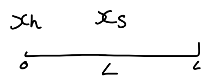
PDE

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + u(x,t)$$

1D 假设房间 $\Omega=[0,L]$



Boundary condition: 绝热墙壁 (insulated ends)

$(x=0)$: $\left.\frac{\partial T}{\partial x}\right|_{x=0}=0$ $\quad$ $(x=L)$: $\left.\frac{\partial T}{\partial x}\right|_{x=L}=0$

x_h 加热器 $\quad$ x_s 温控器

$$u(x,t)=\begin{cases} u_{max} & \text{if } T(x_s,t)<T_{set} \\ 0 & \text{if } T(x_s,t)\geq T_{set} \end{cases}$$

将时间t划分为间隔为Δt小段,

将空间x划分为间隔为Δx小段,

T_i^n 表示在第n个时间步,第i个空间位置的温度

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1}-T_i^n}{\Delta t}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{\frac{T_{i+1}^n-T_i^n}{\Delta x}-\frac{T_i^n-T_{i-1}^n}{\Delta x}}{\Delta x} = \frac{T_{i+1}^n-2T_i^n+T_{i-1}^n}{(\Delta x)^2}$$

$$\frac{T_i^{n+1}-T_i^n}{\Delta t} = \alpha\left(\frac{T_{i+1}^n-2T_i^n+T_{i-1}^n}{(\Delta x)^2}\right)+u_i^n$$

$$T_i^{n+1}=T_i^n+\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{i+1}^n-2T_i^n+T_{i-1}^n\right)+\Delta t\, u_i^n$$

无论是否隔热

☆ interior Nodes 的通式

($i=1,\dots N-1$)

同时是端点式子推导的基础

当暖气片(热源)安装在墙边时,

对于内部点式子中, $u_i^n=0$.

当暖气片(热源)安装在左墙

左边界点的 u_0^n 就代表暖气的输入功率.

左边界 ($i=0$) $T_{-1}=T_1$

$$T_0^{n+1}=T_0^n+\frac{\alpha\Delta t}{(\Delta x)^2}\left(2T_1^n-2T_0^n\right)+\Delta t\cdot u_0^n$$

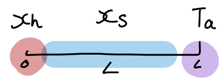
右边界 ($i=N$) $T_{N+1}=T_{N-1}$

$$T_N^{n+1}=T_N^n+\frac{\alpha\Delta t}{(\Delta x)^2}\left(2T_{N-1}^n-2T_N^n\right)+\Delta t\cdot u_N^n$$

内部节点 (interior Nodes)

$$T_i^{n+1}=T_i^n+\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{i+1}^n-2T_i^n+T_{i-1}^n\right)$$

1D assume $\Omega=[0,L]$



Boundary condition: non-insulated wall

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = -\frac{h}{k}(T_N^n - T_a)$$ Robin condition

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{h}{k}(T_0^n - T_a)$$

When $i=N$

$$\frac{\partial T}{\partial x} \approx \frac{T_{N+1}^n - T_{N-1}^n}{2\Delta x} = -\frac{h}{k}(T_N^n - T_a)$$

$$T_{N+1}^n = T_{N-1}^n - 2\Delta x \frac{h}{k}(T_N^n - T_a) \quad *$$

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t\, u_i^n$$

把*代入到前一页计算的通式中

$$T_N^{n+1} = T_N^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left[2T_{N-1}^n - 2T_N^n - 2\Delta x \frac{h}{k}(T_N^n - T_a)\right] + \Delta t\, u_N^n$$

同理可以推 $i=0$.

$$T_0^{n+1} = T_0^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left[2T_1^n - 2T_0^n - 2\Delta x \frac{h}{k}(T_0^n - T_a)\right] + \Delta t\, u_0^n$$

关于内部点式子，与隔热墙推导思路与结论相同

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t\, u_i^n$$

☆☆☆☆

1D情况下通式

散热因子γ $\begin{cases} \text{insulated}: \gamma=0 \\ \text{non-insulated}: \gamma=\frac{2h\Delta x}{k} \end{cases}$

