On-Shell Methods for Tree-Level Amplitudes in (De)Constructed Gauge Theory

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Contents

Why We Study Scattering Amplitudes?

1. **Bridge between theory and experiment**
   - Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
   - Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections

Why We Study Scattering Amplitudes?

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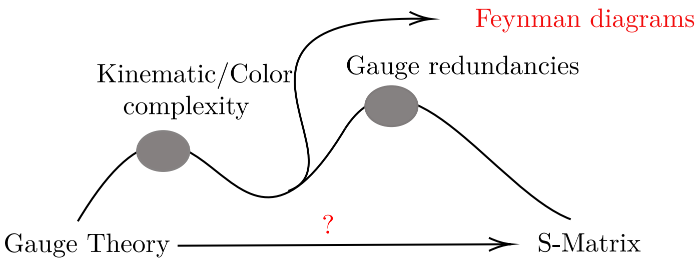
    - Core prediction targets for high-energy collider experiments such as the LHC , especially for high multiplicity amplitudes.
    - Any new theory (SUSY, GUTs, extra dimensions) must predict observable cross sections
2. **Reveal deep structures of quantum field theory**
    - Amplitudes exhibit hidden symmetries (e.g., dual conformal, Yangian) not visible in the Lagrangian
    - These symmetries suggest deeper theoretical frameworks, such as amplituhedra or holographic principle (celestial duality).

Challenges we face before



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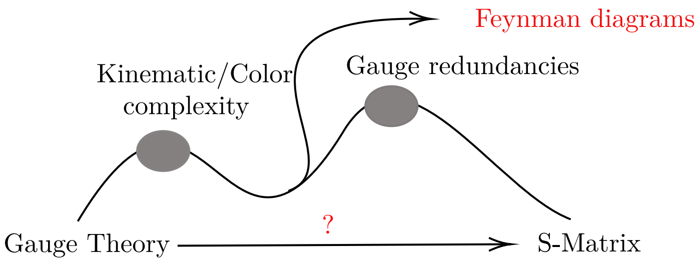
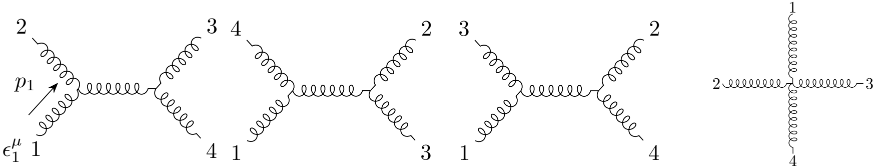


n pt. amplitudes	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

The number of Feyman diagrams grow quite rapidly!

Conventional Computation

Usually, when we compute the gluon amplitudes by using Feynman diagram, we will obtain something like



s-channel t-channel u-channel Contact Term

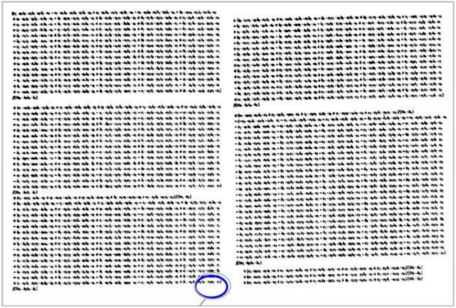
$$\mathcal{M}_s(p_1 p_2 \to p_3 p_4) = -\frac{g_s^2}{s} f^{abe} f^{cde}$$

$$\times \Big\{ -4\,(\epsilon_1 \cdot \epsilon_3^*)\,(\epsilon_2 \cdot p_1)\,(\epsilon_4^* \cdot p_3) + 2\,(\epsilon_1 \cdot \epsilon_2)\,(\epsilon_3^* \cdot p_1)\,(\epsilon_4^* \cdot p_3)$$

$$-2\,(\epsilon_1 \cdot p_4)\,(\epsilon_2 \cdot p_1)\,(\epsilon_3^* \cdot \epsilon_4^*) + (\epsilon_1 \cdot \epsilon_2)\,(p_4 \cdot p_1)\,(\epsilon_3^* \cdot \epsilon_4^*) + 10 \text{ terms} \Big\}$$

If you consider 5point case, it will become worse:

★ We have 25 diagrams and nearly 10000 terms!



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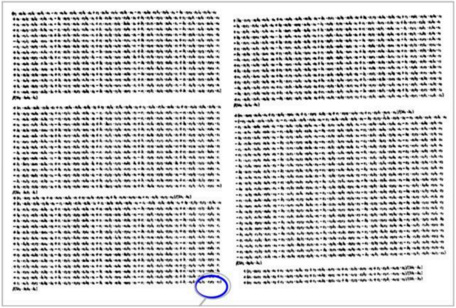


simple action $\longrightarrow$ **complicated** feynman rules and **non gauge-invariant** green function $\xrightarrow{\text{LSZ.}}$ **on-shell** helicity amplitude

$$S = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu}$$

$\mathcal{M} = -ig_s^3\{$  $+ \cdots \}$

$$\frac{\langle\rangle^4}{\langle\rangle\langle\rangle\langle\rangle\langle\rangle}$$

Can we use gauge invariant quantity here?

