

On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

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June 11th, 2025

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- 1 Motivation
- 2 Preliminary
- 3 Model and Computation
- 4 Some problems and extends
- 5 Summary

Why We Study Scattering Amplitudes?

1. **Bridge between theory and experiment**

- Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
- Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections

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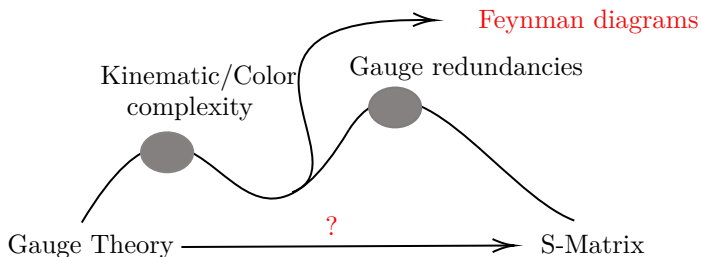
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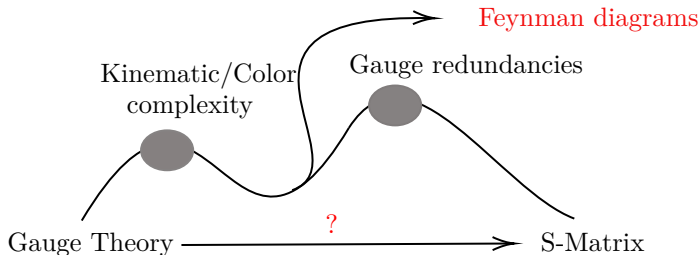
2. Reveal deep structures of quantum field theory

- Amplitudes exhibit hidden symmetries (e.g., dual conformal, Yangian) not visible in the Lagrangian
- These symmetries suggest deeper theoretical frameworks, such as amplituhedra or AdS/CFT correspondence

Challenges we face before



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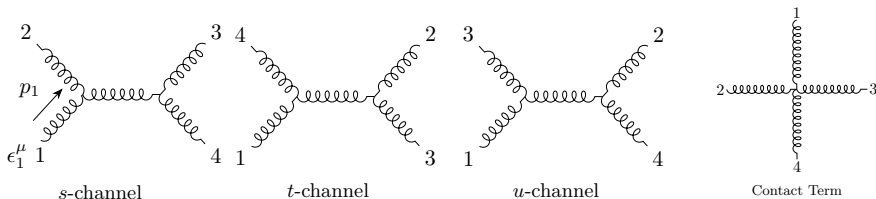


n pt. amplitudes	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

The number of Feynman diagrams grow quite rapidly!

Conventional Computation

Usually, when we compute the gluon amplitudes by using Feynman diagram, we will obtain something like



$$\begin{aligned} \mathcal{M}_s(p_1 p_2 \rightarrow p_3 p_4) = & -\frac{g_s^2}{s} f^{abe} f^{cde} \\ & \times \left\{ -4 (\epsilon_1 \cdot \epsilon_3^*) (\epsilon_2 \cdot p_1) (\epsilon_4^* \cdot p_3) + 2 (\epsilon_1 \cdot \epsilon_2) (\epsilon_3^* \cdot p_1) (\epsilon_4^* \cdot p_3) \right. \\ & \left. - 2 (\epsilon_1 \cdot p_4) (\epsilon_2 \cdot p_1) (\epsilon_3^* \cdot \epsilon_4^*) + (\epsilon_1 \cdot \epsilon_2) (p_4 \cdot p_1) (\epsilon_3^* \cdot \epsilon_4^*) + 10 \text{ terms} \right\} \end{aligned}$$

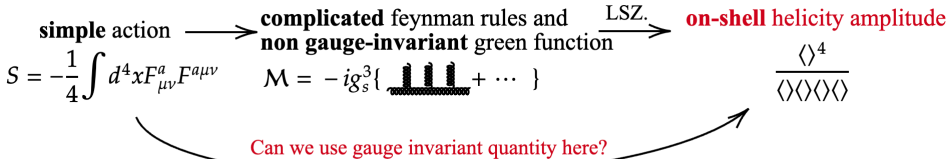
If you consider 5point case, it will become worse:

★ We have 25 diagrams and nearly 10000 terms!



