

# On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

**Su Yingze**

E Lab, Nagoya University

July 1st, 2025

# Contents

- 1 Motivation
- 2 Preliminary
- 3 Model and Computation
- 4 Some problems and extends
- 5 Summary

# Why We Study Scattering Amplitudes?

## 1. Bridge between theory and experiment

- Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
- Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections

# Why We Study Scattering Amplitudes?

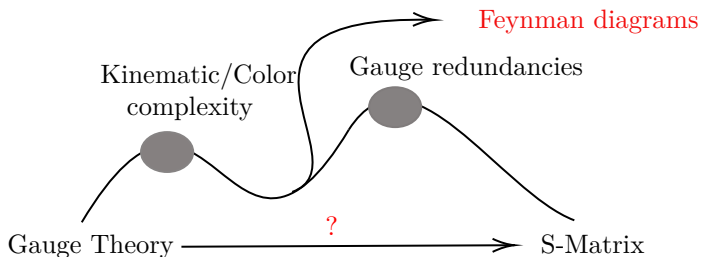
## 1. Bridge between theory and experiment

- Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
- Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections

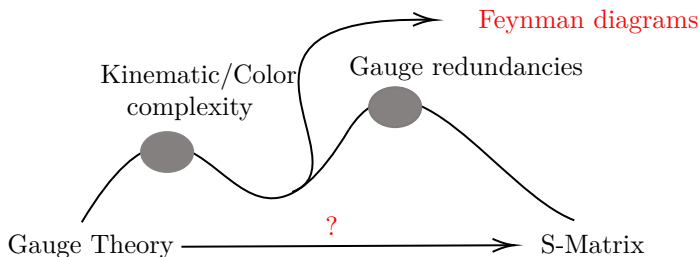
## 2. Reveal deep structures of quantum field theory

- Amplitudes exhibit hidden symmetries (e.g., dual conformal, Yangian) not visible in the Lagrangian
- These symmetries suggest deeper theoretical frameworks, such as amplituhedra or holographic principle (celestial duality).

# Challenges we face before



# Challenges we face before

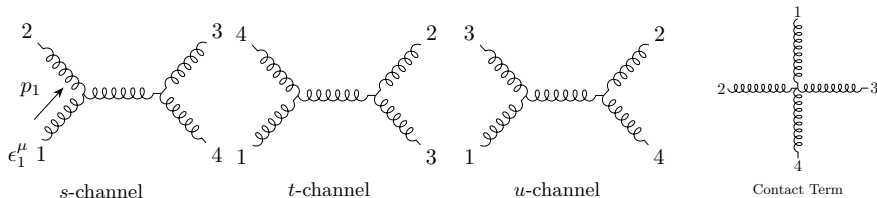


$n$ pt. amplitudes	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

The number of Feynman diagrams grow quite rapidly!

# Conventional Computation

Usually, when we compute the gluon amplitudes by using Feynman diagram, we will obtain something like



$$\begin{aligned} \mathcal{M}_s(p_1 p_2 \rightarrow p_3 p_4) = & -\frac{g_s^2}{s} f^{abe} f^{cde} \\ & \times \left\{ -4 (\epsilon_1 \cdot \epsilon_3^*) (\epsilon_2 \cdot p_1) (\epsilon_4^* \cdot p_3) + 2 (\epsilon_1 \cdot \epsilon_2) (\epsilon_3^* \cdot p_1) (\epsilon_4^* \cdot p_3) \right. \\ & \left. - 2 (\epsilon_1 \cdot p_4) (\epsilon_2 \cdot p_1) (\epsilon_3^* \cdot \epsilon_4^*) + (\epsilon_1 \cdot \epsilon_2) (p_4 \cdot p_1) (\epsilon_3^* \cdot \epsilon_4^*) + 10 \text{ terms} \right\} \end{aligned}$$

If you consider 5point case, it will become worse:

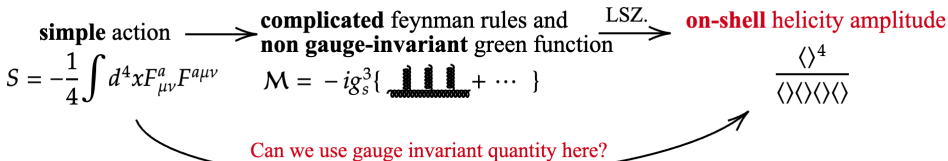
★ We have 25 diagrams and nearly 10000 terms!





If you consider 5point case, it will become worse:

★ We have 25 diagrams and nearly 10000 terms!