Contents

Analyticity

Amplitudes are analytic functions

- Tree level scattering amplitudes are rational functions of Lorentz invariants, such as $\boldsymbol{p_{i\mu} p_j^\mu}$, $\boldsymbol{p_{i\mu} \epsilon_j^\mu}$.
- **Locality** tells us that any pole of a tree-level amplitude must correspond to a on-shell propagating particle.
- There's only single pole, no branch cuts (logs, square roots, etc) at tree level.

Historically, when people consider 4-point scattering

$$p_1, p_2, p_3, p_4, \quad \text{satisfying} \quad p_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{4} p_i = 0$$

$$\text{d.o.f} = 4n - 4 - n = 3n - 4 \quad \text{independent complexed d.o.f}$$

This seems to be too much, so people change to use Mandelstam variables

$$s, t, u (\text{for} \;\; \pi\pi \;\; \text{scattering}) \quad \text{satisfying} \quad s + t + u = 0$$

Also for earliest String Amplitude (Venezaino Amplitude)

$$A_{\text{open}}(s,t) = g^2 \frac{\Gamma(-\alpha' s)\,\Gamma(-\alpha' t)}{\Gamma(-\alpha' s - \alpha' t)}$$

But the problem is that Mandelstam variables only apply to 2→2 scattering, how about higher point?

single complexed variable like $\quad A(z)$

This is the starting point of BCFW recursion relation!

Poincaré Invariance and Little Group Scaling

We consider a Poincaré invariant field theory. The one-particle state are defined from a reference momentum k like this

$$|p;\sigma\rangle = U(L(p;k))|k;\sigma\rangle$$

Then the general Lorentz transformation acts like

$$\begin{aligned} U(\Lambda)|p,\sigma\rangle &= U(\Lambda)U(L(p;k))|k,\sigma\rangle \\ &= U(L(\Lambda p;k))U(L^{-1}(\Lambda p;k)\Lambda L(p;k))|k,\sigma\rangle \end{aligned}$$

here $W = L^{-1}(\Lambda p;k)\Lambda L(p;k)$ is not a general transformation but keep k invariant, called little group. Thus we have

$$U(W(\Lambda,p,k))|k;\sigma\rangle = D_{\sigma\sigma'}(W(\Lambda,p,k))|k,\sigma'\rangle$$

so one-particle state with momentum p transformed like

$$U(\Lambda)|p,\sigma\rangle = D_{\sigma\sigma'}(W(\Lambda,p,k))|\Lambda p,\sigma'\rangle$$

One-particle state transformed under the little group!

For massless particle, the little group is $SO(2) \backsimeq U(1)$, so the representation matrix is just a phase. The Poincaré invariance of S-matrix requires

$$\mathcal{M}^{\Lambda}(p_a, \sigma_a) = \prod_a (D_{\sigma\sigma'})\mathcal{M}((\Lambda p)_a, \sigma_a^{'})$$

- **Massless Case**

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

$$\lambda \to t^{-1}\lambda, \qquad \tilde{\lambda} \to t\tilde{\lambda}, \qquad t \in \mathbb{C}$$

same for

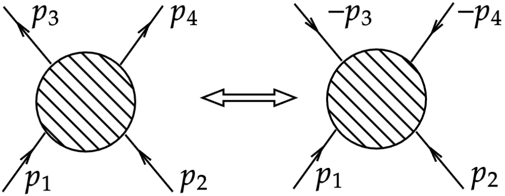
$$|\lambda\rangle \to t^{-1}|\lambda\rangle, \qquad |\lambda] \to t|\lambda]$$

The scattering amplitudes should transform covariantly under little group scaling:

$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \ldots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \ldots) = t_i^{2h_i}\mathcal{A}_n$$

Crossing Symmetry and MHV Classification

Crossing symmetry is a result from analyticity, unitarity and Lorentz invariance.



Modern Amplitude Method

The answer is On-shell method.

$$\text{Gauge Theory} \xrightarrow{\text{On-shell method}} \text{Helicity Amplitude}$$

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$$M_5 = \underbrace{A_5[12345]}_{\text{Color-ordered Amplitudes}} \quad \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_5}] + \text{permutations}$$

Parke–Taylor Formula (MHV amplitudes):

$$A_5[1^+2^+3^+4^+5^+] = 0 \qquad (+,- : \text{ helicity};$$

$$A_5[1^-2^+3^+4^+5^+] = 0 \qquad 1, 2, \cdots, n : \text{ particle labels})$$

$$A_5[1^-2^-3^+4^+5^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \qquad \text{(first non-trival one)}$$

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$$\text{Gauge Theory} \xrightarrow{\text{On-shell method}} \text{Helicity Amplitude}$$