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Gauge Theory $\xrightarrow{\text{On-shell method}}$ Helicity Amplitude

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$$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \text{Tr}[T^{a_1} T^{a_2} \dots T^{a_5}] + \text{permutations}$$

Parke–Taylor Formula (MHV amplitudes):

$$\begin{aligned} A_5[1^+ 2^+ 3^+ 4^+ 5^+] &= 0 & (+, - : \text{ helicity}; \\ A_5[1^- 2^+ 3^+ 4^+ 5^+] &= 0 & 1, 2, \dots, n : \text{ particle labels}) \end{aligned}$$

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Color-ordered Amplitudes

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$$A_5[1^- 2^- 3^- 4^- 5^-] = 0$$

$$A_5[1^+ 2^- 3^- 4^- 5^-] = 0$$

$$A_5[1^+ 2^+ 3^- 4^- 5^-] = \frac{[12]^4}{[12][23][34][45][51]}$$

Color-ordering for Yang-Mills

Consider the Yang-Mills lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The 3 point and 4 point vertices include \tilde{f}^{abc} and $\tilde{f}^{abe}\tilde{f}^{cde} + \text{perms.}$
(With a different convention, $\text{Tr}[T^a T^b] = \delta^{ab}$ and $[T^a, T^b] = i\tilde{f}^{abc}T^c$)

We have

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}$$

and the color factor can be rewritten by the trace of product of generators

$$i\tilde{f}^{abc} = \text{Tr}([T^a, T^b]T^c),$$

Moreover, in $SU(N)$, we have a Fierz identity

$$\sum_a T_{ij}^a T_{kl}^a = \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl}. \quad (1)$$

This identity is easier understood as matrix form like

$$\begin{aligned}\mathrm{Tr}\{T^a A\}\mathrm{Tr}\{T^a B\} &= \mathrm{Tr}\{AB\} - \frac{1}{N}\mathrm{Tr}\{A\}\mathrm{Tr}\{B\}, \\ \mathrm{Tr}\{AT^a BT^a\} &= \mathrm{Tr}\{A\}\mathrm{Tr}\{B\} - \frac{1}{N}\mathrm{Tr}\{AB\}.\end{aligned}$$

So, the 4 gluon s-channel gives us

$$\begin{aligned}\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} &= \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - \mathrm{Tr}(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) \\ &\quad - \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) + \mathrm{Tr}(T^{a_2} T^{a_1} T^{a_4} T^{a_3}).\end{aligned}$$

Therefore, the full 4-point amplitude can be rewritten like

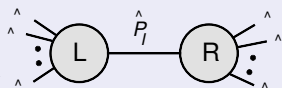
$$\mathcal{A}_{4,\mathrm{tree}} = g^2 (A_4[1234] \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{perms of } (234))$$

here the subamplitudes $A_4[1234]$, $A_4[1243]$, etc. are called **color-ordered amplitudes**. This concept can be easily generalized to tree-level n-point case

$$\mathcal{A}_{n,\mathrm{tree}} = g^{n-2} \sum_{\sigma} A_n[1, \sigma(2, 3 \cdots n)] \mathrm{Tr}(T^{a_1} T^{\sigma(a_2 \cdots a_n)})$$

The Power of BCFW Recursion Relation

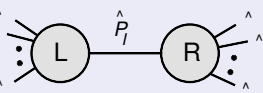
BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$


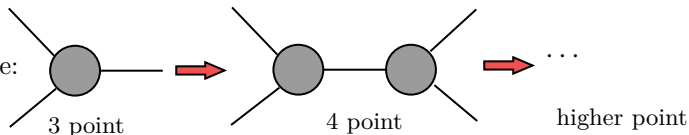
The diagram illustrates the BCFW recursion relation. It shows two sub-diagrams, L and R, represented as circles. Sub-diagram L has several external lines, and sub-diagram R also has several external lines. They are connected by a horizontal line representing a propagator, labeled with \hat{P}_I above it. The entire expression is summed over all possible diagrams I.

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More understandable:



★ From lower point on-shell amp. to higher point on-shell amp.!!

