Supplement for "Linearized Interval Power Flow in Distribution Grids Under DER Uncertainty: A Fast Affine Arithmetic Approach"

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June 25, 2024

1 Taylor Series-Based Linearization

The Taylor-series-based linear approximation is proposed in [1] for power flow calculation in the per-unit system. A theoretical proof for the error analysis of this linear approximation, which is missing in [1], is proposed here.

1.1 Applying to A-Phase Voltages

In premise, define $\Delta v_k^a = 1 - v_k^a$, where v_k^a is the voltage phasor at bus k. The nonlinear term only appears on the left-hand side. Thus, the Taylor expansion of the complex-valued Δv_k^a within $|\Delta v_k^a| < 1$ is adopted:

$$\frac{1}{v_k^a} = \frac{1}{1 - \Delta v_k^a} = \sum_{n=0}^{\infty} (\Delta v_k^a)^n \qquad \forall |\Delta v_k^a| < 1 \tag{1}$$

Ignoring the high-order terms ($n \geq 2$) and substituting the definition of Δv_k^a to (1), the following linear approximation is obtained:

$$\frac{1}{v_k^a} \approx 1 + \Delta v_k^a = 2 - v_k^a \tag{2}$$

Moreover, the errors introduced by the first-order Taylor expansion on v_k^a can be explicitly illustrated by:

$$F(v_k^a) = |1/v_k^a - (2 - v_k^a)| \tag{3}$$

Let $v_k^a = A\angle\alpha$, where A denotes the voltage magnitude and α denotes the voltage phase angle, the following expression of (3) is deducted:

$$\begin{aligned}
F(v_k^a) &= |\frac{1}{A}(\cos\alpha - j\sin\alpha) + A(\cos\alpha + j\sin\alpha) - 2| \\
&= |(\frac{1}{A} + A)\cos\alpha - 2 + j(A - \frac{1}{A})\sin\alpha| \\
&= \sqrt{[(\frac{1}{A} + A)\cos\alpha - 2]^2 + [(A - \frac{1}{A})\sin\alpha]^2} \\
&= \sqrt{(\frac{1}{A} + A)^2\cos^2\alpha - 4(\frac{1}{A} + A)\cos\alpha + 4 + [(A - \frac{1}{A})\sin\alpha]^2} \\
&= \sqrt{\frac{1}{A^2} + A^2 + 2\cos^2\alpha - 2\sin^2\alpha - 4(A + \frac{1}{A})\cos\alpha + 4} \\
&= \sqrt{\frac{1}{A^2} + A^2 + 4\cos^2\alpha - 4(A + \frac{1}{A})\cos\alpha + 2} \\
&= \sqrt{(\frac{1}{A} + A - 2\cos\alpha)^2} = |\frac{1}{A} + A - 2\cos\alpha|
\end{aligned} \tag{4}$$

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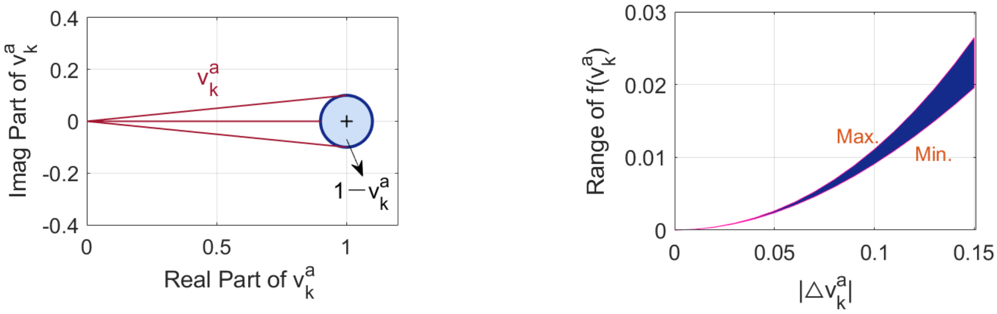

(a) (b)

Figure 1: (a) Schematic diagram of the proposed linearization loss $F(v_k^a)$ [1] b) Error band of linearization errors

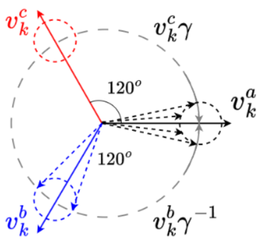

Figure 2: 120-degree phase shift and the error ranges for B- and C-phase rotation for linearization approximation

Lemma 1 Errors of A-Phase Voltages: Assume the phase angle difference along the distribution feeder is small [2] and in per unit system, the phasor angel of the substation voltage is set as 0, then $\cos\alpha \approx 1$. (4) can be approximated as

$$F(v_k^a) = |\frac{1}{A} + A - 2| \tag{5}$$

Based on (5), considering the normal operating range in power systems, for example, $A \in [0.95, 1.05]$, the errors $F(v_k^a)$ are constrained within [0,0.0026]; $A \in [0.9, 1.1]$, the errors $F(v_k^a)$ are constrained within [0,0.011].

