

# Supplement for “Linearized Interval Power Flow in Distribution Grids Under DER Uncertainty: A Fast Affine Arithmetic Approach”

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## 1 Taylor Series-Based Linearization

The Taylor-series-based linear approximation is proposed in [1] for power flow calculation in the per-unit system. A theoretical proof for the error analysis of this linear approximation, which is missing in [1], is proposed here.

### 1.1 Applying to A-Phase Voltages

In premise, define  $\Delta v_k^a = 1 - v_k^a$ , where  $v_k^a$  is the voltage phasor at bus  $k$ . The nonlinear term only appears on the left-hand side. Thus, the Taylor expansion of the complex-valued  $\Delta v_k^a$  within  $|\Delta v_k^a| < 1$  is adopted:

$$\frac{1}{v_k^a} = \frac{1}{1 - \Delta v_k^a} = \sum_{n=0}^{\infty} (\Delta v_k^a)^n \quad \forall |\Delta v_k^a| < 1 \quad (1)$$

Ignoring the high-order terms ( $n \geq 2$ ) and substituting the definition of  $\Delta v_k^a$  to (1), the following linear approximation is obtained:

$$\frac{1}{v_k^a} \approx 1 + \Delta v_k^a = 2 - v_k^a \quad (2)$$

Moreover, the errors introduced by the first-order Taylor expansion on  $v_k^a$  can be explicitly illustrated by:

$$F(v_k^a) = |1/v_k^a - (2 - v_k^a)| \quad (3)$$

Let  $v_k^a = A\angle\alpha$ , where  $A$  denotes the voltage magnitude and  $\alpha$  denotes the voltage phase angle, the following expression of (3) is deducted:

$$\begin{aligned} F(v_k^a) &= \left| \frac{1}{A}(\cos \alpha - j \sin \alpha) + A(\cos \alpha + j \sin \alpha) - 2 \right| \\ &= \left| \left( \frac{1}{A} + A \right) \cos \alpha - 2 + j \left( A - \frac{1}{A} \right) \sin \alpha \right| \\ &= \sqrt{\left[ \left( \frac{1}{A} + A \right) \cos \alpha - 2 \right]^2 + \left[ \left( A - \frac{1}{A} \right) \sin \alpha \right]^2} \\ &= \sqrt{\left( \frac{1}{A} + A \right)^2 \cos^2 \alpha - 4 \left( \frac{1}{A} + A \right) \cos \alpha + 4 + \left[ \left( A - \frac{1}{A} \right) \sin \alpha \right]^2} \\ &= \sqrt{\frac{1}{A^2} + A^2 + 2 \cos^2 \alpha - 2 \sin^2 \alpha - 4 \left( A + \frac{1}{A} \right) \cos \alpha + 4} \\ &= \sqrt{\frac{1}{A^2} + A^2 + 4 \cos^2 \alpha - 4 \left( A + \frac{1}{A} \right) \cos \alpha + 2} \\ &= \sqrt{\left( \frac{1}{A} + A - 2 \cos \alpha \right)^2} = \left| \frac{1}{A} + A - 2 \cos \alpha \right| \end{aligned} \quad (4)$$

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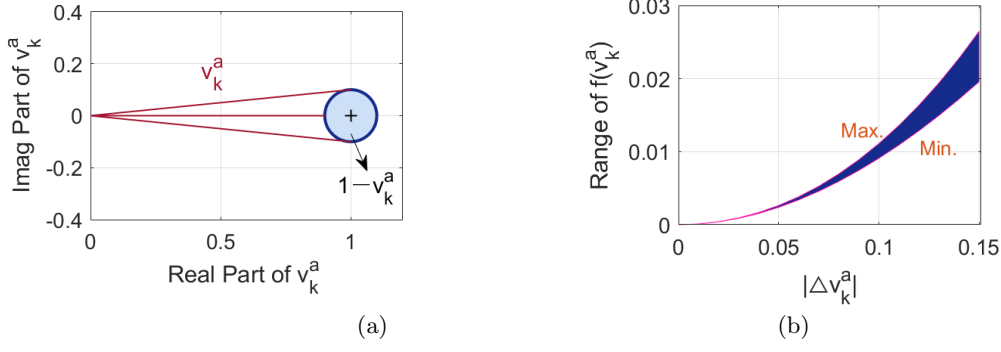


Figure 1: (a) Schematic diagram of the proposed linearization loss  $F(v_k^a)$  [1] (b) Error band of linearization errors

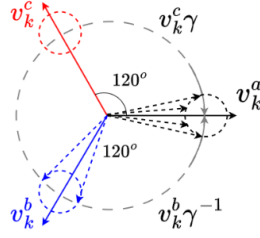


Figure 2: 120-degree phase shift and the error ranges for B- and C-phase rotation for linearization approximation

**Lemma 1 Errors of A-Phase Voltages:** Assume the phase angle difference along the distribution feeder is small [2] and in per unit system, the phasor angel of the substation voltage is set as 0, then  $\cos \alpha \approx 1$ . (4) can be approximated as

$$F(v_k^a) = \left| \frac{1}{A} + A - 2 \right| \quad (5)$$

Based on (5), considering the normal operating range in power systems, for example,  $A \in [0.95, 1.05]$ , the errors  $F(v_k^a)$  are constrained within  $[0, 0.0026]$ ;  $A \in [0.9, 1.1]$ , the errors  $F(v_k^a)$  are constrained within  $[0, 0.011]$ .

