On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

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Contents

Why We Study Scattering Amplitudes?

1. **Bridge between theory and experiment**
   - Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
   - Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections

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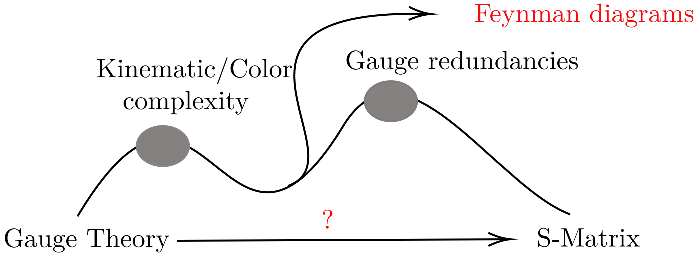
   - Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
   - Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections
2. **Reveal deep structures of quantum field theory**
   - Amplitudes exhibit hidden symmetries (e.g., dual conformal, Yangian) not visible in the Lagrangian
   - These symmetries suggest deeper theoretical frameworks, such as amplituhedra or AdS/CFT correspondence

Challenges we face before



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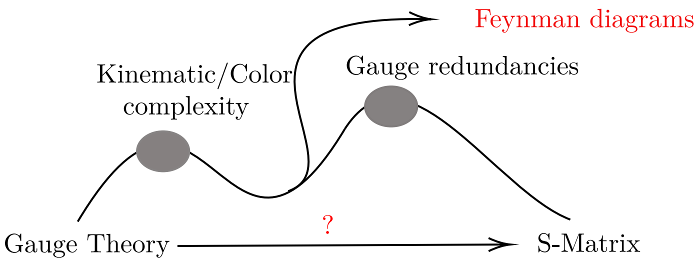
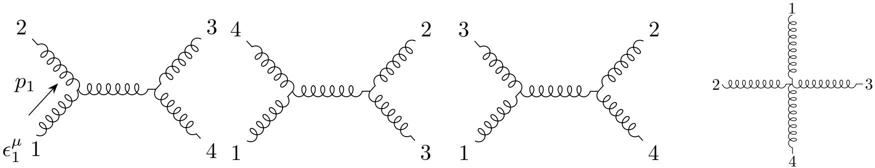


n pt. amplitudes	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

The number of Feyman diagrams grow quite rapidly!

Conventional Computation

Usually, when we compute the gluon amplitudes by using Feynman diagram, we will obtain something like

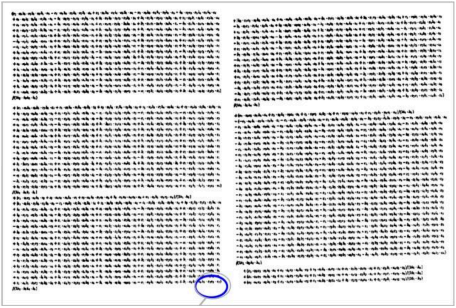


s-channel t-channel u-channel Contact Term

$$\mathcal{M}_s(p_1 p_2 \to p_3 p_4) = -\frac{g_s^2}{s} f^{abe} f^{cde}$$
$$\times \Big\{ -4\,\epsilon_1 \cdot \epsilon_3^*\, \epsilon_2 \cdot p_1\, \epsilon_4^* \cdot p_3 + 2\,\epsilon_1 \cdot \epsilon_2\, \epsilon_3^* \cdot p_1\, \epsilon_4^* \cdot p_3$$
$$-2\,\epsilon_1 \cdot p_4\, \epsilon_2 \cdot p_1\, \epsilon_3^* \cdot \epsilon_4^* + \epsilon_1 \cdot \epsilon_2\, p_4 \cdot p_1\, \epsilon_3^* \cdot \epsilon_4^* + \text{remaining 10 terms}\Big\}$$

If you consider 5point case, it will become worse:

★ We have 25 diagrams and nearly 10000 terms!



