Some Geometry of Local Systems

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Abstract

These notes consist of two parts. In the first part we discuss some equivalent characterizations for local systems and their generalizations to higher cases. In the second part we study the (derived) moduli theory for local systems. And there are some questions I want to figure out in my future study.

Contents

1 Some Characterizations for Local Systems

1.1 Results from Algebraic Topology

Generally speaking for a topological space X a *local system* F on X is a *locally constant sheaf* on the site $\mathcal{U}(X)$ which means there is a covering $\{U_i \to X | i \in I\}$ such that on each U_i, F is a *constant sheaf*.

Suppose A is any set, abelian group or commutative ring. Then there is a notion of constant presheaf which sends any open subset of X to A and constant sheaves are the shifification of constant presheaves. If U is an open subset of X, we see sections of the constant sheaf A_X of A on U consist of sections $U \to \coprod_{x\in X} A \approx A \times X$ where A is equipped with discrete topology. Since they should be continuous, we have

$$A_X(U) = \{\text{locally constant functions } s : U \to A\}$$

There is another characterization for local systems. We know a sheaf can be viewed as a bundle over X which is just a map from some top space to the base space X. For a sheaf F on X its associated bundle is $\coprod_{x\in X} F_x \to X$. And for any bundle $E \to X$ we can have a presheaf sending every open subset U of X to the set of sections $U \to E$. From these we obtain the following theorem.

Theorem 1.1.1. *There is an equivalence between sheaves and* ***étale maps*** *over X i.e.* $\mathrm{Shv}(X) \cong$ *Étale$/X$. Moreover stalks of sheaves are isomorphic to fibers of étale maps.*

Definition 1.1.2 (étale map)**.** A continuous map $p : E \to X$ is an *étale map* if it's a *local homeomorphism* which means for any point $e \in E$ there is an open subset $e \in V \subseteq E$ such that $p|V : V \to p(V)$ is a homeomorphism with $p(V)$ open in X.

We have already known local systems are special sheaves so that they correspond with some certain type of étale maps. Look at constant sheaves. Their bundles are like $A \times X \approx \coprod_{a\in A} X \to X$. So the bundle of a local system will look like this locally.

Definition 1.1.3 (covering space)**.** A continuous map $p : \tilde{X} \to X$ is a *covering projection* if for any point $x \in X$ there exists an open subset $x \in U \subseteq X$ such that $p^{-1}(U)$ is the disjoint union of open subsets in $\tilde{X}$ each of which is mapped homeomorphically onto U by p.

Theorem 1.1.4. *There is an equivalence between local systems on X and covering spaces on X i.e.* $\mathrm{LocSys(X)} \cong \mathrm{Cov}/X$.

Sketch of the proof. To see this we first suppoe $p : \tilde{X} \to X$ is a covering projection. Then for $x \in U \subseteq X$, we have $p^{-1}(U) = \coprod_{i\in I} V_i \approx I \times V$ where $V_i = V \approx U$. Then sections $U \to I \times V$ will just be locally constant functions $U \to I$. So that restriction to U the sheaf associated with p is a constant sheaf of I.

Conversely let F be a local system on X. Then on $U \subseteq X$, F is a constant sheaf with respect to I. For $p : \coprod_{x\in X} F_x \to X$, we have $p^{-1}(U) = \coprod_{x\in U} F_x \approx I \times U \approx \coprod_{i\in I} U$. □

The identification above makes it possible for us to use techniques from algebraic topology to study local systems.

In algebraic topology a *Hurewicz fibration* or simply a *fibration* is a continuous map $p : E \to X$ such that it has the right lifting property with respect to (RLP wrt) all $i_0 : Y \to Y \times I$ where $I = [0, 1]$ is the unit interval. We say it has the *unique path lifting property* if with respect to $* \to I$ the lifting map is unique. In [Spa, Thm. 2 and 3 in P67] we see that a covering projection is a fibration with the unique path lifting property. Now using this property we can compare fibers of a covering space.

Let $p : \tilde{X} \to X$ be a covering projection and ω a path from x_0 to x_1. From ω we have a function

$$F_\omega : p^{-1}(x_0) \times I \to X,\ (\tilde{x}, t) \mapsto \omega(t)$$

Then with respect to $i_0 : p^{-1}(x_0) \to p^{-1}(x_0) \times I$,

$$\begin{tikzcd}
p^{-1}(x_0) \arrow[r] \arrow[d, "i_0"'] & \tilde{X} \arrow[d, "p"] \\
p^{-1}(x_0) \times I \arrow[r, "F_\omega"'] \arrow[ur, dashed, "G_\omega"] & X
\end{tikzcd}$$

there is a lifting $G_\omega : p^{-1}(x_0) \times I \to \tilde{X}$ such that $p \circ G_\omega(\tilde{x}, t) = \omega(t)$. Therefore $G_\omega(\tilde{x}, 1)$ is an element in $p^{-1}(x_1)$ and this defines a function $G_\omega(-, 1) : p^{-1}(x_0) \to p^{-1}(x_1)$.

For any second path ω' from x_0 to x_1 such that $\omega \simeq \omega'$ rel ∂I, we will obtain the same point in $p^{-1}(x_1)$ since their liftings will be homotopic relative to ∂I in $\tilde{X}$[1] And one can prove we have defined a functor $\Pi_1(X) \to \mathbf{Set}$ sending a point $x \in X$ to $p^{-1}(x)$ where $\Pi_1(X)$ is the fundamental groupoid of X. So that the obtained map $G_\omega(-, 1)$ is an isomorphism.

The argument above shows for a local system on X if there is a path from x_0 to x_1 then their stalks will be isomorphic to each other. This is similar to the notion of *parallel transport* in differential geometry which we will discuss later.

[1] [Spa, Lem. 3 in P72]

Now naturally there is a question whether every functor $\Pi_1(X) \to \mathbf{Set}$ can be obtained in such a way. The answer is positive if X is good enough[2].

Theorem 1.1.5. *If X is good enough, then there is an equivalence between local systems on X and functors $\Pi_1(X) \to \mathbf{Set}$, i.e.* $\mathrm{LocSys(X)} \cong \mathrm{Funct}(\Pi_1(X), \mathbf{Set})$.

Sketch of the proof. Given a functor $F : \Pi_1(X) \to \mathbf{Set}$ we can define a presheaf $\tilde{F}$ on X such that for every open subset $U \subseteq X$, $\tilde{F}(U)$ consists of all functions $f : X \to \coprod_{x \in X} F(x)$ satisfying $f(x) \in F(x)$ and for any path ω in U we have $f(\omega(1)) = F(\omega)(f(\omega(0)))$. Clearly $\tilde{F}$ is a sheaf from the paracompactness. Moreover by considering how $\tilde{F}$ behaves on connected open subsets we can see it's actually a local system. $\square$

Question 1. What are ∞-local systems? For a topological space X, $\mathrm{Sing}_\bullet(X)$ is a Kan complex i.e. an ∞-groupoid which acts like $\Pi_\infty(X)$. A generalization for $\mathrm{Funct}(\Pi_1(X), \mathbf{Set})$ is ∞-$\mathrm{Funct}(\mathrm{Sing}_\bullet(X), \mathbf{sSet})$. But on the other hand a local system is a locally constant sheaf so that its natural generalization should be a locally constant ∞-stack $\mathcal{U}(X)^{op} \to \mathbf{sSet}$. Are they still equivalent at the infinite level? The nlab page of locally constant ∞-stack is helpful to answer this question.

1.1.1 Representations of Fundamental Groups

Here we give another characterization for local systems.

Now if we fix a point x_0 in X, then there is a functor of fibers

$$\mathrm{Fib}_{x_0} : \mathrm{Cov}/X \to \mathbf{Set},\ p \mapsto p^{-1}(x_0)$$

As we have seen, a covering space $p : \tilde{X} \to X$ induces a functor $\Pi_1(X) \to \mathbf{Set}$. Every loop ω in $\pi_1(X, x_0)$ will correspond to an isomorphism $p^{-1}(x_0) \to p^{-1}(x_0)$. The image of $\tilde{x}$ via this isomorphism is denoted by $\omega \cdot \tilde{x}$. Then we can see $p^{-1}(x_0)$ is actually a $\pi_1(X, x_0)$-set and the action is called the *monodromy* action on the fiber.

Definition 1.1.6. A *representation* for a group G is a group homomorphism $G \to \mathrm{Aut}(M)$ where M is some set. This is equivalent to a G-set structure on M. So that $\mathbf{Rep}(G) = G$-$\mathbf{Set}$. If A is a commutative ring, an A-*representation* for G is a group homomorphism $G \to \mathrm{Aut}_A(M)$ for some A-module M and we have $\mathbf{Rep}_A(G) = G$-$\mathbf{Mod}_A$.

Theorem 1.1.7. *If X is good enough[3], the functor*

$$\mathrm{Fib}_{x_0} : \mathrm{Cov}/X \to \mathbf{Rep}(\pi_1(X, x_0)),\ \ p \mapsto p^{-1}(x_0)$$

is an equivalence between covering spaces on X and representations of $\pi_1(X, x_0)$. Moreover we can see $\mathrm{Aut}(\mathrm{Fib}_{x_0}) \cong \pi_1(X, x_0)$.

Sketch of the proof. To see this, given any $\pi_1(X, x_0)$-set S we decompose it as a disjoint union of $\pi_1(X, x_0)$-orbits. Then for every orbit we have a quotient space of the universal covering space $\tilde{X}_u$ by the stabilizer of some point in the orbit. Then the disjoint union of these quotients is just what we want. Fully faithfulness and details can be found in [Sza, Thm. 2.3.4]. $\square$

We summarize results we have obtained until now as follows.

Theorem 1.1.8. *If X is good enough, the following categories are equivalent:*

1. *local systems on X i.e.* $\mathrm{LocSys}(X)$*;*

[2] According to [Spa, Ex. F.4 in P360], the space X should be paracompact, Hausdorff, connected, locally path connected and semilocally 1-connected, e.g. a connected complex manifold.

[3] At least X should be connected, locally path connected and semilocally 1-connected, so that it will admit the universal covering space.

2. *covering spaces on X i.e.* Cov/X*;*

3. *functors from the fundamental groupoid of X to sets i.e.* $\mathrm{Funct}(\Pi_1(X), \mathbf{Set})$*;*

4. *representations of the fundamental group i.e.* $\mathbf{Rep}(\pi_1(X, x_0))$*.*

Note that we can replace $\mathbf{Set}$ by $\mathbf{Mod}_A$ for any commutative ring A in which case an A-local system should be a locally constant A_X-sheaf where A_X is the constant sheaf of A.

