# ECS509U- Probability & Matrices

Tassos Tombros

*Week 6*

# Week 6: Learning Objectives

- At the end of Week 6 you should be able to:
  - Perform basic matrix arithmetic operations
  - Apply properties of matrix algebra for doing basic calculations
  - Determine whether a 2x2matrix is invertible, and if yes, to find its inverse matrix
  - Explain how matrices and linear algebra are related

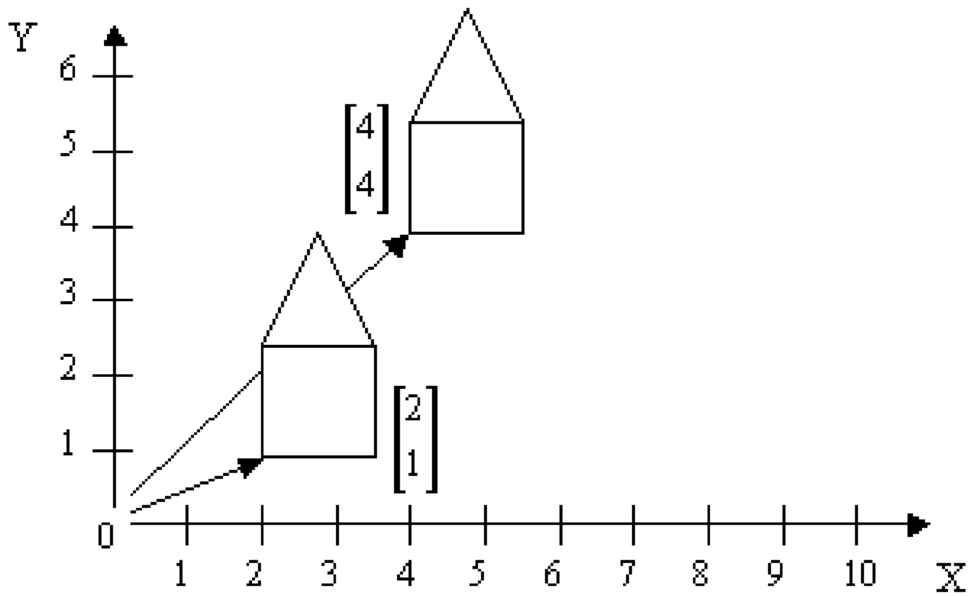
# Why Matrices & linear algebra?

- Many reasons, these are very important topics in a CS degree
- **In the context of your studies at QMUL?**
  - **Computer graphics, Image Processing**
  - Data Mining, Big Data Processing
- More generally, matrix notation and linear algebra allow you to model, or to look at problems, from a different angle that:
  - is mathematically sound and provides you with solid tools to solve problems
  - offers you clear, simple methodologies for getting from A to B

# Simple example in graphics

$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{t}, \qquad \boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \boldsymbol{t} = \begin{bmatrix} dx \\ dy \end{bmatrix}, \qquad \begin{cases} x' = x + dx \\ y' = y + dy \end{cases}$$

