This notes consists of two parts. In the first part we discuss some equivalent characterizations for local systems and their genelizations to higher cases. In the second part we consider the moduli theory for local systems.

Some Characterizations for Local Systems

Main References:
[1]: Spanier, *Algebraic Topology*
[2]: Tamas Szamuely, *Galois Groups and Fundamental Groups*

Generally speaking for a topological space X a **local system** F on X is a *locally constant sheaf* on the site $\mathcal{U}(X)$ which means there is a covering $\{U_i \to X | i \in I\}$ such that on each U_i, F is a constant sheaf. Suppose A is any set, abelian group or commutative ring. Then there is a notion of constant presheaf which sends any open subset of X to A and constant sheaves are the shifification of constant presheaves. If U is an open subset of X, we see sections of the constant sheaf A_X of A on U consist of sections $U \to \coprod_{x \in X} A \approx A \times X$ where A is equipped with discrete topology. Since they should be continuous, we have

$$A_X(U) = \{\text{locally constant functions } s : U \to A\}$$

There is another characterization for local systems. We know a sheaf can be viewed as a bundle over X which is just a map from some top space to the base space X. For a sheaf F on X its associated bundle is $\coprod_{x \in X} F_x \to X$. And for any bundle $E \to X$ we can have a presheaf sending every open subset U of X to the set of sections $U \to E$. From these we obtain the following theorem.

Theorem: There is an equivalence between sheaves and *étale maps* over X i.e. $\mathrm{Shv}(X) \cong \mathrm{\acute{E}tale}/X$. Moreover stalks of sheaves are isomorphic to fibers of étale maps.

Definition (étale map): A continuous map $p: E \to X$ is an **étale map** if it's a *local homeomorphism* which means for any point $e \in E$ there is an open subset $e \in V \subseteq E$ such that $p|V : V \to p(V)$ is a homeomorphism with $p(V)$ open in X.

We have already known local systems are special sheaves so that they correspond with some certain type of étale maps. Look at constant sheaves. Their bundles are like $A \times X \approx \coprod_{a \in A} X \to X$. So the bundle of a local system will look like this locally.

Definition (covering space): A continuous map $p : \tilde{X} \to X$ is a **covering projection** if for any point $x \in X$ there exists an open subset $x \in U \subseteq X$ such that $p^{-1}(U)$ is the disjoint union of open subsets in $\tilde{X}$ each of which is mapped homeomorphically onto U by p.

Theorem: There is an equivalence between local systems on X and covering spaces on X i.e. $\mathrm{LocSys(X)} \cong \mathrm{Cov}/X$.

To see this we first suppoe $p : \tilde{X} \to X$ is a covering projection. Then for $x \in U \subseteq X$, we have $p^{-1}(U) = \coprod_{i \in I} V_i \approx I \times V$ where $V_i = V \approx U$. Then sections $U \to I \times V$ will just be locally constant functions $U \to I$. So that restriction to U the sheaf associated with p is a constant sheaf of I.

Conversely let F be a local system on X. Then on $U \subseteq X$, F is a constant sheaf with respect to I. For $p : \coprod_{x \in X} F_x \to X$, $p^{-1}(U) = \coprod_{x \in U} F_x \approx I \times U \approx \coprod_{i \in I} U$.

Results from Algebraic Topology

The identification above makes it possible for us to use techniques from algebraic topology to study local systems.

In algebraic topology a *Hurewicz fibration* or simply a *fibration* is a continuous map $p : E \to X$ such that it has the right lifting property with respect to all $i_0 : Y \to Y \times I$ where $I = [0, 1]$ is the unit interval. We say it has the *unique path lifting property* if with respect to $* \to I$ the lifting map is unique. In [1, Thm. 2 and 3 in P67] we see that a covering projection is a fibration with the unique path lifting property. Now using this property we can compare fibers of a covering space.

Let $p : \tilde{X} \to X$ be a covering projection and ω a path from x_0 to x_1. From ω we have a function

$$F_\omega : p^{-1}(x_0) \times I \to X,\ (\tilde{x}, t) \mapsto \omega(t)$$

Then with respect to $i_0 : p^{-1}(x_0) \to p^{-1}(x_0) \times I$, there is a lifting $G_\omega : p^{-1}(x_0) \times I \to \tilde{X}$ such that $p \circ G_\omega(\tilde{x}, t) = \omega(t)$. Therefore $G_\omega(\tilde{x}, 1)$ is an element in $p^{-1}(x_1)$ and this defines a function $G_\omega(-, 1) : p^{-1}(x_0) \to p^{-1}(x_1)$.

For any second path ω' from x_0 to x_1 such that $\omega \simeq \omega' \operatorname{rel} \partial I$, we will obtain the same point in $p^{-1}(x_1)$ since their liftings will be homotopic relative to ∂I in $\tilde{X}$[1]. And one can prove we have defined a functor $\Pi_1(X) \to \mathbf{Set}$ sending a point $x \in X$ to $p^{-1}(x)$ where $\Pi_1(X)$ is the fundamental groupoid of X. So that the obtained map $G_\omega(-, 1)$ is an isomorphism.

The argument above shows for a local system on X if there is a path from x_0 to x_1 then their stalks will be isomorphic to each other. This is similar to the notion of **parallel transport** in differential geometry which we will discuss later.

