ECS509U - Probability & Matrices

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Week 9

What we have covered in matrices

- In Weeks 6 & 8 we covered:
  - Basics of Matrix Algebra
    - Addition, subtraction, multiplication by a number, equation between matrices
    - Identity matrix, transpose matrices
    - Inverse matrix, how to find the inverse of a 2x2 matrix
  - Solutions of systems of linear equations
    - Applying elementary row operations on the augmented matrix (Gauss and Gauss-Jordan)
    - Solving 2x2 systems of equations using the matrix form AX=B, where $X=A^{-1}B$

Week 9: Learning Objectives

- Part I: **Inverses of any size of square matrices**. You should be able to:
  - find inverses of square matrices of any size
  - solve square systems of equations of any size using the matrix form AX=B, where X= $A^{-1}B$
  - calculate the determinant of 3x3 matrices
- Part II: Introduction to **vectors & vector arithmetic**. You should be able to:
  - explain the concept of a vector in terms of its dimensions
  - perform basic vector arithmetic operations
  - calculate the norm of a vector
  - compute unit vectors
  - interpret various vector operations geometrically

The main story for today

- We will see **a "quick recipe" for finding inverse matrices** by using elementary row operations and matrices
- And **a quick recipe for calculating the determinant of a 3x3 matrix**
- There is a fair bit of theory behind these 'recipes', but I omit it as it does not add significantly to the practical side of the material
  - anyone who is interested to read more, can look up ***elementary matrices***, and ***inverse elementary row operations***

How to find the inverse of **any** square matrix (not only 2x2)

- Let us assume that A is an invertible nxn matrix. We then **form a new matrix [A | I_n]**
  - essentially we append I_n after A
  - so the new extended matrix will have ***n*+*n* = 2*n* columns**
- All we need to do is find a series of row operations **that will convert the '*A*' portion** (the 'left' nxn part) **of this extended matrix to I_n,** making sure to **apply the operations to the whole extended matrix**
- Once we have done this, **the extended matrix will then look like: [I_n | A^{-1}]**, provided of course that A is invertible
  - so the 'left' nxn part of the matrix be the identity matrix, and then whatever is at **the 'right' *n*x*n* part of the matrix will be the inverse matrix A^{-1}**

A simple example

- Find the inverse of the following 2x2 matrix using the new method

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$ We first form the extended matrix: $$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix}$$

We will try to convert the left 2 columns of the matrix to I_2 and then what results in the right 2 columns will be the inverse A^{-1}

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

At this point we see that we have reached the desired form. The right 2 columns of the extended matrix give us the inverse:

$$A^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$$

You can confirm that this is the right result by: $AA^{-1} = A^{-1}A = I_2$, and also by finding the inverse using the week 6 method

A longer example

Determine the inverse of the following matrix, given that it is invertible

$$C = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix}$$

We first form the new matrix:

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

We will try to convert the left 3 columns of the matrix to I_3 and then what results in the right 3 columns will be the inverse C^{-1}

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 2 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 + R_1 \\ R_3 - 5R_1 \\ \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 7 & 6 & 1 & 3 & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right]$$

Example continued

$$\overset{R_3+25R_2}{\rightarrow}\left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} & -\frac{10}{7} & \frac{5}{7} & 1 \end{array}\right] \overset{\frac{7}{3}R_3}{\rightarrow}\left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right]$$

$$\overset{\substack{R_2-\frac{6}{7}R_3 \\ R_1-4R_3}}{\rightarrow}\left[\begin{array}{ccc|ccc} 1 & 5 & 0 & \frac{43}{3} & -\frac{14}{3} & -\frac{28}{3} \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right] \overset{R_1-5R_2}{\rightarrow}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right]$$

So we have reduced the left 3 columns of the matrix to I_3 and so what is now in the right 3 columns is the inverse C^{-1}

$$C^{-1}=\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 3 & -1 & -2 \\ -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{bmatrix}$$

You can verify on your own that C^{-1} is actually the inverse of C: $CC^{-1} = C^{-1}C = I_3$

What if there is no inverse?