左边界 ($i = 0$) —— 散热 + 热源：

$$T_0^{n+1} = T_0^n + \frac{\alpha\Delta t}{(\Delta x)^2}\left[2T_1^n - 2T_0^n - \gamma(T_0^n - T_a)\right] + \Delta t u_0^n$$

分析： 这里既有向室外散失的热量（γ 项），也有暖气片补进来的热量（u_0 项）。

内部节点 ($i = 1 \dots N-1$) —— 纯传导：

$$T_i^{n+1} = T_i^n + \frac{\alpha\Delta t}{(\Delta x)^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t u_i^n$$

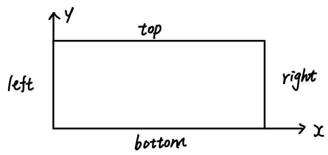
分析： 通常设 $u_i^n = 0$。但如果房间中间有一个火炉，你就可以令对应的 $u_i^n > 0$。

右边界 ($i = N$) —— 散热：

$$T_N^{n+1} = T_N^n + \frac{\alpha\Delta t}{(\Delta x)^2}\left[2T_{N-1}^n - 2T_N^n - \gamma(T_N^n - T_a)\right] + \Delta t u_N^n$$

分析： 通常右边界没有暖气，设 $u_N^n = 0$。

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + u(x,y,t)$$



u 热源位置　　(x_s, y_s) 温控器位置

$T(x_s, y_s, t)$

2D 假设房间 $\Omega = [0, L] \times [0, H]$

Boundary condition: 绝热墙壁 : $\frac{\partial T}{\partial n} = 0$

Boundary condition: 非绝热墙壁 $-k\frac{\partial T}{\partial n} = h(T - T_a) \Rightarrow \frac{\partial T}{\partial n} = -\frac{h}{k}(T - T_a)$

在2D中 $T_{i,j}$ 受到 $T_{i+1,j}$ $T_{i-1,j}$ $T_{i,j-1}$ $T_{i,j+1}$ 四个方向的传导

(x方向)　　(y方向)

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}\right) + \frac{\alpha \Delta t}{(\Delta y)^2}\left(T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}\right) + \Delta t u_{i,j}^{n}$$

assume $\Delta x = \Delta y$

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4T_{i,j}^{n}\right) + \Delta t u_{i,j}^{n}$$

参照 1D 的推导

B. 四个边界的更新式（以 $\Delta x = \Delta y$ 为例）

$\gamma = 0$ for insulated

$\gamma = \frac{2h\Delta x}{K}$ for non-insulated

1. **左边界 ($i = 0$):**

$$T_{0,j}^{n+1} = T_{0,j}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left[2T_{1,j}^{n} + T_{0,j+1}^{n} + T_{0,j-1}^{n} - 4T_{0,j}^{n} - \gamma_{left}(T_{0,j}^{n} - T_a)\right] + \Delta t u_{0,j}^{n}$$

2. **右边界 ($i = N$):**

$$T_{N,j}^{n+1} = T_{N,j}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left[2T_{N-1,j}^{n} + T_{N,j+1}^{n} + T_{N,j-1}^{n} - 4T_{N,j}^{n} - \gamma_{right}(T_{N,j}^{n} - T_a)\right] + \Delta t u_{N,j}^{n}$$

3. **底边界 ($j = 0$):**

$$T_{i,0}^{n+1} = T_{i,0}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left[T_{i+1,0}^{n} + T_{i-1,0}^{n} + 2T_{i,1}^{n} - 4T_{i,0}^{n} - \gamma_{bottom}(T_{i,0}^{n} - T_a)\right] + \Delta t u_{i,0}^{n}$$

4. **顶边界 ($j = M$):**

$$T_{i,M}^{n+1} = T_{i,M}^{n} + \frac{\alpha \Delta t}{(\Delta x)^2}\left[T_{i+1,M}^{n} + T_{i-1,M}^{n} + 2T_{i,M-1}^{n} - 4T_{i,M}^{n} - \gamma_{top}(T_{i,M}^{n} - T_a)\right] + \Delta t u_{i,M}^{n}$$