On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

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Contents

Why We Study Scattering Amplitudes?

1. **Bridge between theory and experiment**
   - Core prediction targets for high-energy collider experiments such as the LHC
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Why We Study Scattering Amplitudes?

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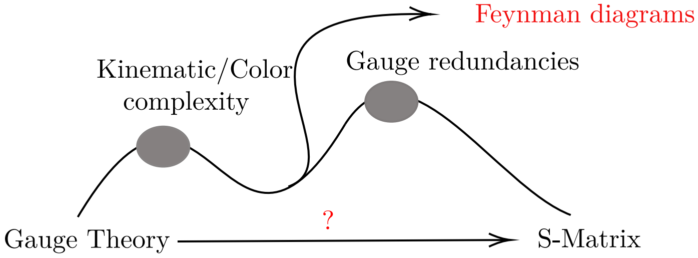
   - Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections
2. **Reveal deep structures of quantum field theory**
   - Amplitudes exhibit hidden symmetries (e.g., dual conformal, Yangian) not visible in the Lagrangian
   - These symmetries suggest deeper theoretical frameworks, such as amplituhedra or AdS/CFT correspondence

Challenges we face before



n pt. amplitudes	4	5	6	7	8	9	10
# of diag.	4	25	220	2485	34300	559405	10525900

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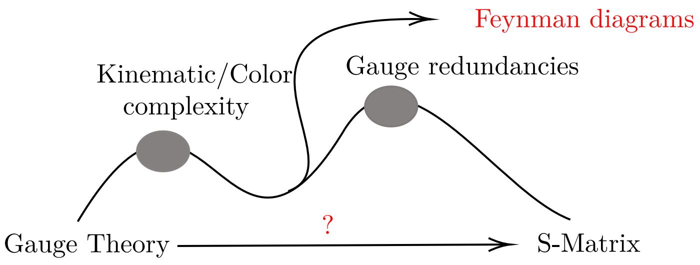

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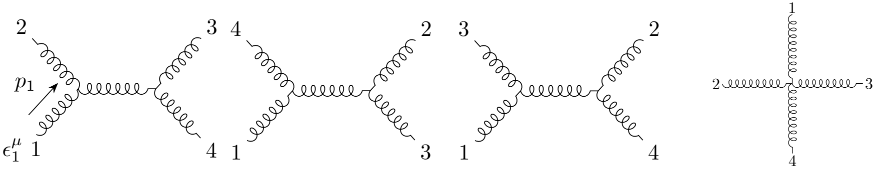
The number of Feyman diagrams grow quite rapidly!

Conventional Computation

Usually, when we compute the gluon amplitudes by using Feynman diagram, we will obtain something like

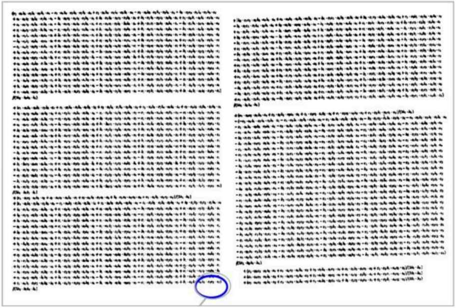


s-channel t-channel u-channel Contact Term

$$\mathcal{M}_s(p_1p_2 \to p_3p_4) = -\frac{g_s^2}{s} f^{abe} f^{cde}$$
$$\times \Big\{ -4\,\epsilon_1 \cdot \epsilon_3^* \,\epsilon_2 \cdot p_1 \,\epsilon_4^* \cdot p_3 + 2\,\epsilon_1 \cdot \epsilon_2 \,\epsilon_3^* \cdot p_1 \,\epsilon_4^* \cdot p_3$$
$$-2\,\epsilon_1 \cdot p_4 \,\epsilon_2 \cdot p_1 \,\epsilon_3^* \cdot \epsilon_4^* + \epsilon_1 \cdot \epsilon_2 \,p_4 \cdot p_1 \,\epsilon_3^* \cdot \epsilon_4^* + \text{remaining 10 terms} \Big\}$$

If you consider 5point case, it will become worse:

★ We have 25 diagrams and nearly 10000 terms!



