Linear Quadratic Regulator (LQR) - State Feedback Design

A system is expressed in state variable form as

$$\dot{x} = Ax + Bu$$

with $x(t) \in R^n, u(t) \in R^m$ and the initial condition $x(0) = x_0$.

A. The stabilization problem using state variable feedback.

The following formulates the *stabilization problem using state variable feedback*.

We assume that all the states are measurable (or observable) and seek to find a state-variable feedback (SVFB) control

$$u = -Kx + v$$

which guarantees obtaining the desired closed-loop properties.

The closed-loop system using this control becomes

$$\dot{x} = (A - BK)x + Bv = A_c x + Bv$$

with A_c the closed-loop plant matrix and v(t) the new input for the closed loop system.
The stabilization problem has solution if and only if the system is stabilizable (reachable), i.e. the reachability matrix is invertible.

If there is only one input so that m=1, then Ackermann's formula gives a state variable feedback gain K that places the poles of the closed-loop system as desired (such that the closed loop system is stable and has the desired performances specified by the engineer).

We now introduce a method of designing controllers which produce optimal response of the closed loop system, in the sense of minimizing a certain performance index. Unlike Ackermann's formula, that can only be used to determine a controller for a single-input system, the technique that we now introduce works for any number of inputs.

B. Optimal stabilization problem using state variable feedback

To design a state variable feedback gain that is optimal, we define the *performance index (PI)*

$$J = \int_0^\infty x^T Q x + u^T R u \, dt .$$

The objective of optimal state variable feedback design is to select the state variable feedback gain K which *stabilizes* the system and also *minimizes* the performance index J.

Since the plant is linear and the performance index is quadratic, the problem of determining the state variable feedback which minimizes J is called the *Linear Quadratic Regulator (LQR)*. The word 'regulator' refers to the fact that the function of this feedback is to regulate the states to zero. Thus in this case we assume that the input signal v(t) is equal to zero since our only concern here are the internal stability properties of the closed-loop system.

Compared to the state feedback stabilization problem, where the performance of the closed loop system was specified by desired positions of the poles of the closed loop, in this case the desired performance of the system is given by the values of the two matrices Q and R present in the performance index to be minimized.

The performance index J can be interpreted as an energy function, such that making it small keeps small the total energy of the closed-loop system. Note that both the state x(t) and the control input u(t) are weighted in J, so that if J is small, then neither x(t) nor u(t) can be too large. Also, if J is minimized, then it is certainly finite, and since it is an infinite integral of x(t), this implies that $x(t) \to 0$ as $t \to \infty$. This guarantees that the closed-loop system will be *stable.*

C. What is the intuition behind choosing the two matrices Q and R?

This performance index describes a balance between the amount of energy used to achieve desired performances and the specified desired performances. This balance is described by the choice of the two matrices Q (an $n \times n$ matrix) and R (an $m \times m$ matrix) which are selected by the design engineer. Depending on how these design parameters are selected, the closed-loop system will exhibit a different response.

a. Selecting Q large means that, in order to keep J small, the state x(t) must be smaller. This means that larger values of Q generally result in the poles of the closed-loop system matrix $A_c = (A - BK)$ being further left in the s-plane so that the state decays faster to zero.

b. Selecting R large means that the control input u(t) must be smaller to keep J small. Larger R means that less control effort is used, so that the poles are generally slower, resulting in larger values of the state x(t).

One should select Q to be *positive semi-definite* and R to be *positive definite*. This means that
- the scalar quantity $x^T Qx$ is always positive or zero at each time t for *all functions x(t)*, and the scalar quantity $u^T Ru$ is always positive at each time t for *all* values of u(t).
- in terms of eigenvalues, the eigenvalues of Q should be non-negative, while those of R should be positive.
- if both matrices are selected diagonal, then all the entries of R must be positive while those of Q should be positive, with possibly some zeros on its diagonal.

Note that if R is chosen positive definite then <u>R is invertible</u>.

D. Under which conditions can we find a controller with a finite value for the cost function?

Before proceeding to the derivation of the optimal state feedback control policy which minimizes the infinite horizon performance index, first we would like to know under which conditions we would be able to derive a controller which would result in a finite value for the cost function.
In other words, if for any initial value for the state and for any control u(t) we obtain that the cost function is equal to infinity, then the problem of obtaining an optimal state feedback controller which minimizes J would have no meaning. When is the optimization problem solvable?

The condition which makes possible obtaining a state feedback control policy which has a finite cost is that the system is reachable (i.e. the reachability matrix is invertible, or (A,B) - reachable).

