybrid arcs and the concept of (τ, ε)-closeness between hybrid arcs, as used in the "Basic Mathematical Properties" section.

For references using set-valued analysis in dynamical systems, in particular, in differential inclusions, see [S22], [S16], and [S24]; for relevance to optimal control, see [S23].

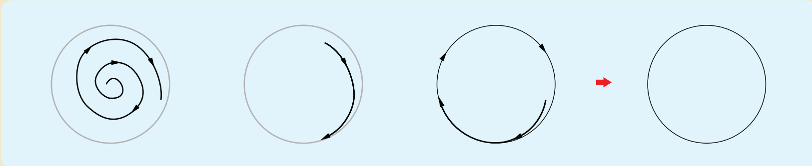

FIGURE S7 A solution to a differential equation approaching a periodic solution that covers a circle. The circle is the omega-limit of the solution. Tails of the solution converge, in the sense of set convergence, to the circle.

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[S23] F. H. Clarke, *Optimization and Nonsmooth Analysis.* Philadelphia: SIAM, 1990.

[S24] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, vol. 41, *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 2002.

$$\dot{p} = v, \quad \dot{v} = \kappa(p, v, p^*), \qquad (p, v) \in C. \tag{12}$$

Assume that if $C = \mathbb{R}^2$ then the point $(p^*, 0)$ is locally asymptotically stable with basin of attraction $\mathcal{B}$. Now let $C \subset \mathcal{B}$ and notice that this choice eliminates each solution of (12) with $C = \mathbb{R}^2$ that does not start in $\mathcal{B}$. It follows, for the system (12) with $C \subset \mathcal{B}$, that the point $(p^*, 0)$ is globally pre-asymptotically stable. A generalization of this observation appears in Theorem S10 of "Why 'Pre'-Asymptotic Stability?"

Now consider the hybrid system from Example 1, with state $x = (p, v, q)$ satisfying

$$\dot{p} = v, \quad \dot{v} = \kappa_q(p, v, p^*), \quad \dot{q} = 0, \quad (p, v) \in C_q, \tag{13}$$

$$p^+ = p, \quad v^+ = v, \quad q^+ = 3 - q, \quad (p, v) \in D_q. \tag{14}$$

According to the assumptions of Example 1 and the conclusions drawn above about the system (12), for each $i \in \{1, 2\}$ the point $(p^*, 0)$ is globally pre-asymptotically stable for the system

$$\dot{p} = v, \quad \dot{v} = \kappa_i(p, v, p^*) \quad (p, v) \in C_i. \tag{15}$$

Moreover, each solution of (15) with $i = 2$ that starts in D_1, which contains a neighborhood of $(p^*, 0)$ and is contained in C_2, does not reach the boundary of C_2. Also, $C_1 = \overline{\mathbb{R}^2 \setminus D_1}$ and $D_2 = \overline{\mathbb{R}^2 \setminus C_2}$.

Global asymptotic stability of the point $(p^*, 0, 2)$ for the system (13), (14) then follows from the global pre-asymptotic stability of $(p^*, 0)$ for (15) with $i \in \{1, 2\}$ together with the

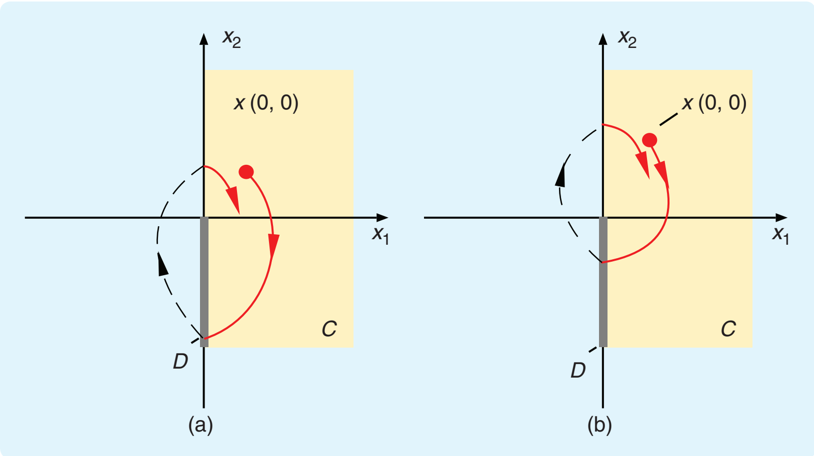

FIGURE 16 Flow and jump sets for the hybrid system in Example 9, with typical solutions to the system. Solutions flow clockwise in the right-half plane and jump when x_1 is zero and x_2 is nonpositive. In (a), the norm of the solution increases during flows and decreases at jumps. In (b), the norm decreases during flows and increases at jumps. In both cases, the origin is globally asymptotically stable.

fact that the maximum number of jumps a solution of the system (13), (14) experiences is two. To see the latter property, note that if there is more than one jump for a solution x of (13), (14) then one of the first two jumps must be from $q=1$ to $q=2$. At this jump, we must have $(p,v)\in D_1$ and, since p and v do not change during a jump, $(p,v)\in D_1$ after the jump. Since a solution of (15) with $i=2$ that starts in D_1 does not reach D_2, the solution x of (13), (14) does not jump again after a jump from $q=1$ to $q=2$.