More generally if we consider a discrete group G not a commutative ring A and replace $\mathbf{Set}$ by the category of G-sets where G acts transitively, then we obtain three equivalent characterizations as follows:

1. locally constant sheaves on X such that G_X acts on them transitively;

2. principal G-bundles over X such that parallel transports induced by homotopy classes of paths in X are G-equivariant;

3. functors from the fundamental groupoid of X to G-sets where G acts transitively.

We call these geometric objects *G-local systems*.

If we set $G = GL_n(k)$ for some field k, since for a sheaf F, $\mathrm{Aut}(F)$-torsors are equivalent to $\mathrm{Twist}(F)$ twists of F which are locally isomorphic to F[4], a $GL_n(k)$-local system is just a locally constant sheaf of k-vector spaces of rank n.

Question 2. Are there local systems of some type equivalent to representations of étale fundamental groups i.e. Galois representations? I think this can help understand how we can get geometric Langlands conjecture from arithmetic Langlands conjecture on the Galois side.

1.2 Riemann-Hilbert Correspondence

1.2.1 Classical Case

As we have seen a local system is roughly a collection of objects with the notion of parallel transports within them. In differential geometry parallel transports are induced by *connections*[5] so that some certain kind of connections may be equivalent to local systems. Now we can suppose X is a connected complex manifold or a smooth projective algebraic variety over $\mathbb{C}$ and in the latter case we need to consider the complex analytic space X^{an} of X.

A *holomorphic vector bundle* over X is a complex manifold E with a holomorphic map $\pi : E \to X$ such that its fiber $\pi^{-1}(x)$ is isomorphic to a finite dimensional $\mathbb{C}$-vector space for every $x \in X$ and satisfies some certain compatible conditions. And this definition will be equivalent to *locally free $\mathcal{O}_X$-sheaves* where $\mathcal{O}_X$ is the sheaf of homolomorphic functions. And we will not distinguish the two definitions.

Definition 1.2.1 (connections)**.** A *(holomorphic) connection* on X is a pair (E, ∇) where E is a holomorphic vector bundle and $\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ is a map of sheaves such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for $f \in \mathcal{O}_X$ and $s \in E$. This map extends naturally to

$$\nabla^n : \Omega^n_X \otimes_{\mathcal{O}_X} E \to \Omega^{n+1}_X \otimes_{\mathcal{O}_X} E,\ \ \omega \otimes s \mapsto d\omega \otimes s + (-1)^n \omega \wedge \nabla(s)$$

A connection is called *flat* or *integrable* if $\nabla^{n+1} \circ \nabla^n = 0$ which is equivalent to $\nabla^1 \circ \nabla^0 = 0$.

[4]They are both isomorphic to the first cohomology group of $\mathrm{Aut}(F)$. It's a standard theorem in algebraic geometry, see e.g. Ulrich Gortz and Torsten Wedhorn, *Algebraic Geometry I*, Sec. 11.6

[5]There is a concept of *Ehresmann connection* in differential geometry. Details can be found in the Chapter 12 of Jeffrey M. Lee, *Manifolds and Differential Geometry*. Theorem 12.20 there gives such a parallel translation for a connection and a smooth path.

Theorem 1.2.2 (Riemann-Hilbert Correspondence). [6] *The category of finite dimensional complex local systems on X is equivalent to the category of holomorphic vector bundles on X with a flat connection i.e.* $\mathrm{LocSys}_{\mathbb{C}}(X) \cong \mathrm{Flat}(X)$.

Sketch of the proof. For a flat connection (E, ∇) its kernel $\ker \nabla$ will be a complex local system with an isomorphism $\mathcal{O}_X \otimes_{\mathbb{C}_X} \ker \nabla \xrightarrow{\sim} E$ where $\mathbb{C}_X$ is the constant sheaf of $\mathbb{C}$ on X. Conversely for a complex local system F, we can consider $\mathcal{O}_X \otimes_{\mathbb{C}_X} F$ with the connection $d_X \otimes \mathrm{id}_F$. □

This result depends heavily on the complex topology of X. JIn particular this correspondence tells us that there is an equivalence between flat connections and finite dimensional complex representations of $\pi_1(X, x_0)$.

1.2.2 Derived Case

In [Lur1] Lurie says we can drop the assumption of "projective" and just suppose X is a smooth algebraic variety over $\mathbb{C}$. Then we can embed X into a smooth projective algebraic variety $\overline{X}$. From this, a holomorphic vector bundle on X^{an} can be equipped with an algebraic structure by restriction due to Serre's GAGA theorem for $\overline{X}^{an}$ and $\overline{X}$ i.e. it corresponds with an algebraic vector bundle on X which is moreover compatible with connections.[7]

So that the classical Riemann-Hilbert correspondence gives an equivalence between complex local systems on X^{an} and algebraic vector bundles on X with flat (regular) connections. In this case complex local systems are locally free $\mathbb{C}_X$-vector spaces and algebraic vector bundles are locally free $\mathcal{O}_X$-modules. If we want to drop the condition of "locally free", we will need to go much further than the classical case.

Let E be a quasi-coherent $\mathcal{O}_X$-module on X. Then a flat connection $\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ is equivalent to a map $(\Omega^1_X)^\vee \to \mathrm{End}_{\mathbb{C}}(E)$ satisfying certain conditions. Note that

$$(\Omega^1_X)^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(E, \mathcal{O}_X) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

$\mathcal{D}_X$ is defined to be the subalgebra (not necessarily commutative) of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $(\Omega^1_X)^\vee$. Then a flat connection will be equivalent to a left algebraic $\mathcal{D}_X$-module structure on E[8].

On the other side we generalize local systems on X^{an} to *constructible sheaves* which can be roughly understood as "locally" locally constant sheaves.

Definition 1.2.3 (constructible sheaf). For a complex analytic space X^{an} a *stratification* is a locally finite partition $X^{an} = \coprod_{\alpha \in A} X^{an}_\alpha$ by locally closed complex analytic subsets X^{an}_α such that every closure $\overline{X^{an}_\alpha}$ is a disjoint union of some X^{an}_β's.

A $\mathbb{C}_{X^{an}}$-module is a *constructible sheaf* if when restricted to some stratification on X^{an}, it's a locally constant sheaf on every X^{an}_α. Note that the existence of stratification is due to Whitney.

Passing to derived categories there are *de Rham* and *solution* functors i.e.

$$DR_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}}),\ \ M \mapsto \omega^{an}_X \otimes^{\mathbb{L}}_{D_{X^{an}}} M^{an}$$

and

$$\mathrm{Sol}_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}})^{op},\ \ M \mapsto \mathbb{R}\mathcal{H}\mathrm{om}_{D_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}})$$

where ω_X is the canonical bundle of X.

Riemann-Hilbert correspondence says they are equivalences from $D^b_{rh}(D_X)$ the subcategory of $D^b(D_X)$ consisting of bounded complexes of D-modules whose cohomology groups are *regular holonomic* D-modules

[6] [Con, Thm. 2.6]

[7] Deligne has proved the analytification functor gives an equivalence between (regular) algebraic flat connections on X and analytic or holomorphic flat connections on X^{an}. See e.g. [HTT, Thm. 5.3.8].

[8] [HTT, Lem. 1.2.1]

to $D^b_c(X)$ the subcategory of $D^b(\mathbb{C}_{X^{an}})$ consisting of bounded complexes of $\mathbb{C}_{X^{an}}$-modules whose cohomologpy groups are constructible sheaves. Moreover the image of $D_X\text{-Mod}_{rh}$ by DR_X is an abelian category in $D^b_c(X)$ consisting of *perverse sheaves*[9].

1.2.3 Differential Graded Case

As we have seen, classical Riemann-Hilbert correspondence establishes an equivalence between flat connections and complex representations of $\pi_1(X, x_0)$ for a connected complex manifold X. But it seems it's only a truncated version because $\pi_1(X, x_0)$ or $\Pi_1(X)$ does not contain all homotopy information of X. So the higher version of Riemann-Hilbert correspondences tries to get a similar relationship for higher flat connections and ∞-local systems. In [BS] the authors work on a real manifold but this makes no big difference when considering a complex manifold (see [CHL, Sec. 8.2]) and we only need to replace the de Rham algebra by the Dolbeault algebra. For simplicity we also take X as a (connected) real manifold.

In the classical theory a local system is a functor $\Pi_1(X) \to \mathbf{Set}$. Here we replace the fundamental groupoid $\Pi_1(X)$ by the smooth ∞-groupoid $\Pi_\infty(X) := \mathrm{Sing}^\infty_\bullet(X)$ whose sections consist of smooth simplices. Since we consider real local systems, we need to replace $\mathbf{Set}$ by some ∞-category of real vector spaces. In [BS] this desired ∞-category $\mathcal{C}_\infty$ is obtained from $\mathcal{C} := \mathbf{Ch}^\bullet(\mathbb{R})$ by taking $H^{\leq 0}$ on internal hom-complexes, Dold-Kan equivalence and simplicial nerve (see [BS, Rem. 2.5]). Then *real ∞-local systems* are defined to be ∞-functors i.e. maps of simplicial sets from $\Pi_\infty(X)$ to $\mathcal{C}_\infty$. This category is denoted by $\mathrm{LocSys}^\infty_{\mathcal{C}}(X)$ which is actually a dg-category ([BS, Prop. 2.8]).

On the other side we need to extend the usual concept of connections to *superconnections*. Let $\Omega = (\Omega^\bullet(X), d)$ be the de Rham differential graded algebra on X. In the classical case a connection is a locally free sheaf E with a derivation map on $E \otimes \Omega^\bullet_X$ of degree 1. And in algebraic geometry we know a locally free sheaf of finite rank is equivalent to a finite projective module. So that here we suppose $E^\bullet$ is a bounded complex of finitely generated projective $\Omega^0(X)$-modules and view it as a *dg-vector bundle*.

Definition 1.2.4. We say a pair $(E^\bullet, \mathbb{E})$ is a *cohesive module* or a *dg-vector bundle with a superconnection* if $\mathbb{E}$ is a *$\mathbb{Z}$-graded connection* on $E^\bullet \otimes_{\Omega^0(X)} \Omega^\bullet(X)$ of degree 1 satisfying the usual Leibniz condition i.e.

$$\mathbb{E}(e\omega) = (\mathbb{E}(e \otimes 1))\omega + (-1)^{\deg(e)} e d\omega$$

This superconnection is *flat* if $d\mathbb{E} + \mathbb{E}^2 = 0$. The category of cohesive modules is denoted by $\mathcal{P}_\Omega$.

In [BS] the authors define a really complicated functor $\mathcal{RH} : \mathcal{P}_\Omega \to \mathrm{LocSys}^\infty_{\mathcal{C}}(X)$[10] and prove it's a *quasi-equivalence* of dg-categories so that they are equivalent in $\mathrm{Ho}(\mathbf{dgCat}_{\mathbb{R}})$.