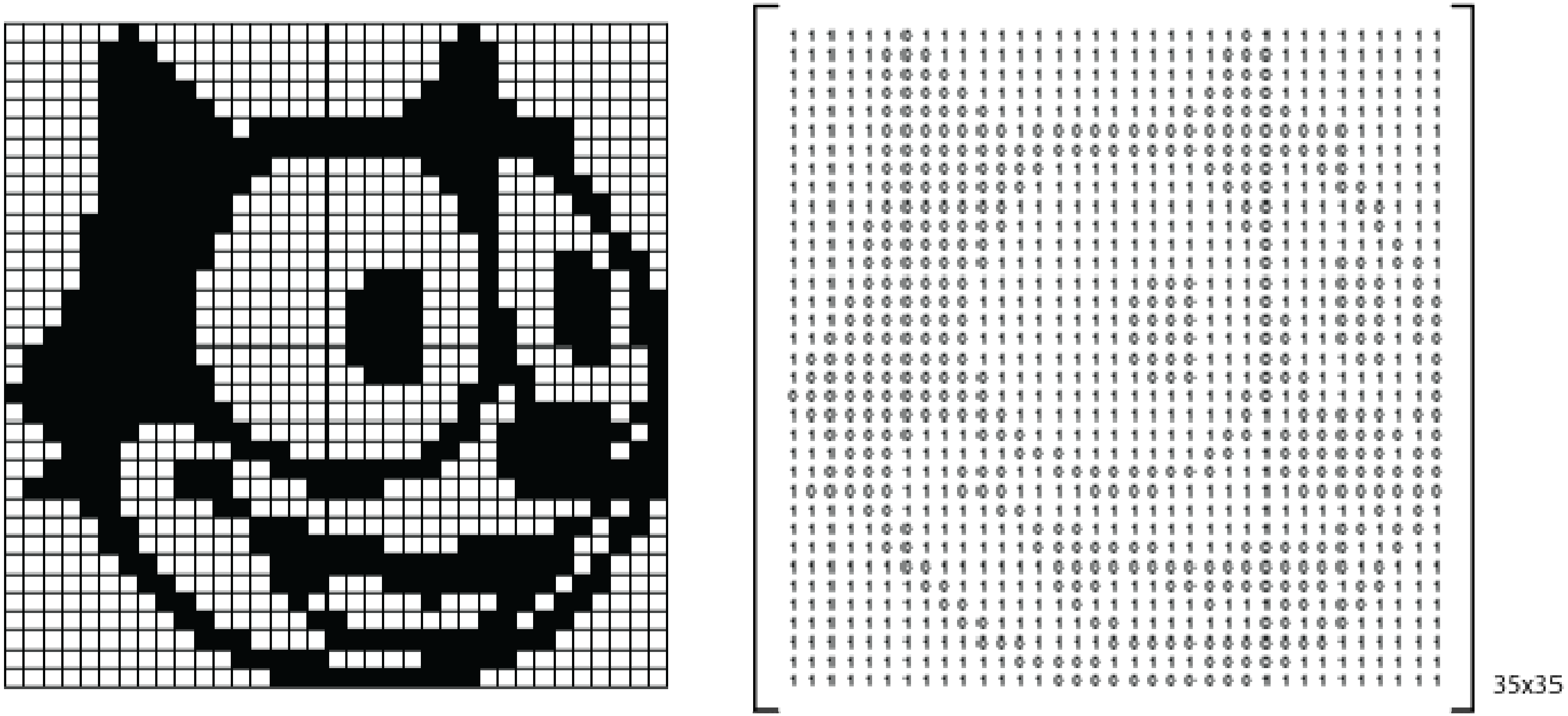
dx = 2
dy = 3




This would be an example of moving the shape from the initial position (2,1) to (4,4)

We would need to calculate the transformation matrix *t*

# Images are matrices




This is a simplified example only in B&W but full colour works similarly (e.g. one matrix each for each of the RGB values, etc.)

# Another example

- Consider an e-commerce site like Amazon, which is based on ratings of products by users. It is possible to construct matrices that allow us to see patterns of similar behaviour between users

Columns can be items (e.g. films, CDs, etc.)

$$UserRatingsMatrix = \begin{bmatrix} 3 & 0 & 4 & 5 & 2 \\ -1 & 3 & 2 & -1 & -1 \\ -1 & 0 & 1 & 2 & 2 \\ -1 & 3 & 4 & 5 & 5 \\ 0 & 2 & 4 & 5 & 4 \end{bmatrix}$$

User$_3$ rated item$_3$ with 1 star

Each row corresponds to a user

# Basic matrix definitions

- A **matrix**, is a rectangular array of numbers usually presented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The matrix has ***m* rows** and ***n* columns**, we will use $\mathbf{A}_{\mathbf{mxn}}$ to represent it

The element in the *i-th* row and *j-th* column is represented by $\mathbf{a}_{\mathbf{ij}}$

We can denote the entire matrix as $A=[a_{ij}]$

A matrix with one row is called a **row matrix**, or **row vector**

A matrix with one column is called a **column matrix** or **column vector**

# Examples

$$A = \begin{bmatrix} 9 & 7 & 0 \\ -1 & 3 & 7 \end{bmatrix} \text{is a 2x3 matrix (2 rows, 3 columns). The element } a_{23} = 7$$

$$B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 7 & -9 & 12 \end{bmatrix} \text{is a 3x3 matrix}$$