Now naturally there is a question whether every functor $\Pi_1(X) \to \mathbf{Set}$ can be obtained in such a way? The answer is positive if X is good enough[2].

Theorem: If X is good enough, then there is an equivalence between local systems on X and functors $\Pi_1(X) \to \mathbf{Set}$, i.e. $\mathrm{LocSys(X)} \cong \mathrm{Fun}(\Pi_1(X), \mathbf{Set})$.

Given a functor $F : \Pi_1(X) \to \mathbf{Set}$ we can define a presheaf $\tilde{F}$ on X such that for every open subset $U \subseteq X$, $\tilde{F}(U)$ consists of all functions $f : X \to \coprod_{x \in X} F(x)$ satisfying $f(x) \in F(x)$ and for any path ω in U we have $f(\omega(1)) = F(\omega)(f(\omega(0)))$. Clearly $\tilde{F}$ is a sheaf from the paracompactness. Moreover by considering how $\tilde{F}$ behaves on connected open subsets we can see it's actually a local system.

Question 1: What are ∞-local systems? For a topological space X, $\mathrm{Sing}_\bullet(X)$ is a Kan complex i.e. an ∞-groupoid which acts like $\Pi_\infty(X)$. A generalization for $\mathrm{Fun}(\Pi_1(X), \mathbf{Set})$ is ∞-$\mathrm{Fun}(\mathrm{Sing}_\bullet(X), \mathbf{sSet})$. But on the other hand a local system is a locally constant sheaf so that its natural genelization should be a locally constant ∞-stack $\mathcal{U}(X)^{op} \to \mathbf{sSet}$. Are they still equivalent at the infinite level? Maybe the nlab page of locally constant ∞-stack is helpful to answer this question.

Representations of Fundamental Groups

Here we give another characterization for local systems.

Now if we fix a point x_0 in X, then there is a functor of fibers

$$\mathrm{Fib}_{x_0} : \mathrm{Cov}/X \to \mathbf{Set},\ p \mapsto p^{-1}(x_0)$$

As we have seen a covering space $p : \tilde{X} \to X$ induces a functor $\Pi_1(X) \to \mathbf{Set}$. Every loop ω in $\pi_1(X, x_0)$ will correspond with an isomorphism $p^{-1}(x_0) \to p^{-1}(x_0)$. The image of $\tilde{x}$ via this isomorphism is denoted by $\omega \cdot \tilde{x}$. Then we can see $p^{-1}(x_0)$ is actually a $\pi_1(X, x_0)$-set and the action is called the *monodromy* action on the fiber.

Definition: A *representation* for a group G is a group homomorphism $G \to \mathrm{Aut}(M)$ where M is some set which is equivalent to a G-set structure on M. So that $\mathbf{Rep}(G) = G\text{-}\mathbf{Set}$.
If A is a commutative ring, an A-representation for G is a group homomorphism $G \to \mathrm{Aut}_A(M)$ for some A-module M and we have $\mathbf{Rep}_A(G) = G\text{-}\mathbf{Mod}_A$.

Theorem[3]: If X is good enough[4], the functor $\mathrm{Fib}_{x_0} : \mathrm{Cov}/X \to \mathbf{Rep}(\pi_1(X, x_0))$ is an equivalence between covering spaces on X and representations of $\pi_1(X, x_0)$. Moreover we can see $\mathrm{Aut}(\mathrm{Fib}_{x_0}) \cong \pi_1(X, x_0)$.

To see this given any $\pi_1(X, x_0)$-set S we decompose it as a disjoint union of $\pi_1(X, x_0)$-orbits. Then for every orbit we have a quotient space of the universal covering space $\tilde{X}_u$ by the stabilizer of some point in the orbit. Then the disjoint union of these quotients is just what we want. For fully faithfulness and details can be found in [2, Thm. 2.3.4].

We summarize results we have obtained until now as follows.

Theorem: If X is good enough, the following categories are equivalent:

1. local systems on X i.e. $\mathrm{LocSys}(X)$;
2. covering spaces on X i.e. Cov/X;
3. functors from the fundamental groupoid of X to sets i.e. $\mathrm{Fun}(\Pi_1(X), \mathbf{Set})$;
4. representations of the fundamental group i.e. $\mathbf{Rep}(\pi_1(X, x_0))$.

Note that we can replace $\mathbf{Set}$ by $\mathbf{Mod}_A$ for any commutative ring A in which case an A-local system should be a locally constant A_X-sheaf where A_X is the constant sheaf of A.

More generally if we consider a discrete group G not a commutative ring A and replace $\mathbf{Set}$ by the category of G-sets where G acts transitively, then we obtain three equivalent characterizations as follows:

1. locally constant sheaves on X such that G_X acts on them transitively;
2. principal G-bundles over X such that parallel transports induced by homotopy classes of paths in X are G-equivariant;
3. functors from the fundamental groupoid of X to G-sets where G acts transitively.

We call these geometric objects G**-local systems**.

If we set $G = GL_n(k)$ for some field k, since for a sheaf F, $\mathrm{Aut}(F)$-torsors are equivalent to $\mathrm{Twist}(F)$ twists of F which are locally isomorphic to F[5], a $GL_n(k)$-local system is just a locally constant sheaf of k-vector spaces locally of rank n.

