Scattering of massless particles: scalars, gluons and gravitons

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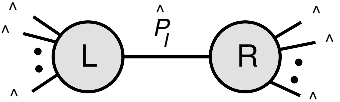
Contents

Some developments of amplitudes

- BCFW recursion relation.

$$A_n = \sum_{\text{diagrams } I} \hat{A}_{\mathsf{L}}(z_I)\,\frac{1}{P_I^2}\,\hat{A}_{\mathsf{R}}(z_I) = \sum_{\text{diagrams } I}$$



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- Dual superconformal symmetry of $\mathcal{N}=4$ Super Yang-Mills amplitude.
- Generalized unitarity
- Positive geometry and Amplituhedra.
- ...

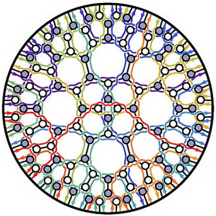

Figure: Positive Grassmanian

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Gauge theory color structure

At tree level, with particles in the adjoint representaion of gauge group SU(N), the amplitude can be decomposed as

$$\mathcal{A}_n^{tree}(1,2,3,\ldots,n) = \sum_{\mathcal{P}(2,3,\ldots,n)} Tr[T^{a_1}T^{a_2}T^{a_3}\ldots T^{a_n}]A_n^{tree}[1,2,3,\ldots,n]$$

here we ommit the coupling constant g, and A_n^{tree} is called tree-level color-ordered partial amplitude. Notice that this basis includes $(n-1)!$ independent amplitudes.

Color-ordered partial amplitudes satisfy a set of well-known properties,

- Cyclic:

$$A_n^{tree}[1,2,3,\dots,n] = A_n^{tree}[2,3,\dots,n,1]$$

- Reflection:

$$A_n^{tree}[1,2,3,\dots,n] = (-1)^n A_n^{tree}[n,\dots,3,2,1]$$

- "photon" decoupling:

$$\sum_{\sigma\in\text{cyclic}} A_n^{tree}[1,\sigma(2,3,\dots,n)] = 0$$

- KK(Kleiss-Kuijf) relation:

$$A_n^{tree}[1,\alpha,n,\beta] = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in OP(\{\alpha\},\{\beta\}^T)} A_n^{tree}[1,\{\sigma\}_i,n]$$

where the OP means ordered permutaions, that is all permutations of $\{\alpha\}\bigcup\{\beta\}^T$ that matains the order of individual elements of each set.

For example, a five point amplitude $A_n^{tree}(1,\{2,3\},5,\{4\})$, we have

$$A_n^{tree}[1,2,3,5,4] = -A_n^{tree}[1,2,3,4,5] - A_n^{tree}[1,2,4,3,5] - A_n^{tree}[1,4,2,3,5]$$

The other five point relations can be obtained by permuting legs 2,3,4 and using cyclic and reflection properties.

This means that the six amplitudes $A_n^{tree}(1,\mathcal{P}\{2,3,4\},5)$ form a basis of remaining five-point partial amplitudes. More generally, for multiplicity n, the KK relation can be used to rewrite any color-ordered partial amplitude in terms of only $(n-2)!$ basis partial amplitudes, where two legs are fixed(usually choose 1 and n).

A Similar Structure

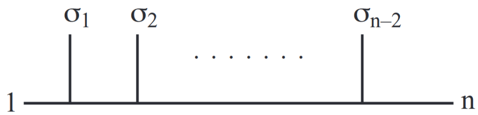
It can be proved that tree level amplitudes can also be decomposed like

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma_1} b_1} \tilde{f}^{b_1 a_{\sigma_2} b_2} \ldots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n} \tilde{A}_n(1, \sigma_1, \sigma_2, \ldots, \sigma_{n-2}, n)$$

The basis is called DDM and more convenient in parctice. It is easy to realize that the number of basis amplitudes is also $(n-2)!$ with fiexed particle labels 1 and n.

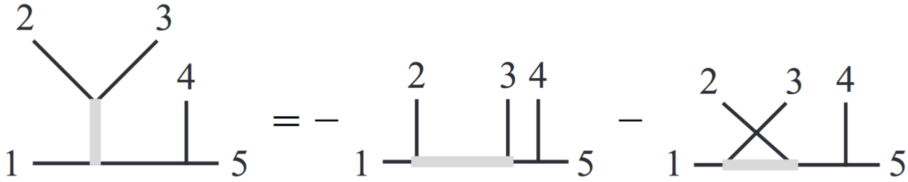
How can we realize the existence of this basis?

We just need to notice the contribution from ladder diagram



$$\rightarrow \quad \tilde{f}^{a_1 a_{\sigma_1} b_1}\, \tilde{f}^{b_1 a_{\sigma_2} b_2} \ldots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n}$$

And any (three vertex) diagram, by using the Jacobbi identity like



can be transformed to the ladder diagram. It has been proved that $\tilde{A}_n(\ldots)$ here is identical to $A_n^{tree}[\ldots]$ in the trace basis.

