

Scattering of massless particles: scalars, gluons and gravitons

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July 23rd 2024

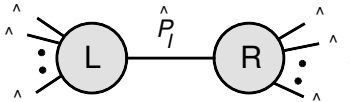
1 Why the S -matrix is so important?

2 Review of some pertinent results

3 CHY Formula

Some developments of amplitudes

- BCFW recursion relation.

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$


- Dual superconformal symmetry of $\mathcal{N} = 4$ Super Yang-Mills amplitude.
- Generalized unitarity
- Positive geometry and Amplituhedra.
- ...

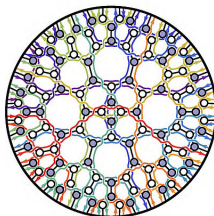


Figure: Positive Grassmanian

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Gauge theory color structure

At tree level, with particles in the adjoint representation of gauge group $SU(N)$, the amplitude can be decomposed as

$$\mathcal{A}_n^{tree}(1, 2, 3, \dots, n) = \sum_{\mathcal{P}(2,3,\dots,n)} \text{Tr}[T^{a_1} T^{a_2} T^{a_3} \dots T^{a_n}] A_n^{tree}[1, 2, 3, \dots, n]$$

here we omit the coupling constant g , and A_n^{tree} is called tree-level color-ordered partial amplitude. Notice that this basis includes $(n-1)!$ independent amplitudes.

Color-ordered partial amplitudes satisfy a set of well-known properties,

- Cyclic:

$$A_n^{tree}[1, 2, 3, \dots, n] = A_n^{tree}[2, 3, \dots, n, 1]$$

- Reflection:

$$A_n^{tree}[1, 2, 3, \dots, n] = (-1)^n A_n^{tree}[n, \dots, 3, 2, 1]$$

- "photon" decoupling:

$$\sum_{\sigma \in \text{cyclic}} A_n^{tree}[1, \sigma(2, 3, \dots, n)] = 0$$

- KK(Kleiss-Kuijf) relation:

$$A_n^{tree}[1, \alpha, n, \beta] = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in OP(\{\alpha\}, \{\beta\}^T)} A_n^{tree}[1, \{\sigma\}_i, n]$$

where the OP means ordered permutations, that is all permutations of $\{\alpha\} \cup \{\beta\}^T$ that maintains the order of individual elements of each set.

For example, a five point amplitude $A_n^{tree}(1, \{2, 3\}, 5, \{4\})$, we have

$$A_n^{tree}[1, 2, 3, 5, 4] = -A_n^{tree}[1, 2, 3, 4, 5] - A_n^{tree}[1, 2, 4, 3, 5] - A_n^{tree}[1, 4, 2, 3, 5]$$

The other five point relations can be obtained by permuting legs 2,3,4 and using cyclic and reflection properties.

This means that the six amplitudes $A_n^{tree}(1, \mathcal{P}\{2, 3, 4\}, 5)$ form a basis of remaining five-point partial amplitudes. **More generally, for multiplicity n , the KK relation can be used to rewrite any color-ordered partial amplitude in terms of only $(n-2)!$ basis partial amplitudes, where two legs are fixed (usually choose 1 and n).**

A Similar Structure

It can be proved that tree level amplitudes can also be decomposed like

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma_1} b_1} \tilde{f}^{b_1 a_{\sigma_2} b_2} \dots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n} \tilde{A}_n(1, \sigma_1, \sigma_2, \dots, \sigma_{n-2}, n)$$

The basis is called DDM and more convenient in practice. It is easy to realize that the number of basis amplitudes is also $(n-2)!$ with fixed particle labels 1 and n .

How can we realize the existence of this basis?

We just need to notice the contribution from ladder diagram

$$\begin{array}{c} \sigma_1 \quad \sigma_2 \quad \dots \quad \sigma_{n-2} \\ | \quad | \quad \dots \quad | \\ 1 \text{ --- } \text{ --- } \text{ --- } n \end{array} \rightarrow \tilde{f}^{a_1 a_{\sigma_1} b_1} \tilde{f}^{b_1 a_{\sigma_2} b_2} \dots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n}$$

And any (three vertex) diagram, by using the Jacobi identity like

$$\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ | \\ 1 \text{ --- } \text{ --- } 5 \\ | \\ 4 \end{array} = - \begin{array}{c} 2 \quad 3 \quad 4 \\ | \quad | \quad | \\ 1 \text{ --- } \text{ --- } 5 \end{array} - \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \quad | \\ 1 \text{ --- } \text{ --- } 5 \end{array}$$

can be transformed to the ladder diagram. It has been proved that $\tilde{A}_n(\dots)$ here is identical to $A_n^{tree}[\dots]$ in the trace basis.

Color-Kinematics Duality

The Color-Kinematics Duality here is not the complete version, but we can catch some points. At four points we have

$$A_4^{\text{tree}}(1, 2, 3, 4) + A_4^{\text{tree}}(1, 3, 2, 4) + A_4^{\text{tree}}(1, 4, 2, 3) = 0.$$

We recognize that the only nontrivial way the equation holds according to $s + t + u = 0$

$$\begin{aligned} A_4^{\text{tree}}(1, 2, 3, 4) + A_4^{\text{tree}}(1, 3, 2, 4) + A_4^{\text{tree}}(1, 4, 2, 3) \\ = (s + t + u)\chi = 0 \end{aligned}$$

Considering that $A_4^{\text{tree}}(1, 2, 3, 4)$ treats s and t the same, and similar to the other two, we make the identification

$$A_4^{\text{tree}}(1, 2, 3, 4) = u\chi, \quad A_4^{\text{tree}}(1, 3, 2, 4) = t\chi, \quad A_4^{\text{tree}}(1, 4, 2, 3) = s\chi$$