The approach to the problem taken by [CHL] is a bit different where ∞-local systems are regarded as some kind of *locally constant dg-sheaves* and cohesive modules are replaced by *perfect twisted modules* which are actually shown equivalent to the former [CHL, Cor. 3.1].

1.3 Crystals

We should remember a local system is roughly a collection of objects within which there exist parallel transports for paths. But in a smooth algebraic variety X it's equipped with Zariski topology which is much different from usual topologies like complex topology so that the usual concept of paths is not valid here.

For a complex manifold it's semilocally 1-connected or locally simply connected which means every point in it admits a simply connected open neighborhood. Since parallel transports are given by homotopy classes of paths, fibers over this simply connected open neighborhood will be canonically isomorphic to each other. But this is not the case in the Zariski topology. In [Lur2] Lurie says there is no simply connected

[9] There is another description for perverse sheaves in [HTT, Sec. 8] using t-structures.
[10] This requires a deep analysis of (higher) holonomy.

open subsets in a smooth curve of genus > 0 (**need a proof!**). So that we need a new way to say two points are close enough i.e. "in a simply connected open subset", and the parallel transport is canonical. In Grothendieck's language the former is to say they are *infinitesimally close* and the latter is equivalent to the *cocycle condition*.

We suppose X is a smooth algebraic variety over a field k of characteristic zero e.g. $\mathbb{C}$. For a commutative k-algebra R, its *nilradical ideal* is $I_R = \bigcap_{\mathfrak{p}\in\mathrm{Spec}R} \mathfrak{p}$ consisting of nilpotent elements. We say two R-valued points $x, y \in X(R)$ are *infinitesimally close* if they have the same image under the map $X(R) \to X(R/I_R)$. Note that since $\mathrm{Spec}R$ and $\mathrm{Spec}R/I_R$ are homeomorphic in terms of underlying topological spaces, two infinitesimally close points x, y induce the same map $\mathrm{Spec}R \to X$ on underlying topological spaces.

Definition 1.3.1 (crystal)**.** A *crystal of quasi-coherent sheaf* on X is a pair $(\mathscr{F}, \eta_{x,y})$ where $\mathscr{F}$ is a quasi-coherent sheaf on X and $\eta_{x,y} : x^*\mathscr{F} \to y^*\mathscr{F}$ is an isomorphism of quasi-coherent sheaves on $\mathrm{Spec}R$ for two infinitesimally close points $x, y \in X(R)$ such that

1. for a map $R \to R'$, x, y induce two new infinitesimally close points $x', y' \in X(R')$ and they satisfy

$$\eta_{x',y'} : x'^*\mathscr{F} \simeq x^*\mathscr{F} \otimes_R R' \to y^*\mathscr{F} \otimes_R R' \simeq y'^*\mathscr{F}$$

   is obtained from $\eta_{x,y}$ by tensoring R';

2. $\eta_{x,y}$ satisfies the *cocycle condition* i.e. $\eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y}$ and $\eta_{x,x} = \mathrm{id}$.

Note that $x^*\mathscr{F}$ is obtained by the pullback of $\mathscr{F}$ along x. We also write $\mathscr{F}(x)$ for $x^*\mathscr{F}$.

Theorem 1.3.2. *The category of crystals of quasi-coherent sheaves on X is equivalent to the category of quasi-coherent left D_X-modules.*

We know flat connections are equivalent to D_X-module structures. Hence crystals of quasi-coherent sheaves are then also equivalent to flat connections as an algebraic replacement. From this viewpoint crystals are a good notion for algebraic parallel transports.

Sketch of the proof. We sketch the idea here and details can be found in [Lur2, Thm. 0.4].

Suppose $\mathscr{F}$ is a quasi-coherent sheaf on X. For two points $x, y \in X(R)$ they induce a map $\mathrm{Spec}R \to X \times X$. They are infinitesimally close iff the induced map $\mathrm{Spec}R/I_R \to X \times X$ forctors through the diagonal map $\Delta : X \to X \times X$. By considering the formal completion $(X \times X)^\vee$ of the locally closed immersion Δ i.e. the *ind-scheme* or "colimit" of $X \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq X \times X$, Lurie shows an isomorphism $\eta_{x,y}$ is equivalent to a map $\mathscr{F} \to pr_{1,*}^{(n)} pr_2^{(n),*}\mathscr{F}$ and then equivalent to $D_X^{\leq n} \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{F}$ where $pr_i^{(n)}$'s are projections from $(X \times X)^\vee$ to X when restriction to $X^{(n)}$. And the cocycle condition will be equivalent to the associative algebra axiom of the D_X-action. □

We can define the concept of quasi-coherent sheaves on an arbitrary functor $X : \mathbf{Alg}_k \to \mathbf{Set}$ which is a pair $(\mathscr{F}(x), \alpha_{x,x'})$ where for any point $x \in X(R)$, $\mathscr{F}(x)$ is an R-module and if $R \to R'$ is a map of k-algebras and $x' \in X(R')$ is the image of x, then $\alpha_{x,x'} : \mathscr{F}(x') \simeq \mathscr{F}(x) \otimes_R R'$ is an isomorphism of R'-modules. Note that this pair should satisfy the compatible condition for $R \to R' \to R''$. If X is actually a scheme, then it will be covered by open affine schemes and the quasi-coherent sheaf on it will be glued by these modules on an open affine covering. Hence this definition generalizes the classical definition for quasi-coherent sheaves.

With this definition we can see a crystal on X is actually equivalent to a quasi-coherent sheaf on the *de Rham stack* X_{dR} which sends any algebra R to the set $X(R/I_R)$. Note that $\eta_{x,y}$'s identify an R/I_R-module canonically and we can lift such a module to be an R-module. Moreover for a smooth scheme X, $X(R) \to X(R/I_R)$ is surjective according to the lifting property of smoothness. Therefore the natural map $X \to X_{dR}$ is an epimorphism and in this sense we can view X_{dR} as a quotient of X by the relation of "infinitesimal closeness".

This feature can help us generalize the concept of crystals to the higher case. For a prestack $X : \mathbf{cdgA}_k^{\leq 0} \to \mathbf{sSet}$, where $\mathbf{cdgA}_k^{\leq 0}$ is the category of cdgas (commutative differential graded algebras centered in degree $n \leq 0$), its *de Rham prestack* X_{dR} can be defined such that

$$X_{dR}(A^\bullet) = X(H^0(A^\bullet)/I_{H^0(A^\bullet)})$$

In [GR] Gaitsgory and Rozenblyum say **"the key idea is that one should regard higher homotopy groups of a derived ring as a generalization of nilpotent elements."** So that we do not let $X_{dR}(A^\bullet)$ be $X(A^\bullet/I_{A^\bullet})$. Then a *(left) crystal* on X will be defined as a quasi-coherent sheaf on X_{dR}. And they also generalize the two phenomena we talked above to this derived case. They show if X is good enough i.e. *eventually coconnective* which means $H^n(\mathcal{O}_X) \neq 0$ for only finitely many n's, then this definition for crystals will be equivalent to Grothendieck's version of infinitesimal closeness (see [GR, Prop. 3.4.3]). And they describe relations between crystals and D-modules as well in this derived case (see [GR, Sec. 5]).

Question 3. In the above we have seen the differential graded Riemann-Hilbert correspondence on a smooth or complex manifold. Does it still hold on a *higher analytic stack*? Moreover by the higher GAGA theorem described in [PY], whether we can obtain a result similar to Deligne's theorem in the classical case. In other words, what's the relation between crystals on a higher geometric stack X and flat superconnections or ∞-local systems on its analytification X^{an}? Another version of Riemann-Hilbert correspondence in [TV3] may be helpful (not sure).

Question 4. What are local systems on an analytic stack? Recently there is a theory of *analytic stacks* developed by Clausen and Scholze. An analytic stack is a prestack $\mathbf{AnRing} \to \mathbf{sSet}$ on analytic rings satisfying the descent for !-hypercovers. This concept unifies two notions in [PY] i.e. higher complex/non-achimedean analytic stack in the sense that there are natural analytic rings corresponding to (compact Stein) complex-analytic spaces, and affinoid Berkovich analytic spaces; and open covers or quasi-étale maps go to !-covers. So whether there is a good theory of local systems on this analytic stack?

2 Moduli Theory for Local Systems

2.1 Some Classical Approaches

Via GIT In [Simp1] Simpson uses the technique of Mumford's GIT theory to construct *Betti* moduli space M_{B}, *Dolbeault* moduli space M_{Dol} and *de Rham* moduli space M_{dR} parametrizing representations of the fundamental group, Higgs bundles and vector bundles with a flat connection respectively for a complex manifold or a smooth projective algebraic variety X on $\mathbb{C}$. Note that these moduli spaces are coarse in the sense of Mumford. He also shows some topological equivalences among these moduli spaces. The classical Riemann-Hilbert correspondence can therefore give an equivalence $M_{\mathrm{B}}^{an}(X) \simeq M_{\mathrm{dR}}^{an}(X)$ of complex analytic spaces.

Remark 2.1.1. These moduli spaces are based on the (coarse) moduli space of coherent sheaves. But to obtain the latter we need to only consider about (semi-)stable sheaves since they form a *bounded family* but all sheaves without the (semi-)stable condition are not bounded. For example we can consider the set $\{\mathcal{O}(n) \oplus \mathcal{O}(-n) | n \geq 0\}$ on $\mathbb{P}^1$ and there is no family over a scheme of finite type gluing all of them. So here M_{Dol} and M_{dR} actually parameterize (semi-)stable bundles.

Via Stacks Although [CW] deals with Higgs bundles, similar arguments hold as well for vector bundles with a flat connection. Suppose X is a smooth projective curve over $\mathbb{C}$. Then the stack parametrizes pairs (E, ∇) where E is a vector bundle of rank r and degree d. ∇ can be a flat connection or a *Higgs field* i.e. $\nabla : E \to E \otimes \Omega_X^1$ with $\nabla \wedge \nabla = 0$. A morphism of pairs makes them form a pullback. Then this can be an *algebraic stack* [CW, Thm. 7.18]. And the semi-stability will be an open condition. We can therefore obtain an open substack of semi-stable bundles whose map to the coarse moduli scheme talked above is initial among all morphisms to schemes.

In Gauge Theory The moduli space of G-local systems up to gauge equivalence is considered in mathematical physics. On a smooth manifold X a G-local system can be viewed as a principal G-bundle with a flat connection (to induce the parallel transport). Then this moduli space will be

$$M_G(X) \cong \frac{\mathrm{Hom}(\pi_1(X,x),G)}{G}$$

consisting of group homomorphims $\pi_1(X,x) \to G$ up to conjugation. And this set can be equipped with a manifold structure but not necessarily smooth. Moreover if X is a compact orientable surface and the corresponding Lie algebra of G satisfies some more conditions, then the moduli space $M_G(X)$ will be a *symplectic reduction*. In this specific example we can think about more structures like symplectic structure or Poisson structure on it and this idea is already generalized in derived algebraic (symplectic) geometry.

