

Tree level scattering amplitude in (De)constructed gauge theory

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- 4 Scattering amplitudes from BCFW

Why we need new method?

Feynman diagram is a brilliant method without doubt, but it also faces many challenges.

- How to compute high multiplicity amplitudes, like n gluons scattering.
- Gauge redundancies bring us many structural complications and non-physical degrees of freedom.
- Some symmetries are not manifest in the lagrangian level, like the dual superconformal symmetry for $\mathcal{N} = 4$ SYM, etc.
- The unexpected simplicity of some amplitudes has led to the speculation that local QFT may not be the most fundamental framework.

From Frynman diagram to On-shell method

On-shell here means that all quantities we use are gauge invariant and satisfy the on-shell condition. Specifically, there are many ingredients under this frame

- The analytic continuation for S-matrix.
- The color-ordered amplitudes.
- The BCFW recursion relation.
- The spinor helicity discription for amplitudes.

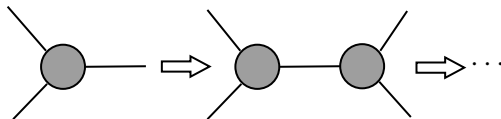
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A brief introduction to BCFW

BCFW recursion relation is a method to compute scattering amplitude recursively, means that you can compute higher point amplitudes from lower point.

- Ruth Britto
- Freddy Cachazo
- Bo Feng
- Edward Witten



From real to complex – Analytic Continuation

Why can we conduct analytic continuation?

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as $p_{i\mu}p_j^\mu$, $p_{i\mu}\epsilon_j^\mu$.
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.



Amplitudes can be shifted to complex plane

Momentum Shift in BCFW

What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \dots, n$. We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$

Note that,

- Momentum conservation holds for shifted momentum: $\sum_{i=1}^n \hat{p}_i^\mu = 0$,
- Each shifted momentum is still on-shell: $\hat{p}_i^2 = 0$
- For a non-trivial subset of generic momenta $\{p_i\}_{i \in I}$, define $P_I^\mu = \sum_{i \in I} p_i^\mu$, \hat{P}_I^2 is **linear** in z :

$$\hat{P}_I^2 = \left(\sum_{i \in I} \hat{p}_i \right)^2 = P_I^2 - 2z P_I \cdot k = -\frac{P_I^2}{z_I} (z - z_I) \quad \text{with} \quad z_I = \frac{P_I^2}{2P_I \cdot k}$$

Fantastic result from Cauchy Theorem

As a result, we can consider amplitude A_n in terms of shifted momentum \hat{p}_i^μ instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

and we have known the possible positions of single poles, z_I , different propagators give us different single poles in the z -plane.

If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\text{Res}|_{z=0} \frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n$$

It means that the original amplitude equals to the residue at origin.

From Cauchy Theorem, we can obtain

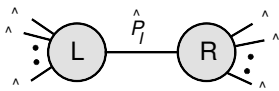
$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.

Then, at a z_I pole, the propagator \hat{P}_I^2 goes to on-shell. In that limit, the shifted amplitude **factorizes** into to on-shell parts (Unitarity)

$$\hat{A}_n(z) \xrightarrow{z \text{ near } z_I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) = - \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$- \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) =$$


Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

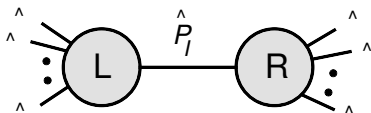
$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	z^3
-	$1/z$	$1/z$

Spinor-Helicity formalism

In the part of introduction to BCFW



→ No helicity appears here

but the S-matrix is a function of momentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

The answer is **Spinor-Helicity formalism** → Catch p_i and h_i at the same time.

Spinor-helicity formalism

■ Massless Case

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \rightarrow t^{-1}\lambda, \quad \tilde{\lambda} \rightarrow t\tilde{\lambda}, \quad t \in \mathbb{C}$$

same as

$$|\lambda\rangle \rightarrow t^{-1}|\lambda\rangle, \quad |\lambda] \rightarrow t|\lambda]$$

The scattering amplitudes should transform covariantly under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \dots, \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \dots) = t_i^{2h_i} \mathcal{A}_n$$

■ Massive Case

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

On-shell 3-point can be completely determined

On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \quad p_2 = (1 - \kappa)p_3 \quad (\text{Collinear})$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

You can not obtain any notrvial thing from 3-point!

Of coure, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\square \Psi)$$

but it still equals to 0 in the on-shell case.