# Contents

- 1 Motivation
- 2 Preliminary
- 3 Model and Computation
- 4 Some problems and extends
- 5 Summary

# Modern Amplitude Method

The answer is On-shell method.

Gauge Theory  $\xrightarrow{\text{On-shell method}}$  Helicity Amplitude

# Modern Amplitude Method

The answer is On-shell method.

Gauge Theory  $\xrightarrow{\text{On-shell method}}$  Helicity Amplitude

$$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \text{Tr}[T^{a_1} T^{a_2} \dots T^{a_5}] + \text{permutations}$$

Color-ordered Amplitudes

**Parke–Taylor Formula (MHV amplitudes):**

$$\begin{aligned} A_5[1^+ 2^+ 3^+ 4^+ 5^+] &= 0 & (+, - : \text{helicity}; \\ A_5[1^- 2^+ 3^+ 4^+ 5^+] &= 0 & 1, 2, \dots, n : \text{particle labels}) \end{aligned}$$

# Modern Amplitude Method

The answer is On-shell method.

Gauge Theory  $\xrightarrow{\text{On-shell method}}$  Helicity Amplitude

$$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \quad \text{Tr}[T^{a_1} T^{a_2} \dots T^{a_5}] + \text{permutations}$$

Color-ordered Amplitudes

**Parke–Taylor Formula (MHV amplitudes):**

$$\begin{aligned} A_5[1^+ 2^+ 3^+ 4^+ 5^+] &= 0 & (+, - : \text{helicity}; \\ A_5[1^- 2^+ 3^+ 4^+ 5^+] &= 0 & 1, 2, \dots, n : \text{particle labels}) \end{aligned}$$

$$A_5[1^- 2^- 3^- 4^- 5^-] = 0$$

$$A_5[1^+ 2^- 3^- 4^- 5^-] = 0$$

$$A_5[1^+ 2^+ 3^- 4^- 5^-] = \frac{[12]^4}{[12][23][34][45][51]}$$

# Color-ordering for Yang-Mills

Consider the Yang-Mills lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The 3 point and 4 point vertices include  $\tilde{f}^{abc}$  and  $\tilde{f}^{abe}\tilde{f}^{cde} + \text{perms.}$   
(With a different convention,  $\text{Tr}[T^a T^b] = \delta^{ab}$  and  $[T^a, T^b] = i\tilde{f}^{abc}T^c$ )

We have

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}$$

and the color factor can be rewritten by the trace of product of generators

$$i\tilde{f}^{abc} = \text{Tr}([T^a, T^b]T^c),$$

Moreover, in  $SU(N)$ , we have a Fierz identity

$$\sum_a T_{ij}^a T_{kl}^a = \delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}. \quad (1)$$

This identity is easier understood as matrix form like

$$\text{Tr}\{T^a A\}\text{Tr}\{T^a B\} = \text{Tr}\{AB\} - \frac{1}{N}\text{Tr}\{A\}\text{Tr}\{B\},$$

$$\text{Tr}\{AT^a BT^a\} = \text{Tr}\{A\}\text{Tr}\{B\} - \frac{1}{N}\text{Tr}\{AB\}.$$

So, the 4 gluon s-channel gives us

$$\begin{aligned}\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - \text{Tr}(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) \\ &\quad - \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) + \text{Tr}(T^{a_2} T^{a_1} T^{a_4} T^{a_3}).\end{aligned}$$

Therefore, the full 4-point amplitude can be rewritten like

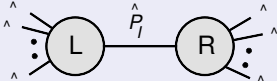
$$\mathcal{A}_{4,\text{tree}} = g^2 (A_4[1234] \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{perms of } (234))$$

here the subamplitudes  $A_4[1234]$ ,  $A_4[1243]$ , etc. are called **color-ordered amplitudes**. This concept can be easily generalized to tree-level n-point case

$$\mathcal{A}_{n,\text{tree}} = g^{n-2} \sum_{\sigma} A_n[1, \sigma(2, 3 \cdots n)] \text{Tr}(T^{a_1} T^{\sigma(a_2 \cdots a_n)})$$

# The Power of BCFW Recursion Relation

## BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$


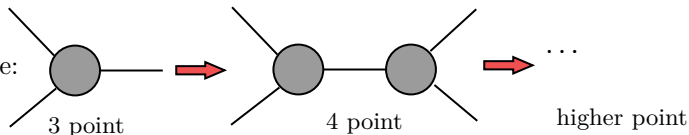


# The Power of BCFW Recursion Relation

## BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \hat{A}_L^{\wedge} \hat{P}_I \hat{A}_R^{\wedge}$$

More understandable:



★ From lower point on-shell amp. to higher point on-shell amp.!!