This is the **diagonal**, $b_{11}$, $b_{22}$, $b_{33}$, in general $b_{ij}$ where *i=j*. It only makes sense in square matrices

(square matrix because number of rows = number of columns)

$$C = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 3 \end{bmatrix} \text{is 4x1 matrix, a column vector, it has a single column}$$

$$D = \begin{bmatrix} -1 & 6 & 6 & 0 & 1 \end{bmatrix} \text{is a 1x5 matrix, a row vector, it has a single row}$$

# Matrix arithmetic & operations

- **Matrix equality:** If A and B are both *m*x*n* matrices, then we say A=B provided that all corresponding entries from each matrix are equal. **Matrices of different sizes can never be equal**
  - A=B provided $a_{ij}=b_{ij}$ for all possible *i* and *j*
- **Matrix addition: If A and B are both *m*x*n* matrices**, then A+B is a new mxn matrix that is found by adding the corresponding entries from each matrix
  - **Matrices of different sizes can never be added**
- Similar definition for **matrix subtraction**
- **Matrix multiplication by a scalar:** If A is any matrix, and *c* is any number, then the product (or scalar multiple) *cA,* is a new matrix, the same size as A, where its entries are found by multiplying the original entries of A by *c*

# Examples

$$A=\begin{bmatrix}1 & x\\ 3 & -2\end{bmatrix}, \mathrm{B}=\begin{bmatrix}1 & -9\\ 3 & -2\end{bmatrix}, \mathrm{C}=\begin{bmatrix}1\\ 3\end{bmatrix}$$

For these matrices, A ≠ C and B ≠ C since they are different sizes. A = B only if x = -9

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$$A=\begin{bmatrix}2 & 0 & -3 & 2\\ -1 & 8 & 10 & -5\end{bmatrix}, \mathrm{B}=\begin{bmatrix}0 & -4 & -7 & 2\\ 12 & 3 & 7 & 9\end{bmatrix}, \mathrm{C}=\begin{bmatrix}2 & 0 & 2\\ -4 & 9 & 5\\ 6 & 0 & -6\end{bmatrix}$$

$A+B=\begin{bmatrix}2 & -4 & -10 & 4\\ 11 & 11 & 17 & 4\end{bmatrix}$, $\mathrm{B}\text{-}\mathrm{A}=\begin{bmatrix}-2 & -4 & -4 & 0\\ 13 & -5 & -3 & 14\end{bmatrix}$, addition/subtraction between C and A or B can not happen since C has different size from A and B

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$$\mathrm{A}=\begin{bmatrix}0 & 9\\ 2 & -3\\ -1 & 1\end{bmatrix}, \mathrm{B}=\begin{bmatrix}8 & 1\\ -7 & 0\\ 4 & -1\end{bmatrix}, \mathrm{C}=\begin{bmatrix}2 & 3\\ -2 & 5\\ 10 & -6\end{bmatrix}$$

$$3A+2B-1/2C=\begin{bmatrix}0 & 27\\ 6 & -9\\ -3 & 3\end{bmatrix}+\begin{bmatrix}16 & 2\\ -14 & 0\\ 8 & -2\end{bmatrix}-\begin{bmatrix}1 & 3/2\\ -1 & 5/2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}15 & 55/2\\ -7 & -23/2\\ 0 & 4\end{bmatrix}$$

# Mona Lego Lisa

**A= M(0)** **M(0.50)** **Z= M(1)**

A simple linear transformation **M(t) = (1-t)A + t Z,** $\forall t \in [0,1]$

# Some properties (they all make sense)

- Consider any matrices A, B, C (with the same size), and any numbers c and d:
  - $(A+B)+C = A+(B+C)$
  - $A+0 = 0+A = A$
  - $A+(-A) = (-A)+A = 0$, where 0 is the **zero matrix** whose entries are all equal to zero
  - $A+B = B+A$
  - $c(A+B)= cA+cB$
  - $(c+d)A=cA + dA$
  - $(cd)A = c(dA)$
  - $1.A=A$

# Matrix multiplication - almost

- Before we move to full scale matrix multiplication, let's start with a simple example:

$$A = \begin{bmatrix} 4 & -10 & 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 \\ 3 \\ 8 \\ 2 \end{bmatrix}. \text{ A is 1x4 matrix (row vector) and}$$

B is 4x1 (column vector).

