1 Relevant Background

Recall that the inner product of $a(t)$ and $b(t)$ is defined as

$$\langle a|b\rangle = \int_{-\infty}^{\infty} a^*(\tau)b(\tau)d\tau$$

Also recall that the convolution of $a(t)$ and $b(t)$ is defined as

$$(a * b)(t) = \int_{-\infty}^{\infty} a(\tau)b(t-\tau)d\tau$$

In a similar way, the cross-correlation of $a(t)$ and $b(t)$ is defined as

$$R_{ab}(t) = \int_{-\infty}^{\infty} a^*(\tau - t)b(\tau)d\tau$$

where the star denotes a complex conjugate. The auto-correlation function of $a(t)$ is simply defined as $R_a(t) \equiv R_{aa}(t)$.

(a) Show that for real functions, $R_{ab}(t) = a(t) * b(-t)$. If we wanted to implement cross-correlation in practice, what could we do instead?

(b) Visually compare the inner product vs cross-correlation defintions. What do you notice? Why is the cross-correlation sometimes called the "sliding inner product"?

(c) Show that the auto-correlation function is hermitian. That is, $R_a^*(t) = R_a(-t)$.

1.a

RHS $a(t) * b(-t) = b(-t) * a(t)$ (by commutative) $= \int_{-\infty}^{\infty} a(t-\tau)b(-\tau)d\tau$

Since a is real, we have LHS $R_{ab}(t) = \int_{-\infty}^{\infty} a(\tau - t)b(\tau)d\tau$. Flip the sign doesn't change the result since we integrate from minus infinity to positive infinity, so we have $R_{ab}(t) = \int_{-\infty}^{\infty} a(t-\tau)b(-\tau)d\tau =$ RHS.

In practice, we can reverse one of the signals and then compute the convolution with the other signal. This is equivalent to the cross-correlation.

1.b

The inner product is a measure of how similar two signals are, while the cross-correlation is a measure of how much one signal is shifted with respect to the other. The cross-correlation is called the "sliding inner product" because it is essentially the inner product of two signals as one of them is shifted along the time axis.

1.c

Proof.

$$R_a(t) = \int_{-\infty}^{\infty} a^*(\tau - t)a(\tau)d\tau$$

Thus, we have

$$\begin{aligned}
R_a^*(t) &= \int_{-\infty}^{\infty} a(\tau - t)a^*(\tau)d\tau \\
&= \int_{-\infty}^{\infty} a^*(\tau)a(\tau - t)d\tau \\
&= \int_{-\infty}^{\infty} a^*(\tau + t)a(\tau)d\tau \\
&= R_a(-t)
\end{aligned}$$

□

Derivation

Matched filters are used extensively in signal analysis, particularly when you have a known signal that you want to find in a very noisy data set. The basic idea behind it is that you compare a template signal to your data $x(t)$ which contains some true signal $s(t)$ and some noise $n(t)$, such that the data you observe is $x(t) = s(t) + n(t)$. To derive how to extract where our template best matches the true signal in the data, we will do some operation on the given data $x(t)$ with some general filter $h(t)$ (called the matched filter), getting a new, convolved output $y(t) = (x * h)(t)$ which somehow distinguishes where the signal is in the data. The way to do this is to choose the filter $h(t)$ that maximizes the signal to noise ratio (and thus is a better fit).

We will start the derivation by noting that we can split the output into a signal and noise part. Namely, $y(t) = (s * h)(t) + (n * h)(t) \equiv y_s(t) + y_n(t)$.

(d) Show that $y_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega$, where $H(\omega)$ and $S(\omega)$ are the Fourier transforms of $h(t)$ and $s(t)$, respectively. Hint: Recall that convolution in the time domain is equivalent to multiplication in the frequency domain.

Now we will consider minimizing the signal to noise ratio (SNR) at some time t_0. This is simply the ratio of the power of output that is due to the signal compared to that of the averaged noise. Namely,

$$SNR(t_0) = \frac{|y_s(t_0)|^2}{E\{|y_n(t)|^2\}}$$

Using the Wiener-Khinchin theorem, we can write

$$E\{|n(t)|^2\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)d\omega$$

where $S_n(\omega)$ is the power spectral density of the noise.

(e) Using the above expression, rewrite the SNR as

$$SNR(t_0) = \frac{1}{2\pi}\frac{|\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega|^2}{\int_{-\infty}^{\infty} |H(\omega)|^2 S_n(\omega)d\omega}$$

Hint: The expectation value here acts only on $n(t)$, not $h(t_0)$.

