

# Tree level scattering amplitude in (De)constructed gauge theory

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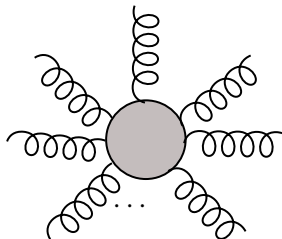
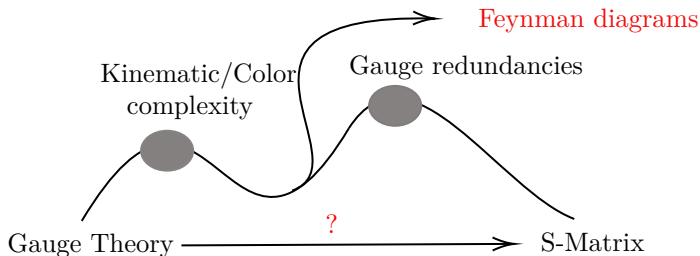
April 21st 2025

# Contents

- 1 Why we need new method?
- 2 Preliminary
- 3 Introduction of quiver gauge theory
- 4 Scattering amplitudes from BCFW
- 5 Summary and Outlook

# Why we need new method?

Feynman diagram is a brilliant method without doubt, helping us compute the scattering process perturbatively.



Can we really compute this by hand?

# From Frynman diagram to On-shell method

The answer is On-shell method.



On-shell here means that all quantities we use are gauge invariant and satisfy the on-shell condition. Specifically, there are many ingredients under this frame

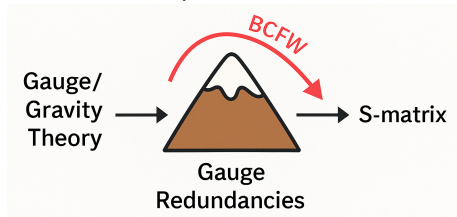
- The analytic continuation for S-matrix.
- The color-ordered amplitudes.
- The BCFW recursion relation.
- The spinor helicity discription for amplitudes.

# Contents

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- 2 Preliminary
- 3 Introduction of quiver gauge theory
- 4 Scattering amplitudes from BCFW
- 5 Summary and Outlook

# A brief introduction to BCFW

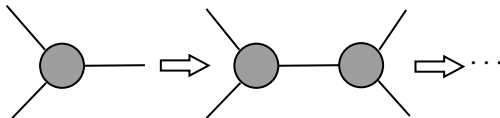
BCFW helps us solve one of the problems



with the cost of introducing **complexed momentum**.

BCFW is a method to compute amplitudes recursively, proposed by

- Britto, Cachazo, Feng, arXiv: hep-th/0412308
- Britto, Cachazo, Feng, Witten, arXiv: hep-th/0501052



# From real to complex – Analytic Continuation

## Why can we conduct analytic continuation?

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as  $p_{i\mu}p_j^\mu$ ,  $p_{i\mu}\epsilon_j^\mu$ .
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.



Amplitudes can be shifted to complex plane

# Momentum Shift in BCFW

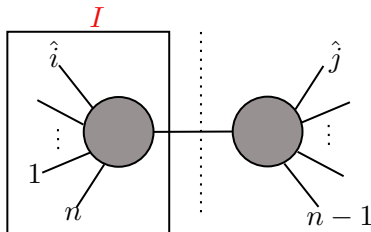
## What did BCFW do to make the shift?

Here we consider the case in which all particles are massless,  $p_i^2 = 0$  for all  $i = 1, 2, \dots, n$ . We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$



For a non-trivial subset of generic momenta  $\{p_i\}_{i \in I}$

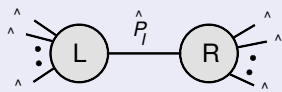
$$\hat{P}_I^2 = P_I^2 - 2z P_I \cdot k = -\frac{P_I^2}{z_I} (z - z_I)$$

with  $z_I = \frac{P_I^2}{2P_I \cdot k}$ .