The above construction can also be applied to obtain the Betti moduli space M_{B}. In [Simp2] Simpson deals with the case where $G = GL_n$. We suppose the topological space X is good enough e.g. the complex analytic space of a smooth projective variety over $\mathbb{C}$ and fix a point $x \in X$. Let $\Gamma = \pi_1(X,x)$ which is finitely generated. Then we can define a functor from $\mathbf{Sch}_{\mathbb{C}}$ to $\mathbf{Set}$ sending any scheme S to the set of group homomorphisms $\mathrm{Hom}(\Gamma, GL(n, \mathcal{O}_S(S)))$. This functor is represented by an affine scheme $\mathbf{R}(\Gamma,n)$ which can be constructed by relations in Γ as a closed subscheme in the product of the general linear algebraic group GL_n. In this sense $\mathbf{R}(\Gamma,n) = \mathrm{Hom}(\Gamma, GL_n)$. Considering the conjugate action of GL_n on $\mathbf{R}(\Gamma,n)$, we get the (coarse) Betti moduli space $M_{\mathrm{B}}(X,n)$ via GIT. (see [Simp2, Prop. 6.1])

This construction can be generalized for any reductive affine algebraic group G. And we have the (coarse) Betti moduli space $M_G(X)$ whose $\mathbb{C}$-points are in one-to-one correspondence with isomorphism classes of semi-simple locally constant principal G-bundles.

From the stacky viewpoint we can also get a quotient stack $\mathcal{M}_G(X) := [\mathbf{R}_G(\Gamma)/G]$ whose map to $M_G(X)$ is initial among morphisms to schemes.

2.2 Derived Betti Moduli Stack

We introduce the derived Betti moduli stack of G-local systems. In the following we suppose k is a field of characteristic 0 e.g. $\mathbb{C}$ and X is a good enough topological space e.g. a connected complex manifold. There are some ways to describe this derived moduli stack. We deal with the case $G = \mathrm{GL}_n$ first.

At the beginning we take a moduli theory viewpoint to see which geometric objects it really classifies. Here we extend complex vector spaces or vector bundles to be differential graded modules (dg-modules). Again we assume X is good enough and A is a cdga over k. A-$\mathbf{Mod}_X^{\leq 0}$ denotes the category of presheaves of dg-A-modules on X. It's equipped with a *global projective model structure* and is a dg-category. Apply *left Bousfield localization* to this model category we can get a new model category where weak equivalences are *local weak equivalences* such that they induce quai-isomorphisms of dg-A-modules on stalks.

Definition 2.2.1. We say a presheaf $\mathcal{F}$ of dg-A-modules on X is *locally on $X \times A_{\acute{e}t}$ equivalent to A^n* if for any point $x \in X$ there exists an open neighborhood U of x and an étale covering $\{\mathrm{Spec}B_i \to \mathrm{Spec}A | i \in I\}$ for the *affine derived stack* $\mathrm{Spec}A$ such that $\mathcal{F}_{|U} \otimes_A B_i$ is weakly equivalent to B^n as presheaves.

Then the category $\mathrm{LocSys}_n(X;A)$ of local systems of dg-A-modules of rank n consists of cofibrant objects locally on $X \times A_{\acute{e}t}$ equivalent to A^n and morphisms local weak equivalences. Note that we can regard this category as a simplicial set by applying the dg-nerve functor (in [Lur3, Construction 1.3.1.6]). Then we get a derived prestack $\mathbb{R}\mathrm{LocSys}_n(X)$ sending any cdga A to the simplicial set $\mathrm{N_{dg}}(\mathrm{LocSys}_n(X;A))$. It's shown in [TV2, Prop. 2.2.6.5] this derived prestack is a *derived stack*.

There is also another way to describe this derived stack. We have already known a local system or a locally constant sheaf of sets is equivalent to a functor from $\Pi_1(X)$ to $\mathbf{Set}$. Therefore a natual viewpoint tells us the higher moduli space of local systems should parametrize ∞-local systems and be a mapping space from $\Pi_\infty(X)$ to some infinite category. For complex local systems $\mathbf{Set}$ is replaced by complex vector spaces. Therefore here we consider the derived moduli stack $\mathbb{R}\mathscr{B}un_n$ of vector bundles of rank n and the derived stack of local systems of rank n is then defined as

$$\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), \mathbb{R}\mathscr{B}un_n)$$

the mapping stack in the category $\mathbf{dStk}_k = \mathbf{dAff}^{\sim}_{\text{ét}}$ of derived stacks where $\Pi_\infty(X)$ is the constant derived stack associated to the singular Kan complex $\mathrm{Sing}_\bullet(X)$. Also note that a classical theorem tells us a vector bundle of rank n is actually equivalent to a GL_n-torsor. Therefore $\mathbb{R}\mathscr{B}un_n$ is equivalent to the classifying derived stack $B\mathrm{GL}_n$. This version can be adapted to any other suitable group G and we can define the derived moduli stack of G-local systems as $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG)$.

Theorem 2.2.2. [11] *$\mathbb{R}\mathrm{LocSys}_n(X)$ is equivalent to $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), \mathbb{R}\mathscr{B}un_n)$.*

Question 5. Whether there is a good homotopy theory for geometric objects like general (derived) schemes such that a local system can be identified with a functor on this homotopy type? So that we can define the Betti moduli stack for (derived) schemes as the derived mapping stack on homotopy types. And also we can study its relations to the de Rham moduli stack which is defined to be $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X_{dR}, BG)$ for any prestack X. Note that for a smooth projective variety X over $\mathbb{C}$, the two stacks are equivalent on the truncation by Riemann-Hilbert correspondence which is a result of our characterizations before.

The final approach to getting the derived moduli stack of local systems is similar to the method we discussed before to obtain the quotient stack $\mathcal{M}_G(X) := [\mathbf{R}_G(\Gamma)/G]$ where $\Gamma = \pi_1(X, x)$. In the theory of simplicial sets there is a Quillen pair between simplicial sets and simplicial groups [12]. The *loop group construction* sends every simplicial set to a weakly equivalent simplicial group. Then for our space X, it can be associated with a simplicial group GX whose geometric realization is weakly homotopy equivalent to X i.e. having the same homotopy type. If it's necessary, we can resolve GX further to obtain a simplicial group $\Gamma_\bullet$ such that every Γ_n is finitely free. In this case $\mathbf{R}_G(\Gamma_\bullet)$ will be a *cosimplicial affine scheme* which is actually an affine derived stack. Actually if we use the model of cdga, we need to apply the normalization functor to the simplicial commutative algebra structure on $\mathbf{R}_G(\Gamma_\bullet)$ to obtain a cdga $\mathscr{A}_G(X)$. Then $\mathbb{R}\mathrm{LocSys}_G(X)$ is defined as the *derived quotient stack* $[\mathrm{Spec}\mathscr{A}_G(X)/G]$ (see [TV2, Sec. 1.3.4]).

Remark 2.2.3. To see $[\mathrm{Spec}\mathscr{A}_G(X)/G]$ is equivalent to $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG)$ we follow [AGKRRV, Prop. 4.5.4]. Choose a point $\mathrm{pt} \to X$ and this induces a map

$$\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG) \to \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\mathrm{pt}, BG) \simeq BG$$

Consider the following fiber product

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt} \arrow[d] \\ \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG) \arrow[r] & BG \end{tikzcd}$$

where $\mathscr{X} = \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG) \times_{BG} \mathrm{pt}$. Since BG is the derived stack classifying principal G-bundles, the map $\mathscr{X} \to \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG)$ will be a G-fibration with a G-action and $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG)$ is then a quotient of $\mathscr{X}$. So we only need to see $\mathscr{X}$ is affine which suffices to prove the truncation $t_0\mathscr{X}$ is a classical affine scheme.

To see this, for any classical affine scheme S we have

$$\begin{aligned} \mathrm{Map}(S, t_0\mathscr{X}) &\simeq \mathrm{Map}(S, t_0\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\Pi_\infty(X), BG)) \times_{\mathrm{Map}(S,BG)} \mathrm{Map}(S, \mathrm{pt}) \\ &\simeq \mathrm{Map}(S, \mathbf{Map}_{\mathbf{Stk}}(\Pi_\infty(X), BG)) \times_{BG(S)} \mathrm{pt} \\ &\simeq \mathrm{Map}(S \times \Pi_\infty(X), BG) \times_{BG(S)} \mathrm{pt} \\ &\simeq \mathrm{Map}(\Pi_\infty(X), \mathrm{Map}(S, BG)) \times_{BG(S)} \mathrm{pt} \\ &\simeq \mathrm{Map}(\Pi_\infty(X), BG(S)) \times_{BG(S)} \mathrm{pt} \end{aligned}$$

[11] [TV2, Prop. 2.2.6.5]

[12] See e.g. [GJ, Theorem 7.8]

Since $\mathrm{pt} \to \Pi_\infty(X)$ is a cofibration and $BG(S)$ is a Kan complex, $\mathrm{Map}(\Pi_\infty(X), BG(S)) \to BG(S)$ is then a Kan fibration. Moreover since **sSet** is proper, the homotopy fiber product above is an actual fiber product which will be the fiber consisting of elements satisfying the group law i.e. group homomorphisms from $\pi_1(X, \mathrm{pt})$ to $G(S)$.

To see this precisely, we can replace $\Pi_\infty(X)$ by its fundamental groupoid $\Pi_{\leq 1}(X)$ since $BG(S)$ is 1-truncated, which implies $\mathrm{Map}(\Pi_\infty(X), BG(S))$ will also be 1-truncated. And from connected properties of X, $\Pi_{\leq 1}(X)$ is equivalent to $B(\pi_1(X, \mathrm{pt}))$.

Therefore $t_0\mathscr{X}$ is equivalent to $\mathbf{R}_G(\Gamma)$ where $\Gamma = \pi_1(X, \mathrm{pt})$ as we talked before in the classical case which is an affine scheme.

Question 6. Is there any description in the moduli theory sense that the de Rham stack classifies something like *flat superconnections*, just similar to the Betti case? And how can we develop a derived moduli theory for Higgs bundles on the Dolbeault side such that there is a good (derived) non-abelian Hodge theory in it? These should be the generalization of Simpson's work.

2.3 Derived Symplectic Geometry

The symplectic structure on the stack of local systems can also be generalized to this derived case which covers the classical case when X is a compact orientable surface.

