Scattering of massless particles: scalars, gluons and gravitons

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July 23rd 2024

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At tree level, with particles in the adjoint representaion of gauge group SU(N), the amplitude can be decomposed as

$$\mathcal{A}_n^{tree}(1,2,3,\ldots,n) = \sum_{\mathcal{P}(2,3,\ldots,n)} Tr[T^{a_1}T^{a_2}T^{a_3}\ldots T^{a_n}]A_n^{tree}[1,2,3,\ldots,n]$$

here we ommit the coupling constant g, and A_n^{tree} is called tree-level color-ordered partial amplitude. Notice that this basis includes $(n-1)!$ independent amplitudes.

Color-ordered partial amplitudes satisfy a set of well-known properties,

- Cyclic:
$$A_n^{tree}[1, 2, 3, \ldots, n] = A_n^{tree}[2, 3, \ldots, n, 1]$$

- Reflection:
$$A_n^{tree}[1, 2, 3, \ldots, n] = (-1)^n A_n^{tree}[n, \ldots, 3, 2, 1]$$

- "photon" decoupling:
$$\sum_{\sigma \in \text{cyclic}} A_n^{tree}[1, \sigma(2, 3, \ldots, n)] = 0$$

- KK(Kleiss-Kuijf) relation:
$$A_n^{tree}[1, \alpha, n, \beta] = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in OP(\{\alpha\}, \{\beta\}^T)} A_n^{tree}[1, \{\sigma\}_i, n]$$

where the OP means ordered permutaions, that is all permutations of $\{\alpha\}$, $\{\beta\}^T$ that matains the order of individual elements of each set.

For example, a five point amplitude $A_n^{tree}(1, \{2,3\}, 5, \{4\})$, we have

$$A_n^{tree}[1,2,3,5,4] = -A_n^{tree}[1,2,3,4,5] - A_n^{tree}[1,2,4,3,5] - A_n^{tree}[1,4,2,3,5]$$

The other five point relations can be obtained by permuting legs 2,3,4 and using cyclic and reflection properties.

This means that the six amplitudes $A_n^{tree}(1, \mathcal{P}\{2,3,4\}, 5)$ form a basis of remaining five-point partial amplitudes. More generally, for multiplicity n, the KK relation can be used to rewrite any color-ordered partial amplitude in terms of only $(n-2)!$ basis partial amplitudes, where two legs are fixed(usually choose 1 and n).
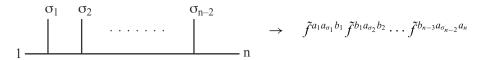
A Similar Structure

It can be proved that tree level amplitudes can also be decomposed like

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma_1} b_1} \tilde{f}^{b_1 a_{\sigma_2} b_2} \cdots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n} \tilde{A}_n(1, \sigma_1, \sigma_2, \ldots, \sigma_{n-2}, n)$$
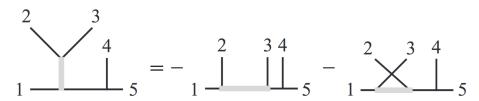
It is easy to realize that the number of basis amplitudes is also $(n-2)!$.

How can we realize the existence of this basis?

We just need to notice the contribution from ladder diagram

$$\rightarrow \quad \tilde{f}^{a_1 a_{\sigma_1} b_1} \tilde{f}^{b_1 a_{\sigma_2} b_2} \dots \tilde{f}^{b_{n-3} a_{\sigma_{n-2}} a_n}$$

And any (three vertex) diagram, by using the Jacobbi identity like

can be transformed to the ladder diagram. It has been proved that $\tilde{A}_n(\dots)$ here is identical to $A_n^{tree}[\dots]$ in the trace basis.

The Color-Kinematics Duality here is not the complete version, but we can sketch some point from lower point case

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Scattering equations

It has benn proposed that there is connection between the scattering data of n massless particles and the n-punctured sphere from a rational map

$$k_\mu^a = \frac{1}{2\pi i} \oint_{|z-\sigma_a|} dz \frac{p^\mu(z)}{\prod_{b=1}^n (z-\sigma_b)}$$

