Tree level scattering amplitude in (De)constructed gauge theory

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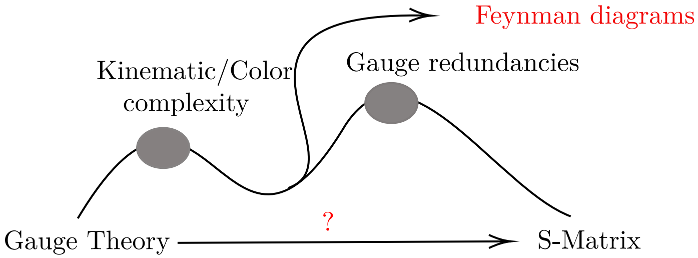
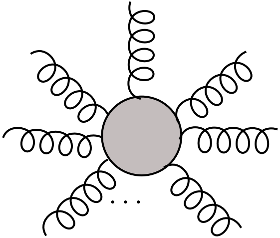
Contents

Why we need new method?

Feynman diagram is a brilliant method without doubt, helping us compute the scattering process pertubatively.





Can we really compute this by hand?

From Frynman diagram to On-shell method

The answer is On-shell method.

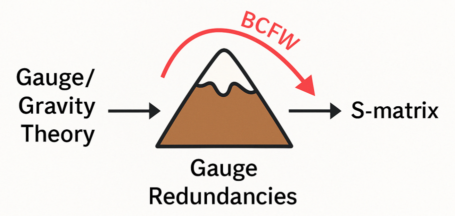
$$\text{Gauge Theory} \xrightarrow{\text{On-shell method}} \text{S-Matrix}$$

On-shell here means that all quantities we use are gauge invariant and satisfy the on-shell condition. Specifically, there are many ingredients under this frame

- The analytic continuation for S-matrix.
- The color-ordered amplitudes.
- The BCFW recursion relation.
- The spinor helicity discription for amplitudes.

Contents

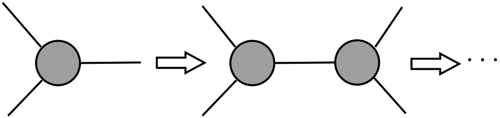
BCFW helps us solve one of the problems



with the cost of introducing complexed momentum.

BCFW is a method to compute amplitudes recursively, proposed by

- Britto, Cachazo, Feng, arXiv: hep-th/0412308
- Britto, Cachazo, Feng, Witten, arXiv: hep-th/0501052



From real to complex – Analytic Continuation

Why can we conduct analytic continuation?

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as $p_{i\mu}p_j^{\mu}$, $p_{i\mu}\epsilon_j^{\mu}$.
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.

$$\Downarrow$$

Ampltudes can be shifted to complex plane

Momentum Shift in BCFW

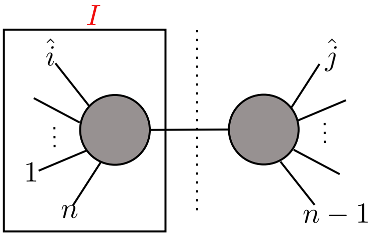
What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \ldots, n$. We choose two momentum to be shifted oppositely

$$p_i \to \hat{p}_i(z) \equiv p_i - zk, \qquad p_j \to \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \qquad p_i \cdot k = 0, \qquad p_j \cdot k = 0$$



For a non-trival subset of generic momenta $\{p_i\}_{i \in I}$

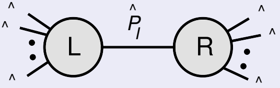
$$\hat{P}_I^2 = P_I^2 - 2zP_I \cdot k = -\frac{P_I^2}{z_I}(z - z_I)$$

with $\quad z_I = \frac{P_I^2}{2P_I \cdot k}$.