- Of course there will be cases where a matrix is not invertible (i.e. it is singular)
- In such cases **the series of elementary row operations will not be able to give us the identity matrix** on the left part of the 'extended' matrix, and so we will not be able to find the inverse
- We will see such cases at the tutorial on Friday

OK, but how can we quickly know <u>if</u> a matrix is invertible?

- We know how to deal with the 2x2 case: we check the matrix's determinant, and if it is non-zero then we know the matrix is invertible
- There are a few other tricks, but **we will also use the determinant for the 3x3 case**
- I will also put on the web site material that shows you how to find higher order determinants (e.g. 4x4, 5x5, etc.) - optional reading, not examinable

3x3 determinant

- In the 2x2 case we used this "formula":

$$\begin{array}{c} + \qquad\qquad - \\ \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \end{array} \qquad a_{11}a_{22} - a_{21}a_{12}$$

- In the 3x3 case, things can be a bit more complicated, but we can use this trick to easily get the determinant:

$$\begin{array}{c} + \qquad\qquad\qquad\qquad - \\ \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right. \end{array}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Compute the determinant of the following matrix:

$$B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$$

$$\begin{array}{cccccc} & & + & & & - \\ \det(B) = & \left| \begin{array}{ccc} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{array} \right. & \begin{array}{cc} 3 & 5 \\ -2 & -1 \\ -11 & 1 \end{array} \end{array}$$

$$\det(B) = (3)(-1)(7) + (5)(8)(-11) + (4)(-2)(1) - (5)(-2)(7) - (3)(8)(1) - (4)(-1)(-11)$$

$$= -467$$

Another example

- Consider the following 3x3 system of equations. Does it have a unique solution?

$$\begin{aligned} x - 2y + z &= -1 \\ 3x + y + z &= 0 \\ -2x + 4y - 2z &= 2 \end{aligned}$$

Since it is a square system, we can answer this by bringing the system in the matrix form **AX=B**. **We can then look at the coefficient matrix A and see if it is invertible or not.** The system will have a unique solutions **if and only if A is invertible, so if and only if det(A)≠0**

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 1 \\ -2 & 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{so } A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 1 \\ -2 & 4 & -2 \end{bmatrix} \text{ and}$$

$$\det(A) = (1)(1)(-2) + (-2)(1)(-2) + (1)(3)(4) - (-2)(3)(-2) - (1)(1)(4) - (1)(1)(-2) =$$

$$-2 + 4 + 12 - 12 - 4 + 2 = 0, \text{ so the system does not have a unique solution}$$

Introduction to Vectors

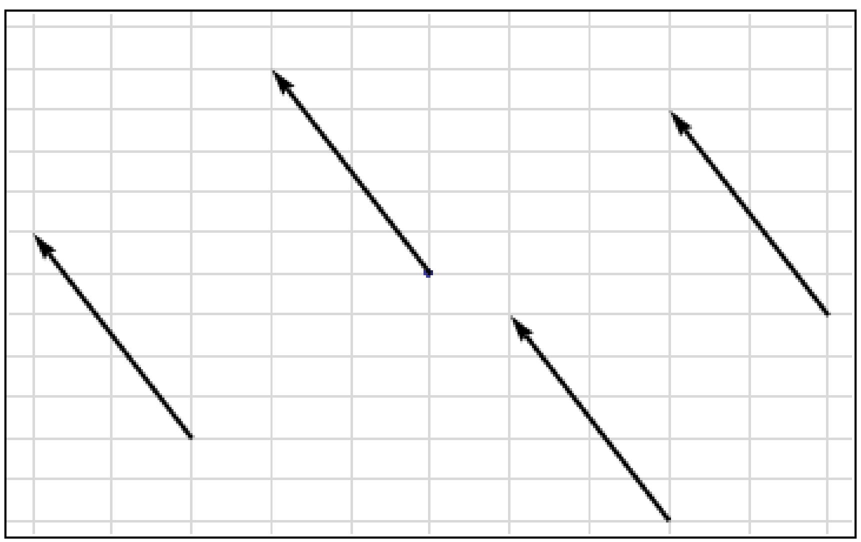
- Vectors are characterised by **length** (**magnitude**) and **direction**
- They can be represented as **directed line segments** that start at a point A and end at a point B
- We will use boldface lower case letters to denote vectors, e.g. **v, w, a, b**
- It helps to talk about vectors in terms of a 2-dimensional space, and in terms of a 3-dimensional space (easier to visualise)
  - but, as we will see, they can also be defined in spaces with *n*-dimensions