Question 2: Are there local systems of some type equivalent to representations of étale fundamental groups i.e. Galois representations? I guess this can help understand how we can get geometric Langlands conjecture from arithmetic Langlands conjecture on the Galois side.

Riemann-Hilbert Correspondence

Classical Case

Main References:
[3]: Brian Conrad, *Classical Motivation for the Riemann-Hilbert Correspondence*

As we have seen a local system is roughly a collection of objects with the notion of parallel transports within them. In differential geometry parallel transports are induced by *connections*[6] so that some certain kind of connections may be equivalent to local systems. Now we can suppose X is a connected complex manifold or a smooth projective algebraic variety over $\mathbb{C}$ and in the latter case we need to consider the complex analytic space X^{an} of X.

A **holomorphic vector bundle** over X is a complex manifold E with a holomorphic map $\pi : E \to X$ such that its fiber $\pi^{-1}(x)$ is isomorphic to a finite dimensional $\mathbb{C}$-vector space for every $x \in X$ and satisfies some certain compatible conditions. And this definition will be equivalent to **locally free $\mathcal{O}_X$-sheaves** where $\mathcal{O}_X$ is the sheaf of homolomorphic functions. And we will not distinguish the two definitions.

Definition (conncetion): A **(holomorphic) connection** on X is a pair (E, ∇) where E is a holomorphic vector bundle and $\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ is a map of sheaves such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for $f \in \mathcal{O}_X$ and $s \in E$. This map extends naturally to

$$\nabla^n : \Omega^n_X \otimes_{\mathcal{O}_X} E \to \Omega^{n+1}_X \otimes_{\mathcal{O}_X} E, \ \ \omega \otimes s \mapsto d\omega \otimes s + (-1)^n \omega \wedge \nabla(s)$$

A connection is called **flat** or **integrable** if $\nabla^{n+1} \circ \nabla^n = 0$ which is equivalent to $\nabla^1 \circ \nabla^0 = 0$.

Theorem (Riemann-Hilbert Correspondence)[7]: The category of finite dimensional complex local systems on X is equivalent to the category of holomorphic vector bundles on X with a flat connection i.e. $\mathrm{LocSys}_{\mathbb{C}}(X) \cong \mathrm{Flat}(X)$.

For a flat connection (E, ∇) its kernel $\ker \nabla$ will be a complex local system with an isomorphism $\mathcal{O}_X \otimes_{\mathbb{C}_X} \ker \nabla \xrightarrow{\sim} E$ where $\mathbb{C}_X$ is the constant sheaf of $\mathbb{C}$ on X. Conversely for a complex local system F we can consider $\mathcal{O}_X \otimes_{\mathbb{C}_X} F$ with the connection $d_X \otimes \mathrm{id}_F$.

This result depends heavily on the complex topology of X. Especially this correspondence tells us that there is an equivalence between flat connections and finite dimensional complex representations of $\pi_1(X, x_0)$.

Derived Case

Main References:
[4]: Ryoshi Hotta, Kiyoshi Takeuchi Toshiyuki Tanisaki, *D-modules, Perverse Sheaves, and Representation Theory*
[5]: Jacob Lurie, Youtube Videos on "A Riemann-Hilbert Correspondence in p-adic Geometry" Part 1 in MPIM or bilibili
The following two Harvard minor thesises may be readable.
[6]: Maxim Jeffs, *Perverse Sheaves and D-modules*
[7]: Charles Wang, *D-modules and the Riemann-Hilbert Correspondence*

In [5] Lurie says we can drop the assumption of "projective" and just suppose X is a smooth algebraic variety over $\mathbb{C}$. Then we can embed X into a smooth projective algebraic variety $\overline{X}$. From this a holomorphic vector bundle on X^{an} can be equipped with an algebraic structure by restriction due to Serre's GAGA theorem for $\overline{X}^{an}$ and $\overline{X}$ i.e. it corresponds with an algebraic vector bundle on X which is moreover compatible with connections.[8]

So that the classical Riemann-Hilbert correspondence gives an equivalence between complex local systems on X^{an} and algebraic vector bundles on X with flat (regular) connections. In this case complex local systems are locally free $\mathbb{C}_X$-modules and algebraic vector bundles are locally free $\mathcal{O}_X$-modules. If we want to drop the condition of "locally free", we will need to go much further than the classical case.

Let E be a quasi-coherent $\mathcal{O}_X$-module on X. Then a flat connection $\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ is equivalent to a map $(\Omega^1_X)^\vee \to \mathrm{End}_{\mathbb{C}}(E)$ satifying certain conditions. Note that $(\Omega^1_X)^\vee = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$. $\mathcal{D}_X$ is defined to be the subalgebra (not necessarily commutative) of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $(\Omega^1_X)^\vee$. Then a flat connection will be equivalent to a left algebraic $\mathcal{D}_X$-module structure on E[9].

On the other side we generalize local systems on X^{an} to *constructible sheaves* which can be roughly understood as "locally" locally constans sheaves.

Definition (constructible sheaf): For a complex analytic space X^{an} a **stratification** is a locally finite partition $X^{an} = \coprod_{\alpha \in A} X_\alpha^{an}$ by locally closed complex analytic subset X_α^{an} such that every closure $\overline{X_\alpha^{an}}$ is a disjoint union of some X_β^{an}'s. A $\mathbb{C}_{X^{an}}$-module is a **constructible sheaf** if when restriction to some stratification on X^{an}, it's a locally constant sheaf on every X_α^{an}.