Momentum Shift in BCFW

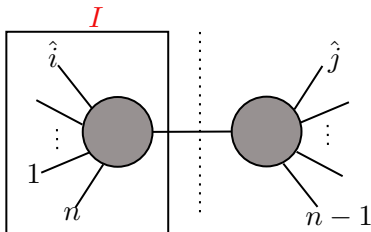
What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \dots, n$. We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$



For a non-trivial subset of generic momenta $\{p_i\}_{i \in I}$

$$\hat{P}_I^2 = P_I^2 - 2z P_I \cdot k = -\frac{P_I^2}{z_I}(z - z_I)$$

with $z_I = \frac{P_I^2}{2P_I \cdot k}$.

Brief explanation: We choose two momentum to be shifted oppositely

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If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane. From Cauchy Theorem, we can obtain

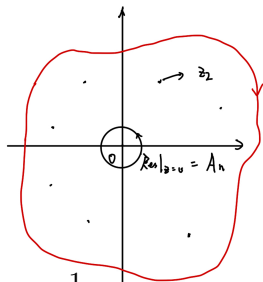
$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.

From Cauchy Theorem, we can obtain

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.



$$\hat{A}_n(z) \xrightarrow{z \text{ near } z_I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) = - \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$-\text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \frac{(z - z_I)z_I}{z(z - z_I)} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)|_{z=z_I} = \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	z^3
-	$1/z$	$1/z$

proved by using background field expansion (N. Arkani-Hamed and J. Kaplan, [arXiv:0801.2385 [hep-th]].)

■ Massless Case

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \rightarrow t^{-1}\lambda, \quad \tilde{\lambda} \rightarrow t\tilde{\lambda}, \quad t \in \mathbb{C}$$

same for

$$|\lambda\rangle \rightarrow t^{-1}|\lambda\rangle, \quad [\lambda] \rightarrow t[\lambda]$$

The scattering amplitudes should transform **covariantly** under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \dots, \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \dots) = t_i^{2h_i} \mathcal{A}_n$$

■ Massive Case

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

On-shell 3-point can be completely determined

Another necessity to introduce complex momentum If the momentum is complexed, we have

$$\langle 12 \rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or square brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}} \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

Then all index can be solved from this system of equations, so that

$$\begin{aligned} A_3^{h_1 h_2 h_3} &= c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3} & h_1 + h_2 + h_3 < 0 \\ A_3^{h_1 h_2 h_3} &= c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} & h_1 + h_2 + h_3 > 0 \end{aligned}$$

★ **All massless on-shell 3-point amplitudes are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g' \frac{[13][23]}{[12]^3}$$

but actually it comes from the **non-local** interaction $g' A A \frac{\partial}{\square} A$, so we discard it.

So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
 - **Pure Yang-Mills** theory
 - Tree-level MHV amplitudes
 - Color-ordered partial amplitudes

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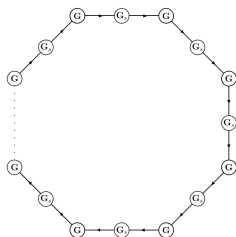
Next: Realistic Models

- Move beyond massless gauge theory
- Consider:
 - **(De)constructed gauge theories**
- Key questions:
 - Can BCFW still apply?
 - What new structures emerge?

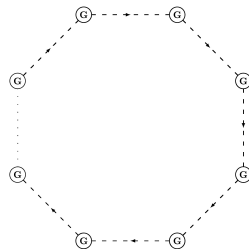
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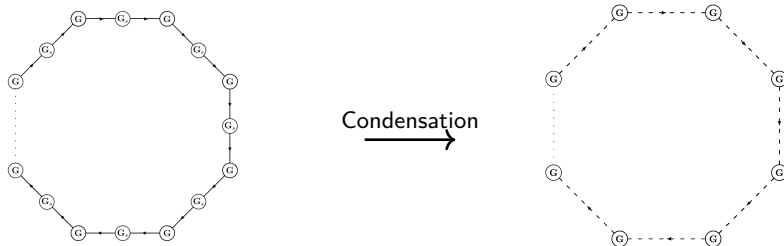
Introduction of (De)Constructed gauge theory



Condensation
→



Introduction of (De)Constructed gauge theory



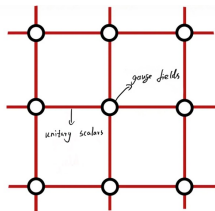
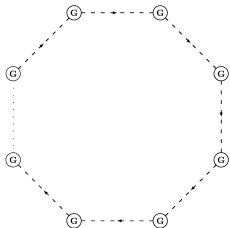
The Lagrangian can be written like

$$\mathcal{L} = - \sum_{i=1}^N \frac{1}{2} \text{Tr}(F_i^2) + \sum_{i=1}^N \text{Tr} \left[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) \right],$$