Color-Kinematics Duality

The Color-Kinematics Duality here is not the complete version, but we can catch some points. At four points we have

$$A_4^{\text{tree}}(1,2,3,4) + A_4^{\text{tree}}(1,3,2,4) + A_4^{\text{tree}}(1,4,2,3) = 0.$$

We recognize that the only nontrival way the equation holds accoring to $s+t+u=0$

$$\begin{aligned}&A_4^{\text{tree}}(1,2,3,4) + A_4^{\text{tree}}(1,3,2,4) + A_4^{\text{tree}}(1,4,2,3)\\ &= (s+t+u)\chi = 0\end{aligned}$$

Considering that $A_4^{\text{tree}}(1,2,3,4)$ treats s and t the same, and similar to the other two, we make the identification

$$A_4^{\text{tree}}(1,2,3,4) = u\chi, \quad A_4^{\text{tree}}(1,3,2,4) = t\chi, \quad A_4^{\text{tree}}(1,4,2,3) = s\chi$$

We obtain

$$
\begin{aligned}
tA_4^{\text{tree}}(1,2,3,4) &= uA_4^{\text{tree}}(1,3,4,2), \\
sA_4^{\text{tree}}(1,2,3,4) &= uA_4^{\text{tree}}(1,4,2,3), \\
tA_4^{\text{tree}}(1,2,3,4) &= sA_4^{\text{tree}}(1,3,4,2).
\end{aligned}
$$

And in order to obtain the knematic analog of Jacobbi identity, it is convenient to express the amplitudes in terms of poles

$$
\begin{aligned}
A_4^{\text{tree}}(1,2,3,4) &\equiv \frac{n_s}{s} + \frac{n_t}{t}, \\
A_4^{\text{tree}}(1,3,4,2) &\equiv -\frac{n_u}{u} - \frac{n_s}{s}, \\
A_4^{\text{tree}}(1,4,2,3) &\equiv -\frac{n_t}{t} + \frac{n_u}{u}.
\end{aligned}
$$

the relative sign here is just a convention.

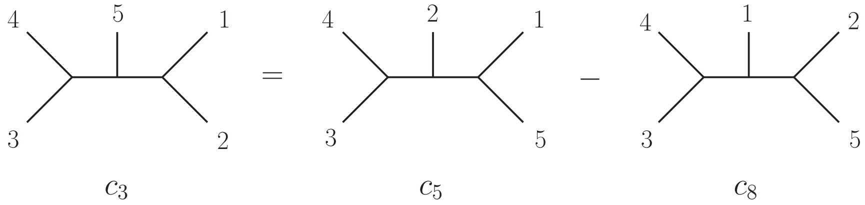
Combining the two relation above gives the desired identity

$$n_u = n_s - n_t$$

this is exactly the same form of Jacobbi identity for the color factors. The full amplitude can be written like

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_u c_u}{u} + \frac{n_t c_t}{t} \right)$$

For the five point case, an example is



we have the corresponding color factor Jacobbi identities like

$$c_3 = c_5 - c_8$$

so we need to have the knematic analog

$$n_3 = n_5 - n_8$$

The explict forms of all these factors can be found in the original paper **New relations for gauge theory amplituds, PRD 78, 085011**

Full amplitudes of 5 points are

$$\begin{aligned}\mathcal{A}_5^{\text{tree}} \quad &= \quad g^3\left(\frac{n_1c_1}{s_{12}s_{45}}+\frac{n_2c_2}{s_{23}s_{51}}+\frac{n_3c_3}{s_{34}s_{12}}+\frac{n_4c_4}{s_{45}s_{23}}+\frac{n_5c_5}{s_{51}s_{34}}+\frac{n_6c_6}{s_{14}s_{25}}\right.\\ &\quad+\frac{n_7c_7}{s_{32}s_{14}}+\frac{n_8c_8}{s_{25}s_{43}}+\frac{n_9c_9}{s_{13}s_{25}}+\frac{n_{10}c_{10}}{s_{42}s_{13}}+\frac{n_{11}c_{11}}{s_{51}s_{42}}+\frac{n_{12}c_{12}}{s_{12}s_{35}}\\ &\quad\left.+\frac{n_{13}c_{13}}{s_{35}s_{24}}+\frac{n_{14}c_{14}}{s_{14}s_{35}}+\frac{n_{15}c_{15}}{s_{13}s_{45}}\right),\end{aligned}$$

This relation had been proved upto 8 points at the time and I think it has been improved to arbitrary points.

Contents

Scattering equations

It has benn proposed that there is connection between the scattering data of n massless particles and the n-punctured sphere from a rational map

$$k_\mu^a = \frac{1}{2\pi i}\oint_{|z-\sigma_a|} dz \frac{p^\mu(z)}{\prod_{b=1}^n (z-\sigma_b)}$$

To discribe the n-punctured sphere more properly, we can introduce the Riemann sphere as

$$\mathbb{CP}^1 \cong \mathcal{S}^2 \cong \mathbb{C} \cup \{\infty\}$$

and n-punctured Riemann sphere can be discribed by $SL(2,\mathbb{C})$ affine coordinates $\sigma_1, \sigma_2, \ldots, \sigma_n$, that is to say we have a equvilance relation

$$\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \sim \{\psi(\sigma_1), \psi(\sigma_2), \ldots, \psi(\sigma_n)\},$$
$$\psi(\sigma) := \frac{\alpha\sigma + \beta}{\gamma\sigma + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma = 1$$

because of the redundancy of $\mathrm{SL}(2,\mathbb{C})$, only $n-3$ of them are independent.

From this map, we can easily obtain the main ingredients of this report

Scattering equations

$$\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \qquad a \in \{1, \ldots, n\}$$

It has been proved that the number of solutions in any dimension is $(n-3)!$, and only $n-3$ of the equtions are independent, so we can rewrite the scattering equations as following

$$\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad a \in \{4, 5, \ldots, n\} \quad and \quad \sigma_1 \to \infty, \sigma_2 = 0, \sigma_3 = 1$$

KLT Orthogonality

- KLT orthogonality is a striking property of the solutions to scattering equations.

