

## Motivating Stability of Sets

Intuition built on the theory of linear systems conditions us to think of asymptotic stability as a property of an equilibrium point. However, it is also natural to consider the asymptotic stability of a set. Conceptually, set stability places many apparently different phenomena under one umbrella. Set stability is needed for systems that include timers, counters, and other discrete states that do not converge. In addition, set stability is helpful for characterizing systems that exhibit complicated asymptotic behavior.

Asymptotic stability of a set is defined in the main text. Roughly speaking, a set is globally asymptotically stable if each solution that starts close to the set remains close to the set, each solution is bounded, and each solution with an unbounded time domain converges to the set.

To illustrate set stability, we consider a linear, sampled-data control system. We group the plant state, hold state, and controller state into the state  $\xi \in \mathbb{R}^n$ . Between sampling events, the state  $\xi$  evolves according to a closed-loop, linear differential equation  $\dot{\xi} = F\xi$  for some matrix  $F$ . For each state component that remains constant between sampling events, the corresponding row of  $F$  is filled with zeros. At sampling events, the state  $\xi$  is updated according to the equation  $\xi^+ = J\xi$  for some matrix  $J$ . For each state component that does not change at sampling events, the corresponding row of  $J$  is filled with zeros except for a one in the appropriate column. The interaction between the continuous evolution  $\dot{\xi} = F\xi$  and the discrete evolution  $\xi^+ = J\xi$  is scheduled by a timer state  $\tau$  that evolves according to  $\dot{\tau} = 1$  between sampling events and is reset to zero at sampling events. Sampling events occur whenever the timer state reaches the value  $T > 0$ , which denotes the sampling period of the system. Thus, the sampled-data control system can be written as a hybrid system with state  $x = (\xi, \tau)$  satisfying

$$\begin{cases} \dot{\xi} = F\xi \\ \dot{\tau} = 1 \end{cases} \quad x \in C := \mathbb{R}^n \times [0, T],$$

$$\begin{cases} \xi^+ = J\xi \\ \tau^+ = 0 \end{cases} \quad x \in D := \mathbb{R}^n \times \{T\}.$$

The matrices  $F$  and  $J$  are designed so that  $\xi$  converges to zero. However, the timer state  $\tau$  does not converge. Letting  $\tau_0 \in [0, T]$  denote the initial value of  $\tau$ , the solution satisfies  $\tau(t, j) = t - (jT - \tau_0)$  for all positive integers  $j$  and all  $t \in [jT - \tau_0, (j+1)T - \tau_0]$ . Consequently,  $\tau$  revisits every point in the interval  $[0, T]$ . For this reason, the hybrid system does not possess an asymptotically stable equilibrium point. On the other hand, the compact set  $\mathcal{A} := \{0\} \times [0, T]$  is globally

and the corresponding basin of pre-attraction are preserved under the perturbation  $\sigma$ . This result provides a key step in the proof of the existence of a smooth Lyapunov function for a hybrid system with a pre-asymptotically stable compact set. For more information, see "Converse Lyapunov Theorems."

asymptotically stable. In particular, each solution to the hybrid system converges to the compact set  $\mathcal{A}$ . This property captures the fact that the state  $\xi$  converges to zero.

As another example of set stability, consider a system switching among a family of asymptotically stable linear systems under an average dwell-time constraint, as discussed in "Switching Systems." The corresponding hybrid system has the form

$$\begin{cases} \dot{\xi} = F_q \xi \\ \dot{q} = 0 \\ \dot{\tau} \in [0, \delta] \end{cases} \quad C := \mathbb{R}^n \times Q \times [0, N],$$

$$\begin{cases} \xi^+ = \xi \\ q^+ \in Q \\ \tau^+ = \tau - 1 \end{cases} \quad D := \mathbb{R}^n \times Q \times [1, N],$$

where  $Q \subset \mathbb{R}$  is compact and the eigenvalues of  $F_q$  have negative real part for each  $q \in Q$ . When the average dwell-time parameter  $\delta > 0$  is small enough, the compact set  $\{0\} \times Q \times [0, N]$  is asymptotically stable. In particular, the state  $\xi$  converges to zero. The states  $q$  and  $\tau$  may or may not converge, depending on the particular solution considered.

In some situations, hybrid systems admit complicated, asymptotically stable compact sets that can be characterized through the concept of an  $\Omega$ -limit set [S33].  $\Omega$ -limit sets are exploited for nonlinear control design, for example for nonlinear output regulation [S30]. Results pertaining to  $\Omega$ -limit sets for hybrid systems are given in [S32] and [S31]. In [S32, Cor. 3] and [S31, Thm. 1], conditions are given under which the  $\Omega$ -limit set is an asymptotically stable compact set. In particular, suppose there exist  $T > 0$  and compact sets  $K_0$  and  $K_T$ , with  $K_T$  contained in the interior of  $K_0$ , such that each solution  $x$  starting in  $K_0$  is bounded and, for all  $(t, j) \in \text{dom } x$  satisfying  $t + j \geq T$ , we have  $x(t, j) \in K_T$ . Then, either the  $\Omega$ -limit set from  $K_0$  is empty or it is the smallest compact, asymptotically stable set contained in  $K_T$ .

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## Theorem 15 [14, Theorem 7.9]

For the hybrid system  $\mathcal{H}$  satisfying the Basic Assumptions, suppose that a compact set  $\mathcal{A} \subset \mathbb{R}^n$  is pre-asymptotically stable with basin of pre-attraction  $\mathcal{B}_{\mathcal{A}}$ . Then there exists a continuous function  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $\sigma(x) > 0$  for

## Why “Pre”-Asymptotic Stability?

In engineered hybrid systems, it is reasonable to insist that each maximal solution that starts sufficiently near an asymptotically stable set has an unbounded time domain. Recall that a solution with an unbounded time domain is called a complete solution. However, a complete solution does not necessarily have a time domain that is unbounded in the  $t$  direction; see “Zeno Solutions” for more information on systems with complete solutions having domains that are bounded in the  $t$  direction.

The definition of global asymptotic stability used in the main text, which we call global pre-asymptotic stability, does not stipulate completeness of each maximal solution. A compact set  $\mathcal{A}$  is globally pre-asymptotically stable if each solution that starts close to  $\mathcal{A}$  remains close to  $\mathcal{A}$ , each solution is bounded, and each complete solution converges to  $\mathcal{A}$ . Thus, solutions do not need to be complete, but complete solutions must converge to  $\mathcal{A}$ . A set is globally asymptotically stable when it is globally pre-asymptotically stable and all solutions are complete.

Completeness of solutions is not required in the pre-asymptotic stability definition as a matter of convenience. For example, since completeness does not need to be verified, sufficient conditions for pre-asymptotic stability are simpler than they would be otherwise. When completeness is important, it can be established by verifying pre-asymptotic stability together with local existence of solutions.

Conceptually, pre-asymptotic stability offers advantages over asymptotic stability. On the one hand, pre-asymptotic stability is all that is needed for the existence of smooth Lyapunov functions. See “Converse Lyapunov Theorems” for more details. Pre-asymptotic stability also makes it easy to express local pre-asymptotic stability in terms of global pre-asymptotic stability, as in the next theorem [14], by considering a reduced system that is contained in the original system. The system  $(C_{\circ}, F_{\circ}, D_{\circ}, G_{\circ})$

is contained in the system  $(C, F, D, G)$  if  $C_{\circ} \subset C$ ,  $F_{\circ}(x) \subset F(x)$  for all  $x \in C_{\circ}$ ,  $D_{\circ} \subset D$  and  $G_{\circ}(x) \subset G(x)$  for all  $x \in D_{\circ}$ . Completeness of solutions for  $(C, F, D, G)$  does not guarantee completeness of solutions for  $(C_{\circ}, F_{\circ}, D_{\circ}, G_{\circ})$ .

### Theorem S10

Suppose that, for the hybrid system  $(C, F, D, G)$ , the compact set  $\mathcal{A}$  is pre-asymptotically stable with basin of pre-attraction  $B_{\mathcal{A}}$ . Then, for each hybrid system  $(C_{\circ}, F_{\circ}, D_{\circ}, G_{\circ})$  that is contained in  $(C, F, D, G)$ , the set  $\mathcal{A}$  is pre-asymptotically stable with basin of pre-attraction containing  $B_{\mathcal{A}}$ . In particular, if there exists a compact set  $K \subset B_{\mathcal{A}}$  such that  $C_{\circ} \cup D_{\circ} \subset K$ , then the set  $\mathcal{A}$  is globally pre-asymptotically stable for  $(C_{\circ}, F_{\circ}, D_{\circ}, G_{\circ})$ .

One particular application of Theorem S10 says that if the hybrid system  $\mathcal{H}$  has the compact set  $\mathcal{A}$  pre-asymptotically stable, then so does the hybrid system that uses only the continuous-time data  $(C, F)$  and so does the hybrid system that uses only the discrete-time data  $(D, G)$ . The converse of this assertion holds only when the separate systems admit a common Lyapunov function. See “Converse Lyapunov Theorems” for more details.

It is reasonable to question the utility of studying systems for which no solutions are complete. One motivation comes from the results discussed in the section “Hybrid Feedback Control Based on Limited Events”, where it is established that pre-asymptotic stability of a compact set for a system with events inhibited implies pre-asymptotic stability of the compact set for the system with events allowed as long as events are not too frequent. In this statement, pre-asymptotic stability is useful since the system with inhibited events may not exhibit complete solutions, whereas the system with events allowed may exhibit complete solutions.

all  $x \in B_{\mathcal{A}} \setminus \mathcal{A}$  such that, for the hybrid system  $\mathcal{H}_{\sigma}$ , the compact set  $\mathcal{A}$  is pre-asymptotically stable with basin of pre-attraction  $B_{\mathcal{A}}$ .

The idea in the next example is related to Lebesgue sampling presented in [3] and to the ideas in [80] used for control in the presence of communication or power constraints. The idea exploits the robustness described in Theorem 15, but for the special case of classical systems, to explain the asymptotic stability induced by a particular hybrid control strategy.

### Example 16: Control Through Event-Based Hold Updates

Consider the continuous-time, nonlinear control system  $\dot{\xi} = f(\xi, u)$ ,  $(\xi, u) \in \mathbb{R}^n \times \mathbb{R}^m$ , with the continuous state feedback  $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that stabilizes a compact set  $\mathcal{A}$  with basin of attraction  $B_{\mathcal{A}} \subset \mathbb{R}^n$ . Suppose the feedback is implemented by keeping the control value constant until an event triggers a change in the control value. The trigger comes

from sensing that the state deviates from the state used to compute the control value by an amount that is significant enough to warrant a change in the input value. This amount is determined by a continuous function  $\sigma_1 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  related to the function  $\sigma$  in Theorem 15.

The closed-loop system can be modeled as a hybrid system with the data

$$\begin{aligned} C &:= \{x \in \mathbb{R}^{2n} : |x_1 - x_2| \leq \sigma_1(x_1)\}, \\ F(x) &:= \begin{bmatrix} f(x_1, \kappa(x_2)) \\ 0 \end{bmatrix} \text{ for all } x \in C, \\ D &:= \{x \in \mathbb{R}^{2n} : |x_1 - x_2| \geq \sigma_1(x_1)\}, \\ G(x) &:= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \text{ for all } x \in D, \end{aligned}$$

where  $\sigma_1$  is specified as follows. Let  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  satisfy the conditions of Theorem 15 for the (classical) system

## Converse Lyapunov Theorems

**D**espite the Lyapunov-based sufficient conditions for pre-asymptotic stability given by Theorem 32 and [6, 19, 90], it is a mistake to conclude that Lyapunov functions do not exist for asymptotically stable hybrid systems. Early results on the existence of Lyapunov functions for hybrid systems can be found in [90]. More recently, [14] establishes that a hybrid system with a pre-asymptotically stable compact set admits a smooth ( $C^\infty$ ) Lyapunov function as long as the hybrid system satisfies the Basic Assumptions. Converse theorems of this type are typically established theoretically, constructing smooth Lyapunov functions from the system's solutions, which are not usually available explicitly. Thus, converse theorems are of limited help in constructing Lyapunov functions. Nevertheless, converse theorems justify searching for Lyapunov functions that can be constructed without knowledge of the system's trajectories.