# Momentum Shift in BCFW

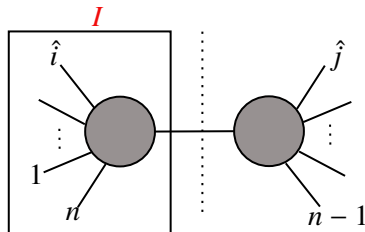
## What did BCFW do to make the shift?

Here we consider the case in which all particles are massless,  $p_i^2 = 0$  for all  $i = 1, 2, \dots, n$ . We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$



For a non-trivial subset of generic momenta  $\{p_i\}_{i \in I}$

$$\hat{P}_I^2 = P_I^2 - 2z P_I \cdot k = -\frac{P_I^2}{z_I}(z - z_I)$$

with  $z_I = \frac{P_I^2}{2P_I \cdot k}$ .

**Brief explanation:** We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$

We consider amplitude  $A_n$  in terms of shifted momentum  $\hat{p}_i^\mu$  instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

**Brief explanation:** We choose two momentum to be shifted oppositely

$$p_i \rightarrow \hat{p}_i(z) \equiv p_i - zk, \quad p_j \rightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \quad p_i \cdot k = 0, \quad p_j \cdot k = 0$$

We consider amplitude  $A_n$  in terms of shifted momentum  $\hat{p}_i^\mu$  instead of original real momentum.

$$A_n \longrightarrow \hat{A}_n(z)$$

If we consider the meromorphic function  $\frac{\hat{A}_n(z)}{z}$  in the complex plane. From Cauchy Theorem, we can obtain

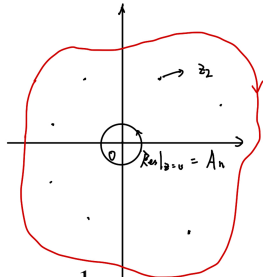
$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where  $B_n$  is the residue of the pole at  $z = \infty$ , called boundary term.

From Cauchy Theorem, we can obtain

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where  $B_n$  is the residue of the pole at  $z = \infty$ , called boundary term.



$$\hat{A}_n(z) \xrightarrow{z \text{ near } z_I} \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I) = - \frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at  $z = z_I$

$$-\text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} = \frac{(z - z_I)z_I}{z(z - z_I)} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)|_{z=z_I} = \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

# Large $z$ behavior

In the BCFW formula, the boundary term  $B_n$  affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications, one assumes or much better, proves  $B_n = 0$ . This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large  $z$  behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	$z^3$
-	$1/z$	$1/z$

proved by using background field expansion (N. Arkani-Hamed and J. Kaplan, [arXiv:0801.2385 [hep-th]].)

# Little group scaling

- **Massless Case**

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \rightarrow t^{-1}\lambda, \quad \tilde{\lambda} \rightarrow t\tilde{\lambda}, \quad t \in \mathbb{C}$$

same for

$$|\lambda\rangle \rightarrow t^{-1}|\lambda\rangle, \quad [\lambda] \rightarrow t[\lambda]$$

The scattering amplitudes should transform **covariantly** under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \dots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \dots) = t_i^{2h_i} \mathcal{A}_n$$

- **Massive Case**

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

# On-shell 3-point can be completely determined

**Another necessity to introduce complex momentum** If the momentum is complexed, we have

$$\langle 12 \rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or square brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}} \langle 12 \rangle^{x_{12}} \langle 13 \rangle^{x_{13}} \langle 23 \rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \quad 2h_3 = -x_{13} - x_{23}.$$



Then all index can be solved from this system of equations, so that

$$\boxed{\begin{aligned} A_3^{h_1 h_2 h_3} &= c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3} & h_1 + h_2 + h_3 < 0 \\ A_3^{h_1 h_2 h_3} &= c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} & h_1 + h_2 + h_3 > 0 \end{aligned}}$$

★ **All massless on-shell 3-point amplitudes are completely determined by little group scaling!**

**Example:** 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g' \frac{[13][23]}{[12]^3}$$

but actually it comes from the **non-local** interaction  $g' A \frac{\partial}{\square} A$ , so we discard it.

