Notes on Permutations and Combinations

Permutations and combinations allow us to count the **total number of ways in which we can select *r* items out of a total *n* items**, depending on whether the **order** in which we select the *r* items matters or not. In all cases $n \geq r$.

1. Order matters: Permutations

If the order in which we select the *r* items matters then we what we have to count is permutations.
Notation: P(n, r)

Examples: Choosing 4 digits for your bank card's pin, selecting letters, digits and other characters for your password, choosing *r* letters of the alphabet to form strings / words, etc.

If for example you consider the three letters a, b, c then the number of different ways in which you can arrange them to form 3-letter strings are: abc, acb, bac, bca, cab, cba – clearly each one of these is a different string even if they contain the same letters, so **order matters**.

We can have permutations where the selected items can be repeated (**with repetitions**), or cannot be repeated (**without repetitions**).

1.1 Permutations with repetitions

Example: You select 4 digits for a safety code from the numbers 0-9. Repetitions are allowed, so the code 3212 is valid, as is the code 3333, etc.

The total number of ways in which we can select *r* items out of *n* with repetitions when order matters is $\mathbf{P(n, r) = n^r}$.

In the example above n=10 (the 10 numbers 0-9), and r=4 (4 digits in the code), so P(10,4) = 10^4 = 10,000.

A further way to understand this category (still using the example above), is to see it like this:

10 options	10 options	10 options	10 options
Digit 1	**Digit 2**	**Digit 3**	**Digit 4**

For every digit there are 10 options (the numbers 0-9) because repetitions are allowed. The total number of codes is given by the product: $10 \times 10 \times 10 \times 10 = 10^4$.

1.2 Permutations without repetitions

Example: You have 3 chairs and 5 students to sit in these chairs. Order matters as it matters which student will sit in which chair (the 1st, 2nd, 3rd). Obviously repetitions are not allowed because the same student can not sit in two chairs.

The total number of ways in which we can select *r* items out of *n* without repetitions when order matters is $\boldsymbol{P(n, r) = \frac{n!}{(n-r)!}}$

In the example above, n=5, r=3, so $\boldsymbol{P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60.}$

A further way to understand this category (still using the example above), is to see it like this:

5 options	4 options	3 options
Chair1	**Chair2**	**Chair3**

For every chair the number of choices is different (decreasing), as once a student is seated in a chair the number of choices for the next chair is reduced by one. The total number of arrangements of the 5 students in 3 chairs is given by the product $5 \times 4 \times 3 = 60$.

You should note that when you want to calculate the total number of ways in which you can select $\boldsymbol{n}$ **items chosen out of** $\boldsymbol{n}$, the above formula becomes: $\boldsymbol{P(n,n)} = \frac{\boldsymbol{n}!}{(\boldsymbol{n-n})!} = \boldsymbol{n}!$. So there are **n!** ways in which you can select *n* items chosen out of *n* when order matters and when repetitions are not allowed.

For example, if you had 4 chairs and 4 students to sit, there would be 4! = 24 different ways in which you could arrange the students in the chairs (4 options for the 1st char, 3 for the 2nd, 2 for the 3rd and 1 for the 4th).

1.3 Permutations without repetitions when some items are the same (Permutations of indistinguishable objects – Mississippi)

There are cases where we need to count permutations, but some of our *n* items are the same. In the two cases above (1.1 and 1.2) the *n* items were all distinct to each other (e.g. the numbers 0-9, 5 different students, etc.).

Example: Consider the letters a, a, b. How many different 3-letter strings can we form from these 3 letters without repetitions? (Without repetitions means we can have aab as a string, but not aaa, or not bba, etc.). Order clearly matters as baa is a different string from aba, etc. (This is the same as the Mississippi example we saw in the lectures but in smaller scale so we can follow all possible cases).

The problem is P(n,n) (P(3,3) in the example) but some of the *n* items are not distinct: we have *n*=3 letters to choose from, but some of them (the 2 a's) are the same. So we can identify two groups of letters: $\boldsymbol{n_1}$**=2** (the a's) and $\boldsymbol{n_2}$**=1** (the b). Check that $\boldsymbol{n_1 + n_2 = n}$.