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$$A_5[1^+2^+3^-4^-5^-] = \frac{[12]^4}{[12][23][34][45][51]}$$

Color-ordering for Yang-Mills

Consider the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

The 3 point and 4 point vertices include $\tilde{f}^{abc}$ and $\tilde{f}^{abe}\tilde{f}^{cde}$ + perms.
(With a different convention,$\mathrm{Tr}[T^aT^b] = \delta^{ab}$ and $[T^a, T^b] = i\tilde{f}^{abc}T^c$)
We have

$$c_s = \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}, \quad c_t = \tilde{f}^{a_4a_1b}\tilde{f}^{ba_2a_3}, \quad c_u = \tilde{f}^{a_1a_3b}\tilde{f}^{ba_2a_4}$$

and the color factor can be rewritten by the trace of product of generators

$$i\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c),$$

Moreover, in $SU(N)$, we have a Fierz identity

$$\sum_a T^a_{ij}T^a_{kl} = \delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{1}$$

This identity is easier understood as matrix form like

$$\mathrm{Tr}\{T^a A\}\mathrm{Tr}\{T^a B\} = \mathrm{Tr}\{AB\} - \frac{1}{N}\mathrm{Tr}\{A\}\mathrm{Tr}\{B\},$$
$$\mathrm{Tr}\{AT^a BT^a\} = \mathrm{Tr}\{A\}\mathrm{Tr}\{B\} - \frac{1}{N}\mathrm{Tr}\{AB\}.$$

So, the 4 gluon s-channel gives us

$$\begin{aligned}\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3}T^{a_4}) \\ &\quad - \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_4}T^{a_3}).\end{aligned}$$

Therefore, the full 4-point amplitude can be rewritten like

$$\mathcal{A}_{4,\mathrm{tree}} = g^2(A_4[1234]\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{perms of }(234))$$

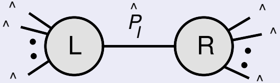
here the subamplitudes $A_4[1234]$, $A_4[1243]$, etc. are called **color-ordered ampltitudes**. This concept can be easily generalized to tree-level n-point case

$$\mathcal{A}_{n,\mathrm{tree}} = g^{n-2}\sum_{\sigma} A_n[1,\sigma(2,3\cdots n)]\mathrm{Tr}(T^{a_1}T^{\sigma(a_2\cdots}T^{a_n)})$$

The Power of BCFW Recursion Relation

BCFW recursion relation

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I) = \sum_{\text{diagrams } I}$$



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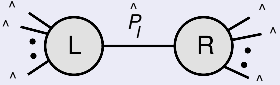
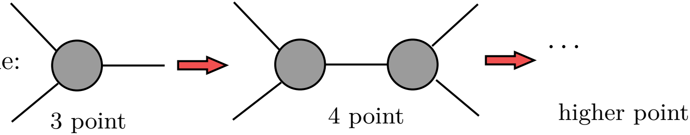


More understandable:



★ From lower point on-shell amp. to higher point on-shell amp.!!

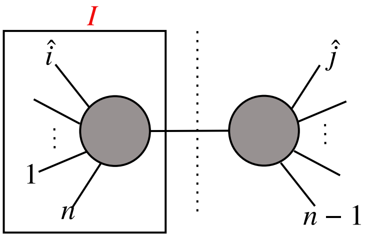
What did BCFW do to make the shift?

Here we consider the case in which all particles are massless, $p_i^2 = 0$ for all $i = 1, 2, \ldots, n$. We choose two momentum to be shifted oppositely

$$p_i \longrightarrow \hat{p}_i(z) \equiv p_i - zk, \qquad p_j \longrightarrow \hat{p}_j(z) \equiv p_j + zk$$

satisfying

$$k^2 = 0, \qquad p_i \cdot k = 0, \qquad p_j \cdot k = 0$$



For a non-trival subset of generic momenta $\{p_i\}_{i\in I}$

$$\hat{P}_I^2 = P_I^2 - 2zP_I \cdot k = -\frac{P_I^2}{z_I}(z - z_I)$$

with $z_I = \frac{P_I^2}{2P_I \cdot k}$.

Brief explaination: We choose two momentum to be shifted oppositely

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We consider amplitude A_n in terms of shifted momentum $\hat{p}_i^\mu$ instead of original real momentum.

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$$A_n \longrightarrow \hat{A}_n(z)$$

If we consider the meromorphic function $\frac{\hat{A}_n(z)}{z}$ in the complex plane. From Cauchy Theorem, we can ontain

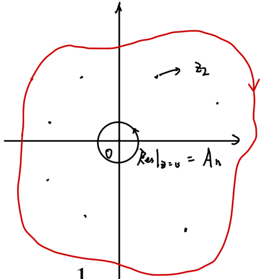
$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

where B_n is the residue of the pole at $z = \infty$, called boundary term.

