On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

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Abstract

This paper mainly show the computation for scattering amplitudes in a kind of (De)Constructed gauge theory, by using the so called on shell method. As we have known, under the conventional quantum field theory frame, Feynman proposed a brilliant method – Feynman diagrams, to help us perturbatively compute scattering amplitude by a diagrammatic method. However, this method faces many challenges during the improvements of physical theory and complexity of construction for model building, there are many amplitudes hard to compute by hand or even impossible to compute. Hence, it is quite necessary to introduce a new method.

Contents

1 Introduction

As we have known, under the conventional quantum field theory frame, Richard Feynman proposed a brilliant method – Feynman diagrams, to help us perturbatively compute scattering amplitude by a diagrammatic method. However, this method faces many challenges during the improvements of physical theory and complexity of construction for model building, there are many amplitudes hard to compute by hand or even impossible to compute. In gauge theory, there are huge number of gauge redundancies, making the computation quite complicated. We need to address with this kind of unphysical degree of freedom, otherwise we can not obtain the correct physical quantities. Also, we have to address with kinematic factor and color factor simultaneously in the nonabelian gauge theory. And it has been known that the amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory obtain the symmetry – dual superconformal symmetry [1], which is not reflected in conventional Feynman diagram method.

Hence, all of these factors impetus physicists to propose something new, then the BCFW recursion relation is the first product as a new method to compute amplitudes . Historically, the original recursion relation is for gluon scattering amplitudes, coming from Britto, Cachazo and Feng [2], and it can be seen as the first breakthrough for modern amplitude method. They explored the analytic properties of amplitudes when extended into complex momentum space. In particular, they considered deforming two external momenta by a complex parameter z, in such a way that the on-shell conditions and momentum conservation are preserved. Through this deformation, they observed that tree-level amplitudes exhibit simple pole structures in the complex z-plane, corresponding to internal propagators going on-shell. This analytic structure allowed them to derive a recursion relation that expresses an n-point amplitude in terms of lower-point amplitudes. Edward Witten subsequently noticed that his Twistor String Theory [3], which reveals the hidden symmetry and geometry structure of amplitude, implied that the scattering amplitudes has stronger analytic property. Their collaboration led to a general formulation of the recursion relations, now known as the BCFW recursion relations [4], named after Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. In particular, Witten helped clarify the large – z behavior of the amplitudes under complex momentum shifts — a crucial condition ensuring the validity of the recursion.

The starting point comes from the precise cancellation in scattering for longitudinal modes of massive spin-2 Kaluza-Klein(KK) states. While individual contributions grow as $\mathcal{O}(s^5)$, $\mathcal{O}(s^4)$, $\mathcal{O}(s^3)$ and $\mathcal{O}(s^2)$, it has been proved that these contributions are cancelled with each other in a quite intricate way [5], and the final results only grow as $\mathcal{O}(s)$. But it is quite difficult to compute this kind of scattering amplitudes, so if we can obtain some clues for this KK scattering amplitudes from other aspects, it may help us to understand this cancellation in another way. This paper is motivated by a kind of (De)constructed gauge theory, proposed by Nima, Cohen and Georgi [6]. They constructed a renormalizable, asymptotically free, four dimension gauge theories that dynamically generate a fifth dimension. In this paper, the authors proposed

that the " Condensed " theory actually discretized a five dimension gauge theory with gauge group $SU(m)$. After higgsing, the Kaluza-Klein spectrum for S^1 compactification appears. It encourages to compute scattering amplitudes in this model. This paper mainly contributes to the computation for scattering amplitudes in the simplest 2-site model by utilizing BCFW recursion relation and other related method, such as color-ordered amplitudes, spinor-helicity formalism, etc.

2 Review of BCFW recursion relation and others

Traditionally, one relies on Feynman diagrams to calculate scattering amplitudes. Feynman diagrams provide a clear picture of physics and a systematic procedure of calculations. They are in textbooks and widely used. But Feynman diagrams are not efficient in complicated calculations for high energy physics. Increasing the number of particles in a scattering, the number of Feynman diagrams increase exponentially. If gauge fields are involved, one easily encounters thousands of diagrams. For example, for pure gluon case, the number of Feynman diagrams for n-gluons at tree-level is given by

n=	4	5	6	7	8	9	10
	4	25	220	2485	34300	559405	10525900

(These numbers are counted with the inclusion of 4 point interaction.)

Not only with huge number of diagrams, the expression for a single Feynman diagram can also be very complicated. For example, the three-graviton vertex has almost 100 terms. It is almost impossible to calculate scattering amplitudes of gravitons directly from Feynman diagrams. For gauge theories, single Feynman diagram usually depends on the gauge. Many terms cancel with each other at the end of process of calculation. In practice, one does not even know where to start most times.

BCFW are devised to solve some of these problems. So in the following part, I will give a systematic introduction to BCFW recursion relation and other necessary tools. This section is mainly based on the excellent review by Elvang and Huang [7].

2.1 Spinor-Helicity Formalisim for Massless Particles

2.1.1 Brief introduction of spinor-helicity formalism

The spinor-helicity formalism just told us that a light-like Lorentz 4-vector can be decomposed to the product of two Weyl spinor. It is quite natural to see it from the representation of Lorentz group. A Lorentz 4-vector lives in $(\frac{1}{2}, \frac{1}{2})$ representation, which can be decomposed to $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$. We have known that the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ correspond to left handed Weyl spinor and right handed Weyl spinor respectively.

Given a null momentum p_μ in four dimension spacetime, we can define a 2×2 matrix by sigma matrix

$$p_{\alpha\dot\alpha} = p_\mu \sigma^\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}$$

note that $\det p_{\alpha\dot\alpha} = 0$ for massless particles, so it is always possible find two Weyl spinor(two components quantity) satsfying the following equation

$$p_\mu \sigma^\mu = p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} = |\lambda\rangle[\lambda| \tag{2.1}$$

and similarly we can obtain define

$$p_\mu \bar\sigma^\mu = p^{\dot\alpha\alpha} = \tilde\lambda^{\dot\alpha} \lambda^\alpha = |\lambda]\langle\lambda|, \tag{2.2}$$

For general complex momenta, the λ_α and $\tilde\lambda_{\dot\alpha}$ are independent two dimensional complex vectors. For real momenta, the matrix is Hermitian and so we have $\tilde\lambda_{\dot\alpha} = (\pm)(\lambda_\alpha)^*$.

Thet satisfy the following Weyl equation

$$p_{\alpha\dot\alpha}|p]^{\dot\alpha} = 0, \quad [p|_{\dot\alpha}p^{\dot\alpha\alpha} = 0, \quad p^{\dot\alpha\alpha}|p\rangle_\alpha, \quad \langle p|^\alpha p_{\alpha\dot\alpha} = 0 \tag{2.3}$$

and we can use two-dimension antisymmetric tensor to raise or lower the indices

$$[p|_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}|p]^{\dot\beta}, \qquad \langle p|^\alpha = \varepsilon^{\alpha\beta}|p\rangle_\beta \tag{2.4}$$

Then,

> The angle and square spinors are the core of **spinor-helicity formalism**.

Here, it is also necessary to introduce the **angle spinor bracket** $\langle pq \rangle$ and **square spinor bracket** $[pq]$, it is the key ingredient for writing amplitudes in terms of spinor-helicity variable.

$$\langle pq \rangle = \langle p|^\alpha |q\rangle_\alpha, \qquad [pq] = [p|_{\dot\alpha}|q]^{\dot\alpha}. \tag{2.5}$$

Since the indices are raised and lowed by antisymmetric tensor, so the brackets are antisymmetric:

$$\langle pq \rangle = -\langle qp \rangle, \qquad [pq] = -[qp]. \tag{2.6}$$

There are no $\langle pq]$ brackets, because the indices cannot contract with each other to form a Lorentz scalar.