Note that the existence of stratification is due to Whitney.

Passing to derived categories there are *de Rham* and *solution* funcrtors i.e.

$$DR_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}}), \;\; M_\bullet \mapsto \Omega_{X^{an}} \otimes^{\mathbb{L}}_{D_{X^{an}}} M_\bullet^{an}$$

and

$$\mathrm{Sol}_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}})^{op}, \;\; M_\bullet \mapsto \mathbb{R}\mathrm{Hom}_{D_{X^{an}}}(M_\bullet^{an}, \mathcal{O}_{X^{an}})$$

Riemann-Hilbert correspondence says they are equivalences from $D^b_{rh}(D_X)$ the subcategory of $D^b(D_X)$ consisting of bounded complexes of D-modules whose cohomology groups are *regular holonomic* D-modules to $D^b_c(X)$ the subcategory of $D^b(\mathbb{C}_{X^{an}})$ consisting of bounded complexes of $\mathbb{C}_{X^{an}}$-modules whose cohomolopgy groups are constructible sheaves. Morever the image of $D_X\text{-}\mathrm{Mod}_{rh}$ by DR_X is an abelian category in $D^b_c(X)$ consisting of **perverse sheaves**[10].

Differential Graded Case

Main References:
[8]: Jonathan Block, Aaron M. Smith, *A Riemann–Hilbert correspondence for infinity local systems*, arXiv:0908.2843
[9]: Joseph Chuang, Julian Holstein, Andrey Lazarev, *Maurer-Cartan moduli and theorems of Riemann-Hilbert type*, arXiv:1802.02549

As we have seen classical Riemann-Hilbert correspondence establishes an equivalence between flat connections and complex representations of $\pi_1(X, x_0)$ for a connected complex manifold X. But it seems it's only a truncated version since $\pi_1(X, x_0)$ or $\Pi_1(X)$ does not contain all homotopy information of X. So the higher version of Riemann-Hilbert correspondences tries to get a similar relationship for higher flat connections and ∞-lcoal systems. In [8] the authors work on a real manifold but this makes no big difference when considering a complex manifold (see [9, Sec. 8.2]) and we only need to replace the de Rham algebra by the Dolbeault algebra. For simplicity we also take X as a (connected) real manifold.

In the classical theory a local system is a functor $\Pi_1(X) \to \mathbf{Set}$. Here we replace the fundamental groupoid $\Pi_1(X)$ by the smooth ∞-groupoid $\Pi_\infty(X) := \mathrm{Sing}^\infty_\bullet(X)$ whose sections consist of smooth simplices. Since we consider real local systems, we need to replace $\mathbf{Set}$ by some ∞-category of real vector spaces. In [8] this desired ∞-category $\mathcal{C}_\infty$ is obtained from $\mathcal{C} := \mathbf{Ch}^\bullet(\mathbb{R})$ by taking H^0 on internal hom-complexes, Dold-Kan equivalence and simplicial nerve (see [8, Rem. 2.5]). Then **real ∞-local systems** are defined to be ∞-functors i.e. maps of simplicial sets from $\Pi_\infty(X)$ to $\mathcal{C}_\infty$. This category is denoted by $\mathrm{LocSys}^\infty_{\mathcal{C}}(X)$ which is actually a dg-category ([8, Prop. 2.8]).

On the other side we need to extend the usual concept of connections to *superconnections*. Let $\Omega = (\Omega^\bullet(X), d)$ be the de Rham differential graded algebra on X. In the classical case a connection is a locally free sheaf E with a derivation map on $E \otimes \Omega^\bullet_X$ of degree 1. And in algebraic geometry we know a locally free sheaf of finite rank is equivalent to a finite projective module. So that here we suppose $E^\bullet$ is a bounded complex of finitely generated projective $\Omega^0(X)$-modules and view it as a *dg-vector bundle*. We say a pair $(E^\bullet, \mathbb{E})$ is a **cohesive module** or a *dg-vector budle with a superconnection* if $\mathbb{E}$ is a *$\mathbb{Z}$-graded connection* on $E^\bullet \otimes_{\Omega^0(X)} \Omega^\bullet(X)$ of degree 1 satisfying the usual Leibniz condition i.e.

$$\mathbb{E}(e\omega) = (\mathbb{E}(e \otimes 1))\omega + (-1)^{\deg(e)} e d\omega$$

This superconnection is flat if $d\mathbb{E} + \mathbb{E}^2 = 0$. The category of cohesive modules is denoted by $\mathcal{P}_\Omega$.

In [8] the authors define a really complicated functor $\mathcal{RH} : \mathcal{P}_\Omega \to \mathrm{LocSys}^\infty_{\mathcal{C}}(X)$[11] and prove it's a *quasi-equivalence* of dg-categories so that they are equivalent in $\mathrm{Ho}(\mathbf{dgCat}_\mathbb{R})$.

The approach to the problem taken by [9] is a bit different where ∞-local systems are regarded as some kind of *locally constant dg-sheaves* and cohesive modules are replaced by *perfect twisted modules* which are actually shown equivalent to the former [9, Cor. 3.1].