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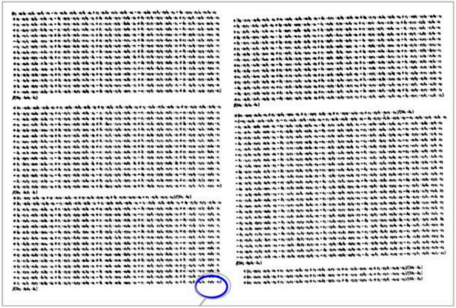

simple action → **complicated** feynman rules and **non gauge-invariant** green function $\xrightarrow{\text{LSZ.}}$ **on-shell** helicity amplitude

$$S = -\frac{1}{4}\int d^4x F^a_{\mu\nu} F^{a\mu\nu}$$

$$\mathcal{M} = -ig_s^3\{$$  $$+ \cdots \}$$

$$\frac{\langle\rangle^4}{\langle\rangle\langle\rangle\langle\rangle\langle\rangle}$$

Can we use gauge invariant quantity here?

Contents

Modern Amplitude Method

The answer is On-shell method.

$$\text{Gauge Theory} \xrightarrow{\color{red}{\text{On-shell method}}} \text{Helicity Amplitude}$$

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$$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \quad \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_5}] + \text{permutations}$$

Parke–Taylor Formula (MHV amplitudes):

$$A_5[1^+2^+3^+4^+5^+] = 0 \qquad (+,- : \text{ helicity;}$$

$$A_5[1^-2^+3^+4^+5^+] = 0 \qquad 1,2,\cdots,n : \text{ particle labels)}$$

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$$A_5[1^-2^-3^+4^+5^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \quad \text{(first non-trivial one)}$$

$$(p_{i\mu}\sigma^\mu)_{\alpha\dot\alpha} = (|p_i\rangle)_\alpha([p_i|)_{\dot\alpha}, \quad |p_i\rangle = |i\rangle$$

$$\sigma^\mu = (1,\vec\sigma), \quad |i\rangle, |i] : \text{Weyl Spinors}$$

$$\langle ij\rangle = \langle i|^\alpha|j\rangle_\alpha = \varepsilon^{\alpha\beta}|i\rangle_\beta|j\rangle_\alpha, \quad \varepsilon^{\alpha\beta} : \text{antisymmetric tensor}$$

Modern Amplitude Method

The answer is On-shell method.

$$\text{Gauge Theory} \xrightarrow{\text{On-shell method}} \text{S-Matrix}$$

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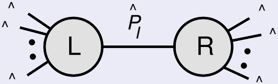
$$A_5[1^-2^-3^-4^-5^-] = 0$$
$$A_5[1^+2^-3^-4^-5^-] = 0$$
$$A_5[1^+2^+3^-4^-5^-] = \frac{[12]^4}{[12][23][34][45][51]}$$

BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_{\text{diagrams } I}$$



The Power of BCFW Recursion Relation

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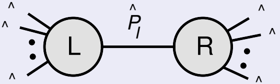
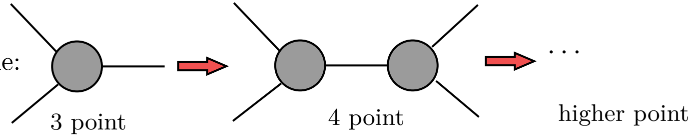


More understandable:



★ From lower point on-shell amp. to higher point on-shell amp.!!

More specifically, it helps us to prove P. T. formula

$$3\text{pt.} \longrightarrow 4\text{pt.} \longrightarrow 5\text{pt.} \longrightarrow \cdots$$

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

More specifically, it helps us to prove P. T. formula

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$$\boxed{\Rightarrow: \quad A[1^+ \cdots\ i^-(i+1)^+ \cdots\ j^-(j+1)^+ \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}}$$

★ This is the power of BCFW recursion relation.