The principle behind this asymptotic stability result is generalized in the section "Stability Analysis Through Limited Events." ■

Example 2 Revisited: Asymptotic Stability

Consider the hybrid system in Example 2. We study asymptotic stability of the origin. The origin is not globally asymptotically stable since the solution starting at $(2,0)$ does not converge to the origin. On the other hand, the solution starting at $(1,0)$ satisfies $|x(t,j)|\le|x(0,0)|$ for all $(t,j)\in \operatorname{dom} x$ and $\lim_{t+j\to\infty}|x(t,j)|=0$. More generally, constructing solutions as in Example 2, it follows that the origin is asymptotically stable with basin of attraction given by

$$\{x\in C: x_2<3,\ x_1+x_2\in(-2,2)\}\cup \{x\in D: x_1\in(-1,1)\}.$$ ■

A consequence of Theorem 4 appears in Theorem 10, which states that pre-asymptotic stability of a compact set is implied by forward invariance, as defined in Proposition 6, together with uniform pre-attractivity. A compact set $\mathcal{A}$ is *uniformly pre-attractive* from a set K if for each $\varepsilon>0$ there exists $T>0$ such that $x(0,0)\in K$, $(t,j)\in \operatorname{dom} x$, and $t+j\ge T$ imply $|x(t,j)|_{\mathcal{A}}\le\varepsilon$. The next section establishes the implication that is opposite to the one stated in Theorem 10.

Theorem 10 [26, Prop. 6.1]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, if the compact set $\mathcal{A}$ is forward invariant and uniformly pre-attractive from a compact set containing a neighborhood of $\mathcal{A}$, then the set $\mathcal{A}$ is pre-asymptotically stable.

Example 11: An Impulsive Observer with Finite-Time Convergence

This example comes from [67], which addresses impulsive observers for linear systems. Consider a linear, continuous-time system $\dot\xi=F\xi+v$, where ξ belongs to a compact set $K_1\subset\mathbb{R}^n$ and v belongs to a compact, convex set $K_2\subset\mathbb{R}^n$. Let $H\in\mathbb{R}^{r\times n}$ and assume we have measurements of the output vector $H\xi$ and the input vector v. The pair (H,F) is *observable* if $\dot\xi=F\xi$ and $H\xi(t)\equiv0$ imply $\xi(t)\equiv0$. This property enables assigning the spectra of the matrix $F-LH$ arbitrarily through the matrix L. In particular, a classical dynamical system with state $\hat\xi$ can be constructed so that $\xi(t)-\hat\xi(t)$ approaches zero as $t\to\infty$. Such a dynamical system is called an observer. A classical observer has the form $\dot{\hat\xi}=(F-LH)\hat\xi+LH\xi+v$, where L is chosen so that $F-LH$ is Hurwitz, meaning that each eigenvalue of $F-LH$ has negative real part. This choice gives the observation error equation $\dot e=(F-LH)e$, where $e:=\xi-\hat\xi$. Since $F-LH$ is Hurwitz, the error e converges to zero exponentially.

We consider a hybrid observer that reconstructs the state ξ in finite time. The first thing to note is that the observability of the pair (H,F) permits finding matrices L_1 and L_2 such that, for almost all $\delta>0$, the matrix $I-\exp((F-L_2H)\delta)\exp(-(F-L_1H)\delta)$ is invertible [67, Remark 1]. Define $F_i:=F-L_iH$ and henceforth assume that $\delta>0$, L_1 and L_2 are such that $I-\exp(F_2\delta)\exp(-F_1\delta)$ is invertible.

Consider a hybrid system with state $x=(\xi,\hat\xi_1,\hat\xi_2,\tau)$, flow set $C:=K_1\times\mathbb{R}^n\times\mathbb{R}^n\times[0,\delta]$, jump set $D:=K_1\times\mathbb{R}^n\times\mathbb{R}^n\times\{\delta\}$, flow map

$$F(x)=\left\{\begin{bmatrix}F\xi+v\\ F_1\hat\xi_1+(F-F_1)\xi+v\\ F_2\hat\xi_2+(F-F_2)\xi+v\\ 1\end{bmatrix}: v\in K_2\right\},$$

and jump map

$$G(x)=\begin{bmatrix}\xi\\ G_1\hat\xi_1+G_2\hat\xi_2\\ G_1\hat\xi_1+G_2\hat\xi_2\\ 0\end{bmatrix},$$

where

Many engineering systems experience impacts; walking and jumping robots, juggling systems, billiards, and a bouncing ball are examples.

$$[G_1 \quad G_2] := (I - \exp(F_2\delta)\exp(-F_1\delta))^{-1} \times [-\exp(F_2\delta)\exp(-F_1\delta) \;\; I].$$

This hybrid system contains two different continuous-time observers, of the form $\dot{\hat{\xi}}_i = (F - L_i H)\hat{\xi}_i + L_i H\xi + v$, the states of which make jumps every δ seconds according to the rule specified by the jump map.