Proposition 1

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \delta_{ij}$$

First we need to define the Jacobian matrix associated to the scattering equations

$$\boxed{\Phi_{ab} \equiv \partial \left(\sum_{c \neq a} \frac{s_{ac}}{\sigma_a - \sigma_c} \right) / \left(\partial \sigma_b\right) = \begin{cases} \frac{s_{ab}}{(\sigma_a - \sigma_b)^2}, & a \neq b, \\ -\sum_{c \neq a} \Phi_{ac}, & a = b \end{cases}}$$

As mentioned above only n-3 of the scattering equations are independent so the matrix Φ has rank n-3. (This matrix was first encountered in the gravity amplitudes constructed from gauge theory using KLT relation)

Consider a generalization of Φ_{ab}

$$\Psi_{ab,a\neq b} \equiv \frac{s_{ab}}{(\sigma_a - \sigma_b)(\sigma'_a - \sigma'_b)}, \quad \Psi_{aa} \equiv -\sum_{c\neq a} \Psi_{ac}.$$

Proposition 2

$$\operatorname{rank} \Psi(\{\sigma\}, \{\sigma'\}) = \begin{cases} n-4, \{\sigma\} \neq \{\sigma'\} \\ n-3, \{\sigma\} = \{\sigma'\} \end{cases}$$

σ and σ' are assmused to be solutions to scattering equation.

Prove of KLT orthogonality

For the purpose of proving KLT orthogonality, we can construct a n! dimension vector for each solution

$$\frac{1}{(\sigma_{\omega(1)} - \sigma_{\omega(2)})(\sigma_{\omega(2)} - \sigma_{\omega(3)}) \cdots (\sigma_{\omega(n)} - \sigma_{\omega(1)})}$$

Not so obvious is the fact that we can fix the positon of 3 labels, which we choose 1,n-1,n, give rise to the KK relation and BCJ relation.

Now the vectors become (n-3)! dimension, and even aftering selecting three lables, we still have the freedom of where to put them. Here we only use two choices :

$$(1, \omega(2), \ldots, \omega(n-2), n-1, n) \quad and \quad (1, \omega(2), \ldots, \omega(n-2), n, n-1)$$

The corresponding two vectors are

$$V(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_{n-1})(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)},$$
$$U(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_n)(\sigma_n - \sigma_{n-1})(\sigma_{n-1} - \sigma_1)}.$$

In this language, we can construct a bilinear form

$$S[\alpha|\beta] = \prod_{i=2}^{n-2} \left(s_{1,\alpha_i} + \sum_{j=2}^{i-1} \theta(\alpha(j), \alpha(i))_\beta s_{\alpha(j),\alpha(i)} \right)$$

where $\alpha, \beta \in S_{n-3}$,$\theta(i,j)_\beta = 1$if the order of i,j is the same in both permutations $\alpha(2,3,\dots,n-2)$ and $\beta(2,3,\dots,n-2)$, and 0 otherwise. S is usually called Momentum Kernel.

Given any two solutions of scatering equations,

$$\{\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_n^{(i)}\} \quad and \quad \{\sigma_1^{(j)}, \sigma_2^{(j)}, \ldots, \sigma_n^{(j)}\}$$

define two vectors, $V(\alpha)^{(i)}$ and $U(\beta)^{(j)}$, i,j are choices of solutions and α, β are the choices of permutaions, the number of both is (n-3)!.

A natural inner product can be defined as

$$(i,j) := \sum_{\alpha,\beta \in S_{n-3}} V^{(i)}(\alpha) S[\alpha|\beta] U^{(j)}(\beta)$$

Knowing all definetions above, we can proceed to prove KLT orthogonality.

The starting point is to notice that

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \delta_{ij}$$

is clearly invariant under $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Partially fixing both $SL(2,\mathbb{C})$ redundancies with convenient choice $\sigma^{(i)}_{n-1} = \sigma^{(j)}_{n} = \infty$ and $\sigma^{(i)}_{n} = \sigma^{(j)}_{n-1} = 1$ and define

$$K_n(\{\sigma\},\{\sigma'\}) \equiv \sum_{\alpha,\beta\in S_{n-3}} \frac{1}{\sigma_{1,\alpha(2)}\cdots\sigma_{\alpha(n-3),\alpha(n-2)}} S[\alpha|\beta] \frac{1}{\sigma'_{1,\beta(2)}\cdots\sigma'_{\beta(n-3),\beta(n-2)}}$$

The motivation for this definition is that K_n appears in the numerator of KLT orthogonality.

It is also convenient to define an auxiliary co-rank one $(n-2) \times (n-2)$ matrix $\psi^{(n)}$

$$\psi_{ab, a \neq b} = \frac{s_{ab}}{\sigma_{ab}\sigma'_{ab}}, \quad \psi_{aa} = -\sum_{b \neq a} \psi_{ab}, \quad a, b = 1, \ldots, n-2$$

It can be proven that any $(n-3) \times (n-3)$ minors of $\psi^{(n)}$ are the same, and we denote such a minor as $\det'\psi^{(n)}$, that is to say, the determinat of the matrix after removing any row and colum.