Note that from 1.2 previously, the total number of permutations of *n* items chosen out of *n* without repetitions is n! (if all *n* items were distinct). So in our example we could have 3! = 6 strings of length 3 from the 3 letters a a b. But is this the case? Let's see what these 6 strings would be:

aab	
aab	I swapped places between the 2 a's – did you notice a difference?
aba	
aba	I swapped places between the 2 a's – did you notice a difference?
baa	
baa	I swapped places between the 2 a's – did you notice a difference?

What you should notice is that half of the permutations result in the same string because the letters a, a are the same. So instead of having n! = 6 distinct strings, we have 3 (half the number).

The general formula that allows us to calculate the total number of distinct permutations of *n* items where n_1, n_2, … n_k of these items are the same, is:

$$\frac{\boldsymbol{n}!}{\boldsymbol{n_1}!\boldsymbol{n_2}!\ldots\boldsymbol{n_k}!}, \boldsymbol{where\ n_1 + n_2 + \cdots + n_k = n.}$$

Note that the numerator of this formula is the total number of permutations of *n* items chosen out of *n*. The denominator excludes the total number of ways in which each of the groups of similar items can be rearranged (if you have n_1 similar items, they can be rearranged in n_1! ways, etc.). This quantity is excluded because it would result in the same (i.e. not distinct) output (see Table above).

In our example above, the total number of distinct strings is 3!/(2!1!) = 3. The denominator tells us to exclude half of the strings as they would correspond to the same output (see Table above).

2. Order does not matter: Combinations

If the order in which we select the *r* items does not matter then we what we have to count is combinations. **Notation:** C(n, r).

Examples: Choosing the 6 winning numbers of the lottery out of 49 numbers, choosing 10 students from a class to form a committee, picking 3 pens to take with you from your desk, etc.

Note that since order does not matter, **the number of combinations will be no more than the number of permutations for the same parameters *n, r*.**

We will only study combinations **without repetitions**. There is a method to count combinations with repetitions but we will not cover this.

2.1 Combinations without repetitions

Example: There are 49 numbers from which you may pick the 6 winning numbers in a lottery draw. Order does not matter, as all you have to do is choose the 6 winning numbers from the 49, no matter the order.

The total number of ways in which we can select *r* items out of *n* without repetitions when order does not matter is $\boldsymbol{C(n,r) = \frac{n!}{(n-r)!r!}}$

In our lottery example we would have C(49,6) = 49! / (43!6!) = 13,983,816.

If you compare the formula for counting combinations with the respective formula for permutations from 1.2, you will notice **that C(*n*,*r*) is r! times less than P(*n*,*r*).** The total number of ways in which you could reorder the *r* items is r! , but in combinations we do not care about this reordering (order does not matter, remember?), so we exclude this component from the calculation by dividing by r!.

To explain this further by example consider that there are 4 students and wish to select 3 of them to form a committee. The total number of ways to do this is C(4,3) = 4! / 1!3! = 4. We do not care about the order, so the only 4 possible committees are formed as such: (1,2,3), (1,2,4), (2,3,4), (1,3,4). Contrast this with permutations where order matters (e.g. 4 students to sit in 3 chairs) – P(4,3) = 24. The difference in the two counts is a factor of 6 (3!) which corresponds to all possible ways in which we could reorder the 3 students. It is just that when we form a committee we do not care about the reordering!!!

You should also note that when you want to calculate the total number of ways in which you can select ***n* items chosen out of *n*** when order does not matter, the above formula gives: $\boldsymbol{C(n,n) = \frac{n!}{(n-n)!n!} = 1}$. So there is only 1 way in which you can select *n* items chosen out of *n* when order does not matter and when repetitions are not allowed. Think about it, it makes sense: You have 3 pens in front of you and you want to pick all 3 of them at once (you don't care about which one you pick first, which second, etc.) – there is only one way to do it, you just pick all 3 of them at once.