We show that the compact set $\mathcal{A} := \{(\xi, \hat{\xi}_1, \hat{\xi}_2) \in K_1 \times \mathbb{R}^n \times \mathbb{R}^n : \xi = \hat{\xi}_1 = \hat{\xi}_2\} \times [0, \delta]$ is globally asymptotically stable using Theorem 10. The set $\mathcal{A}$ is forward invariant since $G(\mathcal{A} \cap D) \subset \mathcal{A}$ and, during flows, the errors $e_i := \xi - \hat{\xi}_i$ satisfy $\dot{e}_i = F_i e_i$. Moreover, the set $\mathcal{A}$ is globally uniformly attractive. In particular, $(t, j) \in \operatorname{dom} x$ and $t \geq 2\delta$ imply $j \geq 2$ and $x(t, j) \in \mathcal{A}$. The condition on j follows from the nature of the data of the hybrid system. The condition $x(t, j) \in \mathcal{A}$ follows from the fact that, when there exists w such that ξ, $\hat{\xi}_1$, and $\hat{\xi}_2$ satisfy

$$\hat{\xi}_1 = \xi + \exp(F_1\delta)w, \quad \hat{\xi}_2 = \xi + \exp(F_2\delta)w,$$

the jump map sends the state to $\mathcal{A}$. The given relations are satisfied after one jump followed by a flow interval of length δ. In this case, w is equal to the difference between ξ and $\hat{\xi}_1$ immediately after the jump. We conclude from Theorem 10 that the set $\mathcal{A}$ is globally asymptotically stable. Moreover, the analysis above shows that the convergence to $\mathcal{A}$ is in finite time $t \leq 2\delta$. ∎

Equivalence with Uniform Asymptotic Stability

In this section, we describe uniform pre-asymptotic stability on compact subsets of the basin of pre-attraction and point out that this property is equivalent to pre-asymptotic stability. The first step in stating this characterization is to make an observation about the basin of pre-attraction that extends classical results [42, pp. 69–71] for differential and difference equations to the hybrid setting. This result, and all of the subsequent results in this section, depend on Theorem 4.

Theorem 12 [26, Prop. 6.4], [14, Thm. 3.14]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, the basin of pre-attraction for a compact, pre-asymptotically stable set $\mathcal{A}$ is an open, forward invariant set containing a neighborhood of $\mathcal{A}$.

This theorem helps in developing some concepts used to express the fact that pre-asymptotic stability is equivalent to uniform pre-asymptotic stability on compact subsets of the basin of pre-attraction. In that direction, the next result states that excursions away from and convergence toward a pre-asymptotically stable compact set are uniform over compact subsets of the basin of pre-attraction.

Theorem 13 [26, Prop. 6.3]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, let the compact set $\mathcal{A}$ be pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_{\mathcal{A}}$. For each compact set $K_0 \subset \mathcal{B}_{\mathcal{A}}$, the compact set $\mathcal{A}$ is uniformly pre-attractive from K_0, and there exists a compact set $K_1 \subset \mathcal{B}_{\mathcal{A}}$ such that solutions starting in K_0 satisfy $x(t, j) \in K_1$ for all $(t, j) \in \operatorname{dom} x$.

The properties established in Theorem 13, of uniform overshoot and uniform convergence from compact subsets of the basin of pre-attraction, can be expressed in terms of a single bound on solutions starting in the basin of pre-attraction. To that end, let $\mathcal{A}$ be compact and let $\mathcal{O}$ be an open set containing $\mathcal{A}$. A continuous function $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is called a *proper indicator for* $\mathcal{A}$ *on* $\mathcal{O}$ if $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and also $\omega(x_i)$ tends to infinity when x_i tends to infinity or tends to the boundary of $\mathcal{O}$. Every open set $\mathcal{O}$ and compact set $\mathcal{A} \subset \mathcal{O}$ admit a proper indicator. Thus, using Theorem 12, for each pre-asymptotically stable set $\mathcal{A}$ there exists a proper indicator for $\mathcal{A}$ on its basin of pre-attraction. The function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\omega(x) := |x|_{\mathcal{A}}$ for all $x \in \mathbb{R}^n$ is a proper indicator for $\mathcal{A}$ on $\mathbb{R}^n$. For a general open set $\mathcal{O}$, it is always possible to take $\omega(x) = |x|_{\mathcal{A}}$ for x sufficiently close to $\mathcal{A}$. The concept of a proper indicator function first appears in conjunction with stability theory in [42]. A typical formula for a proper indicator is given in that work and in [39, (C.14)]. A sublevel set of a proper indicator on $\mathcal{O}$ is a compact subset of $\mathcal{O}$.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it is continuous; for each $s \geq 0$, $r \mapsto \beta(r, s)$ is nondecreasing and zero at zero; and, for each $r \geq 0$, $s \mapsto \beta(r, s)$ is nonincreasing and tends to zero when s tends to infinity. Class-$\mathcal{KL}$ functions are featured prominently in [31] and have become familiar to the nonlinear systems and control community through their use in the input-to-state stability property [76]. The next result is a generalization to hybrid systems of a result contained in [42] for continuous differential equations.

