hybrid feedback. See, for example, [35]. We construct a robustly stabilizing hybrid feedback using the idea of PCLFs.

Define $r(\xi) := \sqrt{\xi_1^2 + \xi_2^2}$ and let the constants $\rho > 1$ and $\varepsilon > 0$ satisfy $\rho(1+\varepsilon) < 2$. Let $V_1, V_2 : \mathbb{R}^3 \to \mathbb{R}$ and $\Omega_1', \Omega_1, \Omega_2', \Omega_2 \subset \mathbb{R}^3$ be given by

$$\begin{aligned} V_1(\xi) &:= (1+\varepsilon)\sqrt{\rho|\xi_3|} - \xi_1, & V_2(\xi) &:= \frac{1}{2}\xi^\top \xi, \\ \Omega_1' &:= \mathbb{R}^3, & \Omega_2' &:= \{r(\xi)^2 > \rho|\xi_3|\}, \\ \Omega_1 &:= \mathbb{R}^3, & \Omega_2 &:= \{r(\xi)^2 > |\xi_3|\}. \end{aligned}$$

We show that these choices correspond to a PCLF with respect to $\mathcal{A} = \{0\}$ for the nonholonomic integrator control system.

Observe that $\overline{\Omega_2'} \setminus \mathcal{A} \subset \Omega_2$. Also, the unit outward normal vector to Ω_2, given by

$$n_2(\xi) := \frac{(-2\xi_1, \ -2\xi_2, \ \mathrm{sgn}(\xi_3))}{\sqrt{4r(\xi)^2 + 1}},$$

is continuous on $\partial\Omega_2 \setminus \mathcal{A}$ since $\xi_3 \neq 0$ in that set. For the same reason, V_1 is smooth on an open set containing $\overline{\mathbb{R}^3 \setminus \Omega_2'} \setminus \mathcal{A}$.

Next we bound V_1 and V_2, examine their directional derivatives, and determine whether it is possible to pick the controls to make these functions decrease while also satisfying the appropriate condition with respect to the unit outward normal vectors.

Since, for each $\xi \in \mathbb{R}^3 \setminus \Omega_2'$, we have $\sqrt{\rho|\xi_3|} \geq r(\xi) \geq \xi_1$, it follows that

$$0.5\varepsilon(\sqrt{\rho|\xi_3|} + r(\xi)) \leq V_1(\xi) \leq (1+\varepsilon)(\sqrt{\rho|\xi_3|} + r(\xi)), \quad \xi \in \mathbb{R}^3 \setminus \Omega_2'.$$

This condition implies the existence of class-$\mathcal{K}_\infty$ functions $\underline{\gamma}_1$ and $\overline{\gamma}_1$ such that $\underline{\gamma}_1(|\xi|) \leq V_1(\xi) \leq \overline{\gamma}_1(|\xi|)$ for all $\xi \in \Omega_1 \setminus \Omega_2'$. Also, $0.5|\xi|^2 \leq V_2(\xi) \leq 0.5|\xi|^2$ for all $\xi \in \Omega_2$.

Now we consider possible control choices. Take $u_{1,\xi} = (1, 0)$ for all $\xi \in \mathbb{R}^3$. Then, for all $\xi \in \mathbb{R}^3 \setminus (\Omega_2' \cup \mathcal{A})$,

$$\begin{aligned} \langle \nabla V_1(\xi), f_p(\xi, u_{1,\xi}) \rangle &= -1 + 0.5(1+\varepsilon)\rho \frac{-\xi_2}{\sqrt{\rho|\xi_3|}} \\ &\leq -1 + 0.5\,(1+\varepsilon)\,\rho < 0. \end{aligned}$$

Since there is no unit outward normal for Ω_1, this condition is all that needs to be checked for $q = 1$.