The most important point here is that

$$\boxed{\mathrm{Res}|_{z=0}\frac{\hat{A}_n(z)}{z} = \hat{A}_n(0) = A_n}$$

It means that the original amplitude equals to the residue at origin.



$$\hat{A}_n(z) \xrightarrow{z \text{ near } z_I} \hat{A}_L(z_I)\frac{1}{\hat{P}_I^2}\hat{A}_R(z_I) = -\frac{z_I}{z - z_I}\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

This makes it easy to evaluate the residue at $z = z_I$

$$-\mathrm{Res}|_{z=z_I}\frac{\hat{A}_n(z)}{z} = \frac{(z-z_I)z_I}{z(z-z_I)}\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)|_{z=z_I} = \hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = -\sum_{z_I} \text{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j\rangle$	$+$	$-$
$+$	$1/z$	z^3
$-$	$1/z$	$1/z$

proved by using background field expansion (N. Arkani-Hamed and J. Kaplan, [arXiv:0801.2385 [hep-th]].)

Little group scaling

- **Massless Case**

$$p_\mu \sigma^\mu = p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} = |\lambda\rangle[\lambda|$$

There is an ambiguity for the definition, the momentum is invariant under the following redefinition

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same for

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$$\mathcal{A}_n(\{|1\rangle, |1], h_1\}, \ldots \{t_i^{-1}|i\rangle, t_i|i], h_i\}, \ldots) = t_i^{2h_i}\mathcal{A}_n$$

- **Massive Case**

It can also be handled in terms of spinor-helicity variable, see also arXiv:1709.04891 [hep-th] (Nima Arkani-Hamed, Tzu-Chen Huang, Yu-tin Huang).

On-shell 3-point can be completely determined

Another necessarity to introduce complex momentum If the momentum is complexed, we have

$$\langle 12\rangle \neq [21]^*$$

Then we can obtain

$$|1\rangle \propto |2\rangle \propto |3\rangle \qquad or \qquad |1] \propto |2] \propto |3]$$

It means that 3-point amplitude depends only on angle brackets or squar brackets. Here I choose the first case to give an example

$$A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}},$$

Little group scaling tells us that

$$t_1^{2h_1} A_3(1^{h_1}, 2^{h_2}, 3^{h_3}) = c t_1^{-x_{12}} t_1^{-x_{13}}\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}.$$

We can obtain

$$2h_1 = -x_{12} - x_{13}$$

Similarly, we can also obtain

$$2h_2 = -x_{12} - x_{23}, \qquad 2h_3 = -x_{13} - x_{23}.$$

Then all index can be solved from this system of equations, so that

$$\boxed{\begin{array}{ll} A_3^{h_1h_2h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} & h_1+h_2+h_3<0 \\ A_3^{h_1h_2h_3} = c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & h_1+h_2+h_3>0 \end{array}}$$

⋆ **All massless on-shell 3-point ampltides are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

There's another possibility

$$A_3(g_1^-, g_2^-, g_3^+) = g'\frac{[13][23]}{[12]^3}$$

but actually it comes from the non-local interaction $g'AA\frac{\partial}{\Box}A$, so we discard it.

So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
- Applied to:
  - **Pure Yang-Mills** theory
  - Tree-level MHV amplitudes
  - Color-ordered partial amplitudes

From Review to Applications

So far: Foundations

- Reviewed the structure of **BCFW recursion relation**
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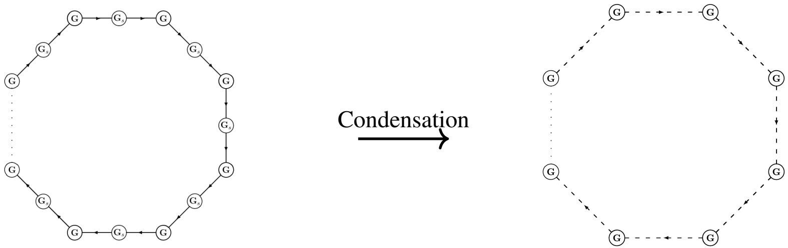
Next: Realistic Models

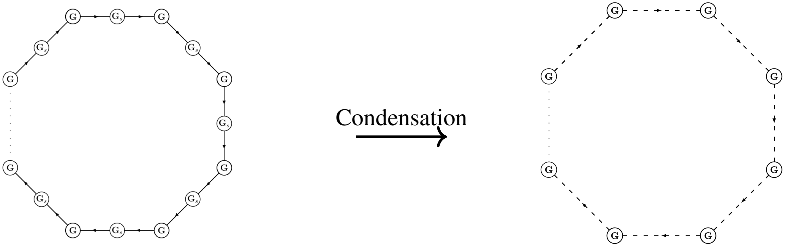
- Move beyond massless gauge theory
- Consider:
  - **(De)constructed gauge theories**
- Key questions:
  - Can BCFW still apply?
  - What new structures emerge?