## So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
  - **Pure Yang-Mills** theory
  - Tree-level MHV amplitudes
  - Color-ordered partial amplitudes

## So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
  - **Pure Yang-Mills** theory
  - Tree-level MHV amplitudes
  - Color-ordered partial amplitudes

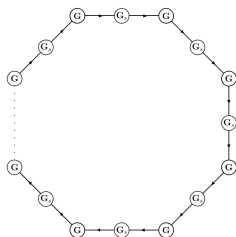
## Next: Realistic Models

- Move beyond massless gauge theory
- Consider:
  - **(De)constructed gauge theories**
- Key questions:
  - Can BCFW still apply?
  - What new structures emerge?

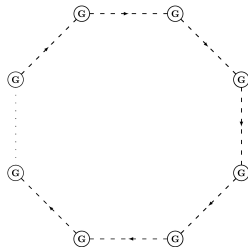
# Contents

- 1 Motivation
- 2 Preliminary
- 3 Model and Computation**
- 4 Some problems and extends
- 5 Summary

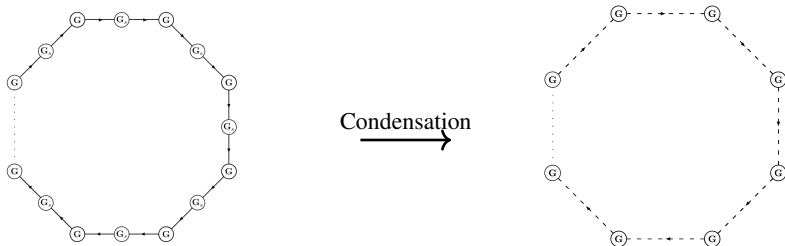
# Introduction of (De)Constructed gauge theory



Condensation  
 $\longrightarrow$



# Introduction of (De)Constructed gauge theory



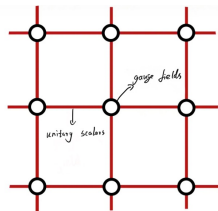
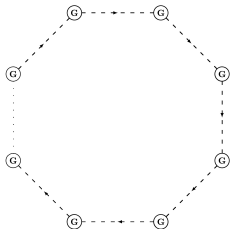
The Lagrangian can be written like

$$\mathcal{L} = - \sum_{i=1}^N \frac{1}{2} \text{Tr}(F_i^2) + \sum_{i=1}^N \text{Tr} \left[ (D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) \right],$$

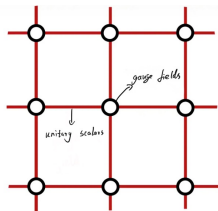
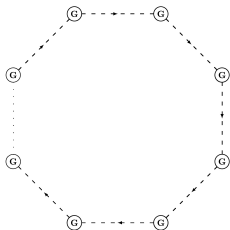
here  $F_i$  refers to the  $i$ th gauge field strength. The scalar field  $\Phi_i$  transforms under the **bi-fundamental** representation, and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - ig_i A_{i\mu} \Phi_i + ig_{i+1} \Phi_i A_{i+1\mu}.$$

It has been proposed that this model actually discretized a **five-dimension gauge theory** with gauge group  $SU(m)$ , where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.



It has been proposed that this model actually discretized a **five-dimension gauge theory** with gauge group  $SU(m)$ , where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.



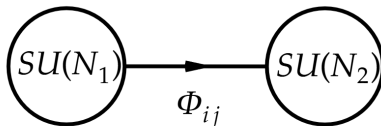
After higgsing the scalar field, we can obtain a spectrum

$$M_k^2 = 4g^2 f_s^2 \sin^2 \left( \frac{\pi k}{N} \right)$$

This is precisely the **Kaluza-Klein** spectrum under  $S^1$  compactification.



# Amplitudes from BCFW



$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_1)^2 - \frac{1}{2}\text{Tr}(F_2)^2 + \text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

From the Lagrangian, we have known that there are only two kinds of 3 point amplitude (+, - : helicity  $\Phi, \Phi^\dagger$  : charge of scalar)

$$A[1^\Phi 2^{\Phi^\dagger} 3^+] = \frac{[23][31]}{[12]}, \quad A[1^\Phi 2^{\Phi^\dagger} 3^-] = \frac{\langle 23 \rangle \langle 31 \rangle}{\langle 12 \rangle}$$
$$A[3^+ 4^+ 5^-] = \frac{[34]^3}{[45][53]}, \quad A[3^- 4^- 5^+] = \frac{\langle 34 \rangle^3}{\langle 45 \rangle \langle 53 \rangle}$$

By using the 3 point building block, we can construct 4 point colorordered amplitudes from BCFW recursion relation.