Fantasitic result from Cauchy Theorem

BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_{\text{diagrams } I}$$



Brief proof:

We consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

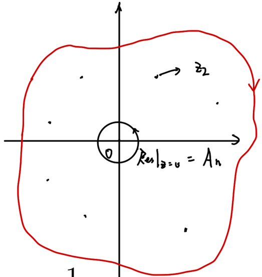
If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

From Cauchy Theorem, we can ontain

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.



$$\hat{A}_n(z) \quad \xrightarrow{z \text{ near } z_I} \quad \hat{A}_L(z_I)\frac{1}{\hat{P}_I^2}\hat{A}_R(z_I) = -\frac{z_I}{z - z_I}\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$-\mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

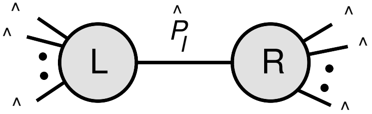
$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j\rangle$	$+$	$-$
$+$	$1/z$	z^3
$-$	$1/z$	$1/z$

Spinor-Helicity formalisim

In the part of introduction to BCFW



$\rightarrow$ No helicity appears here

but the S-matrix is a function of moentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

The answer is **Spinor-Helicity formalism** $\rightarrow$ Catch p_i and h_i at the same time.

- **Massless Case**

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \to t^{-1}\lambda, \qquad \tilde{\lambda} \to t\tilde{\lambda}, \qquad t \in \mathbb{C}$$

same for

$$|\lambda\rangle \to t^{-1}|\lambda\rangle, \qquad |\lambda] \to t|\lambda]$$

The scattering amplitudes should transform covariantly under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \ldots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \ldots) = t_i^{2h_i}\mathcal{A}_n$$

- **Massive Case**

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

On-shell 3-point can be completely determined

On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \qquad p_2 = (1-\kappa)p_3 \quad \text{(Collinear)}$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) \quad = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

We can not obtain any thing nontrival from 3-point!

Of coure, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\Box \Psi)$$

but it still equals to 0 under the on-shell condition.

Another necessarity to introduce complex momentum

If the momentum is complexed, we have

$$\langle 12\rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \qquad or \qquad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or squar brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}} \langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \qquad 2h_3 = -x_{13} - x_{23}.$$

Then all index can be solved from this system of equations, so that

$$\boxed{\begin{array}{ll} A_3^{h_1h_2h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} & h_1+h_2+h_3<0 \\ A_3^{h_1h_2h_3} = c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & h_1+h_2+h_3>0 \end{array}}$$

$\star$ **All massless on-shell 3-point ampltides are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g'\frac{[13][23]}{[12]^3}$$

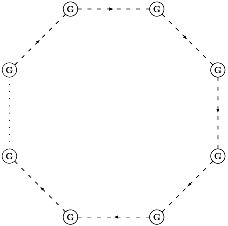
but actually it comes from the non-local interaction $g'AA\frac{\partial}{\Box}A$, so we discard it.

Contents

Introduction of Quiver or Moose gauge theory

Quiver: A container for carring arrows
Moose: A kind of deer with large horns

In the language of field theories, quiver gauge theories contain gauge fields and bi-fundamental scalars, summarized in a pictotial representaion.



Moose diagram

N-sided polygon
G : gauge group $\quad SU(m)$
$\rightarrow$: Unitary scalar fields $\quad \Phi_{ij}$

Why we foucs on quiver gauge theory?

The lagrangian can be written like

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2}\mathrm{Tr}(F_i)^2 + \sum_{i=1}^{N} \mathrm{Tr}[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)],$$

here F_i refers to the ith gauge field strength, scalar field Φ_i transformed under the bi-fundamental representation and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - i g_i A_{i\mu} \Phi_i + i g_{i+1} \Phi_i A_{i+1\mu}.$$

Here, gauge field and scalar field transformed like

$$\boldsymbol{A_{i\mu}} \to U_i(x) \boldsymbol{A_{i\mu}} U_i^-(x) - \frac{i}{g_i}(\partial \mu U) U^{-1}, \qquad \Phi_i \to U_i(x) \Phi_i U_{i+1}^-(x)$$

It is easy to confirm that this theory is invariant under $\prod_1^N SU(m)$ gauge group.

