Note for Scattering Amplitude Computation

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1 3-point building block

We have known that the on-shell 3-point amplitudes can be completely determined by the little group scaling, according to the following formulas

$$A_3^{h_1h_2h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} \qquad h_1+h_2+h_3<0$$
$$A_3^{h_1h_2h_3} = c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \qquad h_1+h_2+h_3>0$$

Because of the specialty of this kind of 2 site gauge theory, there are no direct interaction between gauge boson and scalar, so there are only two kinds of 3-point amplitudes.

- 2 scalar 1 gauge boson

$$A[1,2,3^+] = \frac{[23][31]}{[12]}, \qquad A[1,2,3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

- 3 gauge boson

$$A[3^+,4^+,5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^-,4^-,5^+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

If there are no exceptions, 1 and 2 always represent the antiscalar and scalar respectively, other number represent the gauge boson.

2 Gauge boson sector

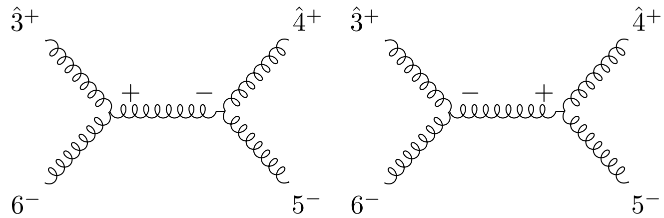
In this section, we will show how to build the gauge boson scattering. Because there's no direct interaction between gauge boson 1 and gauge boson 2, so we only need to compute one of them. And although we have already known the formulas for MHV color-ordered amplitudes for gluon scattering – Parke-Talyor Formula

$$A[\cdots,i^-,\cdots,j^-,\cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}$$

also for the anti-MHV amplitudes

$$A[\cdots,i^+,\cdots,j^+,\cdots] = \frac{[ij]^4}{[12][34]\cdots[n-1,n][n1]}$$

Here, we will give an concrete example to show how to use BCFW to compute the 4-point amplitudes $A[3^+,4^+,5^-,6^-]$



4pt gluon

We choose the $[3,4\rangle$ shift, and it has been proved that $[+,+\rangle$ shift is valid

$$|\hat{3}] = |3] - z|4], \qquad |\hat{4}\rangle = |4\rangle + z|3\rangle$$
$$|\hat{3}\rangle = |3\rangle, \qquad |\hat{4}] = |4].$$

The first diagram can be evaluated

$$A_1 = \frac{[\hat{3}\hat{I}]^3}{[\hat{I}6][6\hat{3}]} \times \frac{1}{s_{36}} \times \frac{\langle 5\hat{I}\rangle^3}{\langle \hat{I}\hat{4}\rangle\langle \hat{4}5\rangle}$$

The point here is that

$$\text{Pole position}: \quad \hat{P}_{34}^2 = 0 = \langle 36\rangle[6\hat{3}] \quad \Rightarrow \quad [6\hat{3}] = 0,$$

for the similar reason, we can obtain $[\hat{I}6] = [6\hat{3}] = 0$, so we conclude that the left part

$$\frac{[\hat{3}\hat{I}]^3}{[\hat{I}6][6\hat{3}]} = 0$$

so the first channel is vanishing.

The second diagram can be similarly evaluated

$$\begin{aligned} A_2 &= \frac{\langle \hat{I}6\rangle^3}{\langle 6\hat{3}\rangle\langle \hat{3}\hat{I}\rangle} \times \frac{1}{s_{36}} \times \frac{[\hat{I}\hat{4}]^3}{[\hat{4}5][5\hat{I}]} \\ &= \frac{[34]^3}{[34][45][56][61]} \end{aligned}$$

From this, we can conclude that the color-ordered amplitude equals to

$$A[3^+,4^+,5^-,6^-] = \frac{[34]^3}{[34][45][56][63]}.$$

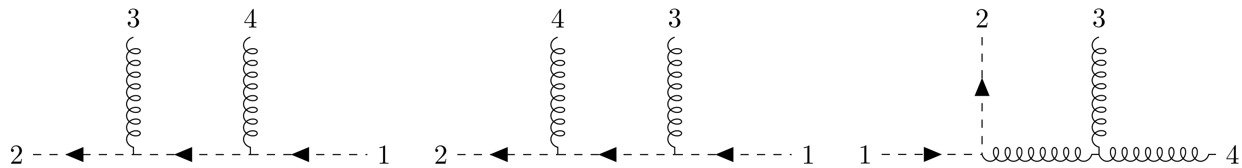
and it can also be expressed by angle brackets interchangeably

$$A[3^+,4^+,5^-,6^-] = \frac{\langle 56\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 63\rangle}$$

Then, 5-point , 6-point can be recursively computed by using BCFW recursion relation.