Crystals

Main References:
[10]: Dennis Gaitsgory, Nick Rozenblyum, *Crystals and D-modules*, arXiv:1111.2087
[11]: Jacob Lurie, *Notes on Crystals and Algebraic D-modules*

We should remember a local system is roughly a collection of objects within which there exist parallel transports for paths. But in a smooth algebraic variety X it's equipped with Zartiski topology which is much different from usual topologies like complex topology so that the usual concept of paths is not valid here.

For a complex manifold it's semilocally 1-connected or locally simply connected which means every point in it admits a simply connected open neighborhood. Since parallel transports are given by homotopy classes of paths, fibers over this simply connected open neighborhood will be canonically isomorphic to each other. But this is not the case in the Zariski topology. In [11] Lurie says there is no simply connected open subsets in a smooth curve of genus > 0 (**need a proof!**). So that we need a new way to say two points are close enough i.e. "in a simply connected open subset", and the parallel transport is canonical. In Grothendieck's language the former is to say they are *infinitesimally close* and the latter is equivalent to the *cocycle condition*.

We suppose X is a smooth algebraic variety over a field k of characteristic zero e.g. $\mathbb{C}$. For a commutative k-algebra R, its *nilradical ideal* is $I_R = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$ consisting of nilpotent elements. We say two R-valued points $x, y \in X(R)$ are **infinitesimally close** if they have the same image under the map $X(R) \to X(R/I_R)$. Note that since $\operatorname{Spec} R$ and $\operatorname{Spec} R/I_R$ are homeomorphic in terms of underlying topological spaces, two infinitesimally close points x, y induce the same map $\operatorname{Spec} R \to X$ on unerlying topological spaces.

Definition (crystal): A **crystal of quasi-coherent sheaf** on X is a pair $(\mathscr{F}, \eta_{x,y})$ where $\mathscr{F}$ is a quasi-coherent sheaf on X and $\eta_{x,y} : x^*\mathscr{F} \to y^*\mathscr{F}$ is an isomorphism of quasi-coherent sheaves on $\operatorname{Spec} R$ for two infinitesimally close points $x, y \in X(R)$ such that

1. for a map $R \to R'$, x, y induce two new infinitesimally close points $x', y' \in X(R')$ and they satisfy

$$\eta_{x',y'} : x'^*\mathscr{F} \simeq x^*\mathscr{F} \otimes_R R' \to y^*\mathscr{F} \otimes_R R' \simeq y'^*\mathscr{F}$$

   is obtained from $\eta_{x,y}$ by tensoring R';
2. $\eta_{x,y}$ satisfies the *cocycle condition* i.e. $\eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y}$ and $\eta_{x,x} = \mathrm{id}$.

Note that $x^*\mathscr{F}$ is obtained by the pullback of $\mathscr{F}$ along x. We also write $\mathscr{F}(x)$ for $x^*\mathscr{F}$.

Theorem[12]: The category of crystals of quasi-coherent sheaves on X is equivalent to the category of quasi-coherent left D_X-modules.

We know flat connections are equivalent to D_X-module structures. Hence crystals of quasi-coherent sheaves are then also equivalent to flat connections as an algebraic replacement. From this point crystals are a good notion for algebraic parallel transports.

We sketch the ideal here and details can be found in [11, Thm. 0.4]. Suppose $\mathscr{F}$ is a quasi-coherent sheaf on X. For two points $x, y \in X(R)$ they induce a map $\mathrm{Spec}R \to X \times X$. They are infinitesimally close iff the induced map $\mathrm{Spec}R/I_R \to X \times X$ forctors through the diagonal map $\Delta : X \to X \times X$. By considering the formal completion $(X \times X)^\vee$ of the locally closed immersion Δ i.e. the *ind-scheme* or "colimit" of $X \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq X \times X$, Lurie shows an isomorphism $\eta_{x,y}$ is equivalent to a map $\mathscr{F} \to pr_{1,*}^{(n)} pr_2^{(n),*} \mathscr{F}$ and then equivalent to $D_X^{\leq n} \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{F}$ where $pr_i^{(n)}$'s are projections from $(X \times X)^\vee$ to X when restriction to $X^{(n)}$. And the cocycle condition will be equivalent to the associative axiom of the D_X-action.

We can define the concept of quasi-coherent sheaves on an arbitrary functor $X : \mathbf{Alg}_k \to \mathbf{Set}$ which is a pair $(\mathscr{F}(x), \alpha_{x,x'})$ where for any point $x \in X(R)$, $\mathscr{F}(x)$ is an R-module and if $R \to R'$ is a map of k-algebras and $x' \in X(R')$ is the image of x, then $\alpha_{x,x'} : \mathscr{F}(x') \simeq \mathscr{F}(x) \otimes_R R'$ is an isomorphism of R'-modules. Note that this pair should satisfy the compatible condition for $R \to R' \to R''$. If X is actaully a scheme, then it will be covered by open affine schemes and the quasi-coherent sheaf on it will be glued by these modules on open affine schemes. Hence this definition generalizes the classical definition for quasi-coherent sheaves.