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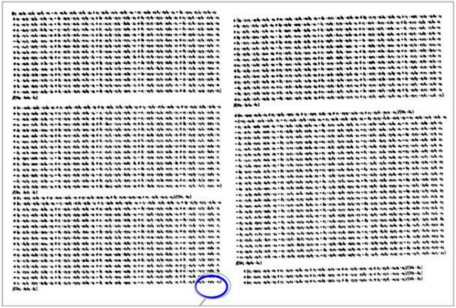

simple action $\longrightarrow$ **complicated** and **redundant** Feynman rules $\longrightarrow$ **simple** amplitude squares and cross sections

$$S=-\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^{a\mu\nu} \longrightarrow \mathcal{M}=-ig_s^2\left\{\text{horrible mess}\quad\right\} \longrightarrow \overline{|\mathcal{M}|}^2=\frac{9g_s^4}{2}\left(3-\frac{su}{t^2}-\frac{ut}{s^2}-\frac{st}{u^2}\right)$$

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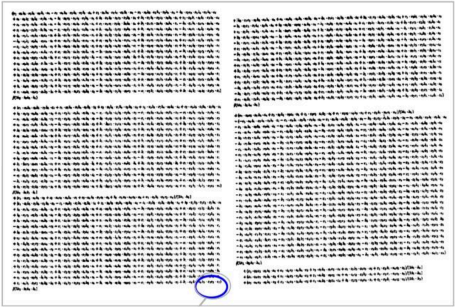

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We need to cancel many useless information because of gauge redundancies!

Contents

Modern Amplitude Method

The answer is On-shell method.

$$\text{Gauge Theory} \xrightarrow{\color{red}{\text{On-shell method}}} \text{S-Matrix}$$

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$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \quad \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_5}] + \text{perms}$, $\quad +, - :$ helicity

Parke–Taylor Formula (MHV amplitudes):

$$A_5[1^+, 2^+, 3^+, 4^+, 5^+] = 0$$

$$A_5[1^-, 2^+, 3^+, 4^+, 5^+] = 0$$

$$A_5[1^-, 2^-, 3^+, 4^+, 5^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$ (first non-trival one)

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$$p_\mu\sigma^\mu = p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha} = |\lambda\rangle_\alpha[\lambda|_{\dot\alpha}, \quad \sigma^\mu = (1,\vec\sigma),$$

$|\lambda\rangle, |\lambda]$: Weyl Spinors

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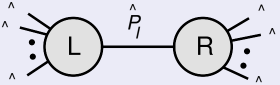
$$A_5[1^+,2^+,3^-,4^-,5^-] = \frac{[12]^4}{[12][23][34][45][51]}$$

BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_{\text{diagrams } I}$$



The Power of BCFW Recursion Relation

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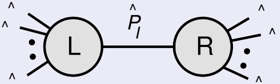
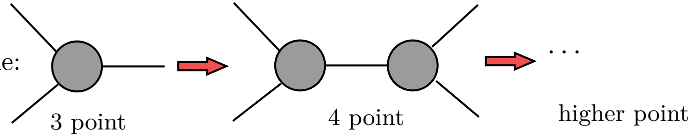


More understandable:



★ From lower point on-shell amp. to higher point on-shell amp.!!

More specifically,

$$\text{3pt.} \longrightarrow \text{4pt.} \longrightarrow \text{5pt.} \longrightarrow \cdots$$

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

More specifically,

$$3\text{pt.} \longrightarrow 4\text{pt.} \longrightarrow 5\text{pt.} \longrightarrow \cdots$$

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$$\boxed{\Rightarrow: \quad A[1^+ \cdots \ i^-, (i+1)^+ \cdots \ j^- \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}}$$

★ This is the power of BCFW recursion relation.