Converse theorems also play a role in establishing stabilization results for nonlinear control systems. For example, early results [S40, Theorem 4] and [S34, Theorem 2.1] on backstepping, that is, proving that smooth stabilizability is not destroyed by adding an integrator, construct a stabilizing feedback by using the gradient of a Lyapunov function whose existence is guaranteed by a converse theorem. Also, the result in [76] showing that smooth stabilization implies coprime factorization, that is, input-to-state stabilization with respect to disturbances that add to the control variable, exploits a converse Lyapunov theorem. Similar results apply for hybrid systems. For example, in [14] converse theorems are developed and used to show that smooth stabilization with logic-based feedback implies smooth,

logic-based input-to-state stabilization with respect to matched disturbances. Converse Lyapunov theorems can also be used to establish various forms of robustness, including robustness to small, persistent perturbations.

The converse theorems for hybrid systems given in [14] draw heavily from the literature on converse theorems for continuous-time and discrete-time systems. We recount some of the major milestones in the development of converse Lyapunov theorems for time-invariant, finite-dimensional dynamical systems having compact, asymptotically stable sets.

In his 1892 Ph.D. dissertation, in particular [S38, Section 20, Theorem II], Lyapunov provides the first contribution to converse theorems, where he addresses asymptotically stable linear systems. Generalizations of this result to nonlinear systems did not appear until the 1940s and 1950s. For example, the 1949 paper [S39] provides a converse Lyapunov theorem for time-invariant, continuously differentiable systems having a locally asymptotically stable equilibrium.

Essentially all of the converse Lyapunov theorems until [42] pertain only to dynamical systems with unique solutions. In contrast, [42] establishes the first converse Lyapunov theorems for differential equations with continuous right-hand side without assuming uniqueness of solutions. This extension is significant in anticipation of hybrid systems where nonuniqueness of solutions is not uncommon. The contribution of [42] to the development of converse theorems is immense; it appears to be the first work that relies explicitly on robustness and advanced smoothing techniques to establish the existence of smooth Lyapunov functions.

$\dot{\xi} = \tilde{F}(\xi) := f(\xi, \kappa(\xi))$ . Take  $\sigma_1: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  to be continuous, positive on  $\mathcal{B}_A \setminus \mathcal{A}$ , zero on  $\mathcal{A}$ , such that  $\sigma_1(x_1) \leq \sigma(x_1)$  for all  $x_1 \in \mathcal{B}_A$  and

$$|f(x_1, \kappa(x_2)) - f(x_2, \kappa(x_2))| \leq \sigma(x_1) \\ \text{for all } x_1 \in \mathcal{B}_A, x_2 \in x_1 + \sigma_1(x_1)\mathbb{B}.$$

For all  $x \in C$ , it follows that

$$f(x_1, \kappa(x_2)) = f(x_2, \kappa(x_2)) + f(x_1, \kappa(x_2)) - f(x_2, \kappa(x_2)) \\ \in \overline{\text{con}} \tilde{F}(x_1 + \sigma(x_1)\mathbb{B}) + \sigma(x_1)\mathbb{B}.$$

The jumps do not change  $x_1$ . Thus, according to Theorem 15,  $x_1$  stays close to  $\mathcal{A}$  and converges to  $\mathcal{A}$  whenever the solution has a hybrid time domain that is unbounded in the  $t$  direction. For such solutions,  $x_2$  also stays close to  $\mathcal{A}$  and converges to  $\mathcal{A}$ . For solutions with a hybrid time domain bounded in the  $t$  direction, notice that when  $x \in D$  but  $x_1 \notin \mathcal{A}$ , it follows that  $G(x) \notin D$ . Thus, the only way that a solution can have a domain bounded in the  $t$  direction is if  $x_1$  converges to  $\mathcal{A}$ . Whenever  $x_1$  converges to  $\mathcal{A}$ , so

does  $x_2$ . It thus follows for the closed-loop system that the set  $\mathcal{A} \times \mathcal{A}$  is asymptotically stable with basin of attraction  $\mathcal{B}_A \times \mathbb{R}^n$ . ■

For problems where the size of the perturbations does not become arbitrarily small as the state approaches the set  $\mathcal{A}$  or the boundary of the basin of pre-attraction, the following theorem is relevant.

### Theorem 17 [26, Theorem 6.6]

For a hybrid system  $\mathcal{H}$  satisfying the Basic Assumptions, suppose that a compact set  $\mathcal{A}$  is pre-asymptotically stable with basin of pre-attraction  $\mathcal{B}_A$ . In particular, suppose that there exist  $\beta \in \mathcal{KL}$  and a proper indicator function  $\omega$  for  $\mathcal{A}$  on  $\mathcal{B}_A$  such that, for all solutions starting in  $\mathcal{B}_A$ ,

$$\omega(x(t, j)) \leq \beta(\omega(x(0, 0)), t + j) \quad \text{for all } (t, j) \in \text{dom } x.$$

Then, for each  $\varepsilon > 0$  and compact set  $K \subset \mathcal{B}_A$ , there exists  $\delta > 0$  such that, with the function  $\sigma$  defined as  $\sigma(x) = \delta$  for all  $x \in \mathbb{R}^n$ , each solution to the hybrid system  $\mathcal{H}_\sigma$  starting in  $K$  satisfies

The converse theorem of [S41] is noteworthy for its smoothing technique, which is adopted in more recent converse results. We emphasize that continuously differentiable Lyapunov functions are needed to establish robustness in a straightforward manner, yet establishing the existence of continuously differentiable Lyapunov functions is challenging for systems with nonunique solutions.

Continuing down the path of systems with nonunique solutions, in the context of the input-to-state stability property, the work in [S37] establishes a converse Lyapunov theorem for locally Lipschitz differential inclusions, establishing smoothness of the Lyapunov function. This work is extended in [S36] to differential inclusions satisfying assumption (A2) of the Basic Assumptions with  $C = \mathbb{R}^n$ . The approach of [S36] again emphasizes the fundamental link between robustness and smoothness of the Lyapunov function.

The results in [14] draw heavily on earlier smoothing techniques, especially the smoothing techniques of [S36] adapted to discrete-time inclusions, as in [S27]. Here we state a simple version of a converse Lyapunov theorem in [14].

#### **Theorem S11 [14, Thm. 3.14]**

For the hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfying the Basic Assumptions, if the compact set  $\mathcal{A}$  is globally pre-asymptotically stable, then there exist a  $C^\infty$  function  $V: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$\omega(x(t, j)) \leq \beta(\omega(x(0, 0)), t + j) + \varepsilon \quad \text{for all } (t, j) \in \text{dom } x. \quad (22)$$

The property concluded in Theorem 17 is referred to as either semiglobal, practical robustness to persistent perturbations or semiglobal, practical pre-asymptotic stability. “Semiglobal” refers to the fact that the bound (22) can be achieved from an arbitrary, compact subset of the basin of pre-attraction. “Practical” refers to the fact that the value of  $\varepsilon$  in (22) can be made arbitrarily small.

We now discuss two applications of Theorem 17. An additional application can be found in “Zeno Solutions,” where temporal regularization is discussed.

#### **Example 18: Hybrid Control with Slowly Varying Parameters**

Consider the parameterized nonlinear control system  $\dot{\xi} = F_1(\xi, p, u)$ ,  $\xi \in \mathbb{R}^{n_1}$ , where, for now,  $p$  represents a constant parameter taking values in a compact set  $\mathcal{P} \subset \mathbb{R}^{n_2}$ . Consider also a dynamic, hybrid controller with state  $\eta \in \mathbb{R}^{n_3}$  perhaps consisting of timers, discrete states, and continuous states, designed under the assumption that

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq -V(x) && \text{for all } x \in C, f \in F(x), \\ V(g) &\leq \frac{V(x)}{2} && \text{for all } x \in D, g \in G(x). \end{aligned}$$

Extensions of this result to more general asymptotic stability notions are given in [S35]. A version for local pre-asymptotic stability is given in [14], which also specializes Theorem S11 to the case of hybrid systems with discrete states and gives a Lyapunov-based proof of robustness to various sources of perturbations, including slowly varying parameters.

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$p$  is constant. The resulting hybrid closed-loop system with state  $x = (\xi, p, \eta)$  has the data  $(C, F, D, G)$ , where  $\dot{x} \in F(x)$  implies  $\dot{p} = 0$ ,  $x^+ \in G(x)$  implies  $p^+ = p$ , and  $C \cup D \subset \mathbb{R}^{n_1} \times \mathcal{P} \times \mathbb{R}^{n_3}$ . Moreover, suppose that the compact set  $\mathcal{A} \subset \mathbb{R}^{n_1} \times \mathcal{P} \times \mathbb{R}^{n_3}$  is asymptotically stable. We now consider what happens when  $p$  can vary, slowly during flows and with small changes during jumps. Such variations are captured within the modeling of perturbations in (16)–(19) with  $\sigma(x) \equiv \bar{\sigma} > 0$  since  $\dot{x} \in F_\sigma(x)$  implies  $\dot{p} \in \bar{\sigma}\mathbb{B}$  and  $x^+ \in G_\sigma(x)$  implies  $p^+ \in p + \bar{\sigma}\mathbb{B}$ . Hence, we conclude with the help of Theorem 17 that the pre-asymptotic stability of the compact set  $\mathcal{A}$  is semiglobally, practically robust to small parameter variations. Similar results for differential equations are pointed out in [40] using results from [37]. ■

The following corollary of Theorem 17 is a reduction result that is related to results for continuous-and discrete-time nonlinear systems in [38] and [28], respectively.

#### **Corollary 19**

Consider a hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfying the Basic Assumptions. If the compact set  $\mathcal{A}_1$  is globally

pre-asymptotically stable for  $\mathcal{H}$  and the compact set  $\mathcal{A}_2 \subset \mathcal{A}_1$  is globally pre-asymptotically stable for  $\mathcal{H}|_{\mathcal{A}_1} := (C \cap \mathcal{A}_1, F, D \cap \mathcal{A}_1, G \cap \mathcal{A}_1)$ , then  $\mathcal{A}_2$  is globally pre-asymptotically stable for  $\mathcal{H}$ .

Corollary 19 follows from Theorem 17 since solutions that start close to  $\mathcal{A}_2$  start close to  $\mathcal{A}_1$  and thus stay close to  $\mathcal{A}_1$ . Thus the solutions of  $\mathcal{H}$  are contained in a small perturbation of  $\mathcal{H}|_{\mathcal{A}_1}$ . Moreover, these perturbations vanish with time since  $\mathcal{A}_1$  is assumed to be globally pre-asymptotically stable.