- $nV_1$  or  $nV_2$

This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\text{Parke - Talyor Formula : } A[\cdots i^- \cdots j^- \cdots] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

The color factor in this sector looks like

$$(T^{a_1} T^{a_2} \dots T^{a_n})_{ij}$$

so we need to notice is just the order of gauge boson.

The amplitudes can be computed like

- $\Phi^\dagger V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^-] = (-1) \frac{\langle 14 \rangle^2 \langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad (\text{Parke -Talyor like Formula})$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^+ 5^-] = \frac{\langle 15 \rangle^2 \langle 25 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1^\Phi 2^{\Phi^\dagger} \cdots (n+2)^-] = (-1)^{n+1} \frac{\langle 1, n+2 \rangle^2 \langle 2, n+2 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n+1, n+2 \rangle \langle n+2, 1 \rangle}$$

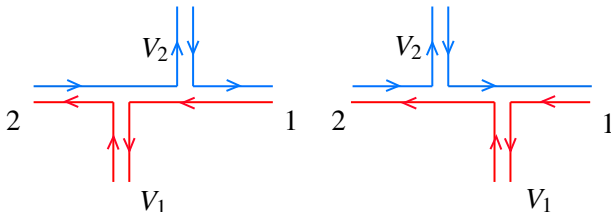
★ Bonus relation:

$$A[1^\Phi 2^{\Phi^\dagger} 3^+ 4^+] = 0 \quad \Rightarrow \quad A[1^\Phi 2^{\Phi^\dagger} 3^+ \cdots n^+] = 0$$

For the amplitude  $\Phi(nV_2)\Phi^\dagger$ , we can obtain nearly the same expression.

# Pure 2-site amplitude

The straightforward way to observe the color structure in this case is double line notation as follows



The color factor here have special form like

$$(T_1^{a_1} T_1^{a_2} \cdots T_1^{a_{n_1}})_{ij} (T_2^{b_1} T_2^{b_2} \cdots T_2^{b_{n_2}})_{\bar{j}\bar{i}}$$

we can notice that the relative order between two gauge group do not affect the color structure, but the order inside the gauge group matters.

So we introduce the **OPP (Order Preserving Permutation)**

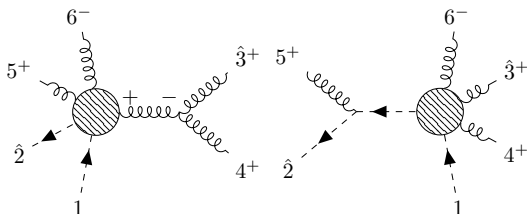
- $\Phi V_2 \Phi^\dagger V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_2^-] = \frac{\langle 14 \rangle \langle 24 \rangle}{\langle 13 \rangle \langle 23 \rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1^\Phi 2^{\Phi^\dagger} 3_1^+ 4_1^+ 5_2^-] = (-1) \frac{\langle 25 \rangle^2 \langle 15 \rangle^2}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 25 \rangle \langle 51 \rangle}$$

Here I show the concrete computation process



$$\begin{aligned}
 A_1 &= \frac{(-1)\langle\hat{2}6\rangle^2\langle16\rangle^2}{\langle25\rangle\langle56\rangle\langle61\rangle\langle\hat{2}\hat{1}\rangle\langle\hat{1}1\rangle} \times \frac{1}{s_{34}} \times \frac{[\hat{3}4]^3}{[4\hat{1}][\hat{1}\hat{3}]} \\
 &= \frac{\langle26\rangle^2\langle16\rangle}{\langle25\rangle\langle56\rangle\langle\hat{2}\hat{1}\rangle\langle\hat{1}1\rangle} \times \frac{1}{s_{34}} \times \frac{[34]^3}{[4\hat{1}][\hat{1}\hat{3}]} \\
 &= \frac{\langle26\rangle^2\langle16\rangle\cancel{[34]^3\langle42\rangle}}{\langle25\rangle\langle56\rangle\langle41\rangle\langle32\rangle\langle43\rangle\cancel{[43][43]}\cancel{[34]}\cancel{\langle24\rangle}} \\
 &= \frac{\langle26\rangle^2\langle16\rangle^2}{\langle23\rangle\langle34\rangle\langle41\rangle\langle25\rangle\langle56\rangle\langle61\rangle}
 \end{aligned}$$

- Compact formula for general case

$$A = \frac{\langle 2a \rangle^2 \langle 1a \rangle^2}{\underbrace{\langle 2\star \rangle \cdots \langle \star 1 \rangle}_{SU(N_1)} \underbrace{\langle 2* \rangle \cdots \langle *1 \rangle}_{SU(N_2)}}$$