Robustness and Generalized Solutions

When designing control systems, engineers must ensure that the closed-loop behavior is robust to reasonable levels of measurement noise, plant uncertainty, and environmental disturbances.

For linear systems, eigenvalues of matrices change continuously with parameters, and asymptotic stability of an equilibrium point is an open-set condition on eigenvalues. It follows that sufficiently small perturbations to the system matrices do not change stability properties. Similarly, sufficiently small additive disturbances lead to small excursions from the equilibrium point.

Classical results for ordinary differential and difference equations with nonlinear but continuous right-hand sides establish that small perturbations to the right-hand side result in small changes to the solutions on compact time intervals. This property is described for hybrid systems in theorems 5 and 8. If an equilibrium point or, more generally, a compact set is asymptotically stable for the nominal system, then these compact time intervals can be stitched together to establish that sufficiently small disturbances lead to small excursions from the asymptotically stable compact set. This idea is behind the results in theorems 15 and 17. This behavior is the essence of the total stability property described in [31, Sec. 56], and also contained in the local version of the input-to-state stability property [76].

For differential equations and difference equations with a discontinuous right-hand side, asymptotic stability of a compact set is not necessarily robust to arbitrary small perturbations [S26], [S27], [S25]. The lack of robustness motivates generalized notions of solutions. For differential equations, [S20] establishes the connection between the generalized notion of solution in [S26], expressed in terms of small state perturbations, and the generalized notion of solution given in [S28], expressed in terms of set-valued dynamics. The set-valued dynamics arise when a discontinuous right-hand side of a differential, or difference, equation is converted into an inclusion. The resulting system is the regularization of the original system, that is, it is a *regularized system*.

The conversion of a differential, or difference, equation with a discontinuous right-hand side to an inclusion is done by considering a closure of the graph of the discontinuous right-hand side and, in the differential equation case, by taking the convex hull of the values of the right-hand side. For an illustration, see Figure S8. The price paid for such a conversion is the introduction of additional solutions, some of which may not behave well. However, these extra solutions are meaningful since they arise from arbitrarily small state perturbations that converge to zero asymptotically. Moreover, asymptotic stability in the original, possibly discontinuous, system is robust if and only if asymptotic stability holds for the regularized system.

Anticipating that a similar result holds for hybrid systems and desiring robustness in asymptotically stable hybrid systems, we consider the stability properties of regularized systems. Following the lead of discontinuous continuous-time and discrete-time systems, we convert discontinuous flow and jump equations to inclusions. Interpreting these inclusions as closure operations on the discontinuous flow map and jump map as Figure S8 indicates, we also take the closures of the flow and jump sets. As for continuous- and discrete-time systems, these operations may introduce new solutions. However, the regularized hybrid system satisfies the Basic Assumptions, and therefore, the results of the main text are applicable. In particular, if the regularized system has an asymptotically stable compact set $\mathcal{A}$, then $\mathcal{A}$ is robustly asymptotically stable for the regularized system, and consequently, $\mathcal{A}$ is robustly asymptotically stable for the original system.

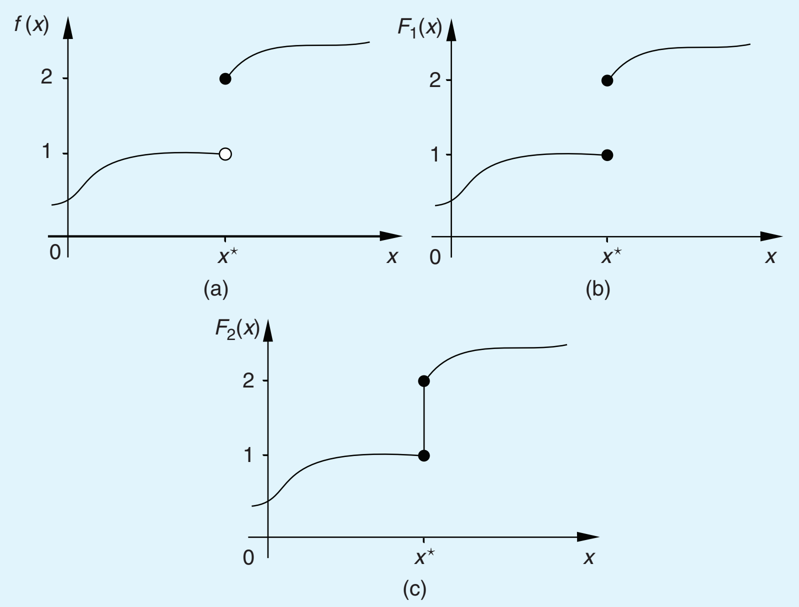

FIGURE S8 Regularization of a discontinuous function. (a) The function $f:\mathbb{R}\to\mathbb{R}$ has a discontinuity at $x=x^*$. (b) The set-valued mapping $F_1:\mathbb{R}\rightrightarrows\mathbb{R}$ is the mapping whose graph is the closure of the graph of f. Therefore, $F_1(x^*)$ is the set $\{1, 2\}$. (c) The set-valued mapping $F_2:\mathbb{R}\rightrightarrows\mathbb{R}$ is the mapping obtained by taking the convex hull of the values of the mapping in (b), at each $x\in\mathbb{R}$. Therefore, $F_2(x^*)$ is the interval $[1, 2]$.