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group $SU(m)$, where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.

- If $SU(m)_1$ and $SU(m)_N$ are connected $\longrightarrow$ S^2 compactification
- If not connected $\longrightarrow \text{Interval}$ compactification

After higgsing the scalar field, we can obtain a spectrum

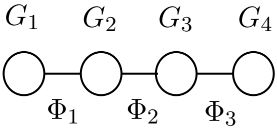
$$M_k^2 = 4g^2 f_s^2 \sin^2\left(\frac{\pi k}{N}\right)$$

This is precisely the Kaluza-Klein spectrum under S^2 compactification.

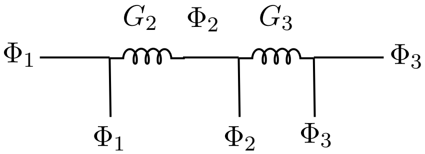
What is relation to scattering amplitude?

The critical point is locality.

- Space-Time Locality $\longrightarrow$ local field theories
- Theory Space Locality $\longrightarrow$ Discretized theory space



If we change this to a scattering diagram, and compute the large-z behavior



$\sim 1/z^{4}$

Contents

Classification of Scattering Amplitudes

For simplicity, we start from the two-site gauge theory with gauge fields V_1, V_2 and scalar fields Φ, $\Phi^\dagger$. The amplitudes are classified by their multiplicity:

3-point	**4-point**	**5-point**	**6-point**
$\boldsymbol{V_1\Phi\Phi^\dagger}$	$\boldsymbol{V_1V_1V_1V_1}$	$\boldsymbol{V_1V_1V_1V_1V_1}$	$\boldsymbol{V_1V_1V_1V_1V_1V_1}$
$\boldsymbol{V_2\Phi\Phi^\dagger}$	$\boldsymbol{V_2V_2V_2V_2}$	$\boldsymbol{V_2V_2V_2V_2V_2}$	$\boldsymbol{V_2V_2V_2V_2V_2V_2}$
$\boldsymbol{V_1V_1V_1}$	$\boldsymbol{\Phi^\dagger V_1V_1\Phi}$	$\boldsymbol{\Phi^\dagger V_1V_1V_1\Phi}$	$\boldsymbol{\Phi^\dagger V_1V_1V_1V_1\Phi}$
$\boldsymbol{V_2V_2V_2}$	$\boldsymbol{\Phi V_2V_2\Phi^\dagger}$	$\boldsymbol{\Phi V_2V_2V_2\Phi^\dagger}$	$\boldsymbol{\Phi V_2V_2V_2V_2\Phi^\dagger}$
	$\boldsymbol{\Phi V_2\Phi^\dagger V_1}$	$\boldsymbol{V_2\Phi^\dagger V_1V_1\Phi}$	$\boldsymbol{V_2V_2\Phi^\dagger V_1V_1\Phi}$
	$\boldsymbol{\Phi\Phi^\dagger\Phi\Phi^\dagger}$	$\boldsymbol{\Phi V_2V_2\Phi^\dagger V_1}$	$\boldsymbol{\Phi V_2V_2\Phi^\dagger V_1V_1}$
		$\Phi\Phi^\dagger\Phi\Phi^\dagger V_1$	$\vdots$
		$\Phi\Phi^\dagger\Phi\Phi^\dagger V_2$	$\vdots$

Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$A[1,2,3^+] = \frac{[23][31]}{[12]}, \qquad A[1,2,3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

$$A[3^+,4^+,5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^-,4^-,5^+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

By using the 3 point building block, we can construct 4 point color-ordered amplitudes from BCFW recursion relation.