It is very easy to derive the following important relation:

$$\langle pq \rangle [pq] = 2p \cdot q = (p + q)^2 \tag{2.7}$$

by using (2.1) and

$$\mathrm{Tr}(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu}.$$

Here, we list other identities without any provement

$$[k|\gamma^\mu|p\rangle = \langle p|\gamma^\mu|k], \tag{2.8}$$

$$[k|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|k\rangle \qquad \text{(for real moemnta)}. \tag{2.9}$$

and **Fierz identity**

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24] \tag{2.10}$$

will be used in several times. In amplitude calculations, **momentum conservation** is imposed on n particles as $\sum_{i=1}^{n} p_i^\mu = 0$(here we consider all particles ingoing). Translating by spinor-helicity variable, it becomes

$$\sum_{i=1}^{n} |i\rangle[i| = 0, \quad \text{i.e.} \quad \sum_{i=1}^{n} \langle qi\rangle[ik] = 0, \tag{2.11}$$

here q and k are arbitrary light-like vectors.

We end this subsection by introducing one more identity: **Schouten Identity**. It comes from a rather trivial fact: there are no three independent 2-dimensional vectors. So if we have three 2 components angle spinors $|i\rangle$, $|j\rangle$ and $|k\rangle$, we can write one of them as a linear combination of two others

$$|k\rangle = a|i\rangle + b|j\rangle, \qquad \text{for complex a and b.} \tag{2.12}$$

One can contract a $|i\rangle$ and a $|b\rangle$ with the both sides, then a,b can be solved. (2.12) can be cast to the form

$$|i\rangle\langle jk\rangle + |k\rangle\langle ij\rangle + |j\rangle\langle ki\rangle = 0, \tag{2.13}$$

This is Schouten identity and often written with a fourth spinor $\langle r|$

$$\langle ri\rangle[jk] + \langle rk\rangle\langle ij\rangle + \langle rk\rangle\langle ki\rangle = 0. \tag{2.14}$$

We have a similar Schouten identity holding for square spinors

$$[ri][jk] + [rk][ij] + [rj][ki] = 0. \tag{2.15}$$

There is also a important result can be obtained, the **3-particle soecial kinematics**. If we have three light-like vectors satisfying momentum conservation $p_1^\mu + p_2^\mu + p_3^\mu = 0$. Then

$$\langle 12\rangle\langle 21\rangle = 2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_3^2 = 0, \tag{2.16}$$

so either $\langle 12 \rangle$ or $[12]$ equals to 0. If we suppose $\langle 12 \rangle \neq 0$, then from $\langle 12 \rangle [23] = \langle 1|p_2|3] = -\langle 1|p_1 + p_3|3] = 0$, we can conclude that $[23] = 0$. Similarly, we can also obtain $[31] = 0$. Thus, $[12] = [23] = [31] = 0$, which means that the three square spinors are proportional with each other

$$|1] \propto |2] \propto |3] \tag{2.17}$$

or another possibility

$$|1\rangle \propto |2\rangle \propto |3\rangle. \tag{2.18}$$

As a consequence,

1. A non-vanishing on-shell 3-particle amplitude depends only on square brackets or angle brackets.

2. Since for real momenta, angle brackets are complex conjugated with square brackets, so *on-shell 3 point amplitudes are only meanful for complex momenta*(unless it is a constant, like ϕ^3 theory).

2.1.2 Yang-Mills and Color-ordering

Let us consider the Yang-Mills lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.19}$$

with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}} [A_\mu, A_\nu]$, and $A_\mu = A_\mu^a T^a$. Gauge fields belong to adjoint representation, so the index a runs over $1, 2 \cdots N^2 - 1$ in $SU(N)$ case. The generators are normalized like $\mathrm{Tr}[T^a T^b] = \delta^{ab}$ and $[T^a, T^b] = i\tilde{f}^{abc} T^{c}$[1].

The amplitude-friendly gauge choice is *Gervais-Neveu gauge* with gauge fixing term $\mathcal{L}_{gf} = -\frac{1}{2}\mathrm{Tr}(H_\mu^\mu) = 0$, here $H_{\mu\nu} = \partial_\mu A_n u - \frac{ig}{\sqrt{2}} A_\mu A_\nu$. After gauge fixing, the lagrangian becomes

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu - i\sqrt{2}g \partial^\mu A^\nu A_\nu A_\mu + \frac{g^2}{4} A^\mu A^\nu A_\mu A_\nu \right) \tag{2.20}$$

The 3- and 4-gluon vertices involve $\tilde{f}^{abc}$ and $\tilde{f}^{abi}\tilde{f}^{icd}$+permutations, respectively, each dressed up with kinematic factors. The amplitudes constructed from these rules can be organized into different group theory structures. For example, the color factors of the s-, t-, and u-channel diagram of the 4-gluon tree amplitude are

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4} \tag{2.21}$$

and the four point interaction just gives a sum of contributions from c_s, c_t and c_u. And because

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[1] In the usual QFT textbook, $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$ and $[T^a, T^b] = if^{abc} T^c$, with $\tilde{f}^{abc} = \sqrt{2} f^{abc}$ are common choice.

of the Jacobi identity, we have

$$c_s = c_t + c_u. \tag{2.22}$$

And the color factor can be written by the trace of product of generators

$$i\tilde{f}^{abc} = \text{Tr}([T^a, T^b]T^c), \tag{2.23}$$

where T^a are generators of fundamental representation. Moreover, in $SU(N)$, we have a Fierz identity

$$\sum_a T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{2.24}$$

This identity is easier understood as matrix form like

$$\text{Tr}\{T^a A\}\text{Tr}\{T^a B\} = \text{Tr}\{AB\} - \frac{1}{N}\{A\}\text{Tr}\{B\}, \tag{2.25}$$

and

$$\text{Tr}\{AT^a BT^a\} = \text{Tr}\{A\}\text{Tr}\{B\} - \frac{1}{N}\text{Tr}\{AB\}. \tag{2.26}$$

Then it can be used to simplify the calculation.

For example, the 4 gluon s-channel gives us

$$\tilde{f}^{a_1 a_2 b}\tilde{f}^{ba_3 a_4} = \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \text{Tr}(T^{a_2}T^{a_1}T^{a_3}T^{a_4}) - \text{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) + \text{Tr}(T^{a_2}T^{a_1}T^{a_4}T^{a_3}). \tag{2.27}$$

Similarly, three other diagrams can also be written in terms of single trace. Therefore, the full 4-point amplitude can be rewritten like

$$\mathcal{A}_{4,\text{tree}} = g^2(A_4[1234]\text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{perms of }(234)) \tag{2.28}$$

here the subamplitudes $A_4[1234]$, $A_4[1243]$, etc. are called **color-ordereed ampltitudes**. This concept can be easily generalized to tree-level n-point case

$$\mathcal{A}_{n,\text{tree}} = g^{n-2}\sum_\sigma A_n[1, \sigma(2, 3\cdots n)]\text{Tr}(T^{a_1}T^{\sigma(a_2\cdots}T^{a_n)}) \tag{2.29}$$

where the sun is taken over the $(n-1)!$ trace basis (considering the cyclic property of trace). Actually, the number of independent basis can be reduced to $(n-3)!$, called Del Duca-Dixon-Maltoni (DDM) color decomposition [8]. But it has no tight relation with this paper, so here we do not offer more detailed explanation for it.