For $q = 2$, consider the control choice

$$u_{2,\xi} = \begin{bmatrix} -\xi_1 + 4\dfrac{\xi_2\xi_3}{r(\xi)^2} \\ -\xi_2 - 4\dfrac{\xi_1\xi_3}{r(\xi)^2} \end{bmatrix} \quad \text{for all } \xi \in \Omega_2 \setminus \mathcal{A}$$

which is continuous on $\Omega_2 \setminus \mathcal{A}$ and approaches zero as ξ approaches $\mathcal{A}$. We then have

$$\langle \nabla V_2(\xi), f_p(\xi, u_{2,\xi}) \rangle = -\xi_1^2 - \xi_2^2 - 4\xi_3^2 \quad \text{for all } \xi \in \Omega_2 \setminus \mathcal{A}$$

and, for all $\xi \in \partial\Omega_2 \setminus \mathcal{A}$,

$$\langle n_2(\xi), f_p(\xi, u_{2,\xi}) \rangle = \frac{2\xi_1^2 + 2\xi_2^2 - 4|\xi_3|}{\sqrt{4r(\xi)^2 + 1}} = -\frac{r(\xi)^2 + |\xi_3|}{\sqrt{4r(\xi)^2 + 1}}.$$

These calculations verify that the given patches and functions constitute a PCLF for the nonholonomic integrator system. Then, to construct a hybrid feedback control algorithm for this system following the presentation on PCLFs, we take $\Psi_1 = C_1 := \overline{\mathbb{R}^3 \setminus \Omega_2'}$, $C_2 := \Omega_2$, $\Psi_2 = \Omega_2'$, $\kappa_1(\xi) = u_{1,\xi}$ for all $\xi \in C_1$, $\kappa_2(\xi) = u_{2,\xi}$ for all $\xi \in C_2 \setminus \mathcal{A}$ and $\kappa_2(0) = 0$. Finally, the hybrid feedback stabilizer is defined using Ψ_q, C_q, and κ_q, $q \in \{1, 2\}$. ■

AN EXAMPLE BASED ON MULTIPLE LYAPUNOV FUNCTIONS

The final example uses many of the analysis tools that have been presented in this article. In particular, it uses Lyapunov functions, the invariance principle, results on stability with a finite number of events, and Theorem 32 based on the idea of multiple Lyapunov functions.

Example 37 Revisited: Stabilization on the Three-Sphere with Restricted Controls

We again consider the problem of stabilizing the point $\xi^* := (0, 0, 0, 1) \in \mathbb{S}^3$ for the kinematic equations in Example 37. This time, we restrict our attention to controls in the set $\mathcal{U} := \{u \in \mathbb{R}^3 : u_3 = u_1 u_2 = 0, \ |u| \leq 1\}$. This problem can be associated with stabilizing a desired orientation for an underactuated rigid body. For example, see [16]. We focus only on the local asymptotic stabilization problem. Following the ideas outlined in this article, this solution can be combined with other hybrid controllers to achieve global asymptotic stabilization. The ideas used here are taken from [84], where global asymptotic stabilization is achieved.

Controller Specification

We use a hybrid controller with state $q \in Q = \{1, 2, 3, 4, 5, 6\}$, a timer state $\tau \in [0, 1]$, a state $\varsigma \in \{-1, 1\}$, and a state $\varphi \in [0, \pi/4]$. The closed-loop system state is $x := (\xi, q, \tau, \varsigma, \varphi) \in \mathbb{R}^8$, constrained to the set $K := \{\xi \in \mathbb{S}^3 : \xi_4 \geq \varepsilon\} \times Q \times [0, 1] \times \{-1, 1\} \times [0, \pi/4]$, where $\varepsilon \in (0, 1)$. Our goal is to globally pre-asymptotically stabilize the set $\mathcal{A} := \{\xi^*\} \times Q \times [0, 1] \times \{-1, 1\} \times [0, \pi/4]$, resulting in local asymptotic stability of $\mathcal{A}$ when the constraint $\xi_4 \geq \varepsilon$ is removed.