(f) Using the Cauchy-Schwarz inequality $|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle$, show that the signal to noise can be given the upper bound

$$SNR(t_0) \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{S_n(\omega)}d\omega$$

Thus, the $h(t)$ that maximizes the SNR will be the one where the SNR equals this bound.

(g) Show that the upper bound is met if we set $H(\omega) = Ae^{i\omega t_0}\frac{S^*(\omega)}{S_n(\omega)}$ for an arbituary constant A.

Considering white noise now (i.e. uncorrelated, zero mean noise) and a real signal $s(t)$, we can simplify this to $H(\omega) = e^{i\omega t_0}S(-\omega)$.

(h) Plug this expression for $H(\omega)$ back into $y_s(t)$ and show that $y_s(t) = R_s(t)$ (with a change of variable $t - t_0 \to t$), a simple autocorrelation function.

(i) Therefore, show that $y(t) = x(t) * s(-t)$ and thus infer what $h(t)$ equals.

To recap, we found that the filter $h(t)$ required to maximize the signal to noise ratio (SNR; the power of the signal $s(t)$ to the power of the noise $n(t)$ in the observed data) in the convolution of the observed data $x(t)$ with that filter, is simply the time-inversion of the signal we expect to see somewhere in the data. Another way of saying this is that, to extract the location of a template in a dataset, we compute the cross-correlation of the template with that dataset. If we divide this by the noise estimate, we get the signal to noise of our template in the dataset. An observed signal that appears like the template in the dataset will appear as a peak in this SNR plot. You will now apply this with something like ‘np.correlate‘ below.

1.d

Since $y_s(t) = (s * h)(t)$, we have

$$\begin{aligned} y_s(t) &= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} s(\tau)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega(t-\tau)}d\omega\right)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\left(\int_{-\infty}^{\infty} s(\tau)e^{i\omega(t-\tau)}d\tau\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega \end{aligned}$$

1.e

$$\begin{aligned} SNR(t_0) &= \frac{|y_s(t_0)|^2}{E\{|y_n(t)|^2\}} \\ &= \frac{|\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega|^2/(2\pi)^2}{\frac{1}{2\pi}|H(\omega)|^2\int_{-\infty}^{\infty} S_n(\omega)d\omega} \\ &= \frac{1}{2\pi}\frac{|\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega|^2}{\int_{-\infty}^{\infty}|H(\omega)|^2 S_n(\omega)d\omega} \end{aligned}$$

1.f

$$\begin{aligned} SNR(t_0) &\le \frac{1}{2\pi}\frac{|\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega|^2}{\int_{-\infty}^{\infty}|H(\omega)|^2 S_n(\omega)d\omega} \\ &\le \frac{1}{2\pi}\frac{\int_{-\infty}^{\infty}|H(\omega)|^2 S(\omega)S^*(\omega)d\omega}{\int_{-\infty}^{\infty}|H(\omega)|^2 S_n(\omega)d\omega} \\ &\le \frac{1}{2\pi}\frac{\int_{-\infty}^{\infty}|S(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} S_n(\omega)d\omega} \end{aligned}$$

1.g

The maximum value of the SNR is achieved when the equality holds in the Cauchy-Schwarz inequality. Thus, we have

$$H(\omega) = Ae^{i\omega t_0}\frac{S^*(\omega)}{S_n(\omega)}$$

Take $A = 1$, we have $H(\omega) = e^{i\omega t_0}\frac{S^*(\omega)}{S_n(\omega)}$. Since $S_n(\omega)$ is real, we have $H(\omega) = e^{i\omega t_0}S(-\omega)$.

1.h

Plug this back to $y_s(t)$, we have

$$\begin{aligned}
y_s(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t_0}S(-\omega)S(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(-\omega)S(\omega)e^{i\omega(t-t_0)}d\omega \\
&= R_s(t-t_0) \\
&= R_s(t) \text{ (by change of variable)}
\end{aligned}$$

1.i

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega(t-\tau)}d\omega\right)d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\left(\int_{-\infty}^{\infty} x(\tau)e^{i\omega(t-\tau)}d\tau\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{i\omega t}d\omega
\end{aligned}$$

Thus, we have $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(-\omega)e^{i\omega t}d\omega = s(-t)$. Then we have $y(t) = x(t) * h(t) = x(t) * s(-t)$.