Proposition 3

The two functions defined above are identical up to a sign.

$$K_n(\{\sigma\}, \{\sigma'\}) = (-1)^n \det'\psi^{(n)}$$

The final step is put all pieces together. With the choice $\sigma^{(i)}_{n-1} = \sigma^{(j)}_n = \infty$ and $\sigma^{(i)}_n = \sigma^{(j)}_{n-1} = 1$, we have

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \frac{K_n(\{\sigma^{(i)}\},\{\sigma^{(j)}\})}{K_n^{\frac{1}{2}}(\{\sigma^{(i)}\},\{\sigma^{(i)}\})K_n^{\frac{1}{2}}(\{\sigma^{(j)}\},\{\sigma^{(j)}\})}$$

In addition, one finds that the minor of ψ obtained by removing the first row and colum is identical to that of $\Psi(\{\sigma\},\{\sigma'\})$ after removing rows and colums $\{1, n-1, n\}$. We denote them respectively $|\psi^{(n)}|^1_1$ and $|\Psi|^{1,n-1,n}_{1,n-1,n}$. Then,

$$\begin{aligned}
\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} &= \frac{K_n(\{\sigma^{(i)}\},\{\sigma^{(j)}\})}{K_n^{\frac{1}{2}}(\{\sigma^{(i)}\},\{\sigma^{(i)}\})K_n^{\frac{1}{2}}(\{\sigma^{(j)}\},\{\sigma^{(j)}\})} \\
&= \frac{(-1)^n|\psi^{(n)}|^1_1}{(-1)^n|\psi^{(n)}|^{1\,\frac{1}{2}}_1|\psi^{(n)}|^{1\,\frac{1}{2}}_1} \\
&= \frac{|\Psi(\{\sigma^{(i)}\},\{\sigma^{(j)}\})|^{1,n-1,n}_{1,n-1,n}}{(|\Psi(\{\sigma^{(i)},\sigma^{(i)}\}|^{1,n-1,n}_{1,n-1,n})^{\frac{1}{2}}(|\Psi(\{\sigma^{(j)}\},\{\sigma^{(j)}\}|^{1,n-1,n}_{1,n-1,n})^{\frac{1}{2}}}
\end{aligned}$$

Fianlly, we just need to use Proposition 2.

- If $i = j$, the rank of matrix Ψ is $n-3$ and the minor is nonzero, we obtain

$$\frac{(i,i)}{(i,i)^{\frac{1}{2}}(i,i)^{\frac{1}{2}}} = \frac{|\Psi(\{\sigma^{(i)}\},\{\sigma^{(j)}\})|^{1,n-1,n}_{1,n-1,n}}{|\Psi(\{\sigma^{(i)}\},\{\sigma^{(j)}\})|^{1,n-1,n}_{1,n-1,n}} = 1$$

- If $i \neq j$, the rank of matrix is $n-4$, so any minor with volume more than $n-4$ equals 0.

$$|\Psi(\{\sigma^{(i)}\},\{\sigma^{(j)}\})|^{1,n-1,n}_{1,n-1,n} = 0 \Rightarrow \frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = 0$$

Up to now, we conclude the proof of KLT orthogonality.

Attempt to construct S-matrix — Towards CHY

Thanks to the excellent properties of scattering equations, it is very tempting to propose that the solutions to scattering equations should be used to construct scattering amplitudes.

The first two constructed are YM and gravity amplitudes in any dimensions

$$M_n^{\text{YM}}(1,2,\ldots,n) = \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\right) \frac{E_n(\{k,\epsilon,\sigma\})}{\sigma_{12}\ldots\sigma_{n1}},$$

$$M_n^{\text{gravity}} = \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\right) E_n(\{k,\epsilon,\sigma\})^2$$

The measure is defined as following

$$\prod_a{}' \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\right) := \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\right)$$

The reason we extract 3 indices from delta equation is the fact that only $n-3$ scattering equations are independent. This from can be proved to be independent of choice of i, j, k, therefore permutaion invariant. We also have

$$\sigma_a \to \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad d\mu_n \to \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^{-4} d\mu_n$$

$E_n(\{k, \epsilon, \sigma\})$ itself is permutaion invariant with resprct to σ_a, k_a^μ and ϵ_a^μ. The $SL(2, \mathbb{C})$ invariance of amplitude also constraints the form of $E_n(\{k, \epsilon, \sigma\})$

$$\sigma_a \to \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad E_n(\{k, \epsilon, \sigma\}) \to E_n(\{k, \epsilon, \sigma\}) \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^2$$

The form of measure

It is worth to computed the measure explicitly.Aftering "gauge fixing" the $SL(2,\mathbb{C})$ redundancy, one finds

$$\int \prod_{c\neq p,q,r} d\sigma_c (\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\right)$$

The delta functions completely localize all integrals and the answer is evaluating a Jacobian defined above.

$$\Phi_{ab} \equiv \partial \left(\sum_{c\neq a} \frac{s_{ac}}{\sigma_a - \sigma_c}\right) / (\partial \sigma_b) = \begin{cases} \frac{s_{ab}}{(\sigma_a-\sigma_b)^2}, & a \neq b, \\ -\sum_{c\neq a} \Phi_{ac}, & a = b \end{cases}$$

Then, we obtain the measure

$$\boxed{\sum_{\{\sigma\}\in \text{solutions}} \frac{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{|\Phi|^{ijk}_{pqr}}}$$

Always denoted by

$$\boxed{\det{}'\Phi := \frac{|\Phi|^{ijk}_{pqr}}{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}}$$

$|\Phi|^{ijk}_{pqr}$ means that we need to delete the rows $\{i, j, k\}$ and the colums $\{p, q, r\}$, of course it is free to choose which inedex refers to row or colum (Φ is a symmetric matrix).