# Fantastic result from Cauchy Theorem

## BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \hat{P}_I^{\mu} \hat{A}_R(z_I)$$


Brief proof:

We consider amplitude  $A_n$  in terms of shifted momentum  $\hat{p}_i^\mu$  instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

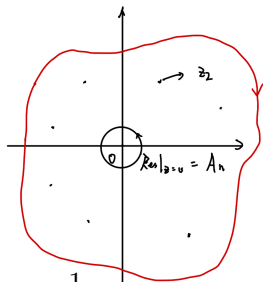
If we consider the meromorphic function  $\frac{\hat{A}_n(z)}{z}$  in the complex plane, pick a contour that surrounds the single pole at the origin. ★ The most important point here is that

$$\text{Res}|_{z=0} \frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n$$

From Cauchy Theorem, we can obtain

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where  $B_n$  is the residue of the pole at  $z = \infty$ , called boundary term.



$$\hat{A}_n(z) \xrightarrow{z \text{ near } z_I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) = - \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at  $z = z_I$

$$-\text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

# Large $z$ behavior

In the BCFW formula, the boundary term  $B_n$  affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves  $B_n = 0$ . This is often justified by declaring a stronger condition

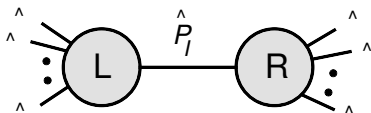
$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large  $z$  behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	$z^3$
-	$1/z$	$1/z$

# Spinor-Helicity formalism

In the part of introduction to BCFW



→ No helicity appears here

but the S-matrix is a function of momentum  $p_i$  and helicity  $h_i$

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

The answer is **Spinor-Helicity formalism** → Catch  $p_i$  and  $h_i$  at the same time.

# Spinor-helicity formalism

## ■ Massless Case

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \rightarrow t^{-1}\lambda, \quad \tilde{\lambda} \rightarrow t\tilde{\lambda}, \quad t \in \mathbb{C}$$

same for

$$|\lambda\rangle \rightarrow t^{-1}|\lambda\rangle, \quad |\lambda] \rightarrow t|\lambda]$$

The scattering amplitudes should transform covariantly under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \dots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \dots) = t_i^{2h_i} \mathcal{A}_n$$

## ■ Massive Case

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

# On-shell 3-point can be completely determined

## On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \quad p_2 = (1 - \kappa)p_3 \quad (\text{Collinear})$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

**We can not obtain any thing nontrivial from 3-point!**

Of course, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\square \Psi)$$

but it still equals to 0 under the on-shell condition.

## Another necessity to introduce complex momentum

If the momentum is complexed, we have

$$\langle 12 \rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or square brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}} \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \quad 2h_3 = -x_{13} - x_{23}.$$

Then all index can be solved from this system of equations, so that

$$\begin{aligned} A_3^{h_1 h_2 h_3} &= c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3} & h_1 + h_2 + h_3 < 0 \\ A_3^{h_1 h_2 h_3} &= c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} & h_1 + h_2 + h_3 > 0 \end{aligned}$$

★ **All massless on-shell 3-point amplitudes are completely determined by little group scaling!**

**Example:** 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g' \frac{[13][23]}{[12]^3}$$

but actually it comes from the **non-local** interaction  $g' A A \frac{\partial}{\square} A$ , so we discard it.



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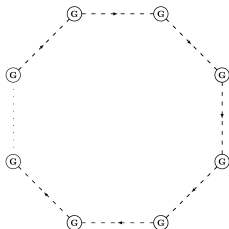
# Introduction of Quiver or Moose gauge theory

Quiver: A container for carrying arrows

Moose: A kind of deer with large horns



In the language of field theories, quiver gauge theories contain gauge fields and bi-fundamental scalars, summarized in a pictorial representation.



Moose diagram

$N$ -sided polygon

$G$  : gauge group  $SU(m)$

$\rightarrow$ : Unitary scalar fields  $\Phi_{ij}$

# Why we focus on quiver gauge theory?

The lagrangian can be written like

$$\mathcal{L} = - \sum_{i=1}^N \frac{1}{2} \text{Tr}(F_i)^2 + \sum_{i=1}^N \text{Tr}[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)],$$

here  $F_i$  refers to the  $i$ th gauge field strength, scalar field  $\Phi_i$  transformed under the **bi-fundamental** representation and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - ig_i A_{i\mu} \Phi_i + ig_{i+1} \Phi_i A_{i+1\mu}.$$

Here, gauge field and scalar field transformed like

$$A_{i\mu} \rightarrow U_i(x) A_{i\mu} U_i^{-1}(x) - \frac{i}{g_i} (\partial_\mu U) U^{-1}, \quad \Phi_i \rightarrow U_i(x) \Phi_i U_{i+1}^{-1}(x)$$

It is easy to confirm that this theory is invariant under  $\prod_1^N SU(m)$  gauge group.