**Green:** Particle with – helicity

**Blue:** Particle belongs to the first gauge group

**Red:** Particle belongs to the second gauge group

**★:** Order for gauge group 1

**\***: Order for gauge group 2



- Compact formula for general case

$$A = \frac{\langle 2a \rangle^2 \langle 1a \rangle^2}{\underbrace{\langle 2\star \rangle \cdots \langle \star 1 \rangle}_{SU(N_1)} \underbrace{\langle 2* \rangle \cdots \langle *1 \rangle}_{SU(N_2)}}$$

**Green:** Particle with – helicity

**Blue:** Particle belongs to the first gauge group

**Red:** Particle belongs to the second gauge group

**★:** Order for gauge group 1

**\***: Order for gauge group 2

For example, if we want to compute  $A[1^\Phi 2^{\Phi^\dagger} 5_1^+ 3_1^+ 4_1^- 7_2^+ 6_2^+ 8_2^+]$ :

$$A = \frac{\langle 24 \rangle^2 \langle 14 \rangle^2}{\langle 25 \rangle \langle 53 \rangle \langle 34 \rangle \langle 41 \rangle \langle 27 \rangle \langle 76 \rangle \langle 68 \rangle \langle 81 \rangle}$$

If you use Feynman diagrams, it may take several days to accomplish the computation.

# Contents

- 1 Motivation
- 2 Preliminary
- 3 Model and Computation
- 4 Some problems and extends**
- 5 Summary

# How about NMHV?

First, let us review the NMHV amplitudes for gluon scattering.

Still we begin with the simplest case – **Split-helicity** NMHV like  $A_6[1^- 2^- 3^- 4^+ 5^+ 6^+]$ .

Here we choose  $[1, 2\rangle$  shift

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A} + \text{diagram B}$$

Diagram A: A t-channel exchange diagram. The left vertex has legs  $\hat{1}^-$  and  $6^+$ . The right vertex has legs  $\hat{2}^-$ ,  $3^-$ ,  $4^+$ , and  $5^+$ . The propagator is labeled  $\hat{P}_{16}$  with a minus sign on the left and a plus sign on the right.

Diagram B: A u-channel exchange diagram. The left vertex has legs  $\hat{1}^-$  and  $5^+$ . The right vertex has legs  $\hat{2}^-$ ,  $3^-$ ,  $4^+$ , and  $6^+$ . The propagator is labeled  $\hat{P}_{156}$  with a minus sign on the left and a plus sign on the right.

# How about NMHV?

First, let us review the NMHV amplitudes for gluon scattering.

Still we begin with the simplest case – **Split-helicity** NMHV like  $A_6[1^- 2^- 3^- 4^+ 5^+ 6^+]$ .

Here we choose  $[1, 2\rangle$  shift

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A} + \text{diagram B}$$

Diagram A: A t-channel exchange between legs (1,6) and (2,3,4,5). The propagator is labeled  $\hat{P}_{16}$ . The external legs are labeled  $\hat{1}^-$ ,  $6^+$ ,  $\hat{2}^-$ ,  $3^-$ ,  $4^+$ , and  $5^+$ . The internal lines are labeled  $-$  and  $+$ .

Diagram B: A u-channel exchange between legs (1,5) and (2,3,4,6). The propagator is labeled  $\hat{P}_{156}$ . The external legs are labeled  $\hat{1}^-$ ,  $5^+$ ,  $\hat{2}^-$ ,  $3^-$ ,  $4^+$ , and  $6^+$ . The internal lines are labeled  $-$  and  $+$ .

- Diagram B includes a propagator  $1/P_{156}^2$ , so there is a 3-particle pole  $P_{156}^2 = 0$ . But by inspecting the external order, it seems that there's no difference between  $(-++)$  channel 561 and 345. We should expect the amplitude to have a pole also at  $P_{345}^2 = 0$ .

$$\text{diagram A} = \frac{\langle \hat{1} \hat{P}_{16} \rangle^3}{\langle \hat{P}_{16} 6 \rangle \langle 6 \hat{1} \rangle} \times \frac{1}{P_{16}^2} \times \frac{\langle \hat{2} 3 \rangle^3}{\langle 3 4 \rangle \langle 4 5 \rangle \langle 5 \hat{P}_{16} \rangle \langle \hat{P}_{16} 2 \rangle}$$

$$\langle \hat{2}\hat{P}_{16} \rangle [\hat{P}_{16}3] = \langle 21 \rangle [\hat{1}3] + \langle \hat{2}6 \rangle [63]$$

It follows from  $\hat{P}_{16}^2 = 0$  that  $z_{16} = -\frac{[16]}{[26]}$ , so

$$\langle \hat{2}\hat{P}_{16} \rangle [\hat{P}_{16}3] = -\frac{[36]}{[26]} (\langle 12 \rangle [12] + \langle 16 \rangle [16] + \langle 26 \rangle [26]) = -\frac{[36]}{[26]} P_{126}^2$$

The 3-particle pole  $P_{126}^2$  is encoded inside the BCFW channel !