We obtain

$$\begin{aligned}tA_4^{\text{tree}}(1, 2, 3, 4) &= uA_4^{\text{tree}}(1, 3, 4, 2), \\sA_4^{\text{tree}}(1, 2, 3, 4) &= uA_4^{\text{tree}}(1, 4, 2, 3), \\tA_4^{\text{tree}}(1, 2, 3, 4) &= sA_4^{\text{tree}}(1, 3, 4, 2).\end{aligned}$$

And in order to obtain the knematic analog of Jacobbi identity, it is convenient to express the amplitudes in terms of poles

$$\begin{aligned}A_4^{\text{tree}}(1, 2, 3, 4) &\equiv \frac{n_s}{s} + \frac{n_t}{t}, \\A_4^{\text{tree}}(1, 3, 4, 2) &\equiv -\frac{n_u}{u} - \frac{n_s}{s}, \\A_4^{\text{tree}}(1, 4, 2, 3) &\equiv -\frac{n_t}{t} + \frac{n_u}{u}.\end{aligned}$$

the relative sign here is just a convention.

Combining the two relation above gives the desired identity

$$n_u = n_s - n_t$$

this is exactly the same form of Jacobi identity for the color factors. The full amplitude can be written like

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_u c_u}{u} + \frac{n_t c_t}{t} \right)$$

For the five point case, an example is

$$c_3 = c_5 - c_8$$

we have the corresponding color factor Jacobi identities like

$$c_3 = c_5 - c_8$$

so we need to have the kinematic analog

$$n_3 = n_5 - n_8$$

The explicit forms of all these factors can be found in the original paper
New relations for gauge theory amplitudes, PRD 78, 085011

Full amplitudes of 5 points are

$$\mathcal{A}_5^{\text{tree}} = g^3 \left(\frac{n_1 c_1}{s_{12} s_{45}} + \frac{n_2 c_2}{s_{23} s_{51}} + \frac{n_3 c_3}{s_{34} s_{12}} + \frac{n_4 c_4}{s_{45} s_{23}} + \frac{n_5 c_5}{s_{51} s_{34}} + \frac{n_6 c_6}{s_{14} s_{25}} \right. \\ \left. + \frac{n_7 c_7}{s_{32} s_{14}} + \frac{n_8 c_8}{s_{25} s_{43}} + \frac{n_9 c_9}{s_{13} s_{25}} + \frac{n_{10} c_{10}}{s_{42} s_{13}} + \frac{n_{11} c_{11}}{s_{51} s_{42}} + \frac{n_{12} c_{12}}{s_{12} s_{35}} \right. \\ \left. + \frac{n_{13} c_{13}}{s_{35} s_{24}} + \frac{n_{14} c_{14}}{s_{14} s_{35}} + \frac{n_{15} c_{15}}{s_{13} s_{45}} \right),$$

This relation had been proved upto 8 points at the time and I think it has been improved to arbitrary points.

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Scattering equations

It has been proposed that there is connection between the scattering data of n massless particles and the n -punctured sphere from a rational map

$$k_{\mu}^a = \frac{1}{2\pi i} \oint_{|z-\sigma_a|} dz \frac{p^{\mu}(z)}{\prod_{b=1}^n (z - \sigma_b)}$$

To describe the n -punctured sphere more properly, we can introduce the Riemann sphere as

$$\mathbb{CP}^1 \cong \mathcal{S}^2 \cong \mathbb{C} \cup \{\infty\}$$

and n -punctured Riemann sphere can be described by $SL(2, \mathbb{C})$ affine coordinates $\sigma_1, \sigma_2, \dots, \sigma_n$, that is to say we have a equivalence relation

$$\{\sigma_1, \sigma_2, \dots, \sigma_n\} \sim \{\psi(\sigma_1), \psi(\sigma_2), \dots, \psi(\sigma_n)\},$$
$$\psi(\sigma) := \frac{\alpha\sigma + \beta}{\gamma\sigma + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma = 1$$

because of the redundancy of $SL(2, \mathbb{C})$, only $n - 3$ of them are independent.

From this map, we can easily obtain the main ingredients of this report

Scattering equations

$$\sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad a \in \{1, \dots, n\}$$

It has been proved that the number of solutions in any dimension is $(n-3)!$, and only $n-3$ of the equations are independent, so we can rewrite the scattering equations as following

$$\sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad a \in \{4, 5, \dots, n\} \quad \text{and} \quad \sigma_1 \rightarrow \infty, \sigma_2 = 0, \sigma_3 = 1$$

- KLT orthogonality is a striking property of the solutions to scattering equations.

Proposition 1

$$\frac{(i, j)}{(i, i)^{\frac{1}{2}} (j, j)^{\frac{1}{2}}} = \delta_{ij}$$

First we need to define the Jacobian matrix associated to the scattering equations

$$\Phi_{ab} \equiv \partial \left(\sum_{c \neq a} \frac{s_{ac}}{\sigma_a - \sigma_c} \right) / (\partial \sigma_b) = \begin{cases} \frac{s_{ab}}{(\sigma_a - \sigma_b)^2}, & a \neq b, \\ -\sum_{c \neq a} \Phi_{ac}, & a = b \end{cases}$$

As mentioned above only $n-3$ of the scattering equations are independent so the matrix Φ has **rank $n-3$** . (This matrix was first encountered in the gravity amplitudes constructed from gauge theory using KLT relation)

Consider a generalization of Φ_{ab}

$$\Psi_{ab, a \neq b} \equiv \frac{s_{ab}}{(\sigma_a - \sigma_b)(\sigma'_a - \sigma'_b)}, \quad \Psi_{aa} \equiv - \sum_{c \neq a} \Psi_{ac}.$$

Proposition 2

$$\text{rank } \Psi(\{\sigma\}, \{\sigma'\}) = \begin{cases} n - 4, \{\sigma\} \neq \{\sigma'\} \\ n - 3, \{\sigma\} = \{\sigma'\} \end{cases}$$

σ and σ' are assumed to be solutions to scattering equation.