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group  $SU(m)$ , where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.

- If  $SU(m)_1$  and  $SU(m)_N$  are connected  $\longrightarrow S^2$  compactification
- If not connected  $\longrightarrow$  Interval compactification

After higgsing the scalar field, we can obtain a spectrum

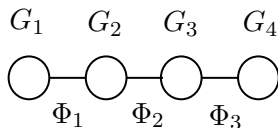
$$M_k^2 = 4g^2 f_s^2 \sin^2 \left( \frac{\pi k}{N} \right)$$

This is precisely the **Kaluza-Klein** spectrum under  $S^2$  compactification.

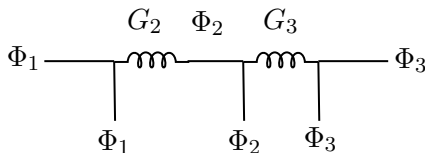
# What is relation to scattering amplitude?

The critical point is **locality**.

- Space-Time Locality  $\longrightarrow$  local field theories
- Theory Space Locality  $\longrightarrow$  Discretized theory space



If we change this to a scattering diagram, and compute the large- $z$  behavior



$$\sim 1/z^{\textcircled{4}}$$

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# Classification of Scattering Amplitudes

For simplicity, we start from the two-site gauge theory with gauge fields  $V_1$ ,  $V_2$  and scalar fields  $\Phi$ ,  $\Phi^\dagger$ . The amplitudes are classified by their multiplicity:

3-point	4-point	5-point	6-point
$V_1 \Phi \Phi^\dagger$	$V_1 V_1 V_1 V_1$	$V_1 V_1 V_1 V_1 V_1$	$V_1 V_1 V_1 V_1 V_1 V_1$
$V_2 \Phi \Phi^\dagger$	$V_2 V_2 V_2 V_2$	$V_2 V_2 V_2 V_2 V_2$	$V_2 V_2 V_2 V_2 V_2 V_2$
$V_1 V_1 V_1$	$\Phi^\dagger V_1 V_1 \Phi$	$\Phi^\dagger V_1 V_1 V_1 \Phi$	$\Phi^\dagger V_1 V_1 V_1 V_1 \Phi$
$V_2 V_2 V_2$	$\Phi V_2 V_2 \Phi^\dagger$	$\Phi V_2 V_2 V_2 \Phi^\dagger$	$\Phi V_2 V_2 V_2 V_2 \Phi^\dagger$
	$\Phi V_2 \Phi^\dagger V_1$	$V_2 \Phi^\dagger V_1 V_1 \Phi$	$V_2 V_2 \Phi^\dagger V_1 V_1 \Phi$
	$\Phi \Phi^\dagger \Phi \Phi^\dagger$	$\Phi V_2 V_2 \Phi^\dagger V_1$	$\Phi V_2 V_2 \Phi^\dagger V_1 V_1$
		$\Phi \Phi^\dagger \Phi \Phi^\dagger V_1$	$\vdots$
		$\Phi \Phi^\dagger \Phi \Phi^\dagger V_2$	$\vdots$

## Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$\begin{aligned} A[1, 2, 3^+] &= \frac{[23][31]}{[12]}, & A[1, 2, 3^-] &= \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle} \\ A[3^+, 4^+, 5^-] &= \frac{[34]^3}{[45][53]}, & A[3^-, 4^-, 5^+] &= \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle} \end{aligned}$$

By using the 3 point building block, we can construct 4 point color-ordered amplitudes from BCFW recursion relation.



- $nV_1$  or  $nV_2$

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\text{Parke - Talyor Formula : } A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

- $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude  $A[1, 2, 3^+, 4^-]$ . We choose  $[2, 3\rangle$  shift

$$\begin{aligned} |\hat{2}\rangle &= |2\rangle - z|3\rangle, & |\hat{2}\rangle &= |2\rangle \\ |\hat{3}\rangle &= |3\rangle, & |\hat{3}\rangle &= |3\rangle + z|2\rangle \end{aligned}$$

The amplitudes can be computed

$$A[1, 2, 3^+, 4^-] = (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1, 2, 3^+, 4^+, 5^-] = \frac{\langle 15 \rangle^2 \langle 25 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1, 2, \dots, (n+2)^-] = (-1)^{n+1} \frac{\langle 1, n+2 \rangle^2 \langle 2, n+2 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n+1, n+2 \rangle \langle n+2, 1 \rangle}$$

★ Bonus relation:

$$A[1, 2, 3^+, 4^+] = 0 \quad \Rightarrow \quad A[1, 2, 3^+, \dots, n^+] = 0$$

For the amplitude  $\Phi(nV_2)\Phi^\dagger$ , we can obtain nearly the same expression.