- Full expression

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \frac{\langle 3|1+2|6 \rangle^3}{P_{126}^2 [21][16] \langle 34 \rangle \langle 45 \rangle \langle 5|1+6|2 \rangle} + \frac{\langle 1|5+6|4 \rangle^3}{P_{156}^2 [23][34] \langle 56 \rangle \langle 61 \rangle \langle 5|1+6|2 \rangle}.$$

The factor  $\langle 5|1+6|2 \rangle$  does not correspond to a physical pole of the scattering amplitude: it is a **spurious pole**.

There has been interesting paper investigating how to systematically cancel the spurious poles, like "A. Hodges, JHEP 1305, 35 (2013) [arXiv:0905.1473 [hep-th]]."

- We utilize the  $[1, 2\rangle$  shift before, what happens if we change to  $[2, 1\rangle$  shift?

$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+] = \text{diagram A'} + \text{diagram B'} + \text{diagram C'}$$

diagram A' anti-MHV  $\times$  NMHV

diagram B' MHV  $\times$  MHV

diagram C' MHV  $\times$  MHV

diagram A' = anti-MHV  $\times$  NMHV, as opposed to diagram A = MHV  $\times$  MHV.

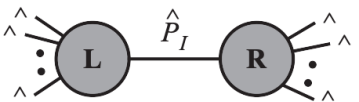
The equivalence between two different shift is related to powerful residue theorem (N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, [arXiv:0907.5418 [hep-th]].) and Grassmannian.

# CSW(Cachazo-Svrcek-Witten) expansion

We can also consider a shift that is implemented via a “holomorphic” square-spinor shift:

$$|\hat{i}\rangle = |i\rangle + zc_i |X], \quad |\hat{i}\rangle = |i\rangle$$

Here  $|X]$  is an arbitrary reference spinor and the coefficients  $c_i$  satisfy  $\sum_{i=1}^n c_i |i\rangle = 0$ .

$$A_n^{\text{NMHV}} = \sum_{\text{diagrams } I} \text{Diagram } I$$
A Feynman diagram representing a term in the CSW expansion. It consists of two circular vertices, labeled 'L' and 'R', connected by a horizontal line. The line connecting them is labeled with a momentum vector  $\hat{P}_I$ . Vertex 'L' has three external lines extending to the left, each ending in a small triangle pointing left. Vertex 'R' has three external lines extending to the right, each ending in a small triangle pointing right. The diagram is part of a sum over all such diagrams 'I'.

There are two possibilities: anti-MHV<sub>3</sub>(= 0) × NMHV or MHV × MHV.

For example, the 6pt split NMHV amplitude

$$\begin{aligned}
 A_n[1^-2^-3^-4^+5^+6^+] = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \\
 & + \text{Diagram 7} \\
 & = \frac{\langle 1\hat{P}_I \rangle^4}{\langle 1\hat{P}_I \rangle \langle \hat{P}_I 5 \rangle \langle 56 \rangle \langle 61 \rangle} \frac{1}{P_{156}^2} \frac{\langle 23 \rangle^4}{\langle 23 \rangle \langle 34 \rangle \langle 4\hat{P}_I \rangle \langle \hat{P}_I 2 \rangle} .
 \end{aligned}$$

The diagrams represent various tree-level Feynman diagrams for the 6-point split NMHV amplitude. Each diagram consists of two vertices connected by a line, with external legs labeled with momenta and helicities. The labels include hats and superscripts to denote specific helicity configurations.