The color-ordered amplitude $A_n[1, 2...n]$ is calculated in terms of diagrams with no lines crossing(planar diagrams) and the ordering of the external lines fixed as given 1, 2, 3,..., n. Here, we directly give the final result for 3-point color-ordered amplitudes without any intermediate calculating process. And in this full paper, we mainly consider the helicity amplitudes which will be explained later, so we need to clarify the helicity configuration.

For 3-point, there are only two non-vanishing configurations

$$A_3[1^-, 2^-, 3^+] = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \tag{2.30}$$

and

$$A_3[1^+, 2^+, 3^-] = \frac{[12]^3}{[23][31]}. \tag{2.31}$$

It has been known that there is a compact formula for n-point gluon color-ordered amplitudes — **Parke - Talyor Formula**

$$A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \tag{2.32}$$

We will prove this formula in the next subsection.

The color-ordered amplitudes have a number of properties

1. *Cyclic*: It follows from the cyclic property for trace that $A_n[12 \cdots n] = A_n[2n \cdots 1]$

2. *Reflection*: $A_n[12 \cdots n] = (-1)^n A_{[}n \cdots 21]$

3. The $U(1)$ *decoupling identity*:

$$A_n[123 \cdots n] + A_n[213 \cdots n] + A_n[231 \cdots n] + \cdots + A_n[23 \cdots 1n] = 0 \tag{2.33}$$

The trace basis (2.29) is overcomplete implies that there are further linear relations among these subamplitudes, which are called **Kleiss - Kuiff(KK) relations**

$$A_n[1, \{\alpha\}, n, \{\beta\}] = (-1)^{|\beta|} \sum_{\sigma \in \text{OP}(\{\alpha\}, \{\beta^T\})} A_n[1, \sigma, n] \tag{2.34}$$

where β^T denotes the reverse ordering of the labels $\{\beta\}$ and the sum is over ordered permutations "OP", means permutations of the labels in the joined set $\{\alpha\} \cup \{\beta^T\}$ in which the ordering within $\{\alpha\}$ and $\{\beta\}^T$ is preserved.

Consider a 5-point case as an example. Taking the LHS to be $A_5[1, \{2\}, 5, \{3, 4\}]$, we have $\{\alpha\} \cup \{\beta^T\} = \{2\} \cup \{4, 3\}$, so the ordered permutations σ refers to $\{243\}, \{423\}, \{432\}$. Thus the KK relation reads

$$A_5[12534] = A_5[12435] + A_5[14235] + A_5[114325]. \tag{2.35}$$

The KK relations combine with cyclic, reflection, and $U(1)$ decoupling identities reduce the number of independent basis to $(n-2)!$. However, there are further linear relations called **BCJ relations** – named after Bern, Carrasco and Johansson, which further reduce the number to $(n-3)!$.

Although I will not show the derivation of BCJ relations, I would like to give some examples

$$s_{14}A_4[1234] - s_{13}A_4[1243] = 0, \tag{2.36}$$

$$s_{12}A_5[21345]s_{23}A_5[13245] - (s_{23} + s_{24})A_5[13425] = 0. \tag{2.37}$$

2.2 Little group scalling

The rich structure of scattering amplitudes can be originated from a fundamental question — "What is a particle?" — which relates directly to Wigner's Little group notation, characterizing how particles transform under Lorentz symmetry. Following this thought, we think particle as a unitary representation of Poincaré group, which is the symmetry group of our spacetime. We use two labels to denote a one-particle state, one is the momentum p^μ, another is σ representing other labels the particle ca carry.

We start from a one-particle state $|k; \sigma\rangle$ with reference momentum k. Another momentum p is related by the lorentz transformation $L(p; k)$, $p = L(p; k)k$. Obviously, this Lorentz transformation is not unique because there exist Lorentz transformations keeping p invariant, called "Little group transformation". We assume that there exist a unitary representation for the Lorentz group, the operator in Hillbert space are denoted like $U(\Lambda)$.

Then we simply *define* the one-particle state with momentum p as

$$|p; \sigma\rangle = U(L(p; k))|k; \sigma\rangle. \tag{2.38}$$

Having made this definition, we can compute how the state transformed under a general Lorentz transformation

$$U(\Lambda)|p, \sigma\rangle = U(\Lambda)U(L(p; k))|k, \sigma\rangle = U(L(\Lambda p; k))U(L^{-1}(\Lambda p; k)\Lambda L(p; k))|k, \sigma\rangle. \tag{2.39}$$

here we use the fact $U(\Lambda_1 \Lambda_2) = U(\Lambda_1)U(\Lambda_2)$. Notice that $W(\Lambda, p, k)$ is not a general transformation but keep the momentum k invariant $Wk = k$. This subgroup of Lorentz group is called little group. Thus we must have

$$U(W(\Lambda, p, k))|k; \sigma\rangle = D_{\sigma\sigma'}(W(\Lambda, p, k))|k, \sigma'\rangle, \tag{2.40}$$

where $D_{\sigma\sigma'}$ is the representation for little group. Then, p should transformed like

$$U(\Lambda)|p, \sigma\rangle = D_{\sigma\sigma'}(W(\Lambda, p, k))|\Lambda p, \sigma'\rangle \tag{2.41}$$

The scattering amplitudes for n particle should keep invariant under Poincaré transforma-

tion – translation invariance and Lorentz invariance

$$\mathcal{M}(p_a, \sigma_a) = \delta^D(p_1^\mu + \cdots + p_n^\mu)\mathcal{M}(p_a, \sigma_a) \tag{2.42}$$

$$\mathcal{M}^\Lambda(p_a, \sigma_a) = \prod_a (D_{\sigma\sigma'})\mathcal{M}((\Lambda p)_a, \sigma_a') \tag{2.43}$$

In D spacetime dimensions, the little group for massive particles is $SO(D-1)$. For massless particles the little group is the the group of Euclidean symmetries in (D - 2) dimensions, which is $SO(D-2)$. In this article, we mainly discuss massless particles scattering in 4 dimension spacetime, so the little group is $SO(2) \simeq U(1)$.

Then from the definition of spinor-helicity variable(2.1), we can notice that there is an ambiguity here. The momentum is invariant under the following redefinition

$$\lambda \to t^{-1}\lambda, \qquad \tilde{\lambda} \to t\tilde{\lambda}, \qquad t \in \mathbb{C} \tag{2.44}$$

same for

$$|\lambda\rangle \to t^{-1}|\lambda\rangle, \qquad |\lambda] \to t|\lambda] \tag{2.45}$$

. This scale perfectly matches the $U(1)$ little group transformation, so we can identify this rescale as the little group transformation. The scattering amplitudes should transform covariantly under little group scaling like (2.43), so

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \ldots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \ldots) = t_i^{2h_i}\mathcal{A}_n \tag{2.46}$$

where h_i refers to the helicity of particles. As an example, consider the QED amplitude, $A_3(f^-\bar{f}^+\gamma^-) = e\frac{\langle 13\rangle^2}{\langle 12\rangle}$. For the negative helicity photon (particle 3), we obtain $t_3^{-2} = t_3^{2(-1)}$. We will see that little group scaling plays a significant role in 3-particle amplitudes.