To discribe the n-punctured sphere more properly, we can introduce the Riemann sphere as

$$\mathbb{CP}^1 \cong \mathcal{S}^2 \cong \mathbb{C} \cup \{\infty\}$$

and n-punctured Riemann sphere can be discribed by $SL(2,\mathbb{C})$ affine coordinates $\sigma_1, \sigma_2, \ldots, \sigma_n$, that is to say we have a equvilance relation

$$\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \sim \{\psi(\sigma_1), \psi(\sigma_2), \ldots, \psi(\sigma_n)\},$$
$$\psi(\sigma) := \frac{\alpha\sigma + \beta}{\gamma\sigma + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma = 1$$

because of the redundancy of

From this map, we can easily obtain the main ingredients of this report

Scattering equations

$$\sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \qquad a \in \{1, \ldots, n\}$$

It has been proved that the number of solutions in any dimension is $(n-3)!$, and only n-3 of the equtions are independent, so we can rewrite the scattering equations as following

$$\sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad a \in \{4, 5, \ldots, n\} \quad and \quad \sigma_1 \to \infty, \sigma_2 = 0, \sigma_3 = 1$$

KLT Relation

So-called KLT relation, in the language of field theory, refers to the decomposition of gravity amplitudes to two gauge theory amplitues.

$\mathcal{N}$	Factors	Supergravity
8	$\mathcal{N} = 4SYM \otimes \mathcal{N} = 4SYM$	pure $\mathcal{N} = 8$SG
6	$\mathcal{N} = 4SYM \otimes \mathcal{N} = 2SYM$	pure $\mathcal{N} = 6$SG
5	$\mathcal{N} = 4SYM \otimes \mathcal{N} = 1SYM$	pure $\mathcal{N} = 5$SG
4	$\mathcal{N} = 4SYM \otimes (\mathcal{N} = 0YM + n_\nu scalars)$	$\mathcal{N} = 4$SG,$n_\nu$vector multiplets
4	$\mathcal{N} = 2SYM \otimes \mathcal{N} = 2SYM$	$\mathcal{N} = 4$SG,2 vector multiplets
3	$\mathcal{N} = 2SYM \otimes \mathcal{N} = 1SYM$	$\mathcal{N} = 3$SG,1 vector multiplet
2	$\mathcal{N} = 2SYM \otimes (\mathcal{N} = 0YM + n_\nu scalars)$	$\mathcal{N} = 2$SG,n_ν multiplets+1 vector multiplets
2	$\mathcal{N} = 1SYM \otimes \mathcal{N} = 1SYM$	$\mathcal{N} = 2$SG ,1 hypermultiplet
1	$\mathcal{N} = 1SYM \otimes (\mathcal{N} = 0YM + n_\nu scalars)$	$\mathcal{N} = 1$SG, n_ν vector and 1 chiral multiplet

KLT Orthogonality

• KLT orthogonality is a striking property of the solutions to scattering equations.

Proposition 1

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \delta_{ij}$$

First we need to define the Jacobian matrix associated to the scattering equations

$$\Phi_{ab} \equiv \partial \left(\sum_{c \neq a} \frac{s_{ac}}{\sigma_a - \sigma_c} \right) / (\partial \sigma_b) = \begin{cases} \frac{s_{ab}}{(\sigma_a - \sigma_b)^2}, & a \neq b, \\ -\sum_{c \neq a} \Phi_{ac}, & a = b \end{cases}$$

As mentioned above only n-3 of the scattering equations are independent so the matrix Φ has rank n-3. (This matrix was first encountered in the gravity amplitudes constructed from gauge theory using KLT relation)

Consider a generalization of Φ_{ab}

$$\Psi_{ab,a\neq b} \equiv \frac{s_{ab}}{(\sigma_a - \sigma_b)(\sigma'_a - \sigma'_b)}, \quad \Psi_{aa} \equiv -\sum_{c\neq a} \Psi_{ac}.$$

Proposition 2

$$\operatorname{rank} \Psi(\{\sigma\}, \{\sigma'\}) = \begin{cases} n - 4, \{\sigma\} \neq \{\sigma'\} \\ n - 3, \{\sigma\} = \{\sigma'\} \end{cases}$$

σ and σ' are assmused to be solutions to scattering equation.

Prove of KLT orthogonality

For the purpose of proving KLT orthogonality, we can construct a n! dimension vector for each solution

$$\frac{1}{(\sigma_{\omega(1)} - \sigma_{\omega(2)})(\sigma_{\omega(2)} - \sigma_{\omega(3)}) \cdots (\sigma_{\omega(n)} - \sigma_{\omega(1)})}$$

Not so obvious is the fact that we can fix the positon of 3 labels, which we choose 1,n-1,n, give rise to the KK relation and BCJ relation.