Prove of KLT orthogonality

For the purpose of proving KLT orthogonality, we can construct a $n!$ dimension vector for each solution

$$\frac{1}{(\sigma_{\omega(1)} - \sigma_{\omega(2)})(\sigma_{\omega(2)} - \sigma_{\omega(3)}) \cdots (\sigma_{\omega(n)} - \sigma_{\omega(1)})}$$

Not so obvious is the fact that we can fix the position of 3 labels, which we choose 1, $n-1$, n , give rise to the KK relation and BCJ relation.

Now the vectors become $(n-3)!$ dimension, and even after selecting three labels, we still have the freedom of where to put them. Here we only use two choices :

$$(1, \omega(2), \dots, \omega(n-2), n-1, n) \quad \text{and} \quad (1, \omega(2), \dots, \omega(n-2), n, n-1)$$

The corresponding two vectors are

$$V(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_{n-1})(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)},$$

$$U(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_n)(\sigma_n - \sigma_{n-1})(\sigma_{n-1} - \sigma_1)}.$$

In this language, we can construct a bilinear form

$$S[\alpha|\beta] = \prod_{i=2}^{n-2} \left(s_{1,\alpha_i} + \sum_{j=2}^{i-1} \theta(\alpha(j), \alpha(i))_{\beta} s_{\alpha(j),\alpha(i)} \right)$$

where $\alpha, \beta \in S_{n-3}$, $\theta(i, j)_{\beta} = 1$ if the order of i, j is the same in both permutations $\alpha(2, 3, \dots, n-2)$ and $\beta(2, 3, \dots, n-2)$, and 0 otherwise. S is usually called **Momentum Kernel**.

Given any two solutions of scattering equations,

$$\{\sigma_1^{(i)}, \sigma_2^{(i)}, \dots, \sigma_n^{(i)}\} \quad \text{and} \quad \{\sigma_1^{(j)}, \sigma_2^{(j)}, \dots, \sigma_n^{(j)}\}$$

define two vectors, $V(\alpha)^{(i)}$ and $U(\beta)^{(j)}$, i, j are choices of solutions and α, β are the choices of permutations, the number of both is $(n-3)!$.

A natural inner product can be defined as

$$(i, j) := \sum_{\alpha, \beta \in S_{n-3}} V^{(i)}(\alpha) S[\alpha|\beta] U^{(j)}(\beta)$$

Knowing all definitions above, we can proceed to prove KLT orthogonality.

The starting point is to notice that

$$\frac{(i, j)}{(i, i)^{\frac{1}{2}}(j, j)^{\frac{1}{2}}} = \delta_{ij}$$

is clearly invariant under $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Partially fixing both $SL(2, \mathbb{C})$ redundancies with convenient choice $\sigma_{n-1}^{(i)} = \sigma_n^{(j)} = \infty$ and $\sigma_n^{(i)} = \sigma_{n-1}^{(j)} = 1$ and define

$$K_n(\{\sigma\}, \{\sigma'\}) \equiv \sum_{\alpha, \beta \in S_{n-3}} \frac{1}{\sigma_{1, \alpha(2)} \cdots \sigma_{\alpha(n-3), \alpha(n-2)}} S[\alpha|\beta] \frac{1}{\sigma'_{1, \beta(2)} \cdots \sigma'_{\beta(n-3), \beta(n-2)}}$$

The motivation for this definition is that K_n appears in the numerator of KLT orthogonality.

It is also convenient to define an auxiliary co-rank one $(n-2) \times (n-2)$ matrix $\psi^{(n)}$

$$\psi_{ab, a \neq b} = \frac{s_{ab}}{\sigma_{ab} \sigma'_{ab}}, \quad \psi_{aa} = - \sum_{b \neq a} \psi_{ab}, \quad a, b = 1, \dots, n-2$$

It can be proven that any $(n-3) \times (n-3)$ minors of $\psi^{(n)}$ are the same, and we denote such a minor as $\det' \psi^{(n)}$, that is to say, the determinant of the matrix after removing any row and column.

Proposition 3

The two functions defined above are identical up to a sign.

$$K_n(\{\sigma\}, \{\sigma'\}) = (-1)^n \det' \psi^{(n)}$$

The final step is put all pieces together. With the choice $\sigma_{n-1}^{(i)} = \sigma_n^{(j)} = \infty$ and $\sigma_n^{(i)} = \sigma_{n-1}^{(j)} = 1$, we have

$$\frac{(i, j)}{(i, i)^{\frac{1}{2}}(j, j)^{\frac{1}{2}}} = \frac{K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})}{K_n^{\frac{1}{2}}(\{\sigma^{(i)}\}, \{\sigma^{(i)}\})K_n^{\frac{1}{2}}(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})}$$