In this case (more later) the product can be defined, and it is equal to:

$$AB = \begin{bmatrix} 4x(-4) + (-10)x3 + 3x8 + 0x2 \end{bmatrix} = \begin{bmatrix} -22 \end{bmatrix}$$

AB is a 1x1 matrix, whose only element $a_{11} = -22$

# Matrix multiplication proper

- If **A is an m×p** matrix, and **B is a p×n matrix**, then the product **AB is a new matrix with size m×n** whose $ij^{th}$ entry is found by **multiplying row i of A by column j of B**
- **Note** that if the number of columns of the first matrix is not equal to the number of rows of the second matrix, **multiplication is not defined!!!!**

| A | B | = | AB |
|---|---|---|---|
| *m×p* | *p×n* | | *m×n* |

# Examples

$$A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}, \mathrm{B}=\begin{bmatrix}2 & 0 & -4\\ 5 & -2 & 6\end{bmatrix} \text{A is } 2\times 2\text{, B is } 2\times 3\text{, so AB will be a } 2\times 3 \text{ matrix}$$

$$\mathrm{AB}=\begin{bmatrix}1\times 2+3\times 5 & 1\times 0+3\times(-2) & 1\times(-4)+3\times 6\\ 2\times 2+(-1)\times 5 & 2\times 0+(-1)\times(-2) & 2\times(-4)+(-1)\times 6\end{bmatrix}=\begin{bmatrix}17 & -6 & 14\\ -1 & 2 & -14\end{bmatrix}$$

Note that we can not define BA, because B is $2\times 3$, A is $2\times 2$ and so B's columns are not the same as A's rows and therefore the product is not defined

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$$A=\begin{bmatrix}0 & 1 & 4\\ -1 & 0 & 2\\ 0 & 1 & 2\end{bmatrix}, \mathrm{C}=\begin{bmatrix}3 & 0\\ 1 & 0\\ 2 & -1\end{bmatrix} \text{A is } 3\times 3 \text{ matrix and C is } 3\times 2 \text{ so AC is defined and is}$$

a $3\times 2$ matrix. CA can not be defined.

$$\mathrm{AC}=\begin{bmatrix}(0\times 3)+(1\times 1)+(4\times 2) & (0\times 0)+(1\times 0)+(4\times -1)\\ (-1\times 3)+(0\times 1)+(2\times 2) & (-1\times 0)+(0\times 0)+(2\times -1)\\ (0\times 3)+(1\times 1)+(2\times 2) & (0\times 0)+(1\times 0)+(2\times -1)\end{bmatrix}=\begin{bmatrix}9 & -4\\ 1 & -2\\ 5 & -2\end{bmatrix}$$

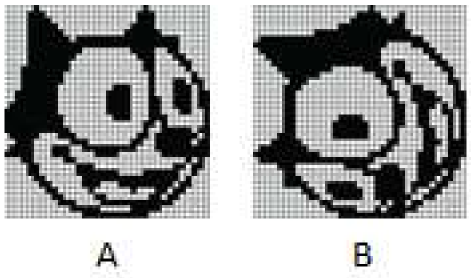
# Some properties

- Let A, B, C be matrices, and *k* any number. Whenever products and sums are defined amongst these matrices:

  - $(AB)C = A(BC)$
  - $A(B+C) = AB + AC$
  - $(B+C)A = BA + CA$
  - $k(AB)=(kA)B=A(kB)$
  - $0A=0=A0$, where 0 is the zero matrix

# Transpose matrix

- The **transpose** of a matrix $A_{m\times n}$, written $\mathbf{A^T}$, is an $n\times m$ matrix obtained by **interchanging the rows and columns of A**
- Some properties of the transpose:
  - Let A and B be matrices, and $k$ any number. Whenever the sum and product are defined:
    - $(A+B)^T=A^T+B^T$
    - $(A^T)^T=A$
    - $(kA)^T=kA^T$
    - $(AB)^T=B^TA^T$