3 SQCD like sector

First we explain the color decomposition in this sector by 4 point amplitude $\Phi^\dagger V_1 V_1 \Phi$



s channel u channel t channel

The color factor can be written respectively as following

$$r_s = \text{Tr}[\Phi_2^\dagger T^{a_3} T^{a_4} \Phi_1],\ r_u = \text{Tr}[\Phi_2^\dagger T^{a_4} T^{a_3} \Phi_1],\ r_t = \text{Tr}[\Phi_2^\dagger [T^{a_3}, T^{a_4}] \Phi_1]$$

We can easily obtain a similar Jacobbi relation

$$r_t = r_s - r_u$$

Then we can accomplish the color decomposition and define the corressponding color-ordered amplitudes.

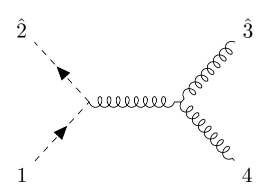
For example, in the 4pt. case, the full amplitude can be decomposed to the following form

$$\begin{aligned} \mathcal{A}_4(\Phi^\dagger V_1 V_1 \Phi) &= A_s r_s + A_u r_u + A_t r_t \\ &= A_s r_s + A_u r_u + A_t (r_s - r_u) \\ &= (A_s + A_t) r_s + (A_u - A_t) r_u \end{aligned}$$

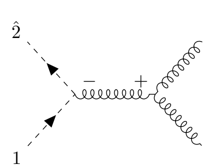
The two subamplitudes can be defined as color-ordered amplitude with order [1,2,3,4] and [1,2,4,3] respectively. Of course, for the type $\Phi^\dagger (nV_1) \Phi$ and $\Phi (nV_2) \Phi^\dagger$, we can do the same thing to define the color-ordered amplitudes.

We start from the 4-point color-ordered $A[1,2,3^+,4^-]$ again

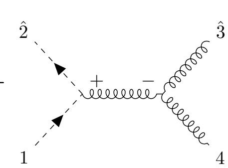
$$A[1,2,3^+,4^-] = \sum_h$$



$$=$$



$$+$$



4 4-point case

For the four point case $\mathcal{A}(V_2\Phi^\dagger V_1\Phi)$, we can construct the color-ordered amplitude from the residue. First, we consider the $(+,-)$ helicity configuration. There are two feynman diagrams contributing to the color-ordered amplitude.

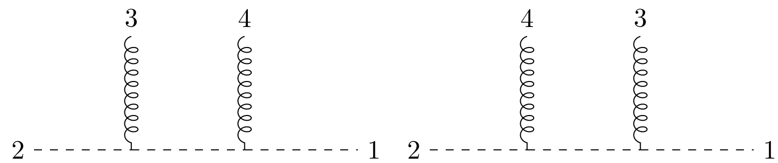


Figure 1: 4pt.

For the first diagram, the residue equals to

$$\mathcal{R}es|_{s_{12}=0} = \frac{[3I][23]}{[I2]} \times \frac{\langle I4\rangle\langle 41\rangle}{\langle 1I\rangle} = \frac{\langle 24\rangle[31]\langle 41\rangle[23]}{[42]\langle 24\rangle}$$

Similarly, the second one is

$$\mathcal{R}es|_{s_{13}=0} = \frac{\langle 4I\rangle\langle 24\rangle}{\langle I2\rangle} \times \frac{[31][I3]}{[1I]} = \frac{\langle 24\rangle[31]\langle 41\rangle[23]}{\langle 32\rangle[23]}$$

Then we can conclude that the four-point color-ordered amplitude $A[1,2,3^+,4^-]$ equals to

$$A[1,2,3^+,4^-] = \frac{\langle 24\rangle[31]\langle 41\rangle[23]}{\langle 32\rangle[23][42]\langle 24\rangle} = \frac{\langle 24\rangle\langle 14\rangle}{\langle 13\rangle\langle 23\rangle}$$

$\star Bonus$

It is still necessary to prove the color-ordered amplitude $A[1,2,3^+,4^+]$ equals to 0. Here we can use the color ordered Feynman rules to show the result.

$$A[1,2,3^+,4^+] \propto \frac{(\epsilon_3\cdot p_2)(\epsilon_4\cdot p_1)}{s_{23}} + \frac{(\epsilon_4\cdot p_2)(\epsilon_3\cdot p_1)}{s_{24}}$$

Here we can utilize the spinor-helicity variable to express polarization vector

$$\epsilon_2^{+\mu} = \frac{\langle r_1|\gamma^\mu|3]}{\sqrt{2}\langle r_1 3\rangle}, \qquad \epsilon_4^{+\mu} = \frac{\langle r_2|\gamma^\mu|4]}{\sqrt{2}\langle r_2 4\rangle}$$

here r_1 and r_2 represent the reference spinor.