In addition, one finds that the minor of ψ obtained by removing the first row and column is identical to that of $\Psi(\{\sigma\}, \{\sigma'\})$ after removing rows and columns $\{1, n-1, n\}$. We denote them respectively $|\psi^{(n)}|_1^1$ and $|\Psi|_{1, n-1, n}^{1, n-1, n}$. Then,

$$\begin{aligned} \frac{(i, j)}{(i, i)^{\frac{1}{2}}(j, j)^{\frac{1}{2}}} &= \frac{K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})}{K_n^{\frac{1}{2}}(\{\sigma^{(i)}\}, \{\sigma^{(i)}\})K_n^{\frac{1}{2}}(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})} \\ &= \frac{(-1)^n |\psi^{(n)}|_1^1}{(-1)^n |\psi^{(n)}|_1^{\frac{1}{2}} |\psi^{(n)}|_1^{\frac{1}{2}}} \\ &= \frac{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})|_{1, n-1, n}^{1, n-1, n}}{(|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(i)}\})|_{1, n-1, n}^{1, n-1, n})^{\frac{1}{2}} (|\Psi(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})|_{1, n-1, n}^{1, n-1, n})^{\frac{1}{2}}} \end{aligned}$$

Fianlly, we just need to use Proposition 2.

- If $i = j$, the rank of matrix Ψ is $n - 3$ and the minor is nonzero, we obtain

$$\frac{(i, i)}{(i, i)^{\frac{1}{2}} (i, i)^{\frac{1}{2}}} = \frac{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})|_{1, n-1, n}^{1, n-1, n}}{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})|_{1, n-1, n}^{1, n-1, n}} = 1$$

- If $i \neq j$, the rank of matrix is $n - 4$, so any minor with volume more than $n - 4$ equals 0.

$$|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})|_{1, n-1, n}^{1, n-1, n} = 0 \Rightarrow \frac{(i, j)}{(i, i)^{\frac{1}{2}} (j, j)^{\frac{1}{2}}} = 0$$

Up to now, we conclude the proof of KLT orthogonality.

Attempt to construct S-matrix — Towards CHY

Thanks to the excellent properties of scattering equations, it is very tempting to propose that the solutions to scattering equations should be used to construct scattering amplitudes.

The first two constructed are YM and gravity amplitudes in any dimensions

$$M_n^{\text{YM}}(1, 2, \dots, n) = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) \frac{E_n(\{k, \epsilon, \sigma\})}{\sigma_{12} \dots \sigma_{n1}},$$
$$M_n^{\text{gravity}} = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) E_n(\{k, \epsilon, \sigma\})^2$$

The measure is defined as following

$$\prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) := \sigma_{ij} \sigma_{jk} \sigma_{ki} \prod_{a \neq i, j, k} \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right)$$

The reason we extract 3 indices from delta equation is the fact that only $n - 3$ scattering equations are independent. This from can be proved to be **independent of choice of i, j, k** , therefore permutaion invariant. We also have

$$\sigma_a \rightarrow \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad d\mu_n \rightarrow \prod_{a=1}^n (\gamma\sigma_a + \delta)^{-4} d\mu_n$$

$E_n(\{k, \epsilon, \sigma\})$ itself is permutaion invariant with resprct to σ_a, k_a^μ and ϵ_a^μ . The $SL(2, \mathbb{C})$ invariance of amplitude also constraints the form of $E_n(\{k, \epsilon, \sigma\})$

$$\sigma_a \rightarrow \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad E_n(\{k, \epsilon, \sigma\}) \rightarrow E_n(\{k, \epsilon, \sigma\}) \prod_{a=1}^n (\gamma\sigma_a + \delta)^2$$

The form of measure

It is worth to computed the measure explicitly. Aftering "gauge fixing" the $SL(2, \mathbb{C})$ redundancy, one finds

$$\int \prod_{c \neq p, q, r} d\sigma_c (\sigma_{pq} \sigma_{qr} \sigma_{rp}) (\sigma_{ij} \sigma_{jk} \sigma_{ki}) \prod_{a \neq i, j, k} \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right)$$

The delta functions completely localize all integrals and the answer is evaluating a Jacobian defined above.

$$\Phi_{ab} \equiv \partial \left(\sum_{c \neq a} \frac{s_{ac}}{\sigma_a - \sigma_c} \right) / (\partial \sigma_b) = \begin{cases} \frac{s_{ab}}{(\sigma_a - \sigma_b)^2}, & a \neq b, \\ -\sum_{c \neq a} \Phi_{ac}, & a = b \end{cases}$$

Then, we obtain the measure

$$\sum_{\{\sigma\} \in \text{solutions}} \frac{(\sigma_{pq} \sigma_{qr} \sigma_{rp}) (\sigma_{ij} \sigma_{jk} \sigma_{ki})}{|\Phi|_{pqr}^{ijk}}$$

Always denoted by

$$\det' \Phi := \frac{|\Phi|_{pqr}^{ijk}}{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}$$

$|\Phi|_{pqr}^{ijk}$ means that we need to delete the rows $\{i, j, k\}$ and the columns $\{p, q, r\}$, of course it is free to choose which index refers to row or column (Φ is a symmetric matrix).

The form of $E_n(\{k, \epsilon, \sigma\})$

In order to present the explicit form of $E_n(\{k, \epsilon, \sigma\})$, first define the following $2n \times 2n$ antisymmetric matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

where A, B and C are $n \times n$ matrices, defined as

$$A_{ab} = \begin{cases} \frac{s_{ab}}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases} \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases}$$

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b, \\ -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{\sigma_a - \sigma_c} & a = b. \end{cases}$$

The first important observation is that while the Pfaffian of Ψ is 0, but after removing any rows i, j and columns i, j with $1 \leq i < j \leq n$, the new matrix Ψ_{ij}^{ij} have nonzero Pfaffian and we define the corresponding reduced Pfaffian as

$$\text{Pf}'\Psi := \frac{(-1)^{i+j}}{(\sigma_i - \sigma_j)} \text{Pf}(\Psi_{ij}^{ij})$$

It can be proved that the reduced Pfaffian is invariant under permutation of **particle labels**.

