The axioms of probability and their consequences

In the lecture we saw the three basic axioms of Probability theory, **Kolmogorov's axioms**, named after the Russian mathematician Andrey Kolmogorov who first stated these axioms. These axioms can not be proven, they are statements, or assumptions, that form the building blocks of the Probability theory that we will cover in the module.

What is really important about these three axioms, is that based on them, we can derive some resulting Probability laws, which we can call Theorems (theorems are things that can be proven based on existing laws/axioms).
In the lecture slides I have included some of these resulting laws, and here we will see how these can be proven based on the Probability axioms and based on other axioms and laws from set theory (hopefully some of these will be known to you from Logic and Discrete structures).

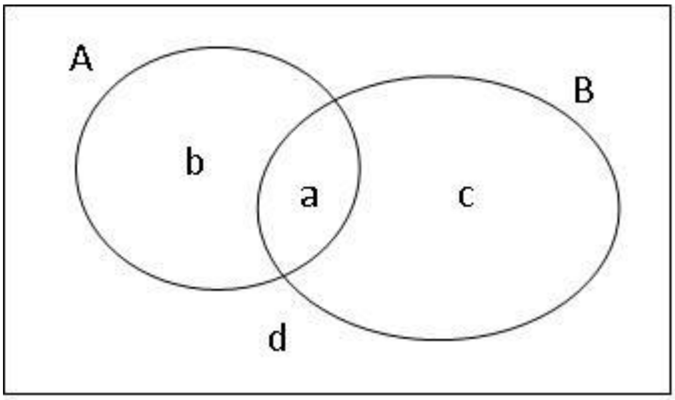
Proofs of Probability theorems

Theorem 1. $P(\emptyset)=0$. The impossible event (the empty set) has a probability of zero.
Proof: For any event A we have that $A \cup \emptyset = A$, where A and $\emptyset$ are disjoint (mutually exclusive). By using the 3rd axiom of probability, $P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$.
At this last equation we add $(-P(A))$ in both sides of the equation, and we therefore get: $P(A) - P(A) = P(A) + P(\emptyset) - P(A)$, from where it follows that $0 = P(\emptyset)$.

Theorem 2. For any even A, $P(A') = 1 - P(A)$. This is the complement rule, and it tells us that the probability of the complement of an event is equal to 1 minus the probability of the event itself.
Proof: By definition of the complement, the sample space $S = A \cup A'$, where A and A' are disjoint (mutually exclusive). From the axioms of Probability we know that $P(S)=1$. We now again apply the 3rd axiom of probability, and we get:
$P(S) = P(A) \cup P(A') \Leftrightarrow 1 = P(A) \cup P(A') \Leftrightarrow P(A') = 1 - P(A)$.
Note: This can also be written as: $P(A) = 1 - P(A')$.

Theorem 3. For any event A, we have $0 \leq P(A) \leq 1$. This tells us that the probability of any event will always lie between 0 and 1.
Proof: From the axioms of Probability we know that $P(A) \geq 0$. We therefore need only to show that $P(A) \leq 1$.
Because $S = A \cup A'$, where A and A' are disjoint (mutually exclusive), we get $1 = P(S) = P(A) \cup P(A') = P(A) + P(A') \Leftrightarrow 1 = P(A) + P(A')$.
Adding $-P(A')$ to both sides of this gives us that $P(A) = 1 - P(A')$. Since (by axiom) $P(A') \geq 0$, we get that $P(A) \leq 1$ as required.

Theorem 4. For any two events A, B we have that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ - this is the general addition rule.
Proof: The proof of this Theorem can be derived using the Venn diagram of the two sets A and B, with A and B intersecting.



From the diagram above we see that: $P(A) + P(B) - P(A \cap B) = (b+a)+(a+c)-a = b + a + c = P(A \cup B)$, which is what we wanted to show.