Example S6: A Frictionless Ball and Two Rooms Separated by a Zero-Width Wall

Consider a particle that moves, with no friction, in one of two rooms separated by a thin wall. A hybrid system describing this situation has the state variable $x=(\xi,q)\in\mathbb{R}^2$ and data

$$\begin{aligned} C&=([-1,1]\setminus\{0\})\times\{-1,1\},\\ F(x)&=(q,0) \text{ for all } x\in C,\\ D&=\{(-1,-1)\}\cup(\{0\}\times\{-1,1\})\cup\{(1,1)\},\\ G(x)&=(\xi,-q) \text{ for all } x\in D. \end{aligned}$$

That is, the particle, whose position is denoted by ξ and velocity by q, moves with speed one in the interval $[-1,0]$ or $[0,1]$, and experiences a reversal of the velocity when at any of the boundary points: -1, 0, or 1.

Each of the sets $[-1,0]\times\{-1,1\}$ and $[0,1]\times\{-1,1\}$ is forward invariant. (When the solution definition is modified so that multiple jumps at the same ordinary time instant are not allowed and the flow constraint is relaxed to $x(t,j)\in C$ for all t in the interior of I_j, each maximal solution is complete and its time domain is unbounded in the ordinary time direction.) However, this forward invariance behavior is not robust, even approximately, to an arbitrarily small inflation of the set C. In particular, when C is changed to $\overline{C}=[-1,1]\times\{-1,1\}$, there exist solutions starting from $x=(1,-1)$ that reach $x=(-1,-1)$ in two time units.

At least two ways can be used to induce the desired behavior while using closed flow and jump sets. One natural approach can be interpreted as thickening the wall between the two rooms. In particular, letting $\varepsilon\in(0,1)$, the data

$$\begin{aligned} C&=([-1,1]\setminus(-\varepsilon,\varepsilon))\times\{-1,1\},\\ F(x)&=(q,0) \text{ for all } x\in C,\\ D&=\{(-1,-1)\}\cup\{(-\varepsilon,1)\}\cup\{(\varepsilon,-1)\}\cup\{(1,1)\},\\ G(x)&=(\xi,-q) \text{ for all } x\in D, \end{aligned}$$

results in solutions, as defined in the main text, rendering each of the sets $[-1,-\varepsilon]\times\{-1,1\}$ and $[\varepsilon,1]\times\{-1,1\}$ robustly forward invariant.

When we insist on a wall of zero width, an extra state variable should be added that gives the system information about how it arrived at the point $(0,q)$, whether from the room on the left or the one on the right. For example, we can consider the hybrid system with state $(\xi,q,r)\in\mathbb{R}^3$ with data

$$\begin{aligned} C&=\{(\xi,q,r)\in[-1,1]\times\{-1,1\}\times\{-1,1\}:\xi r\geq 0\},\\ F(x)&=(q,0,0) \text{ for all } x\in C,\\ D&=\{(-1,-1,-1)\}\cup\{(1,1,1)\}\\ &\quad\cup\{(0,1,-1)\}\cup\{(0,-1,1)\},\\ G(x)&=(\xi,-q,r) \text{ for all } x\in D. \end{aligned}$$

The hybrid system with this data is such that the set C is forward invariant. In particular, since r is constant and belongs to the set $\{-1,1\}$, the state ξ cannot change sign. ■

The next example is discussed in [14].

Example S7: Asymptotic Stability Without Robustness

Consider the hybrid system with data

$$\begin{aligned} C&=[0,1],\ F(x)=-x \text{ for all } x\in C,\\ D&=(1,2],\ G(x)=1 \text{ for all } x\in D. \end{aligned}$$

Solutions that start in $(1,2]$ jump to one and then, not being in the jump set D, flow according to the differential equation $\dot{x}=-x$ toward the origin. The origin is thus globally asymptotically stable. But notice that when the jump map is replaced by $G(x)=1+\varepsilon$, where $\varepsilon>0$, the point $x=1+\varepsilon$ is an equilibrium. In this sense, the global asymptotic stability of the origin is not robust. This lack of robustness can be seen in the regularized system $(C,F,\overline{D},G)$, which exhibits a countable number of solutions from the initial condition $x=1$. These solutions remain at the value 1 for n jumps, where n is any nonnegative integer, and then flow toward the origin. One additional solution remains at one through an infinite number of jumps. This behavior is depicted in Figure S9.