Let $\sigma_i : \mathbb{N} \to \{0, 1\}$ satisfy $\sigma_i(i) = 1$ and $\sigma_i(j) = 0$ for $j \neq i$. For $q \in \{1, 2\}$, we use $u_i = -\sigma_i(q)\xi_i$. For $q \in \{3, 4, 6\}$, we

Synchronization in groups of biological oscillators occurs in swarms of fireflies, groups of crickets, ensembles of neuronal oscillators, and groups of heart muscle cells.

use $u_i = \varsigma\sigma_i(1)$. For $q = 5$, we use $u_i = \varsigma\sigma_i(2)$. For each $q \in Q$, the resulting feedback law is denoted $u = \kappa_q(x)$.

For $q \in \{1, 2\}$, we define $C_q := \{x \in K : \xi_q^2 \geq \mu(\xi_{3-q}^2 + \xi_3^2)\}$, where $\mu \in (0, 1/2)$. For $q \in \{3, 6\}$, we define $C_q := \{x \in K : -\varsigma\xi_1 \geq 0\}$. We also define $C_5 := \{x \in K : -\varsigma\xi_2 \geq 0\}$ and $C_4 := \{x \in K : \tau \in [0, \min\{\sqrt{1 - \xi_4}, \varphi\}]\}$. For $q \in Q$, we define $D_q = \overline{K \setminus C_q}$.

We pick the flow map so that $\dot{q} = \dot{\varsigma} = \dot{\varphi} = 0$. In addition, $\dot{\tau} = 1$ for $q = 4$ while $\dot{\tau} = 0$ for $q \neq 4$.

We pick the jump map G so that $x^+ \in G(x)$ gives the following relationships: $\xi^+ = \xi$; $\tau^+ = 0$; for $q \in \{3, 4, 5\}$, $q^+ = q + 1$; for $q \in \{1, 2, 6\}$, $q^+ = 3$ when $\xi_1^2 + \xi_2^2 < \hat{\mu}\xi_3^2$, where $\hat{\mu} > 2\mu/(1-\mu)$, $q^+ \in \mathcal{I}(\xi)$ when $\xi_1^2 + \xi_2^2 > \hat{\mu}\xi_3^2$, where $\mathcal{I}(\xi) := \{i \in \{1, 2\} : \xi_i^2 = \max\{\xi_1^2, \xi_2^2\}\}$, and $q^+ = \{3\} \cup \mathcal{I}(\xi)$ when $\xi_1^2 + \xi_2^2 = \hat{\mu}\xi_3^2$. The possible mode transitions are indicated in Figure 21.

When $q \in \{1, 2, 5, 6\}$, $\varsigma^+ \in \{s \in \{-1, 1\} : -s\xi_1 \geq 0\}$, when $q = 3$, $\varsigma^+ = \operatorname{sgn}(\xi_2\xi_3)$ for $\xi_2\xi_3 \neq 0$ and $\varsigma^+ = \{-1, 1\}$ for $\xi_2\xi_3 = 0$; when $q = 4$, $\varsigma^+ \in \{s \in \{-1, 1\} : s\xi_2 \geq 0\}$, When $q \neq 3$, $\varphi^+ = \varphi$; when $q = 3$, $\varphi^+ = 0.5\cot^{-1}(|\xi_2/\xi_3|)$ for $\xi_3 \neq 0$, $\varphi^+ = 0$ for $\xi_3 = 0$ and $\xi_2 \neq 0$, and $\varphi^+ \in [0, \pi/4]$ for $\xi_2 = \xi_3 = 0$.

Verifying the Conditions of Theorem 32 for the Closed-Loop System

Condition 1

Since $\xi^+ = \xi$, it follows from the definitions above that $G(D \cap \mathcal{A}) \subset \mathcal{A}$.

Condition 2

We associate events with transitions to modes $q = 3$. Events are indicated by red arrows in Figure 21. To assess stability with these primary events inhibited, we define secondary events, corresponding to transitions to modes $q \in \{4, 5, 6\}$. With primary events inhibited, there are no more than three secondary events. Therefore, according to Theorem 31, the system with primary events inhibited has $\mathcal{A}$ pre-asymptotically stable if the flow dynamics on C_q, $q \in \{3, 4, 5, 6\}$ has $\mathcal{A}$ pre-asymptotically stable, and switching back and forth between between $q = 1$ and $q = 2$ results in $\mathcal{A}$ being pre-asymptotically stable.