For example, the 6pt split NMHV amplitude

$$A_n[1^-2^-3^-4^+5^+6^+] =$$

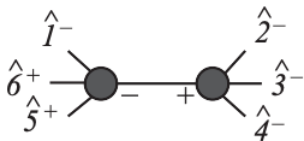
$$= \frac{\langle 1\hat{P}_I \rangle^4}{\langle 1\hat{P}_I \rangle \langle \hat{P}_I 5 \rangle \langle 56 \rangle \langle 61 \rangle} \frac{1}{P_{156}^2} \frac{\langle 23 \rangle^4}{\langle 23 \rangle \langle 34 \rangle \langle 4\hat{P}_I \rangle \langle \hat{P}_I 2 \rangle}$$

We can write

$$|\hat{P}_I\rangle \frac{[\hat{P}_I X]}{[\hat{P}_I X]} = \hat{P}_I |X\rangle \frac{1}{[\hat{P}_I X]} = P_I |X\rangle \frac{1}{[\hat{P}_I X]}$$

We can use the prescription

$$|\hat{P}_I\rangle \rightarrow P_I|X]$$



$$= \frac{\langle 1|P_{156}|X\rangle^4}{\langle 1|P_{156}|X\rangle\langle 5|P_{156}|X\rangle\langle 56\rangle\langle 61\rangle} \cdot \frac{1}{P_{156}^2} \cdot \frac{\langle 23\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4|P_{156}|X\rangle\langle 2|P_{156}|X\rangle}.$$

# Contents

- 1 Motivation
- 2 Preliminary
- 3 Model and Computation
- 4 Some problems and extends
- 5 Summary**

- Introduce the on-shell method, including BCFW recursion relation, spinor-helicity formalism, etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

Thanks for your attention!

## Helicity

**Helicity** is defined as the projection of a particle's spin vector  $\vec{S}$  onto the direction of its momentum  $\vec{p}$ :

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

# Spinor-Helicity Formalism

## Helicity

**Helicity** is defined as the projection of a particle's spin vector  $\vec{S}$  onto the direction of its momentum  $\vec{p}$ :

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

S-matrix is a function of momentum  $p_i$  and helicity  $h_i$

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

# Spinor-Helicity Formalism

## Helicity

**Helicity** is defined as the projection of a particle's spin vector  $\vec{S}$  onto the direction of its momentum  $\vec{p}$ :

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

S-matrix is a function of momentum  $p_i$  and helicity  $h_i$

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

## Massless Case:

- Momenta in spinor form:

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}} = |p\rangle[p|$$



# Large $z$ behavior

In the BCFW formula, the boundary term  $B_n$  affects a lot

$$A_n = - \sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications, one assumes or much better, proves  $B_n = 0$ . This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \rightarrow 0 \quad \text{for} \quad z \rightarrow \infty$$

Here I show the large  $z$  behavior for gluon scattering

$[i \setminus j]$	+	-
+	$1/z$	$z^3$
-	$1/z$	$1/z$

## On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \quad p_2 = (1 - \kappa)p_3 \quad (\text{Collinear})$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

**We can not obtain any thing nontrivial from 3-point!**

Of course, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\square \Psi)$$

but it still equals to 0 under the on-shell condition.

# On-shell 3-point can be completely determined

For the complex momentum, we have

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]$$

$A_3^{h_1 h_2 h_3} = c \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 31 \rangle^{h_2 - h_1 - h_3} \langle 23 \rangle^{h_1 - h_2 - h_3}$	$h_1 + h_2 + h_3 < 0$
$A_3^{h_1 h_2 h_3} = c' [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2}$	$h_1 + h_2 + h_3 > 0$

★ **All massless on-shell 3-point amplitudes are completely determined by little group scaling!**

**Example:** 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

# Scattering Amplitudes from BCFW

For simplicity, we start from the two-site gauge theory with gauge fields  $V_1, V_2$  and scalar fields  $\Phi, \Phi^\dagger$ .

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_1)^2 - \frac{1}{2}\text{Tr}(F_2)^2 + \text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

We only focus on the following amplitudes:

$$nV_1, \quad nV_2, \quad \Phi^\dagger nV_1 \Phi, \quad \Phi nV_2 \Phi^\dagger, \quad \Phi^\dagger \Phi \Phi^\dagger \Phi$$

here  $n$  can be any positive integer.

More specifically, it helps us to prove P. T. formula

$$\begin{array}{ccccc}
 3\text{pt.} & \longrightarrow & 4\text{pt.} & \longrightarrow & 5\text{pt.} & \longrightarrow & \cdots \\
 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} & & \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} & & \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} & & 
 \end{array}$$

More specifically, it helps us to prove P. T. formula

$$\begin{array}{c}
 3\text{pt.} \longrightarrow 4\text{pt.} \longrightarrow 5\text{pt.} \longrightarrow \cdots \\
 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}
 \end{array}$$

$$\Rightarrow: \quad A[1^+ \cdots i^- (i+1)^+ \cdots j^- (j+1)^+ \cdots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

★ This is the power of BCFW recursion relation.