Contents

Introduction of (De)Constructed gauge theory



Introduction of (De)Constructed gauge theory



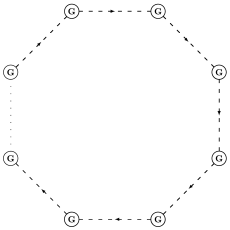
The Lagrangian can be written like

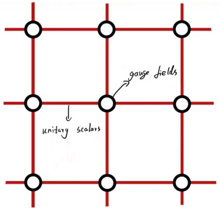
$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{2}\mathrm{Tr}(F_i^2) + \sum_{i=1}^{N} \mathrm{Tr}\left[(D_\mu \Phi_i)^\dagger (D^\mu \Phi_i)\right],$$

here F_i refers to the ith gauge field strength. The scalar field Φ_i transforms under the bi-fundamental representation, and the covariant derivative equals to

$$D_\mu \Phi_i = \partial_\mu \Phi_i - i g_i A_{i\mu} \Phi_i + i g_{i+1} \Phi_i A_{i+1\mu}.$$

It has been proposed that this model actually discretized a five-dimension gauge theory with gauge group $SU(m)$, where only the fifth dimension are latticed. So it is an effective theory for 5d gauge theory.





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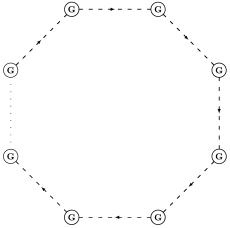
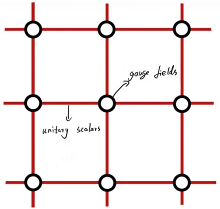


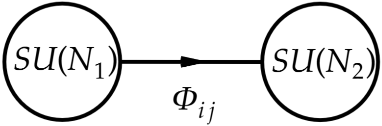


After higgsing the scalar field, we can obtain a spectrum

$$M_k^2 = 4g^2 f_s^2 \sin^2\left(\frac{\pi k}{N}\right)$$

This is precisely the Kaluza-Klein spectrum under S^1 compactification.

Amplitudes from BCFW



$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_1)^2 - \frac{1}{2}\mathrm{Tr}(F_2)^2 + \mathrm{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

From the Lagrangian, we have known that there are only two kinds of 3 point amplitude ($+, -$: helicity $\quad \Phi, \Phi^\dagger$: charge of scalar)

$$A[1^\Phi 2^{\Phi^\dagger} 3^+] = \frac{[23][31]}{[12]}, \qquad A[1^\Phi 2^{\Phi^\dagger} 3^-] = \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle}$$

$$A[3^+4^+5^-] = \frac{[34]^3}{[45][53]}, \qquad A[3^-4^-5^+] = \frac{\langle 34\rangle^3}{\langle 45\rangle\langle 53\rangle}$$

By using the 3 point building block, we can construct 4 point colorordered amplitudes from BCFW recursion relation.

Gauge boson sector

- nV_1 or nV_2
  This part is completely the same as the pure gluon amplitude, so we can directly borrow the existing results.

$$\boxed{\text{Parke - Talyor Formula}: \quad A[\cdots i^{-}\cdots j^{-}\cdots] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}}$$

Notice that this formula only applies to MHV amplitudes, although the NMHV can be completely solved.

SQCD like sector

The color factor in this sector looks like

$$(T^{a_1}T^{a_2}\cdots T^{a_n})_{ij}$$

so we we need to notice is just the order of gauge boson.

The amplitudes can be computed like

- $\Phi^\dagger V_1 V_1 \Phi$

$$A[1^{\Phi}2^{\Phi^\dagger}3^+4^-] = (-1)\frac{\langle 14\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \text{(Parke -Talyor like Formula)}$$

- $\Phi^\dagger V_1 V_1 V_1 \Phi$

$$A[1^{\Phi}2^{\Phi^\dagger}3^+4^+5^-] = \frac{\langle 15\rangle^2\langle 25\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

- $\Phi^\dagger(nV_1)\Phi$

$$A[1^{\Phi}2^{\Phi^\dagger}\cdots(n+2)^-] = (-1)^{n+1}\frac{\langle 1, n+2\rangle^2\langle 2, n+2\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n+1, n+2\rangle\langle n+2, 1\rangle}$$

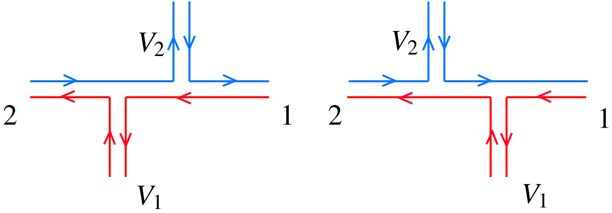
⋆ Bonus relation:

$$A[1^{\Phi}2^{\Phi^\dagger}3^+4^+] = 0 \quad \Rightarrow \quad A[1^{\Phi}2^{\Phi^\dagger}3^+\cdots n^+] = 0$$

For the amplitude $\Phi(nV_2)\Phi^\dagger$, we can obtain nearly the same expression.

