A complete solution for scattering in a kind of quiver gauge theory

Su Yingze

Nagoya University

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Contents

A brief introduction to BCFW

BCFW recursion relation is a method to compute scattering amplitude, especially in Yang-Mills theory and gravity.

- Ruth Britto
- Freddy Cachazo
- Bo Feng
- Edward Witten

From real to complex – Analytic Continuation

Why is analytic continuation valid?

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as $p_{i\mu}p_j^{\mu}$, $p_{i\mu}\epsilon_j^{\mu}$.
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.

$$\Downarrow$$

Ampltudes can be shifted to complex plane

Momentum Shift in BCFW

What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \ldots, n$. Then introduce n complex-valued vectors r_i^μ.

(i) $\sum_{i=1}^{n} r_i^\mu = 0$,

(ii) $r_i \cdot r_j = 0$ for all $i, j = 1, 2, \ldots, n$. In particular $r_i^2 = 0$,

(iii) $p_i \cdot r_i = 0$ for each i (no sum).

These vectors r_i are used to define n shifted momenta

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \qquad \text{with} z \in \mathcal{C}$$

Note that,

(A) By property (i), momentum conservation holds for the shifted momenta: $\sum_{i=1}^{n} \hat{p}_i^\mu = 0$,

(B) By (ii) and (iii), we have $\hat{p}_i^2 = 0$, so each shifted momentum is on-shell,

(C) For a non-trival subset of generic momenta $\{p_i\}_{i\in I}$, define $P_I^\mu = \sum_{i\in I} p_i^\mu$.

Then, $\hat{P}_I^2$ is linear in z:

$$\hat{P}_I^2 = \left(\sum_{i\in I} \hat{p}_i\right)^2 = P_I^2 + 2zP_I \cdot R_I \quad \text{with} \quad R_I = \sum_{i\in I} r_i,$$

because the z^2 term vanishes by property (ii). We can write

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{with} \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I}$$

Fantasitic result from Cauchy Theorem

As a result of (A) and (B) (momentum conservation and on-shell), we can consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

and we have known the possible positions of single poles, z_I, different propagators give us different single poles in the z-plane.

If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane, pick a contour that surrounds the simple pole at the origin. ★ The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0} \frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

It means that the original amplitude equals to the residue at origin.

From Cauchy Theorem, we can ontain

$$A_n = -\sum_{z_I} Res|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.

Then, at a z_I pole, the propagator $\hat{P}_I^2$ goes to on-shell. In that limit, the shifted amplitude factorizes into to on-shell parts (Unitarity)

$$\hat{A}_n(z) \quad \xrightarrow{z \text{ near } z_I} \quad \hat{A}_L(z_I)\frac{1}{\hat{P}_I^2}\hat{A}_R(z_I) = -\frac{z_I}{z - z_I}\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$-Res|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) =$$

Little Group

In the context of relativistic QFT, particles are classified according to the unitary irreducible representations of the Poincaré group.

A crucial concept in this classification is the

> **Little Group:** The subgroup of Lorentz transformations that leaves a given four-momentum invariant.

- **Massless Case**
  For a massless particle with representative momentum

$$p^\mu = (E, 0, 0, E)$$

  the little group is $SO(2) \simeq U(1)$.
  In terms of spinor-helicity variables, the massless momentum can be written as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

The action of the little group is:

$$\lambda \to t^{-1}\lambda, \qquad \tilde{\lambda} \to t\tilde{\lambda}, \qquad t \in \mathbb{C}^*$$

same as

$$|\lambda\rangle \to t|\lambda\rangle, \qquad |\lambda] \to t|\lambda]$$

The scattering amplitudes should transform covariantly under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \ldots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \ldots) = t_i^{2h_i}\mathcal{A}_n$$

- **Massive Case**
  It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

3-point can be completely determined

• 3-particle special kinematics determines an on-shell 3-point amplitude with massless particles depends only on either angle or square brackets of the external momenta.

Let us suppose thet it depends only on angle brackets, then we can write down the general ansatz

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = ct_1^{-x_{12}} t_1^{-x_{13}}\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \qquad 2h_3 = -x_{13} - x_{23}.$$

Then all index can be solved from this system of equations, so that

$$A_3^{h_1 h_2 h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} \quad h_1+h_2+h_3<0.$$

Also for square brackets case, we can obtain

$$A_3^{h_1 h_2 h_3} = c^{'}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \quad h_1+h_2+h_3>0.$$

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

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Introduction to quiver gauge theory

The lagrangian can be written like

$$\mathcal{L} = -\sum_{i=1}^{k} \frac{1}{2}\mathrm{Tr}(F_i)^2 + \sum_{i=1}^{k-1} \mathrm{Tr}[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)],$$

here F_i refers to the ith gauge field strength, scalar field Φ_i transformed under the bi-fundamental representation and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - i g_i V_{i\mu} \Phi_i + i g_{i+1} \Phi_i V_{i+1\mu}.$$

It is easy to confirm that this theory is invariant under $SU(N_1) \times SU(N_2) \times \cdots \times SU(N_k)$.