Example S8: Robust Asymptotic Stability

Consider the hybrid system with state $x=(\xi,q)\in\mathbb{R}^2$, data

$$\begin{aligned} C&:=\{(\xi,q)\in\mathbb{R}\times\{-1,1\}:2\xi q\geq -1\},\\ F(\xi,q)&:=\begin{bmatrix}-\xi+q\\0\end{bmatrix} \text{ for all } (\xi,q)\in C,\\ D&:=\{(\xi,q)\in\mathbb{R}\times\{-1,1\}:2\xi q< -1\},\\ G(\xi,q)&:=\begin{bmatrix}\xi\\-q\end{bmatrix} \text{ for all } (\xi,q)\in D. \end{aligned}$$

This system can be associated with a hybrid control algorithm for the continuous-time, linear system $\dot{\xi}=u$, which uses an internal state $q\in\{-1,1\}$ and aims to globally asymptotically stabilize the two point set $\mathcal{A}:=\{(-1,-1),(1,1)\}$. This set is globally asymptotically stable since $\mathcal{A}$ is contained in C, the flow stabilizes the point $\xi=q$, $\mathcal{A}$ is disjoint from D, the quantity $2\xi q$ increases along flows when it has the value -1, and jumps from D are mapped to points in C. Note, however, that the jump set D is not closed. Consider the effect of closing D. For the hybrid system with data $(C,F,\overline{D},G)$, solutions starting from points where $2x_1x_2=-1$ are no longer unique. From such points, flowing is possible, as is a single jump followed by flow. Nevertheless, the set $\mathcal{A}$ is still globally asymptotically stable for $(C,F,\overline{D},G)$. Thus, the asymptotic stability of $\mathcal{A}$ in the original system is robust. The lack of unique solutions in the regularized system can be associated with the fact that very different solutions arise when starting from $2\xi q=-1$ and considering arbitrarily small measurement noise on the state ξ in the closed-loop control system. ■

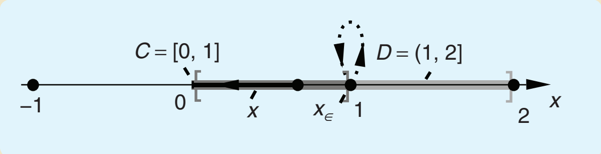

FIGURE S9 The effect of state perturbations in Example S7 showing that the global asymptotic stability of the origin is not robust. The solution x starts in $(-\infty, 1)$ and flows according to the differential equation $\dot{x} = -x$ toward the origin. The solution x_ε starts at $x_\varepsilon(0, 0) = 1$ and is obtained under the presence of a perturbation e of size ε. The perturbation is such that $x_\varepsilon(t, j) + e(t, j) \in D$ for all $(t, j) \in \operatorname{dom} x_\varepsilon$. Hence, as denoted with dotted line, the solution x_ε jumps from 1 to 1 infinitely many times, indicating that the origin is not robustly asymptotically stable to small perturbations.

A mathematical description of the phenomenon displayed in examples S6 and S7 is summarized as follows. Consider a hybrid system for which the data (C, f, D, g) do not meet all of the Basic Assumptions. That is, the flow set C is possibly not closed, the flow map $f : C \to \mathbb{R}^n$ is possibly not continuous, the jump set D is possibly not closed, and the jump map $g : D \to \mathbb{R}^n$ is possibly not continuous. Single-valued mappings f and g are considered here for simplicity; a more general result involving set-valued mappings is possible.

We consider two notions of generalized solutions to hybrid systems. One notion uses a hybrid system with the data $(\overline{C}, F, \overline{D}, G)$, which is obtained from the data (C, f, D, g) by taking the closures of C and D and defining

$$F(x) = \bigcap_{\delta>0} \overline{\operatorname{con}}\, f\left((x+\delta\mathbb{B}) \cap C\right) \quad \text{for all } x \in \overline{C}, \tag{S6}$$

$$G(x) = \bigcap_{\delta>0} \overline{g\left((x+\delta\mathbb{B}) \cap D\right)} \quad \text{for all } x \in \overline{D}. \tag{S7}$$

The mapping G defined in (S7) is the mapping whose graph is the closure of the graph of the function g. When the function f is locally bounded, the mapping F in (S6) is obtained by first considering the closure of the graph of f and then taking the pointwise convex hull. Figure S8 illustrates these two constructions. An alternative interpretation of $(\overline{C}, F, \overline{D}, G)$ is that it represents the smallest set of data that meets the Basic Assumptions and contains the data (C, f, D, g). The other notion of generalized solutions considers the effects of vanishing perturbations on the state. More precisely, it considers the graphical limits of sequences of solutions generated with state perturbations, as the perturbation size decreases to zero. The two notions of generalized solutions turn out to be equivalent, as the following result states.

Theorem 14 [26, Thm. 6.5], [14, Prop. 7.3]

For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, if the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction given by $\mathcal{B}_\mathcal{A}$ then, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{B}_\mathcal{A}$, there exists $\beta \in \mathcal{KL}$ such that each solution x starting in $\mathcal{B}_\mathcal{A}$ satisfies $\omega(x(t, j)) \le \beta(\omega(x(0, 0)), t + j)$ for all $(t, j) \in \operatorname{dom} x$.