Example

ending point

v is an example of a vector in 2-space

starting point



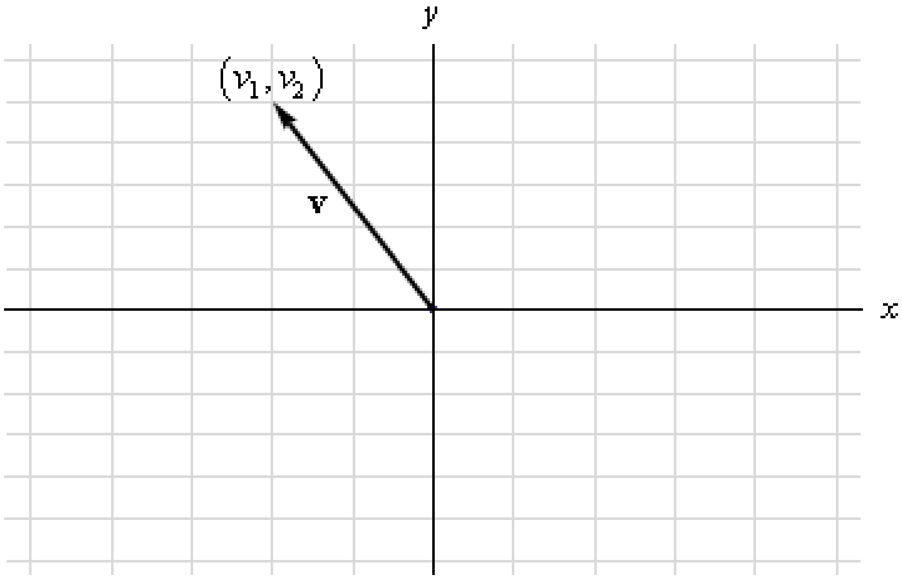
These are examples of **equivalent vectors**: they have the **same direction** and the **same** magnitude (**length**)

It does not matter that they have different starting and ending points

Representing vectors

- Vectors make sense to be defined in what are called **vector spaces** (more on these next week)
  - Examples of vector spaces are $\mathbf{R}^2$ (standard 2-dimensional space), $\mathbf{R}^3$ (standard 3-diemensional space), etc.
  - For example all points in $\mathbf{R}^2$ can be defined as ordered tuples of real numbers (v_1, v_2), each corresponding to one dimension of the space
- Suppose the origin of the co-ordinate system of the space is chosen as the starting point for a vector in $\mathbf{R}^2$, and its ending point is given by co-ordinates (v_1, v_2)