With this definition we can see a crystal on X is actually equivalent to a quasi-coherent sheaf on the **de Rham stack** X_{dR} which sends any algebra R to the set $X(R/I_R)$. Note that $\eta_{x,y}$'s indentify an R/I_R-module canonically and we can lift such a module to be an R-module. Moreover for a smooth scheme X, $X(R) \to X(R/I_R)$ is surjective if R is finitely generated according to the lifting property of formal smoothness. Therefore the natural map $X \to X_{dR}$ is an epimorphism of sheaves and in this sense we can view X_{dR} as a quotient of X by the relation of “infinitesimal closeness”.

This feature can help us generalize the concept of crystals to higher case. For a prestack $X : \mathbf{cdgA}_k^{\leq 0} \to \mathbf{sSet}$, its **de Rham prestack** X_{dR} can be defined such that

$$X_{dR}(A^\bullet) = X(H^0(A^\bullet)/I_{H^0(A^\bullet)})$$

In [10] Gaitsgory and Rozenblyum say "the key idea is that one should regard higher homotopy groups of a derived ring as a generalization of nilpotent elements." So that we do not let $X_{dR}(A^\bullet)$ be $X(A^\bullet/I_{A^\bullet})$. Then a *(left) crystal* on X will be defined as a quasi-coherent sheaf on X_{dR}. And they also generalize the two phenomena we talked above to this derived case. They show if X is good enough i.e. *eventually coconnective*, then this definition for crystals will be equivalent to Grothendieck's version by infinitesimal closeness (see [10, Prop. 3.4.3]). And they describe the relation between crystals and D-modules as well in this derived case (see [10, Sec. 5]).

Question 3: In the above we have seen the differential graded Riemann-Hilbert correspondence on a smooth or complex manifold. Does it still hold on a *higher analytic stack*? Moreover by the higher

GAGA theorem described in Mauro Porta and Tony Yue Yu's paper *Higher Analytic Stacks and GAGA Theorems*, arXiv:1412.5166 whether we can obtain a result similar to Deligne's theorem in the classical case. In other words, what's the relation between crystals on a smooth higher algebriac stack X and flat superconnections or ∞-local systems on its analytification X^{an}? A related reference may be Bertrand Toën, Gabriele Vezzosi, Algebraic Foliations and Derived Geometry: the Riemann-Hilbert Correspondence, arXiv:2001.05450.

Moduli Theory for Local Systems

Via GIT

References:
[12]: Carlos T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety I
[13]: Carlos T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety II

In [12] Simpson uses the technique of Mumford's GIT theory to construct *Betti* moduli space M_{B}, *Dolbeault* moduli space M_{Dol} and *de Rham* moduli space M_{dR} parametrizing representations of the fundamental group, Higgs bundles and vector bundles with a flat connection respectively for a complex manifold or a smooth projective algebraic variety X on $\mathbb{C}$. Note that these moduli spaces are coarse in the sense of Mumford. He also shows some topological equivalences among these moduli spaces. The classical Riemann-Hilbert correspondence can therefore give an equivalence $M_{\mathrm{B}}^{an}(X) \simeq M_{\mathrm{dR}}^{an}(X)$ of complex analytic spaces.

Remark: These moduli spaces are based on the (coarse) moduli space of coherent sheaves. But to obtain the latter we need to only consider about (semi-)stable sheaves since they form a *bounded* family but all sheaves without the (semi-)stable condition are not bounded. For example we can consider the set $\{\mathcal{O}(n) \oplus \mathcal{O}(-n) | n \geq 0\}$ on $\mathbb{P}^1$ and there is no family over a scheme of finite type gluing all of them. So here M_{Dol} and M_{dR} actually parameterize (semi-)stable bundles.

Via Stacks

References:
[14]: Sebastian Casalaina-Martin, Jonathan Wise, An introduction to moduli stacks, with a view towards Higgs bundles on algebraic curves, arXiv:1708.08124

Although [14] deals with Higgs bundles, similar arguments hold as well for vector bundles with a flat connection. Suppose X is a smooth projective curve over $\mathbb{C}$. Then the stack parametrizes pairs

(E, ∇) where E is a vector bundle of rank r and degree d. ∇ can be a flat connection or a *Higgs field* i.e. $\nabla : E \to E \otimes \Omega_X^1$ with $\nabla \wedge \nabla = 0$. A morphism of pairs makes them form a pullback. Then this can be an *algebraic stack* [14, Thm. 7.18]. And the semi-stability will be an open condition. We can therefore obtain an open substack of semi-stable bundles whose map to the coarse moduli scheme talked above is initial among all morphisms to schemes.

From Mathematical Physics

References:
[15]: Daan Michiels, Moduli Spaces of Flat Connections, Master Thesis Leuven 2013

The moduli space of G-local systems up to gauge equivalence is considered in mathematical physics. On a smooth manifold X a G-local system can be viewed as a principal G-bundle with a flat connection (to induce the parallel transport). Then this moduli space will be

$$M_G(X) \cong \frac{\mathrm{Hom}(\pi_1(X, x), G)}{G}$$

consisting of group homomorphisms $\pi_1(X, x) \to G$ up to conjugation. And this set can be equipped with a manifold structure but not necessarily smooth. Moreover if X is a compact orientable surface and the corresponding Lie algebra of G satisfies some more conditions, then the moduli space $M_G(X)$ will be a **symplectic reduction**. In this specific example we can think about more structures like symplectic structure or Poisson structure on it and this idea is generalized in derived algebraic geometry recently by Pantev and Toën.

