Clustering under Perturbation Resilience

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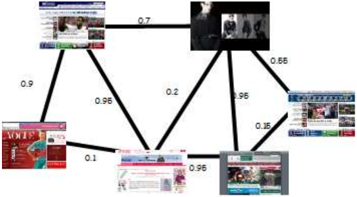
Clustering Comes Up Everywhere

- Cluster news articles or web pages by topic

- Cluster images by who is in them

Standard Theoretical Approach

- View objects as nodes in weighted graph based on distances



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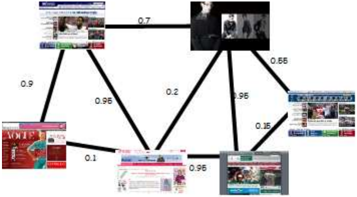

- Pick some objective to optimize
  - *k*-median: find centers $\{c_1, \ldots, c_k\}$ to minimize $\sum_i \sum_{p \in C_i} d(p, c_i)$
  - Min-sum: find partition $\{C_1, \ldots, C_k\}$ to minimize $\sum_i \sum_{p,q \in C_i} d(p, q)$

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- k-median: NP-hard to approximate within $(1 + 1/e)$; can be approximated within a $(1 + \sqrt{3} + \epsilon)$ factor
- Min-sum: NP-hard to optimize; can be approximated within a $\log n$ factor

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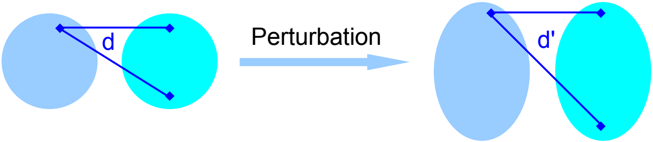
  can be approximated within a $(1 + \sqrt{3} + \epsilon)$ factor
- Min-sum: NP-hard to optimize;
  can be approximated within a $\log n$ factor
- Cool new direction: exploit additional properties of the data to circumvent lower bounds

α-Perturbation Resilience

α-PR [Bilu and Linial, 2010, Awasthi et al., 2012]

A clustering instance (S, d) is α-perturbation resilient to a given objective function Φ if for any function $d' : S \times S \to R_{\geq 0}$ s.t. $\forall p, q \in S, d(p, q) \leq d'(p, q) \leq \alpha d(p, q)$, there is a unique optimal clustering $\mathcal{OPT}'$ for Φ under d' and this clustering is equal to the optimal clustering $\mathcal{OPT}$ for Φ under d.



Main Results

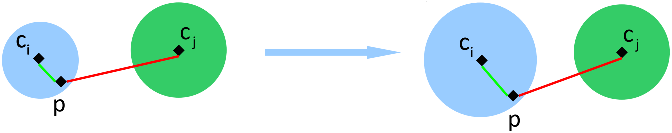
- Polynomial time algorithm for finding $\mathcal{OPT}$ for α-PR *k*-median instances when $\alpha \geq 1 + \sqrt{2}$
  - It works for any center-based objective function, e.g. *k*-means
- Polynomial time algorithm for a generalization (α, ϵ)-PR
- Polynomial time algorithm for finding $\mathcal{OPT}$ for α-PR min-sum instances when $\alpha \geq 3\frac{\max_i |C_i|}{\min_i |C_i|}$

Structure Properties of α-PR k-Median Instance

Claim

α-PR for k-median implies that $\forall p \in C_i, \alpha d(p, c_i) < d(p, c_j)$.

- Blow up all pairwise distances within the optimal cluster by α
- The $\mathcal{OPT}$ does not change, so $\forall p \in C_i, d'(p, c_i) < d'(p, c_j)$
- $d'(p, c_i) = \alpha d(p, c_i) < d'(p, c_j) = d(p, c_j)$



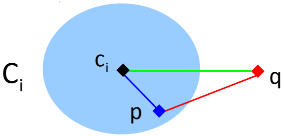
Structure Properties of α-PR k-Median Instance

Claim

α-PR for k-median implies that $\forall p \in C_i, \alpha d(p, c_i) < d(p, c_j)$.

