ECS509U - Probability & Matrices

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Week 8

Week 8: Learning Objectives

- **At the end of Week 8 you should be able to:**
  - discuss the main properties of linear equations and of systems of linear equations
  - work with the augmented and the coefficient matrix of a system of linear equations
  - apply elementary row operations to matrices to bring them to row-echelon and reduced row-echelon forms
  - solve systems of linear equations by applying these transformations
  - represent and solve simple 2x2 systems of linear equations as matrix equations

Linear equations

- What is a linear equation? You all know what they look like, e.g. $2x+3y=9$
- Is there a general form in which we can write them?

  $$\mathbf{a_1x_1 + a_2x_2 + \ldots + a_nx_n} = \boldsymbol{b}$$

  where there are n ***unknowns*** $x_1, x_2, \ldots, x_n$, and $a_1, a_2, \ldots, a_n$ *known numbers*, the ***coefficients***
- A solution to a linear equation is a set of values for the n unknowns that satisfy the equation
  - For example, how many solutions are there to this linear equation? $\mathbf{2}\boldsymbol{x}\mathbf{+3}\boldsymbol{y}\mathbf{=9}$

Systems of equations

- **A system of equations** is nothing more than a collection of 2 or more linear equations
- You all know such systems, and can possibly solve simple systems, e.g.:

  $2x+3y=9$ $\qquad\qquad$ $x-2y=-13$

- Can we generalise these for ***m* equations with *n* unknowns**?

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}}$$

Systems of equations

- We will call such systems *m* by *n* ($m \times n$)
- A system of equations will be ***square*** if $m=n$
  - If we have the same number of equations and unknowns
- A system of equations will be ***homogeneous*** if all the constant terms are zero, $b_1=b_2=\ldots=b_m=0$
- It will be ***nonhomogeneous*** otherwise
- A ***solution*** to a system of equations is a list of values for the *n* unknowns that is **a solution for each of the equations of the system**

Solutions of systems of equations

- Solutions to systems of equations **either exist or do not exist**
  - If a system has **no solution**, it is called ***inconsistent***
  - If a system has a solution, ***consistent system***, then:
    - It either has a **unique solution**, **OR**
    - It has an **infinite number of solutions**

A system of linear equations has either:(i) **a unique solution,** (ii) **no solution** or (iii) **an infinite number of solutions**

Some examples

$$2x + 3y = 9$$
$$x - 2y = -13$$
(1)

$$x - 4y = 10$$
$$x - 4y = -3$$
(2)

$$-2x + y = 8$$
$$8x - 4y = -32$$
(3)

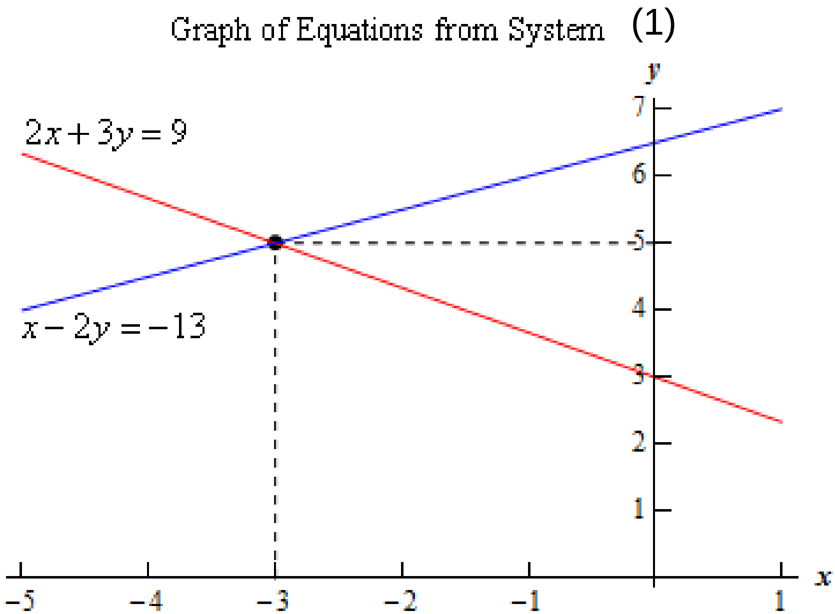
Can you see in which categories (in terms of solutions) these three systems fall?