3-particle amplitudes

By 3-particle special kinematics, we have known that on-shell 3-point amplitudes depends on either square brackets or angle brackets. We suppose that it only depends on angle brackets. We can write down the general ansatz

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}, \tag{2.47}$$

where c is just a constant. Little group scaling (2.46) tell us that

$$t_1^{2h_1}A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = ct_1^{-x_{12}}t_1^{-x_{13}}\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}. \tag{2.48}$$

We can obtain

$$2h_1 = -x_{12} - x_{13} \tag{2.49}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \qquad 2h_3 = -x_{13} - x_{23}. \tag{2.50}$$

Then all index can be solved from this system of equations, so that

$$A_3^{h_1 h_2 h_3} = c\langle 12\rangle^{h_3 - h_1 - h_2}\langle 31\rangle^{h_2 - h_1 - h_3}\langle 23\rangle^{h_1 - h_2 - h_3} \qquad h_1 + h_2 + h_3 < 0 \tag{2.51}$$

$$A_3^{h_1 h_2 h_3} = c'[12]^{h_1 + h_2 - h_3}[23]^{h_2 + h_3 - h_1}[31]^{h_3 + h_1 - h_2} \qquad h_1 + h_2 + h_3 > 0 \tag{2.52}$$

This means that **all massless 3-point amplitudes are completely fixed by little group scaling!**

We have shown the 3-point QED amplitudes. So now we can consider something different – 3-point gluon amplitudes with 2 same helicities and 1 different helicity. Then we can determine the amplitudes from (2.51) (2.52)

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \quad A_3(g_1^+, g_2^+, g_3^-) = g\frac{[12]^3}{[23][31]} \tag{2.53}$$

Here it need some efforts to explain about the condition after equation (2.51) and (2.52).If we assume the amplitude $A_3(g_1^-, g_2^-, g_3^+)$ depending on square brackets, so can obtain a different expression

$$A_3(g_1^-, g_2^-, g_3^+) = g'\frac{[31][23]}{[12]^3}, \tag{2.54}$$

to distinguish these two possibilities, we need to use **dimension analysis**. From (2.7), we can know that both square brackets and angle brackets have mass-dimension 1. Thus the kinematic factor of amplitude (2.53) has mass-dimension 1, compatible with the interaction term $AA\partial A$ in $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$. But the mass-dimension of kinematic factor in (2.54) is (-1), so it should come from the interaction term like $g'AA\frac{1}{\Box}A$. Here, we only consider the local field theory, so this kind of interaction should be discarded.

The combination of **little group scaling** and **locality** uniquely fixes the massless 3-point amplitudes. In general,

$$\text{An n-point amplitude in D=4 spacetime have mass-dimension 4 - n.} \tag{2.55}$$

Although we discard the amplitude (2.54), the coupling s¥constant g' have mass-dimension 2, so (2.54) has the correct mass-dimension as (2.53).

How about the gluon amplitude with all minus helicities? The formula (2.51) immediately tells that it should equal to

$$A_3(g_1^-, g_2^-, g_3^-) = g''\langle 12\rangle\langle 23\rangle\langle 31\rangle. \tag{2.56}$$

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The kinematic part has mass-dimension 3 and reveals that:

1. The coupling constant g'' must have mass-dimension -2.

2. This must come from an interaction term with 3 derivatives like $(\partial A)^3$.

Furthermore, kinematic part has the antisymmetry under the exchange of gluon momenta, so the coupling constant should also have this antisymmetry in order to satisfy the Bose statics as is of course the pure Yang-Mills case. The natural candidate of this amplitude is a dimension 6 operator $\text{Tr} F_\nu^\mu F_\lambda^\nu F_\mu^\lambda$.

2.3 MHV Classification

It has been well known that the all-plus tree-level gluon amplitudes vanish

$$\text{tree-level gluon amplitues}: \quad A_n[1^+, 2^+, \cdots, n^+] = 0 \quad \text{and} \quad A_n[1^+, 2^+, \cdots, n^-] = 0 \quad (2.57)$$

This conclusion holds for the case with all helicities flipped:

$$\text{tree-level gluon amplitues}: \quad A_n[1^-, 2^-, \cdots, n^-] = 0 \quad \text{and} \quad A_n[1^-, 2^-, \cdots, n^+] = 0 \quad (2.58)$$

In supersymmetric Yang-Mills theories, these results hold for all loop order, so actually it is a special case for SYM.

If we flip one more helicit, the situation becomes quite different. Amplitudes $A_n[1^-, 2^-, 3^+, \cdots, n^+]$ does not equal to 0 and is quite important called **MHV amplitudes**. The gluon amplitudes with two negative helicities and n - 2 positive helicities are called ***maximally helicity violating*** – or simply ***MHV*** for short. The concept "maximally helicity violating" comes from thinking a $2 \to n - 2$ scattering process. Because of crossing symmetry, the outgoing particle with positive(negtive) helicity is equivalent with an ingoing particle with negative(positive) helicty. So if we consider the process $A_n[1^+, 2^+, \cdots, n^+]$ with all particle ingoing. It crosses over to $1^+ 2^- + \to 3^- \cdots n^-$ with two ingoing particles and n-2 outgoing particles, that's the reason we call it "helicity violating". We have known that the process in which the most we can violate helicity is $1^+ 2^+ \to 3^+ 4^+ 5^- \cdots n^-$, which is equivalent to amplitudes $A_n[1^+ 2^+ 3^- 4^- 5^+ \cdots n^+]$, therefore it is the *maximally helicity violating* process.

As we have shown in the last subsection, the MHV gluon tree amplitudes are given by Parke - Talyor formula

$$A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (2.59)$$

The MHV amplitudes are the important blocks for scattering amplitudes in (super)Yang-Mills theory. They are simplest amplitudes, and the next-to simplest are called **Next-to-MHV** amplitudes, or just **NMHV**. This refers to scattering process for which two 2 positive gluons

scatter into 3 positive gluons and n-5 negative gluons. Similarly, we can define N^k**MHV** amplitudes. For anti-MHV amplitudes with all helicities flipping, we have a formula

$$A_n[1^- \cdots i^+ \cdots j^+ \cdots n^-] = \frac{[ij]^4}{[12][23] \cdots [n1]}. \qquad (2.60)$$

2.4 BCFW recursion relation

The BCFW recursion relation was introduced in 2005 by Britto, Cachazo, and Feng, and later extended with Witten. It provided a novel way to compute tree-level scattering amplitudes by analytically continuing external momenta into the complex plane and expressing higher-point amplitudes recursively in terms of lower-point ones. This on-shell approach uncovered surprising simplicity in gauge theory amplitudes and offered an alternative to the traditional Feynman diagram expansion. The method was inspired in part by Witten's twistor string theory and the MHV diagram program. In parallel, off-shell recursion relations were also being explored, especially in background field methods and Berends-Giele recursion, which organize amplitudes based on off-shell currents. While off-shell methods remain useful in certain computational frameworks, the BCFW recursion highlighted the power of staying strictly on-shell, leading to a deeper understanding of symmetry, analyticity, and the geometry of scattering processes. It has since become a cornerstone of modern amplitude research.

Beyond BCFW, other on-shell recursion relations have also emerged—for instance, the KLT relations linking gauge and gravity amplitudes, and the double copy structure as well as recursion for theories with special helicity configurations or massive particles. Together, these developments form the foundation of modern amplitude methods, emphasizing physical principles like unitarity, analyticity, and factorization over traditional Lagrangian-based techniques.

2.4.1 Complex shift and Relation from Simple Cauchy's Theorem

An on-shell amplitude(each external leg satisfies on-shell condition) is labeled by their external momentum p_μ and hilicity h_i. Here we only consider the massless particles $p_i^2 = 0$, for $i = 1, 2, \cdots, n$. And momentum conservation $\sum_{i=1}^n p_i^\mu = 0$ should hold.

First, let us consider the most general case. We need to introduce n complex-valued vectors r_i^μ subject to

- $\sum_{i=1}^n r_i^\mu = 0$,

- $r_i \cdot r_j = 0$ for all $i, j = 1, 2, \cdots, n$. In particular, $r_i^2 = 0$,

- $p_i \cdot r_i = 0$ for each i(no sum).