Corollary 19 can be applied to analyze cascaded systems [75]. For example, consider the classical system with state  $x = (x_1, x_2)$  and dynamics

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2) \end{cases} = F(x), \quad (23)$$

where the functions  $f_1$  and  $f_2$  are continuous, the origin of  $\dot{x}_2 = f_2(x_2)$  is globally asymptotically stable, and the origin of  $\dot{x}_1 = f_1(x_1, 0)$  is globally asymptotically stable. For the system

$$\dot{x} = F(x), \quad x \in C = M\mathbb{B} \times M\mathbb{B}, \quad M > 0, \quad (24)$$

it follows that the compact set  $\mathcal{A}_1 = M\mathbb{B} \times \{0\}$  is globally pre-asymptotically stable and that the system  $\dot{x} = F(x)$ ,  $x \in C \cap \mathcal{A}_1$  has the compact set  $\mathcal{A}_2 = \{(0, 0)\}$  globally pre-asymptotically stable. We conclude from Corollary 19 that the origin of the system (24) is globally pre-asymptotically stable. This conclusion means that, for the original system (23), the origin is asymptotically stable. Moreover, since  $M$  is arbitrary in (24), the basin of attraction for the origin for the system (23) contains the set of initial conditions from which each solution of (23) is bounded.

## STABILITY ANALYSIS USING LYAPUNOV FUNCTIONS

Lyapunov functions are familiar to most control engineers. These functions provide a way of establishing asymptotic stability without having to construct the system's solutions explicitly, a daunting task for almost anything other than a linear system. Sufficient Lyapunov conditions for asymptotic stability are well known for both continuous- and discrete-time systems. These conditions amount to finding a nonnegative-valued function that is strictly decreasing along solutions. In the case of a smooth Lyapunov function for a continuous-time system, the decrease condition can be verified from negative definiteness of the inner product of the Lyapunov function's gradient and the vector field that generates solutions. The standard Lyapunov-based sufficient conditions for asymptotic stability are covered in the version below for hybrid dynamical systems; see also [72, Cor. 7.7]. First, we give a definition.

Given the hybrid system  $\mathcal{H}$  with data  $(C, F, D, G)$  and the compact set  $\mathcal{A} \subset \mathbb{R}^n$ , the function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a *Lyapunov-function candidate* for  $(\mathcal{H}, \mathcal{A})$  if i)  $V$  is continuous and nonnegative on  $(C \cup D) \setminus \mathcal{A} \subset \text{dom } V$ , ii)  $V$  is

continuously differentiable on an open set  $\mathcal{O}$  satisfying  $C \setminus \mathcal{A} \subset \mathcal{O} \subset \text{dom } V$ , and iii)

$$\lim_{\{x \rightarrow \mathcal{A}, x \in \text{dom } V \cap (C \cup D)\}} V(x) = 0.$$

Conditions i) and iii) hold when  $\text{dom } V$  contains  $\mathcal{A} \cup C \cup D$ ,  $V$  is continuous and nonnegative on its domain, and  $V(z) = 0$  for all  $z \in \mathcal{A}$ . These conditions are typical of Lyapunov-function candidates for discrete-time systems. Condition ii) holds when  $V$  is continuously differentiable on an open set containing  $C \setminus \mathcal{A}$ , which is typical of Lyapunov-function candidates for continuous-time systems. We impose continuous differentiability for simplicity, but it is possible to work with less regular Lyapunov functions and their generalized derivatives. When  $x = (\xi, q) \in \mathbb{R}^n \times Q$ , where  $Q$  is a discrete set, it is natural to define  $V$  only on  $\mathbb{R}^n \times Q$ . To satisfy condition ii), the definition of  $V$  can be extended to a neighborhood of  $\mathbb{R}^n \times Q$ , with  $V(\xi, q) = V(\xi, q_0)$  for all  $q$  near  $q_0 \in Q$ .

We now state a hybrid Lyapunov theorem.

## Theorem 20

Consider the hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfying the Basic Assumptions and the compact set  $\mathcal{A} \subset \mathbb{R}^n$  satisfying  $G(\mathcal{A} \cap D) \subset \mathcal{A}$ . If there exists a Lyapunov-function candidate  $V$  for  $(\mathcal{H}, \mathcal{A})$  such that

$$\begin{aligned} \langle \nabla V(x), f \rangle &< 0 & \text{for all } x \in C \setminus \mathcal{A}, f \in F(x), \\ V(g) - V(x) &< 0 & \text{for all } x \in D \setminus \mathcal{A}, g \in G(x) \setminus \mathcal{A}, \end{aligned}$$

then the set  $\mathcal{A}$  is pre-asymptotically stable and the basin of pre-attraction contains every forward invariant, compact set.

A consequence of Theorem 20 is that the compact set  $\mathcal{A}$  is globally pre-asymptotically stable if  $C \cup D$  is compact or the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact. A sublevel set of  $V|_{\text{dom } V \cap (C \cup D)}$  is the set  $\{x \in \text{dom } V \cap (C \cup D) : V(x) \leq c\}$ , where  $c \geq 0$ .

Theorem 20 encompasses classical Lyapunov theorems, both for continuous- and discrete-time systems. For example, consider the case where  $F$  is a continuous function, and suppose there exist a continuously differentiable, positive semidefinite function  $V$  and a compact neighborhood  $C$  of the origin such that  $\langle \nabla V(x), F(x) \rangle < 0$  for all  $x \in C \setminus \{0\}$ . According to Theorem 20, the origin of  $\dot{x} = F(x)$ ,  $x \in C$ , is globally pre-asymptotically stable. Then, since  $C$  contains a neighborhood of the origin, it follows that the origin is asymptotically stable for  $\dot{x} = F(x)$ . This conclusion parallels the conclusion of [39, Thm. 4.1]. Similarly, for the case where  $G$  is a continuous, single-valued mapping, if  $G(0) = 0$  and there exist a continuous, positive semidefinite function  $V$  and a compact neighborhood  $D$  of the origin such that  $V(G(x)) < V(x)$  for all  $x \in D \setminus \{0\}$ , then the origin of  $x^+ = G(x)$  is asymptotically stable.

The following examples illustrate the use of Theorem 20.

### Example 2 Revisited: Lyapunov Analysis

Consider the hybrid model in Example 2. Note that  $g(0) = 0$ . Now consider the piecewise quadratic Lyapunov-function candidate

$$V(x) = \begin{cases} \max\{2x_1^2, x_2^2\}, & \text{for all } x_1 \geq 0, x_2 \geq 0, \\ \max\{x_1^2, 2x_2^2\}, & \text{for all } x_1 \leq 0, x_2 \leq 0, \\ 2x_1^2 + 2x_2^2, & \text{for all } x_1 \geq 0, x_2 \leq 0, \\ x_1^2 + x_2^2, & \text{for all } x_1 \leq 0, x_2 \geq 0, \end{cases}$$

where  $\text{dom } V = \mathbb{R}^2$ . The function  $V$  is continuous on its domain, and the sublevel sets of  $V$  are compact. Define  $C_\odot = \{x \in C : x_2 \leq c\}$  where  $c \in (0, 3)$ , and note that  $x \in C_\odot \setminus \{0\}$  implies that  $x_1 \geq x_2$ . It follows that  $V$  is continuously differentiable on an open set containing  $C_\odot \setminus \{0\}$ . Then, through routine calculations, we obtain  $\langle \nabla V(x), f(x) \rangle < 0$  for all  $C_\odot \setminus \{0\}$ . Next define  $D_\odot = \{x \in D : |x_1| \leq d\}$  where  $d \in (0, \sqrt{1/6})$ . Then, for  $x \in D_\odot \setminus \{0\}$  such that  $g(x) \neq 0$ , we obtain  $V(g(x)) \leq 2(2x_1^2)^2 \leq 6d^2x_1^2 < x_1^2 \leq V(x)$ . These calculations and Theorem 20 establish that the origin of the system  $(C_\odot, f, D_\odot, g)$  is globally pre-asymptotically stable. Since there exists a neighborhood  $K$  of the origin such that  $C \cap K \subset C_\odot$  and  $D \cap K \subset D_\odot$ , the origin is pre-asymptotically stable for the system  $(C, f, D, g)$ . ■

### Example 3 Revisited: Lyapunov Analysis of the Bouncing Ball System

For the hybrid bouncing ball model of Example 3, we establish global asymptotic stability of the origin using Theorem 20. First note that  $g(0) = 0$ . Now consider the Lyapunov-function candidate

$$V(x) = x_2 + k\sqrt{\frac{1}{2}x_2^2 + \gamma x_1}$$

where  $k > \sqrt{2}(1 + \rho)/(1 - \rho)$  and  $\text{dom } V = \{x \in \mathbb{R}^2 : (1/2)x_2^2 + \gamma x_1 \geq 0\}$ . This choice for  $V$  is motivated by the ideas in [45]. We obtain  $\langle \nabla V(x), f(x) \rangle = -\gamma < 0$  for all  $x \in C \setminus \{0\}$  and, since  $x \in D \setminus \{0\}$  implies  $x_1 = 0$  and  $x_2 \neq 0$ , it follows that

$$\begin{aligned} V(g(x)) &= -\rho x_2 + k\sqrt{\frac{1}{2}\rho^2 x_2^2} \\ &\leq \rho \left(1 + \frac{k}{\sqrt{2}}\right) |x_2| \\ &< \left(-1 + \frac{k}{\sqrt{2}}\right) |x_2| \quad \text{for all } x \in D \setminus \{0\} \\ &\leq V(x). \end{aligned}$$

It now follows from Theorem 20 that the origin is globally pre-asymptotically stable. Since the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact, the origin is globally pre-asymptotically stable. It also can be shown that nontrivial solutions exist from each point in  $C \cup D$ . See “Existence, Uniqueness, and Other Well-Posedness Issues.” Therefore, the origin is globally asymptotically stable. ■

### Example 9 Revisited: Lyapunov Analysis of a Planar System

For the system in Example 9, assume  $\exp(\alpha\pi/\omega)\gamma < 1$ . Let  $\alpha^*$  satisfy  $\varepsilon_c := \alpha^* - \alpha > 0$  and  $\varepsilon_d := 1 - \exp(2\alpha^*\pi/\omega)\gamma^2 > 0$ . Then consider the Lyapunov-function candidate  $V(x) := \exp(2\alpha^* T(x))|x|^2$ , where  $T$  is a continuously differentiable function on an open set containing  $C \setminus \{(0, 0)\}$  and, for all  $x \in C$ , is equal to the time required to go from  $x$  to  $D$ . Equivalently, for  $x \in C$ ,  $T(x)$  is equal to  $\omega^{-1}$  times the angle of  $x$  in the counterclockwise direction with the angle equal to zero on the negative  $x_2$ -axis and equal to  $\pi$  on the positive  $x_2$ -axis. The function  $V$  is continuous and continuously differentiable on an open set containing  $C \setminus \{(0, 0)\}$ . Moreover, the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact. The function  $T$  satisfies  $\langle \nabla T(x), f(x) \rangle = -1$  for all  $x \in C \setminus \{(0, 0)\}$ . Also note that  $\langle \nabla |x|^2, f(x) \rangle = 2\alpha|x|^2$  for all  $x \in C$ . It follows that

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &= -2\alpha^* V(x) + 2\alpha V(x) \\ &= -2\varepsilon_c V(x) \quad \text{for all } x \in C \setminus \{(0, 0)\}. \end{aligned}$$

In addition,

$$\begin{aligned} V(g(x)) &= \exp(2\alpha^*\pi/\omega)\gamma^2|x|^2 \leq (1 - \varepsilon_d)|x|^2 \\ &= (1 - \varepsilon_d)V(x) \quad \text{for all } x \in D. \end{aligned}$$

It follows from Theorem 20 that the origin is globally asymptotically stable. In fact, the origin is globally exponentially stable since  $V(x(t, j)) \leq \exp(-\lambda(t + j))V(x(0, 0))$ , where  $\lambda = \min\{2\varepsilon_c, -\ln(1 - \varepsilon_d)\}$ , for all solutions  $x$  and all  $(t, j) \in \text{dom } x$ , and  $|x|^2 \leq V(x) \leq \exp(2\alpha^*\pi/\omega)|x|^2$  for all  $x \in C \cup D$ . ■