From Review to Applications

So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
  - **Pure Yang-Mills** theory
  - Tree-level MHV amplitudes
  - Color-ordered partial amplitudes

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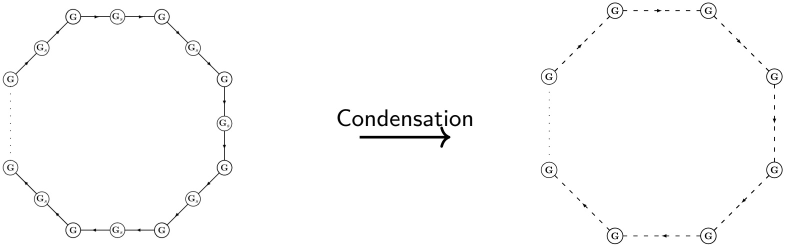
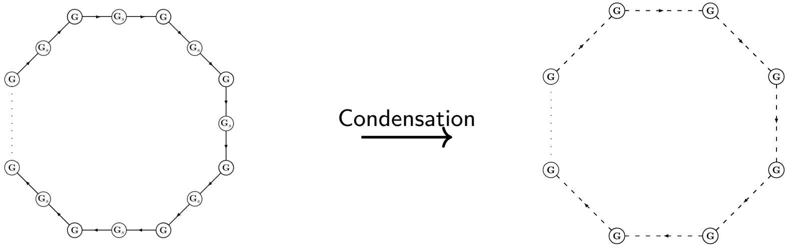
Next: Realistic Models

- Move beyond massless gauge theory
- Consider:
  - **(De)constructed gauge theories**
- Key questions:
  - Can BCFW still apply?
  - What new structures emerge?

Contents

Introduction of (De)Constructed gauge theory





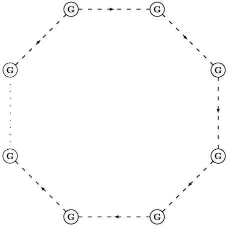
The Lagrangian can be written like

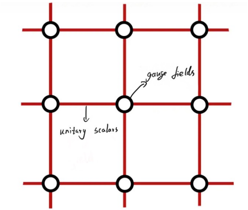
$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2}\mathrm{Tr}(F_i^2) + \sum_{i=1}^{N} \mathrm{Tr}\left[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)\right],$$

here F_i refers to the ith gauge field strength. The scalar field Φ_i transforms under the bi-fundamental representation, and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - i g_i A_{i\mu} \Phi_i + i g_{i+1} \Phi_i A_{i+1\mu}.$$

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group $SU(m)$, where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.





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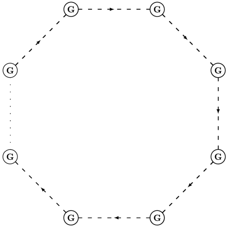
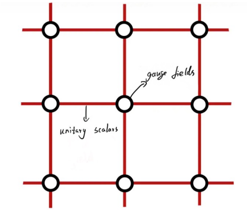


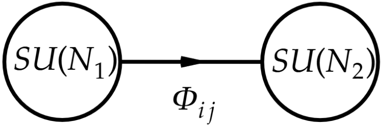


After higgsing the scalar field, we can obtain a spectrum

$$M_k^2 = 4g^2 f_s^2 \sin^2\left(\frac{\pi k}{N}\right)$$

This is precisely the Kaluza-Klein spectrum under S^1 compactification.

Amplitudes from BCFW



From the Lagrangian, we have known that there are only two kinds of 3 point amplitude ($+, -$: helicity $\quad \Phi, \Phi^\dagger$: charge of scalar)

$$A[1^\Phi 2^{\Phi^\dagger} 3^+] = \frac{[23][31]}{[12]}, \qquad A[1^\Phi 2^{\Phi^\dagger} 3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

$$A[3^+ 4^+ 5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^- 4^- 5^+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

By using the 3 point building block, we can construct 4 point colorordered amplitudes from BCFW recursion relation.