Say that the system is reachable, then there exists a state feedback controller $u=-Kx$ such that the closed loop system is stable. Considering that the initial state of the system is x_0 then the trajectories of the states of the closed loop system would be $x(t)=e^{(A-BK)t}x_0$ and the control input $u(t)=-Ke^{(A-BK)t}x_0$. Then the cost function (performance index) becomes

$$J = x_0^T \int_0^\infty e^{(A-BK)^T t}(Q+K^T RK)e^{(A-BK)t}\, dt x_0$$

Define $M=\int_0^\infty e^{(A-BK)^T t}(Q+K^T RK)e^{(A-BK)t}\, dt$. Then, if the closed loop system is stable,

$$0 \le J = x_0^T M x_0 < \infty \text{ (the cost is finite)}$$

E. How can we calculate the optimal controller?

One can show that if the system is reachable (i.e. the reachability matrix is invertible) then the minimum value of the performance index, for any initial condition of the state vector, is

$$J^*(x_0) = x_0^T P x_0$$

where P is the positive semi-definite solution of the ***algebraic Riccati equation (ARE)***

$$A^T P + PA + Q - PBR^{-1}B^T P = 0\,.$$

(Showing this is beyond the objective of this lecture.)

Using the above result we will now derive the optimal state feedback gain.
Say that $u=-K^*x$ is the optimal state feedback control policy which determines the minimum value of the performance index and stabilizes the system.

$$J^*(x_0) = \int_0^\infty x^T(Q+K^{*T}RK^*)x\, dt$$

Notice that

$$J^*(x_0) = x_0^T P x_0 = -\int_0^\infty (x^T P x)'\, dt$$

where we assumed that the closed-loop system is stable so that $x(t) \to 0$ as $t \to \infty$.

Then we can write $x^T(Q + K^{*T}RK^*)x = -(x^TPx)'$ which is

$$\dot{x}^T Px + x^T P\dot{x} + x^T Qx + x^T K^{*T} RK^* x = 0$$

Knowing that $\dot{x} = (A - BK^*)x = A_c x$ then we obtain

$$x^T A_c{}^T Px + x^T PA_c x + x^T Qx + x^T K^{*T} RK^* x = 0$$

$$x^T(A_c{}^T P + PA_c + Q + K^{*T} RK^*)x = 0$$

The last equation has to hold for every x(t). Therefore, the term in brackets must be identically equal to zero. Thus one sees that

$$(A - BK)^T P + P(A - BK) + Q + K^{*T} RK^* = 0$$

$$A^T P + PA + Q + K^{*T} RK^* - K^{*T} B^T P - PBK^* = 0 \; .$$

This is a *matrix quadratic equation* which we can write as

$$\begin{aligned} &A^T P + PA + Q - PBR^{-1}B^T P \\ &\qquad + PBR^{-1}B^T P + K^{*T} RK^* - K^{*T} B^T P - PBK^* = 0 \; . \end{aligned}$$

Now remember that

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \,,$$

then

$$PBR^{-1}B^T P + K^{*T} RK^* - K^{*T} B^T P - PBK^* = 0 \,,$$

which can be written as

$$(K^* - R^{-1}B^T P)^T R(K^* - R^{-1}B^T P) = 0 \,.$$

Since R is a positive definite matrix then this relation will hold only if

$$K^* - R^{-1}B^T P = 0$$

Thus the optimal state feedback controller which minimizes the given quadratic cost index is given by

$$K^* = R^{-1}B^T P$$

where the P matrix is the solution of the ARE

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \,.$$

Notice that the minimal value of the performance index, obtained using the optimal controller gain, is $J^*(x_0) = x_0{}^T Px_0$ and only depends on the initial condition and the solution of the ARE. This means that the cost of using the optimal state variable feedback can be computed from the initial conditions *before the control is ever applied to the system*.

F. When is the optimal control policy is also a stabilizing policy?

It can be shown that even if the optimal value of the cost function is finite the optimal trajectories of the states could go to infinity.

Take for example the system $\dot{x} = x + u$

and the cost function to be minimized is $J = \int_0^\infty u^2 \, dt$.

Then the ARE is $2p - p^2 = 0$ and has solutions $p = 0, p = 2$.
Taking $p = 0$ then $u = 0$ is the optimal controller for any initial conditions and the optimal value of the performance index is 0.
However the trajectory of the state is $x(t) = e^t x_0$, and goes to infinity with time. This shows that the optimal controller is not a stabilizing one.

Now we ask the question: What condition has to be satisfied such that the optimal control policy is also a stabilizing policy?
The answer is

If (A, B) is reachable and $(\sqrt{Q}, A)$ is observable then the optimal state feedback controller $K^* = R^{-1}B^T P$, with P a symmetric positive semidefinite solution of the ARE, is also a stabilizing control policy.

Recall that reachability can be verified by checking that the reachability matrix $U = [B \;\; AB \;\; A^2 B \cdots A^{n-1}B]$ has full rank n.

Recall that observability can be verified by checking that the observability matrix $V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank n. If V is not square then recall the use of the observability grammian.

G. Design procedure for finding the LQR feedback gain K

The design procedure for finding the LQR feedback gain K is:

1. Verify if the system is reachable.
    a. If it is NOT then LQR design is impossible. – STOP
2. If YES, the select design parameter matrices Q and R.
3. Verify if the pair $(\sqrt{Q}, A)$ is observable.
    a. If NOT then go back to step 2 and choose a different Q matrix
4. If YES, solve the algebraic Riccati equation for P (the unique positive semidefinite solution)
5. Find the optimal state variable feedback using $K^* = R^{-1}B^T P$

There exist very good numerical procedures for solving the ARE.
The MATLAB routine that performs this is named *lqr(A,B,Q,R)*.