To assess pre-asymptotic stability of the flow dynamics on C_4, we use the Lyapunov function $W(x) = \rho\sqrt{V(\xi)} - \tau$, where $V(\xi) = 1 - \xi_4$ and $\rho \in (1, \sqrt{2})$. For all $x \in C_4 \setminus \mathcal{A}$, it follows that $(\rho - 1)\sqrt{V(\xi)} \leq W(x) \leq \rho\sqrt{V(\xi)}$. Also, since

$$\langle \nabla V(\xi), \Lambda(\kappa_4(x))\xi \rangle \leq |\xi_1| \leq \sqrt{2}\sqrt{V(\xi)},$$

for all $x \in C_4 \setminus \mathcal{A}$ it follows that

$$\langle \nabla W(x), F_4(x) \rangle \leq \frac{\rho}{\sqrt{2}} - 1 < 0,$$

where F_4 is the closed-loop flow map when $q = 4$.

To assess pre-asymptotic stability of the flow dynamics on C_3, C_5 and C_6 we use $W(x) = V(\xi)$. We obtain

$$\langle \nabla V(\xi), \Lambda(\kappa_q(x))\xi \rangle = \varsigma\xi_j \leq 0,$$

where $j = 2$ for $q = 5$ and $j = 1$ for $q \in \{3, 6\}$. This property establishes stability of $\mathcal{A}$, and since ξ_4 is bounded away from zero, there are no complete flowing solutions on C_3, C_5 or C_6, which implies that $\mathcal{A}$ is pre-asymptotically stable.

To assess pre-asymptotic stability of the combined flow and jump dynamics C_q, D_q, $q \in \{1, 2\}$, we use the Lyapunov function $W(x) = V(\xi)$. We obtain

$$\langle \nabla V(\xi), \Lambda(\kappa_q(x))\xi \rangle = -\xi_q^2 \leq -\frac{\mu}{\mu + 1}(1 - \xi_4^2) < 0$$
$$\text{for all } x \in C_q \setminus \mathcal{A}.$$

In addition, $V(\xi)$ does not change during jumps. So, to establish pre-asymptotic stability of $\mathcal{A}$ we just need to rule out complete solutions that jump only and do not converge. This behavior is ruled out by the fact that a jump to mode $q \in \{1, 2\}$ from a point not in $\mathcal{A}$ means that $\xi_q^2 \geq \xi_{3-q}^2 > 0$ and $\hat{\mu}\xi_3^2 \leq \xi_q^2 + \xi_{3-q}^2$, which implies that

$$\begin{aligned} \mu(\xi_{3-q}^2 + \xi_3^2) &\leq \mu(\xi_{3-q}^2 + \hat{\mu}^{-1}(\xi_{3-q}^2 + \xi_q^2)) \\ &\leq \mu(1 + 2\hat{\mu}^{-1})\xi_q^2 < \xi_q^2. \end{aligned}$$

In other words, a jump from $D_{3-q} \setminus \mathcal{A}$ lands at a point not in D_q.

These calculations establish condition 2) of Theorem 32.

Conditions 3 and 4

We take $W(x) = 1 - \xi_4$, which satisfies Condition 3 with $\alpha_1(s) = \alpha_2(s) = 2\alpha_3(s) = 2s$ for all $s \geq 0$. Now we establish Condition 4. We start from a hybrid time where a

The concepts of average dwell-time switching and multiple Lyapunov functions, which are applicable to switched systems, extend to hybrid systems.