Gauge boson sector

- nV_1 or nV_2
  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\boxed{\text{Parke - Talyor Formula :} \quad A[\cdots i^- \cdots j^- \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

SQCD like sector

The amplitudes can be computed like

- $\Phi^\dagger V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^-] = (-1)\frac{\langle 14\rangle^2 \langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad (\text{Parke -Talyor like Formula})$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^+ 5^-] = \frac{\langle 15\rangle^2 \langle 25\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1^{\Phi}2^{\Phi^\dagger}\cdots(n+2)^-] = (-1)^{n+1}\frac{\langle 1,n+2\rangle^2\langle 2,n+2\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n+1,n+2\rangle\langle n+2,1\rangle}$$

$\star$ Bonus relation:

$$A[1^{\Phi}2^{\Phi^\dagger}3^+4^+] = 0 \quad \Rightarrow \quad A[1^{\Phi}2^{\Phi^\dagger}3^+\cdots n^+] = 0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

- $\Phi V_2 \Phi^\dagger V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_2^-] = \frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_1^+ 5_2^-] = (-1)\frac{\langle 25\rangle^2\langle 15\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 51\rangle}$$

- $\Phi V_2 V_2 \Phi^\dagger V_1 V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_1^+ 5_2^+ 6_2^-] = \frac{\langle 26\rangle^2\langle 16\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 56\rangle\langle 61\rangle}$$

- Compact formula for general case

$$A = \frac{\langle 2a\rangle^2 \langle 1a\rangle^2}{\underbrace{\langle 2\star\rangle \cdots \langle \star 1\rangle}_{SU(N_1)} \underbrace{\langle 2*\rangle \cdots \langle * 1\rangle}_{SU(N_2)}}$$

Green: Particle with $-$ helicity

Blue: Particle belongs to the first gauge group

Red: Particle belongs to the second gauge group

$\star$: Order for gauge group 1

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For example, if we want to compute $A[1^{\Phi} 2^{\Phi^\dagger} 5_1^+ 3_1^+ 4_1^- 7_2^+ 6_2^+ 8_2^+]$:

$$A = \frac{\langle 24\rangle^2 \langle 14\rangle^2}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle\langle 27\rangle\langle 76\rangle\langle 68\rangle\langle 81\rangle}$$

If you use Feynman diagrams, it may take sevral days to accomplish the computation.

Contents

Summary

- Introduce the on-shell method, including BCFW recursion relation, spinor-helicity formalism, etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

Thanks for your attention!

Spinor-Helicity Formalism

Helicity

Helicity is defined as the projection of a particle's spin vector $\vec{S}$ onto the direction of its momentum $\vec{p}$:

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

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S-matrix is a function of momentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

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Massless Case:

- Momenta in spinor form:

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}} = |p\rangle [p|$$

Brief explaination: We choose two momentum to be shifted oppositely

$$p_i \to \hat{p}_i(z) \equiv p_i - zk, \qquad p_j \to \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \qquad p_i \cdot k = 0, \qquad p_j \cdot k = 0$$

We consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

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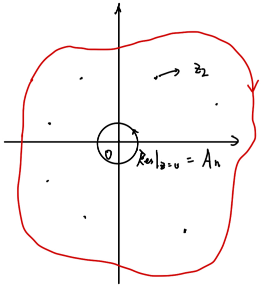
If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane. From Cauchy Theorem, we can ontain

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.



★ The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

and

$$\mathrm{Res}|_{z=z_I} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j\rangle$	$+$	$-$
$+$	$1/z$	z^3
$-$	$1/z$	$1/z$

On-shell 3-point can be completely determined

For the complex momentum, we have

$$|1\rangle \propto |2\rangle \propto |3\rangle \qquad or \qquad |1] \propto |2] \propto |3]$$

$$\boxed{\begin{array}{ll} A_3^{h_1 h_2 h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} & h_1+h_2+h_3<0 \\ A_3^{h_1 h_2 h_3} = c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & h_1+h_2+h_3>0 \end{array}}$$

$\star$ **All massless on-shell 3-point ampltides are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

Scattering Amplitudes from BCFW

For simplicity, we start from the two-site gauge theory with gauge fields V_1, V_2 and scalar fields Φ, $\Phi^\dagger$.

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_1)^2 - \frac{1}{2}\mathrm{Tr}(F_2)^2 + \mathrm{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

We only foucus on the following amplitudes:

$$nV_1, \qquad nV_2, \qquad \Phi^\dagger nV_1\Phi, \qquad \Phi nV_2\Phi^\dagger, \qquad \Phi^\dagger\Phi\Phi^\dagger\Phi$$

here n can be any positive integer.