Tree Level Scattering Amplitudes in (De)Constructed Gauge Theory

Author:
Su Yingze

Affiliation:
Department of Physics, Faculty of Science
Nagoya University

Major:
Theoretical Physics

Degree:
Master of Science

Submitted on:
June 2025

Abstract

This paper mainly show the computation for scattering amplitudes in a kind of (De)Constructed gauge theory, by using the so called on shell method. As we have known, under the conventional quantum field theory frame, Feynman proposed a brilliant method – Feynman diagrams, to help us perturbatively compute scattering amplitude by a diagrammatic method. However, this method faces many challenges during the improvements of physical theory and complexity of construction for model building, there are many amplitudes hard to compute by hand or even impossible to compute. Hence, it is quite necessary to introduce a new method.

Contents

1 Introduction

As we have known, under the conventional quantum field theory frame, Richard Feynman proposed a brilliant method – Feynman diagrams, to help us perturbatively compute scattering amplitude by a diagrammatic method. However, this method faces many challenges during the improvements of physical theory and complexity of construction for model building, there are many amplitudes hard to compute by hand or even impossible to compute. In gauge theory, there are huge number of gauge redundancies, making the computation quite complicated. We need to address with this kind of unphysical degree of freedom, otherwise we can not obtain the correct physical quantities. Also, we have to address with kinematic factor and color factor simultaneously in the nonabelian gauge theory. And it has been known that the amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory obtain the symmetry – dual superconformal symmetry [1], which is not reflected in conventional Feynman diagram method.

Hence, all of these factors impetus physicists to propose something new, then the BCFW recursion relation is the first product as a new method to compute amplitudes . Historically, the original recursion relation is for gluon scattering amplitudes, coming from Britto, Cachazo and Feng [2], and it can be seen as the first breakthrough for modern amplitude method. They explored the analytic properties of amplitudes when extended into complex momentum space. In particular, they considered deforming two external momenta by a complex parameter z, in such a way that the on-shell conditions and momentum conservation are preserved. Through this deformation, they observed that tree-level amplitudes exhibit simple pole structures in the complex z-plane, corresponding to internal propagators going on-shell. This analytic structure allowed them to derive a recursion relation that expresses an n-point amplitude in terms of lower-point amplitudes. Edward Witten subsequently noticed that his Twistor String Theory [3], which reveals the hidden symmetry and geometry structure of amplitude, implied that the scattering amplitudes has stronger analytic property. Their collaboration led to a general formulation of the recursion relations, now known as the BCFW recursion relations [4], named after Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. In particular, Witten helped clarify the large – z behavior of the amplitudes under complex momentum shifts — a crucial condition ensuring the validity of the recursion.

The starting point comes from the precise cancellation in scattering for longitudinal modes of massive spin-2 Kaluza-Klein(KK) states. While individual contributions grow as $\mathcal{O}(s^5)$, $\mathcal{O}(s^4)$, $\mathcal{O}(s^3)$ and $\mathcal{O}(s^2)$, it has been proved that these contributions are cancelled with each other in a quite intricate way [5], and the final results only grow as $\mathcal{O}(s)$. But it is quite difficult to compute this kind of scattering amplitudes, so if we can obtain some clues for this KK scattering amplitudes from other aspects, it may help us to understand this cancellation in another way. This paper is motivated by a kind of (De)constructed gauge theory, proposed by Nima, Cohen and Georgi [6]. They constructed a renormalizable, asymptotically free, four dimension gauge theories that dynamically generate a fifth dimension. In this paper, the authors proposed that the “ Condensed ” theory actually discretized a five dimension gauge theory with gauge group $SU(m)$. After higgsing, the Kaluza-Klein spectrum for S^1 compactification appears. It encourages to compute scattering amplitudes in this model. This paper mainly contributes to the computation for scattering amplitudes in the simplest 2-site model by utilizing BCFW recursion relation and other related method, such as color-ordered amplitudes, spinor-helicity formalism, etc.

2 Review of BCFW recursion relation and others

Traditionally, one relies on Feynman diagrams to calculate scattering amplitudes. Feynman diagrams provide a clear picture of physics and a systematic procedure of calculations. They are in textbooks and widely used. But Feynman diagrams are not efficient in complicated calculations for high energy physics. Increasing the number of particles in a scattering, the number of Feynman diagrams increase exponentially. If gauge fields are involved, one easily encounters thousands of diagrams. For example, for pure gluon case, the number of Feynman diagrams for n-gluons at tree-level is given by

n=	4	5	6	7	8	9	10
	4	25	220	2485	34300	559405	10525900

(These numbers are counted with the inclusion of 4 point interaction.)