The form of $E_n(\{k, \epsilon, \sigma\})$

In order to present the explict form of $E_n(\{k, \epsilon, \sigma\})$, first define the following $2n \times 2n$ antisymmetric matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

where A, B and C are $n \times n$ matrices, defined as

$$A_{ab} = \begin{cases} \frac{s_{ab}}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases} \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases}$$

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b, \\ -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{\sigma_a - \sigma_c} & a \neq b. \end{cases}$$

The first important observation is that while the Pfaffian of Ψ is 0, but after removing any rows i, j and colums i, j with $1 \le i < j \le n$, the new matrix Ψ^{ij}_{ij} have nonzero Pfaffian and we define the corresponding reduced Pfaffian as

$$\boxed{\mathrm{Pf}'\Psi := \frac{(-1)^{i+j}}{(\sigma_i - \sigma_j)}\mathrm{Pf}(\Psi^{ij}_{ij})}$$

It can be proved that the reduced Pfaffian is invariant under permutaion of particle labels.

Pfaffian

Pfaffian is defined for antisymmetric matrix, usually in two ways as following

- $$\mathrm{Pf}(A)^2 = \det A$$

- $$\mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{\sigma \in S_{2n}} sgn(\sigma)\prod_{i=1}^{n} a_{\sigma_{(2i-1)},\sigma(2i)}$$

Write down the proposal

$$E_n(\{k,\epsilon,\sigma\}) = \mathrm{Pf}'\Psi(k,\epsilon,\sigma)$$

Combine the measure and integrand, we conclude the formula for the tree-level S-matrix of Yang-Mills in any dimension

$$M_n^{\mathrm{YM}}(1,2,\ldots,n) = \frac{1}{\text{vol SL}(2,\mathbb{C})}\int \frac{d^n\sigma}{\sigma_{12}\cdots\sigma_{n1}}\prod_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathrm{Pf}'\Psi$$

And using the KLT constrution, we can construct the formula for tree-level S-matrix of gravity as double copy of that of Yang-Mills

$$M_n^{\mathrm{gravity}} = \frac{1}{\text{vol SL}(2,\mathbb{C})}\int d^n\sigma\prod_a{}'\delta\left(\sum_{b\neq a}\frac{s_{ab}}{\sigma_{ab}}\right)\mathrm{Pf}'\Psi\mathrm{Pf}'\tilde{\Psi}$$

We can also write the amplitude in another form

$$M_n^{\mathrm{YM}} = \sum_{\{\sigma\}\in\mathrm{solutions}} \frac{1}{\sigma_{12}\cdots\sigma_{n1}} \frac{\mathrm{Pf}'\Psi(k,\epsilon,\sigma)}{\det{}'\Phi}$$

$$M_n^{\mathrm{gravity}} = \sum_{\{\sigma\}\in\mathrm{solutions}} \frac{\det{}'\Psi(k,\epsilon,\sigma)}{\det{}'\Phi}$$

where we use the property of Pfaffian $\det{}'\Psi(k,\epsilon,\sigma) = \mathrm{Pf}'\Psi(k,\epsilon,\sigma)$ $\times\mathrm{Pf}'\Psi(k,\epsilon,\sigma)$.

Consistency check

- Gauge invariance

  If we replce the ith polariazation vector ϵ_i^μ with momentum k_i^μ, we find that

$$C_{ii} = -\sum_{c\neq i} \frac{\epsilon_i \cdot k_c}{\sigma_i - \sigma_c} \to -\sum_{c\neq i} \frac{k_i \cdot k_c}{\sigma_i - \sigma_c} = 0$$

  It is easy to discover that the ith and $i+n$th colums become identical, so the determinat and Pfaffian become 0.

Consistency check

- Gauge invariance
  If we replce the ith polariazation vector ϵ_i^μ with momentum k_i^μ, we find that
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  It is easy to discover that the ith and $i+n$th colums become identical, so the determinat and Pfaffian become 0.
- Soft limit Using a special property of Pfaffian
  $$\mathrm{Pf}(E) = \sum_{q=1}^{2n}(-1)^q e_{pq}\mathrm{Pf}(E_{pq}^{pq})$$
  we find the amplitude in the soft limit is
  $$A_n \to \left(\frac{\epsilon_n\cdot k_{n-1}}{k_n\cdot k_{n-1}} + \frac{\epsilon_n\cdot k_1}{k_n\cdot k_1}\right)A_{n-1}$$

CHY form of amplitudes

Both formulas above can be written in this simplest form

$$\mathcal{M}^{(s)} = \int \frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_a{}' \delta\Big(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}}\Big)\Big(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_3})}{(\sigma_1-\sigma_2)\ldots(\sigma_n-\sigma_1)}\Big)^{(2-s)} (\mathrm{Pf}'\Psi)^s$$

with $s=1$ for Yang-Mills and $s=2$ for gravity.

Here we would like to consider that the formula above is not only a convenient way to write Yang-Mills and gravity amplitudes, but can be a definition of S-matrix for spin s particles. This means that

$$s = 0 \quad \rightarrow \quad \text{a corresponding scalar theory}$$

In order to get more general case, the gravity amplitudes actually can be modified to the product of two different Pfaffians,each with own choice of polariazation vector

$$\big(\mathrm{Pf}'\Psi(k,\epsilon,\sigma)\big)^2 \to \mathrm{Pf}'\Psi(k,\epsilon,\sigma)\mathrm{Pf}'\Psi(k,\tilde{\epsilon},\sigma)$$

actually it gives amplitudes with gravitons coupled to dilatons and B-fields.

For the case $s=0$, the similar consequence is

$$\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})}{(\sigma_{12})\ldots(\sigma_{n1})}\right)^2 \to \left(\frac{\mathrm{Tr}(T^{a_1}\ldots T^{a_n})}{(\sigma_{12})\ldots(\sigma_{n1})}\right)\left(\frac{\mathrm{Tr}(\tilde{T}^{b_1}\ldots\tilde{T}^{b_n})}{(\sigma_{12})\ldots(\sigma_{n1})}\right)$$

while the original color group is $U(N)$, the new factors are the product of two different color group $U(N)\times U(\tilde{N})$.