2.3.1 Cotangent Complex

Affine Case The concept of *cotangent complex* is a derived generalization of Ω^1_X for a geometric object X, e.g. a scheme. We know in the classical case, for a k-affine scheme $\mathrm{Spec}\ A$, $\Omega^1_{A/k}$ corepresents the functor $\mathrm{Der}_k(A, -)$ of k-derivations on $\mathbf{Mod}_A$. In fact, for an A-module if we define $A[M] = A \oplus M$ to be its *trivial extension* i.e. $(a, m) \cdot (b, n) = (ab, an + bm)$, then we will have

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \cong \mathrm{Der}_k(A, M) \cong \mathrm{Hom}_{\mathbf{Alg}_k/A}(A, A[M])$$

It's similar for a cdga A that $\mathbb{L}_A$ corepresents the the mapping space functor of k-derivations.

Actually we have a Quillen pair

$$\Omega^1_{-/k} \otimes_- A : \mathbf{cdgA}_k^{\leq 0}/A \rightleftarrows \mathbf{dgMod}_A^{\leq 0} : A[-]$$

Passing to homotopy theories or ∞-categories, the *cotangent complex* $\mathbb{L}_A$ for a cdga A is defined as

$$\mathbb{L}_A := \mathbb{L}(\Omega^1_{-/k} \otimes_- A)(A) = \Omega^1_{QA/k} \times_{QA} A$$

where QA is a cofibrant replacement of A. It satisfies

$$\mathrm{Map}_{\mathbf{dgMod}_A^{\leq 0}}(\mathbb{L}_A, M) \simeq \mathrm{Der}_k(A, M) := \mathrm{Map}_{\mathbf{cdgA}_A^{\leq 0}}(A, A[M])$$

In this sense $\mathbb{L}_A$ corepresents $\mathrm{Der}_k(A, -) : \mathbf{dgMod}_A^{\leq 0} \to s\mathbf{Set}$ the mapping space functor of k-derivations and it can be identified with a homotopy fiber i.e.

$$\mathrm{Der}_k(A, M) \simeq \mathrm{fib}\big(\mathrm{Map}_{\mathbf{cdgA}_k^{\leq 0}}(A, A[M]) \xrightarrow{pr_*} \mathrm{Map}_{\mathbf{cdgA}_k^{\leq 0}}(A, A);\ \mathrm{id}_A\big)$$

Remark 2.3.1. Given a map $f : A \to B$ in $\mathbf{cdgA}_k^{\leq 0}$, we have a Quillen pair

$$f^* := - \otimes_A B : \mathbf{dgMod}_A^{\leq 0} \rightleftarrows \mathbf{dgMod}_B^{\leq 0} : f_*$$

and moreover if f is a quasi-isomorphism, then this will induce an equivalence on the homotopy level. Since f_* creates weak equivalence i.e. a morphism $g \in \mathbf{dgMod}_B^{\leq 0}$ is a weak equivalence if and only if $f_*(g)$

is a weak equivalence, we only need to prove for any cofibrant object M in $\mathbf{dgMod}_A^{\leq 0}$ the adjunction unit $M \to M \otimes_A B$ is a weak equivalence.[13] By the functorial cofibrant replacement construction in the small object argument, we may just suppose M is *quasi-free* i.e. of the form $\oplus_{i\in I} A[n_i]$ for some integers n_i. Then we have an isomorphism

$$\mathrm{Tor}_0^{H^*A}(H^*(M), H^*(B)) \xrightarrow{\simeq} H^*(M \otimes_A B)$$

and moreover since $H^*A \cong H^*B$, the left side above is actually $H^*(M)$. Therefore this proves when QA is a cofibrant replacement of A in $\mathbf{cdgA}_k^{\leq 0}$ and $f : QA \to A$ is a weak equivalence, we have a Quillen equivalence (f^*, f_*), from which we can identify $\Omega^1_{QA/k}$ with $\mathbb{L}_A$.

In this afiine case, we see $\mathbb{L}_A$ lies in the interval $(-\infty, 0]$. For a general scheme X, $\mathbb{L}_X$ is glued by its affine pieces, and L. Avramov has proved a scheme locally of finite type over k either has a *perfect* cotangent complex with amplitude $\subseteq [-1, 0]$ which means it's *quasi-smooth*, or has an unbounded cotangent complex.

Vanishing Properties For an affine scheme $\mathrm{Spec}\ A$, its cotangent complex $\mathbb{L}_A$ contains some information of smoothness.

Theorem 2.3.2. *Suppose A is of finite type over k. Then*

1. *A is étale iff $\mathbb{L}_A \simeq 0$;*
2. *A is smooth iff $\mathbb{L}_A \simeq \Omega^1_{A/k}$ and $\Omega^1_{A/k}$ if finite projective over A. The latter means $\mathbb{L}_A$ is perfect centered at degree 0.*

General Case

Definition 2.3.3. Let $\mathscr{X}$ be a derived prestack or a derived stack. The category of quasi-coherent sheaves (complexes) on it is defined to be $\mathbf{QCoh}(\mathscr{X}) := \lim_{S\to\mathscr{X},\ S\in\mathbf{dAff}_k} \mathbf{QCoh}(S)$ where if $S = \mathrm{Spec}\ A \in \mathbf{dAff}_k$, $\mathbf{QCoh}(S) := \mathbf{dgMod}_A$.

Let x be an A-point of $\mathscr{X}$ i.e. $x : S = \mathrm{Spec}\ A \to \mathscr{X}$. We have a functor of *$k$-derivations at x*

$$\mathrm{Der}_{\mathscr{X},x} : \mathbf{dgMod}_A^{\leq 0} \to s\mathbf{Set}, \ \ M \mapsto \mathrm{Map}_{S/\mathbf{PreStk}}(\mathrm{Spec}\ A[M], \mathscr{X})$$

Definition 2.3.4. We say *$\mathscr{X}$ has a cotangent complex at x* if $\mathrm{Der}_{\mathscr{X},x}$ is corepresented by $\mathbb{L}_{\mathscr{X},x} \in \mathbf{QCoh}(S)$ when restricted to $\mathbf{QCoh}(S)^{\leq 0}$. And $\mathbb{L}_{\mathscr{X},x}$ is called the cotangent complex of $\mathscr{X}$ at x.

$\mathscr{X}$ has a *global cotangent complex* if there exists some $\mathbb{L}_{\mathscr{X}} \in \mathbf{QCoh}(\mathscr{X})$ satisfying $x^*\mathbb{L}_{\mathscr{X}} \simeq \mathbb{L}_{\mathscr{X},x}$ and cocycle conditions.

The *tangent complex $\mathbb{T}_{\mathscr{X},x}$ of $\mathscr{X}$ at x* is the dual of $\mathbb{L}_{\mathscr{X},x}$ i.e. $\mathbb{T}_{\mathscr{X},x} := \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S)}(\mathbb{L}_{\mathscr{X},x}, \mathcal{O}_S)$ and a *global tangent complex* $\mathbb{T}_{\mathscr{X}} \in \mathbf{QCoh}(\mathscr{X})$ satisfies $x^*\mathbb{T}_{\mathscr{X}} \simeq \mathbb{T}_{\mathscr{X},x}$ and cocycle conditions. If $\mathbb{L}_{\mathscr{X}}$ is perfect, $\mathbb{T}_{\mathscr{X}}$ will be the dual of $\mathbb{L}_X$, i.e. $\mathbb{T}_{\mathscr{X}} \simeq \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(\mathscr{X})}(\mathbb{L}_{\mathscr{X}}, \mathcal{O}_{\mathscr{X}})$.

Then a generalization of L. Avramov's theorem is as follows.

Theorem 2.3.5. [14] *If $\mathscr{X}$ is an n-Artin derived stack admitting a global cotangent complex $\mathbb{L}_{\mathscr{X}}$, then $\mathbb{L}_{\mathscr{X}} \in \mathbf{QCoh}(\mathscr{X})^{\leq n}$ i.e. $\mathbb{L}_{\mathscr{X},x} \in \mathbf{QCoh}(S)^{\leq n}$ for any A-point $x : S = \mathrm{Spec}\ A \to \mathscr{X}$. Moreover if $\mathscr{X}$ is locally of finite presentation, then $\mathscr{X}$ is smooth iff $\mathbb{L}_{\mathscr{X}}$ is perfect and lies in $\mathbf{QCoh}(\mathscr{X})^{\geq 0,\ \leq n}$.*

Therefore for an Artin derived stack $\mathscr{X}$, the negative degrees of $\mathbb{L}_X$ are referred to as its **derived degrees** and its positive ones as its **stacky dergrees**.

[13] It's Proposition 2.3 in the nLab page of Quillen equivalence.

[14] [TV2, Prop. 1.4.1.10 and Cor. 2.2.5.3]

2.3.2 (Co)tangent Complexes for Bun_G and LocSys_G

Suppose G is an algebraic group over k and X is a smooth projective variety of dimension d. Then the (derived) moduli stack for principal G-bundles on X is defined as

$$\mathrm{Bun}_G := \mathrm{Bun}_G(X) = \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, BG)$$

and the (derived de Rham) moduli stack for G-local systems (flat connections) on X is defined as

$$\mathrm{LocSys}_G := \mathrm{LocSys}_G(X) = \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X_{dR}, BG)$$

We compute their cotangent complexes here, which needs the following theorem to compute the cotangent complex of a mapping stack and we will prove this theorem later.

Theorem 2.3.6. [15] *Let $\mathscr{X}$ be a derived Artin stack locally of finite presentation and X be a smooth projective variety of dimension d. Suppose $S = \mathrm{Spec}\ A$ is an affine derived scheme. An element $\tilde{f} \in \mathbb{R}\mathbf{Map}_{\mathbf{Stk}}(X, \mathscr{X})(S)$ is equivalent to a morphism $f : X_A \to \mathscr{X}$ where $X_A := X \times S = X \times \mathrm{Spec}\ A$. Then we have*

$$\tilde{f}^*\mathbb{T}_{\mathbb{R}\mathbf{Map}_{\mathbf{Stk}}(X,\mathscr{X})} \simeq \Gamma(X_A, f^*\mathbb{T}_{\mathscr{X}}) = \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(X_A)}(\mathcal{O}_{X_A}, f^*\mathbb{T}_{\mathscr{X}})$$

In particular by Serre duality we also have

$$\tilde{f}^*\mathbb{L}_{\mathbb{R}\mathbf{Map}_{\mathbf{Stk}}(X,\mathscr{X})} \simeq \Gamma(X_A, f^*\mathbb{T}_{\mathscr{X}})^\vee \simeq \Gamma(X_A, f^*\mathbb{L}_{\mathscr{X}} \otimes_{\mathcal{O}_{X_A}} \omega_{X_A})$$

where $\omega_{X_A} = p^!(k)$ is the dualizing sheaf on X_A and $p : X_A \to \mathrm{pt}$. If $A = k$, we will have $\omega_{X_A} = \omega[d]$ where ω is the canonical bundle of X.

