A complete solution for scattering in a kind of quiver gauge theory

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Contents

A brief introduction to BCFW

BCFW recursion relation is a method to compute scattering amplitude, especially in Yang-Mills theory and gravity.

- Ruth Britto
- Freddy Cachazo
- Bo Feng
- Edward Witten

From real to complex – Analytic Continuation

Why is analytic continuation valid?

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as $p_{i\mu}p_j^{\mu}$, $p_{i\mu}\epsilon_j^{\mu}$.
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.

$$\Downarrow$$

Ampltudes can be shifted to complex plane

What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \ldots, n$. Then introduce n complex-valued vectors r_i^μ.

(i) $\sum_{i=1}^n r_i^\mu = 0$,

(ii) $r_i \cdot r_j = 0$ for all $i, j = 1, 2, \ldots, n$. In particular $r_i^2 = 0$,

(iii) $p_i \cdot r_i = 0$ for each i (no sum).

These vectors r_i are used to define n shifted momenta

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \qquad \text{with} z \in \mathcal{C}$$

Note that,

(A) By property (i), momentum conservation holds for the shifted momenta: $\sum_{i=1}^{n} \hat{p}_i^\mu = 0$,

(B) By (ii) and (iii), we have $\hat{p}_i^2 = 0$, so each shifted momentum is on-shell,

(C) For a non-trival subset of generic momenta $\{p_i\}_{i \in I}$, define $P_I^\mu = \sum_{i \in I} p_i^\mu$.

Then, $\hat{P}_I^2$ is linear in z:

$$\hat{P}_I^2 = \left(\sum_{i \in I} \hat{p}_i \right)^2 = P_I^2 + 2z P_I \cdot R_I \quad \text{with} \quad R_I = \sum_{i \in I} r_i,$$

because the z^2 term vanishes by property (ii). We can write

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{with} \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I}$$

Fantasitic result from Cauchy Theorem

As a result of (A) and (B) (momentum conservation and on-shell), we can consider amplitude A_n in terms of shifted momentum $\hat{p}_i^{\mu}$ instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

and we have known the possible positions of single poles, z_I, different propagators give us different single poles in the z-plane.

If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\boxed{Res|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

It means that the orignal ampltude equals to the residue at orign.

From Cauchy Theorem, we can ontain

$$A_n = -\sum_{z_I} Res|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.

Then, at a z_I pole, the propagator $\hat{P}_I^2$ goes to on-shell. In that limit, the shifted amplitude factorizes into to on-shell parts (Unitarity)

$$\hat{A}_n(z) \quad \xrightarrow{z \text{ near } z_I} \quad \hat{A}_L(z_I)\frac{1}{\hat{P}_I^2}\hat{A}_R(z_I) = -\frac{z_I}{z - z_I}\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$-Res|_{z=z_I}\frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) =$$