Pfaffian

Pfaffian is defined for antisymmetric matrix, usually in two ways as following



$$\text{Pf}(A)^2 = \det A$$



$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)}$$

Write down the proposal

$$E_n(\{k, \epsilon, \sigma\}) = \text{Pf}' \Psi(k, \epsilon, \sigma)$$

Combine the measure and integrand, we conclude the formula for the tree-level S-matrix of Yang-Mills in any dimension

$$M_n^{\text{YM}}(1, 2, \dots, n) = \frac{1}{\text{vol SL}(2, \mathbb{C})} \int \frac{d^n \sigma}{\sigma_{12} \cdots \sigma_{n1}} \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) \text{Pf}' \Psi$$

And using the KLT construction, we can construct the formula for tree-level S-matrix of gravity as double copy of that of Yang-Mills

$$M_n^{\text{gravity}} = \frac{1}{\text{vol SL}(2, \mathbb{C})} \int d^n \sigma \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) \text{Pf}' \Psi \text{Pf}' \tilde{\Psi}$$

We can also write the amplitude in another form

$$M_n^{\text{YM}} = \sum_{\{\sigma\} \in \text{solutions}} \frac{1}{\sigma_{12} \cdots \sigma_{n1}} \frac{\text{Pf}' \Psi(k, \epsilon, \sigma)}{\det' \Phi}$$

$$M_n^{\text{gravity}} = \sum_{\{\sigma\} \in \text{solutions}} \frac{\det' \Psi(k, \epsilon, \sigma)}{\det' \Phi}$$

where we use the property of Pfaffian $\det' \Psi(k, \epsilon, \sigma) = \text{Pf}' \Psi(k, \epsilon, \sigma) \times \text{Pf}' \Psi(k, \epsilon, \sigma)$.

Consistency check

- Gauge invariance

If we replace the i th polarization vector ϵ_i^μ with momentum k_i^μ , we find that

$$C_{ii} = - \sum_{c \neq i} \frac{\epsilon_i \cdot k_c}{\sigma_i - \sigma_c} \rightarrow - \sum_{c \neq i} \frac{k_i \cdot k_c}{\sigma_i - \sigma_c} = 0$$

It is easy to discover that the i th and $i + n$ th columns become identical, so the determinant and Pfaffian become 0.

Consistency check

- **Gauge invariance**

If we replace the i th polarization vector ϵ_i^μ with momentum k_i^μ , we find that

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It is easy to discover that the i th and $i + n$ th columns become identical, so the determinant and Pfaffian become 0.

- **Soft limit** Using a special property of Pfaffian

$$\text{Pf}(E) = \sum_{q=1}^{2n} (-1)^q e_{pq} \text{Pf}(E_{pq}^{pq})$$

we find the amplitude in the soft limit is

$$A_n \rightarrow \left(\frac{\epsilon_n \cdot k_{n-1}}{k_n \cdot k_{n-1}} + \frac{\epsilon_n \cdot k_1}{k_n \cdot k_1} \right) A_{n-1}$$

CHY form of amplitudes

Both formulas above can be written in this simplest form

$$\mathcal{M}^{(s)} = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta \left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} \right) \left(\frac{\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_3})}{(\sigma_1 - \sigma_2) \dots (\sigma_n - \sigma_1)} \right)^{(2-s)} (\text{Pf}' \Psi)^s$$

with $s = 1$ for Yang-Mills and $s = 2$ for gravity.

Here we would like to consider that the formula above is not only a convenient way to write Yang-Mills and gravity amplitudes, but can be **a definition of S-matrix for spin s particles**. This means that

$s = 0 \quad \rightarrow \quad$ a corresponding scalar theory

In order to get more general case, the gravity amplitudes actually can be modified to the product of two different Pfaffians, each with own choice of polarization vector

$$(\text{Pf}'\Psi(k, \epsilon, \sigma))^2 \rightarrow \text{Pf}'\Psi(k, \epsilon, \sigma) \text{Pf}'\Psi(k, \tilde{\epsilon}, \sigma)$$

actually it gives amplitudes with gravitons coupled to dilatons and B-fields.

For the case $s = 0$, the similar consequence is

$$\left(\frac{\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n})}{(\sigma_{12}) \dots (\sigma_{n1})} \right)^2 \rightarrow \left(\frac{\text{Tr}(T^{a_1} \dots T^{a_n})}{(\sigma_{12}) \dots (\sigma_{n1})} \right) \left(\frac{\text{Tr}(\tilde{T}^{b_1} \dots \tilde{T}^{b_n})}{(\sigma_{12}) \dots (\sigma_{n1})} \right)$$

while the original color group is $U(N)$, the new factors are the product of two different color group $U(N) \times U(\tilde{N})$.