From the theorem above to compute cotangent complexes for Bun_G and LocSys_G, we need to compute that for BG first.

Cotangent Complex for BG

Proposition 2.3.7. $\mathbf{QCoh}(BG) \simeq \mathbf{Rep}(G)$

Proof. Let $\sigma : \mathrm{pt} \to BG$ corresponds to the trivial principal bundle $G \to \mathrm{pt}$. Then we have a pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{pt} \arrow[d, "\sigma"] \\ \mathrm{pt} \arrow[r, "\sigma"'] & BG \end{tikzcd}$$

Since $\mathrm{char}\ k = 0$, the algebraic group G is smooth (see e.g. Lemma 047N in Stacks Project). Then $\sigma : \mathrm{pt} \to BG$ is a smooth atlas. And note that $\mathbf{QCoh}(-)$ can be glued by smooth morphisms, elements in $\mathbf{QCoh}(BG)$ will be equivalent to elements in $\mathbf{QCoh}(\mathrm{pt}) = \mathbf{Vect}$[16] together with isomorphisms over G satisfying some certain cocycle conditions which are actually G-equivariant sheaves and they are equivalent to G-representations.

Here we can see the induced map $\mathbf{QCoh}(BG) \to \mathbf{Vect}$ is a forgetful functor. □

For the algebraic group G, its associated Lie algebra $\mathfrak{g}$ is the tangent space at its identity element $e : \mathrm{pt} \to G$. Note that since G is smooth, $\mathbb{L}_G \simeq \Omega^1_{G/k}$ which implies $e^*\mathbb{L}_G = \mathfrak{g}^\vee$ and implies $\mathbb{L}_G \simeq \mathfrak{g}^\vee \otimes \mathcal{O}_G$.

[15] [TV2, Cor. 2.2.6.14]

[16] It's the category of complexes of k-vector spaces

Consider the pullback diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\sigma"] \\ \mathrm{pt} \arrow[r, "\sigma"'] & BG \end{tikzcd}$$

and we see $f^*\mathbb{L}_{\mathrm{pt}/BG} \simeq \mathbb{L}_{G/\mathrm{pt}} = \mathbb{L}_G = \mathfrak{g}^\vee \otimes \mathcal{O}_G$. Next from $f \circ e = \mathrm{id}$,

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "e"] \arrow[dd, bend right=60, "\mathrm{id}"'] & e^*f^*\mathbb{L}_{\mathrm{pt}/BG} \\ G \arrow[d, "f"] & f^*\mathbb{L}_{\mathrm{pt}/BG} = \mathfrak{g}^\vee \otimes \mathcal{O}_G \\ \mathrm{pt} & \mathbb{L}_{\mathrm{pt}/BG} \end{tikzcd}$$

we have $\mathbb{L}_{\mathrm{pt}/BG} = e^*f^*\mathbb{L}_{pt/BG} = f^*\mathbb{L}_{\mathrm{pt}/BG} \otimes_{\mathcal{O}_G} k = \mathfrak{g}^\vee \otimes_k \mathcal{O}_k \otimes_{\mathcal{O}_G} k = \mathfrak{g}^\vee$

The map σ induces a fiber sequence or a distinguished triangle

$$\sigma^*\mathbb{L}_{BG} \to \mathbb{L}_{\mathrm{pt}} = 0 \to \mathbb{L}_{\mathrm{pt}/BG}$$

which implies $\mathbb{L}_{\mathrm{pt}/BG} = \sigma^*\mathbb{L}_{BG}[1]$. Hence $\mathbb{L}_{BG} = \mathfrak{g}^\vee[-1]$ and $\mathbb{T}_{BG} = \mathfrak{g}[1]$.

Then by Theorem 2.3.6, for a map $\tilde{g} : \mathrm{pt} \to \mathrm{Bun}_G$ and $\tilde{f} : \mathrm{pt} \to \mathrm{LocSys}_G$, we'll have

$$\begin{aligned} \tilde{g}^*\mathbb{T}_{\mathrm{Bun}_G} &= \Gamma(X, g^*\mathbb{T}_{BG}) = \Gamma(X, g^*\mathfrak{g}[1])) \\ \tilde{f}^*\mathbb{T}_{\mathrm{LocSys}_G} &= \Gamma(X_{dR}, f^*\mathfrak{g}[1]) = \Gamma_{dR}(X, f^*\mathfrak{g}[1]) = \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(X_{dR})}(\mathcal{O}_{X_{dR}}, f^*\mathfrak{g}[1]) \end{aligned}$$

where Γ_{dR} means this computes the de Rham cohomology.

De Rham Cohomology We explain why Γ_{dR} computes the de Rham cohomology.

Let $\mathcal{F}_{dR} \in \mathbf{Cry}(X) := \mathbf{QCoh}(X_{dR})$ and it's equivalent to $\mathcal{F} \in \mathbf{DMod}(X)$. And from [GR, 6.5.1] the *de Rham cohomology* of $\mathcal{F}$ is defined to be

$$\Gamma_{dR}(X, \mathcal{F}) := \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(X_{dR})}(\mathcal{O}_{X_{dR}}, \mathcal{F}) \simeq \mathbb{R}\mathrm{Hom}_{\mathbf{DMod(X)}}(\mathcal{O}_X, \mathcal{F})$$

In [HTT, Lem. 1.5.27], there are resolutions for left D_X-module $\mathcal{O}_X$ and right D_X-module ω_X

$$\begin{aligned} &0 \to D_X \otimes_{\mathcal{O}_X} \bigwedge^d T_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \bigwedge^0 T_X \to \mathcal{O}_X \to 0 \\ &0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^d \otimes_{\mathcal{O}_X} D_X \to \omega_X \to 0 \end{aligned}$$

where $T_X = (\Omega_X^1)^\vee$ and $\omega_X = \Omega_X^{\dim X}$. Then we have

$$\begin{aligned} \mathbb{R}\mathrm{Hom}_{\mathbf{DMod}(X)}(\mathcal{O}_X, D_X) &\simeq \Big(\mathrm{Hom}_{D_X}(D_X \otimes_{\mathcal{O}_X} \bigwedge^0 T_X, D_X) \to \cdots \to \cdots \to \mathrm{Hom}_{D_X}(D_X \otimes_{\mathcal{O}_X} \bigwedge^d T_X, D_X)\Big) \\ &\simeq \Big(\mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^0 T_X, D_X) \to \cdots \to \mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^d T_X, D_X)\Big), \text{ by adjointness} \\ &\simeq \Big(\Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^d \otimes_{\mathcal{O}_X} D_X\Big), \text{ since } \Omega_X^n \text{ and } \bigwedge^n T_X \text{ are locally free} \\ &\simeq \omega_X[-d] \end{aligned}$$

This gives $\mathbb{R}\mathrm{Hom}_{\mathbf{DMod}(X)}(\mathcal{O}_X, \mathcal{F}) \simeq \mathbb{R}\mathrm{Hom}_{\mathbf{DMod}(X)}(\mathcal{O}_X, D_X) \otimes_{\mathcal{D}_X} \mathcal{F} \simeq \omega_X \otimes_{\mathcal{D}_X} \mathcal{F}[-d]$.

If $\mathcal{F}$ is a usual sheaf i.e. $\mathcal{F} \in \mathbf{DMod}(X)^{\heartsuit}$, it will the usual de Rham complex with coefficients in $\mathcal{F}$

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{F} \to \cdots \to \Omega_X^d \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

And the *algebraic de Rham cohomology* is defined to be the *hypercohomology* of this complex.

Hidden Smoothness Principle Still from Theorem 2.3.6, we see

$$\tilde{g}^*\mathbb{L}_{\mathrm{Bun}_G} = \Gamma(X, g^*\mathfrak{g}[1])^\vee \simeq \Gamma(X, g^*\mathfrak{g}^\vee \otimes \omega_X)[d-1]$$

Then it's clear if X is a smooth projective curve i.e. $\dim X = d = 1$, then Bun_X is a smooth 1-Artin stack and for a higher dimensional variety X of dimension d, $\mathbb{L}_{\mathrm{Bun}_G}$ will then lie in $[1-d, 1]$ and it will not be smooth when $d \geq 2$.

Hidden smoothness principle says for the moduli space of a reasonable moduli problem should be smooth and if it's not smooth, then there should exist a new "smooth" moduli space such that the original space is the truncation of it. In general the new moduli space is a derived stack. But here we can see even if Bun_G exists as a derived stack, it's still not smooth when $d \geq 2$. Actually the smoothness for a derived stack we considered is *strong smoothness* (see e.g [TV2, Def. 2.2.2.3]). In fact, a more reasonable smoothness in the sense of hidden smoothness principle should be *"homotopical smoothness"* which means the cotangent complex is prefect of finite Tor-amplitude and it's satisfied in the case Bun_G for higher dimensional variety X.

As for LocSys_G, we have

$$\tilde{f}^*\mathbb{L}_{\mathrm{LocSys}_G} = \Gamma_{dR}(X, f^*\mathfrak{g}[1])^\vee \simeq \Gamma_{dR}(X, f^*\mathfrak{g}^\vee[-1] \otimes \mathcal{O}_X[2d]) = \Gamma_{dR}(X, f^*\mathfrak{g}^\vee)[2d-1]$$

Even for a smooth curve X, we see LocSys_G is not smooth, but it's *quasi-smooth* which plays an important role on the definition of *singular support* in [AG].

To see in this case the dualizing complex on de Rham prestack X_{dR} i.e. in $\mathbf{DMod}(X)$ is $\mathcal{O}_X[2d]$, we need to know for a smooth projective variety $p : X \to \mathrm{Spec}\, k$ of dimension d, $p^!(k) = k_X[2d]$[17] in the derived category $D(k_X)$ which by Riemann-Hilbert correspondence is equivalent to $\mathcal{O}_X \otimes_{k_X} k_X[2d] = \mathcal{O}_X[2d]$ in $\mathbf{DMod}(X)$.

Next we start to prove Theorem 2.3.6.