Representing vectors

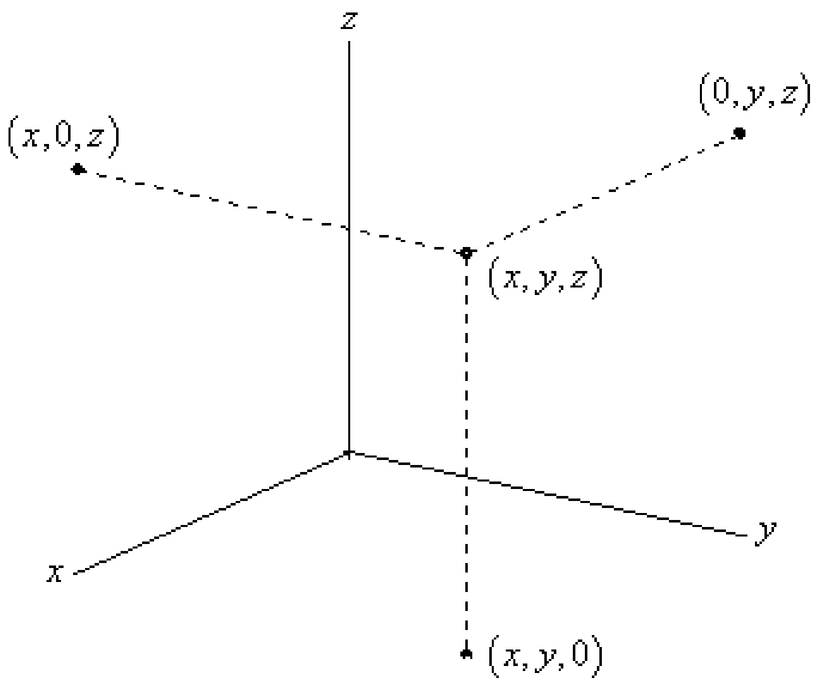


In such cases we call the co-ordinates of the terminal point the **components of v** and write:

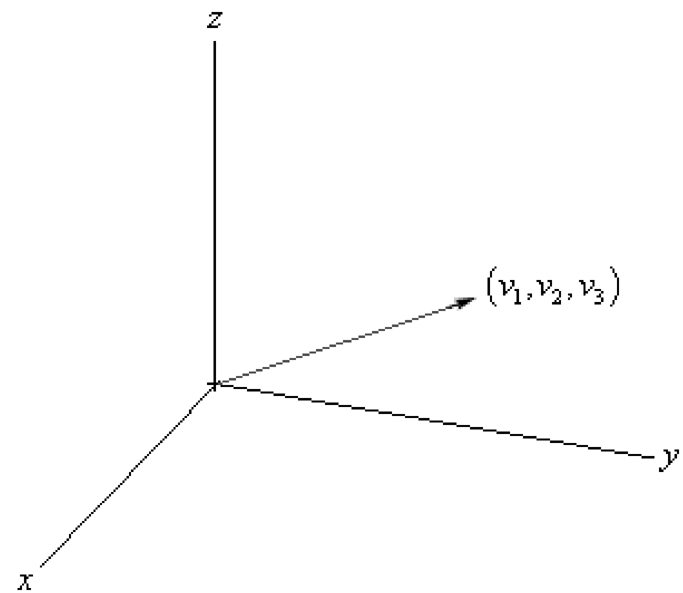
$\mathbf{v}=(v_1, v_2)$

We can extend these definitions in 3-dimensional spaces (e.g. $\mathbf{R}^3$), 4-dimensional, …, *n*-dimensional

Examples in 3-space



Points in 3-space (e.g. in **R**^{3})



A vector **v** in 3-space (e.g. in **R**^{3}): **v**=(v_1, v_2, v_3)