Another necessity to introduce complex momentum

If the momentum is complexed, we have

$$\langle 12 \rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or square brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}} \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \quad 2h_3 = -x_{13} - x_{23}.$$

Then all index can be solved from this system of equations, so that

$$\begin{aligned} A_3^{h_1 h_2 h_3} &= c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3} & h_1 + h_2 + h_3 < 0 \\ A_3^{h_1 h_2 h_3} &= c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} & h_1 + h_2 + h_3 > 0 \end{aligned}$$

★ **All massless 3-point amplitudes are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g' \frac{[13][23]}{[12]^3}$$

but actually it comes from the **non-local** interaction $g' A A \frac{\partial}{\square} A$, so we discard it.

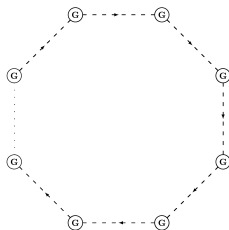
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Introduction of Quiver or Moose gauge theory

- Quiver: A container for carrying arrows
- Moose: A kind of deer with large horns

In the language of field theories, quiver gauge theories contain gauge fields and fermions, summarized in a pictorial representation.



Moose diagram

N -sided polygon

G : gauge group $SU(m)$

\rightarrow : Unitary scalar fields Φ_{ij}

Why we focus on quiver gauge theory?

The lagrangian can be written like

$$\mathcal{L} = - \sum_{i=1}^N \frac{1}{2} \text{Tr}(F_i)^2 + \sum_{i=1}^k \text{Tr}[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)],$$

here F_i refers to the i th gauge field strength, scalar field Φ_i transformed under the **bi-fundamental** representation and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - ig_i A_{i\mu} \Phi_i + ig_{i+1} \Phi_i A_{i+1\mu}.$$

Here, gauge field and scalar field transformed like

$$A_{i\mu} \rightarrow g_i(x) A_{i\mu} g_i^{-1}(x), \quad \Phi_i \rightarrow g_i(x) \Phi_i g_{i+1}^{-1}(x)$$

It is easy to confirm that this theory is invariant under $\prod_1^N SU(m)$ gauge group.

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group $SU(m)$, where only the fifth dimension are latticed.

- If $SU(m)_1$ and $SU(m)_N$ are connected $\longrightarrow S^2$ compactification
- If not connected \longrightarrow Interval compactification

After higgsing, we can obtain a spectrum

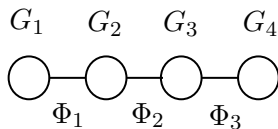
$$M_k^2 = 4g^2 g_s^2 \sin^2 \left(\frac{\pi k}{N} \right)$$

This is precisely the **Kaluza-Klein** spectrum under S^2 compactification.

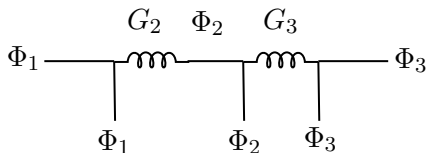
What is relation to scattering amplitude?

The critical point is **locality**.

- Space-Time Locality \longrightarrow local field theories
- Theory Space Locality \longrightarrow Discretized theory space



If we change this to a scattering diagram, and compute the large- z behavior



$$\sim 1/z^{\textcircled{4}}$$

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Classification of scattering amplitude

For the simplicity, we started from the two-site gauge theory, means that there are only two gauge fields V_1 , V_2 and one scalar field Φ , Φ^\dagger . The amplitudes in this theory can be classified to many types

- 3-point

- $V_1 \Phi \Phi^\dagger$
- $V_2 \Phi \Phi^\dagger$
- $V_1 V_1 V_1$
- $V_2 V_2 V_2$

- 4-point

- $V_1 V_1 V_1 V_1$
- $V_2 V_2 V_2 V_2$
- $\Phi^\dagger V_1 V_1 \Phi$
- $\Phi V_2 V_2 \Phi^\dagger$
- $\Phi V_2 \Phi^\dagger V_1$
- $\Phi \Phi^\dagger \Phi \Phi^\dagger$

- 5-point

- $V_1 V_1 V_1 V_1 V_1$
- $V_2 V_2 V_2 V_2 V_2$
- $\Phi^\dagger V_1 V_1 V_1 \Phi$
- $\Phi V_2 V_2 V_2 \Phi^\dagger$
- $V_2 \Phi^\dagger V_1 V_1 \Phi$
- $\Phi V_2 V_2 \Phi^\dagger V_1$
- $\Phi \Phi^\dagger \Phi \Phi^\dagger V_1$ (Not yet solved)
- $\Phi \Phi^\dagger \Phi \Phi^\dagger V_2$ (Not yet solved)