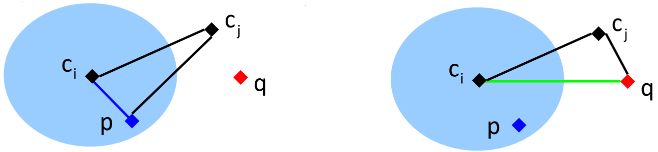
Implication:

- if $\alpha \geq 1 + \sqrt{2}, \forall p \in C_i, q \notin C_i$,
  $d(c_i, p) < d(c_i, q)$ and $d(c_i, p) < d(p, q)$



Structure Properties of α-PR k-Median Instance

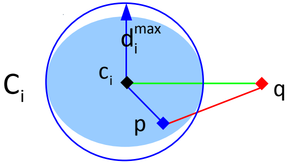
- Implication 1: if $\alpha \geq 1 + \sqrt{2}, \forall p \in C_i, q \notin C_i$, $d(c_i, p) < d(c_i, q)$
  - $d(c_i, c_j) \geq d(p, c_j) - d(p, c_i) > (\alpha - 1) d(p, c_i)$
  - $d(c_i, c_j) \leq d(q, c_i) + d(q, c_j) < (1 + \frac{1}{\alpha}) d(q, c_i)$



- Implication 2: a similar argument shows $d(c_i, p) < d(p, q)$

Structure Properties of α-PR k-Median Instance

- Let $d_i^{max} = \max_{p \in C_i} d(p, c_i)$. Construct a ball $B(c_i, d_i^{max})$
  - the ball covers exactly C_i
  - points inside are closer to the center than to points outside, i.e. $\forall p \in B(c_i, d_i^{max}), q \notin B(c_i, d_i^{max}), d(p, c_i) < d(p, q)$

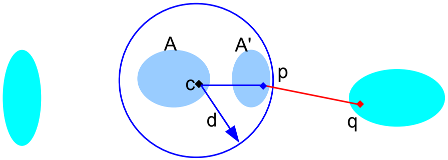


Closure Distance

Closure Distance

The closure distance $d_S(A, A')$ between two subsets A and A' is the minimum d, such that there exists a point $c \in A \cup A'$ satisfying:

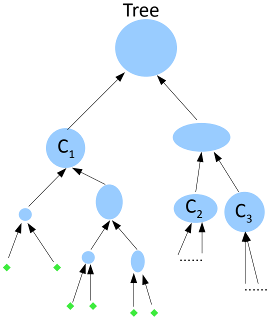
- **coverage condition**: the ball $B(c, d)$ covers $A \cup A'$;
- **margin condition**: points inside are closer to the center than to points outside, i.e. $\forall p \in B(c, d), q \notin B(c, d), d(c, p) < d(p, q)$.



Algorithm for α-PR k-median

Closure Linkage

- Begin with each point being a cluster
- Repeat until one cluster remains:
  merge the two clusters with
  minimum closure distance
- Output the tree with points as leaves
  and merges as internal nodes



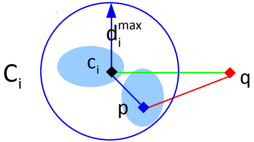
Theorem

If $\alpha \geq 1 + \sqrt{2}$, the tree output contains $\mathcal{OPT}$ as a pruning.

Proof

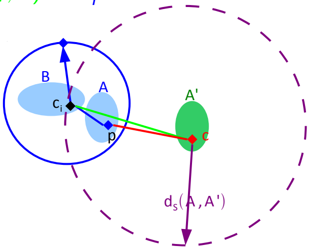
By induction, we show that the algorithm will not merge a strict subset $A \subset C_i$ with a subset A' outside C_i.

- Pick $B \subset C_i \setminus A$ such that $c_i \in A \cup B$
- $d_S(A, B) \leq d_i^{max} = \max_{p \in C_i} d(p, c_i)$
  - d_i^{max} and $c_i \in A \cup B$ satisfy the two conditions of closure distance



Proof

- $d_S(A, A') > d_i^{max}$
  - Suppose the center c for the ball defining $d_S(A, A')$ is from A'
  - Since $c \notin C_i$, $d(c_i, p) < d(p, c)$ for arbitrary $p \in A$.
    By margin condition,
    $c_i \in B(c, d_S(A, A')), i.e.\ d_S(A, A') \geq d(c_i, c)$
  - Since $c \notin C_i$, $d(c_i, c) > d_i^{max}$



  - A similar argument holds for the case $c \in A$

(α, ϵ)-Perturbation Resilience

- α-PR imposes a strong restriction that the $\mathcal{OPT}$ does not change after perturbation
- We propose a more realistic relaxation

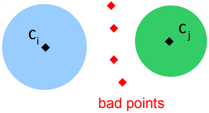
(α, ϵ)-Perturbation Resilience

A clustering instance (S, d) is (α, ϵ)-perturbation resilient to a given objective function Φ if for any function $d' : S \times S \to R_{\geq 0}$ s.t. $\forall p, q \in S, d(p, q) \leq d'(p, q) \leq \alpha d(p, q)$, the optimal clustering $\mathcal{OPT}'$ for Φ under d' is ϵ-close to the optimal clustering $\mathcal{OPT}$ for Φ under d.