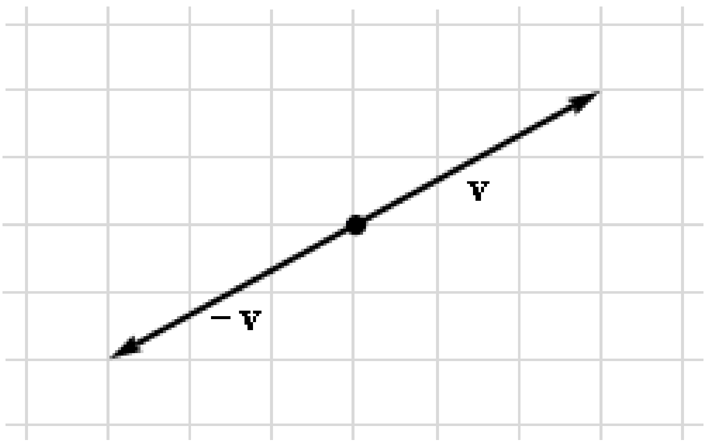
Arithmetic of vectors

- There are two important operations: **addition of two vectors**, and multiplication of a vector by a real number (**scalar multiplication**)
- We will look at both an arithmetic and a geometric interpretation of these
- **Arithmetic interpretation:**
  - Consider two vectors **v** and **u**:

    $\mathbf{v}=(v_1, v_2, \ldots, v_n)$, $\mathbf{u}=(u_1, u_2, \ldots, u_n)$
  - **Vector addition: u+v** = $(v_1+u_1, v_2+u_2, \ldots, v_n+u_n)$, the sum of two vectors is a vector, and can only be defined if the two vectors have the same number of components (i.e. are of the same dimension)
  - **Scalar multiplication:** If c is a real number, then $c\mathbf{v} = (cv_1, cv_2, \ldots, cv_n)$ and the result is a vector with the same number of components as the original vector **v**

Some more vector arithmetic

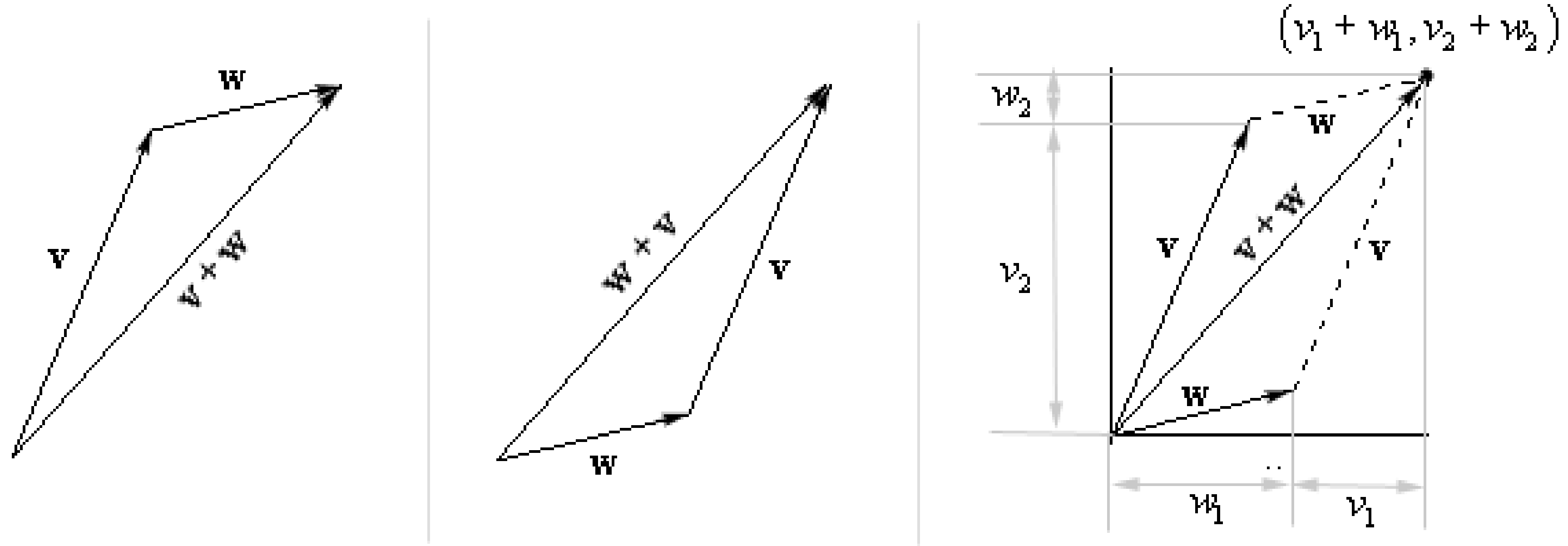
- The ***negative*** of a vector **v**, denoted by **-v**, is a vector with the same length as **v** but has the opposite direction to **v**, as shown below