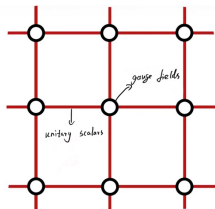
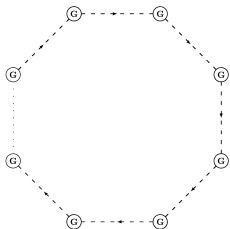
here F_i refers to the i th gauge field strength. The scalar field Φ_i transforms under the **bi-fundamental** representation, and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - ig_i A_{i\mu} \Phi_i + ig_{i+1} \Phi_i A_{i+1\mu}.$$

It has been proposed that this model actually discretized a **five-dimension gauge theory** with gauge group $SU(m)$, where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.



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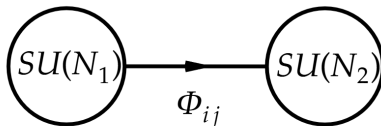


After higgsing the scalar field, we can obtain a spectrum

$$M_k^2 = 4g^2 f_s^2 \sin^2 \left(\frac{\pi k}{N} \right)$$

This is precisely the **Kaluza-Klein** spectrum under S^1 compactification.

Amplitudes from BCFW



$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_1)^2 - \frac{1}{2}\text{Tr}(F_2)^2 + \text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

From the Lagrangian, we have known that there are only two kinds of 3 point amplitude (+, - : helicity Φ, Φ^\dagger : charge of scalar)

$$A[1^\Phi 2^{\Phi^\dagger} 3^+] = \frac{[23][31]}{[12]}, \quad A[1^\Phi 2^{\Phi^\dagger} 3^-] = \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle}$$

$$A[3^+ 4^+ 5^-] = \frac{[34]^3}{[45][53]}, \quad A[3^- 4^- 5^+] = \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle}$$

By using the 3 point building block, we can construct 4 point colorordered amplitudes from BCFW recursion relation.

- nV_1 or nV_2

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\text{Parke - Talyor Formula : } A[\cdots i^- \cdots j^- \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

The color factor in this sector looks like

$$(T^{a_1} T^{a_2} \dots T^{a_n})_{ij}$$

so we need to notice is just the order of gauge boson.

The amplitudes can be computed like

- $\Phi^\dagger V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^-] = (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad (\text{Parke -Talyor like Formula})$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^+ 5^-] = \frac{\langle 15 \rangle^2 \langle 25 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1^\Phi 2^{\Phi^\dagger} \cdots (n+2)^-] = (-1)^{n+1} \frac{\langle 1, n+2 \rangle^2 \langle 2, n+2 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n+1, n+2 \rangle \langle n+2, 1 \rangle}$$

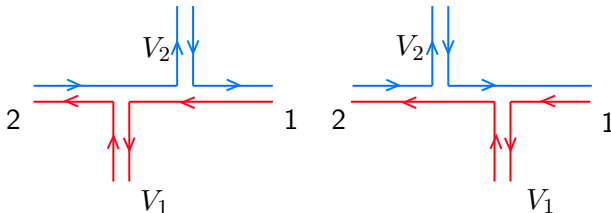
★ Bonus relation:

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^+] = 0 \quad \Rightarrow \quad A[1^\Phi 2^{\Phi^\dagger} 3^+ \cdots n^+] = 0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

The straightforward way to observe the color structure in this case is double line notation as follows



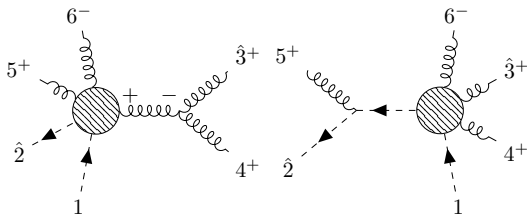
- $\Phi V_2 \Phi^\dagger V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_2^-] = \frac{\langle 14 \rangle \langle 24 \rangle}{\langle 13 \rangle \langle 23 \rangle}$$