Pure 2-site amplitude

The straightforward way to observe the color structure in this case is double line notation as follows



The color factor here have special form like

$$(T_1^{a_1} T_1^{a_2} \cdots T_1^{a_{n_1}})_{ij} (T_2^{b_1} T_2^{b_2} \cdots T_2^{b_{n_2}})_{\bar{j}\bar{i}}$$

we can notice that the relative order between two gauge group do not affect the color structure, but the order inside the gauge group matters.

So we introduce the **OPP (Order Preserving Permutation)**

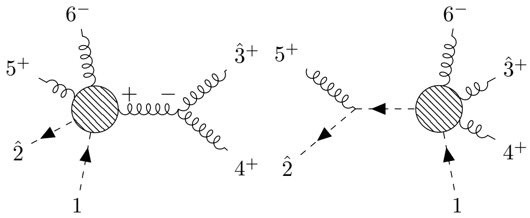
- $\Phi V_2 \Phi^\dagger V_1$

$$A[1^{\Phi} 2^{\Phi^\dagger} 3_1^+ 4_2^-] = \frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle}$$

- $\Phi V_2 \Phi^\dagger V_1 V_1$

$$A[1^{\Phi} 2^{\Phi^\dagger} 3_1^+ 4_1^+ 5_2^-] = (-1)\frac{\langle 25\rangle^2\langle 15\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 25\rangle\langle 51\rangle}$$

Here I show the concrete computation process



$$
\begin{aligned}
A_1 &= \frac{(-1)\langle \hat{2}6\rangle^2\langle 16\rangle^2}{\langle 25\rangle\langle 56\rangle\langle 61\rangle\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[\hat{3}4]^3}{[4\hat{I}]\,[\hat{I}\hat{3}]} \\
&= \frac{\langle 26\rangle^2\langle 16\rangle}{\langle 25\rangle\langle 56\rangle\langle \hat{2}\hat{I}\rangle\langle \hat{I}1\rangle} \times \frac{1}{s_{34}} \times \frac{[34]^3}{[4\hat{I}]\,[\hat{I}3]} \\
&= \frac{\langle 26\rangle^2\langle 16\rangle\cancel{[34]^3}\cancel{\langle 42\rangle}}{\langle 25\rangle\langle 56\rangle\langle 41\rangle\langle 32\rangle\langle 43\rangle\cancel{[43][43][34]}\cancel{\langle 24\rangle}} \\
&= \frac{\langle 26\rangle^2\langle 16\rangle^2}{{\color{blue}\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\color{red}\langle 25\rangle\langle 56\rangle\langle 61\rangle}}
\end{aligned}
$$

- Compact formula for general case

$$A = \frac{\langle 2a\rangle^2 \langle 1a\rangle^2}{\underbrace{\langle 2\star\rangle \cdots \langle \star 1\rangle}_{SU(N_1)} \underbrace{\langle 2*\rangle \cdots \langle * 1\rangle}_{SU(N_2)}}$$

Green: Particle with – helicity

Blue: Particle belongs to the first gauge group

Red: Particle belongs to the second gauge group

$\star$: Order for gauge group 1

$*$: Order for gauge group 2

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Green: Particle with – helicity
Blue: Particle belongs to the first gauge group
Red: Particle belongs to the second gauge group
$\star$: Order for gauge group 1
$*$: Order for gauge group 2

For example, if we want to compute $A[1^{\Phi}2^{\Phi^\dagger}5_1^+3_1^+4_1^-7_2^+6_2^+8_2^+]$:

$$A = \frac{\langle 24\rangle^2\langle 14\rangle^2}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle\langle 27\rangle\langle 76\rangle\langle 68\rangle\langle 81\rangle}$$

If you use Feynman diagrams, it may take sevral days to accomplish the computation.

Contents

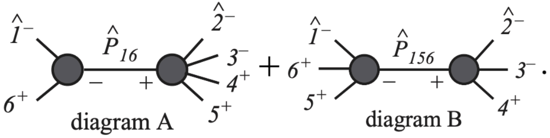
How about NMHV?

First, let us review the NMHV amplitudes for gluon scattering.

Still we begin with the simplest case – **Split-helicity** NMHV like $A_6[1^- \, 2^- \, 3^- \, 4^+ \, 5^+ \, 6^+]$.