- The **zero vector** denoted by **0** is a vector with no length, and by convention, we say that it can have any direction

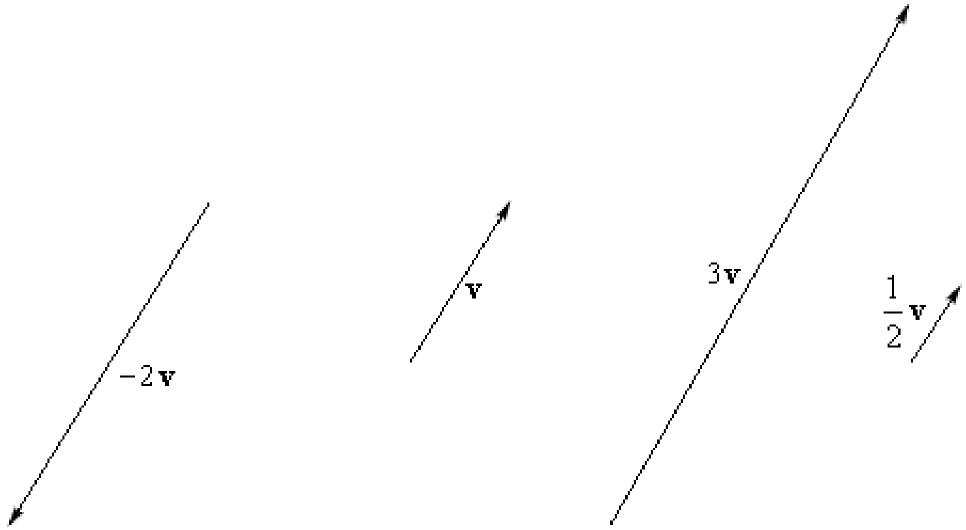
Vector arithmetic: Geometric interpretation

- A geometric interpretation of vector addition:



Vector arithmetic: Geometric interpretation

- A geometric interpretation of scalar multiplication:



Also, -**v**= (-1)**v** = $(-v_1, -v_2, \ldots, -v_n)$

Laws of vector addition and scalar multiplication

- For any vectors **u, v** and **w** in a vector space (e.g. 2- or 3- dimensional), and *c* and *k* scalars (i.e. real numbers):
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\mathbf{u} + (\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v}) + \mathbf{w}$
  - $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - $\mathbf{u} - \mathbf{u} = \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  - $1\mathbf{u} = \mathbf{u}$
  - $(ck)\mathbf{u} = c(k\mathbf{u}) = k(c\mathbf{u})$
  - $(c+k)\mathbf{u} = c\mathbf{u} + k\mathbf{u}$
  - $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

Examples

- Given the following vectors, compute the indicated quantity where possible:

  $\mathbf{a}=(4, -6)$ $\mathbf{b}=(-3, -7)$ $\mathbf{c}=(-1,5)$

  $\mathbf{u}=(1,-2,6)$ $\mathbf{v}=(0,4,-1)$ $\mathbf{w}=(9,2,-3)$

i) $-\mathbf{w}$	$-\mathbf{w}=(-9, -2, 3)$
ii) $\mathbf{a}+\mathbf{b}$	$\mathbf{a}+\mathbf{b}=(4+(-3), (-6)+(-7)) = (1,-13)$
iii) $\mathbf{a}-\mathbf{c}$	$\mathbf{a}-\mathbf{c}=(4-(-1), (-6)-5) = (5,-11)$
iv) $\mathbf{a}-3\mathbf{w}+4\mathbf{v}$	Can not be defined as **a** is in 2 dimensions
v) $\mathbf{a}-3\mathbf{b}+10\mathbf{c}$	$=(4,-6) - (-9, -21) + (-10, 50) = (3,65)$
vi) $4\mathbf{u}+\mathbf{v}-2\mathbf{w}$	$=(4,-8,24)+(0,4,-1)-(18,4,-6) = (-14,-8,29)$