Structure Property of (α, ϵ)-PR k-median

Theorem

Assume $\min_i |C_i| = O(\epsilon n)$. Except for $\leq \epsilon n$ bad points, any other point is α times closer to its own center than to other centers.



Keypoint of the Proof

- Carefully construct a perturbation that forces all the bad points move
- By (α, ϵ)-PR, there could be at most ϵn bad points

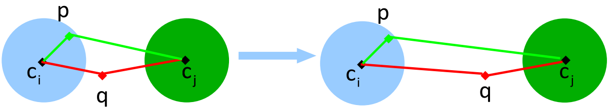
Proof of Property of (α, ϵ)-PR

B_i: bad points in C_i.

For technical reasons, select $\min\{|B_i|, \epsilon n + 1\}$ bad points from B_i.

Perturbation: blow up all pairwise distances by α, except

- between selected bad points and their second nearest centers
- between the other points and their own centers



p : good point; q : selected bad point.

Proof of Property of (α, ϵ)-PR

B_i: bad points in C_i.
For technical reasons, select $\min\{|B_i|, \epsilon n + 1\}$ bad points from B_i.

Perturbation: blow up all pairwise distances by α, except

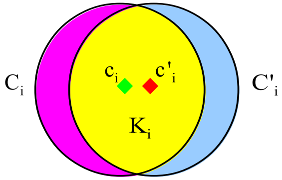
- between selected bad points and their second nearest centers
- between the other points and their own centers

Intuition: ideally, after the perturbation,

- selected bad points assigned to their second nearest centers
- all the other points stay

Proof of Property of (α, ϵ)-PR: centers after perturbation

Let c'_i be the new center for the new i-th cluster C'_i.
Sufficient to show: $c'_i \neq c_i$ leads to a contradiction.



Algorithm for (α, ϵ)-PR k-median

A robust version of Closure Linkage algorithm can be used to show:

Theorem

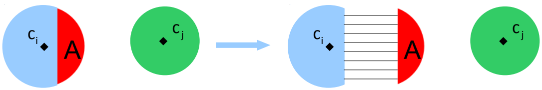
Assume $\min_i |C_i| \geq c\epsilon n$. If $\alpha \geq 2 + \sqrt{7}$, then the tree output contains a pruning that is ϵ-close to the optimal clustering. Moreover, the cost of this pruning is $(1 + O(\epsilon/\rho))$-approximation where $\rho = \min_i |C_i|/n$.

Structure Property of α-PR Min-Sum

Claim

α-PR implies $\forall A \subseteq C_i, \alpha d(A, C_i \setminus A) < d(A, C_j)$.

Proof: blow up the distances between A and $C_i \setminus A$ by α



Structure Property of α-PR Min-Sum

Claim

α-PR implies $\forall A \subseteq C_i, \alpha d(A, C_i \setminus A) < d(A, C_j)$.

Implications when $\alpha \geq 3\frac{\max_i |C_i|}{\min_i |C_i|}$:

(1) For any point, its $\min_i |C_i|/2$ nearest neighbors are from the same optimal cluster

(2) Any strict subset of an optimal cluster has smaller average distance to the other points in the same cluster than to those in other clusters

Algorithm for α-PR Min-Sum

- Connect each point with its $\min_i |C_i|/2$ nearest neighbors
- Perform average linkage on the components

Theorem

If $\alpha \geq 3\frac{\max_i |C_i|}{\min_i |C_i|}$, then the tree output contains $\mathcal{OPT}$ as a pruning.

Keypoints of the proof

- Implication (1) guarantees that the components are pure
- Implication (2) guarantees that no strict subset of an optimal cluster will be merged with a subset outside the cluster

Conclusion

- Polynomial time algorithm for finding (nearly) optimal solutions for perturbation resilient instances.
- Also consider a more realistic relaxation (α, ϵ)-PR

Thanks!

Awasthi, P., Blum, A., and Sheffet, O. (2012).
Center-based clustering under perturbation stability.
Inf. Process. Lett., 112(1-2):49–54.

Bilu, Y. and Linial, N. (2010).
Are stable instances easy?
In *Innovations in Computer Science*.