(1) Has a unique solution

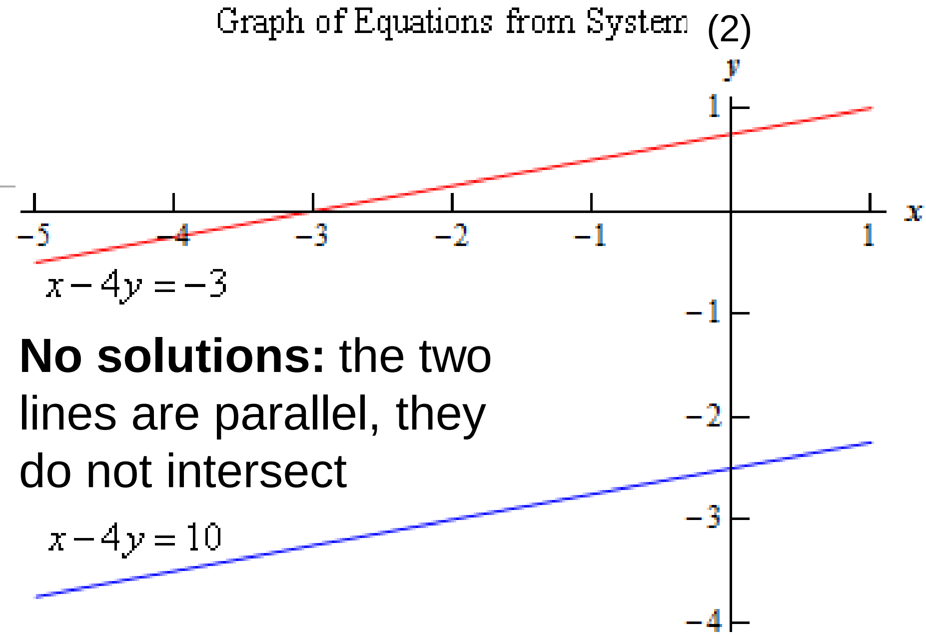
(2) Has no solution (can you see why?)

(3) Has an infinite number of solutions (can you see why?)

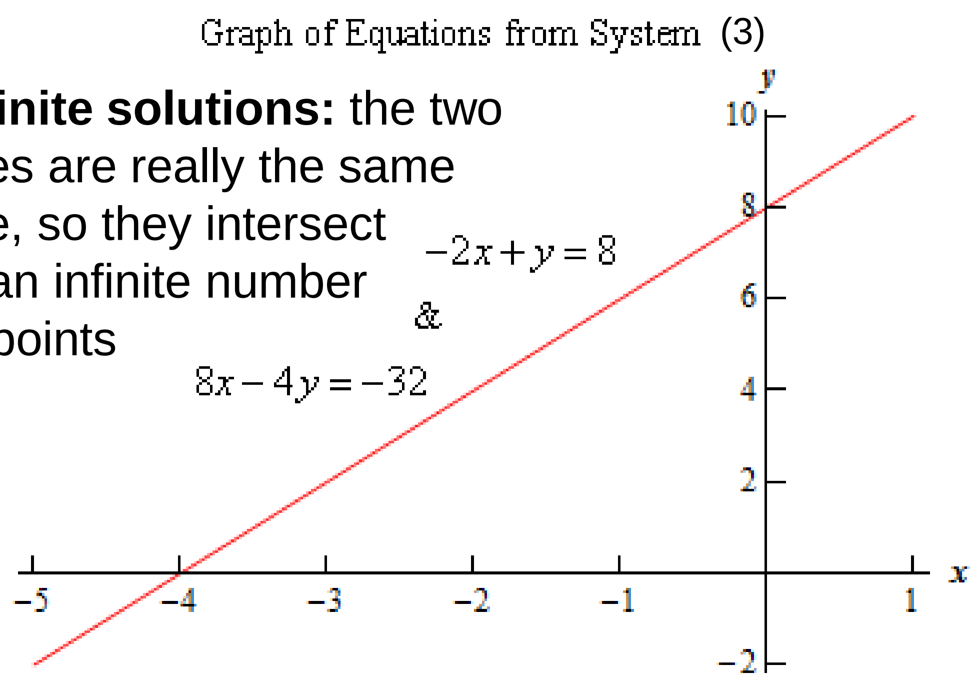
A graphical view



Unique solution: lines intersect in a single point



No solutions: the two lines are parallel, they do not intersect



Infinite solutions: the two lines are really the same line, so they intersect in an infinite number of points

Matrix representation of a system of equations: Augmented matrix

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

- Any such system of equations can be written as an ***augmented matrix***. Here is the augmented matrix for the system above:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