Gauge boson sector

- nV_1 or nV_2
  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\boxed{\text{Parke - Talyor Formula :} \quad A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}}$$

Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

SQCD like sector

- $\Phi^\dagger V_1 V_1 \Phi$
  Here we compute the color-ordered amplitude $A[1,2,3^+,4^-]$. We choose $[2,3\rangle$ shift

$$|\hat{2}] = |2] - z|3], \qquad |\hat{2}\rangle = |2\rangle$$
$$|\hat{3}] = |3], \qquad |\hat{3}\rangle = |3\rangle + z|2\rangle$$

  The amplitudes can be computed

$$A[1,2,3^+,4^-] = (-1)\frac{\langle 14\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1,2,3^+,4^+,5^-] = \frac{\langle 15\rangle^2\langle 25\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1,2,\cdots,(n+2)^-] = (-1)^{n+1}\frac{\langle 1,n+2\rangle^2\langle 2,n+2\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n+1,n+2\rangle\langle n+2,1\rangle}$$

$\star$ Bonus relation:

$$A[1,2,3^+,4^+] = 0 \quad \Rightarrow \quad A[1,2,3^+,\cdots,n^+] = 0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

- $\Phi V_2 \Phi^\dagger V_1$

$$A[1,2,3_1^+,4_2^-] = \frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1,2,3_1^+,4_1^+,5_2^-] = (-1)\frac{\langle 25\rangle^2\langle 15\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 51\rangle}$$

- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$

$$A[1,2,3_1^+,4_1^+,5_2^+,6_2^-] = \frac{\langle 26\rangle^2\langle 16\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 56\rangle\langle 51\rangle}$$

- Compact formula for general case

$$A = \frac{\langle 2a\rangle^2 \langle 1a\rangle^2}{\underbrace{\langle 2\star\rangle \cdots \langle \star 1\rangle}_{SU(N_1)} \underbrace{\langle 2*\rangle \cdots \langle * 1\rangle}_{SU(N_2)}}$$

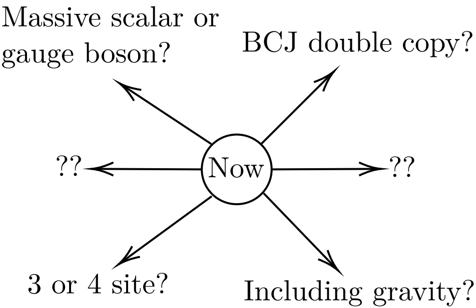
Green: Particle with $-$ helicity
Blue: Particle belongs to the first gauge group
Red: Particle belongs to the second gauge group
$\star$: Order for gauge group 1
$*$: Order for gauge group 2

Contents

Summary

- Introduce the on-shell method, including BCFW recursion relation, color-ordered amplitudes. etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- The locality plays an important role to relate the spacetime locality and field space locality.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas can be obtained.

Possible future work



Thanks for your attention!

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  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

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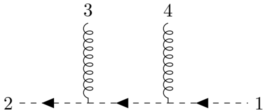
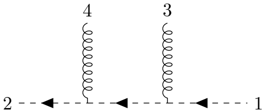
Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

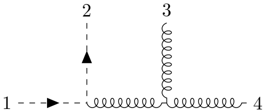
- $\Phi^\dagger V_1 V_1 \Phi$



(a) s channel



(b) u channel



(c) t channel

The color factor can be written respectively as following

$r_s = \mathrm{Tr}[\Phi_2^\dagger T^{a_3} T^{a_4} \Phi_1]$, $r_u = \mathrm{Tr}[\Phi_2^\dagger T^{a_4} T^{a_3} \Phi_1]$, $r_t = \mathrm{Tr}[\Phi_2^\dagger [T^{a_3}, T^{a_4}] \Phi_1]$

We can easily obtain a similar Jacobbi relation

$$r_t = r_s - r_u$$

Then we can accomplish the color decomposition and define the corressponding color-ordered amplitudes.

For example, in the 4pt. case, the full amplitude can be decomposed to the following form

$$\begin{aligned}\mathcal{A}_4(\Phi^\dagger V_1 V_1 \Phi) &= A_s r_s + A_u r_u + A_t r_t \\ &= A_s r_s + A_u r_u + A_t(r_s - r_u) \\ &= (A_s + A_t) r_s + (A_u - A_t) r_u\end{aligned}$$

The two subamplitudes can be defined as color-ordered amplitude with order [1,2,3,4] and [1,2,4,3] respectively.