Not only with huge number of diagrams, the expression for a single Feynman diagram can also be very complicated. For example, the three-graviton vertex has almost 100 terms. It is almost impossible to calculate scattering amplitudes of gravitons directly from Feynman diagrams. For gauge theories, single Feynman diagram usually depends on the gauge. Many terms cancel with each other at the end of process of calculation. In practice, one does not even know where to start most times.

BCFW are devised to solve some of these problems. So in the following part, I will give a systematic introduction to BCFW recursion relation and other necessary tools. This section is mainly based on the excellent review by Elvang and Huang [7].

2.1 Spinor-Helicity Formalisim for Massless Particles

2.1.1 Brief introduction of spinor-helicity formalism

The spinor-helicity formalism just told us that a light-like Lorentz 4-vector can be decomposed to the product of two Weyl spinor. It is quite natural to see it from the representation of Lorentz group. A Lorentz 4-vector lives in $(\frac{1}{2}, \frac{1}{2})$ representation, which can be decomposed to $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$. We have known that the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ correspond to left handed Weyl spinor and right handed Weyl spinor respectively.

Given a null momentum p_μ in four dimension spacetime, we can define a 2×2 matrix by sigma matrix

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}$$

note that $\det p_{\alpha\dot{\alpha}} = 0$ for massless particles, so it is always possible find two Weyl spinor(two components quantity) satsfying the following equation

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda| \tag{2.1}$$

For general complex momenta, the λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are independent two dimensional complex vectors. For real momenta, the matrix is Hermitian and so we have $\tilde{\lambda}_{\dot{\alpha}} = (\pm)(\lambda_\alpha)^*$.

Thet satisfy the following Weyl equation

$$p_{\alpha\dot{\alpha}}|p]^{\dot{\alpha}} = 0, \quad [p|_{\dot{\alpha}} p^{\dot{\alpha}\alpha} = 0, \quad p^{\dot{\alpha}\alpha}|p\rangle_\alpha, \quad \langle p|^\alpha p_{\alpha\dot{\alpha}} = 0 \tag{2.2}$$

and we can use two-dimension antisymmetric tensor to raise or lower the indices

$$[p|_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}|p]^{\dot{\beta}}, \qquad \langle p|^\alpha = \varepsilon^{\alpha\beta}|p\rangle_\beta \tag{2.3}$$

Then,

The angle and square spinors are the core of **spinor-helicity formalism**.

Here, it is also necessary to introduce the **angle spinor bracket** $\langle pq\rangle$ and **square spinor bracket** $[pq]$, it is the key ingredient for writing amplitudes in terms of spinor-helicity variable.

$$\langle pq\rangle = \langle p|^\alpha |q\rangle_\alpha, \qquad [pq] = [p|_{\dot{\alpha}}|q]^{\dot{\alpha}}. \tag{2.4}$$

Since the indices are raised and lowed by antisymmetric tensor, so the brackets are antisymmetric:

$$\langle pq\rangle = -\langle qp\rangle, \qquad [pq] = -[qp]. \tag{2.5}$$

There are no $\langle pq]$ brackets, because the indices cannot contract with each other to form a Lorentz scalar.

It is very easy to derive the following important relation:

$$\langle pq\rangle[pq] = 2p\cdot q = (p+q)^2 \tag{2.6}$$

by using (2.1) and

$$\mathrm{Tr}(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu}.$$

In amplitude calculations, **momentum conservation** is imposed on n particles as $\sum_{i=1}^n p_i^\mu =$

0(here we consider all particles ingoing). Translating by spinor-helicity variable, it becomes

$$\sum_{i=1}^{n} |i\rangle[i| = 0, \quad \text{i.e.} \quad \sum_{i=1}^{n} \langle qi\rangle[ik] = 0, \tag{2.7}$$

here q and k are arbitrary light-like vectors.