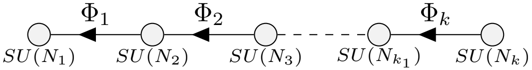


Figure: Quiver gauge theory

Classification of scattering amplitude

For the simplicity, we started front the two-site gauge theory, means that there are only two gauge fields V_1, V_2 and one scalar field Φ, $\Phi^\dagger$. The amplitudes in this theory can be classified to many types

- 3-point
  - $\boldsymbol{V_1 \Phi \Phi^\dagger}$
  - $\boldsymbol{V_2 \Phi \Phi^\dagger}$
  - $\boldsymbol{V_1 V_1 V_1}$
  - $\boldsymbol{V_2 V_2 V_2}$
- 4-point
  - $\boldsymbol{V_1 V_1 V_1 V_1}$
  - $\boldsymbol{V_2 V_2 V_2 V_2}$
  - $\boldsymbol{\Phi^\dagger V_1 V_1 \Phi}$
  - $\boldsymbol{\Phi V_2 V_2 \Phi^\dagger}$
  - $\boldsymbol{\Phi V_2 \Phi^\dagger V_1}$
  - $\boldsymbol{\Phi \Phi^\dagger \Phi \Phi^\dagger}$

- 5-point
  - $\boldsymbol{V_1 V_1 V_1 V_1 V_1}$
  - $\boldsymbol{V_2 V_2 V_2 V_2 V_2}$
  - $\boldsymbol{\Phi^\dagger V_1 V_1 V_1 \Phi}$
  - $\boldsymbol{\Phi V_2 V_2 V_2 \Phi^\dagger}$
  - $\boldsymbol{V_2 \Phi^\dagger V_1 V_1 \Phi}$
  - $\boldsymbol{\Phi V_2 V_2 \Phi^\dagger V_1}$
  - $\Phi\Phi^\dagger\Phi\Phi^\dagger V_1$ (Not yet solved)
  - $\Phi\Phi^\dagger\Phi\Phi^\dagger V_2$ (Not yet solved)
- 6-point
  - $\boldsymbol{V_1 V_1 V_1 V_1 V_1 V_1}$
  - $\boldsymbol{V_2 V_2 V_2 V_2 V_2 V_2}$
  - $\boldsymbol{\Phi^\dagger V_1 V_1 V_1 V_1 \Phi}$
  - $\boldsymbol{\Phi V_2 V_2 V_2 V_2 \Phi^\dagger}$
  - $\boldsymbol{V_2 V_2 \Phi^\dagger V_1 V_1 \Phi}$
  - $\boldsymbol{\Phi V_2 V_2 \Phi^\dagger V_1 V_1}$

$$\vdots$$

$$\vdots$$

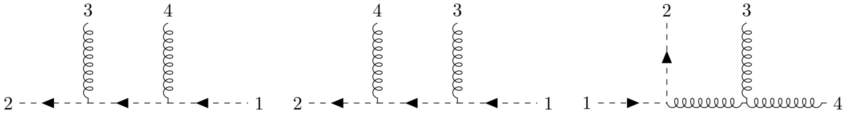
Color structure

- nV_1 or nV_2
  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\boxed{\text{Parke - Talyor Formula :} \quad A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}}$$

Notice that this formula only applies for MHV amplitudes, although the NMHV can be completely solved.

- $\Phi^\dagger V_1 V_1 \Phi$



(a) s channel (b) u channel (c) t channel

The color factor can be written respectively as following

$r_s = \mathrm{Tr}[\Phi_2^\dagger T^{a_3} T^{a_4} \Phi_1]$, $r_u = \mathrm{Tr}[\Phi_2^\dagger T^{a_4} T^{a_3} \Phi_1]$, $r_t = \mathrm{Tr}[\Phi_2^\dagger [T^{a_3}, T^{a_4}] \Phi_1]$

We can easily obtain a similar Jacobbi relation

$$r_t = r_s - r_u$$

Then we can accomplish the color decomposition and define the corressponding color-ordered amplitudes.