- 6-point

- $V_1 V_1 V_1 V_1 V_1 V_1$
- $V_2 V_2 V_2 V_2 V_2 V_2$
- $\Phi^\dagger V_1 V_1 V_1 V_1 \Phi$
- $\Phi V_2 V_2 V_2 V_2 \Phi^\dagger$
- $V_2 V_2 \Phi^\dagger V_1 V_1 \Phi$
- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$
- \vdots
- \vdots
- \vdots

Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$\begin{aligned} A[1, 2, 3^+] &= \frac{[23][31]}{[12]}, & A[1, 2, 3^-] &= \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle} \\ A[3^+, 4^+, 5^-] &= \frac{[34]^3}{[45][53]}, & A[3^-, 4^-, 5^+] &= \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle} \end{aligned}$$

By using the 3 point building block, we can construct 4 point color-ordered amplitudes from BCFW recursion relation.

- nV_1 or nV_2

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\text{Parke - Talyor Formula : } A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

- $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude $A[1, 2, 3^+, 4^-]$. We choose $[2, 3\rangle$ shift

$$\begin{aligned} |\hat{2}\rangle &= |2\rangle - z|3\rangle, & |\hat{2}\rangle &= |2\rangle \\ |\hat{3}\rangle &= |3\rangle, & |\hat{3}\rangle &= |3\rangle + z|2\rangle \end{aligned}$$

The amplitudes can be computed

$$A[1, 2, 3^+, 4^-] = (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1, 2, 3^+, 4^+, 5^-] = \frac{\langle 15 \rangle^2 \langle 25 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1, 2, \dots, (n+2)^-] = (-1)^{n+1} \frac{\langle 1, n+2 \rangle^2 \langle 2, n+2 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n+1, n+2 \rangle \langle n+2, 1 \rangle}$$

★ Bonus relation:

$$A[1, 2, 3^+, 4^+] = 0 \quad \Rightarrow \quad A[1, 2, 3^+, \dots, n^+] = 0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

- $\Phi V_2 \Phi^\dagger V_1$

$$A[1, 2, 3_1^+, 4_2^-] = \frac{\langle 14 \rangle \langle 24 \rangle}{\langle 13 \rangle \langle 23 \rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1, 2, 3_1^+, 4_1^+, 5_2^-] = (-1) \frac{\langle 2\textcolor{green}{5} \rangle^2 \langle 1\textcolor{green}{5} \rangle^2}{\langle \textcolor{blue}{23} \rangle \langle \textcolor{blue}{34} \rangle \langle \textcolor{blue}{41} \rangle \langle \textcolor{red}{25} \rangle \langle \textcolor{red}{51} \rangle}$$

- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$

$$A[1, 2, 3_1^+, 4_1^+, 5_2^+, 6_2^-] = \frac{\langle 2\textcolor{green}{6} \rangle^2 \langle 1\textcolor{green}{6} \rangle^2}{\langle \textcolor{blue}{23} \rangle \langle \textcolor{blue}{34} \rangle \langle \textcolor{blue}{41} \rangle \langle \textcolor{red}{25} \rangle \langle \textcolor{red}{56} \rangle \langle \textcolor{red}{51} \rangle}$$

- Compact formula for general case

$$A = \frac{\langle 2a \rangle^2 \langle 1a \rangle^2}{\underbrace{\langle 2\star \rangle \cdots \langle \star 1 \rangle}_{SU(N_1)} \underbrace{\langle 2* \rangle \cdots \langle *1 \rangle}_{SU(N_2)}}$$

Green: Particle with — helicity

Blue: Particle belongs to the first gauge group

Red: Particle belongs to the second gauge group

\star : Order for gauge group 1

$*$: Order for gauge group 2

5 Appendix

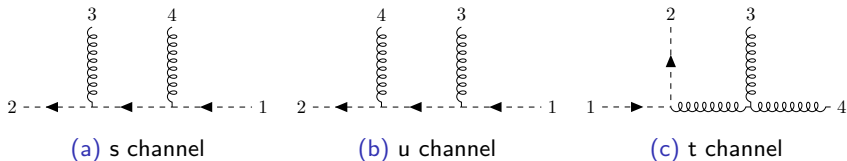
- nV_1 or nV_2

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\text{Parke - Talyor Formula : } A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

■ $\Phi^\dagger V_1 V_1 \Phi$



The color factor can be written respectively as following

$$r_s = \text{Tr}[\Phi_2^\dagger T^{a_3} T^{a_4} \Phi_1], \quad r_u = \text{Tr}[\Phi_2^\dagger T^{a_4} T^{a_3} \Phi_1], \quad r_t = \text{Tr}[\Phi_2^\dagger [T^{a_3}, T^{a_4}] \Phi_1]$$

We can easily obtain a similar Jacobi relation

$$r_t = r_s - r_u$$

Then we can accomplish the color decomposition and define the corresponding color-ordered amplitudes.