- $\Phi V_2 \Phi^\dagger V_1$

$$A[1, 2, 3_1^+, 4_2^-] = \frac{\langle 14 \rangle \langle 24 \rangle}{\langle 13 \rangle \langle 23 \rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1, 2, 3_1^+, 4_1^+, 5_2^-] = (-1) \frac{\langle 2\bar{5} \rangle^2 \langle 1\bar{5} \rangle^2}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 25 \rangle \langle 51 \rangle}$$

- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$

$$A[1, 2, 3_1^+, 4_1^+, 5_2^+, 6_2^-] = \frac{\langle 2\bar{6} \rangle^2 \langle 1\bar{6} \rangle^2}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 25 \rangle \langle 56 \rangle \langle 61 \rangle}$$

- Compact formula for general case

$$A = \frac{\langle 2a \rangle^2 \langle 1a \rangle^2}{\underbrace{\langle 2\star \rangle \cdots \langle \star 1 \rangle}_{SU(N_1)} \underbrace{\langle 2* \rangle \cdots \langle *1 \rangle}_{SU(N_2)}}$$

**Green:** Particle with — helicity

**Blue:** Particle belongs to the first gauge group

**Red:** Particle belongs to the second gauge group

$\star$ : Order for gauge group 1

$*$ : Order for gauge group 2

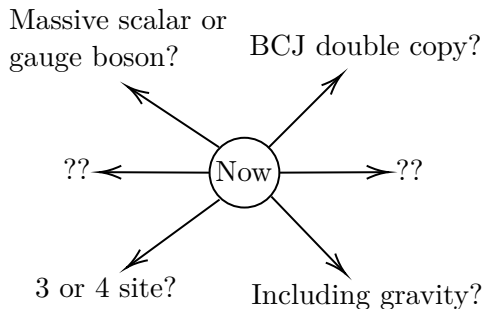
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# Summary

- Introduce the on-shell method, including BCFW recursion relation, color-ordered amplitudes. etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- The locality plays an important role to relate the spacetime locality and field space locality.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

# Possible future work





Thanks for your attention!

## 6 Appendix

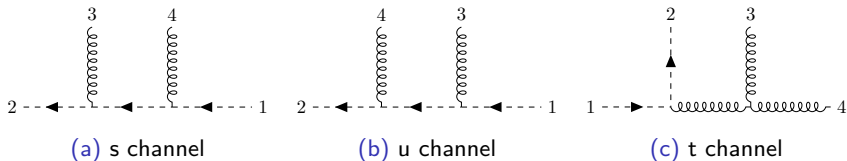
- $nV_1$  or  $nV_2$

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

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Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

- $\Phi^\dagger V_1 V_1 \Phi$



The color factor can be written respectively as following

$$r_s = \text{Tr}[\Phi_2^\dagger T^{a_3} T^{a_4} \Phi_1], \quad r_u = \text{Tr}[\Phi_2^\dagger T^{a_4} T^{a_3} \Phi_1], \quad r_t = \text{Tr}[\Phi_2^\dagger [T^{a_3}, T^{a_4}] \Phi_1]$$

We can easily obtain a similar Jacobbi relation

$$r_t = r_s - r_u$$

Then we can accomplish the color decomposition and define the corresponding color-ordered amplitudes.

For example, in the 4pt. case, the full amplitude can be decomposed to the following form

$$\begin{aligned}
 \mathcal{A}_4(\Phi^\dagger V_1 V_1 \Phi) &= A_s r_s + A_u r_u + A_t r_t \\
 &= A_s r_s + A_u r_u + A_t (r_s - r_u) \\
 &= (A_s + A_t) r_s + (A_u - A_t) r_u
 \end{aligned}$$

The two subamplitudes can be defined as color-ordered amplitude with order  $[1,2,3,4]$  and  $[1,2,4,3]$  respectively.

Of course, for the type  $\Phi^\dagger(nV_1)\Phi$  and  $\Phi(nV_2)\Phi^\dagger$ , we can do the same thing to define the color-ordered amplitudes. It should be noticed that the order only has the relation with the order of external gluon line.