These vectors r_i are used to define n shifted momentum

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \quad \text{with} \quad z \in \mathcal{C} \qquad (2.61)$$

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Note that, by imposing these constraints,

- Momentum conservation holds for the shifted momentum: $\sum_{i=1}^{n} \hat{p}_i^{\mu} = 0$,

- We have $\hat{p}_i^2 = 0$, so each shifted momentum still keep on-shell

- For a non-trival [2] subset of momentum $\{p_i\}_{i\in I}$, define $P_i^{\mu} = \sum_{i\in I} p_i^{\mu}$. Then $\hat{P}_I^2$ is **linear** in z:

$$\hat{P}_I^2 = P_I^2 - 2z P_I \cdot k = -\frac{P_I^2}{z_I}(z - z_I) \tag{2.62}$$

beacuse the z^2 term vanishes. We can write

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \qquad with z_I = \frac{P_I^2}{2 P_I \cdot k} \tag{2.63}$$

If we consider the original amplitude A_n in terms of the shifted momentum $\hat{p}_i^{\mu}$ instead of original momentum. Then the amplitude can be seen as a holomorphic function of z. Actually, if we set $z = 0$, we can obtain the origin ampltide $A_n = \hat{A}_n(z = 0)$.

In this paper, we only focus on the tree-level scatterung amplitudes, in which case the analytic structure becomes quite simple. The tree amplitudes does not have any branch cuts – there are no logs, no square roots, etc.. The tree-level amplitude is a rational function of Lorentz invariant kinematic quantities, like the contraction between momentum or momentum and polarization vector. Therefore, the analytic structure can be completely captured by its single poles in the complex z plane. To see that all of the poles are single, we just need to notice that we can only ontain poles from propagators. But we have seen that all of the poles can be dertermined by (2.63). So $1/\hat{P}_I^2$ gives a single pole at z_I, and for generic momentum, $P_I^2 \neq 0$(propagator is off-shell) so $z_I \neq 0$. Hence, $\hat{A}_n(z)$ only has single poles away from the orignal poine $z = 0$.[3]

Then let us turn to define a new holomorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane. We can pick a contour surrounding the original point $z = 0$ which is a single pole of this function. The residue of this single pole can be easily computed

$$\text{Res}|_{z=0} \frac{\hat{A}_n(z)}{z} = \hat{A}_n(z = 0) = A_n, \tag{2.64}$$

It is just the unshifted amplitude what we want to compute at the begining.

Also, we can choose another countour surrounding all single poles, then Cauchy's theorem just tell us

$$A_n = -\sum_{z_I} \text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n , \tag{2.65}$$

---

[2]non-trival here means at least two and no more than n-2 such that $P_I^2 \neq 0$

[3]Here we need to impose locality to ensure all of the poles can be determined by the propagators.

where the B_n is the residue at $z = \infty$ infinity, called bounday term. By taking $z \to 1/\omega$, it is easy to notice that B_n is the $O(z^0)$ term in the $z \to \infty$ expansion of A_n.
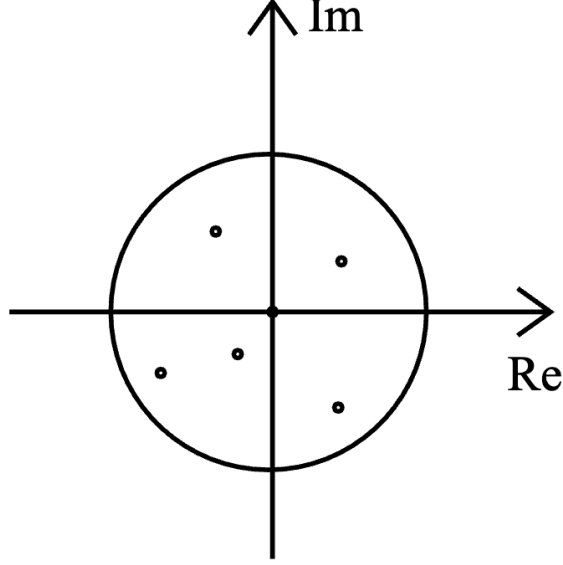

Figure 1: Cauchy Theorem

Now, at the position of z_I pole, the propagator $1/\hat{P}_I^2$ gose on-shell. The unitarity has a tight relation with I poles in the S-matrix with on-shell intermediate states.

$$G_n(p_1, \ldots, p_n) = (2\pi)^4 \, \delta^4 \left(\sum p \right) \frac{i}{p_\Psi^2 - m_\Psi^2 + i\varepsilon} \mathcal{M}_\Psi^{1,r} \, \mathcal{M}_\Psi^{r+1,n\dagger} + \text{extra} \qquad (2.66)$$

where "extra" refers to anything else that contributes. This equation says that Green's functions always have poles when on-shell intermediate particles can be produced.

Then, in the on-shell limit, the shifted amplitude *factorizes* into two on-shell parts

$$\hat{A}_n(z) \xrightarrow{\;z \text{ near } z_I\;} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) = -\frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \qquad (2.67)$$

This makes it easy to compute the residue at $z = z_I$

$$-\text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = -(z - z_I) \frac{z_I}{z(z - z_I)}|_{z=z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \qquad (2.68)$$
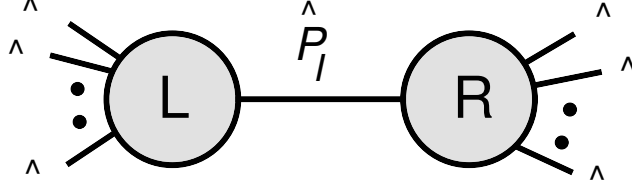
Figure 2: factorization

Note that, as opposed to Feynman diagrams for which the intermediate propagator does not equal to 0, here the intertal line $\hat{P}_I^2$ of Figure 2 does equal to 0. And the blobs which refer to subamplitudes are **on-shell amplitudes**. In the diagrammatic representation Figure 2, the internal line is assigned the usual scalar propagator $1/P_I^2$, evaluated using unshifted momenta. Since each subamplitude involves fewer than n external particles, all the residues at finite z can be expressed entirely in terms of lower-point on-shell amplitudes. This fundamental idea underlies the construction of the recursion relations.

The contribution B_n does affect a lot, and it has no similar general expression in trems of lower-point amplitudes. Although in some cases, the boundary term B_n can be computed, but there is not a general method to compute it. Thus, in most applications, we always assumes, or better, prove that $B_n = 0$. This is often modified by demonstrating stronger statement that

$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty. \tag{2.69}$$

If (2.69) holds, we state the shift is **valid**.

For a valid shift, the n-point amplitudes can be recursively computed interms of lower-point amplitude

$$A_n = \sum_{\text{diagrams } I} \hat{A}_{\text{L}}(z_I) \frac{1}{P_I^2} \hat{A}_{\text{R}}(z_I) = \sum_{\text{diagrams } I} \text{\includegraphics{L R}} . \tag{2.70}$$

where I means all possible factorization channels. We did not use any $D = 4$ spacetime property, so this recursion relation can be utilized to general D dimension spacetime.