It is an **(m x (n+1)) matrix**

Contains **the coefficients and the constants** of the system of equations

Matrix representation of a system of equations: Coefficient matrix

- If we do not add the column with the constants of the system in the augmented matrix, then we have an (m x n) matrix called the ***coefficient matrix***
- The coefficient matrix for the general system of equations is:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is an (m x n) matrix

Contains the **coefficients** of the system of equations

Matrix equation for a system of linear equations

- The general system of *m* equations by *n* unknowns is equivalent to the following matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} \quad \text{or} \quad \mathrm{AX} = \mathrm{B}$$

where A is the coefficient matrix, X is the column vector of unknowns, and B is the column vector of the constants (in some books you will see this as Ax = b)

As an exercise verify this at home by doing the multiplications

Some examples

- **Write down the augmented matrix** for the following system of equations:

$$\begin{aligned} 3x_1 - 10x_2 + 6x_3 - x_4 &= 3 \\ x_1 + 9x_3 - 5x_4 &= -12 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 7 \end{aligned} \longrightarrow \begin{bmatrix} 3 & -10 & 6 & -1 & 3 \\ 1 & 0 & 9 & -5 & -12 \\ -4 & 1 & -9 & 2 & 7 \end{bmatrix}$$

- For the given augmented matrix, **write down the corresponding system of equations and the corresponding matrix equation of the form AX=B**:

$$\begin{bmatrix} 4 & -1 & 1 \\ -5 & -8 & 4 \\ 9 & 2 & -2 \end{bmatrix} \longrightarrow \begin{aligned} 4x_1 - x_2 &= 1 \\ -5x_1 - 8x_2 &= 4 \\ 9x_1 + 2x_2 &= -2 \end{aligned} \longrightarrow \begin{bmatrix} 4 & -1 \\ -5 & -8 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Solving systems of equations

- We will see two different ways of solving systems of linear equations
- **The first method uses the matrix equation form**, and can be used when the system is square (same number of equations and unknowns)
- **The second method uses the augmented matrix,** and through a series of matrix operations tries to bring it to a specific form that gives us the solution
  - it can be used for any kind of systems, not only square systems

Matrix equation for a system of linear equations: How to solve

- If the system $AX=B$ **is square** (i.e. the coefficient matrix A is square), **then the system has a unique solution if, and only if, the matrix *A* is invertible**
- In such a case, **the solution X is** $\mathbf{X=A^{-1}B}$
- Let us consider system (1) again:

$2x + 3y=9$

$x - 2y=-13$

Solve it using the matrix equation AX=B for this system

Note that the system is square because there are as many unknowns as equations (and so the coefficient matrix will be square)

Example continued

The **matrix equation** for system (1) is:

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -13 \end{bmatrix}$$

$$\mathrm{A} \quad \mathrm{X} = \mathrm{B}$$

The determinant of *A* is det(A)=(2)(-2)-(3)(1)=-7, so A is invertible and A^{-1} is:

$$A^{-1} = -1/7 \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/7 & 3/7 \\ 1/7 & -2/7 \end{bmatrix}$$

and then **the solution of the system is given by $A^{-1}B$:**

$$X = A^{-1}B = \begin{bmatrix} 2/7 & 3/7 \\ 1/7 & -2/7 \end{bmatrix} \begin{bmatrix} 9 \\ -13 \end{bmatrix} = \begin{bmatrix} (2/7)(9) + (3/7)(-13) \\ (1/7)(9) + (-2/7)(-13) \end{bmatrix} = \begin{bmatrix} (18-39)/7 \\ (9+26)/7 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Of course it is the same solution like the one we found in the graph form

Another example

Let us now consider system (3) from before:

-2x + y = 8

8x -4y = -32

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -32 \end{bmatrix}$$

$\det(A) = (-2)(-4) - (1)(8) = 0$, so we know that it does

not have a unique solution because there is no inverse for A

Notice that for the columns of A and the column of B:

-2/8 = 1/(-4) = 8/(-32) (one equation is a multiple of the other)

In such cases, a system will have infinite number of solutions

And another example

Let us now consider system (2) from before:

x - 4y = 10

x - 4y = -3

$$\begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

$\det(A) = (1)(-4) - (-4)(1) = 0,$ so we know that it does not have a unique solution because there is no inverse for A

Notice that for the columns of A and the column of B:

$1/1 = (\text{-}4)/(\text{-}4) \neq 10/(\text{-}3)$

In such cases, a system will not have any solutions

The second way of solving systems of linear equations

- The second method for finding solutions **can be applied to any size systems**, not only square systems
- This method works on the **augmented matrix** of the system
- Before we actually see how it works, we need to learn some basics:
  - elementary row operations
  - echelon forms for matrices

Elementary row operations

- The following 3 operations can be applied to the augmented matrix of a system of equations, and result in equivalent matrices of systems of equations that will have the same solutions:

Row operation	**Equation Operation**	**Notation**
Multiply row *i* by the constant *c*	Multiply equation *i* by the non-zero constant c	cR_i
Interchange rows *i* and *j*	Interchange equations *i* and *j*	$R_i \longleftrightarrow R_j$
Add *c* times row *i* to to row *j*	Add *c* times equation *i* to equation *j* (c is non-zero)	$R_j + cR_i$

Examples

- Given the augmented matrix below, perform the indicated elementary row operations: (**for this example, we apply the operations to the initial matrix in every case**)

$$\begin{bmatrix} 2 & 4 & -1 & -3 \\ 6 & -1 & -4 & 10 \\ 7 & 1 & -1 & 5 \end{bmatrix}$$

a) replace R_1 by -3R_1

b) replace R_2 by 1/2R_2

c) interchange rows 1 and 3

d) replace R_2 by R_2+5R_3

$$a)\begin{bmatrix} -6 & -12 & 3 & 9 \\ 6 & -1 & -4 & 10 \\ 7 & 1 & -1 & 5 \end{bmatrix} \quad \mathrm{b)}\begin{bmatrix} 2 & 4 & -1 & -3 \\ 3 & -1/2 & -2 & 5 \\ 7 & 1 & -1 & 5 \end{bmatrix} \quad \mathrm{c)}\begin{bmatrix} 7 & 1 & -1 & 5 \\ 6 & -1 & -4 & 10 \\ 2 & 4 & -1 & -3 \end{bmatrix}$$

$$d)\begin{bmatrix} 2 & 4 & -1 & -3 \\ 41 & 4 & -9 & 35 \\ 7 & 1 & -1 & 5 \end{bmatrix}$$

Method for solving systems of equations using matrices

- We will start with the **augmented matrix** of the system, and by applying ONLY **elementary row operations** we will try to bring it to a form such that:
  - If there are any **rows of all zeros**, then they are at the bottom of the matrix
  - **If a row does not consist of all zeros** then its first non-zero entry (i.e. the left most non-zero entry) is a 1. This 1 is called a leading 1, or a pivot
  - **In any two successive rows**, neither of which consists of all zeroes, **the leading 1 of the lower row is to the right of the leading 1 of the higher row**
- A matrix in such a form is in ***row-echelon form***
- Additionally, if a **column** contains a leading 1, then all the other entries in the column are zero, then the matrix is in ***reduced row-echelon form***

Examples

- All the following examples are in row-echelon form

$$\begin{bmatrix} 1 & -6 & \underline{9} & \underline{1} & 0 \\ 0 & 0 & 1 & \underline{-4} & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \underline{5} \\ 0 & 1 & \underline{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \underline{-8} & 10 & \underline{5} & -3 \\ 0 & 1 & 13 & \underline{9} & 12 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The elements that are stopping them from being in reduced row-echelon are underlined

If these elements were 0, then the matrices would be in reduced row-echelon form

- All the following examples are in reduced row-echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 9 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The only **difference between echelon and reduced echelon** forms is that the former is required to have 0s below a leading 1, where the latter is required to have 0s below AND above a leading 1

Elimination methods

- We start with the augmented matrix and we apply elementary row operations
  - If we bring the matrix to **row-echelon form** and then stop, then our method is called **Gaussian Elimination**
  - If we bring the matrix to **reduced row-echelon form,** our method is called **Gauss-Jordan Elimination**
- Unless an exercise tells us to, we normally start with Gauss, and then, if necessary, we look at Gauss-Jordan
  - Of course, if done correctly, they will give the same solutions

Example

Use **Gaussian elimination AND Gauss-Jordan elimination** to solve the following system of equations:

$$\boxed{\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}} \longrightarrow \begin{bmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

Important: There are many ways you can follow to take the augmented matrix to row-echelon or reduced row-echelon forms.