jump to $q=3$ has just occurred. By the time a jump to $q=4$ occurs, $W(x)$ has not increased and we have $\xi_1=0$ and $\xi_2^2 \le \hat{\mu}\xi_3^2$. When $q=4$, $W(x)$ increases, but we argue that the sequence of modes $q=4$, $q=5$, $q=6$ results in a decrease in $W(x)$. Since $W(x)$ is also strictly decreasing for $q \in \{1,2\}$, this implies that when $q=3$ is revisited, $W(x)$ has decreased. The key to showing this property is to establish that, with $\chi_{i,q}$ denoting ξ_i at the end of mode q, there exists a continuous function ρ that is less than one except when its argument is one, so that $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$. Indeed, using that $\chi_{1,6}=\chi_{2,5}=0$ and $\chi_{2,6}^2+\chi_{3,6}^2=\chi_{2,5}^2+\chi_{2,6}^2$, this gives

$$\begin{aligned}1-\chi_{4,6}^2 &= \chi_{1,6}^2+\chi_{2,6}^2+\chi_{3,6}^2\\ &= \chi_{2,6}^2+\chi_{3,6}^2\\ &= \chi_{2,5}^2+\chi_{3,5}^2\\ &= \chi_{3,5}^2\\ &\le \rho^2(\chi_{4,3})\chi_{3,3}^2\\ &\le \rho^2(\chi_{4,3})(1-\chi_{4,3}^2).\end{aligned}$$

Since $\chi_{4,6}$ and $\chi_{4,3}$ are positive, this implies

$$1-\chi_{4,6} \le \rho^2(\chi_{4,3})(1-\chi_{4,3}).$$

Finally, we establish $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$. Let τ_q, $q \in \{4,5\}$, denote the time spent in mode q. Let ς_q, $q \in \{4,5\}$, denote the value of ς in mode q. A routine calculation involving the solution of a linear, two-dimensional oscillator and using that $\chi_{1,3}=\chi_{2,5}=0$ gives

$$\begin{aligned}\cos(\varsigma_4\tau_4)\chi_{3,5} = \cos(\varsigma_5\tau_5)[&(\cos(\varsigma_4\tau_4)^2-\sin(\varsigma_4\tau_4)^2)\chi_{3,3}\\ &-2\cos(\varsigma_4\tau_4)\sin(\varsigma_4\tau_4)\chi_{2,3}].\end{aligned}$$

Due to the construction of the jump map from mode $q=3$, $\varsigma 4=\operatorname{sgn}(\chi_{2,3}\chi_{3,3})$ when $\chi_{2,3}\chi_{3,3}\neq 0$; otherwise $\varsigma_4 \in \{-1,1\}$. Then, due to the value of φ in mode $q=4$, which limits τ_4, it follows that the sign of $\sin(\varsigma_4\tau_4)\chi_{2,3}$ is the same as the sign of $\chi_{3,3}$. Using $\tau_4 \le \pi/4$ and $\varsigma_4 \in \{-1,1\}$, it follows that

$$|\chi_{3,5}| \le \frac{\cos(\tau_4)^2-\sin(\tau_4)^2}{\cos(\tau_4)}|\chi_{3,3}|.$$

The function involving τ_4 on the right-hand side takes values in the interval $[0,1)$ for all $\tau_4 \in (0,\pi/4]$. Now, by the definition of C_4, κ_4, and the fact that $\chi_{1,3}=0$, τ_4 can be expressed as a continuous function of $\chi_{4,3}$ that is zero when $\chi_{4,3}=1$ and is positive otherwise. In turn, this establishes the bound $|\chi_{3,5}| \le \rho(\chi_{4,3})|\chi_{3,3}|$ for a continuous, nonnegative-valued function ρ that is less than one except when $\chi_{4,3}=1$. ■