The simplest possibility is the theory with only cubic interaction

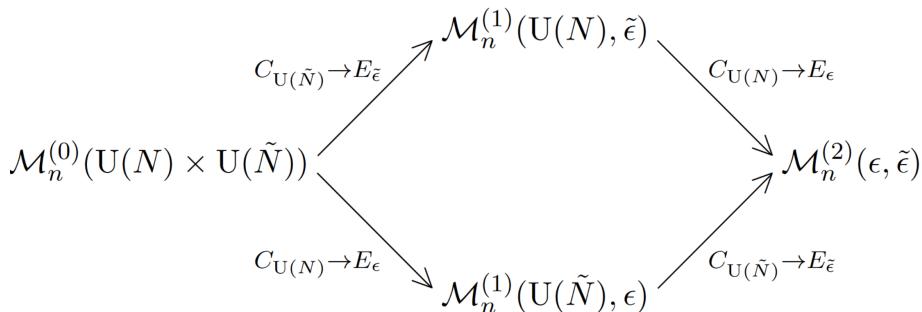
$$-f_{abc}\tilde{f}_{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}$$

All of above leads to the conclusion that the factors

$$\mathcal{C}_{U(N)} \equiv \sum_{\sigma \in S_n/Z_n} \left(\frac{\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n})}{(\sigma_{12}) \dots (\sigma_{n1})} \right) \quad \text{and} \quad E_\epsilon \equiv \text{Pf}' \Psi(\epsilon)$$

are interchangeable and this is a color-Kinematics correspondence which is valid for individual solutions to scattering equations.

The connection of amplitudes between 3 theories can be described by the following diagram



Double partial amplitudes

Because there are two color indices in this scalar theory, so it can be anticipated that the amplitude have double trace decomposition structure

$$\begin{aligned}
 \mathcal{M}_n^{(0)} &= \sum_{\alpha \in S_n/Z_n} \text{Tr}(\tilde{T}^{b_{\alpha(1)}} \tilde{T}^{b_{\alpha(2)}} \dots \tilde{T}^{b_{\alpha(n)}}) M_n^{(0)}(\alpha(1), \alpha(2), \dots, \alpha(n)) \\
 &= \sum_{\alpha, \beta \in S_n/Z_n} \text{Tr}(\tilde{T}^{b_{\alpha(1)}} \tilde{T}^{b_{\alpha(2)}} \dots \tilde{T}^{b_{\alpha(n)}}) \text{Tr}(T^{a_{\beta(1)}} T^{a_{\beta(2)}} \dots T^{a_{\beta(n)}}) \\
 &\quad \times m_n^{(0)}(\alpha|\beta)
 \end{aligned}$$

where the last term $m_n^{(0)}(\alpha|\beta)$ is called **double partial amplitude** and can be read off from the full amplitude

$$\begin{aligned}
 m_n^{(0)}(\alpha|\beta) &= \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \frac{\prod_a' \delta(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}})}{(\sigma_{\alpha(1), \alpha(2)} \cdots \sigma_{\alpha(n), \alpha(1)}) (\sigma_{\beta(1), \beta(2)} \cdots \sigma_{\beta(n), \beta(1)})} \\
 &= \sum_{\{\sigma\} \in \text{solutions}} \frac{1}{\det' \Phi} \frac{1}{(\sigma_{\alpha(1), \alpha(2)} \cdots \sigma_{\alpha(n), \alpha(1)}) (\sigma_{\beta(1), \beta(2)} \cdots \sigma_{\beta(n), \beta(1)})}
 \end{aligned}$$

Likewise the decomposition in the first section, it is more usually to write the amplitudes in terms of colore basis

$$\mathbf{c}_\alpha \equiv \sum_{\mathbf{c}_1, \dots, \mathbf{c}_{n-3}} f_{\mathbf{a}_1 \mathbf{a}_{\alpha(2)} \mathbf{c}_1} \cdots f_{\mathbf{c}_{n-3} \mathbf{a}_{\alpha(n-1)} \mathbf{a}_n}$$

where $\alpha \in S_{n-2}$. The amplitude is

$$\mathcal{M}_n^{(0)} = \sum_{\alpha, \beta \in S_{n-2}} \mathbf{c}_\alpha \tilde{\mathbf{c}}_\beta m_n^{(0)}(\alpha|\beta)$$

Examples

- The simplest example is the 3 point case

$$\mathcal{M}_3^{(0)}(1^{aa',bb',cc'}) = (\sigma_{12}\sigma_{23}\sigma_{31})^2 \frac{f_{abc}f_{a'b'c'}}{(\sigma_{12}\sigma_{23}\sigma_{31})^2} = f_{abc}f_{a'b'c'}$$

It actually gives the correct answer.

- The 4 point case is a little complex. Solving the scattering equations with $\sigma_1 = 0, \sigma_2 = 1, \sigma_3 = \infty$ gives $\sigma_4 = -s_{23}/s_{12}$. Define $s_{12} = s$, $s_{23} = t$, $s_{13} = u$, the color factors are

$$\mathbf{c}_s = \sum_b f_{a_1 a_2 b} f_{b a_3 a_4}, \mathbf{c}_t = \sum_b f_{a_1 a_4 b} f_{b a_3 a_2}, \mathbf{c}_u = \sum_b f_{a_1 a_3 b} f_{b a_2 a_4}$$

Denoting the ordering (1324) as P and computing $\det' \Phi = \frac{s^2}{t} / (\sigma_{12}\sigma_{23}^2\sigma_{31}\sigma_{34}\sigma_{42})$, one gets

$$\begin{aligned}
\mathcal{M}_4^{(0)} &= \mathbf{c}_s \tilde{\mathbf{c}}_s m_4^{(0)}(I; I) + \mathbf{c}_s \tilde{\mathbf{c}}_u m_4^{(0)}(I; P) + \mathbf{c}_u \tilde{\mathbf{c}}_s m_4^{(0)}(P; I) + \mathbf{c}_u \tilde{\mathbf{c}}_u m_4^{(0)}(P; P) \\
&= \mathbf{c}_s \tilde{\mathbf{c}}_s \frac{u}{st} + (\mathbf{c}_s \tilde{\mathbf{c}}_u + \mathbf{c}_u \tilde{\mathbf{c}}_s) \frac{1}{t} + \mathbf{c}_u \tilde{\mathbf{c}}_u \frac{s}{ut} \\
&= -\frac{\mathbf{c}_s \tilde{\mathbf{c}}_s}{s} - \frac{\mathbf{c}_t \tilde{\mathbf{c}}_t}{t} - \frac{\mathbf{c}_u \tilde{\mathbf{c}}_u}{u}
\end{aligned}$$

as expected for a color-dressed cubic theory.