For example, in the 4pt. case, the full amplitude can be decomposed to the following form

$$\begin{aligned}\mathcal{A}_4(\Phi^\dagger V_1 V_1 \Phi) &= A_s r_s + A_u r_u + A_t r_t \\ &= A_s r_s + A_u r_u + A_t(r_s - r_u) \\ &= (A_s + A_t) r_s + (A_u - A_t) r_u\end{aligned}$$

The two subamplitudes can be defined as color-ordered amplitude with order [1,2,3,4] and [1,2,4,3] respectively.

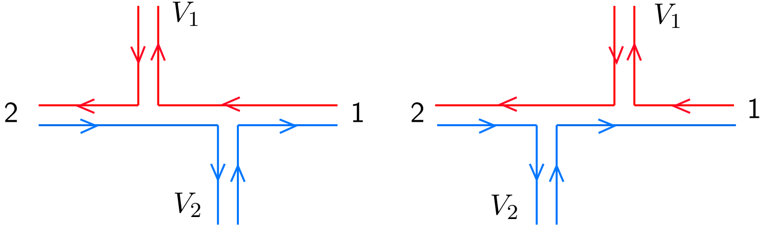
Of course, for the type $\Phi^\dagger(nV_1)\Phi$ and $\Phi(nV_2)\Phi^\dagger$, we can do the same thing to define the color-ordered amplitudes. It should be noticed that the order only has the relation with the order of external gluon line.

$$[1, 2, \sigma(3), \sigma(4), \cdots, \sigma(n)]$$

- $\Phi V_2 \Phi^\dagger V_1$
  The color structure for this kind of amplitude has special form, like

$$(T_1^a)_{ij}(T_2^b)_{\bar{j}\bar{i}}$$

It is more straightforward to observe the color structure in terms of double line notation as follows



OPP(Order preserving permutation)

From the 4 point case, we have known that the relative order between gauge boson 1 and gauge boson 2 does not affect the color structure. Thus, it is necessary to introduce the **OPP(Order preserving permutation)**. For example:

$$(3_1, 4_1, 5_2) \quad (3_1, 5_2, 4_1) \quad (5_2, 3_1, 4_1)$$

These three permutations are different OPP for $(3_1, 4_1, 5_2)$, so that give us the same color factor.

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Basic building block – 3-point

From the previous section, we have known that there are only two kinds of 3 point amplitude

$$A[1,2,3^{+}] = \frac{[23][31]}{[12]}, \qquad A[1,2,3^{-}] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

$$A[3^{+},4^{+},5^{-}] = \frac{[34]^3}{[45][53]}, \qquad A[3^{-},4^{-},5_{+}] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

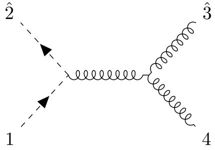
By using the 3 point building block, we can construct 4 point color-ordered amplitudes from BCFW recursion relation.

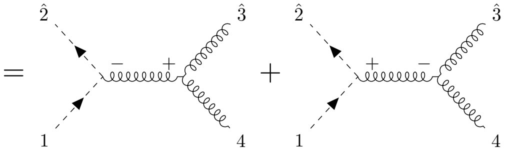
- $\Phi^\dagger V_1 V_1 \Phi$

Here we compute the color-ordered amplitude $A[1,2,3^+,4^-]$. We choose $[2,3\rangle$ shift

$$|\hat{2}] = |2] - z|3], \qquad |\hat{2}\rangle = |2\rangle$$
$$|\hat{3}] = |3], \qquad |\hat{3}\rangle = |3\rangle + z|2\rangle$$

$$A[1,2,3^+,4^-] = \sum_h$$





We denote these two different BCFW channels as A_1 and A_2, then

$$A_1 = \frac{\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle}{\langle 1\hat{I}\rangle} \times \frac{1}{s_{12}} \times \frac{[\hat{I}\hat{3}]^3}{[\hat{3}4][4\hat{I}]}$$

$$= (-1)\frac{\langle 14\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

where we use the fact $|\hat{2}\rangle = |2\rangle$, $|\hat{3}] = |3]$, and the Fierz Identity

$$[ij][kl] + [il][jk] + [ik][lj] = 0$$

Similarly, we can obtain

$$A[1,2,3^-,4^+] = (-1)\frac{\langle 13\rangle^2\langle 23\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

★ Bonus relation

$$A[1,2,3^+,4^+] = A[1,2,3^-,4^-] = 0$$