2.4.2 BCFW recursion relation

In the last section, we introduce the recursion relation with all momentum shifted. But actually, some of the shift vectors r_i^μ can be choose to 0. The BCFW shift is a kind of special shift for which only two momentum are shifted, called i and j. In D=4 spacetime diension, the shift is implemented

$$\hat{p}_i(z) = p_i - zq, \qquad \hat{p}_j(z) = p_j + zq \tag{2.71}$$

so momentum conservation is kept and the hold for on-shell condition require ot impose

$$q^2 = q \cdot p_i = q \cdot p_j = 0 \,. \tag{2.72}$$

This shift can be implemented in terms of spinor-helicity variable as

$$|\hat{i}] = |i] - z\,|j]\,, \qquad |\hat{j}] = |j]\,, \qquad |\hat{i}\rangle = |i\rangle\,, \qquad |\hat{j}\rangle = |j\rangle + z|i\rangle\,. \tag{2.73}$$

No other spinors are shifted, this kind of shift are called $[i,j\rangle$ shift(it is worth to mention that $[i,j\rangle$ shift is completely different with $[j,i\rangle$ shift).Note that $[\hat{i}k]$ and $\langle\hat{j}k\rangle$ are linear in z for $k \neq i,j$, while $\langle\hat{i}\hat{j}\rangle = \langle ij\rangle$, $[\hat{i}\,\hat{j}] = [ij]$, $\langle\hat{i}k\rangle = \langle ik\rangle$, and $[\hat{j}k] = [jk]$ remain unshifted. In the literature, the BCFW deformation is usually written in the above spinor form. However, the momentum form (2.71) provides another perspective.

The BCFW recursion relation takes the form

$$A_n = \sum_{\text{diagrams } I} \hat{A}_{\mathrm{L}}(z_I)\,\frac{1}{P_I^2}\,\hat{A}_{\mathrm{R}}(z_I) = \sum_{\text{diagrams } I} \quad \raisebox{-0.5em}{\includegraphics{}} \tag{2.74}$$

Beforing going into the apllication of BCFW, let us first consider the large-z behavior. In pure Yang-Mills case, it has been proved that the color-ordered gluon tree-level amplitudes behave like the following manner [6]

$$
\begin{array}{ccccc}
[i,j\rangle & [-,-\rangle & [-,+\rangle & [+,+\rangle & [+,-\rangle \\
\hat{A}_n(z) \sim & \frac{1}{z} & \frac{1}{z} & \frac{1}{z} & z^3
\end{array}
$$

If i,j are not adjacent, one obtains an extra $1/z$ in each case. Thus any one of shifts , $[-,-\rangle, [-,+\rangle, [+,+\rangle$ are valid but $[+,-\rangle$ is not valid. This is a quite non-trival result. If we consider more general case [9]

$$p_i(z) = p_i + z\lambda_i\tilde{\lambda}_j, \quad p_j(z) = p_j - z\lambda_i\tilde{\lambda}_j, \Rightarrow \begin{cases} A(z) \to z^{-2}, & i,j \text{ not nearby} \\ A(z) \to z^{-1}, & i,j \text{ nearby} \end{cases}$$

$$p_i(z) = p_i + z\lambda_i\tilde{\lambda}_k \;(k \neq j), \quad p_k(z) = p_k - z\lambda_i\tilde{\lambda}_k, \Rightarrow \begin{cases} A(z) \to z^{-2}, & i,k \text{ not nearby} \\ A(z) \to z^{-1}, & i,k \text{ nearby} \end{cases}$$

$$p_i(z) = p_i + z\lambda_k\tilde{\lambda}_i \ (k \neq j), \quad p_k(z) = p_k - z\lambda_k\tilde{\lambda}_i, \Rightarrow \begin{cases} A(z) \to z^2, & i,k \text{ not nearby} \\ A(z) \to z^3, & i,k \text{ nearby} \end{cases}$$

$$p_k(z) = p_k + z\lambda_l\tilde{\lambda}_k, \quad p_l(z) = p_l - z\lambda_l\tilde{\lambda}_k, \quad k,l \neq i,j \Rightarrow \begin{cases} A(z) \to z^{-2}, & l,k \text{ not nearby} \\ A(z) \to z^{-1}, & l,k \text{ nearby} \end{cases}$$
$$(2.75)$$

where i,j are negetive helicities. Then we can go forward to the provement of Parke - Talyor formula by using BCFW recursion relation. First, we can notice that the formula

$$A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \tag{2.76}$$

holds for $n = 3$ case, and we start the provement by induction from 3-point. For specific n, we suppose (2.76) holds for amplitudes with external legs less than n. Then we can writte down the n-point amplitude $A_n[1^-2^-3^+ \cdots n^-]$ with BCFW shift $[1, 2\rangle$

$$A_n[1^-2^-3^+ \ldots n^+] = \sum_{k=4}^{n} \quad $$

$$= \sum_{k=4}^{n} \sum_{h_I = \pm} \hat{A}_{n-k+3}\left[\hat{1}^-, \hat{P}_I^{h_I}, k^+ \ldots, n^+\right] \frac{1}{P_I^2}$$
$$\times \hat{A}_{k-1}\left[-\hat{P}_I^{-h_I}, \hat{2}^-, 3^+ \ldots, (k-1)^+\right]. \tag{2.77}$$

The internal momentum can be evaluated as $P_I = p_2 + p_3 + \cdots + p_{k-1}$ and $\hat{P}_I = \hat{p}_2 + p_3 + \cdots + p_{k-1}$ if we choos $k = 4, 5, \cdots, n$. Notice that there are no such diagrams in which the $\hat{p}_1$ and $\hat{p}_2$ locate at the same side, whereby there are no z independence, so that we can not obtain any single pole in the complex plane. And because we are computing the color-ordered amplitudes, only diagrams that preserve the color-ordering of the external states are included. Note that we explicitly include the sum over all possible helicity assignments for the particle exchanged on the on-shell internal line. Specifically, if the exchanged gluon appears as an outgoing particle with negative helicity in the left subamplitude, it will correspondingly appear as an outgoing gluon with positive helicity in the right subamplitude.

We have known that the color-amplitudes $A_n[- + + c \cdots +]$ equals to 0, except for $n = 3$

case. So (2.77) can be computed like

$$A_n[1^-2^-3^+ \ldots n^+] = \text{} + \text{}$$

$$= \hat{A}_3\left[\hat{1}^-, -\hat{P}_{1n}^+, n^+\right] \frac{1}{P_{1n}^2} \hat{A}_{n-1}\left[\hat{P}_{1n}^-, \hat{2}^-, 3^+ \ldots (n-1)^+\right]$$

$$+ \hat{A}_{n-1}\left[\hat{1}^-, \hat{P}_{23}^-, 4^+ \ldots, n^+\right] \frac{1}{P_{23}^2} \hat{A}_3\left[-\hat{P}_{23}^+, \hat{2}^-, 3^+\right]. \tag{2.78}$$

Notice that in the first diagram of (2.78), there is a 3-point subamplitude

$$\hat{A}_3\left[\hat{1}^-, -\hat{P}_{1n}^+, n^+\right] = \frac{[\hat{P}_{1n} n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]}. \tag{2.79}$$

here we take the convention of analytic continuation like

$$|-p\rangle = -|p\rangle, \qquad |-p] = +|p]. \tag{2.80}$$

Because the propagator satisfies the on-shell condition, so we can obtain

$$\hat{P}_{1n}^2 = 0 = 2\hat{p}_1 \cdot p_n = \langle\hat{1}n\rangle[n\hat{1}], \tag{2.81}$$

because we take the $[1,2\rangle$ shift, so that $|\hat{1}\rangle = |1\rangle$. And the angle bracket $\langle 1n \rangle$ can not equal to 0, so that

$$[\hat{1}n] = 0 \qquad \Longrightarrow \qquad z = \frac{[1n]}{[2n]}, \tag{2.82}$$

so the potion of pole is determined by the on-shell propagator. Also, we can compute like

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n} n] = -\hat{P}_{1n}|n] = -(\hat{p}_1 + p_n)|n] = |1\rangle[\hat{1}n] = 0, \tag{2.83}$$

we can conclude $[\hat{P}_{1n} n] = 0$ because of $[\hat{1}n] = 0$ and $|\hat{P}_{1n}\rangle \neq 0$. So the 3-point amplitude (2.79) has both a vanishing numerator and denominator, with the numerator vanishing one order faster. We conclude that special 3-point kinematics force $\hat{A}_3\left[\hat{1}^-, -\hat{P}_{1n}^+, n^+\right]$. So the contribution from the first diagram in (2.78)vanishes.