An illustrative diagram for the error analysis of A-phase voltages can be found in Fig. 1a) and b). The errors as the function value with different $|\Delta v_k^a|$ based on an x-coordinate transformation $|\Delta v_k^a| = |1 - v_k^a|$, and the data points with $(|\Delta v_k^a|, F(v_k^a))$ can be depicted as an area with the maximum and minimum as the bounds.

1.2 Extending to B- and C- Phases

Then (2) is extended to the three-phase system by only using the ±120-degree phase shift assumption, which is widely used in distribution system operation research such as the three-phase version DistFlow [3]. This approximation model can consider the line losses and has less reliance on certain assumptions, compared with other linear power flow models that ignore them (e.g., [3]).

Define a phasor rotation operator γ, which rotates a voltage vector counterclockwise by 120 degrees when multiplied by it:

$$\gamma = e^{-j\frac{2}{3}\pi} \tag{6}$$

Similarly, $\gamma^{-1} = e^{j\frac{2}{3}\pi}$.

Take the B-phase as an example, after rotating the original B-phase voltage clockwise by 120 degrees, i.e., $v_k' = \gamma^{-1}v_k^b$, $|1 - v_k'| < 1$ exists. Thus, letting $\Delta v_k' = 1 - \gamma^{-1}v_k^b$, (2) is applied to the rotated B-phase voltage phasor v_k'. Then the following approximation for the pre-rotated B-phase voltage phasor v_k^b is derived:

$$\frac{1}{v_k^b} = \gamma^{-1}(2 - \gamma^{-1} v_k^b) \tag{7}$$

Similarly, for the C-phase voltage, the following equation is obtained:

$$\frac{1}{v_k^c} = \gamma(2 - \gamma v_k^c) \tag{8}$$

Lemma 2 Errors of B-Phase Voltages: Based on the assumption of **Lemma 1** and 120-degree phasor shift, in three-phase balanced distribution systems, define $v_k^b = B\angle\beta$, the approximation error for the B-phase voltage can be written as:

$$F(v_k^b) = |\frac{1}{B} + B - 2\cos(\beta + 120°)| \tag{9}$$

Proof:

$$\begin{aligned} F(v_k^b) &= |\frac{1}{v_k^b} - \gamma^{-1}(2 - \gamma^{-1} v_k^b)| \\ &= |\gamma^{-1}| \cdot |\frac{1}{\gamma^{-1} v_k^b} - (2 - \gamma^{-1} v_k^b)| \\ &= |\frac{1}{\gamma^{-1} v_k^b} - (2 - \gamma^{-1} v_k^b)| \end{aligned} \tag{10}$$

where $\gamma^{-1} v_k^b = B\angle(\beta + 120°)$. In scenario of $\beta = -120°$, $F(v_k^b) = |\frac{1}{B} + B - 2|$.

An illustrative diagram for three-phase voltages can be found in Fig. 2.

Then, the following approximation is deducted for the three-phase voltage vector on bus k:

$$\mathbf{1} \oslash \mathbf{v}_k = \mathbf{t}_k \circ (2 - \mathbf{t}_k \circ \mathbf{v}_k) \tag{11}$$

where $\mathbf{t}_k \in \mathbb{C}^3$, $\mathbf{t}_k = \{1, \gamma^{-1}, \gamma\}$; $\circ$ denotes the element-wise multiplication. The subset of $\mathbf{t}_k$ can implement the rotation operation for a two- or single-phase voltage $\mathbf{v}_k$.

Remarks for Unbalanced Systems: The error analysis works for a three-phase balanced system. As for unbalanced systems, extra errors are produced from imbalances. We have already investigated the accumulated system-wide errors by testing different DER penetration levels, which lead to various imbalance degrees. Through the linear power flow model, the system-wide error is proportional based on the network admittance matrix-based sensitivity.

References

[1] A. Garces, "A linear three-phase load flow for power distribution systems," *IEEE Trans. Power Systems*, vol. 31, no. 1, pp. 827–828, 2016.

[2] L. Yu, D. Czarkowski, and F. De León, "Optimal distributed voltage regulation for secondary networks with dgs," *IEEE Transactions on Smart Grid*, vol. 3, no. 2, pp. 959–967, 2012.

[3] M. D. Sankur, R. Dobbe, E. Stewart, D. S. Callaway, and D. B. Arnold, "A linearized power flow model for optimization in unbalanced distribution systems," *arXiv preprint arXiv:1606.04492*, vol. 144, 2016.