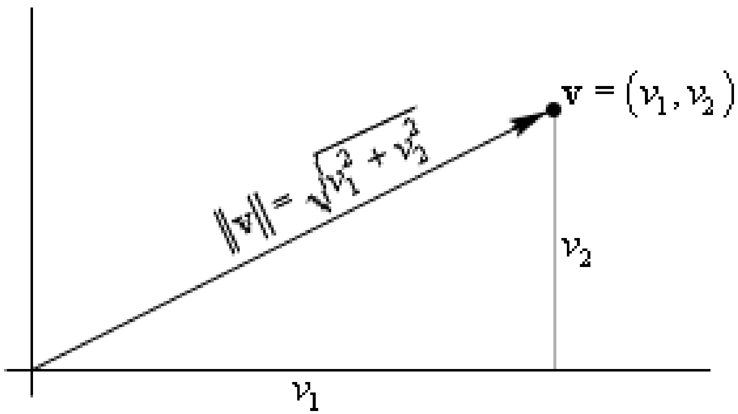
The norm of a vector

- If **v** is a vector, then the magnitude of the vector is called the ***norm*** of the vector and denoted by ||**v**||, and is calculated as:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

- For any vector **v**, ||v||≥0. ||**v**||=0 if and only if **v** = **0** (if and only if **v** is the zero vector)
- If *c* is a scalar, then ||*c***v**|| = *c*||**v**||
- If for a vector **v** its norm ||**v**|| = 1 then **v** is a **unit vector**

The geometric interpretation



We use the Pythagorean theorem to calculate the norm of the vector $\mathbf{v}=(v_1,v_2)$

We know its co-ordinates (v_1 and v_2) that correspond to the lengths of the sides of a triangle as shown

A similar case can be made in 3 dimensions, etc.

Examples

- Compute the norms given the following vectors:
v=(-5,3,9), **j**=(0,1,0), **w**=(3,-4) and 1/5**w**

$$\|v\| = \sqrt{(-5)^2 + 3^2 + 9^2} = \sqrt{115}$$

$$\|j\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1 \text{ (j is a unit vector)}$$

$$\|\mathrm{w}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

$$\left\|\frac{1}{5}w\right\| = (1/5)\|w\| = (1/5)\sqrt{3^2 + (-4)^2} = (1/5)\sqrt{25} = 1$$

so (1/5)w is a unit vector

Some more on the unit vector

- Given a non-zero vector **v**, if we define a new vector **u**

  $$u = \frac{1}{\|v\|} v$$

  then **u** is a **unit vector**.

  Note that what we do is multiply the vector **v** with a positive scalar (1/||**v**|| will always be positive), so **the unit vector will point in the same direction as the original vector**

Example

Given the vector **v** = (-2, 3), find a unit vector that (a) points in the same direction as **v** (b) points in the opposite direction to **v**

(a) $\|v\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ then we can define the unit vector u as :

$\text{u} = \frac{1}{\sqrt{13}}(-2,3) = (\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}})$ and u will pointing in the same direction as v

(b) All we need to do here, is take the negative of u :

$\text{-u} = (\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}})$ and this will be a unit vector pointing in the opposite direction to v

Summary of lecture

- In Week 9 we covered:
  - Recipes for finding inverse matrices of any size
  - Recipe for finding the determinant of 3x3 matrices
  - Introduction to vectors and vector arithmetic
    - Vector addition, scalar multiplication
    - Unit vectors, norms of vectors
- For Friday’s tutorial:
  - Come to the tutorial having attempted the week’s exercises