You should follow the one you find easiest, provided **you only use the 3 elementary row operations**

Example continued

Augmented matrix

$$\begin{bmatrix} \boxed{-2} & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ \boxed{-2} & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ \boxed{3} & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & \boxed{5} & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

Interchanging rows 1 and 2 is the easiest way to get a leading 1 in row 1

We then need to change the numbers in the 1st column under the leading 1 into zeros. We will do this one element at a time in the next 2 operations

Now the first row and first column look ok, we will start working on the second row, and try to change the first 5 into a leading 1

In every repetition, we work on the element indicated in the square box, and try to make it either 1 or 0 depending on the requirements of the echelon form we are trying to achieve

Example continued

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & \boxed{5} & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix} \xrightarrow{1/5\ R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & \boxed{-6} & -8 & -40 \end{bmatrix} \xrightarrow{R_3+6R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & \boxed{-2} & -4 \end{bmatrix} \xrightarrow{-1/2\ R_3} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Having fixed the leading 1 in row 2 we now move down its column to change the rest of the elements into zeroes

Notice that the last step missing to make the matrix into row-echelon form is to change the -2 into a 1

The last matrix is in row-echelon form, so Gauss would stop here. We translate the matrix into equations, and with back substitution we find the values of the unknowns:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ \boxed{x_3 = 2} & \end{aligned}$$

$\boxed{x_2=4}$

$\boxed{x_1=-1}$

We then take the row-echelon form, and we try to put it into reduced row-echelon (because we are asked to, otherwise we would stop here)

Example continued

$$\begin{bmatrix} 1 & \boxed{2} & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1\text{-}2R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \boxed{1} & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2\text{-}R_3} \begin{bmatrix} 1 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1\text{-}R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We now need to change elements above the leading 1s into zeroes for all columns - start with elements over leading 1 of row 2

We now move into the elements over the leading 1 of the 3rd row

There is only one reduced row-echelon form for a matrix, no mater which way we get to it

The last matrix is in reduced row-echelon form, so we stop here. We translate the matrix into equations, and we get the same solution as before:

$$\begin{aligned} x_1 \quad\quad\quad\quad &= -1 \\ x_2 \quad\quad &= 4 \\ x_3 &= 2 \end{aligned}$$

No solutions / infinite solutions

- The previous example ended up nicely in giving us a unique solution that solved all equations
- Consider the following two matrices of some systems of equations in row-echelon form:

$$a)\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

- What about the solutions to these systems? Always, they key to giving the answer is to translate the augmented matrices to the corresponding systems of equations at this stage

Examining System (a)

$$a)\begin{bmatrix}1 & -2 & 3 & 7\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}\Rightarrow\begin{Bmatrix}x-2y+3z=7\\ 0x+y+z=-2\\ 0=0\end{Bmatrix}$$

- The key here is the 3rd equation, which tells us that **0=0** - but that is ALWAYS true, so this last row of the matrix would not 'add' anything to our solutions
- Instead, we now have 2 equations, with 3 unknowns - we can solve by back-substitution:

$$\begin{Bmatrix}\mathrm{x}-2\mathrm{y}+3\mathrm{z}=7\\ \mathrm{y}=-2-\mathrm{z}\end{Bmatrix}\Rightarrow\begin{Bmatrix}\mathrm{x}+4+2\mathrm{z}+3\mathrm{z}=7\\ \mathrm{y}=-2-\mathrm{z}\end{Bmatrix}\Rightarrow\begin{Bmatrix}\mathrm{x}=3-5\mathrm{z}\\ \mathrm{y}=-2-\mathrm{z}\end{Bmatrix}$$

- This means that for **ANY** value of *z*, the system will have a different set of solutions (e.g. try putting z=0, then you get x=3, y=-2, etc.), and therefore the system would have an **infinite number of solutions**

Examining System (b)

$$b)\begin{bmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Remember: All equations in a system must be 'satisfied' (I.e. solved) in order to have a solution

- This is simpler, just look at the last equation, it basically tells us that **0x + 0y + 0z = -3**
- This statement can never be true, **it is inconsistent**, and this system would have no solutions because of this
- Any such inconsistent statements would indicate that the system of equations has **no solutions**
  - Other examples of inconsistent statements would be to have e.g. one row of the matrix saying y=0 and anther saying y=-1, etc.

Summary of lecture

- **Solving Systems of Linear Equations**
- In Week 8 we covered:
  - Augmented matrix and coefficient matrix of a system of equations
  - Matrix equation representation and solution of a system of equations
  - Elementary row operations, echelon forms
  - Solution to systems of equations by Gauss and Gaus-Jordan eliminations