**Felix the cat transposed:** $A^T = B$

# Examples

$$A=\begin{bmatrix}4 & 10 & -7 & 0\\ 5 & -1 & 3 & -2\end{bmatrix}, \mathrm{B}=\begin{bmatrix}3 & 2 & -6\\ -9 & 1 & -7\\ 5 & 0 & 12\end{bmatrix}, \mathrm{C}=\begin{bmatrix}9\\ -1\\ 8\end{bmatrix}, \mathrm{D}=\begin{bmatrix}-12 & -7\\ -7 & 10\end{bmatrix}$$

$$A^T=\begin{bmatrix}4 & 5\\ 10 & -1\\ -7 & 3\\ 0 & -2\end{bmatrix}, \mathrm{B}^\mathrm{T}=\begin{bmatrix}3 & -9 & 5\\ 2 & 1 & 0\\ -6 & -7 & 12\end{bmatrix}, \mathrm{C}^\mathrm{T}=\begin{bmatrix}9 & -1 & 8\end{bmatrix}, \mathrm{D}^\mathrm{T}=\begin{bmatrix}-12 & -7\\ -7 & 10\end{bmatrix}$$

Notice that $\mathbf{D^T=D}$. In these cases the matrix is called **symmetric** (more later)

# A bit more on square matrices

- A **square matrix** is a matrix with the same number of rows as columns. An *nxn* square matrix is said to be of order *n*, and is sometimes called an *n-square matrix*
- **The trace of A, tr(*A*),** is the sum of its diagonal elements
  - $tr(A)=a_{11}+a_{22}+\ldots+a_{nn}$
  - If A and B are n-square matrices and *k* is any number:
    - $tr(A+B)= tr(A)+tr(B)$
    - $tr(kA)=k\, tr(A)$
    - $tr(A^T)=tr(A)$
    - $tr(AB)=tr(BA)$

# The identity matrix

- The identity matrix is a square matrix
- It acts like the number 1 in real number arithmetic for multiplication
- **If *A* is an *m*x*n* matrix** we have:
  - $I_m A = A I_n = A$
  - Note that we need different identity matrices on each side of A depending upon the size of A
  - Normally we just use the letter $I$ to denote the identity matrix
  - The identity matrix has 1's in the diagonal and 0's elsewhere

# Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc.}$$

Exercise:

$A = \begin{bmatrix} 0 & -2 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$ is a 3x2 matrix. Multiply by the appropriate identity matrix to the left and right to verify the identity matrix property

**THINK carefully:** for the matrix multiplication to be allowed, **what should the size of the identity matrix be** when you multiply from the left and what when you multiply from the right?

# Powers of matrices

- Let A be an n-square matrix and n,m integers:
  - $A^0=I$
  - $A^2=AA$, $A^3=AAA=A^2A$, …, $A^{n+1}=A^nA$
  - $A^nA^m=A^{(n+m)}$
  - $(A^n)^m=A^{nm}$
- And we can also calculate **polynomials** on matrix A:
- $p(A)=a_nA^n+a_{n-1}A^{n-1}+\ldots+a_1A+a_0I$ , where $a_n$, $a_{n-1}$, etc. are numbers and $I$ is the identity matrix with the same size as $A$

# Examples

- As exercise, calculate:
  - $p(A)=-A^2+10A-9I$

$$\text{for the matrix} : A = \begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix}$$

$$\text{The answer you should get is} : \begin{bmatrix} -143 & 48 \\ 80 & -15 \end{bmatrix}$$

# Inverse matrices

- An **n-square matrix *A*** is said to be **invertible** (or nonsingular), if there exists a matrix *B* of the same size such that: ***AB=BA=I***
- Such a matrix *B* is unique, it is called the inverse of *A*, and we denote it by $\mathbf{A^{-1}}$

Verify that given the matrix A, the given matrix $A^{-1}$ is its inverse:

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}, \; A^{-1} = \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix}$$