Now the vectors become (n-3)! dimension, and even aftering selecting three lables, we still have the freedom of where to put them. Here we only use two choices :

$$(1, \omega(2), \ldots, \omega(n-2), n-1, n) \quad and \quad (1, \omega(2), \ldots, \omega(n-2), n, n-1)$$

The corresponding two vectors are

$$V(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_{n-1})(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)},$$

$$U(\omega) = \frac{1}{(\sigma_1 - \sigma_{\omega(2)}) \cdots (\sigma_{\omega(n-2)} - \sigma_n)(\sigma_n - \sigma_{n-1})(\sigma_{n-1} - \sigma_1)}.$$

In this language, we can construct a bilinear form

$$S[\alpha|\beta] = \prod_{i=2}^{n-2} \left(s_{1,\alpha_i} + \sum_{j=2}^{i-1} \theta(\alpha(j), \alpha(i))_\beta s_{\alpha(j),\alpha(i)} \right)$$

where $\alpha, \beta \in S_{n-3}, \theta(i,j)_\beta = 1$ if the order of i,j is the same in both permutations $\alpha(2, 3, \ldots, n-2)$ and $\beta(2, 3, \ldots, n-2)$, and 0 otherwise. S is usually called Momentum Kernel.

Given any two solutions of scatering equations,

$$\{\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_n^{(i)}\} \quad and \quad \{\sigma_1^{(j)}, \sigma_2^{(j)}, \ldots, \sigma_n^{(j)}\}$$

define two vectors, $V(\alpha)^{(i)}$ and $U(\beta)^{(j)}$, i,j are choices of solutions and α, β are the choices of permutaions, the number of both is (n-3)!.

A natural inner product can be defined as

$$(i,j) := \sum_{\alpha, \beta \in S_{n-3}} V^{(i)}(\alpha) S[\alpha|\beta] U^{(j)}(\beta)$$

Knowing all definetions above, we can proceed to prove KLT orthogonality.

The starting point is to notice that

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \delta_{ij}$$

is clearly invariant under $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Partially fixing both $SL(2,\mathbb{C})$ redundancies with convenient choice $\sigma_{n-1}^{(i)} = \sigma_n^{(j)} = \infty$ and $\sigma_n^{(i)} = \sigma_{n-1}^{(j)} = 1$ and define

$$K_n(\{\sigma\},\{\sigma'\}) \equiv \sum_{\alpha,\beta \in S_{n-3}} \frac{1}{\sigma_{1,\alpha(2)} \cdots \sigma_{\alpha(n-3),\alpha(n-2)}} S[\alpha|\beta]$$

$$\frac{1}{\sigma'_{1,\beta(2)} \cdots \sigma'_{\beta(n-3),\beta(n-2)}}$$

The motivation for this definition is that K_n appears in the numerator of KLT orthogonality.

It is also convenient to define an auxiliary co-rank one $(n-2) \times (n-2)$ matrix $\psi^{(n)}$

$$\psi_{ab, a \neq b} = \frac{s_{ab}}{\sigma_{ab}\sigma'_{ab}}, \quad \psi_{aa} = -\sum_{b \neq a} \psi_{ab}, \quad a, b = 1, \ldots, n-2$$

It can be proven that any $(n-3) \times (n-3)$ minors of $\psi^{(n)}$ are the same, and we denote such a minor as $\det'\psi^{(n)}$, that is to say, the determinat of the matrix after removing any row and colum.

Proposition 3

The two functions defined above are identical up to a sign.

$$K_n(\{\sigma\}, \{\sigma'\}) = (-1)^n \det'\psi^{(n)}$$

The final step is put all pieces together. With the choice $\sigma_{n-1}^{(i)} = \sigma_n^{(j)} = \infty$ and $\sigma_n^{(i)} = \sigma_{n-1}^{(j)} = 1$, we have

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \frac{K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})}{K_{\bar{n}}^{\frac{1}{2}}(\{\sigma^{(i)}\}, \{\sigma^{(i)}\}) K_{\bar{n}}^{\frac{1}{2}}(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})}$$