For the second diagram in (2.78), we will not obtain the vanishing result, because it is an anti-MHV amplitude with angle spinor $|2\rangle$ shifted. Henceforth, the seemingly complicated summation (2.78) redeuce to a single diagram. This is one merit of BCFW recursion relation.

Then the amplitude can be evaluated like

$$A_n[1^-2^-3^+\ldots n^+] = \hat{A}_{n-1}\left[\hat{1}^-, \hat{P}_{23}^-, 4^+, \ldots, n^+\right] \frac{1}{P_{23}^2} \hat{A}_3\left[-\hat{P}_{23}^+, \hat{2}^-, 3^+\right]. \tag{2.84}$$

By using the lower point Parke - Talyor formula which assune to hold, the amplitude can be written like

$$A_n[1^-2^-3^+\ldots n^+] \;=\; \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle\langle \hat{P}_{23}\,4\rangle\langle 45\rangle\cdots\langle n\hat{1}\rangle} \times \frac{1}{\langle 23\rangle[23]} \times \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\,\hat{2}][\hat{2}3]}. \tag{2.85}$$

This formula can be further simplified by using

$$\langle \hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] = -\langle \hat{1}\hat{P}_{23}\rangle[\hat{P}_{23}3] = -\langle 1|\hat{p}_2 + p_3|3] = -\langle 1|\hat{p}_2|3] = \langle 12\rangle[32] \tag{2.86}$$

here we use the fact $|\hat{1}\rangle = |1\rangle$ and $\langle \hat{1}2\rangle = \langle 12\rangle$. And similarly , for the demoninator we have the similar result

$$\langle \hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] = -\langle 4\hat{P}_{23}\rangle[\hat{P}_{23}\hat{2}] = -\langle 4|\hat{p}_2 + p_3|2] = \langle 34\rangle[32] \tag{2.87}$$

here we use the fact $|\hat{2}] = |2]$.

Then the full amplitude equals to

$$A_n[1^-2^-3^+\ldots n^+] = \frac{\langle 12\rangle^3[32]^3}{\langle 34\rangle[32]\langle 23\rangle[23][23]\langle 45\rangle\cdots\langle n1\rangle} \tag{2.88}$$

$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{2.89}$$

so we finish the provement of Parke - Talyor formula by utlizing BCFW recursion relation.

Notice that this formula only holds for MNH anplitudes. For NMHV amplitudes, of coure we can still use BCFW recusion relation to similarly compute them. Here, I only show the result of 6-gluon NMHV tree amplitude.

$$A_6[1^-2^-3^-4^+5^+6^+] = \tag{2.90}$$

$$= \frac{\langle 3|1+2|6]^3}{P_{126}^2[21][16]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]} + \frac{\langle 1|5+6|4]^3}{P_{156}^2[23][34]\langle 56\rangle\langle 62\rangle\langle 5|1+6|2]} \tag{2.91}$$

Color-ordered tree amplitudes can exhibit physical poles only when the sum of momenta of adjacent external lines goes on-shell. As previously noted, MHV gluon amplitudes feature only

two-particle poles and do not possess multi-particle poles. In contrast, you have now seen that six-gluon NMHV amplitudes contain both two- and three-particle poles. Interestingly, in the BCFW representation, each diagram contributes a peculiar denominator factor of the form $\langle 5|1+6|2]$. This term does not correspond to any physical pole of the scattering amplitude—it is known as a ***spurious pole***. The residue associated with this unphysical pole must vanish, and indeed it does: the spurious contributions cancel out when summing over all relevant BCFW diagrams.

When does it work?

- Yang - Mills theory and gluon scattering.

- Scalar - QED.

- Scalar theory $\lambda\phi^4$.

- $\mathcal{N} = 4$ super Yang - Mills theory.

- Gravity.

3 (De)Constructed Gauge Theory

3.1 Motivation

Our four-dimensional description of spacetime may only be an effective low-energy approximation. While the observable universe appears four-dimensional — with one time and three spatial directions — it is possible that additional spatial dimensions exist but are hidden at quite small scales beyond our current experimental realm. A simple and widely reasearched realization of this thought involves compact extra dimensions, where fields propagate on a higher-dimensional spacetime with some dimensions compactified to small radius. At energies much lower than the compactification scale, such theories reduce to four-dimensional physics as an effetive theory, but at higher energies, the influence of these extra dimensions become manifest, for example, the appearance of Kaluza - Klein excitations(KK modes).

Theoretical interest in extra dimensions arises not only from string theory but also from the desire to better understand the ultraviolet behavior of field theories. Unfortunately, traditional higher-dimensional field theories are nonrenormalizable and contain dimensionful couplings, requiring a cutoff and becoming strongly coupled in the UV. This poses a major obstacle to fully understanding their high-energy behavior.

In response to this, Arkani-Hamed, Cohen, and Georgi proposed the idea of (De)constructed dimensions, in which higher-dimensional behavior emerges dynamically from a renormalizable four-dimensional theory. Their construction discretizes the extra dimension using a chain of gauge groups and link fields, producing a low-energy spectrum that mimics a compactified fifth dimension. Crucially, the theory remains perturbative and controllable in the UV, making it

an ideal framework to study phenomena associated with extra dimensions—such as KK mode scattering—without running into the usual ultraviolet issues.

This paper is motivated by this (De)constructed approach. We focus on computing tree-level scattering amplitudes in the simplest two-site model, employing modern amplitude techniques such as the BCFW recursion relation, spinor-helicity formalism, and color ordering.

3.2 $SU(m) \times SU(n)$ *Moose*

At first, we consider four-dimensional field theories that contain gauge fields and fermions. These theories can be conveniently illustrated using a type of diagram known as a "moose" or "quiver" diagram[4]. In such diagrams, gauge groups are represented by open circles, and fermions are shown as single arrows (directed lines) attached to those circles.
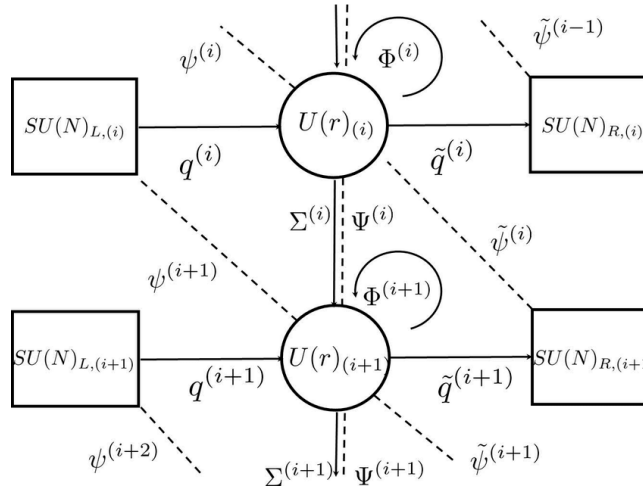

Figure 3: Example of quiver diagram

An arrow pointing away from a circle represents a set of Weyl fermions transforming under the fundamental representation of that gauge group. Conversely, an arrow pointing toward a circle corresponds to fermions transforming in the complex conjugate of fundamental representation.