The simplest possibility is the theory with only cubic interaction

$$-f_{abc}\tilde{f}_{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}$$

All of above leads to the conclusion that the factors

$$\mathcal{C}_{U(N)} \equiv \sum_{\sigma\in S_n/Z_n}\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})}{(\sigma_{12})\ldots(\sigma_{n1})}\right) \quad \text{and} \quad E_\epsilon \equiv \mathrm{Pf}'\Psi(\epsilon)$$

are interchangeable and this is a color-Kinematics corresesopondence which is valid for individual solutions to scattering equations.

The connection of amplitudes between 3 theories can be described by the following diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{M}_n^{(1)}(\mathrm{U}(N), \tilde{\epsilon}) \arrow[dr, "C_{\mathrm{U}(N)} \to E_{\epsilon}"] & \\
\mathcal{M}_n^{(0)}(\mathrm{U}(N) \times \mathrm{U}(\tilde{N})) \arrow[ur, "C_{\mathrm{U}(\tilde{N})} \to E_{\tilde{\epsilon}}"] \arrow[dr, "C_{\mathrm{U}(N)} \to E_{\epsilon}"'] & & \mathcal{M}_n^{(2)}(\epsilon, \tilde{\epsilon}) \\
 & \mathcal{M}_n^{(1)}(\mathrm{U}(\tilde{N}), \epsilon) \arrow[ur, "C_{\mathrm{U}(\tilde{N})} \to E_{\tilde{\epsilon}}"'] &
\end{tikzcd}
$$

Double partial amplitudes

Because there are two color indices in this sclar theory, so it can be anticipateed that the amplitude have double trace decomposition structure

$$\mathcal{M}_n^{(0)} = \sum_{\alpha \in S_n/Z_n} \mathrm{Tr}(\tilde{T}^{\mathsf{b}_{\alpha(1)}} \tilde{T}^{\mathsf{b}_{\alpha(2)}} \cdots \tilde{T}^{\mathsf{b}_{\alpha(n)}}) M_n^{(0)}(\alpha(1), \alpha(2), \ldots, \alpha(n))$$

$$= \sum_{\alpha,\beta \in S_n/Z_n} \mathrm{Tr}(\tilde{T}^{\mathsf{b}_{\alpha(1)}} \tilde{T}^{\mathsf{b}_{\alpha(2)}} \cdots \tilde{T}^{\mathsf{b}_{\alpha(n)}}) \mathsf{Tr}(T^{\mathsf{a}_{\beta(1)}} T^{\mathsf{a}_{\beta(2)}} \cdots T^{\mathsf{a}_{\beta(n)}})$$

$$\times\, m_n^{(0)}(\alpha|\beta)$$

where the last term $m_n^{(0)}(\alpha|\beta)$ is called double partial amplitude and can be read off from the full amplitude

$$m_n^{(0)}(\alpha|\beta) = \int \frac{d^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{\prod_a{}'\delta(\sum_{b\neq a} \frac{s_{ab}}{\sigma_{ab}})}{(\sigma_{\alpha(1),\alpha(2)} \cdots \sigma_{\alpha(n),\alpha(1)})(\sigma_{\beta(1),\beta(2)} \cdots \sigma_{\beta(n),\beta(1)})}$$

$$= \sum_{\{\sigma\} \in \mathrm{solutions}} \frac{1}{\det' \Phi} \frac{1}{(\sigma_{\alpha(1),\alpha(2)} \cdots \sigma_{\alpha(n),\alpha(1)})(\sigma_{\beta(1),\beta(2)} \cdots \sigma_{\beta(n),\beta(1)})}$$

Likewise the decomposition in the first section, it is more usually to write the amplitudes in terms of colore basis

$$\boxed{\mathbf{c}_\alpha \equiv \sum_{\mathsf{c}_1,\ldots,\mathsf{c}_{n-3}} f_{\mathsf{a}_1 \mathsf{a}_{\alpha(2)} \mathsf{c}_1} \cdots f_{\mathsf{c}_{n-3} \mathsf{a}_{\alpha(n-1)} \mathsf{a}_n}}$$

where $\alpha \in S_{n-2}$. The amplitude is

$$\mathcal{M}_n^{(0)} = \sum_{\alpha,\beta \in S_{n-2}} \mathbf{c}_\alpha \tilde{\mathbf{c}}_\beta m_n^{(0)}(\alpha|\beta)$$

Examples

- The simplest example is the 3 point case

$$\mathcal{M}_3^{(0)}(1^{aa',bb',cc'}) = (\sigma_{12}\sigma_{23}\sigma_{31})^2 \frac{f_{abc}f_{a'b'c'}}{(\sigma_{12}\sigma_{23}\sigma_{31})^2} = f_{abc}f_{a'b'c'}$$

  It actually gives the correct answer.