From Review to Applications

So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
  - **Pure Yang-Mills** theory
  - Tree-level MHV amplitudes
  - Color-ordered partial amplitudes

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So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
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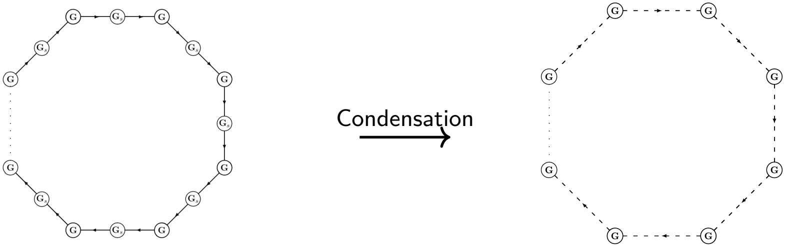
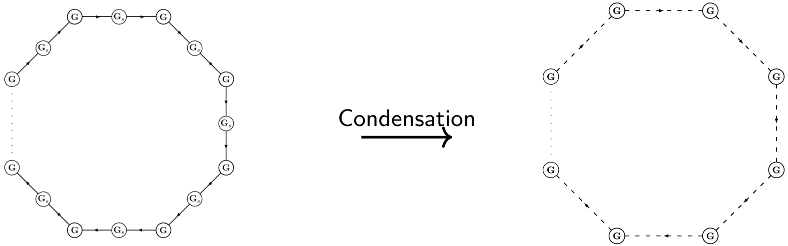
Next: Realistic Models

- Move beyond massless gauge theory
- Consider:
  - **(De)constructed gauge theories**
- Key questions:
  - Can BCFW still apply?
  - How to choose momentum shift?
  - What new structures emerge?

Contents

Introduction of quiver gauge theory





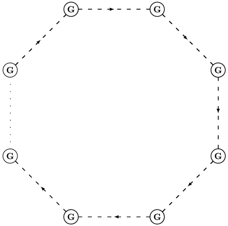
The Lagrangian can be written like

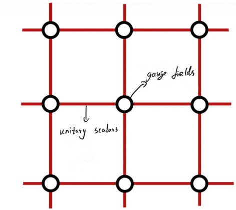
$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2}\mathrm{Tr}(F_i^2) + \sum_{i=1}^{N} \mathrm{Tr}\left[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)\right],$$

here F_i refers to the ith gauge field strength. The scalar field Φ_i transforms under the bi-fundamental representation, and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - i g_i A_{i\mu} \Phi_i + i g_{i+1} \Phi_i A_{i+1\mu}.$$

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group $SU(m)$, where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.





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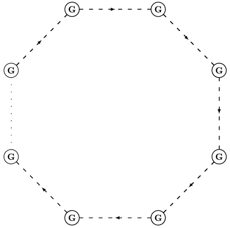
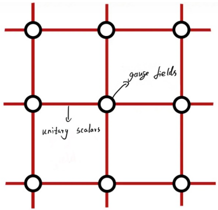


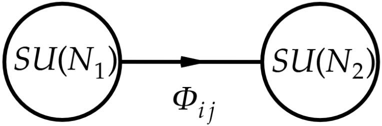


After higgsing the scalar field, we can obtain a spectrum

$$M_k^2 = 4g^2 f_s^2 \sin^2\left(\frac{\pi k}{N}\right)$$

This is precisely the Kaluza-Klein spectrum under S^1 compactification.

Amplitudes from BCFW



From the previous section, we have known that there are only two kinds of 3 point amplitude

$$A[1^{\Phi}, 2^{\Phi^\dagger}, 3^+] = \frac{[23][31]}{[12]}, \qquad A[1^{\Phi}, 2^{\Phi^\dagger}, 3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

$$A[3^+, 4^+, 5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^-, 4^-, 5^+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

By using the 3 point building block, we can construct 4 point colorordered amplitudes from BCFW recursion relation.

Gauge boson sector

- nV_1 or nV_2
  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\boxed{\text{Parke - Talyor Formula :} \quad A[\cdots, i^-, \cdots, j^-, \cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

SQCD like sector

- $\Phi^\dagger V_1 V_1 \Phi$

  Here we compute the color-ordered amplitude $A[1,2,3^+,4^-]$. We choose $[2,3\rangle$ shift

$$|\hat{2}] = |2] - z|3], \qquad |\hat{2}\rangle = |2\rangle$$
$$|\hat{3}] = |3], \qquad |\hat{3}\rangle = |3\rangle + z|2\rangle$$