An illustrative diagram for the error analysis of A-phase voltages can be found in Fig. 1a) and b). The errors as the function value with different  $|\Delta v_k^a|$  based on an x-coordinate transformation  $|\Delta v_k^a| = |1 - v_k^a|$ , and the data points with  $(|\Delta v_k^a|, F(v_k^a))$  can be depicted as an area with the maximum and minimum as the bounds.

## 1.2 Extending to B- and C- Phases

Then (2) is extended to the three-phase system by only using the  $\pm 120$ -degree phase shift assumption, which is widely used in distribution system operation research such as the three-phase version DistFlow [3]. This approximation model can consider the line losses and has less reliance on certain assumptions, compared with other linear power flow models that ignore them (e.g., [3]).

Define a phasor rotation operator  $\gamma$ , which rotates a voltage vector counterclockwise by 120 degrees when multiplied by it:

$$\gamma = e^{-j\frac{2}{3}\pi} \quad (6)$$

Similarly,  $\gamma^{-1} = e^{j\frac{2}{3}\pi}$ .

Take the B-phase as an example, after rotating the original B-phase voltage clockwise by 120 degrees, i.e.,  $v_k' = \gamma^{-1}v_k^b$ ,  $|1 - v_k'| < 1$  exists. Thus, letting  $\Delta v_k' = 1 - \gamma^{-1}v_k^b$ , (2) is applied to the rotated B-phase voltage phasor  $v_k'$ . Then the following approximation for the pre-rotated B-phase

voltage phasor  $v_k^b$  is derived:

$$\frac{1}{v_k^b} = \gamma^{-1}(2 - \gamma^{-1}v_k^b) \quad (7)$$

Similarly, for the C-phase voltage, the following equation is obtained:

$$\frac{1}{v_k^c} = \gamma(2 - \gamma v_k^c) \quad (8)$$

**Lemma 2 Errors of B-Phase Voltages:** Based on the assumption of **Lemma 1** and 120-degree phasor shift, in three-phase balanced distribution systems, define  $v_k^b = B\angle\beta$ , the approximation error for the B-phase voltage can be written as:

$$F(v_k^b) = \left| \frac{1}{B} + B - 2\cos(\beta + 120^\circ) \right| \quad (9)$$

Proof:

$$\begin{aligned} F(v_k^b) &= \left| \frac{1}{v_k^b} - \gamma^{-1}(2 - \gamma^{-1}v_k^b) \right| \\ &= |\gamma^{-1}| \cdot \left| \frac{1}{\gamma^{-1}v_k^b} - (2 - \gamma^{-1}v_k^b) \right| \\ &= \left| \frac{1}{\gamma^{-1}v_k^b} - (2 - \gamma^{-1}v_k^b) \right| \end{aligned} \quad (10)$$

where  $\gamma^{-1}v_k^b = B\angle(\beta + 120^\circ)$ . In scenario of  $\beta = -120^\circ$ ,  $F(v_k^b) = \left| \frac{1}{B} + B - 2 \right|$ .

An illustrative diagram for three-phase voltages can be found in Fig. 2.

Then, the following approximation is deducted for the three-phase voltage vector on bus  $k$ :

$$\mathbf{1} \oslash \mathbf{v}_k = \mathbf{t}_k \circ (2 - \mathbf{t}_k \circ \mathbf{v}_k) \quad (11)$$

where  $\mathbf{t}_k \in \mathbb{C}^3$ ,  $\mathbf{t}_k = \{1, \gamma^{-1}, \gamma\}$ ;  $\circ$  denotes the element-wise multiplication. The subset of  $\mathbf{t}_k$  can implement the rotation operation for a two- or single-phase voltage  $\mathbf{v}_k$ .

**Remarks for Unbalanced Systems:** The error analysis works for a three-phase balanced system. As for unbalanced systems, extra errors are produced from imbalances. We have already investigated the accumulated system-wide errors by testing different DER penetration levels, which lead to various imbalance degrees. Through the linear power flow model, the system-wide error is proportional based on the Jacobian matrix-based sensitivity.

## References

- [1] A. Garces, "A linear three-phase load flow for power distribution systems," *IEEE Trans. Power Systems*, vol. 31, no. 1, pp. 827–828, 2016.
- [2] L. Yu, D. Czarkowski, and F. De León, "Optimal distributed voltage regulation for secondary networks with dgs," *IEEE Transactions on Smart Grid*, vol. 3, no. 2, pp. 959–967, 2012.
- [3] M. D. Sankur, R. Dobbe, E. Stewart, D. S. Callaway, and D. B. Arnold, "A linearized power flow model for optimization in unbalanced distribution systems," *arXiv preprint arXiv:1606.04492*, vol. 144, 2016.