Here we choose $[1,2\rangle$ shift

$$A_6[1^-2^-3^-4^+5^+6^+] =$$



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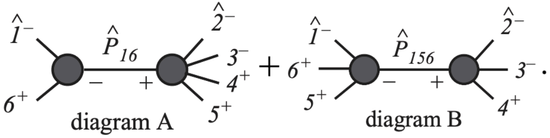


- Diagram B includes a propagator $1/P_{156}^2$, so there is a 3-particle pole $P_{156}^2 = 0$. But by inspecting the external order, it seems that there's no difference between $(-++)$ channel 561 and 345. We should expect the amplitude to have a pole also at $P_{345}^2 = 0$.

$$\text{diagram A} = \frac{\langle \hat{1}\hat{P}_{16}\rangle^3}{\langle \hat{P}_{16}6\rangle\langle 6\hat{1}\rangle} \times \frac{1}{P_{16}^2} \times \frac{\langle \hat{2}3\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 5\hat{P}_{16}\rangle\langle \hat{P}_{16}2\rangle}$$

$$\langle \hat{2}\hat{P}_{16}\rangle[\hat{P}_{16}3] = \langle 21\rangle[\hat{1}3] + \langle \hat{2}6\rangle[63]$$

It follows from $\hat{P}^2_{16} = 0$ that $z_{16} = -\frac{[16]}{[26]}$, so

$$\langle \hat{2}\hat{P}_{16}\rangle[\hat{P}_{16}3] = -\frac{[36]}{[26]}\left(\langle 12\rangle[12] + \langle 16\rangle[16] + \langle 26\rangle[26]\right) = -\frac{[36]}{[26]}P^2_{126}$$

The 3-particle pole P^2_{126} is encoded inside the BCFW channel !

- Full expression

$$A_6[1^-\,2^-\,3^-\,4^+\,5^+\,6^+] = \frac{\langle 3|1+2|6]^3}{P^2_{126}[21][16]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]} + \frac{\langle 1|5+6|4]^3}{P^2_{156}[23][34]\langle 56\rangle\langle 61\rangle\langle 5|1+6|2]}.$$

The factor $\langle 5|1+6|2]$ does not correspond to a physical pole of the scattering amplitude: it is a **spurious pole**.
There has been interesting paper investigating how to systematically cancel the spurious poles, like "A. Hodges, JHEP 1305, 35 (2013) [arXiv:0905.1473 [hep-th]]."

- We utilize the $[1, 2\rangle$ shift before, what happens if we change to $[2, 1\rangle$ shift?

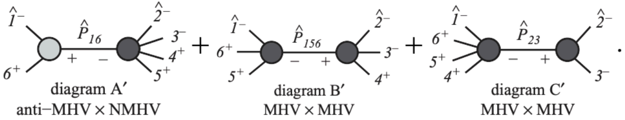
$A_6[1^-2^-3^-4^+5^+6^+] =$ 

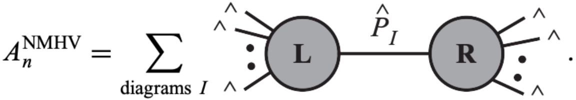
diagram A' = anti-MHV $\times$ NMHV, as opposed to diagram A = MHV $\times$ MHV.

The equivalence between two different shift is related to powerful residue theorem (N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, [arXiv:0907.5418 [hep-th]].) and Grassmannian.

We can also consider a shift that is implemented via a “holomorphic” square-spinor shift:

$$|\hat{i}] = |i] + zc_i|X], \qquad |\hat{i}\rangle = |i\rangle$$

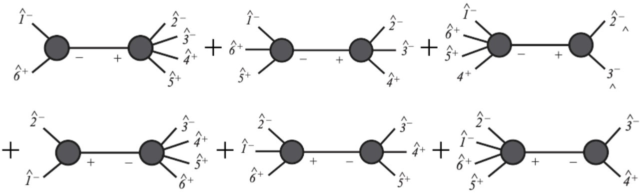
Here $|X]$ is an arbitrary reference spinor and the coefficients c_i satisfy $\sum_{i=1}^{n} c_i|i\rangle = 0$.

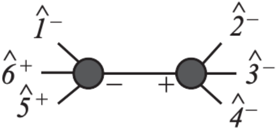


There are two possibilities: anti-MHV$_3$(= 0) $\times$ NMHV or MHV $\times$ MHV.

For example, the 6pt split NMHV amplitude

$A_n[1^-2^-3^-4^+5^+6^+] =$





$$= \frac{\langle 1\hat{P}_I\rangle^4}{\langle 1\hat{P}_I\rangle\langle \hat{P}_I 5\rangle\langle 56\rangle\langle 61\rangle}\frac{1}{P^2_{156}}\frac{\langle 23\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4\hat{P}_I\rangle\langle \hat{P}_I 2\rangle}.$$

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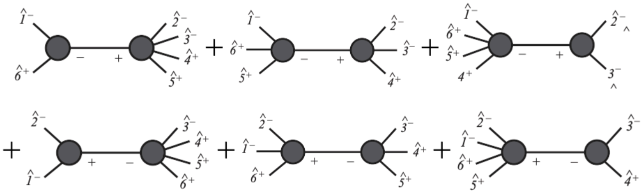
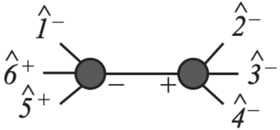