We end this subsection by introducing one more identity: **Schouten Identity**. It comes from a rather trivial fact: there are no three independent 2-dimensional vectors. So if we have three 2 components angle spinors $|i\rangle$, $|j\rangle$ and $|k\rangle$, we can write one of them as a linear combination of two others

$$|k\rangle = a|i\rangle + b|j\rangle, \qquad \text{for complex a and b.} \tag{2.8}$$

One can contract a $|i\rangle$ and a $|b\rangle$ with the both sides, then a,b can be solved. (2.8) can be cast to the form

$$|i\rangle\langle jk\rangle + |k\rangle\langle ij\rangle + |j\rangle\langle ki\rangle = 0, \tag{2.9}$$

This is Schouten identity and often written with a fourth spinor $\langle r|$

$$\langle ri\rangle[jk] + \langle rk\rangle\langle ij\rangle + \langle rk\rangle\langle ki\rangle = 0. \tag{2.10}$$

We have a similar Schouten identity holding for square spinors

$$[ri][jk] + [rk][ij] + [rj][ki] = 0. \tag{2.11}$$

There is also a important result can be obtained, the **3-particle soecial kinematics**. If we have three light-like vectors satisfying momentum conservation $p_1^\mu + p_2^\mu + p_3^\mu = 0$. Then

$$\langle 12\rangle\langle 21\rangle = 2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_3^2 = 0, \tag{2.12}$$

so either $\langle 12\rangle$ or $[12]$ equals to 0. If we suppose $\langle 12\rangle \neq 0$, then from $\langle 12\rangle[23] = \langle 1|p_2|3] = -\langle 1|p_1 + p_3|3] = 0$, we can conclude that $[23] = 0$. Similarly, we can also obtain $[31] = 0$. Thus, $[12] = [23] = [31] = 0$, which means that the three square spinors are proportional with each other

$$|1] \propto |2] \propto |3] \tag{2.13}$$

or another possibility

$$|1\rangle \propto |2\rangle \propto |3\rangle. \tag{2.14}$$

As a consequence,

1. A non-vanishing on-shell 3-particle amplitude depends only on square brackets or angle brackets.

2. Since for real momenta, angle brackets are complex conjugated with square brackets, so *on-shell 3 point amplitudes are only meanful for complex momenta*(unless it is a constant, like ϕ^3 theory).

2.1.2 Yang-Mills and Color-ordering

Let us consider the Yang-Mills lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.15}$$

with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]$, and $A_\mu = A_\mu^a T^a$. Gauge fields belong to adjoint representation, so the index a runs over $1, 2 \cdots N^2 - 1$ in $SU(N)$ case. The generators are normalized like $\mathrm{Tr}[T^a T^b] = \delta^{ab}$ and $[T^a, T^b] = i\tilde{f}^{abc} T^c$[1].

The amplitude-friendly gauge choice is *Gervais-Neveu gauge* with gauge fixing term $\mathcal{L}_{gf} = -\frac{1}{2}\mathrm{Tr}(H_\mu^\mu) = 0$, here $H_{\mu\nu} = \partial_\mu A_n u - \frac{ig}{\sqrt{2}} A_\mu A_\nu$. After gauge fixing, the lagrangian becomes

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - i\sqrt{2} g \partial^\mu A^\nu A_\nu A_\mu + \frac{g^2}{4} A^\mu A^\nu A_\mu A_\nu \right) \tag{2.16}$$

The 3- and 4-gluon vertices involve $\tilde{f}^{abc}$ and $\tilde{f}^{abi}\tilde{f}^{icd}$+permutations, respectively, each dressed up with kinematic factors. The amplitudes constructed from these rules can be organized into different group theory structures. For example, the color factors of the s-, t-, and u-channel diagram of the 4-gluon tree amplitude are

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4} \tag{2.17}$$

and the four point interaction just gives a sum of contributions from c_s, c_t and c_u. And because of the Jacobi identity, we have

$$c_s = c_t + c_u. \tag{2.18}$$

And the color factor can be written by the trace of product of generators

$$i\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b] T^c), \tag{2.19}$$

where T^a are generators of fundamental representation. Moreover, in $SU(N)$, we have a Fierz identity

$$\sum_a T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{2.20}$$

[1] In the usual QFT textbook, $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$ and $[T^a, T^b] = i f^{abc} T^c$, with $\tilde{f}^{abc} = \sqrt{2} f^{abc}$ are common choice.