Theorem 14 contains the result in Theorem 13 that excursions away from and convergence toward a pre-asymptotically stable compact set are uniform over compact subsets of the basin of pre-attraction. In other words, pre-asymptotic stability for compact sets is equivalent to uniform pre-asymptotic stability, which sometimes is called $\mathcal{KL}$-stability. Thus, when we provide sufficient conditions for pre-asymptotic stability, we in fact give sufficient conditions for $\mathcal{KL}$-stability.

When discussing global pre-asymptotic stability, where $\mathcal{B}_\mathcal{A} = \mathbb{R}^n$, we can use $\omega(x) = |x|_\mathcal{A}$. Thus, for global pre-asymptotic stability, the bound in Theorem 14 becomes $|x(t, j)|_\mathcal{A} \le \beta(|x(0, 0)|_\mathcal{A}, t + j)$ for all $(t, j) \in \operatorname{dom} x$. When $\omega(x) = |x|_\mathcal{A}$ and $\beta(s, r) = \gamma s \exp(-\lambda r)$ for some positive real numbers γ and λ, the set $\mathcal{A}$ is globally pre-exponentially stable.

Robustness

A feature of hybrid systems satisfying the Basic Assumptions is that pre-asymptotic stability is robust.

One way to characterize the robustness of pre-asymptotic stability of a compact set $\mathcal{A}$ is to study the effect of state-dependent perturbations on the hybrid system data and show that, when the perturbations are small enough, the pre-asymptotic stability of $\mathcal{A}$ and the basin of pre-attraction are preserved, as in Theorem 15. Typically, these state-dependent perturbations must decrease in size as the state approaches the pre-asymptotically stable set and also as the state approaches the boundary of the basin of pre-attraction.

Another way to characterize robustness is to consider constant perturbation levels and show that these perturbations lead to "practical" pre-asymptotic stability from arbitrarily large subsets of the basin of attraction, as in Theorem 17.

In some cases the nominal system can tolerate state-dependent perturbations that grow without bound when the state grows unbounded. These systems are closely related to hybrid systems having inputs and possessing the input-to-state stability (ISS) property [76]. ISS for hybrid dynamical systems is studied in [13].

For a given hybrid system $\mathcal{H}$ with data (C, F, D, G) and a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, we define the σ-perturbation $\mathcal{H}_\sigma$ of $\mathcal{H}$ through the data

$$C_\sigma := \{x : (x + \sigma(x)\mathbb{B}) \cap C \ne \emptyset\}, \tag{16}$$

$$F_\sigma(x) := \overline{\operatorname{con}} F((x+\sigma(x)\mathbb{B}) \cap C) + \sigma(x)\mathbb{B} \quad \text{for all } x \in C_\sigma, \tag{17}$$

$$D_\sigma := \{x : (x + \sigma(x)\mathbb{B}) \cap D \ne \emptyset\}, \tag{18}$$

$$G_\sigma(x) := \{v : v \in g + \sigma(g)\mathbb{B},\ g \in G((x + \sigma(x)\mathbb{B}) \cap D)\} \quad \text{for all } x \in D_\sigma. \tag{19}$$

Theorem S9 [S29, Thm. 3.1, Rem. 5.4]

Suppose that the functions $f, g: \mathbb{R}^n \to \mathbb{R}^n$ are locally bounded on $\mathbb{R}^n$. Let $x: \operatorname{dom} x \to \mathbb{R}^n$ be a hybrid arc such that $\operatorname{dom} x$ is compact. Then, the following statements are equivalent:

a) x is a solution to the hybrid system $(\overline{C}, F, \overline{D}, G)$;

b) there exist hybrid arcs x_i and functions $e_i: \operatorname{dom} x_i \to \mathbb{R}^n$, $i \in \{1, 2, \ldots\}$, such that $\lim_{i\to\infty} x_i(0,0) = x(0,0)$, the sequence $\{x_i\}_{i=1}^{\infty}$ converges graphically to x, $\lim_{i\to\infty} \sup_{(t,j)\in \operatorname{dom} x_i} |e_i(t,j)| = 0$, and, for every $i \in \{1, 2, \ldots\}$ the following hold:

– For each fixed j, $t \mapsto e_i(t,j)$ is measurable.