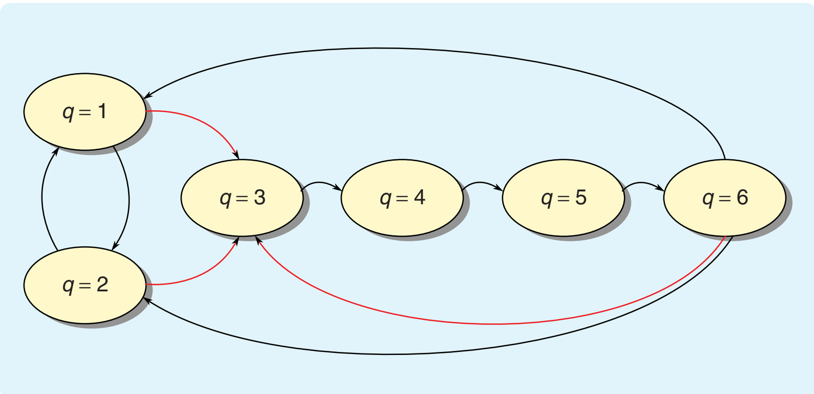

FIGURE 21 Possible mode transitions when stabilizing a point on the three-sphere by hybrid feedback that uses only two angular velocities, one at a time. Normal operation corresponds to jumps between modes $q=1$ and $q=2$. However, at times the sequence of modes $3 \to 4 \to 5 \to 6$ is required. The initiation of such a sequence is associated with an event in the hybrid system. Transitions associated with events are indicated by red arrows in the diagram. The quantity $V(\xi)=1-\xi_4$ does not increase along solutions except when $q=4$. When events are inhibited, so that the mode $q=4$ is reached no more than once, the resulting hybrid system has the correct pre-asymptotically stable set. With events included and $V(\xi)$ considered at event times, the resulting values of $V(\xi)$ constitute a strictly decreasing sequence. Thus, Theorem 32 can be used to establish pre-asymptotic stability for the hybrid system.

CONCLUSIONS

Hybrid dynamical systems combine flows and jumps. They can be modeled in a compact form, and they cover a fascinating variety of dynamic phenomena.

With the use of hybrid time domains and the notion of graphical convergence, sequential compactness of the space of solutions and semicontinuous dependence of solutions on initial conditions and perturbations can be established under mild conditions.

Basic questions about solutions to dynamical systems concern existence, uniqueness, and dependence on initial conditions and other parameters.

The properties of the space of solutions to a hybrid system have important consequences for stability theory. For example, these properties imply that asymptotic stability of a compact set is uniform and robust to perturbations. The properties also facilitate extensions of the classical invariance principle and converse Lyapunov theorems to the hybrid setting. The hybrid invariance principle and Lyapunov functions lend themselves to natural sufficient conditions for asymptotic stability in a hybrid system. Additional stability analysis tools, related to identifying and limiting events in a hybrid system, can also be developed.

The stability analysis tools can be used to predict the behavior of hybrid systems and to design hybrid control algorithms. The examples of hybrid control systems provided in this article only scratch the surface of what is possible using hybrid feedback control. The framework and tools presented in this article may help in the process of discovering new hybrid feedback control ideas.

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Advancing the Field

With no other choice but to live one day at a time and to keep intellectually alive, Tsien continued to work and teach and await the next step, whatever it might be, whenever it might be. He turned to other fields of research, such as the study of games and economic behavior. In 1954, he published a textbook titled *Engineering Cybernetics*, a book on systems of communication and control. It too would be well received.

Years later, Wallace Vander Velde, an MIT professor and renowned expert in cybernetics, would describe the book as "remarkable" and "an extraordinary achievement in its time." Wrote Vander Velde of the book:

> In 1954, a decent theory of feedback control for linear, time invariant systems existed and servomechanism design was an established practice. But Tsien was looking ahead to more complex control and guidance problems—notably the guidance of rocket-propelled vehicles. This stimulated his interest in the systems with time-varying coefficients, time lag and nonlinear behavior. All these topics are treated in this book.

But Tsien went further to deal with optimal control via the variational calculus, optimalizing control and fault-tolerant control systems among other topics! He visualized a theory of guidance and control which would be distinct from, and would support, the practice of these disciplines. This has certainly come to be, and his pioneering effort may be thought of as a major foundation stone of that effort which continues to this day.

—*Thread of the Silkworm*, by Iris Chang, BasicBooks, New York, 1994, pp. 175–176.