Proof of Theorem 2.3.6. For a derived (pre)stack $\mathscr{X}$, its n-th *tangent stack* for $n \geq 0$ is defined to be

$$T^n\mathscr{X} := \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\mathrm{Spec}\, k[\epsilon_n], \mathscr{X})$$

where $k[\epsilon_n] = k \oplus k$ centered at degrees $-n$ and 0 with zero differential. Given a map $u : S = \mathrm{Spec}\, A \to \mathscr{X}$, $T^n\mathscr{X}$ is characterized by the following equations

$$\begin{aligned} \mathrm{Map}_{\mathbf{PreStk}/\mathscr{X}}(S, T^n\mathscr{X}) &\simeq \mathrm{Map}_{\mathbf{PreStk}/\mathscr{X}}(S, \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(\mathrm{Spec}\, k[\epsilon_n], \mathscr{X})) \\ &\simeq \mathrm{Map}_{S/\mathbf{PreStk}}(S \times \mathrm{Spec}\, k[\epsilon_n], \mathscr{X}) \\ &\simeq \mathrm{Map}_{S/\mathbf{PreStk}}(S_{\mathcal{O}_S[n]}, \mathscr{X}) \\ &\simeq \mathrm{Map}_{\mathbf{QCoh}(S)}(u^*\mathbb{L}_{\mathscr{X}}, \mathcal{O}_S[n]) \end{aligned}$$

But

$$\begin{aligned} \mathrm{Map}_{\mathbf{QCoh}(S)}(\mathcal{O}_S[-n], u^*\mathbb{T}_{\mathscr{X}}) &\simeq \mathrm{Map}_{\mathbf{QCoh}(S)}\big(\mathcal{O}_S[-n], \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S)}(u^*\mathbb{L}_{\mathscr{X}}, \mathcal{O}_S)\big) \\ &\simeq \mathrm{Map}_{\mathbf{QCoh}(S)}(u^*\mathbb{L}_{\mathscr{X}}, \mathcal{O}_S[n]) \end{aligned}$$

[17] See e.g. [Ach, Cor. 2.2.10].

So we get

$$\mathrm{Map}_{\mathbf{PreStk}/\mathscr{X}}(S, T^n\mathscr{X}) \simeq \mathrm{Map}_{\mathbf{QCoh}(S)}(u^*\mathbb{L}_{\mathscr{X}}, \mathcal{O}_S[n]) \simeq \mathrm{Map}_{\mathbf{QCoh}(S)}(\mathcal{O}_S[-n], u^*\mathbb{T}_{\mathscr{X}})$$

The n-th tangent stack construction commutes with the mapping stack construction i.e.

$$T^n\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X}) = \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}\big(\mathrm{Spec}\, k[\epsilon_n], \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X})\big) \simeq \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, T^n\mathscr{X})$$

Hence for $\tilde{f} : S \to \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X})$ which corresponds to $f : S \times X \to \mathscr{X}$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathbf{PreStk}/\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X,\mathscr{X})}\big(S, T^n\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X})\big) &\simeq \mathrm{Map}_{\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X,\mathscr{X})}\big(S, \mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, T^n\mathscr{X})\big) \\
&\simeq \mathrm{Map}_{\mathbf{PreStk}/\mathscr{X}}(S \times X, T^n\mathscr{X}) \\
&\simeq \mathrm{Map}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}[-n], f^*\mathbb{T}_{\mathscr{X}}) \\
&\simeq \mathrm{Map}_{\mathbf{QCoh}(S)}(\mathcal{O}_S[-n], \tilde{f}^*\mathbb{T}_{\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X,\mathscr{X})})
\end{aligned}$$

But

$$\begin{aligned}
&\mathrm{Map}_{\mathbf{QCoh}(S)}\big(\mathcal{O}_S[-n], \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}, f^*\mathbb{T}_{\mathscr{X}})\big) \\
\simeq\ & \tau^{\leq n}\mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S)}\big(\mathcal{O}_S, \mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}, f^*\mathbb{T}_{\mathscr{X}})\big)[n] \\
\simeq\ & \tau^{\leq n}\mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{O}_{S\times X}, f^*\mathbb{T}_{\mathscr{X}})[n] \\
=\ & \tau^{\leq n}\mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}, f^*\mathbb{T}_{\mathscr{X}})[n] \\
\simeq\ & \mathrm{Map}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}[-n], f^*\mathbb{T}_{\mathscr{X}}) \\
\simeq\ & \mathrm{Map}_{\mathbf{QCoh}(S)}(\mathcal{O}_S[-n], \tilde{f}^*\mathbb{T}_{\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X,\mathscr{X})})
\end{aligned}$$

Since $n \geq 0$ is arbitrary, we conclude $\mathbb{R}\mathrm{Hom}_{\mathbf{QCoh}(S\times X)}(\mathcal{O}_{S\times X}, f^*\mathbb{T}_{\mathscr{X}}) \simeq \tilde{f}^*\mathbb{T}_{\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X,\mathscr{X})}$.

□

2.3.3 Shifted Symplectic Structures

Affine Case For a cdga A, its differential forms $\Omega^1_{A/k}$ of degree 1 is a differential graded A-module with the derivation $d_{\mathrm{dR}} : A \to \Omega^1_{A/k}$. In this case the de Rham complex

$$\Omega^\bullet_A := A \xrightarrow{d_{\mathrm{dR}}} \Omega^1_{A/k} \xrightarrow{d_{\mathrm{dR}}} \Omega^2_{A/k} \to \cdots$$

where $\Omega^n_{A/k} = \bigwedge^n_A \Omega^1_{A/k}$ is actually the following double complex

$$\begin{array}{ccccccc}
A^0 & \xrightarrow{d_{dR}} & \Omega^{1,0}_{A/k} & \xrightarrow{d_{dR}} & \Omega^{2,0}_{A/k} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
A^{-1} & \xrightarrow{d_{dR}} & \Omega^{1,-1}_{A/k} & \xrightarrow{d_{dR}} & \Omega^{2,-1}_{A/k} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
A^{-2} & \xrightarrow{d_{dR}} & \Omega^{1,-2}_{A/k} & \xrightarrow{d_{dR}} & \Omega^{2,-2}_{A/k} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Here we also write $\Omega^0_{A/k}$ for the cdga A. Note that there is a sign trick. We know d_{dR} is a map of dg-modules and to get the *total complex* $\mathrm{Tot}^{\prod}\Omega^\bullet_A$ i.e.

$$(\mathrm{Tot}^{\prod}\Omega^\bullet_A)^n := \prod_{p+q=n} \Omega^{p,q}_{A/k}$$

the differential on $\Omega^{p,q}_{A/k}$ should actually be $d_{\mathrm{dR}} + (-1)^p d$.

For a complex or a double complex, there is a filtration defined as follows

$$F^p(\cdots \to V^n \to V^{n+1} \to \cdots) = \cdots \to 0 \to V^p \to V^{p+1} \to \cdots$$

Here for algebraic de Rham complex, such filtration is called *Hodge filtration*. Classically for a smooth algebraic variety X over k especially over $\mathbb{C}$, the sheaf $Z^p\Omega^\bullet_X$ will be quasi-isomorphic to $(F^p\Omega^\bullet_X)[p]$ in the sense of *hypercohomology*, which means the sheaf cohomology of $Z^p\Omega^\bullet_X$ will be isomorphic to the hypercohomology of algebraic Hodge filtration $(F^p\Omega^\bullet_X)[p]$.

From Remark 2.3.1 we can identify $\mathbb{L}_A$ with Ω^1_{QA}, so here we use the latter to define *shifted symplectic structures*.

Definition 2.3.8 (closed p-forms)**.** For a cdga A or an affine derived scheme $\mathrm{Spec}A$, the *complex of closed p-forms* is defined to be $\mathcal{A}^{p,cl}(A) := (\mathrm{Tot}^{\prod} F^p\Omega^\bullet_{QA})[p]$.

The shifted version for $\mathrm{Spec}\ A$ is defined as follows

- the *complex of n-shifted p-forms* is $\mathcal{A}^p(A,n) := \Omega^p_{QA}[n]$
- the *complex of n-shifted closed p-forms* is $\mathcal{A}^{p,cl}(A,n) := (\mathrm{Tot}^{\prod} F^p\Omega^\bullet_{QA})[p+n]$

An n-shifted closed p-form ω on $\mathrm{Spec}A$ is an element in $Z^0\mathcal{A}^{p,cl}(A,n)$ and more precisely $\omega = (\omega_i)_{i\geq 0}$ where $\omega_i \in \Omega^{p+i,n-i}_{QA}$ satisfies $d_{dr}\omega_i + d\omega_{i+1} = 0$ and $d\omega_0 = 0$.

Classically a *symplectic structure* ω on a smooth scheme X over k is a closed algebraic 2-form in $\Omega^2_{X/k}(X)$ such that the induced map $\theta_\omega : T_X \to \Omega^1_{X/k}$ is a sheaf isomorphism. We know for a k-algebra R, $\Omega^n_R = \bigwedge^n \Omega^1_{R/k}$ consists of all alternating functions $\prod_{i=1}^n T_R \to R$. Then for Ω^2_R any element in it will define an alternating map $T_R \times T_R \to R$ and especially it induces $T_R \to \Omega^1_{R/k} = T_R^\vee$. This explains how θ_ω works. We can generalize this definition to the derived case.

Definition 2.3.9. An *n-shifted symplectic structure* on A is a *non-degenerate* n-shifted closed 2-form $\omega \in \mathcal{A}^{2,cl}(A,2)$ on A such that the underlying 2-form ω_0 induces an equivalence

$$\theta_\omega : \mathbb{T}_A \xrightarrow{\sim} \mathbb{L}_A[n]$$

General Case In the above we have defined the complex of n-shifted (closed) p-forms i.e. $\mathcal{A}^p(A,n)$ and $\mathcal{A}^{p,cl}(A,n)$ for an affine derived scheme $\mathrm{Spec}\ A$. By Dold-Kan correspondence, they can be translated to be a simplicial set and we will get prestacks

$$\mathcal{A}^p(-,n),\ \mathcal{A}^{p,cl}(-,n) : \mathbf{dAff}_k^{op} \longrightarrow s\mathbf{Set}$$

From [PTVV, Prop. 1.11] the two prestacks are actually derived stacks with respect to étale topology.

Definition 2.3.10. For a general derived stack $\mathscr{X}$,

- the *space of n-shifted p-forms* is $\mathcal{A}^p(\mathscr{X},n) := \mathrm{Map}\big(\mathscr{X}, \mathcal{A}^p(-,n)\big)$
- the *space of n-shifted closed p-forms* is $\mathcal{A}^{p,cl}(\mathscr{X},n) := \mathrm{Map}\big(\mathscr{X}, \mathcal{A}^{p,cl}(-,n)\big)$

Then an n-*shifted symplectic structure* on $\mathscr{X}$ will be an element $\omega \in \pi_0\mathcal{A}^{p,cl}(\mathscr{X}, n)$ such that its underlying 2-form in $\mathcal{A}^p(\mathscr{X}, n)$ is *non-degenerate* i.e. induces an equivalence

$$\theta_\omega : \mathbb{T}_{\mathscr{X}} \xrightarrow{\sim} \mathbb{L}_{\mathscr{X}}[n]$$

in $\mathbf{QCoh}(\mathscr{X})$.

Graded Mixed Complexes There is another approach using the theory of *graded mixed complexes* to describe n-shifted p-forms.