### Example 21: A Bounded-Rate Hybrid System

This example is based on [36, Ex. 1]. Consider the hybrid system with data  $(C, F, D, G)$ , where  $D = \{0\}$ ,  $G(0) = 0$ ,  $C = \mathbb{R}_{\geq 0}^3$ , and

$$F = \overline{\text{con}} \left\{ \begin{array}{l} f_1 := \begin{bmatrix} 100 \\ -90 \\ 1 \end{bmatrix}, \quad f_2 := \begin{bmatrix} -90 \\ 100 \\ -90 \end{bmatrix}, \quad f_3 := \begin{bmatrix} 1 \\ -90 \\ 100 \end{bmatrix}, \\ f_4 := \begin{bmatrix} 1 \\ 50 \\ -90 \end{bmatrix}, \quad f_5 := \begin{bmatrix} -90 \\ 50 \\ 1 \end{bmatrix} \end{array} \right\}.$$

Consider the Lyapunov-function candidate  $V(x) = x_1 + 1.5x_2 + x_3$ , for which the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact. It is easy to verify, for each of the vectors  $f_i$ ,  $i = 1, \dots, 5$ , that  $\langle \nabla V(x), f_i \rangle < 0$ . Then it is a simple calculation to verify that  $\langle \nabla V(x), f \rangle < 0$  for all  $f \in F$ . It follows from Theorem 20 that the origin is globally pre-asymptotically stable. In fact, the “pre” can be dropped since, for each initial point in  $C \cup D$ , there exists a non-trivial solution, and consequently, all maximal solutions to the hybrid system are complete. Indeed, for each  $x \neq 0$  on the boundary of  $C$  there exists  $f \in F$  along which flow in  $C$  is possible. In fact, for each  $x \neq 0$  there exists  $i \in \{1, \dots, 5\}$  such that  $f_i \in T_C(z)$  for all  $z \in C$  near  $x$ . See “Existence, Uniqueness, and Other Well-Posedness Issues” for details. Existence of nontrivial solutions also holds when defining  $F$  to be the convex hull of the vectors  $f_i$ ,  $i = 1, 2, 3$ , which corresponds to the system considered in [36, Ex. 1], but it cannot be verified by using only the three generating vectors at the nonzero points  $(0, 0, x_3)$  and  $(x_1, 0, 0)$ . ■

The next example demonstrates that a Lyapunov function can be zero at points outside of  $\mathcal{A}$ .

### Example 22: Lyapunov Function Can Be Zero at Points Outside of $\mathcal{A}$

Consider the hybrid system with state  $x \in \mathbb{R}^2$  and data

$$C := \mathbb{R}_{\geq 0}^2, \quad f(x) := \begin{bmatrix} x_1^2 + x_2^2 \\ -x_1^2 - x_2^2 \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x \in \mathbb{R}_{\geq 0}^2 : x_2 = 0\}, \quad G(x) := 0 \quad \text{for all } x \in D.$$

We establish pre-asymptotic stability of  $\mathcal{A} = \{0\}$ . Consider the function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $V(x) := x_2$ , which is a Lyapunov-function candidate for  $(\mathcal{H}, \mathcal{A})$ . Note that  $V$  is zero on  $D$  and satisfies

$$\langle \nabla V(x), f(x) \rangle = -x_1^2 - x_2^2 < 0 \quad \text{for all } x \in C \setminus \mathcal{A}.$$

In addition, the set  $G(x) \setminus \mathcal{A}$  is empty. Thus, the origin is pre-asymptotically stable according to Theorem 20. In fact, global pre-asymptotic stability follows from the fact that the compact sets  $\{x \in C \cup D : x_1 + x_2 \leq c\}$ , where  $c > 0$ , cover  $C \cup D$ , and each one is forward invariant. ■

## STABILITY ANALYSIS USING LYAPUNOV-LIKE FUNCTIONS AND AN INVARIANCE PRINCIPLE

Pre-asymptotically stable compact sets always admit Lyapunov functions. See “Converse Lyapunov Theorems.” Nevertheless, it can be challenging to construct a Lyapunov function even for systems that are not hybrid. This fact inspires the development of relaxed, Lyapunov-based conditions for asymptotic stability. Perhaps the most well-known result in this direction is the Barbasin-

Krasovskii theorem [5]. This result guarantees asymptotic stability when the Lyapunov function is not increasing along solutions and the only value of the Lyapunov function that can be constant along solutions is a value of the Lyapunov function taken only on the set  $\mathcal{A}$ . It is a specialization to asymptotic stability questions of the invariance principle [46], [47]. The next result, which can be called a hybrid Barbasin-Krasovskii-LaSalle theorem, is essentially contained in [72, Thm. 7.6] and combines the idea of the Barbasin-Krasovskii theorem with the invariance principle for hybrid systems. For more information on the invariance principle, see “Invariance.”

For  $\mu \in \mathbb{R}$  and a function  $V : \text{dom } V \rightarrow \mathbb{R}$  let  $L_V(\mu) := \{x \in \text{dom } V : V(x) = \mu\}$ .

### Theorem 23

Consider a hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfying the Basic Assumptions and a compact set  $\mathcal{A} \subset \mathbb{R}^n$  satisfying  $G(D \cap \mathcal{A}) \subset \mathcal{A}$ . If there exists a Lyapunov-function candidate  $V$  for  $(\mathcal{H}, \mathcal{A})$  that is positive on  $(C \cup D) \setminus \mathcal{A}$  and satisfies

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq 0 \quad \text{for all } x \in C \setminus \mathcal{A}, f \in F(x), \\ V(g) - V(x) &\leq 0 \quad \text{for all } x \in D \setminus \mathcal{A}, g \in G(x) \setminus \mathcal{A} \end{aligned}$$

then the set  $\mathcal{A}$  is stable. If furthermore there exists a compact neighborhood  $K$  of  $\mathcal{A}$  such that, for each  $\mu > 0$ , no complete solution to  $\mathcal{H}$  remains in  $L_V(\mu) \cap K$ , then the set  $\mathcal{A}$  is pre-asymptotically stable. In this case, the basin of pre-attraction contains every compact set contained in  $K$  that is forward invariant.

A consequence of Theorem 23 is that  $\mathcal{A}$  is globally pre-asymptotically stable if  $K$  can be taken to be arbitrarily large and  $C \cup D$  is compact or the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact.

Realizing the full potential of Theorem 23 hinges on detecting whether  $\mathcal{H}$  has complete solutions in  $L_V(\mu) \cap K$ . This property can be assessed by focusing on a hybrid system that is contained in the original hybrid system. For a vector  $v \in \mathbb{R}^n$ , let  $v^\perp := \{w \in \mathbb{R}^n : \langle v, w \rangle = 0\}$ . For each  $\mu > 0$ , define

$$F_\mu(x) := F(x) \cap \nabla V(x)^\perp \quad \text{for all } x \in C \cap L_V(\mu), \quad (25)$$

$$G_\mu(x) := G(x) \cap L_V(\mu) \quad \text{for all } x \in D \cap L_V(\mu), \quad (26)$$

$$C_\mu := \text{dom } F_\mu, \quad (27)$$

$$D_\mu := \text{dom } G_\mu \cap (\text{dom } F_\mu \cup G_\mu(\text{dom } G_\mu)). \quad (28)$$

Ruling out complete solutions to  $\mathcal{H}$  that remain in  $L_V(\mu) \cap K$  is equivalent to ruling out complete solutions to the hybrid system  $\mathcal{H}_{\mu, K} := (C_\mu \cap K, F_\mu, D_\mu \cap K, G_\mu \cap K)$ .

Sometimes the absence of complete solutions to the system  $\mathcal{H}_{\mu, K}$  can be verified by inspection. Otherwise, we may try to use an auxiliary function  $W$ , which does not need to be sign definite, to rule out complete solutions to

## Hybrid dynamical systems can model a variety of closed-loop feedback control systems.

$\mathcal{H}_{\mu, K}$ . When this auxiliary function is always decreasing along solutions to  $\mathcal{H}_{\mu, K}$ , the system  $\mathcal{H}_{\mu, K}$  cannot have complete solutions since  $(C_\mu \cup D_\mu) \cap K$  is compact for each  $\mu > 0$  and each compact set  $K$ . More generally, Proposition 24 below can be applied iteratively to reduce the size of the flow and jump sets in an attempt to make them empty eventually, thereby ruling out complete solutions. This result is another consequence of the general invariance principle discussed in “Invariance.”

### Proposition 24

Consider the hybrid system  $\mathcal{H} := (C, F, D, G)$  satisfying the Basic Assumptions, and assume that  $C \cup D$  is compact. Let  $W$  be continuously differentiable on an open set containing  $C$  and continuous on  $D$ . Suppose

$$\begin{aligned}\langle \nabla W(x), f \rangle &\leq 0 \quad \text{for all } x \in C, \quad f \in F(x), \\ W(g) - W(x) &\leq 0 \quad \text{for all } x \in D, \quad g \in G(x) \cap (C \cup D).\end{aligned}$$

For each  $\nu \in W(\text{dom } W \cap (C \cup D))$ , define

$$\begin{aligned}F_\nu(x) &:= F(x) \cap \nabla W(x)^\perp \quad \text{for all } x \in C \cap L_W(\nu), \\ G_\nu(x) &:= G(x) \cap L_W(\nu) \quad \text{for all } x \in D \cap L_W(\nu), \\ C_\nu &:= \text{dom } F_\nu, \\ D_\nu &:= \text{dom } G_\nu \cap (\text{dom } F_\nu \cup G_\nu(\text{dom } G_\nu)).\end{aligned}$$

Then the hybrid system  $\mathcal{H}$  has complete solutions if and only if there exists  $\nu \in W(\text{dom } W \cap (C \cup D))$  such that the hybrid system  $\mathcal{H}_\nu := (C_\nu, F_\nu, D_\nu, G_\nu)$  has complete solutions.

Whenever  $\langle \nabla W(x), f \rangle < 0$  for all  $x \in C, f \in F(x)$ , and  $W(g) - W(x) < 0$  for all  $x \in D, g \in G(x) \cap (C \cup D)$ , it follows that  $C_\nu$  and  $D_\nu$  are empty and thus  $\mathcal{H}_\nu$  has no complete solutions. Typically,  $C_\nu$  and  $D_\nu$  have fewer points than  $C$  and  $D$ , and so it may be easier to establish that  $\mathcal{H}_\nu$  has no complete solutions than to establish that  $\mathcal{H}$  has no complete solutions. The combination of Theorem 23 with a repeated application of Proposition 24 is the idea behind Matrosov’s theorem for time-invariant hybrid systems, as presented in [74]. The original Matrosov theorem, conceived for time-varying systems, appears in [53]. Matrosov theorems reach their full potency for time-varying systems, where invariance principles typically do not hold. Refinements of Matrosov’s original idea have appeared over the years. See [60] and [50] and the references therein. A version for time-varying hybrid systems appears in [52].

We illustrate Theorem 23 with several examples.