- The 4 point case is a little complex. Solving the scattering equations with $\sigma_1 = 0, \sigma_2 = 1, \sigma_3 = \infty$ gives $\sigma_4 = -s_{23}/s_{12}$. Define $s_{12} = s$, $s_{23} = t$, $s_{13} = u$, the color factors are

$$\mathbf{c_s} = \sum_b f_{a_1a_2b}f_{ba_3a_4}, \mathbf{c_t} = \sum_b f_{a_1a_4b}f_{ba_3a_2}, \mathbf{c_u} = \sum_b f_{a_1a_3b}f_{ba_2a_4}$$

  Denoting the ordering (1324) as P and comupuling $\det{}'\Phi = \frac{s^2}{t}/(\sigma_{12}\sigma_{23}^2\sigma_{31}\sigma_{34}\sigma_{42})$, one gets

$$
\begin{aligned}
\mathcal{M}_4^{(0)} &= \mathbf{c_s}\tilde{\mathbf{c_s}} m_4^{(0)}(I;I) + \mathbf{c_s}\tilde{\mathbf{c_u}} m_4^{(0)}(I;P) + \mathbf{c_u}\tilde{\mathbf{c_s}} m_4^{(0)}(P;I) + \mathbf{c_u}\tilde{\mathbf{c_u}} m_4^{(0)}(P;P) \\
&= \mathbf{c_s}\tilde{\mathbf{c_s}} \frac{u}{st} + (\mathbf{c_s}\tilde{\mathbf{c_u}} + \mathbf{c_u}\tilde{\mathbf{c_s}}) \frac{1}{t} + \mathbf{c_u}\tilde{\mathbf{c_u}} \frac{s}{ut} \\
&= -\frac{\mathbf{c_s}\tilde{\mathbf{c_s}}}{s} - \frac{\mathbf{c_t}\tilde{\mathbf{c_t}}}{t} - \frac{\mathbf{c_u}\tilde{\mathbf{c_u}}}{u}
\end{aligned}
$$

as expected for a color-dressed cubic theory.

- For the five point, I just give the results of some double partial amplitudes. Denoting the orderings as $I = P_0$, $(13245) = P_1$, $(12435) = P_2$, $(14325) = P_3$, $(13425) = P_4$, $(14235) = P_5$

$$
m_5^{(0)}(I|I) = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}} + \frac{1}{s_{45}s_{12}} + \frac{1}{s_{51}s_{23}},
$$

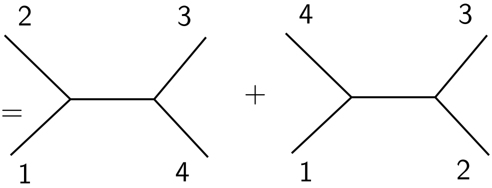
$$
m_5^{(0)}(I|P_1) = -\frac{1}{s_{23}}\left(\frac{1}{s_{45}} + \frac{1}{s_{12}}\right), \quad m_5^{(0)}(I|P_2) = -\frac{1}{s_{34}}\left(\frac{1}{s_{51}} + \frac{1}{s_{12}}\right).
$$

$$m_5^{(0)}(I|P_3) = -\frac{1}{s_{51}}\left(\frac{1}{s_{23}} + \frac{1}{s_{34}}\right), \quad m_5^{(0)}(I|P_4) = -\frac{1}{s_{34}s_{51}},$$
$$m_5^{(0)}(I|P_5) = 0$$

From there examples, it is easy to see that when both permutations in $m_n^{(0)}(\alpha|\beta)$ are the same, then the answer is a sum over all color-orded trivalent diagrams; When the two permutations are different, it gives a subset of terms of $m_n^{(0)}(\alpha|\alpha)$.

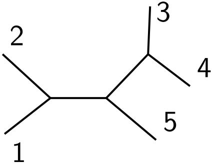
More explicitly,

$m_4^{(0)}(I, I) =$

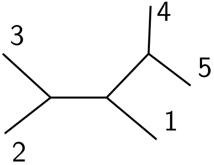


Similarly,

$m_5(I, I) =$



$+$



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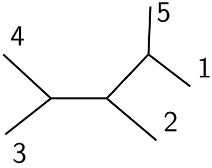

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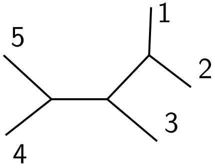
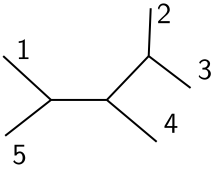


$+$



Trivalent graph expansion

Proposition

The function $m_n^{(0)}(\alpha|\beta)$ computes the sum of all trivalent scalar diagrams which can be regarded as both α color-ordered and β color-orded.

Trivalent graph expansion

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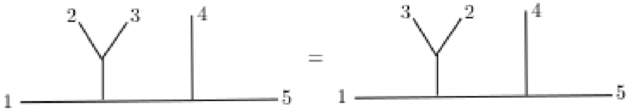
The function $m_n^{(0)}(\alpha|\beta)$ computes the sum of all trivalent scalar diagrams which can be regarded as both α color-ordered and β color-orded.

More explicitly,

$$\boxed{m_n^{(0)}(\alpha|\beta) = (-1)^{n-3+n_{\text{flip}}(\alpha|\beta)} \sum_{g\in\mathcal{T}(\alpha)\cap\mathcal{T}(\beta)} \prod_{e\in E(g)} \frac{1}{s_e}}$$

where the flip$(\alpha|\beta)$ is defined below, $\mathcal{T}(\alpha)$ and $\mathcal{T}(\beta)$ refer to the set of color-ordered diagrams in α and β ordering respectively. To make this expression more clear, see the following diagram

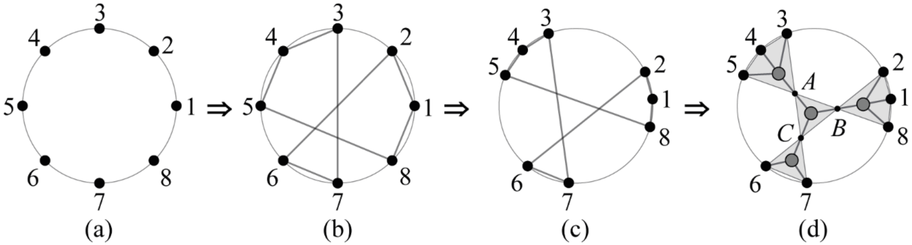


$= \frac{1}{s_{23}} \frac{1}{s_{45}}$

We take $m_8^{(0)}(I;18543762)$ as an example to explain how to compute it in an systematic way