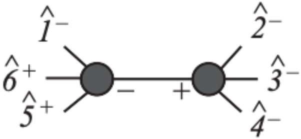
$$= \frac{\langle 1\hat{P}_I\rangle^4}{\langle 1\hat{P}_I\rangle\langle \hat{P}_I 5\rangle\langle 56\rangle\langle 61\rangle} \frac{1}{P_{156}^2} \frac{\langle 23\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4\hat{P}_I\rangle\langle \hat{P}_I 2\rangle}.$$

We can write

$$|\hat{P}_I\rangle \frac{[\hat{P}_I X]}{[\hat{P}_I X]} = \hat{P}_I|X] \frac{1}{[\hat{P}_I X]} = P_I|X] \frac{1}{[\hat{P}_I X]}$$

We can use the prescription

$$|\hat{P}_I\rangle \to P_I|X]$$



$$= \frac{\langle 1|P_{156}|X]^4}{\langle 1|P_{156}|X]\langle 5|P_{156}|X]\langle 56\rangle\langle 61\rangle}\cdot\frac{1}{P_{156}^2}\cdot\frac{\langle 23\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4|P_{156}|X]\langle 2|P_{156}|X]}\,.$$

Contents

Summary

- Introduce the on-shell method, including BCFW recursion relation, spinor-helicity formalism, etc.
- Introduce a (de)constructed gauge theory model, which is an effective field theory for 5 dimension gauge theory.
- Much of the scattering amplitudes in this model can be recursively computed by BCFW, and some compact formulas are offered.

Thanks for your attention!

Spinor-Helicity Formalism

Helicity

Helicity is defined as the projection of a particle's spin vector $\vec{S}$ onto the direction of its momentum $\vec{p}$:

$$h = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

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S-matrix is a function of momentum p_i and helicity h_i

$$\mathcal{M}(p_i, h_i)$$

How can we catch the information of helicity?

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Massless Case:

- Momenta in spinor form:

$$p_\mu \sigma^\mu = p_{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}} = |p\rangle [p|$$

Large z behavior

In the BCFW formula, the boundary term B_n affects a lot

$$A_n = -\sum_{z_I} \mathrm{Res}|_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n,$$

In most applications. one assumes or much better, proves $B_n = 0$. This is often justified by declaring a stronger condition

$$\hat{A}_n(z) \to 0 \quad \text{for} \quad z \to \infty$$

Here I show the large z behavior for gluon scattering

$[i \setminus j\rangle$	$+$	$-$
$+$	$1/z$	z^3
$-$	$1/z$	$1/z$

On-shell 3-point for real momentum

Because of the constrain from momentum conservation and on-shell condition

$$p_1 = \kappa p_3, \qquad p_2 = (1-\kappa) p_3 \quad \text{(Collinear)}$$

All of the contribution

$$(p_1 \cdot p_2), \quad (p_1 \cdot p_3), \quad (p_2 \cdot p_3) \quad = 0$$

In terms of Spinor- Helicity variable, we have

$$2p_1 \cdot p_2 = \langle 12 \rangle [21] = 0 \longrightarrow \langle 12 \rangle = [21]^* = 0$$

We can not obtain any thing nontrival from 3-point!

Of coure, you can introduce non-minimal interaction

$$\mathcal{L}_3 \ni \frac{1}{\Lambda^2} \bar{\Psi} \not{D} (\Box \Psi)$$

but it still equals to 0 under the on-shell condition.

On-shell 3-point can be completely determined

For the complex momentum, we have

$$|1\rangle \propto |2\rangle \propto |3\rangle \qquad \mathit{or} \qquad |1] \propto |2] \propto |3]$$

$$\boxed{\begin{array}{ll} A_3^{h_1 h_2 h_3} = c\langle 12\rangle^{h_3-h_1-h_2}\langle 31\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} & h_1+h_2+h_3<0 \\ A_3^{h_1 h_2 h_3} = c'[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & h_1+h_2+h_3>0 \end{array}}$$

★ **All massless on-shell 3-point ampltides are completely determined by little group scaling!**

Example: 3-gluon amplitude

$$A_3(g_1^-, g_2^-, g_3^+) = g\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

For simplicity, we start from the two-site gauge theory with gauge fields V_1, V_2 and scalar fields Φ, $\Phi^\dagger$.

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_1)^2 - \frac{1}{2}\mathrm{Tr}(F_2)^2 + \mathrm{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)],$$

We only foucus on the following amplitudes:

$$nV_1, \qquad nV_2, \qquad \Phi^\dagger nV_1\Phi, \qquad \Phi nV_2\Phi^\dagger, \qquad \Phi^\dagger\Phi\Phi^\dagger\Phi$$

here n can be any positive integer.

More specifically, it helps us to prove P. T. formula

$$\text{3pt.} \longrightarrow \text{4pt.} \longrightarrow \text{5pt.} \longrightarrow \cdots$$

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

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$$\boxed{\Rightarrow: \quad A[1^+ \cdots i^-(i+1)^+ \cdots j^-(j+1)^+ \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}}$$

★ This is the power of BCFW recursion relation.