  The amplitudes can be computed

$$A[1,2,3^+,4^-] = (-1)\frac{\langle 14\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad (\text{Parke -Talyor like Formula})$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1,2,3^+,4^+,5^-] = \frac{\langle 15\rangle^2\langle 25\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1,2,\cdots,(n+2)^-] = (-1)^{n+1}\frac{\langle 1,n+2\rangle^2\langle 2,n+2\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n+1,n+2\rangle\langle n+2,1\rangle}$$

$\star$ Bonus relation:

$$A[1,2,3^+,4^+]=0 \quad \Rightarrow \quad A[1,2,3^+,\cdots,n^+]=0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

- $\Phi V_2 \Phi^\dagger V_1$

$$A[1,2,3_1^+,4_2^-] = \frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1,2,3_1^+,4_1^+,5_2^-] = (-1)\frac{\langle 25\rangle^2\langle 15\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 51\rangle}$$

- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$

$$A[1,2,3_1^+,4_1^+,5_2^+,6_2^-] = \frac{\langle 26\rangle^2\langle 16\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 56\rangle\langle 61\rangle}$$

- Compact formula for general case

$$A = \frac{\langle 2a\rangle^2 \langle 1a\rangle^2}{\underbrace{\langle 2\star\rangle \cdots \langle \star 1\rangle}_{SU(N_1)} \underbrace{\langle 2*\rangle \cdots \langle *1\rangle}_{SU(N_2)}}$$

Green: Particle with $-$ helicity
Blue: Particle belongs to the first gauge group
Red: Particle belongs to the second gauge group
$\star$: Order for gauge group 1
$*$: Order for gauge group 2

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For example, if we want to compute $A[1, 2, 5_1^+, 3_1^+, 4_1^-, 7_2^+, 6_2^+, 8_2^+]$:

$$A = \frac{\langle 24\rangle^2 \langle 14\rangle^2}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle\langle 27\rangle\langle 76\rangle\langle 68\rangle\langle 81\rangle}$$

Contents

Summary

- Introduce the on-shell method, including BCFW recursion relation, spinor-helicity formalism, etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

Thanks for your attention!

Spinor-Helicity Formalism

Helicity

Helicity is defined as the projection of a particle's spin vector $\vec{S}$ onto the direction of its momentum $\vec{p}$:

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

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S-matrix is a function of momentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

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Massless Case:

- Momenta in spinor form:

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}} = |p\rangle[p|$$

Brief explaination: We choose two momentum to be shifted oppositely

$$p_i \to \hat{p}_i(z) \equiv p_i - zk, \qquad p_j \to \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \qquad p_i \cdot k = 0, \qquad p_j \cdot k = 0$$

We consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

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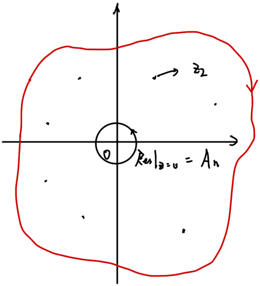
If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane. From Cauchy Theorem, we can ontain

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.



★ The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

and

$$\mathrm{Res}|_{z=z_I} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j\rangle$	$+$	$-$
$+$	$1/z$	z^3
$-$	$1/z$	$1/z$

On-shell 3-point can be completely determined

For the complex momentum, we have

$$|1\rangle \propto |2\rangle \propto |3\rangle \qquad or \qquad |1] \propto |2] \propto |3]$$

$$\boxed{\begin{aligned} A_3^{h_1h_2h_3} &= c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} \qquad && h_1+h_2+h_3<0 \\ A_3^{h_1h_2h_3} &= c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \qquad && h_1+h_2+h_3>0 \end{aligned}}$$

$\star$ **All massless on-shell 3-point ampltides are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

Scattering Amplitudes from BCFW

For simplicity, we start from the two-site gauge theory with gauge fields V_1, V_2 and scalar fields Φ, $\Phi^\dagger$.

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_1)^2 - \frac{1}{2}\mathrm{Tr}(F_2)^2 + \mathrm{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

We only foucus on the following amplitudes:

$$nV_1, \qquad nV_2, \qquad \Phi^\dagger nV_1\Phi, \qquad \Phi nV_2\Phi^\dagger, \qquad \Phi^\dagger\Phi\Phi^\dagger\Phi$$

here n can be any positive integer.