For example, in the 4pt. case, the full amplitude can be decomposed to the following form

$$\begin{aligned}\mathcal{A}_4(\Phi^\dagger V_1 V_1 \Phi) &= A_s r_s + A_u r_u + A_t r_t \\ &= A_s r_s + A_u r_u + A_t (r_s - r_u) \\ &= (A_s + A_t) r_s + (A_u - A_t) r_u\end{aligned}$$

The two subamplitudes can be defined as color-ordered amplitude with order $[1,2,3,4]$ and $[1,2,4,3]$ respectively.

Of course, for the type $\Phi^\dagger(nV_1)\Phi$ and $\Phi(nV_2)\Phi^\dagger$, we can do the same thing to define the color-ordered amplitudes. It should be noticed that the order only has the relation with the order of external gluon line.

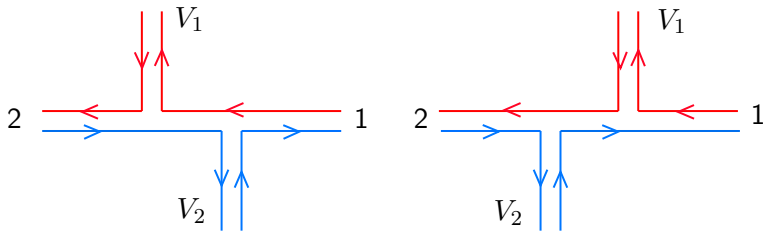
$$[1, 2, \sigma(3), \sigma(4), \cdots, \sigma(n)]$$

- $\Phi V_2 \Phi^\dagger V_1$

The color structure for this kind of amplitude has special form, like

$$(T_1^a)_{ij} (T_2^b)_{\bar{j}\bar{i}}$$

It is more straightforward to observe the color structure in terms of double line notation as follows



OPP(Order preserving permutation)

From the 4 point case, we have known that the relative order between gauge boson 1 and gauge boson 2 does not affect the color structure. Thus, it is necessary to introduce the **OPP(Order preserving permutation)**. For example:

$$(3_1, 4_1, 5_2) \quad (3_1, 5_2, 4_1) \quad (5_2, 3_1, 4_1)$$

These three permutations are different OPP for $(3_1, 4_1, 5_2)$, so that give us the same color factor.

Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$\begin{aligned} A[1, 2, 3^+] &= \frac{[23][31]}{[12]}, & A[1, 2, 3^-] &= \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle} \\ A[3^+, 4^+, 5^-] &= \frac{[34]^3}{[45][53]}, & A[3^-, 4^-, 5_+] &= \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle} \end{aligned}$$

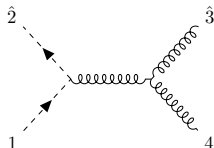
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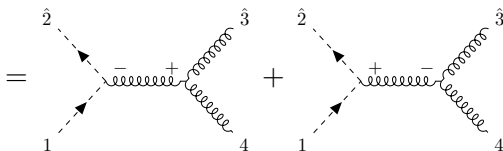
4 point from BCFW

• $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude $A[1, 2, 3^+, 4^-]$. We choose $[2, 3\rangle$ shift

$$\begin{aligned} |\hat{2}\rangle &= |2\rangle - z|3\rangle, & |\hat{2}\rangle &= |2\rangle \\ |\hat{3}\rangle &= |3\rangle, & |\hat{3}\rangle &= |3\rangle + z|2\rangle \end{aligned}$$

$$A[1, 2, 3^+, 4^-] = \sum_h$$




We denote these two different BCFW channels as A_1 and A_2 , then

$$\begin{aligned}
 A_1 &= \frac{\langle \hat{2}\hat{I} \rangle \langle \hat{I}1 \rangle}{\langle 1\hat{I} \rangle} \times \frac{1}{s_{12}} \times \frac{[\hat{I}\hat{3}]^3}{[\hat{3}4][4\hat{I}]} \\
 &= (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}
 \end{aligned}$$

where we use the fact $|\hat{2}\rangle = |2\rangle$, $|\hat{3}\rangle = |3\rangle$, and the **Fierz Identity**

$$[ij][kl] + [il][jk] + [ik][lj] = 0$$

Similarly, we can obtain

$$A[1, 2, 3^-, 4^+] = (-1) \frac{\langle 13 \rangle^2 \langle 23 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

★ Bonus relation

$$A[1, 2, 3^+, 4^+] = A[1, 2, 3^-, 4^-] = 0$$