Notes on Mutually Exclusive and Independent Events

We saw during weeks 1 & 2 two types of events that are of particular interest in probabilistic experiments: mutually exclusive (or *disjoint*) and independent events.

1. Mutually exclusive (or *disjoint*) events

Two events A, B are called mutually exclusive if they cannot occur at the same time, i.e. if one occurs the other does not, and vice versa.

Examples: Consider rolling a single die and two events: A = roll a number greater than 3, B = roll a number less than 2. Clearly A and B cannot both occur.

Based on the definition, **two events A, B are mutually exclusive if and only if** $\mathbf{A \cap B = \emptyset}$ **, and consequently,** $\boldsymbol{P(A \cap B) = 0}$**.**

The definition extends beyond only two events and can be applied to any number of events. In the case of multiple events, a set of events is mutually exclusive if all possible pairs of events in that set are mutually exclusive.

1.1 Testing whether events are mutually exclusive

To examine whether events A, B are mutually exclusive or not, we just need to examine whether they intersect:

$\boldsymbol{A \cap B = \emptyset}$

Yes → A, B are mutually exclusive

No → A, B are not mutually exclusive

Example: Consider rolling a single die and two events: A = roll a number greater than 3, B = roll a number less than 2. Clearly A and B cannot both occur. A={4, 5, 6}, B = {1} and clearly $\boldsymbol{A \cap B = \emptyset}$ so events A and B are mutually exclusive.

2. Independent events

Two events A and B are independent if the fact that A occurs does not affect the probability of B occurring, and vice versa.

Examples: The events A: passing the ECS509U exam and B: the exam is held on a Friday, should be independent (clearly your chances of passing the exam or not should not depend on the day of the week the exam is held). The events A: choosing a 3 from a deck of cards, replacing it and then event B: choosing an ace as the second card, are also independent, etc.

A 'formal' mathematical definition is that two events A, B are independent if **P(A∩B) = P(A)P(B). This is called the multiplication rule and only applies if events A, B are independent** (i.e. we cannot calculate the probability of the intersection as the product of the individual probabilities if the two events A, B are not independent).

Another definition of independence that can be derived from the multiplication rule is that if events A, B are independent, then: **P(A|B) = P(A) and also P(B|A) = P(B).** If you think about what these formulas tell you in simple words you will see that they are closely linked to what it means for events

A, B to be independent: If A and B are independent then the probability of A remains the same whether event B takes place (i.e. P(A|B)), or not (P(A)).

If we combine the two formulas for independence above, we derive the **general multiplication rule,** which tells us that: **P(A∩B) = P(A|B)P(B)**. This, being the general rule, applies regardless of whether events A, B are independent or not (contrast to the 'plain' multiplication rule). Note that if A, B are independent then the general multiplication rules becomes: **P(A∩B) = P(A|B)P(B) = P(A)P(B)** (i.e. it becomes the 'plain' multiplication rule).

Note: You should at this point compare and contrast the addition and general addition rules (week 1 slides) with the multiplication and general multiplication rules.

Addition refers to the union of events (OR) and the 'plain' addition rule assumes that events are mutually exclusive. The general addition rule does not assume mutual exclusivity, but if the events are mutually exclusive then it becomes equivalent to the 'plain' addition rule (general addition rule: for 2 events, P(A∪B) = P(A) + P(B) – P(A∩B) and clearly if A, B are mutually exclusive then P(A∩B)=0 and therefore P(A∪B) = P(A) + P(B) which is the 'plain' addition rule).

Multiplication refers to the intersection of events (AND) and the 'plain' multiplication rule assumes that events are independent. The general multiplication rule does not assume independence, but as we saw if the events are independent then it becomes equivalent to the 'plain' multiplication rule.

The definition of independence can be extended beyond only two events and can be applied to any number of events. Likewise the multiplication rule can be applied to any finite set of events. In this module however, we will not consider such cases, however the extension is straightforward.

2.1 Testing whether events are independent

To examine whether events A, B are independent or not, we can use either one of the following two tests (in an exam you would be free to choose which test to use unless specifically asked to use one or the other (or both)).

2.1.1 Test 1

$\boldsymbol{P(A \cap B) = P(A)P(B)}$

Yes → A, B are independent

No → A, B are not independent

Example: Consider rolling a single die, and the following events: Event A: the die comes up odd; Event B: the die comes up 1.
Step 1: We represent the events in terms of sets (outcomes): A = {1, 3, 5} and B = {1}, with P(A)=3/6 and P(B)=1/6.

Step 2: We find the intersection (A∩B) and its probability P(A∩B): A∩B = {1} and **P(A∩B)= 1/6**.

Step 3: We calculate the product **P(A)P(B)** = (3/6)(1/6) = 3/36 = **1/12.**

Step 4: We compare the two values from Steps 2 & 3 and conclude: **P(A∩B) ≠ P(A)P(B)**, so events A, B are NOT independent.

2.1.2 Test 2

P(A|B) = P(A)

(or equivalently: **P(B|A) = P(B)**)

Yes → A, B are independent

No → A, B are not independent

Example: Consider rolling a single die, and the following events: Event A: the die comes up odd; Event B: the die comes up 1.
Step 1: We represent the events in terms of sets (outcomes): A = {1, 3, 5} and B = {1}, with P(A)=3/6 and P(B)=1/6.

Step 2: We find the intersection ($A \cap B$) and its probability $P(A \cap B)$: $A \cap B = \{1\}$ and **$P(A \cap B)= 1/6$**.

Step 3: We calculate **P(A|B) (or P(B|A)): P(A|B)** = $P(A \cap B)/P(B)$ = (1/6)/(1/6) =**1.**

Step 4: We compare the probability from Step 3 with the respective probability from Step 1. Here we compare P(A|B) =1 with P(A)=3/6. Clearly **$P(A|B) \neq P(A)$**, so events are NOT independent.

3. Relationship between mutually exclusive and independent events

Let's assume we have **two non-empty events A, B** (i.e. $A \neq \emptyset$, $B \neq \emptyset$).

If events A and B are mutually exclusive then they cannot be independent: If A and B are mutually exclusive, then they must have an empty intersection and so $P(A \cap B) = 0$. But we assumed that A and B themselves are $A \neq \emptyset$, $B \neq \emptyset$ (non-empty), so $P(A)P(B) \neq 0$, in other words $P(A \cap B) \neq P(A)P(B)$. Therefore, **two non-empty events A and B that are mutually exclusive cannot be independent.**

Likewise, let's assume that two non-empty events A and B are independent. We know that because they are independent then $P(A \cap B) = P(A)P(B) \neq 0$ and also that $P(A|B) = P(A)$. Clearly then, since A and B are non-empty ($A \neq \emptyset$, $B \neq \emptyset$), they cannot be mutually exclusive. If they were mutually exclusive then $P(A \cap B)$ would be equal to zero, but we know that $P(A \cap B) = P(A)P(B)$ and that $P(A)P(B) \neq 0$ (because A and B are non-empty). Therefore, **two non-empty events A and B that are independent cannot be mutually exclusive.**

The tutorial exercises of Week 2 discussed these two cases. **Please note that in an exam**, if an exercise asks you to examine / show whether events are independent or mutually exclusive you cannot rely on the above two relationships – you would be expected to show / prove whether they are independent or mutually exclusive by applying the tests covered in sections 1 and 2.