Definition 2.3.11. A *graded mixed complexes over* k consists of a $\mathbb{Z}$-families of complexes of k-vector spaces $\{E(p)\}_{p\in\mathbb{Z}}$ equipped with chain maps $\epsilon_p : E(p) \to E(p+1)[1]$ for every $p \in \mathbb{Z}$

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, "\epsilon_p"] & E(p)^n \arrow[r] \arrow[dr, "\epsilon_p"] & E(p)^{n+1} \arrow[r] \arrow[dr, "\epsilon_p"] & \cdots \\
\cdots \arrow[r] & E(p+1)^n \arrow[r] & E(p+1)^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

satisfying $\epsilon \circ \epsilon = 0$ and note that there is a sign trick in $E(p+1)[1]$ for its differential.

A map $f : \{E(p), \epsilon_p\}_{p\in\mathbb{Z}} \to \{\mathcal{F}(p), \delta_p\}_{p\in Z}$ graded mixed complexes consists a $\mathbb{Z}$-families chain maps $f_p : E(p) \to F(p)$ such that the following diagram commutes

$$\begin{tikzcd}
E(p) \arrow[r, "f_p"] \arrow[d, "\epsilon_P"'] & F(p) \arrow[d, "\delta_p"] \\
E(p+1)[1] \arrow[r, "f_{p+1}"'] & F(p+1)[1]
\end{tikzcd}$$

For two graded mixed complexes $\{E(p), \epsilon_p\}_{p\in\mathbb{Z}}$ and $\{F(p), \delta_p\}_{p\in Z}$, their tensor product is

$$(E \otimes F)(p) = \bigoplus_{i+j=p} E(i) \otimes F(j)$$

This defines a *symmetric monoidal model category* ϵ-$\mathbf{dgMod}_k$ whose weak equivalences and cofibrations are defined weight-wise.

Commutative monoid like objects in ϵ-$\mathbf{dgMod}_k$ form a model category (or ∞-category) again, say ϵ-$\mathbf{cdgA}_k^{gr} := \mathrm{Comm}(\epsilon\text{-}\mathbf{dgMod}_k)$ whose weak equivalences and fibrations inherit from the forgetful functor to ϵ-$\mathbf{dgMod}_k$. Objects $\{E(p).\epsilon_p\}_{p\in\mathbb{Z}}$ in ϵ-$\mathbf{cdgA}_k^{gr}$ are called *grade mixed cdga's* and it's equipped with multiplication maps $E(p) \otimes E(q) \to E(p+q)$ compatible with the graded mixed structure ϵ. This implies $E(0)$ will be a cdga.

Since our affine derived schemes are dual to $\mathbf{cdgA}_k^{\leq 0}$, in the following we only consider ϵ-$\mathbf{cdgA}_k^{gr,\,\leq 0}$ which consists of graded mixed cdga's such that $E(0)$ is a non-positively graded cdga i.e. in $\mathbf{cdgA}_k^{\leq 0}$. From our descriptions above we get a functor

$$\epsilon\text{-}\mathbf{cdgA}_k^{gr,\,\leq 0} \longrightarrow \mathbf{cdgA}_k^{\leq 0}, \;\; \{E(p), \epsilon_p\}_{p\in\mathbb{Z}} \mapsto E(0)$$

This functor admits a left adjoint functor

$$\mathbf{DR} : \mathbf{cdgA}_k^{\leq 0} \to \epsilon\text{-}\mathbf{cdgA}_k^{gr,\,\leq 0}, \;\; A \mapsto \mathbf{DR}(A) \simeq \mathrm{Sym}_A(\mathbb{L}_A[-1]) \text{ with } \epsilon_0 = d_{dR}$$

where Sym_A refers to the underived symmetric product of A-dg-modules. Actually on the underlying usual categories we can define a functor DR sending any cdga A to $\mathrm{Sym}_A(\Omega_A^1[-1])$. Then we will have

$$\mathrm{Hom}\big(\mathrm{Sym}_A(\Omega_A^1[-1]), \{E(p)\}_{p\in\mathbb{Z}}\big) = \mathrm{Hom}(A, E(0))$$

due to the following diagram

$$\begin{tikzcd} A \arrow[r, "d_{dR}"] \arrow[d] & \Omega^1_A \arrow[d, dashed, "\exists!"] \\ E(0) \arrow[r] & E(1)[1] \end{tikzcd}$$

Then $DR \dashv (-)(0)$ forms a Quillen pair and $\mathbf{DR}$ is defined to be the left derived funcotor of DR. With the identification between $\mathbb{L}_A$ and Ω^1_{QA}, we see $\mathbf{DR}(A) \simeq \mathrm{Sym}_A(\mathbb{L}_A[-1])$ and it's called the *de Rham algebra of* A.

And for a prestack $\mathscr{X}$ its de Rham algebra will be

$$\mathbf{DR}(\mathscr{X}) := \lim_{\mathrm{Spec}\ A \to \mathscr{X}} \mathbf{DR}(A)$$

Moreover if $\mathscr{X}$ is a derived Artin stack, then $\mathbf{DR}(\mathscr{X}) \simeq \Gamma(\mathscr{X}, \mathrm{Sym}(\mathbb{L}_{\mathscr{X}}[-1]))$.

Definition 2.3.12. For a cdga A, the *space of n-shifted closed p-forms* is

$$\mathcal{A}^{p,cl}(A,n) := \mathrm{Map}_{\epsilon\text{-}\mathbf{dgMod}_k}\big(k(p)[-p-n], \mathbf{DR}(A)\big)$$

where $k(p)[-n]$ is the graded mixed complexes sitting in the weight degree p and the cohomological degree $p+n$. And for a derived Artin stack $\mathscr{X}$, its space of n-shifted closed p-forms will then be

$$\mathcal{A}^{p,cl}(\mathscr{X},n) = \lim_{\mathrm{Spec}\ A \to \mathscr{X}} \mathcal{A}^{p,cl}(A,n) \simeq \mathrm{Map}_{\epsilon\text{-}\mathbf{dgMod}_k}\big(k(p)[-p-n], \mathbf{DR}(\mathscr{X})\big)$$

On Bun_G and LocSys_G One of the main theorems in [PTVV] (Thm. 2.5) says

Theorem 2.3.13. *Let $\mathscr{X}$ be a derived Artin stack with an n-shifted symplectic structure ω. If X is an $\mathcal{O}$-compact derived stack with a d-orientation $\Gamma(X, \mathcal{O}_X) \to k[-d]$ such that $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X})$ is a derived Artin stack locally of finite presentation, then $\mathbb{R}\mathbf{Map}_{\mathbf{dStk}}(X, \mathscr{X})$ will admit a canonical $(n-d)$-shifted symplectic structure.*

So we try to find shifted symplectic structures on BG first. We have already known $\mathbb{L}_{BG} = \mathfrak{g}^\vee[-1]$ and $\mathbb{T}_{BG} = \mathfrak{g}[1]$. Therefore the only possible n for $\mathbb{T}_{BG} \simeq \mathbb{L}_{BG}[n]$ is $n = 2$.

If G is a reductive group, we will have

$$\mathbf{DR}(BG) = \mathrm{Sym}_G \mathbb{L}_{BG}[-1] = \mathrm{Sym}_G \mathfrak{g}^\vee[-2] = \{\mathrm{Sym}^p_k(\mathfrak{g}^\vee)^G[-2p],\ 0\}_{p \geq 0}$$

Actually the double complex for $d_{dR} : \mathcal{O}_{BG} \to \mathbb{L}_{BG}$ looks like

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathrm{Sym}^3_k(\mathfrak{g}^\vee)^G \\ 0 \arrow[r] \arrow[u] & \mathrm{Sym}^2_k(\mathfrak{g}^\vee)^G \arrow[r] \arrow[u] & 0 \arrow[u] \\ \mathrm{Sym}^1_k(\mathfrak{g}^\vee)^G \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\ \mathbb{L}_{BG} & \wedge^2 \mathbb{L}_{BG} & \wedge^3 \mathbb{L}_{BG} \end{tikzcd}$$

And this implies $\mathcal{A}^p(BG,n)=\mathrm{Sym}_k^p(\mathfrak{g}^\vee)^G[n-p]$ and

$$\mathcal{A}^{p,cl}(BG,n)=\Big(0 \longrightarrow \underset{\text{degree}=\,p-n}{\mathrm{Sym}_k^p(\mathfrak{g}^\vee)^G} \longrightarrow 0 \longrightarrow \underset{\text{degree}=\,p-n+2}{\mathrm{Sym}_k^{p+1}(\mathfrak{g}^\vee)^G} \longrightarrow \cdots\Big)$$

This means $\mathcal{A}^{p,cl}(BG,n)=\bigoplus_{i\geq 0}\mathrm{Sym}_k^{p+i}(\mathfrak{g}^\vee)^G[n-p-2i]$. And it will be clear $\pi_0\mathcal{A}^{2,cl}(BG,2)\simeq\mathrm{Sym}_k^2(\mathfrak{g}^\vee)^G$. So a symplectic structure on BG is equivalent to a non-degenerate G-invariant quadratic form on $\mathfrak{g}$ which can induce an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^\vee$. Since our G here is reductive, when we embed G into some GL_n as a closed subgroup scheme and get a faithful G-representation on k^n, the invariant symmetric bilinear form obtained from it will be non-degenerate.

Next to use Theorem 2.3.13, we need to find an $\mathcal{O}$-orientation for X and X_{dR} for a smooth projective variety X of dimension d. For Bun_G, if X is Calabi-Yau which means $\omega_X\simeq\mathcal{O}_X$, then use the trace map tr we get a d-orientation

$$H^d(X,\mathcal{O}) \xrightarrow{\sim} H^d(X,\omega) \xrightarrow{\mathrm{tr}} k$$

So in this case if X is Calabi-Yau Bun_G will admit a $(2-d)$-shifted symplectic structure.

As for X_{dR}, we have already known $\Gamma(X_{dR},\mathcal{O})$ computes the de Rham cohomology of X, and then a fundamental class in $H^{2d}_{dR}(X,\mathcal{O})$ will give a $2d$-orientation which implies LocSys_G admits a $(2-2d)$-shifted symplectic structure. Actually since we have computed

$$\tilde{f}^*\mathbb{L}_{\mathrm{LocSys}_G}\simeq\Gamma_{dR}(X,f^*\mathfrak{g}^\vee)[2d-1], \text{ and } \tilde{f}^*\mathbb{T}_{\mathrm{LocSys}_G}\simeq\Gamma_{dR}(X,f^*\mathfrak{g})[1]$$

the only possible shifted degree n for $\mathbb{T}_{\mathrm{LocSys}}\simeq\mathbb{L}_{\mathrm{LocSys}}[n]$ is $n=2-2d$. And we can find it similar to the case of BG.

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