In addition, one finds that the minor of ψ obtained by removing the first row and colum is identical to that of $\Psi(\{\sigma\}, \{\sigma'\})$ after removing rows and colums $\{1, n-1, n\}$. We denote them respectively $|\psi^{(n)}|_1^1$ and $|\Psi|_{1,n-1,n}^{1,n-1,n}$. Then,

$$\begin{aligned}
\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} &= \frac{K_n(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})}{K_{\bar{n}}^{\frac{1}{2}}(\{\sigma^{(i)}\}, \{\sigma^{(i)}\}) K_{\bar{n}}^{\frac{1}{2}}(\{\sigma^{(j)}\}, \{\sigma^{(j)}\})} \\
&= \frac{(-1)^n |\psi^{(n)}|_1^1}{(-1)^n |\psi^{(n)}|_1^{1\frac{1}{2}} |\psi^{(n)}|_1^{1\frac{1}{2}}} \\
&= \frac{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})|_{1,n-1,n}^{1,n-1,n}}{(|\Psi(\{\sigma^{(i)}, \sigma^{(i)}\}|_{1,n-1,n}^{1,n-1,n})^{\frac{1}{2}} (|\Psi(\{\sigma^{(j)}\}, \{\sigma^{(j)}\}|_{1,n-1,n}^{1,n-1,n})^{\frac{1}{2}}}
\end{aligned}$$

Fianlly, we just need to use Proposition 2.

- If $i = j$, the rank of matrix Ψ is $n - 3$ and the minor is nonzero, we obtain

$$\frac{(i,i)}{(i,i)^{\frac{1}{2}}(i,i)^{\frac{1}{2}}} = \frac{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})||_{1,n-1,n}^{1,n-1,n}}{|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})||_{1,n-1,n}^{1,n-1,n}} = 1$$

- If $i \neq j$, the rank of matrix is $n - 4$, so any minor with volume more than $n - 4$ equals 0.

$$|\Psi(\{\sigma^{(i)}\}, \{\sigma^{(j)}\})||_{1,n-1,n}^{1,n-1,n} = 0 \Rightarrow \frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = 0$$

Up to now, we conclude the proof of KLT orthogonality.

Attempt to construct S-matrix — Towards CHY

Thanks to the excellent properties of scattering equations, it is very tempting to propose that the solutions to scattering equations should be used to construct scattering amplitudes.

The first two constructed are YM and gravity amplitudes in any dimensions

$$M_n^{\text{YM}}(1, 2, \ldots, n) = \int \frac{d^n\sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a {}'\delta\bigg(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\bigg) \frac{E_n(\{k, \epsilon, \sigma\})}{\sigma_{12}\ldots\sigma_{n1}},$$

$$M_n^{\text{gravity}} = \int \frac{d^n\sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a {}'\delta\bigg(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\bigg) E_n(\{k, \epsilon, \sigma\})^2$$

The measure is defined as following

$$\prod_a {}'\delta\left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\right) := \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a \neq i,j,k} \delta\left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\right)$$

The reason we extract 3 indices from delta equation is the fact that only $n-3$ scattering equations are independent. This from can be proved to be independent of choice of i, j, k, therefore permutaion invariant. We also have

$$\sigma_a \to \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad d\mu_n \to \prod_{a=1}^{n}(\gamma\sigma_a + \delta)^{-4}d\mu_n$$

$E_n(\{k, \epsilon, \sigma\})$ itself is permutaion invariant with resprct to σ_a, k_a^μ and ϵ_a^μ. The $SL(2, \mathbb{C})$ invariance of amplitude also constraints the form of $E_n(\{k, \epsilon, \sigma\})$

$$\sigma_a \to \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} : \quad E_n(\{k, \epsilon, \sigma\}) \to E_n(\{k, \epsilon, \sigma\})\prod_{a=1}^{n}(\gamma\sigma_a + \delta)^2$$

The form of measure

It is worth to computed the measure explictly.Aftering "gauge fixing" the $SL(2, \mathbb{C})$ redundancy, one finds

$$\int \prod_{c \neq p,q,r} d\sigma_c (\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \prod_{a \neq i,j,k} \delta\left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\right)$$

The delta functions completely localize all integrals and the answer is evaluating a Jacobian defined above.

$$\Phi_{ab} \equiv \partial \left(\sum_{c \neq a} \frac{s_{ac}}{\sigma_a - \sigma_c}\right) / (\partial \sigma_b) = \begin{cases} \frac{s_{ab}}{(\sigma_a - \sigma_b)^2}, & a \neq b, \\ -\sum_{c \neq a} \Phi_{ac}, & a = b \end{cases}$$

Then, we obtain the measure

$$\boxed{\sum_{\{\sigma\} \in \text{solutions}} \frac{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{|\Phi|_{pqr}^{ijk}}}$$

Always denoted by

$$\det{}'\Phi := \frac{|\Phi|_{pqr}^{ijk}}{(\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}$$

$|\Phi|_{pqr}^{ijk}$ means that we need to delete the rows $\{i, j, k\}$ and the colums $\{p, q, r\}$, of course it is free to choose which inedex refers to row or colum (Φ is a symmetric matrix).