We can freely choose $r_1 = r_2 = 1$ or 2, then $\langle r_1 2\rangle$,$\langle r_2 1\rangle$,$\langle r_1 1\rangle$,$\langle r_2 2\rangle$, two of them equal to 0, so we can conclude that

$$A[1,2,3^+,4^+] = 0$$

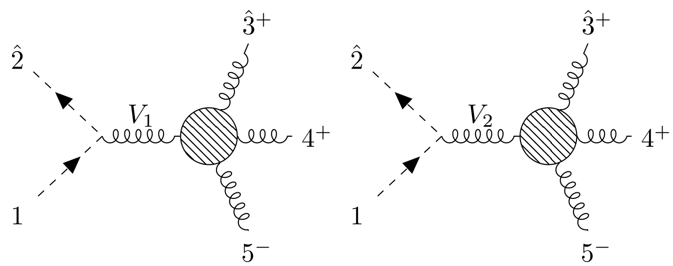


Figure 3: 5pt. 2

5 5-point case

For the 5-point case, we can utilize the BCFW recursion relation which can help us generate higher point amplitude from lower point on-shell subamplitudes. Here, we always consider the MHV (Maximal helicity violation) amplitude.

If there is no special case, we always choose the following BCFW shift

$$|\hat{2}] = |2] - z|3], \qquad |\hat{3}\rangle = |3\rangle + z|3\rangle$$
$$|\hat{2}\rangle = |2\rangle, \qquad |\hat{3}] = |3]$$

where 2 always refers to antiscalar and 3 refers to gauge boson with + helicity.

Let us begin with the simplest case $A[1, 2, 3_1^+, 4_1^+, 5_2^-]$, where the subscript represent which gauge group the particle belongs to. Because of the property of this kind of gauge theory, the color structure is invariant under the OPP (Order Preserving Permutation), in this case, for example,

$$(3_1^+, 4_1^+, 5_2^-) \qquad (3_1^+, 5_2^-, 4_1^+) \qquad (5_2^-, 3_1^+, 4_1^+)$$

give us the same color factor. So in the process of BCFW recursion, these three order offer the same amplitude. We can draw all diagrams contributing to the BCFW process, the first two are following

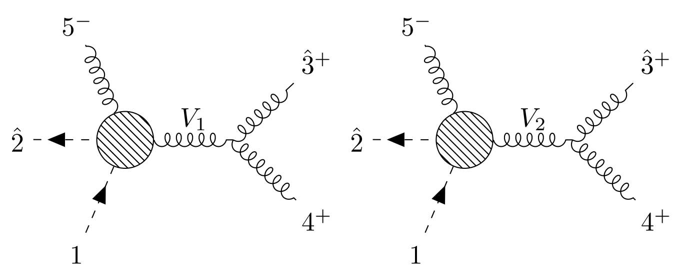


Figure 2: 5pt. 1

It is obvious that the second diagram in Figure 2 equals to 0, because there are no interaction between the two gauge bosons.

Similarly, another two diagrams equal to 0 for the same reason

The last diagram still gives 0 contribution because it includes a subamplitude $A[1, \hat{I}, \hat{3}^+, 4^+] = 0$.

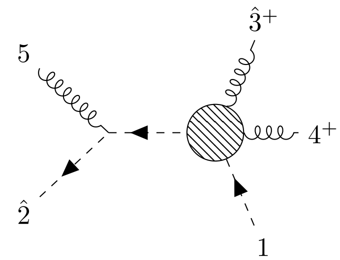


Figure 4: 5pt. 3

Above all. only the first diagram in Figure 1 has non-vanishing contributions, so the full color ordered amplitude equals to