- For the five point, I just give the results of some double partial amplitudes. Denoting the orderings as $I = P_0$, $(13245) = P_1$, $(12435) = P_2$, $(14325) = P_3$, $(13425) = P_4$, $(14235) = P_5$

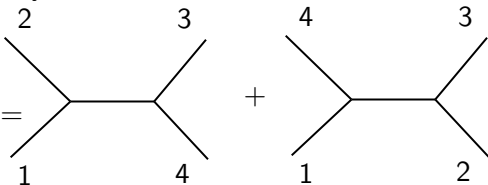
$$\begin{aligned}
m_5^{(0)}(I|I) &= \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}} + \frac{1}{s_{45}s_{12}} + \frac{1}{s_{51}s_{23}}, \\
m_5^{(0)}(I|P_1) &= -\frac{1}{s_{23}} \left(\frac{1}{s_{45}} + \frac{1}{s_{12}} \right), \quad m_5^{(0)}(I|P_2) = -\frac{1}{s_{34}} \left(\frac{1}{s_{51}} + \frac{1}{s_{12}} \right).
\end{aligned}$$

$$m_5^{(0)}(I|P_3) = -\frac{1}{s_{51}} \left(\frac{1}{s_{23}} + \frac{1}{s_{34}} \right), \quad m_5^{(0)}(I|P_4) = -\frac{1}{s_{34}s_{51}},$$

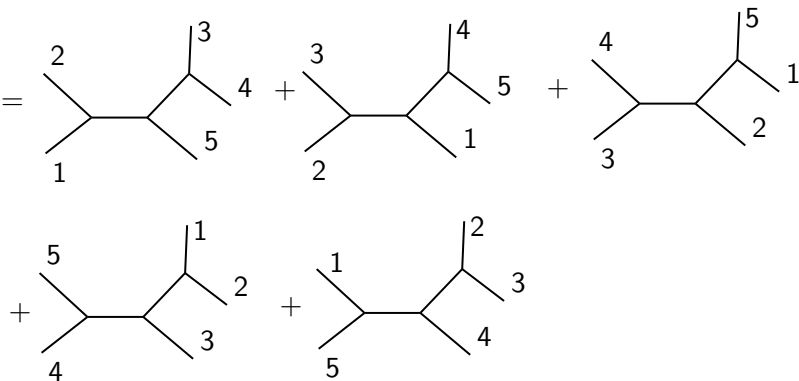
$$m_5^{(0)}(I|P_5) = 0$$

From these examples, it is easy to see that when both permutations in $m_n^{(0)}(\alpha|\beta)$ are the same, then the answer is a sum over all color-ordered trivalent diagrams; When the two permutations are different, it gives a subset of terms of $m_n^{(0)}(\alpha|\alpha)$.

More explicitly,

$$m_4^{(0)}(I, I) =$$


Similarly,

$$m_5(I, I) =$$


The equation shows five diagrams of a 5-pointed star graph (a central horizontal edge with two vertices, each having three other edges) with different labelings of the five outer edges. The diagrams are separated by plus signs.

- Diagram 1: Top-left edge is 2, top-right edge is 3, right edge is 4, bottom-right edge is 5, bottom-left edge is 1.
- Diagram 2: Top-left edge is 3, top-right edge is 4, right edge is 5, bottom-right edge is 1, bottom-left edge is 2.
- Diagram 3: Top-left edge is 4, top-right edge is 5, right edge is 1, bottom-right edge is 2, bottom-left edge is 3.
- Diagram 4: Top-left edge is 5, top-right edge is 1, right edge is 2, bottom-right edge is 3, bottom-left edge is 4.
- Diagram 5: Top-left edge is 1, top-right edge is 2, right edge is 3, bottom-right edge is 4, bottom-left edge is 5.

Trivalent graph expansion

Proposition

The function $m_n^{(0)}(\alpha|\beta)$ computes the sum of all trivalent scalar diagrams which can be regarded as both α color-ordered and β color-ordered.

Trivalent graph expansion

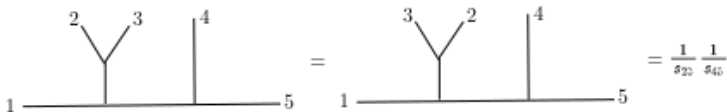
Proposition

The function $m_n^{(0)}(\alpha|\beta)$ computes the sum of all trivalent scalar diagrams which can be regarded as both α color-ordered and β color-ordered.

More explicitly,

$$m_n^{(0)}(\alpha|\beta) = (-1)^{n-3+n_{\text{flip}}(\alpha|\beta)} \sum_{g \in \mathcal{T}(\alpha) \cap \mathcal{T}(\beta)} \prod_{e \in E(g)} \frac{1}{s_e}$$

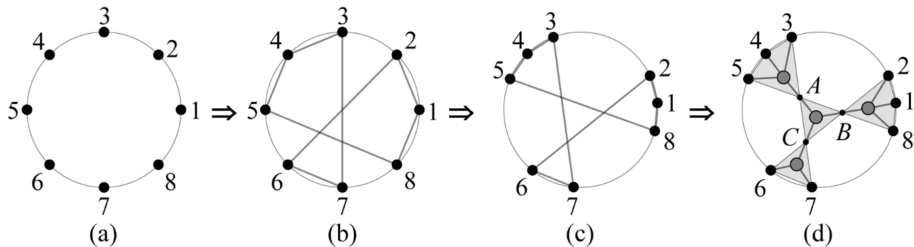
where the $\text{flip}(\alpha|\beta)$ is defined below, $\mathcal{T}(\alpha)$ and $\mathcal{T}(\beta)$ refer to the set of color-ordered diagrams in α and β ordering respectively. To make this expression more clear, see the following diagram