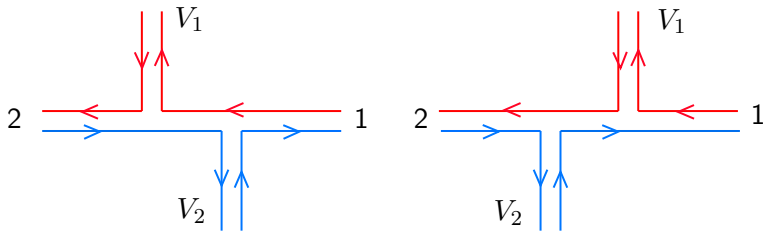
$$[1, 2, \sigma(3), \sigma(4), \dots, \sigma(n)]$$

- $\Phi V_2 \Phi^\dagger V_1$

The color structure for this kind of amplitude has special form, like

$$(T_1^a)_{ij} (T_2^b)_{\bar{j}\bar{i}}$$

It is more straightforward to observe the color structure in terms of double line notation as follows



# OPP(Order preserving permutation)

From the 4 point case, we have known that the relative order between gauge boson 1 and gauge boson 2 does not affect the color structure. Thus, it is necessary to introduce the **OPP(Order preserving permutation)**. For example:

$$(3_1, 4_1, 5_2) \quad (3_1, 5_2, 4_1) \quad (5_2, 3_1, 4_1)$$

These three permutations are different OPP for  $(3_1, 4_1, 5_2)$ , so that give us the same color factor.

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From the previous section, we have known that there are only two kinds of 3 point amplitude

$$\begin{aligned} A[1, 2, 3^+] &= \frac{[23][31]}{[12]}, & A[1, 2, 3^-] &= \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle} \\ A[3^+, 4^+, 5^-] &= \frac{[34]^3}{[45][53]}, & A[3^-, 4^-, 5_+] &= \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle} \end{aligned}$$

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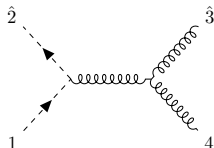


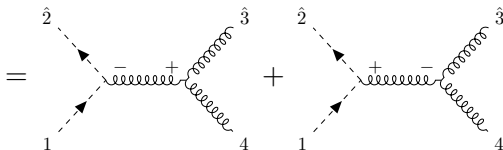
## 4 point from BCFW

- $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude  $A[1, 2, 3^+, 4^-]$ . We choose  $[2, 3\rangle$  shift

$$\begin{aligned} |\hat{2}\rangle &= |2\rangle - z|3\rangle, & |\hat{2}\rangle &= |2\rangle \\ |\hat{3}\rangle &= |3\rangle, & |\hat{3}\rangle &= |3\rangle + z|2\rangle \end{aligned}$$

$$A[1, 2, 3^+, 4^-] = \sum_h$$




We denote these two different BCFW channels as  $A_1$  and  $A_2$ , then

$$\begin{aligned}
 A_1 &= \frac{\langle \hat{2}\hat{I} \rangle \langle \hat{I}1 \rangle}{\langle 1\hat{I} \rangle} \times \frac{1}{s_{12}} \times \frac{[\hat{I}\hat{3}]^3}{[\hat{3}4][4\hat{I}]} \\
 &= (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}
 \end{aligned}$$

where we use the fact  $|\hat{2}\rangle = |2\rangle$ ,  $|\hat{3}\rangle = |3\rangle$ , and the **Fierz Identity**

$$[ij][kl] + [il][jk] + [ik][lj] = 0$$

Similarly, we can obtain

$$A[1, 2, 3^-, 4^+] = (-1) \frac{\langle 13 \rangle^2 \langle 23 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

★ Bonus relation

$$A[1, 2, 3^+, 4^+] = A[1, 2, 3^-, 4^-] = 0$$

# Fantastic result from Cauchy Theorem

As a result, we can consider amplitude  $A_n$  in terms of shifted momentum  $\hat{p}_i^\mu$  instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

and we have known the possible positions of single poles,  $z_I$ , different propagators give us different single poles in the  $z$ -plane.

If we consider the meromorphic function  $\frac{\hat{A}_n(z)}{z}$  in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\text{Res}|_{z=0} \frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n$$

It means that the original amplitude equals to the residue at origin.