### Example 3 Revisited: Lyapunov Analysis with the Invariance Principle.

Consider the hybrid model of a bouncing ball in Example 3. We establish global asymptotic stability of the origin. Let  $\mathcal{A}$  be the origin in  $\mathbb{R}^2$ . The condition  $g(D \cap \mathcal{A}) \subset \mathcal{A}$  holds. Consider a Lyapunov-function candidate  $V: \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $V(x) = \gamma x_1 + (1/2)x_2^2$ . The sublevel sets of  $V|_{C \cup D}$  are compact, and  $V$  satisfies

$$\langle \nabla V(x), f(x) \rangle = \gamma x_2 - \gamma x_2 = 0 \quad \text{for all } x \in C$$

and

$$V(g(x)) = \frac{1}{2}\rho^2 x_2^2 < \frac{1}{2}x_2^2 = V(x) \quad \text{for all } x \in D \setminus \mathcal{A}.$$

The set  $D_\mu$  defined in (25)–(28) is empty for each  $\mu > 0$  since  $\text{dom } G_\mu$  is empty for each  $\mu > 0$ . Thus,  $\mathcal{A}$  is globally asymptotically stable as long as there are no complete solutions that flow only and keep  $V$  equal to a positive constant. It may be possible to see by inspection that no solutions of this type exist. Otherwise, consider using the idea in Proposition 24 to rule out complete solutions to  $\mathcal{H}_\mu$ ,  $\mu > 0$ , that only flow.

Next, applying Proposition 24 with  $C := \{x \in \mathbb{R}^2: x_1 \geq 0, V(x) = \mu\}$ ,  $D$  given by the empty set, and  $W_1(x) = x_2$ , we obtain

$$\langle \nabla W_1(x), f(x) \rangle = -\gamma < 0 \quad \text{for all } x \in C.$$

The set  $C_\nu$  in Proposition 24 is empty for all  $\nu$ , and thus complete solutions that only flow are ruled out. It follows from Theorem 23 that the set  $\mathcal{A}$  is globally asymptotically stable. ■

### Example 25: Interacting Fireflies.

Consider the firefly model given earlier with two fireflies, and suppose each flow map is equal to a constant  $f > 0$ . This choice results in a hybrid system  $\mathcal{H}$  with  $x \in \mathbb{R}^2$  and the data

$$C := [0, 1] \times [0, 1], \quad F(x) := \begin{bmatrix} f \\ f \end{bmatrix} \quad \text{for all } x \in C,$$

$$D := \{x \in C : \max\{x_1, x_2\} = 1\},$$

$$G(x) := \begin{bmatrix} g((1+\varepsilon)x_1) \\ g((1+\varepsilon)x_2) \end{bmatrix} \quad \text{for all } x \in D,$$

## Invariance

The invariance principle has played a fundamental role over the years as a tool for establishing asymptotic stability in nonlinear control algorithms. Prime examples include the Jurdjevic-Quinn approach to nonlinear control design [S45], [S47] as well as general passivity-based control [S42], [S48].

The invariance principle transcends stability analysis by characterizing the nature of the sets to which a bounded solution to a dynamical system converges. The basic invariance principle for dynamical systems with unique solutions is due to LaSalle [47], [S46]. It has been extended to systems with possibly nonunique solutions, in particular, differential inclusions, as well as integral and differential versions with nonsmooth functions [S49]. A version of the invariance principle for hybrid systems with unique solutions and continuous dependence on initial conditions appears in [51]. An invariance principle for left-continuous dynamical systems having unique solutions and quasi-continuous dependence on initial conditions is given in [S43]. An invariance principle for switched systems is given in [S3] and [S44]. The results quoted below are from [72], which contains results analogous to those in [S49] but for hybrid dynamical systems.

### Lemma S12

Suppose that the hybrid system  $\mathcal{H}$  satisfies the Basic Assumptions, and let  $x : \text{dom } x \rightarrow \mathbb{R}^n$  be a complete and bounded solution to  $\mathcal{H}$ . Then the *omega-limit* of  $x$ , that is, the set

$$\omega(x) = \{z \in \mathbb{R}^n : \text{there exists } (t_i, j_i) \in \text{dom } x \text{ there exists } t_i + j_i \rightarrow \infty, x(t_i, j_i) \rightarrow z\},$$

is nonempty, compact, and *weakly invariant* in the sense that the following conditions are satisfied:

- i)  $\omega(x)$  is *weakly forward invariant*, that is, for each  $z \in \omega(x)$  there exists a complete solution  $y$  to  $\mathcal{H}$  such that  $y(0, 0) = z$  and  $y(t, j) \in \omega(x)$  for all  $(t, j) \in \text{dom } y$ .
- ii)  $\omega(x)$  is *weakly backward invariant*, that is, for each  $z \in \omega(x)$  and each  $m > 0$ , there exists a solution  $y$  to  $\mathcal{H}$  such that  $y(t, j) \in \omega(x)$  for all  $(t, j) \in \text{dom } y$  and such that  $y(t_z, j_z) = z$  for some  $(t_z, j_z) \in \text{dom } y$  with  $t_z + j_z > m$ .

Furthermore, the distance from  $x(t, j)$  to  $\omega(x)$  decreases to zero as  $t + j \rightarrow \infty$ , and, in fact,  $\omega(x)$  is the smallest closed set with this property.

where  $\varepsilon > 0$ ,  $g(s) = s$  when  $s < 1$ ,  $g(s) = 0$  when  $s > 1$  and  $g(s) = \{0, 1\}$  when  $s = 1$ .

Note that the compact set  $\mathcal{A} := \{x \in C : x_1 = x_2\}$  represents synchronized flashing. We establish asymptotic stability  $\mathcal{A}$  and characterize the basin of attraction.

Define  $k := \varepsilon/(2 + \varepsilon)$  and consider the Lyapunov-function candidate

$$V(x) := \min \{ |x_1 - x_2|, 1 + k - |x_1 - x_2| \}.$$

### Theorem S13 [72, Thm. 4.3]

Suppose that the hybrid system  $\mathcal{H} = (C, F, D, G)$  satisfies the Basic Assumptions. Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on an open set containing  $C$ , continuous on  $C \cup D$ , and satisfy

$$u_C(x) \leq 0 \text{ for all } x \in C, \text{ where } u_C(x) := \max_{f \in F(x)} \langle \nabla V(x), f \rangle,$$

$$u_D(x) \leq 0 \text{ for all } x \in D, \text{ where } u_D(x) := \max_{g \in G(x)} V(g) - V(x).$$

Then, there exists  $r \in \mathbb{R}$  such that each complete and bounded solution  $x$  to  $\mathcal{H}$  converges to the largest weakly invariant subset of the set

$$\{z : V(z) = r\} \cap (u_C^{-1}(0) \cup (u_D^{-1}(0) \cap G(u_D^{-1}(0)))) , \quad (\text{S8})$$

$$\text{where } u_C^{-1}(0) := \{z \in C : u_C(z) = 0\} \text{ and } u_D^{-1}(0) := \{z \in D : u_D(z) = 0\}.$$

Further information on a particular complete and bounded solution  $x$  may lead to more precise descriptions of the set to which  $x$  converges. For example, if  $x$  is Zeno (see "Zeno Solutions") then both forward and backward weak invariance of  $\omega(x)$  can be verified by a complete solution to  $\mathcal{H}$  that never flows, and, moreover, we can limit our attention to weakly invariant subsets of

$$\{z \in D : u_D(z) = 0\} \cap G(\{z \in D : u_D(z) = 0\}).$$

On the other hand, if there exists  $\tau > 0$  such that all jumps of  $x$  are separated by at least an amount of time  $\tau$ , then we can limit our attention to weakly invariant subsets of

$$\{z \in C : u_C(z) = 0\}.$$

### Example 14: Illustration of the Invariance Principle.

Consider the hybrid system in the plane with data

$$C := \{x \in \mathbb{R}^2 : x_2 \geq 0\}, \quad F(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \text{ for all } x \in C,$$

$$D := \{x \in \mathbb{R}^2 : x_2 \leq 0\}, \quad G(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \text{ for all } x \in D.$$

Solutions flow in the counterclockwise direction when in the closed upper-half plane. From the closed lower-half plane, which is where solutions are allowed to jump, solutions undergo an instantaneous rotation of  $\pi/2$ , also in

This function is continuously differentiable on the open set  $\mathcal{X} \setminus \mathcal{A}$ , where

$$\mathcal{X} := \{x \in \mathbb{R}^2 : V(x) < (1 + k)/2 = (1 + \varepsilon)/(2 + \varepsilon)\}$$

$$= \{x \in \mathbb{R}^2 : |x_1 - x_2| \neq (1 + \varepsilon)/(2 + \varepsilon)\}.$$

Let  $\nu^* = (1 + \varepsilon)/(2 + \varepsilon)$ , let  $\nu \in (0, \nu^*)$ ,  $K_\nu = \{x \in C \cup D : V(x) \leq \nu\}$ , and define  $C_\odot = C \cap K_\nu$  and  $D_\odot = D \cap K_\nu$ . Since  $V$  is a function of only  $x_1 - x_2$  and  $F_1(x) = F_2(x)$ , it follows that

the counterclockwise direction. We analyze the asymptotic behavior of the complete solution  $x$  with  $x(0, 0) = (1, 0)$ . A natural function  $V$  to consider is  $V(x) = |x|^2$ . Then,  $u_C(x) = 0$  for all  $x \in C$  and  $u_D(x) = 0$  for all  $x \in D$ . Since  $F$  and  $G$  are single-valued maps, this property is enough to conclude that  $V(x(t, j)) = V(x(0, 0)) = 1$  for all  $(t, j) \in \text{dom } x$ , and, in particular, that  $x$  is bounded. Now note that the set of all  $z \in C$  such that  $u_C(z) = 0$  is the whole set  $C$ , similarly, the set of all  $z \in D$  such that  $u_D(z) = 0$  is the whole set  $D$ . Furthermore, the set  $G(u_D^{-1}(0)) = G(\{z \in D : u_D(z) = 0\})$  is the closed right-half plane. Hence, Theorem S13 implies that  $x$  converges to the largest weakly invariant subset of

$$S = \{z \in \mathbb{R}^2 : |z| = 1, z_1 \geq 0 \text{ or } z_2 \geq 0\}.$$

This subset turns out to be

$$S^{\text{inv}} = \{z \in \mathbb{R}^2 : |z| = 1, z_2 \geq 0\} \cup (0, -1).$$

As Figure S10 depicts,  $S^{\text{inv}}$  is also exactly the range of the periodic solution  $x$  and thus its omega-limit. Indeed, we have  $x(t, j) = (\cos t, \sin t)$  for  $t \in [0, \pi]$ ,  $x(\pi, 1) = (0, -1)$ , and  $x(t, j) = x(t - \pi, j - 2)$  for  $t \geq \pi$ ,  $j \geq 2$ . In particular,  $S^{\text{inv}}$  is the smallest closed set to which  $x$  converges.