The above construction can also be applied to obtain the Betti moduli space M_{B}. In [13] Simpson deals with the case where $G = \mathrm{GL}_n$. We suppose the topological space X is good enough e.g. the complex analytic space of a smooth projective variety over $\mathbb{C}$ and fix a point $x \in X$. Let $\Gamma = \pi_1(X, x)$ which is finitely generated. Then we can define a functor from $\mathbf{Sch}_{\mathbb{C}}$ to $\mathbf{Set}$ sending any scheme S to the set of group homomorphisms $\mathrm{Hom}(\Gamma, \mathrm{GL}(n, \mathcal{O}_S(S)))$. This functor is represented by an affine scheme $\mathbf{R}(\Gamma, n)$ which can be constructed by relations in Γ as a closed subscheme of the product of the general linear algebriac group GL_n. In this sense $\mathbf{R}(\Gamma, n) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n)$. Considering the conjugate action of GL_n on $\mathbf{R}(\Gamma, n)$, we get the (coarse) Betti moduli space $M_{\mathrm{B}}(X, n)$ via GIT ([14, Prop. 6.1]).

This construction can be generalized for any reductive affine algebraic group G. And we have the (coarse) Betti moduli space $M_G(X)$ whose $\mathbb{C}$-points are in one-to-one correspondence with isomorphism classes of semi-simple locally constant principal G-bundles.

From the stacky viewpoint we can get a quotient stack $\mathcal{M}_G(X) := [\mathbf{R}_G(\Gamma)/G]$ whose map to $M_G(X)$ is initial among morphisms to schemes.

Derived Moduli Stack

References:
[16]: Bertrand Toën, Gabriele Vezzosi, From HAG to DAG: Derived Moduli Stacks
[17]: Bertrand Toën, Gabriele Vezzosi, Homotopical Algebraic Geometry II: Geometric Stacks and Applications
[18]: Tony Pantev, Bertrand Toën, Poisson geometry of the moduli of local systems on smooth varieties, arXiv:1809.03536
[19]: Tony Pantev, Bertrand Toën, Moduli of flat connections on smooth varieties, arXiv:1905.12124

We introduce the derived (Betti) moduli stack of local systems of rank n and suppose k is a field of characteristic 0 e.g. $\mathbb{C}$. There are some ways to describe this derived moduli stack. We deal with the case $G = \mathrm{GL}_n$ first.

At the beginning we take a moduli theory viewpoint to see which geometric objects it really classifies. Here we extend complex vector spaces or vector bundles to be differential graded modules (dg-modules). Again we assume X is good enough and A is a cdga over k. $A\text{-}\mathbf{Mod}_X^{\leq 0}$ denotes the category of presheaves of dg-A-modules on X. It's equipped with a *global projective model structure* and is a dg-category. Apply *Bousfield localization* to this model category we get a new model category where weak equivalences are *local weak equivalences* such that they induce quai-isomorphisms of dg-A-modules on stalks.

We say a presheaf $\mathcal{F}$ of dg-A-modules on X is locally on $X \times A_{\acute{e}t}$ equivalent to A^n if for any point $x \in X$ there exists an open neighborhood U of x and an étale covering $\{\mathrm{Spec}B_i \to \mathrm{Spec}A | i \in I\}$ for the *affine derived stack* $\mathrm{Spec}A$ such that $\mathcal{F}_{|U} \otimes_A B_i$ is weakly equivalent to B^n as presheaves. Then the category $\mathrm{LocSys}_n(X; A)$ of local systems of dg-A-modules of rank n consists of cofibrant objects locally on $X \times A_{\acute{e}t}$ equivalent to A^n and morphisms weak equivalences. Note that we can regard this category as a simplicial set by applying the dg-nerve functor (in Lurie's *Higher Algebra* Construction 1.3.1.6). Then we get a derived prestack $\mathbb{R}\mathrm{LocSys}_n(X)$ sending any cdga A to the simplicial set $\mathrm{N}_{\mathrm{dg}}(\mathrm{LocSys}_n(X; A))$. It's shown in [17, Prop. 2.2.6.5] this derived prestack is a *derived stack*.

There is also another way to describe this derived stack. We have already known a local system or a locally constant sheaf of sets is equivalent to a functor from $\Pi_1(X)$ to $\mathbf{Set}$. Therefore a natual viewpoint tells us the higher moduli space of local systems should parametrize ∞-local systems and be a mapping space from $\Pi_\infty(X)$ to some infinite category. For complex local systems $\mathbf{Set}$ is replaced by complex vector spaces. Therefore here we consider the derived moduli stack $\mathbb{R}\mathscr{B}un_n$ of vector bundles of rank n and the derived stack of local systems of rank n is then defined as

$$\mathbb{R}\mathrm{Map}_{\mathbf{dSt}}(\Pi_\infty(X), \mathbb{R}\mathscr{B}un_n)$$

the mapping stack in the category $\mathbf{dSt}_k = \mathbf{dAff}^{\sim}_{\acute{e}t}$ of derived stacks where $\Pi_\infty(X)$ is the constant derived stack associated to the singular Kan complex $\mathrm{Sing}_\bullet(X)$. Also note that a classical theorem tells us a vector bundle of rank n is actually equivalent to a GL_n-torsor. Therefore $\mathbb{R}\mathscr{B}un_n$ can be replaced by the classifying derived stack $B\mathrm{GL}_n$ as well. This version can be adapted to any other suitable group G and we can define the derived moduli stack of G-local systems as $\mathbb{R}\mathrm{Map}_{\mathbf{dSt}}(\Pi_\infty(X), BG)$.