We will focus on a specific example: a moose diagram shaped like an N-sided polygon, where each node and connection encodes the field content and gauge structure of the theory with gauge group $SU(m)^N \times SU(n)^N$

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[4]Sorry for the ambiguity of different terminology. In this paper, the term "quiver", "moose" and "(De)construction" mainly represent the same meaning
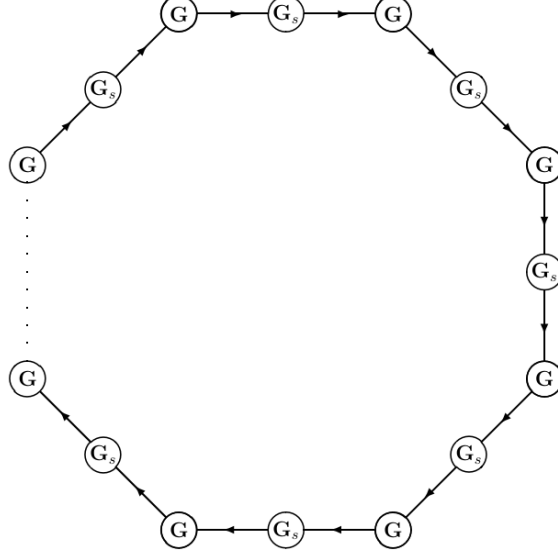
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Figure 4: Moose diagram

We impose a *cyclic symmetry* on the theory so that all $SU(m)$ gauge couplings are set equal to a common value g, and all $SU(n)$ gauge couplings are set equal to g_s. Through *dimensional transmutation*, the theory can equivalently be described by two corresponding *dimensionful parameters*, Λ and Λ_s, which characterize the strong coupling scales of the respective gauge groups.

Each *side of the polygon* in the moose diagram represents two types of fermions, each transforming under the three gauge groups associated with that side: $SU_i(m) \times SU_i(n) \times SU_{i+1}(m)$.
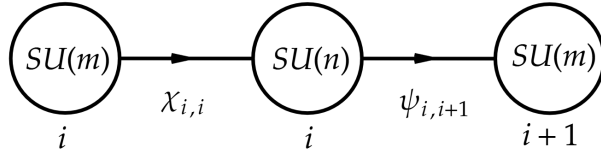
$$\chi_{i,i} \qquad \text{transformed under} \quad (m, \bar{n}, 1) \tag{3.1}$$

$$\psi_{i,i+1} \qquad \text{transformed under} \quad (1, n, \bar{m}) \tag{3.2}$$

where m, n refer to the fundamental representation of gauge group $SU(m)$ and $Su(n)$, $\bar{m}, \bar{n}$ refer to the conjugate of fundamental representation, and 1 just represent the trival representation.

Low-energy behavior. What does the theory look like at longer distances? In the regime where $\Lambda_s \gg \Lambda$, the low-energy behavior of the theory becomes relatively simple. At energy scales around Λ_s, the $SU(m)$ gauge couplings remain weak and can be treated

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perturbatively. In contrast, each $SU(n)$ gauge group becomes strongly coupled at this scale, leading to **fermion condensation**.

Specifically, pairs of fermions associated with each strong gauge group condense, forming nonzero vacuum expectation values:

$$\langle \chi_{i,i}\psi_{i,i+1} \rangle \sim 4\pi f^3 U_{i,i+1}, \quad i = 1, \ldots, N$$

Here, $f \sim \Lambda_s/(4\pi)$ is the characteristic scale of the condensate, and $U_{i,i+1}$ is an $m \times m$ unitary matrix that encodes the orientation of the condensate in field space. And in thefollowing part, we will focus on the model after condensation.

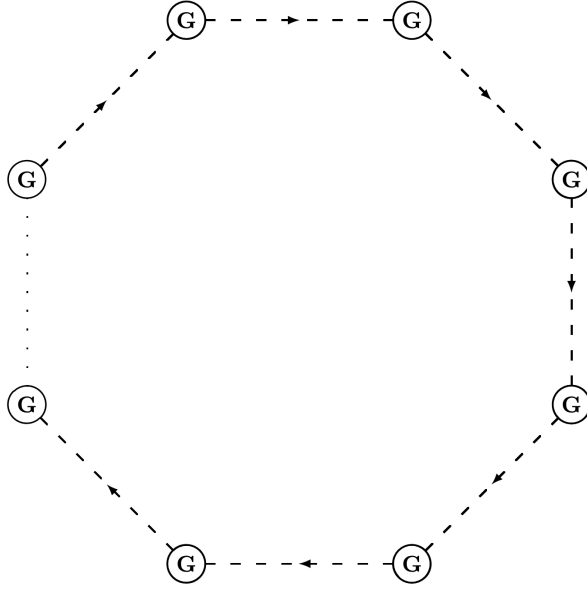We use a similar condensed - moose diagram to represent this condensed theory:

Figure 5: Condensed Moosed diagram

It is described by the following Lagrangian

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2} \mathrm{Tr}(F_i)^2 + \sum_{i=1}^{N} \mathrm{Tr}[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)], \tag{3.3}$$

here F_i refers to the ith gauge field strength, scalar field Φ_i transformed under the bi-fundamental representation and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - ig_i A_{i\mu}\Phi_i + ig_{i+1}\Phi_i A_{i+1\mu}. \tag{3.4}$$

Here, gauge field and scalar field transformed like

$$\boldsymbol{A_{i\mu}} \to U_i(x)\boldsymbol{A_{i\mu}}U_i^-(x) - \frac{i}{g_i}(\partial_\mu U)U^{-1}, \qquad \Phi_i \to U_i(x)\Phi_i U_{i+1}^-(x) \tag{3.5}$$

It is easy to confirm that this theory is invariant under $\prod_1^N SU(m)$ gauge group.

The action is similar to the non-linear sigma model, in which the non-lenear sigma fields effectively connects gauge fields at neighboring sites. In fact, equation (3.3) can be interpreted as a discretized version of a five-dimensional gauge theory with gauge group $SU(m)$, where only the fifth dimension is latticized. The non-linear sigma model fields play the role of **link variables** in a lattice gauge theory, and the condensed moose diagram effectively represents the fifth dimension.

It is quite striking that the moose diagram, which was originally just a tool to keep track of fields and interactions in a four-dimensional theory, **takes on a new physical meaning at low energies**: it effectively describes an extra spatial dimension.

We can higgs the scalar field $U_{i,i+1}$ making the gauge group down to the diagonal subgroup. Its eigenvalues can be easily computed, yielding a discrete mass spectrum labeled by an integer k satisfying $-N/2 < k \leq N/2$. The mass of each mode is given by:

$$M_k^2 = 4g^2 f_s^2 \sin^2\left(\frac{\pi k}{N}\right) \equiv \left(\frac{2}{a}\right)^2 \sin^2\left(\frac{p_5 a}{2}\right), \tag{3.6}$$

where $p_5 \equiv 2\pi k/R$ is the discretized momentum along the fifth dimension. The corresponding eigenvectors take the form $\psi^m \sim \exp(imp_5 a)$.

In the regime $|k| \ll N/2$, the spectrum becomes approximately linear, and the masses simplify to:

$$M_k \simeq |p_k| = \frac{2\pi |k|}{R}. \tag{3.7}$$

This exactly matches the **Kaluza-Klein spectrum** for a five-dimensional gauge boson compactified on a circle of circumference R. The effective four-dimensional gauge coupling of the diagonal subgroup is given by $g_4^2 = g^2/N$. Using the earlier relation in equation (2.7), this implies:

$$\frac{1}{g_4^2} = \frac{R}{g_5^2}, \tag{3.8}$$

which is the standard relation between the gauge couplings in five and four dimensions.

3.3 2-site model

For simplicity, in this paper we only focus on the smallest block in this condensed moose theory – 2-site model, which can be represented by the following diagram
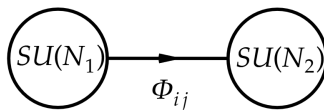

Figure 6: 2-site model

Here we choose a slightly different model, of which the two gauge group are different.

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