$$
\begin{aligned}
A[1,2,3_1^+,4_1^+,5_2^-] &= A[1,2,\hat{I}^+,5^-] \times \frac{1}{s_{34}} \times A[\hat{3}^+,4^+,\hat{I}^-] \\
&= \frac{\langle 15\rangle\langle 25\rangle}{\langle 1\hat{I}\rangle\langle 2\hat{I}\rangle} \times \frac{1}{s_{34}} \times \frac{[\hat{3}4]^3}{[4\hat{I}][\hat{I}\hat{3}]} \\
&= \frac{\langle 15\rangle\langle 25\rangle \cancel{[34]^3}}{\langle 14\rangle\langle 23\rangle\langle 43\rangle \cancel{[43][43][34]}} \\
&= \frac{\langle 15\rangle\langle 25\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\
&= \frac{(-1)\langle 2{\color{green}5}\rangle^2\langle 1{\color{green}5}\rangle^2}{{\color{blue}\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\color{red}\langle 25\rangle\langle 51\rangle}}
\end{aligned}
$$

where we use the fact $|\hat{3}\rangle = |\hat{3}\rangle$ and following identities

$$
\langle 1\hat{I}\rangle[\hat{I}3] = \langle 14\rangle[43], \quad \langle 2\hat{I}\rangle[\hat{I}3] = \langle 24\rangle[43]
$$

and also

$$
\begin{aligned}
\frac{[\hat{I}3]}{[4\hat{I}]} &= -\frac{[3\hat{I}]\langle\hat{I}2\rangle}{[4\hat{I}]\langle\hat{I}2\rangle} = -\frac{[34]\langle 42\rangle}{[43]\langle\hat{3}2\rangle}, \quad (\langle\hat{3}2\rangle = \langle 32\rangle + z\langle 22\rangle = \langle 32\rangle) \\
&= \frac{\langle 42\rangle}{\langle 32\rangle}
\end{aligned}
$$

here green refers to the particle with (-) helicity, red refers to particles belong to gauge group 1, red refers to particles belong to gauge group 2.

Similarly, it is very easy to obtain another color-ordered amplitude $A[1,2,3_1^+,4_1^-,5_2^+]$

$$
A[1,2,3_1^+,4_1^-,5_2^+] = \frac{(-1)\langle 2{\color{green}4}\rangle^2\langle 1{\color{green}4}\rangle^2}{{\color{blue}\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\color{red}\langle 25\rangle\langle 51\rangle}}
$$

and also $A[1,2,3_1^-,4_1^+,5_2^+]$ equals to

$$
A[1,2,3_1^-,4_1^+,5_2^+] = \frac{(-1)\langle 2{\color{green}3}\rangle^2\langle 1{\color{green}3}\rangle^2}{{\color{blue}\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\color{red}\langle 25\rangle\langle 51\rangle}}
$$

But here we need to emphasize that it is necessary to choose another BCFW shift, like $[1,5^+\rangle$, as $[2,3^-\rangle$ is not a valid shift.

6 6-point case

Here we consider $(V_2 V_2 \Phi^\dagger V_1 V_1 \Phi)$ case, the corresponding color-ordered amplitude is $A[1, 2, 3_1^+, 4_1^+, 5_2^+, 6_2^-]$. Similarly, the fowwling orders all give us the same color factor

$$
\begin{array}{ccc}
(3_1^+, 4_1^+, 5_2^+, 6_2^-) & (3_1^+, 5_2^+, 4_1^+, 6_2^-) & (3_1^+, 5_2^+, 6_2^-, 4_1^+) \\
(5_2^+, 3_1^+, 4_1^+, 6_2^-) & (5_2^+, 3_1^+, 6_2^-, 4_1^+) & (5_2^+, 6_2^-, 3_1^+, 4_1^+)
\end{array}
$$

Only two diagrams have semmingly non-zero contributaion,

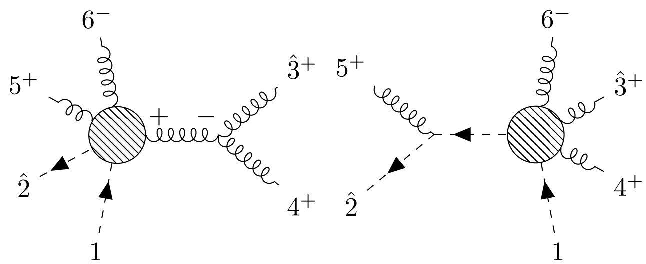


Figure 5: 6pt.