We take $m_8^{(0)}(I; 18543762)$ as an example to explain how to compute it in an systematic way

- First step, draw a disk with n nodes sitting on the boundary in the ordering α , then link the n nodes with a loop of line segments according to the ording β . The line segments from β split the disk into some polygons, like the graph (b). We need to move the external points of every polygon to make them have no common edges, like graph (c).
- Second step, put a point in every polygon, named equivalent vertex, and connect this point to all external points in corresponding area. Lines that connect equivalent vertices in two regions with common vertices are called equivalent propagators. The resulting graph is an equivalent Feynman diagram, as shown in Figure (d).

- Third step, we can read off the corresponding amplitudes from the equivalent Feynman diagram.

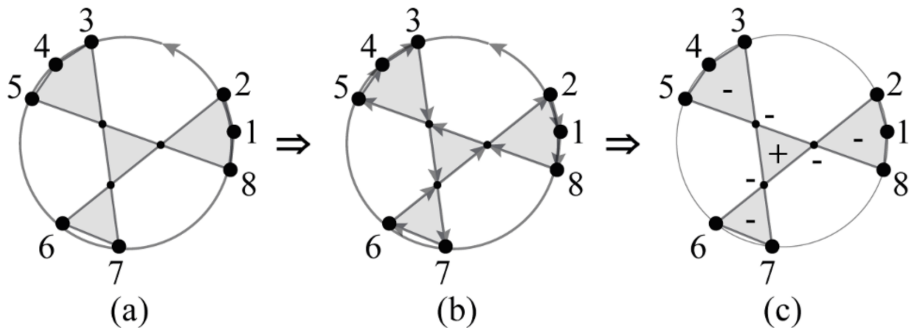


In this example, we can obtain

$$m_8^{(0)}(I|54376218) = (-1)^? \left(\frac{1}{s_{21}} + \frac{1}{s_{18}} \right) \left(\frac{1}{s_{34}} + \frac{1}{s_{45}} \right) \frac{1}{s_{345}s_{812}s_{67}}$$

As for the indefinite sign, there is also a procedure to determine it.

- First step, determine the orientation of the disk by ordering α , and define the loop segments by ordering β , which also determine the orientation of every polygon.
- Second step, (1) each polygon with odd number vertices contributes a plus sign if the orientation is the same as disk, and a minus sign if oppositely; (2) each polygon with even number vertices contribute a minus sign; (3) each intersection point contributes a minus sign.



Relation to KLT matrix

It can be shown that the scalar double partial amplitudes are the same as the inverse of KLT matrix.

$$\begin{aligned}(S_{\text{KLT}}^{-1})_{\beta}^{\alpha} &= (m_{\text{scalar}})_{\beta}^{\alpha} \\ &\equiv m^{(0)}(1, \alpha(2), \dots, \alpha(n-2), n-1, n | 1, \beta(2), \dots, \beta(n-2), n, n-1)\end{aligned}$$

The inverse of KLT matrix have been also discussed in other paper, in which it was related to field theory limit if string disk integrals, so it would be interesting to explore the connection further.

Color-Kinematics Duality again

At the begining, I mentioned that sclar-, gluon- and graviton- amplitudes can be related by simple transformations ($C \rightarrow E$ or $\tilde{C} \rightarrow \tilde{E}$ or both). More explicitly,

$$\mathcal{M}_n^{(0)} = \sum_{I=1}^{(n-3)!} \frac{C(\sigma^{(I)})\tilde{C}(\sigma^{(I)})}{\det'\Phi(\sigma^{(I)})}, \quad \mathcal{M}_n^{(1)} = \sum_{I=1}^{(n-3)!} \frac{C(\sigma^{(I)})\tilde{E}(\sigma^{(I)})}{\det'\Phi(\sigma^{(I)})},$$

$$\mathcal{M}_n^{(2)} = \sum_{I=1}^{(n-3)!} \frac{E(\sigma^{(I)})\tilde{E}(\sigma^{(I)})}{\det'\Phi(\sigma^{(I)})}.$$

If we expand the color factor like

$$C = \sum_{\gamma \in S_{n-2}} \frac{\mathbf{c}_{1\gamma(2)\dots\gamma(n-1)n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1}},$$

It hints the existence of similar form for E . More explicitly, there must functions, denoted as \mathbf{n} , which depends only on kinematic data $\{\epsilon_a^\mu, k_a^\mu\}$

$$E = \text{Pf}'\Psi(\epsilon, k, \sigma) = \sum_{\gamma \in S_{n-2}} \frac{\mathbf{n}_{1\gamma(2)\cdots\gamma(n-1)n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1}}.$$

Now we can unify \mathbf{c} and \mathbf{n} as \mathbf{e} in all three theories, the full amplitude can be written in a unified form

$$\mathcal{M}_n^{(s)} = \sum_{\alpha, \beta \in S_{n-2}} e_\alpha e_\beta m^{(0)}(\alpha|\beta)$$

and the factor satisfies the "BCJ" relation

$$e_{gt} = \pm(e_{gs} - e_{gu})$$

If we contentate on pure Yang-Mills theory, the relation is just the one we list in the first second section.

Conclusion and Outlook