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Here I show the concrete computation process



$$\begin{aligned}
 A_1 &= \frac{(-1)\langle\hat{2}6\rangle^2\langle16\rangle^2}{\langle25\rangle\langle56\rangle\langle61\rangle\langle\hat{2}\hat{I}\rangle\langle\hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[\hat{3}4]^3}{[4\hat{I}][\hat{I}3]} \\
 &= \frac{\langle26\rangle^2\langle16\rangle}{\langle25\rangle\langle56\rangle\langle\hat{2}\hat{I}\rangle\langle\hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[34]^3}{[4\hat{I}][\hat{I}3]} \\
 &= \frac{\langle26\rangle^2\langle16\rangle\cancel{[34]^3}\cancel{\langle42\rangle}}{\langle25\rangle\langle56\rangle\langle41\rangle\langle32\rangle\langle43\rangle\cancel{[43]}\cancel{[43]}\cancel{[34]}\cancel{\langle24\rangle}} \\
 &= \frac{\langle26\rangle^2\langle16\rangle^2}{\langle23\rangle\langle34\rangle\langle41\rangle\langle25\rangle\langle56\rangle\langle61\rangle}
 \end{aligned}$$

- Compact formula for general case

$$A = \frac{\langle 2a \rangle^2 \langle 1a \rangle^2}{\underbrace{\langle 2\star \rangle \cdots \langle \star 1 \rangle}_{SU(N_1)} \underbrace{\langle 2* \rangle \cdots \langle *1 \rangle}_{SU(N_2)}}$$

Green: Particle with – helicity

Blue: Particle belongs to the first gauge group

Red: Particle belongs to the second gauge group

★: Order for gauge group 1

*: Order for gauge group 2

- Compact formula for general case

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★: Order for gauge group 1

*: Order for gauge group 2

For example, if we want to compute $A[1^{\Phi} 2^{\Phi^{\dagger}} 5_1^+ 3_1^+ 4_1^- 7_2^+ 6_2^+ 8_2^+]$:

$$A = \frac{\langle 24 \rangle^2 \langle 14 \rangle^2}{\langle 25 \rangle \langle 53 \rangle \langle 34 \rangle \langle 41 \rangle \langle 27 \rangle \langle 76 \rangle \langle 68 \rangle \langle 81 \rangle}$$

If you use Feynman diagrams, it may take several days to accomplish the computation.

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How about NMHV?

First, let us review the NMHV amplitudes for gluon scattering.

Still we begin with the simplest case – **Split-helicity** NMHV like $A_6[1^- 2^- 3^- 4^+ 5^+ 6^+]$.

Here we choose $[1, 2]$ shift

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A} + \text{diagram B}$$

Diagram A: A t-channel exchange diagram. The left vertex has legs $\hat{1}^-$ and 6^+ . The right vertex has legs $\hat{2}^-$, 3^- , 4^+ , and 5^+ . The internal propagator is labeled \hat{P}_{16} with a minus sign on the left and a plus sign on the right.

Diagram B: A u-channel exchange diagram. The left vertex has legs $\hat{1}^-$ and 5^+ . The right vertex has legs $\hat{2}^-$, 3^- , and 4^+ . The internal propagator is labeled \hat{P}_{156} with a minus sign on the left and a plus sign on the right.

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$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A} + \text{diagram B}$$

Diagram A: A t-channel exchange between legs 1, 2 and 3, 4, 5, 6. The internal propagator is labeled \hat{P}_{16} . The external legs are labeled with helicities: $\hat{1}^-$, 6^+ , 2^- , 3^- , 4^+ , 5^+ . The internal lines are labeled with $-$ and $+$ signs.

Diagram B: A u-channel exchange between legs 1, 2 and 3, 4, 5, 6. The internal propagator is labeled \hat{P}_{156} . The external legs are labeled with helicities: $\hat{1}^-$, 5^+ , 2^- , 3^- , 4^+ , 6^+ . The internal lines are labeled with $-$ and $+$ signs.