The form of $E_n(\{k, \epsilon, \sigma\})$

In order to present the explict form of $E_n(\{k, \epsilon, \sigma\})$, first define the following $2n \times 2n$ antisymmetric matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

where A, B and C are $n \times n$ matrices, defined as

$$A_{ab} = \begin{cases} \frac{s_{ab}}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases} \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases}$$

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b, \\ -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{\sigma_a - \sigma_c} & a \neq b. \end{cases}$$

The first important observation is that while the Pfaffian of Ψ is 0, but after removing any rows i, j and colums i, j with $1 \leq i < j \leq n$, the new matrix Ψ_{ij}^{ij} have nonzero Pfaffian and we define the corresponding reduced Pfaffian as

$$\mathrm{Pf}'\Psi := \frac{(-1)^{i+j}}{(\sigma_i - \sigma_j)}\mathrm{Pf}(\Psi_{ij}^{ij})$$

It can be proved that the reduced Pfaffian is invariant under permutaion of particle labels.

Pfaffian

Pfaffian is defined for antisymmetric matrix, usually in two ways as following

- $$\mathrm{Pf}(A)^2 = \det A$$

- $$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1), \sigma(2i)}$$

Write down the proposal

$$E_n(\{k, \epsilon, \sigma\}) = \mathrm{Pf}'\Psi(k, \epsilon, \sigma)$$

Combine the measure and integrand, we conclude the formula for the tree-level S-matrix of Yang-Mills in any dimension

$$M_n^{\mathrm{YM}}(1, 2, \ldots, n) = \frac{1}{\mathsf{vol}\,\mathsf{SL}(2, \mathbb{C})} \int \frac{d^n\sigma}{\sigma_{12} \cdots \sigma_{n1}} \prod_a{}'\delta\left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\right) \mathrm{Pf}'\Psi$$

And using the KLT constrution, we can construct the formula for tree-level S-matrix of gravity as double copy of that of Yang-Mills

$$M_n^{\mathrm{gravity}} = \frac{1}{\mathsf{vol}\,\mathsf{SL}(2, \mathbb{C})} \int d^n\sigma \prod_a{}'\delta\left(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}}\right) \mathrm{Pf}'\Psi\mathrm{Pf}'\tilde{\Psi}$$

We can also write the amplitude in another form

$$M_n^{\text{YM}} = \sum_{\{\sigma\} \in \text{solutions}} \frac{1}{\sigma_{12} \cdots \sigma_{n1}} \frac{\text{Pf}'\Psi(k, \epsilon, \sigma)}{\det'\Phi}$$

$$M_n^{\text{gravity}} = \sum_{\{\sigma\} \in \text{solutions}} \frac{\det'\Psi(k, \epsilon, \sigma)}{\det'\Phi}$$

where we use the property of Pfaffian $\det'\Psi(k, \epsilon, \sigma) = \text{Pf}'\Psi(k, \epsilon, \sigma)$
$\times \text{Pf}'\Psi(k, \epsilon, \sigma)$.

Consistency check

- ## Gauge invariance
  If we replce the ith polariazation vector ϵ_i^μ with momentum k_i^μ, we find that

  $$C_{ii} = -\sum_{c \neq i} \frac{\epsilon_i \cdot k_c}{\sigma_i - \sigma_c} \rightarrow -\sum_{c \neq i} \frac{k_i \cdot k_c}{\sigma_i - \sigma_c} = 0$$

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  It is easy to discover that the ith and $i + n$th colums become identical, so the determinat and Pfaffian become 0.

- **Soft limit** Using a special property of Pfaffian

  $$\mathrm{Pf}(E) = \sum_{q=1}^{2n} (-1)^q e_{pq} \mathrm{Pf}(E_{pq}^{pq})$$

  we find the amplitude in the soft limit is

  $$A_n \to \left(\frac{\epsilon_n \cdot k_{n-1}}{k_n \cdot k_{n-1}} + \frac{\epsilon_n \cdot k_1}{k_n \cdot k_!} \right) A_{n-1}$$

CHY form of amplitudes

Both formulas above can be written in this simplest form