The two points we disscussed above are equivalent as shown in [17, Prop. 2.2.6.5] .

Question 4: In the Section 10.1 of Dima Arinkin, Dennis Gaitsgory, *Singular support of coherent sheaves, and the geometric Langlands conjecture*, arXiv:1201.6343, Arinkin and Gaitsgory define the derived moduli stack of G-local systems as the mapping stack $\mathrm{Map}(X_{dR}, BG)$ for any DG scheme X where X_{dR} is the de Rham prestack. Why when considering local systems on some space containing “infinitesimal data” we need to replace X by X_{dR} to obtain the correct notion of stack of local systems? How can we know a space contains infinitesimal data or not? Actaully this means what's the difference between the two definitions. Note that our definition above is named as the moduli stack of principal G-bundles in that paper. In fact in our case here X is a good enough topological space and $\Pi_\infty(X)$ is constant so that the two definitions coincide. What's more whether there is a good homotopy theory for geometric objects like (DG) schemes such that a local system can be defined as a functor on this homotopy type as well?

The final approach to getting the derived moduli stack of local systems is similar to the method we discussed before to obtain the quotient stack $\mathcal{M}_G(X) := [\mathbf{R}_G(\Gamma)/G]$ where $\Gamma = \pi_1(X, x)$. In the theory of simplicial sets there is a Quillen pair between simplicial sets and simplicial groups[13]. The *loop group construction* sends every simplicial set to a weakly equivalent simplicial group. Then for our space X, it can be associated with a simplicial group GX whose geometric realization is weakly homotopy equivalent to X i.e. having the same homotopy type. If it's necessary, we can resolve GX further to obtain a simplicial group $\Gamma_\bullet$ such that every Γ_n is finitely free. In this case $\mathbf{R}_G(\Gamma_\bullet)$ will be a *cosimplicial affine scheme* which is an affine derived stack. Actually if we use the model of cdga, we need to apply the normalization functor to the simplicial commutative algebra structure on $\mathbf{R}_G(\Gamma_\bullet)$ to obtain a cdga $\mathscr{A}_G(X)$. Then $\mathbb{R}\mathrm{LocSys}_G(X)$ is defined as the *derived quotient stack* $[\mathrm{Spec}\mathscr{A}_G(X)/G]$[14].

Question 5: How can we show $[\mathrm{Spec}\mathscr{A}_G(X)/G]$ is equivalent to $\mathbb{R}\mathrm{Map}_{\mathbf{dSt}}(\Pi_\infty(X), BG)$?

Using the quotient stack $[\mathrm{Spec}\mathscr{A}_G(X)/G]$, in [18] Pantev and Toën show the derived stack of local systems admits a shifted Poisson structure generalizing the classical case of surfaces.

Question 6: Until now we have only discussed the derived Betti moduli stack. In [19] Pantev and Toën also construct the derived de Rham moduli stack $\mathbb{R}\mathscr{B}un^{\nabla}$ of flat connections. Is there any description in the moduli theory sense that this derived stack classifies something like *flat superconnections*? And maybe there are some relations between the Betti side and the de Rham side in this derived case. So there is also a natural question how we can develop a derived moduli theory for Higgs bundles on the Dolbeault side such that there is a good (derived) non-abelian Hodge theory in it. These should be the generalization of Simpson's work.

1. [1, Lem. 3 in P72] ↩
2. According to [1, Ex. F.4 in P360] the space X should be paracompact, Hausdorff, connected, locally path connected and semilocally 1-connected, e.g. a connected complex manifold. ↩
3. [2, Thm. 2.3.4] ↩
4. At least X should be connected, locally path connected and semilocally 1-connected, so that it will admit a universal covering space. ↩
5. They are both isomorphic to the first cohomology group of $\mathrm{Aut}(F)$. It's a standard theorem in algebraic geometry, see e.g. Ulrich Gortz and Torsten Wedhorn, *Algebraic Geometry I* Sec. 11.6 ↩
6. There is a concept of *Ehresmann connection* in differential geometry. Details can be foound in the Chapter 12 of Jeffrey M. Lee, *Manifolds and Differential Geometry*. Theorem 12.20 there gives such a parallel translation for a connection and a smooth path. ↩
7. [3, Thm. 2.6] ↩
8. Deligne has proved the analytification functor gives an equivalence beween (regular) algebraic flat connections on X and analytic or holomorphic flat connections on X^{an}. See e.g. [4, Thm. 5.3.8]. ↩
9. [4, Lem. 1.2.1] ↩
10. There is another description for perverse sheaves in [4, Sec. 8] using t-structures. ↩
11. This requires a deep analysis of (higher) holonomy. ↩
12. [11, Theorem 0.4] ↩
13. Theorem 7.8 in P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory* ↩
14. See [17, Sec. 1.3.4]. ↩