- Diagram B includes a propagator $1/P_{156}^2$, so there is a 3-particle pole $P_{156}^2 = 0$. But by inspecting the external order, it seems that there's no difference between $(-++)$ channel 561 and 345. We should expect the amplitude to have a pole also at $P_{345}^2 = 0$.

$$\text{diagram A} = \frac{\langle \hat{1} \hat{P}_{16} \rangle^3}{\langle \hat{P}_{16} 6 \rangle \langle 6 \hat{1} \rangle} \times \frac{1}{P_{16}^2} \times \frac{\langle \hat{2} 3 \rangle^3}{\langle 3 4 \rangle \langle 4 5 \rangle \langle 5 \hat{P}_{16} \rangle \langle \hat{P}_{16} 2 \rangle}$$

$$\langle \hat{2}\hat{P}_{16} \rangle [\hat{P}_{16}3] = \langle 21 \rangle [\hat{1}3] + \langle \hat{2}6 \rangle [63]$$

It follows from $\hat{P}_{16}^2 = 0$ that $z_{16} = -\frac{[16]}{[26]}$, so

$$\langle \hat{2}\hat{P}_{16} \rangle [\hat{P}_{16}3] = -\frac{[36]}{[26]} (\langle 12 \rangle [12] + \langle 16 \rangle [16] + \langle 26 \rangle [26]) = -\frac{[36]}{[26]} P_{126}^2$$

The 3-particle pole P_{126}^2 is encoded inside the BCFW channel !

- Full expression

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \frac{\langle 3|1+2|6 \rangle^3}{P_{126}^2 [21][16] \langle 34 \rangle \langle 45 \rangle \langle 5|1+6|2 \rangle} + \frac{\langle 1|5+6|4 \rangle^3}{P_{156}^2 [23][34] \langle 56 \rangle \langle 61 \rangle \langle 5|1+6|2 \rangle}.$$

The factor $\langle 5|1+6|2 \rangle$ does not correspond to a physical pole of the scattering amplitude: it is a **spurious pole**.

There has been interesting paper investigating how to systematically cancel the spurious poles, like "A. Hodges, JHEP 1305, 35 (2013) [arXiv:0905.1473 [hep-th]]."

- We utilize the $[1, 2\rangle$ shift before, what happens if we change to $[2, 1\rangle$ shift?

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A'} + \text{diagram B'} + \text{diagram C'}$$

diagram A' anti-MHV \times NMHV

diagram B' MHV \times MHV

diagram C' MHV \times MHV

diagram A' = anti-MHV \times NMHV, as opposed to diagram A = MHV \times MHV.

The equivalence between two different shift is related to powerful residue theorem (N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, [arXiv:0907.5418 [hep-th]].) and Grassmannian.

CSW(Cachazo-Svrcek-Witten) expansion

We can also consider a shift that is implemented via a “holomorphic” square-spinor shift:

$$|\hat{i}\rangle = |i\rangle + z c_i |X\rangle, \quad |\hat{i}\rangle = |i\rangle$$

Here $|X\rangle$ is an arbitrary reference spinor and the coefficients c_i satisfy $\sum_{i=1}^n c_i |i\rangle = 0$.

$$A_n^{\text{NMHV}} = \sum_{\text{diagrams } I} \text{Diagram with nodes L and R connected by } \hat{P}_I$$

There are two possibilities: $\text{anti-MHV}_3(=0) \times \text{NMHV}$ or $\text{MHV} \times \text{MHV}$.

For example, the 6pt split NMHV amplitude

$$\begin{aligned}
 A_n[1^- 2^- 3^- 4^+ 5^+ 6^+] = & \text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} + \text{Diagram 5} - \text{Diagram 6} \\
 & + \text{Diagram 7} - \text{Diagram 8} + \text{Diagram 9} - \text{Diagram 10} + \text{Diagram 11} - \text{Diagram 12} \\
 & = \frac{\langle 1\hat{P}_I \rangle^4}{\langle 1\hat{P}_I \rangle \langle \hat{P}_I 5 \rangle \langle 56 \rangle \langle 61 \rangle} \frac{1}{P_{156}^2} \frac{\langle 23 \rangle^4}{\langle 23 \rangle \langle 34 \rangle \langle 4\hat{P}_I \rangle \langle \hat{P}_I 2 \rangle} .
 \end{aligned}$$

The diagrams are 6-point tree-level amplitudes with two internal lines. The external legs are labeled with momenta and helicities: $\hat{1}^-$, $\hat{2}^-$, $\hat{3}^-$, $\hat{4}^+$, $\hat{5}^+$, $\hat{6}^+$. The signs $-$ and $+$ between diagrams indicate the relative signs in the sum.