# Some properties

- Suppose that $A$ and $B$ are invertible matrices of the same size:
  - $AB$ is invertible, and $(AB)^{-1}=B^{-1}A^{-1}$
  - $A^{-1}$ is invertible, and $(A^{-1})^{-1}=A$
  - $(A^{n})^{-1}=A^{-n}=(A^{-1})^{n}= A^{-1}A^{-1}A^{-1}\dots A^{-1}$, for n=0, 1, 2, …
  - If $c$ is a non-zero number, then $cA$ is invertible and $(cA)^{-1}=(1/c)A^{-1}$
  - $A^{T}$ is invertible, and $(A^{T})^{-1} = (A^{-1})^{T}$

# Inverse of a 2x2 matrix

- Let *A* be a 2x2 matrix, say: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- We want to find a formula that gives us the elements of its inverse matrix

$$\text{By definition, AA}^{-1} = I \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- We can calculate *a*, *b*, *c*, *d* by solving this system of equations

# Determinant, Inverse

- Let |A| (or *det*(A)) be the **determinant** of matrix *A*
  - **|A| = ad-bc** (or **det(A)** = ad-bc)
- **If |A|=0 then matrix A is not invertible** or we also say that A is **singular**
- **If |A|≠0 then matrix A is invertible**, and its inverse is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Example

- Are the following matrices invertible? If so, find their inverse matrix

$$a)\ \ A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

$$|A| = ad - bc = (-4)(5) - (5)(-2) = -10 \neq 0 \text{ so A is invertible}$$

$$\mathrm{A}^{-1} = \frac{1}{-10}\begin{bmatrix} 5 & 2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/5 \\ 1/2 & 2/5 \end{bmatrix}$$

$$b)\ \mathrm{B} = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix}$$

$$|B| = ad - bc = (-4)(3) - (-2)(6) = -12 + 12 = 0 \text{ so B is not invertible}$$

# Some special square matrices

- A square matrix *D* is called **diagonal** if the only non-zero entries are on the main diagonal

$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix} \text{or a concrete example A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We can represent it as $\mathrm{D} = \mathrm{diag}(\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3, \dots, \mathrm{d}_n)$, so $A = diag(3, -7, 2)$

- A square matrix *A* is called **upper triangular** if all entries below the main diagonal are equal to zero
- Similarly, a square matrix *A* is called **lower triangular** if all entries above its main diagonal are equal to zero

# Some special square matrices

- A square matrix *A* is called **symmetric** if $A^T = A$
  - or in words, if symmetric elements (mirror elements with respect to the diagonal) are equal ($a_{ij} = a_{ji}$)

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}, B = \begin{bmatrix} 6 & -10 & 3 & 0 \\ -10 & 0 & 1 & -4 \\ 3 & 1 & 12 & 8 \\ 0 & -4 & 8 & 5 \end{bmatrix}$$

are examples of symmetric matrices

# Matrices - vectors - linear algebra where is the connection?

- The fundamental problem of linear algebra is solving systems of linear equations, e.g.:
  - $2x - y = 0$
  - $-x + 2y = 3$
- There are two ways to look at this simple 2x2 system of equations:
  - The **row view** and the **column view** (examples of these will be given during the lecture)
- There is also the **matrix view**, which gives us specific tools and conditions under which these systems, and much more complex systems, can be solved
- We will first start learning about solving systems of equations through matrices
- Then we will move into vectors and their properties

# Equations as matrices

- Consider the following system of equations:
  - 3x-2y=4
  - -x+y=2
- Forget the right side of these equations for now
- Can you think of a matrix way to write the left side as a product of two matrices?

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \mathrm{x} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} \text{ Ax is defined (it will be a } 2 \times 1 \text{ matrix):}$$

$$\mathrm{Ax} = \begin{bmatrix} 3\mathrm{x} - 2\mathrm{y} \\ -\mathrm{x} + \mathrm{y} \end{bmatrix}$$

- Solving these kinds of systems using matrix notation will be our focus from Week 8

# Summary of lecture

- **Introduction to Matrices & Linear Algebra**
- In Week 6 we covered:
  - Basic properties of matrices
  - Matrix addition, scalar multiplication
  - Matrix multiplication
  - Transpose matrices
  - The inverse matrix
- For Friday’s tutorial:
  - Come to the tutorial having attempted the week’s exercises