- First step, draw a disk with n nodes sitting on the boundary in the ordering α, then link the n nodes with a loop of line segments according to the ording β. The line segments from β split the disk into some polygons, like the graph (b). We need to move the external points of every polygon to make them have no common edges, like graph (c).
- Second step, put a point in every polygon, named equivalant vertex, and connect this point to all external points in corresponding area. Lines that connect equivalent vertices in two regions with common vertices are called equivalent propagators. The resulting graph is an equivalent Feynman diagram, as shown in Figure (d).

- Third step, we can read off the corresponding amplitudes from the equivalent Feynman diagram.

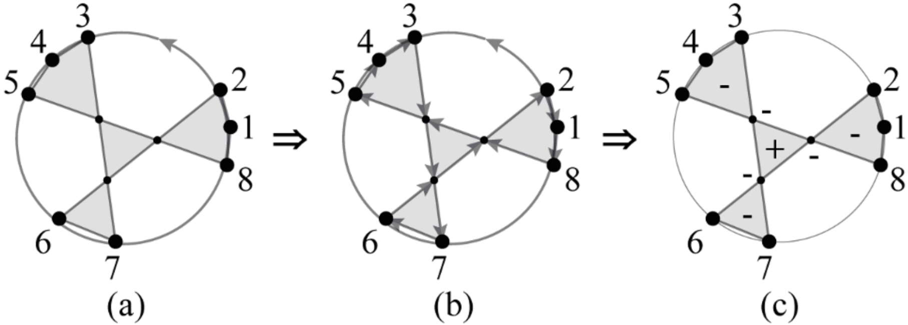


In this example, we can obtain

$$m_8^{(0)}(I|54376218) = (-1)^? \left(\frac{1}{s_{21}} + \frac{1}{s_{18}}\right)\left(\frac{1}{s_{34}} + \frac{1}{s_{45}}\right)\frac{1}{s_{345}s_{812}s_{67}}$$

As for the indefinite sign, there is also a procedure to determine it.

- First step, determine the orientation of the disk by ordering α, and define the loop segments by ordering β, which also determine the orientation of every polygon.
- Second step, (1) each polygon with odd number vertices contributes a plus sign if the orientation is the same as disk, and a minus sign oppositely; (2) each polygon with even number vertices contribute a minus sign; (3) each intersection point contributes a minus sign.



Relation to KLT matrix

It can be shown that the scalar double partial amplitudes are the same as the inverse of KLT matrix.

$$
\begin{aligned}
&(S^{-1}_{\mathrm{KLT}})^{\alpha}_{\beta} = (m_{\mathrm{scalar}})^{\alpha}_{\beta} \\
&\equiv m^{(0)}(1, \alpha(2), \ldots, \alpha(n-2), n-1, n | 1, \beta(2), \ldots, \beta(n-2), n, n-1)
\end{aligned}
$$

The inverse of KLT matrix have been also discussed in other paper, in which it was related to field theory limit if string disk integrals, so it would be interesting to explore the connection further.

Color-Kinematics Duality again

At the begining, I mentioned that sclar-, gluon- and graviton- amplitudes can be related by simple transformations ($C \to E$ or $\tilde{C} \to \tilde{E}$ or both). More explicitly,

$$\mathcal{M}_n^{(0)} = \sum_{I=1}^{(n-3)!} \frac{C(\sigma^{(I)})\tilde{C}(\sigma^{(I)})}{\det{}'\Phi(\sigma^{(I)})}, \quad \mathcal{M}_n^{(1)} = \sum_{I=1}^{(n-3)!} \frac{C(\sigma^{(I)})\tilde{E}(\sigma^{(I)})}{\det{}'\Phi(\sigma^{(I)})},$$

$$\mathcal{M}_n^{(2)} = \sum_{I=1}^{(n-3)!} \frac{E(\sigma^{(I)})\tilde{E}(\sigma^{(I)})}{\det{}'\Phi(\sigma^{(I)})}.$$

If we expand the color factor like

$$C = \sum_{\gamma \in S_{n-2}} \frac{\mathbf{c}_{1\gamma(2)\cdots\gamma(n-1)n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n}\sigma_{n,1}},$$

It hints the exsitence of similar form for E. More explicitly, there must functions, denoted as $\boldsymbol{n}$, which depends only on kinematic data $\{\epsilon_a^\mu, k_a^\mu\}$

$$E = \mathrm{Pf}'\Psi(\epsilon, k, \sigma) = \sum_{\gamma \in S_{n-2}} \frac{\mathbf{n}_{1\gamma(2)\cdots\gamma(n-1)n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n}\sigma_{n,1}}.$$

Now we can unify $\mathbf{c}$ and $\mathbf{n}$ as e in all three theories, the full amplitude can be written in a unified form

$$\mathcal{M}_n^{(s)} = \sum_{\alpha,\beta \in S_{n-2}} e_\alpha e_\beta m^{(0)}(\alpha|\beta)$$

and the factor satisfies the "BCJ" relation

$$e_{g_t} = \pm(e_{g_s} - e_{g_u})$$

If we contentate on pure Yang-Mills theory, the realtion is just the one we list in the first second section.