For example, the 6pt split NMHV amplitude

$$\begin{aligned}
 A_n[1^- 2^- 3^- 4^+ 5^+ 6^+] = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \\
 & + \text{Diagram 7}
 \end{aligned}$$

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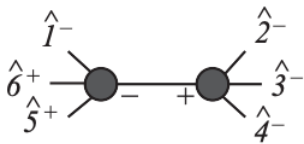
$$\begin{aligned}
 & \hat{6}^+ \text{---} \hat{1}^- \text{---} \hat{2}^- \text{---} \hat{3}^- \text{---} \hat{4}^+ \text{---} \hat{5}^+ \\
 & = \frac{\langle 1\hat{P}_I \rangle^4}{\langle 1\hat{P}_I \rangle \langle \hat{P}_I 5 \rangle \langle 56 \rangle \langle 61 \rangle} \frac{1}{P_{156}^2} \frac{\langle 23 \rangle^4}{\langle 23 \rangle \langle 34 \rangle \langle 4\hat{P}_I \rangle \langle \hat{P}_I 2 \rangle} .
 \end{aligned}$$

We can write

$$|\hat{P}_I \rangle \frac{[\hat{P}_I X]}{[\hat{P}_I X]} = \hat{P}_I |X] \frac{1}{[\hat{P}_I X]} = P_I |X] \frac{1}{[\hat{P}_I X]}$$

We can use the prescription

$$|\hat{P}_I\rangle \rightarrow P_I|X]$$



$$= \frac{\langle 1|P_{156}|X]^4}{\langle 1|P_{156}|X]\langle 5|P_{156}|X]\langle 56\rangle\langle 61\rangle} \cdot \frac{1}{P_{156}^2} \cdot \frac{\langle 23\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4|P_{156}|X]\langle 2|P_{156}|X]}.$$

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Summary

- Introduce the on-shell method, including BCFW recursion relation, spinor-helicity formalism, etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

Thanks for your attention!

Spinor-Helicity Formalism

Helicity

Helicity is defined as the projection of a particle's spin vector \vec{S} onto the direction of its momentum \vec{p} :

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

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S-matrix is a function of momentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

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Massless Case:

- Momenta in spinor form:

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}} = |p\rangle[p|$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	z^3
-	$1/z$	$1/z$

On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \quad p_2 = (1 - \kappa)p_3 \quad (\text{Collinear})$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

We can not obtain any thing nontrivial from 3-point!

Of course, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\square \Psi)$$

but it still equals to 0 under the on-shell condition.

On-shell 3-point can be completely determined

For the complex momentum, we have

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad [1] \propto [2] \propto [3]$$

$$A_3^{h_1 h_2 h_3} = c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3} \quad h_1 + h_2 + h_3 < 0$$

$$A_3^{h_1 h_2 h_3} = c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} \quad h_1 + h_2 + h_3 > 0$$

★ **All massless on-shell 3-point amplitudes are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

Scattering Amplitudes from BCFW

For simplicity, we start from the two-site gauge theory with gauge fields V_1 , V_2 and scalar fields Φ , Φ^\dagger .

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_1)^2 - \frac{1}{2}\text{Tr}(F_2)^2 + \text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

We only focus on the following amplitudes:

$$nV_1, \quad nV_2, \quad \Phi^\dagger nV_1\Phi, \quad \Phi nV_2\Phi^\dagger, \quad \Phi^\dagger\Phi\Phi^\dagger\Phi$$

here n can be any positive integer.

More specifically, it helps us to prove P. T. formula

$$\begin{array}{ccccc}
 3\text{pt.} & \longrightarrow & 4\text{pt.} & \longrightarrow & 5\text{pt.} & \longrightarrow & \cdots \\
 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} & & \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} & & \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} & &
 \end{array}$$

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$$\begin{array}{c}
 3\text{pt.} \longrightarrow 4\text{pt.} \longrightarrow 5\text{pt.} \longrightarrow \cdots \\
 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}
 \end{array}$$

$$\Rightarrow: \quad A[1^+ \cdots i^-(i+1)^+ \cdots j^-(j+1)^+ \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

★ This is the power of BCFW recursion relation.