This identity is easier understood as matrix form like

$$\mathrm{Tr}\{T^a A\}\mathrm{Tr}\{T^a B\} = \mathrm{Tr}\{AB\} - \frac{1}{N}\{A\}\mathrm{Tr}\{B\}, \tag{2.21}$$

and

$$\mathrm{Tr}\{AT^a BT^a\} = \mathrm{Tr}\{A\}\mathrm{Tr}\{B\} - \frac{1}{N}\mathrm{Tr}\{AB\}. \tag{2.22}$$

Then it can be used to simplify the calculation.

For example, the 4 gluon s-channel gives us

$$\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}a_4) - \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3}a_4) - \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}a_3) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_4}a_3). \tag{2.23}$$

Similarly, three other diagrams can also be written in terms of single trace. Therefore, the full 4-point amplitude can be rewritten like

$$\mathcal{A}_{4,\mathrm{tree}} = g^2(A_4[1234]\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}a_4) + \text{perms of } (234)) \tag{2.24}$$

here the subamplitudes $A_4[1234], A_4[1243]$, etc. are called **color-ordereed ampltitudes**. This concept can be easily generalized to tree-level n-point case

$$\mathcal{A}_{n,\mathrm{tree}} = g^{n-2}\sum_{\sigma} A_n[1, \sigma(2, 3\cdots n)]\mathrm{Tr}(T^{a_1}T^{\sigma(a_2\cdots}T^{a_n)}) \tag{2.25}$$

where the sun is taken over the $(n-1)!$ trace basis (considering the cyclic property of trace). Actually, the number of independent basis can be reduced to $(n-3)!$, called Del Duca-Dixon-Maltoni (DDM) color decomposition [8]. But it has no tight relation with this paper, so here we do not offer more detailed explanation for it.

The color-ordered amplitude $A_n[1, 2...n]$ is calculated in terms of diagrams with no lines crossing(planar diagrams) and the ordering of the external lines fixed as given 1, 2, 3,..., n. Here, we directly give the final result for 3-point color-ordered amplitudes without any intermediate calculating process. And in this full paper, we mainly consider the helicity amplitudes which will be explained later, so we need to clarify the helicity configuration.

For 3-point, there are only two non-vanishing configurations

$$A_3[1^-, 2^-, 3^+] = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \tag{2.26}$$

and

$$A_3[1^+, 2^+, 3^-] = \frac{[12]^3}{[23][31]}. \tag{2.27}$$

It has been known that there is a compact formula for n-point gluon color-ordered amplitudes

— ***Parke - Talyor Formula***

$$A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}. \tag{2.28}$$

We will prove this formula in the next subsection.

The color-ordered amplitudes have a number of properties

1. *Cyclic*: It follows from the cyclic property for trace that $A_n[12\cdots n] = A_n[2n\cdots 1]$
2. *Reflection*: $A_n[12\cdots n] = (-1)^n A_[n\cdots 21]$
3. The $U(1)$ *decoupling identity*:

$$A_n[123\cdots n] + A_n[213\cdots n] + A_n[231\cdots n] + \cdots + A_n[23\cdots 1n] = 0 \tag{2.29}$$

2.2 BCFW recursion relation

References

[1] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev. Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory. *Nucl. Phys. B*, 828:317–374, 2010.

[2] Ruth Britto, Freddy Cachazo, and Bo Feng. New recursion relations for tree amplitudes of gluons. *Nucl. Phys. B*, 715:499–522, 2005.

[3] Edward Witten. Perturbative gauge theory as a string theory in twistor space. *Commun. Math. Phys.*, 252:189–258, 2004.

[4] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys. Rev. Lett.*, 94:181602, 2005.

[5] R. Sekhar Chivukula, Dennis Foren, Kirtimaan A. Mohan, Dipan Sengupta, and Elizabeth H. Simmons. Scattering amplitudes of massive spin-2 Kaluza-Klein states grow only as $\mathcal{O}(s)$. *Phys. Rev. D*, 101(5):055013, 2020.

[6] Nima Arkani-Hamed, Andrew G. Cohen, and Howard Georgi. (De)constructing dimensions. *Phys. Rev. Lett.*, 86:4757–4761, 2001.

[7] Henriette Elvang and Yu-tin Huang. Scattering Amplitudes. 8 2013.

[8] Vittorio Del Duca, Lance J. Dixon, and Fabio Maltoni. New color decompositions for gauge amplitudes at tree and loop level. *Nucl. Phys. B*, 571:51–70, 2000.