– For all $j \in \mathbb{N}$ such that $I_{i,j} := \{t : (t,j) \in \operatorname{dom} x_i\}$ has nonempty interior,

$$\begin{aligned} \dot{x}_i(t,j) &= f(x_i(t,j) + e_i(t,j)) \quad \text{for almost all } t \in I_{i,j}, \\ x_i(t,j) + e_i(t,j) &\in C \quad \text{for almost all } t \in [\min I_{i,j}, \sup I_{i,j}). \end{aligned}$$

– For all $(t,j) \in \operatorname{dom} x_i$ such that $(t, j+1) \in \operatorname{dom} x_i$,

$$\begin{aligned} x_i(t, j+1) &= g_i(x_i(t,j) + e_i(t,j)), \\ x_i(t,j) + e_i(t,j) &\in D. \end{aligned}$$

A result corresponding to Theorem S9 is given in [S25, Thm. 3.2] for hybrid systems for which the perturbations e_i enter the closed-loop system through feedback, and do not affect all of the dynamics. This result considers equations $\dot{x}_i = f'(x_i, u_c)$, $x_i^+ = g'(x_i, u_d)$ in (b) above with state-feedback laws $u_c = k_c(x_i + e_i)$ and $u_d = k_d(x_i + e_i)$, and poses stronger continuity assumptions on the functions f' and g', but allows the functions k_c and k_d to be discontinuous.

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These definitions match the definitions (8)–(11) with $\delta = 1$. Figure 15 illustrates the idea behind perturbations of the sets C and D. Note that $C \subset C_\sigma$, $D \subset D_\sigma$, $F(x) \subset F_\sigma(x)$ for all $x \in C$ and $G(x) \subset G_\sigma(x)$ for all $x \in D$. Since the data of $\mathcal{H}_\sigma$ contain the data of $\mathcal{H}$, the solutions of $\mathcal{H}$ are also solutions to $\mathcal{H}_\sigma$. On the other hand, $\mathcal{H}_\sigma$ typically exhibits solutions that are not solutions to $\mathcal{H}$.

The extra solutions of $\mathcal{H}_\sigma$ can be linked to solutions that arise due to parameter variations, measurement noise in control systems, and external disturbances. For a link to solutions that arise from parameter variations, see Example 18. For the case of external disturbances d that are bounded in norm by a value $M > 0$, observe that the solutions of $\dot{x} = F(x) + d$ are contained in the set of solutions of $\dot{x} \in F(x) + M\mathbb{B}$ and, taking $\sigma(x) = M$ for all x, $F(x) + M\mathbb{B} \subset F_\sigma(x)$. The same link between external disturbances and system perturbations holds for $x^+ = G(x) + d$.

Now consider the case of measurement noise in a hybrid control system of the form discussed in the section "Hybrid Controllers for Nonlinear Systems," given as

$$\left.\begin{aligned} \dot{x}_p &= f_p(x_p, \kappa_c(x_p + e, x_c)) \\ \dot{x}_c &= f_c(x_p + e, x_c) \end{aligned}\right\} \quad (x_p + e, x_c) \in C, \tag{20}$$

$$\left.\begin{aligned} x_p^+ &= x_p \\ x_c^+ &\in G_c(x_p + e, x_c) \end{aligned}\right\} \quad (x_p + e, x_c) \in D, \tag{21}$$

where e represents measurement noise, assumed to be bounded in norm by $M > 0$. When the function f_p is continuous, there exists a continuous function $\tilde{\sigma}$ such that $\tilde{\sigma}(0, x) = 0$ for all $x \in \mathbb{R}^{n+m}$ and, for all $e \in \mathbb{R}^n$ satisfying $|e| \le M$ and all $x = (x_p, x_c) \in \mathbb{R}^{n+m}$,

$$\begin{aligned} |f_p(x_p, \kappa_c(x_p + e, x_c)) - f_p(x_p + e, \kappa_c(x_p + e, x_c))| \\ \le \tilde{\sigma}(M, x). \end{aligned}$$

In this case, the hybrid control system (20)–(21) can be written as

$$\begin{aligned} \dot{x} &= F(x + d_1) + d_2, & x + d_1 &\in C, \\ x^+ &\in G(x + d_1), & x + d_1 &\in D, \end{aligned}$$

where F and G are defined in (6) and (7), respectively, $|d_1| \le M$, and $|d_2| \le \tilde{\sigma}(M, x)$. Therefore, the solutions of the hybrid control system (20)–(21) are contained in the solutions of $(C_\sigma, F_\sigma, D_\sigma, G_\sigma)$ where $\sigma(x) := \max\{M, \tilde{\sigma}(M, x)\}$ for all $x \in \mathbb{R}^{n+m}$.

The mappings F_σ and G_σ may be set valued at points x where $\sigma(x) > 0$, even when F and G are single-valued mappings. Also, when $x \in C \cap D$ and $\sigma(x) > 0$, the point x belongs to the interior of both C_σ and D_σ. Thus, at such points, the system $\mathcal{H}_\sigma$ has solutions that initially flow and also solutions that initially jump.

For a hybrid system $\mathcal{H}$ having a compact set $\mathcal{A}$ that is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_\mathcal{A}$, Theorem 15 below asserts the existence of a continuous function σ that is positive on $\mathcal{B}_\mathcal{A} \backslash \mathcal{A}$ so that, for the hybrid system $\mathcal{H}_\sigma$, the compact set $\mathcal{A}$ is pre-asymptotically stable with basin of pre-attraction $\mathcal{B}_\mathcal{A}$. In other words, pre-asymptotic stability