Note that asserting that the periodic solution  $x$  converges to  $S^{\text{inv}}$  is possible only by considering both weak forward and backward invariance. Indeed, the largest weakly forward invariant subset of  $S$  is  $S$  itself. ■

In the conclusion of Theorem S13 we can equivalently consider the largest weakly invariant subset of

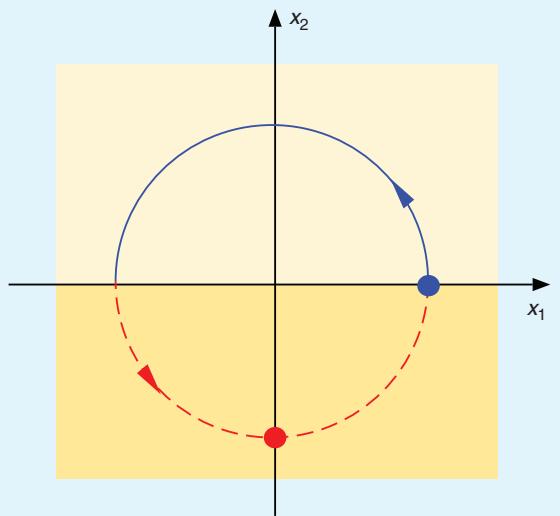
$$\{z : V(z) = r\} \cap (u_C^{-1}(0) \cup u_D^{-1}(0)).$$

This set has a simpler description but is usually larger than the set used in Theorem S13. More extensive investigation using invariance properties is then needed to obtain the same set to which solutions converge. If the term  $G(u_D^{-1}(0))$  in (S8) were to be omitted when applying Theorem S13 to the hybrid system in Example 14, then the set  $S$  would be larger. In particular,  $S$  would be the whole unit circle. Nevertheless, considering forward and backward weak invariance leads to the same invariant subset  $S^{\text{inv}}$ .

$$\langle \nabla V(x), F(x) \rangle = 0 \quad \text{for all } x \in C_{\odot} \setminus \mathcal{A}.$$

Now consider  $x \in D_{\odot}$ . By symmetry, without loss of generality we can consider  $x = (1, x_2)$ , where  $x_2 \in [0, 1] \setminus \{1/(2 + \varepsilon)\}$ . We obtain

$$\begin{aligned} V(x) &= \min\{1 - x_2, k + x_2\}, \\ V(G(x)) &= \min\{g((1 + \varepsilon)x_2), 1 + k - g((1 + \varepsilon)x_2)\}. \end{aligned}$$



**FIGURE S10** A solution to the hybrid system in Example S14. All solutions to this system flow in the counterclockwise direction when in the closed upper-half plane. From the closed lower-half plane, all solutions jump and undergo a rotation by  $\pi/2$ , also in the counterclockwise direction. An application of Theorem S13 indicates that all solutions converge to the set  $S^{\text{inv}} = \{z \in \mathbb{R}^2 : |z| = 1, z_2 \geq 0\} \cup (0, -1)$ . This set coincides with the range of the solution that starts flowing from  $x(0, 0) = (1, 0)$ .

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When  $g((1 + \varepsilon)x_2) = 0$ , it follows that  $V(G(x)) = 0$ . Now consider the case where  $g((1 + \varepsilon)x_2) = (1 + \varepsilon)x_2$ . There are two possibilities, namely,  $x_2 < 1/(2 + \varepsilon)$  and  $x_2 > 1/(2 + \varepsilon)$ . In the first situation,

$$V(x) = k + x_2 > (1 + \varepsilon)x_2 \geq V(G(x)).$$

In the second situation,  $V(x) = 1 - x_2 > 1 + k - (1 + \varepsilon)x_2 \geq V(G(x))$ . It now follows from Theorem 23 that the set

$\mathcal{A}$  is globally pre-asymptotically stable for the system  $(C_\circ, F, D_\circ, G)$ . To see this fact, note that the calculations above imply that the set  $\text{dom } G_\mu$ , and thus also  $D_\mu$ , used in (25)–(28) are empty for each  $\mu > 0$ . Furthermore, complete solutions that only flow are impossible. This fact can be seen by inspection or by applying Proposition 24 with  $W(x) = -x_1 - x_2$ .

Since  $\nu \in (0, \nu^*)$  used in the definitions of  $C_\circ$  and  $D_\circ$  is arbitrary, and the sets  $K_\nu$  are compact and forward invariant, the basin of attraction of  $\mathcal{A}$  for the system  $(C, F, D, G)$  contains  $\mathcal{X}$ . In fact, solutions starting from the condition  $|x_1 - x_2| = (1 + \varepsilon)/(2 + \varepsilon)$  do not converge to  $\mathcal{A}$ . This behavior can be seen by noting that  $V(G(x)) = V(x)$  when  $x_1 = 1$  and  $|x_1 - x_2| = (1 + \varepsilon)/(2 + \varepsilon)$ . It then follows that  $\mathcal{X}$  is the basin of attraction for  $\mathcal{A}$  for the system  $(C, F, D, G)$ . ■

The final two illustrations of Theorem 23 address stability analysis for general classes of systems where sampling is involved. The first application pertains to classical sampled-data systems. The second application covers nonlinear networked control systems, which are becoming more prevalent due to the ubiquity of computers and communication networks.

#### Example 26: Absolute Stability for Sampled-Data Systems

In this example we study the absolute stability of periodic jump linear systems, including sampled-data systems. Absolute stability refers to asymptotic stability in the presence of arbitrary time-varying, sector-bounded nonlinearities [39, Sec. 7.1]. It is possible to model sector-bounded nonlinearities through a differential inclusion.

Consider the class of hybrid systems with state  $x \in \mathbb{R}^{n+1}$ , decomposed as  $x = (\xi, \tau)$ , where  $\xi \in \mathbb{R}^n$ , and data

$$\begin{aligned} C &:= \{x : \tau \in [0, T]\}, \\ F(x) &:= \left\{ \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, w: \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} M_1 & M_3 \\ M_3^\top & M_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \leq 0 \right\} \\ &\quad \text{for all } x \in C, \\ D &:= \{x : \tau = T\}, \\ G(x) &:= \left\{ \begin{bmatrix} J\xi + Lw \\ 0 \end{bmatrix}, w: \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} N_1 & N_3 \\ N_3^\top & N_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \leq 0 \right\} \\ &\quad \text{for all } x \in D, \end{aligned}$$

where  $M_1, M_2, N_1$ , and  $N_2$  are symmetric. We assume that the eigenvalues of  $M_1$  and  $N_1$ , which are real, are nonnegative; in other words,  $M_1$  and  $N_1$  are positive-semidefinite matrices. The matrices  $M_2$  and  $N_2$  are positive definite, meaning that their eigenvalues are positive. These conditions guarantee that the Basic Assumptions hold. The state  $\tau$  corresponds to a timer state that forces jumps every  $T$  seconds. With  $w$  constrained to zero, the system is a simple periodic jump linear system with flow equation  $\dot{\xi} = A\xi$  and jump equation  $\xi^+ = J\xi$ . More generally,  $w$  is constrained to satisfy a quadratic constraint that is a function of the state

$\xi$ , thereby modeling a sampled-data linear control system with sector-bounded nonlinearities.

To establish asymptotic stability of the compact set  $\mathcal{A} := \{x : \xi = 0, \tau \in [0, T]\}$ , consider a Lyapunov-function candidate of the form  $V(x) := \xi^\top P(\tau)\xi$ , where  $P : [0, T] \rightarrow \mathcal{P}_n$  and  $\mathcal{P}_n$  denotes the set of symmetric, positive definite matrices. The function  $P$  is chosen to satisfy

$$\begin{aligned} \nabla P(\tau) &= -A^\top P(\tau) - P(\tau)A + M_1 \\ &\quad -(P(\tau)B - M_3)M_2^{-1}(B^\top P(\tau) - M_3^\top), \end{aligned} \quad (29)$$

where we assume that  $P(T) = P^\top(T) > 0$  can be chosen to guarantee that  $P(\tau)$  exists and is positive definite for all  $\tau \in [0, T]$ . For more on this condition, see below. With this choice for  $V$ , for all  $x \in C$  and  $f \in F(x)$ , we obtain

$$\begin{aligned} \langle \nabla V(x), f \rangle &= 2\xi^\top P(\tau)Bw + \xi^\top M_1\xi - \xi^\top(P(\tau)B - M_3) \\ &\quad \times M_2^{-1}(B^\top P(\tau) - M_3^\top)\xi \\ &\leq 2\xi^\top P(\tau)Bw + \xi^\top M_1\xi - \xi^\top(P(\tau)B - M_3) \\ &\quad \times M_2^{-1}(B^\top P(\tau) - M_3^\top)\xi \\ &\quad - \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} M_1 & M_3 \\ M_3^\top & M_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix}. \end{aligned}$$

Completing squares leads to the conclusion that

$$\langle \nabla V(x), f \rangle \leq 0 \quad \text{for all } x \in C, f \in F(x).$$

We now turn to the change in  $V$  during jumps. Asymptotic stability of the set  $\mathcal{A}$  follows from Theorem 23 when there exists  $\varepsilon > 0$  such that

$$\begin{aligned} &(\xi^\top J^\top + w^\top L^\top)P(0)(J\xi + Lw) - \xi^\top P(T)\xi \\ &- \begin{bmatrix} \xi \\ w \end{bmatrix}^\top \begin{bmatrix} N_1 & N_3 \\ N_3^\top & N_2 \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \leq -\varepsilon \xi^\top \xi. \end{aligned} \quad (30)$$

To check this condition, we need to relate  $P(0)$  and  $P(T)$ . The solution to the matrix differential equation (29) can be written explicitly by forming the Hamiltonian matrix

$$H = \begin{bmatrix} A - BM_2^{-1}M_3^\top & BM_2^{-1}B^\top \\ M_1 - M_3M_2^{-1}M_3^\top & -(A - BM_2^{-1}M_3^\top)^\top \end{bmatrix},$$

forming the matrix exponential  $\Phi(\tau) := \exp(-H\tau)$ , partitioning  $\Phi$  in the same way that  $H$  is partitioned, and verifying that, where  $P$  is defined,

$$P(T - \tau) = (\Phi_{21}(\tau) + \Phi_{22}(\tau)P(T))(\Phi_{11}(\tau) + \Phi_{12}(\tau)P(T))^{-1}.$$

When  $\Phi_{11}(t)$  is invertible for all  $t \in [0, T]$ , the quantities  $-\Phi_{11}(t)^{-1}\Phi_{12}(t)$  and  $\Phi_{21}(t)\Phi_{11}(t)^{-1}$  are positive semidefinite for all  $t \in [0, T]$ . Also, defining  $\Psi := \Phi(T)$ ,  $X := P(T)$ , and letting  $S$  be a matrix satisfying  $SS^\top = -\Psi_{11}^{-1}\Psi_{12}$ ,

$P(t)$  is defined and positive definite for all  $t \in [0, T]$  when  $X = X^\top > 0$  and  $S^\top XS < I$ . These conditions on  $X$  and  $S$  are guaranteed by what follows.

Using the fact that  $\Psi$  is symplectic, that is,  $\Psi^\top \Omega \Psi = \Omega$ , where  $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ , the value  $P(0)$  is related to  $X$  through the formula

$$P(0) = \Psi_{11}^{-1}[X + XS(I - S^\top XS)^{-1}S^\top X]\Psi_{11}^{-T} + \Psi_{21}\Psi_{11}^{-1}.$$

Substituting for  $P(0)$  in (30) and using Schur complements, it follows that asymptotic stability of the set  $\mathcal{A}$  is guaranteed when  $\Phi_{11}(t)$  is invertible for all  $t \in [0, T]$  and the matrix inequality

$$\begin{bmatrix} J^\top \\ L^\top \\ 0 \end{bmatrix} (\Psi_{11}^{-1}X\Psi_{11}^{-T} + \Psi_{21}\Psi_{11}^{-1}) [J \quad L \quad 0] - \begin{bmatrix} X + N_1 & N_3 & J^\top \Psi_{11}^{-1}XS \\ N_3^\top & N_2 & L^\top \Psi_{11}^{-1}XS \\ S^\top X\Psi_{11}^{-T}J & S^\top X\Psi_{11}^{-T}L & (I - S^\top XS) \end{bmatrix} < 0 \quad (31)$$

is satisfied for some  $X = X^\top > 0$ .

### Example 27: Networked Nonlinear Control Systems.

A wide variety of interesting control problems are associated with networked control systems. Stability analysis for some of these problems is similar to stability analysis for sampled-data systems. The primary differences are the following:

- » The length of time between updates can be unpredictable, due to network variability.
- » The update rules are often time varying, typically periodic as when using a round-robin protocol, or nonlinear, as when using the “try-once-discard” protocol presented in [89]. These attributes arise from communication constraints that limit how much of the state can be updated at a given time. Protocols are developed to make a choice about which component of the state to update at the current time.