Of course, for the type $\Phi^\dagger(nV_1)\Phi$ and $\Phi(nV_2)\Phi^\dagger$, we can do the same thing to define the color-ordered amplitudes. It should be noticed that the order only has the relation with the order of external gluon line.

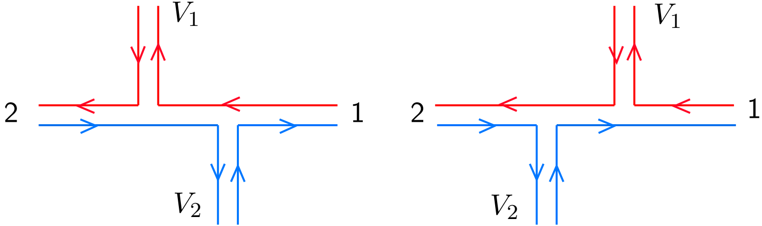
$$[1, 2, \sigma(3), \sigma(4), \cdots, \sigma(n)]$$

- $\Phi V_2 \Phi^\dagger V_1$

  The color structure for this kind of amplitude has special form, like

$$(T_1^a)_{ij}(T_2^b)_{\bar{j}\bar{i}}$$

  It is more straightforward to observe the color structure in terms of double line notation as follows



OPP(Order preserving permutation)

From the 4 point case, we have known that the relative order between gauge boson 1 and gauge boson 2 does not affect the color structure. Thus, it is necessary to introduce the **OPP(Order preserving permutation)**. For example:

$$(3_1, 4_1, 5_2) \quad (3_1, 5_2, 4_1) \quad (5_2, 3_1, 4_1)$$

These three permutations are different OPP for $(3_1, 4_1, 5_2)$, so that give us the same color factor.

Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$A[1,2,3^+] = \frac{[23][31]}{[12]}, \qquad A[1,2,3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$
$$A[3^+,4^+,5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^-,4^-,5_+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

By using the 3 point building block, we can construct 4 point color-ordered amplitudes from BCFW recursion relation.

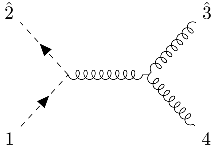
4 point from BCFW

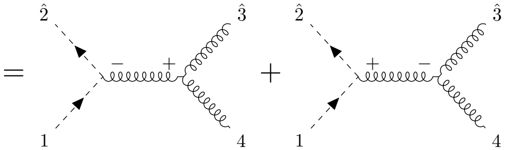
- $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude $A[1,2,3^+,4^-]$. We choose $[2,3\rangle$ shift

$$|\hat{2}] = |2] - z|3], \qquad |\hat{2}\rangle = |2\rangle$$
$$|\hat{3}] = |3], \qquad |\hat{3}\rangle = |3\rangle + z|2\rangle$$

$$A[1,2,3^+,4^-] = \sum_h$$





We denote these two different BCFW channels as A_1 and A_2, then

$$
\begin{aligned}
A_1 &= \frac{\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle}{\langle 1\hat{I}\rangle} \times \frac{1}{s_{12}} \times \frac{[\hat{I}\hat{3}]^3}{[\hat{3}4][4\hat{I}]} \\
&= (-1)\frac{\langle 14\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}
$$

where we use the fact $|\hat{2}\rangle = |2\rangle$, $|\hat{3}] = |3]$, and the Fierz Identity

$$
[ij][kl] + [il][jk] + [ik][lj] = 0
$$

Similarly, we can obtain

$$A[1,2,3^-,4^+] = (-1)\frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

★ Bonus relation

$$A[1,2,3^+,4^+] = A[1,2,3^-,4^-] = 0$$

Fantasitic result from Cauchy Theorem

As a result, we can consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

and we have known the possible positions of single poles, z_I, different propagators give us different single poles in the z-plane.

If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

It means that the original amplitude equals to the residue at origin.