so the full color ordered amplitude equals to

$$
\begin{aligned}
A_1 &= \frac{(-1)\langle \hat{2}6\rangle^2 \langle 16\rangle^2}{\langle 25\rangle\langle 56\rangle\langle 61\rangle\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[\hat{3}4]^3}{[4\hat{I}][\hat{I}\hat{3}]} \\
&= \frac{\langle 26\rangle^2 \langle 16\rangle}{\langle 25\rangle\langle 56\rangle\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[34]^3}{[4\hat{I}][\hat{I}3]} \\
&= \frac{\langle 26\rangle^2 \langle 16\rangle \cancel{[34]^3\langle 42\rangle}}{\langle 25\rangle\langle 56\rangle\langle 41\rangle\langle 32\rangle\langle 43\rangle \cancel{[43][43][34]\langle 24\rangle}} \\
&= \frac{\langle 26\rangle^2 \langle 16\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 56\rangle\langle 51\rangle}
\end{aligned}
$$

where we have used the fact $|\hat{2}\rangle = |2\rangle$, $|\hat{3}] = |3]$, and the following identities

$$
\langle 2\hat{I}\rangle[\hat{I}3] = \langle 24\rangle[43], \qquad [4\hat{I}]\langle \hat{I}1\rangle = [43]\langle \hat{3}1\rangle
$$

The point here is that we first $\langle \hat{3}1\rangle$ which does not appear in 5-point case, so we nned to compute it carefully

$$
\begin{aligned}
\text{pole position :} \quad & \hat{P_{34}}^2 = 0 = 2P_3 \cdot P_4 = \langle 4\hat{3}\rangle[34] \qquad \Rightarrow \qquad \langle 4\hat{3}\rangle = 0 \\
& \langle 43\rangle + z\langle 42\rangle = 0 \qquad \Rightarrow \qquad z = -\frac{43}{42}
\end{aligned}
$$

then

$$\langle\hat{3}1\rangle = \langle 31\rangle + z\langle 21\rangle = \langle 31\rangle - \frac{\langle 43\rangle}{\langle 42\rangle}\langle 21\rangle$$
$$= \frac{\langle 42\rangle\langle 31\rangle - \langle 43\rangle\langle 21\rangle}{42}$$
$$= \frac{\langle 41\rangle\langle 32\rangle}{\langle 42\rangle}$$

where we have used the Fierz identity

$$\langle 42\rangle\langle 31\rangle + \langle 41\rangle\langle 23\rangle + \langle 43\rangle\langle 12\rangle = 0.$$

Simiraly, we can compute the second diagram

$$A_2 = \frac{[\hat{2}5][5\hat{I}]}{[\hat{I}\hat{2}]} \times \frac{1}{s_{25}} \times \frac{(-1)\langle 16\rangle^2\langle\hat{I}6\rangle^2}{\langle\hat{I}\hat{3}\rangle\langle\hat{3}4\rangle\langle 41\rangle\langle\hat{I}6\rangle\langle 61\rangle}$$

but from the pole position

$$\hat{P_{25}}^2 = 0 = 2P_2 \cdot P_5 = \langle 52\rangle[\hat{2}5] \qquad \Rightarrow \qquad [\hat{2}5] = 0,$$

and simiraly

$$[\hat{2}\hat{I}] = [5\hat{I}] = 0.$$

Then we can conclude that the left part of the amplitude equals to 0 so $A_2 = 0$. Finally, we obtain the color-ordered amplitude

$$A[1, 2, 3_1^+, 4_1^+, 5_2^+, 6_2^-] = A_1 + A_2 = \frac{\langle 26\rangle^2\langle 16\rangle^2}{\color{blue}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\color{red}{\langle 25\rangle\langle 56\rangle\langle 51\rangle}}.$$

7 n-point case

Here, we first propose a compact formula for the color-ordered amplitude

$$A = \frac{\langle 2a\rangle^2\langle 1a\rangle^2}{\underbrace{\langle 2\star\rangle\cdots\langle\star 1\rangle}_{SU(N_1)}\underbrace{\langle 2*\rangle\cdots\langle * 1\rangle}_{SU(N_2)}}$$

where a refer to the particle with - helicity, whichever gauge group it belongs to. And, '$\star$' refers to the ordering for the first gauge group, '$*$' refers to the ordering for the second gauge group. We suppose there are n_1 gauge boson 1, n_2 gauge boson 2, so the n-point means that $n = n_1 + n_2 + 2$.

The usual way to prove this kind of compact formula is deduction. First we suppose that all of the amplitudes with external point lower than n satisfy the compact formula. And although there are $\frac{(n_1+n_2)!}{n_1!n_2!}$ OPP, we just need to consider some of them.