Time-varying update rules can be addressed in the framework of sampled-data systems [20]. Uncertain and variable update times also can be addressed in the framework of sampled-data systems when we are satisfied with a common quadratic Lyapunov function for the various discrete-time systems that emerge from the variable update times. When we move to nonlinear dynamics and nonlinear updates, it becomes more challenging to exploit exact knowledge of the functions that describe the evolution of networked control system. In this situation, a reasonable approach is to analyze the closed-loop behavior using coarse information about the functions involved.

Consider the networked control system with state  $x = (x_1, x_2, \tau) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}$  and data

$$C := \{x = (x_1, x_2, \tau) : \tau \in [0, T]\},$$

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ 1 \end{bmatrix} \text{ for all } x \in C,$$

$$D := \{x = (x_1, x_2, \tau) : \tau = T\},$$

$$g(x) = \begin{bmatrix} x_1 \\ g_2(x_1, x_2) \\ [0, \varepsilon T] \end{bmatrix} \text{ for all } x \in D,$$

where  $\varepsilon \in [0, 1)$ . The quantity  $x_1$  denotes physical states in the plants that are being controlled over the communication network. The quantity  $x_2$  denotes states associated with communication. For example, the state  $x_2$  may denote mismatch between ideal control actions and control actions achieved by the network. Uncertainty and variability in the transmission times are captured by the set-valued nature of the update rule for the timer state  $\tau$ , which can be updated to any value in the interval  $[0, \varepsilon T]$ . For simplicity, we consider the case where the desired steady-state behavior corresponds to  $(x_1, x_2) = (0, 0)$ . In a more general analysis, both  $x_1$  and  $x_2$  may contain states that don't tend to zero.

The function  $g_2$  addresses the communication protocol used to update  $x_2$  at transmission times. We are considering time-invariant protocols here, since  $g_2$  does not depend on time. However, all of the ideas below extend to the time-varying, periodic case.

We assume there exist continuous functions  $\phi_1$  and  $\phi_2$  that are zero at zero and positive otherwise, positive numbers  $\varepsilon, k_{11}, k_{12}, k_{21}, k_{22}$ , and  $k_3$ , and continuously differentiable functions  $V_1, V_2$  that are zero at zero and positive otherwise, with compact sublevel sets, such that

$$\langle \nabla V_1(x_1), f_1(x_1, x_2) \rangle \leq -(\varepsilon + k_{11})\phi_1(x_1)^2 + k_{12}\phi_1(x_1)\phi(x_2), \quad (32)$$

$$\langle \nabla V_2(x_2), f_2(x_1, x_2) \rangle \leq k_{21}\phi(x_1)\phi(x_2) + k_{22}\phi_2(x_2)^2, \quad (33)$$

$$\max \{k_3\phi(x_2)^2, \lambda^{-2}V_2(g_2(x_1, x_2))\} \leq V_2(x_2). \quad (34)$$

Possible choices for the functions  $\phi_1$  and  $\phi_2$  include the Euclidean norms of the arguments. Condition (32) includes the assumption that the origin is globally asymptotically stable in the case of perfect communication, that is,  $x_2 \equiv 0$ . Condition (34) includes the assumption that the communication protocol, which is determined by the function  $g_2$ , makes  $x_2 = 0$  asymptotically stable when the protocol dynamics are disconnected from the continuous-time dynamics. Condition (33) is a coupling condition between (32) and (34) that bounds the growth of  $x_2$  during flows.

Now consider the Lyapunov-function candidate  $V(x) := V_1(x_1) + p(\tau)V_2(x_2)$ , where  $p: [0, T] \rightarrow \mathbb{R}_{>0}$  satisfies

$$\nabla p(\tau) = -\frac{1}{k_3 k_{11}} \left[ k_{22} k_{11} p(\tau) + \frac{1}{4} (k_{12} + p(\tau) k_{21})^2 \right], \quad (35)$$

and we assume that  $p(T) > 0$  can be chosen so that  $p$  is defined on  $[0, T]$ . This choice is similar to the Lyapunov function considered in [15]. We obtain

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &\leq -(k_{11} + \varepsilon) \phi_1(x_1)^2 + k_{12} \phi_1(x_1) \phi_2(x_2) \\ &\quad + p(\tau) [k_{21} \phi_1(x_1) \phi_2(x_2) + k_{22} \phi_2(x_2)^2] \\ &\quad - \frac{1}{k_3 k_{11}} \left[ k_{22} k_{11} p(\tau) + \frac{1}{4} (k_{12} + p(\tau) k_{21})^2 \right] \\ &\quad \times k_3 \phi_2(x_2)^2 \\ &= \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_2) \end{bmatrix}^\top \\ &\quad \begin{bmatrix} -(k_{11} + \varepsilon) & \frac{1}{2} (k_{12} + p(\tau) k_{21}) \\ \frac{1}{2} (k_{12} + p(\tau) k_{21}) & -\frac{1}{4 k_{11}} (k_{12} + p(\tau) k_{21})^2 \end{bmatrix} \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_2) \end{bmatrix} \\ &\leq -\varepsilon \phi_1(x_1)^2 \leq 0. \end{aligned}$$

We now turn to the change in  $V$  due to jumps. Since  $p(0) \geq p(t)$  for all  $t \in [0, \varepsilon T]$ , we also have

$$\begin{aligned} V(g(x)) &\leq V_1(x_1) + p(0) V_2(g(x_1, x_2)) \\ &\leq V_1(x_1) + p(0) \lambda^2 V_2(x_2). \end{aligned}$$

When  $p(T)$  can be chosen so that  $p(0)\lambda^2 < p(T)$ , global asymptotic stability of the set

$$\mathcal{A} := \{x: x_1 = 0, x_2 = 0, \tau \in [0, T]\}$$

follows from Theorem 23. Indeed, points in the set  $\text{dom } F_\mu$  used in (25) must have  $x_1 = 0$ , while points in the set  $\text{dom } G_\mu$  used in (26) must have  $x_2 = 0$ , and thus points in the set  $G_\mu(\text{dom } G_\mu)$  must have  $x_2 = 0$ . In turn, points in the set  $D_\mu$  defined in (28) must have  $x_1 = 0, x_2 = 0$ , and  $V(x) = \mu$ . Since this is impossible for  $\mu > 0$ ,  $D_\mu$  is empty for  $\mu > 0$ . Therefore, to rule out complete solutions for  $\mathcal{H}_\mu$  we just need to rule out solutions that only flow. These solutions are ruled out by  $\dot{\tau} = 1$  and the fact that  $C_\mu$  bounded in the  $\tau$  direction.

Now we must relate  $p(0)$  to  $p(T)$ . As in the sampled-data problem, the solution to the differential equation (35) can be written explicitly by forming the Hamiltonian matrix

$$H := \begin{bmatrix} -\frac{k_{22}}{2k_3} - \frac{k_{21}k_{22}}{k_3 k_{11}} & \frac{k_{21}^2}{4k_3 k_{11}} \\ -\frac{k_{12}^2}{4k_3 k_{11}} & \frac{k_{22}}{2k_3} + \frac{k_{21}k_{22}}{k_3 k_{11}} \end{bmatrix}$$

and the corresponding matrix exponential  $\Phi(\tau) = \exp(-H\tau)$ , and verifying

$$\begin{aligned} p(T-t) &= (\phi_{21}(t) + \phi_{22}(t)p(T))(\phi_{11}(t) + \phi_{12}(t)p(T))^{-1} \\ \text{for all } t &\in [0, T]. \end{aligned}$$

When  $\phi_{11}(t) > 0$  for all  $t \in [0, T]$ , the quantities  $\phi_{21}(t)$  and  $-\phi_{12}(t)$  are nonnegative for all  $t \in [0, T]$ . When, in addition,  $p(T) > 0$  and  $-p(T)\phi_{12}(T)/\phi_{11}(T) < 1$ , the quantity  $p(t)$  is defined for all  $t \in [0, T]$ . Then, with the definitions  $\Psi := \Phi(T)$ ,  $s := \sqrt{-\psi_{12}/\psi_{11}}$ ,  $r := \sqrt{\psi_{21}/\psi_{11}}$ , and  $q := p(T)$ , it follows that

$$p(0) = r^2 + [q + (1 - s^2 q)^{-1} s^2 q^2]/\psi_{11}^2.$$

In turn, the stability condition  $p(0)\lambda^2 < p(T)$  is guaranteed by the conditions that  $\phi_{11}(t) > 0$  for all  $t \in [0, T]$  and

$$\begin{bmatrix} \lambda^2 \left( r^2 + \frac{q}{\psi_{11}^2} \right) - q & \lambda s q / \psi_{11} \\ \lambda s q / \psi_{11} & -1 + s^2 q \end{bmatrix} < 0, \quad 0 < q.$$

These conditions on  $q$  are feasible for  $T$  sufficiently small since  $\lambda < 1$ , and  $r$  and  $s$  tend to zero and  $\psi_{11}$  tends to one as  $T$  tends to zero. ■

## LYAPUNOV-BASED HYBRID FEEDBACK CONTROL

In a control system, part of the data of the system is free to be designed. Sensors are used to measure state variables, and actuators are used to affect the system's behavior, resulting in a closed-loop dynamical system. When the design specifies regions in the state space where flowing and jumping, respectively, are allowed, it yields hybrid feedback control. In this case, the closed-loop system is a hybrid dynamical system. Since a typical goal of feedback control is asymptotic stability, construction of the hybrid control algorithm is guided by stability analysis tools for hybrid dynamical systems. In this section, we use several examples to illustrate how Lyapunov-based analysis tools, especially the Barbasin-Krasovskii-LaSalle theorem, are used to derive hybrid feedback control laws. In the examples below, the asymptotic stability induced by feedback is robust in the sense of theorems 15 and 17. This robustness is achieved by insisting on regularity of the data of the control laws so that the resulting closed loop hybrid systems satisfy the Basic Assumptions.

### Example 25 Revisited: Impulsive Clock Synchronization Based on the Firefly Model

The synchronicity analysis in Example 25 for a network of fireflies can be thought of as a synchronization control problem where the impulsive control law  $u_i = ex_i$  in the jump equation  $x_i^+ = g(x_i + u_i)$  is chosen to make Theorem 23 applicable, establishing almost global synchronization. Global synchronization can be achieved for two fireflies by redesigning  $u_i$ . Indeed, letting  $k \in (0, 1)$  and picking

# Some asymptotically controllable nonlinear control systems cannot be robustly stabilized to a point using classical, time-invariant state feedback ... hybrid feedback, and supervisory control in particular, makes robust stabilization possible.

$u_i = 0$  when  $x_i < (1 - k)/2$  and  $u_i = 2k$  when  $x_i > (1 - k)/2$  results in global synchronization. This property is established using the same Lyapunov-function candidate used in Example 25.

### Example 26 Revisited: Sampled-Data Feedback Control Design

The design of a linear sampled-data feedback controller can be carried out based on the analysis in Example 26. The matrix  $M$  in Example 26 is used to indicate a desired dissipation inequality for the closed-loop system or a desired stability robustness margin, while the matrices  $L$  and  $N$  are typically zero. In this case, using the definitions of  $S$  and  $\Psi$  given in Example 26, the matrix inequality (31) reduces to

$$\begin{bmatrix} J^\top (\Psi_{11}^{-1} X \Psi_{11}^{-\top} + \Psi_{21} \Psi_{11}^{-1}) J - X & J^\top \Psi_{11}^{-1} X S \\ S^\top X \Psi_{11}^{-\top} J & -I + S^\top X S \end{bmatrix} < 0. \quad (36)$$

In turn, the matrix  $J$  decomposes as  $J = G + H\Theta K$ , where  $G$ ,  $H$ , and  $K$  are fixed and  $\Theta$  is a design parameter corresponding to feedback gains. Then feasibility of the matrix inequality (36) in the parameters  $\Theta$  and  $X = X^\top > 0$  can be identified with a synthesis problem for the discrete-time system

$$\begin{aligned} \xi^+ &= \Psi_{11}^{-\top} (G + H\Theta K) \xi + Sw, \\ y &= Y(G + H\Theta K) \xi, \end{aligned}$$

where  $Y^\top Y = \Psi_{21} \Psi_{11}^{-1}$ . Recall that the matrix  $\Psi_{21} \Psi_{11}^{-1}$  is guaranteed to be positive semidefinite as long as  $\Phi_{11}(t)$  is invertible for all  $t \in [0, T]$ . The synthesis problem corresponds to picking the parameter  $\Theta$  to ensure that the  $\ell_2$ -gain from the disturbance  $w$  to the output  $y$  is less than one, as certified by the energy function  $V(\xi) = \xi^\top X \xi$  through the condition  $V(x^+) - V(x) \leq -\varepsilon |\xi|^2 - |y|^2 + |w|^2$  for some  $\varepsilon > 0$ .

When the parameters  $G$ ,  $H$ , and  $K$  correspond to state feedback or full-order output feedback, the above synthesis problem can be cast as a convex optimization problem in the form of a linear matrix inequality [22]. Generalizations of this sampled-data control approach are developed for multirate sampled-data systems in [44] based on time-varying lifting. The approach discussed above, which is

expressed directly in terms of a Lyapunov analysis and thus avoids lifting, is reminiscent of the approach to sampled-data control taken in [79] and [85].

### Example 28: Resetting Nonlinear Control

This example is based on [11], [29], and [30]. Consider the nonlinear control system  $\dot{\xi} = f(\xi, u)$ , where  $\xi \in \mathbb{R}^n$  and  $f$  is continuous, together with the dynamic controller  $\dot{\eta} = \phi(\eta, \xi)$ ,  $u = \kappa(\eta, \xi)$ , where  $\eta \in \mathbb{R}^m$  and  $\phi$  and  $\kappa$  are continuous. Let  $x = (\xi, \eta)$  and, for each  $x \in \mathbb{R}^{n+m}$ , define  $F(x) := (f(\xi, \kappa(\eta, \xi)), \phi(\eta, \xi))$ . We refer to the system  $\dot{x} = F(x)$  as  $\mathcal{H}_c$ . Let  $V$  be a Lyapunov-function candidate for  $(\mathcal{H}_c, \{0\})$  that is positive on  $\mathbb{R}^{n+m} \setminus \{0\}$  and for which the sublevel sets of  $V$  are compact. Moreover, suppose

R1) The condition  $\langle \nabla V(x), F(x) \rangle \leq 0$  holds for all  $x \in \mathbb{R}^{n+m}$ .

We do not assume that the function  $V$  satisfies the conditions of Theorem 23 for the system  $\mathcal{H}_c$ . Instead, global asymptotic stability of the origin is achieved by resetting the controller state  $\eta$  so that  $V$  decreases at jumps. To that end, suppose that

R2) For each  $\xi \in \mathbb{R}^n$  there exists  $\eta_* \in \mathbb{R}^m$  such that  $V(\xi, \eta_*) < V(\xi, \eta)$  for all  $\eta \neq \eta_*$ .

Since  $V$  is continuous and its sublevel sets are compact, the function  $\xi \mapsto \eta_*(\xi)$  is continuous. Since  $V(x)$  is positive when  $x \neq 0$ ,  $\eta_*(0) = 0$ . As in [29] and [30], the hybrid controller resets  $\eta$  to  $\eta_*(\xi)$  when  $(\xi, \eta)$  belong to the jump set  $D$ . We suppose that

R3) The set  $D \subset \mathbb{R}^{n+m}$  is closed, intersects the set  $\{(\xi, \eta) : \eta = \eta_*(\xi)\}$  only at the origin, and, for each  $c > 0$ , there does not exist a complete solution to

$$\dot{x} = F(x) \quad x \in (\overline{\mathbb{R}^{n+m} \setminus D}) \cap L_V(c),$$

where  $L_V(c)$  denotes the  $c$ -level set of  $V$ .

Under conditions R1), R2), and R3), it follows from Theorem 23 with the Lyapunov-function candidate  $V$  that the hybrid resetting controller

$$\begin{cases} u = \kappa(\eta, \xi) \\ \dot{\eta} = \phi(\eta, \xi) \\ \eta^+ = \eta_*(\xi) \end{cases} \quad \begin{cases} (\xi, \eta) \in \overline{\mathbb{R}^{n+m} \setminus D}, \\ (\xi, \eta) \in D \end{cases}$$

## Zeno Solutions

Zeno of Elea would have liked hybrid systems. According to Aristotle [S59] with embellishment from Simplicius [S57] and the authors of this article, Zeno was the ancient Greek philosopher whose Tortoise bested the swift Achilles in a battle of wits by appealing to hybrid systems theory. Here is our version of the encounter, a modern paraphrase of Zeno's paradox.

*Tortoise:* Hey, Achilles, how about a race?

*Achilles:* Are you kidding me? I'll trounce you!

*Tortoise:* Well, for sure, I need a head start...

*Achilles:* And how much of a head start would you like?

*Tortoise:* Doesn't matter. Any will do.

*Achilles:* And why is that?

*Tortoise:* Because this will be a hybrid race.

*Achilles:* What do you mean by that?

*Tortoise:* Well, Achilles, as you try to catch me, I want you to keep track of where I have been. This will help you see that you are catching up to me. Note where I start. When you reach that point, note where I am again. Keep doing this until you pass me! Surely, your brain can handle this, can't it Achilles?

*Achilles:* So, the race is hybrid because, while my feet are running a race I can win, my mind is racing without end? You know, I just felt a twinge in a tendon near my ankle. Do you think we can postpone this race for a couple of millennia?

Achilles would overtake the Tortoise and win if the race depended only on Achilles' feet. However, the Tortoise's ground rules require Achilles' mind to be involved as well. The mental task that the Tortoise gives to Achilles prevents Achilles from catching the Tortoise. Indeed, no matter how many steps of the mental assignment that Achilles completes, he will still trail the Tortoise physically.

In modern terms, here is the hybrid model that the Tortoise suggests. Let  $\tau \in \mathbb{R}$  denote the position of the Tortoise, which is initialized to zero as a reference point. Let  $\alpha \in \mathbb{R}$  denote the position of Achilles. Due to the Tortoise's head start,  $\alpha$  is initialized to a negative value. The speed of the Tortoise is normalized to one, whereas the speed of Achilles, for sake of convenience, is taken to be two. If this description were the complete model, Achilles would reach the Tortoise at a time proportional to the initial separation between the Tortoise and Achilles, and then Achilles would continue past the Tortoise to victory. But this point is where the Tortoise gets clever. He gives Achilles the following instructions: "Note where I

start. When you reach that point, note where I am again. Keep doing this until you pass me (... if you ever do)!"

To adhere to the Tortoise's ground rules, Achilles needs a state variable  $\rho$  that keeps track of the Tortoise's most recently observed position. Specifically, each time Achilles reaches  $\rho$ , he must update the value of  $\rho$  to be the Tortoise's current position. Thus, we have a hybrid system with the state  $x = (\alpha, \tau, \rho)$  and data

$$C := \{x \in \mathbb{R}^3 : \alpha \leq \rho \leq \tau\}, \quad f(x) := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ for all } x \in C,$$

$$D := \{x \in \mathbb{R}^3 : \alpha = \rho \leq \tau\}, \quad g(x) := \begin{bmatrix} \alpha \\ \tau \\ \tau \end{bmatrix} \text{ for all } x \in D.$$

We call this system the Achilles. Achilles' initial goal is to reach the set  $\mathcal{A} := \{x \in \mathbb{R}^3 : \alpha = \rho = \tau\}$ , but he is observant enough to see that he cannot reach this goal. Indeed, Achilles cannot reach  $\mathcal{A}$  by flowing, since then it would be possible to follow his flowing trajectory backward from a point in  $\mathcal{A}$  to a point outside of  $\mathcal{A}$  while remaining in  $C$ . However, no solution of the equation  $\dot{x} = -f(x)$ ,  $x \in C$  can start in  $\mathcal{A}$  and leave  $\mathcal{A}$ . In fact, satisfying  $\dot{x} = -f(x)$  from any initial condition in  $\mathcal{A}$  immediately would force  $\tau < \rho$ , that is, the Tortoise is behind Achilles' most recent observation of the Tortoise's location, which is outside of  $C$ . Also, Achilles cannot reach  $\mathcal{A}$  by jumping to it. Indeed, being in  $D$  but not in  $\mathcal{A}$  means that  $\alpha = \rho < \tau$ . The value of the state after a jump would satisfy  $\alpha < \rho = \tau$ , which is neither in  $\mathcal{A}$  nor  $D$ .

It is no consolation for Achilles that the set  $\mathcal{A}$  is globally asymptotically stable for the Achilles system. This fact can be established using the Lyapunov-function candidate  $V(x) := 2(x_2 - x_1) + x_2 - x_3$ , which is zero on  $\mathcal{A}$ , positive otherwise, and such that the sublevel sets of  $V|_{\text{dom } V \cap (C \cup D)}$  are compact. The function  $V$  satisfies  $\langle \nabla V(x), f(x) \rangle = -1$  for all  $x \in C$ , and  $V(g(x)) \leq (2/3)V(x)$  for all  $x \in D$ .

Neither is it a consolation for Achilles that the ordinary time  $t$  used to approach  $\mathcal{A}$  is bounded, a property that can be established by integrating the derivative of  $V$  along flows of the solution to the Achilles system. Figure S11 shows typical hybrid time domains for solutions to the Achilles system. The hybrid time domain in Figure S11(a) occurs for solutions starting in  $(C \cup D) \setminus \mathcal{A}$ . The hybrid time domain in Figure S11(b) occurs for solutions starting in  $\mathcal{A}$ .

globally asymptotically stabilizes the origin of the closed-loop system. Indeed, by construction, the Lyapunov-function candidate  $V$  decreases at jumps that occur at points other than the origin, does not increase during flows, and no complete, flowing solution keeps  $V$  equal to a nonzero constant.

Regarding condition R3), consider the case in which  $V(x) = V_1(\xi) + V_2(x)$ , where  $V_1(\xi) > 0$  for  $\xi \neq 0$ ,  $V_2(x) \geq 0$  for all  $x \in \mathbb{R}^{n+m}$ , and  $V_2(x) = 0$  if and only if  $x = (\xi, \eta)$  satisfies  $\eta = \eta_*(\xi)$ . This case is presented in [29] and [30]. In this situation, let

$$D := \{x = (\xi, \eta) : V_2(x) \geq \rho(V_1(\xi)), \langle \nabla V_2(x), F(x) \rangle \leq 0\} \quad (37)$$

for some continuous, nondecreasing function  $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that is zero at zero and positive otherwise. The set  $D$  defined by (37) enables resets when  $\eta$  has moved from  $\eta_*(\xi)$  and  $V_2(x)$  is not increasing along a solution. Since the set  $D$  is closed and intersects the set  $\{(\xi, \eta) : \eta = \eta_*(\xi)\}$ , it follows that R3) reduces to the condition