ECS509U - Probability & Matrices

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Week 2

Reminders

- **Assessed exercises for Week 1:** Deadline on Friday at the start of your exercise class (**Deadline**: Before is fine, after is not fine)
- **QM+:** There is content there for you, use it !!!
  - Solutions, extra notes, etc.
- **Module forum:**
  - Used for announcements, monitoring posts and answering questions when needed
  - Use it for the same reasons, **I assume that in the very least, you read everything that is in the Announcements folder!!!!**

Last week

- In Week 1 we covered:
  - Basic notion of probability
  - Approaches to calculating probability values
  - Sample spaces and events
  - Basic probability laws and how to use these in solving problems
  - Tests for mutual exclusive and independent events

Week 2: Learning Objectives

- **Conditional Probability - Permutations/Combinations**
- At the end of Week 2 you should be able to:
  - Understand the concept of conditional probability
  - Apply conditional probability and event independence for solving problems
  - Appreciate some applications of conditional probability in computer science
  - Understand the concept of permutations and combinations
  - Solve probability problems involving permutations and combinations

Conditional Probability

- The probability that an event *A* occurs given that event *E* has already occurred, is called the **conditional probability of A given E**, and is written: **P(A|E)**
- Some examples:
  - What is the probability that the die will come up 3, given that it has come up odd?
    P(die is 3 | die is odd) = 1/3
  - We roll two dice. Find the probability that one of the dice is a 2, given that the sum is 6.
    P(die is 2 | sum of 2 dice is 6) = 2/5

Conditional Probability

- A formula to help us in the calculations:

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}, \text{ where P(E)} \neq 0$$

  - essentially we are measuring the relative probability of event A with respect to the reduced space E
  - you will also see this with P(E∩A) - this is fine because **P(E∩A) = P(A∩E)**

- As a consequence, if in our probability space all events have equal probability of occurrence:

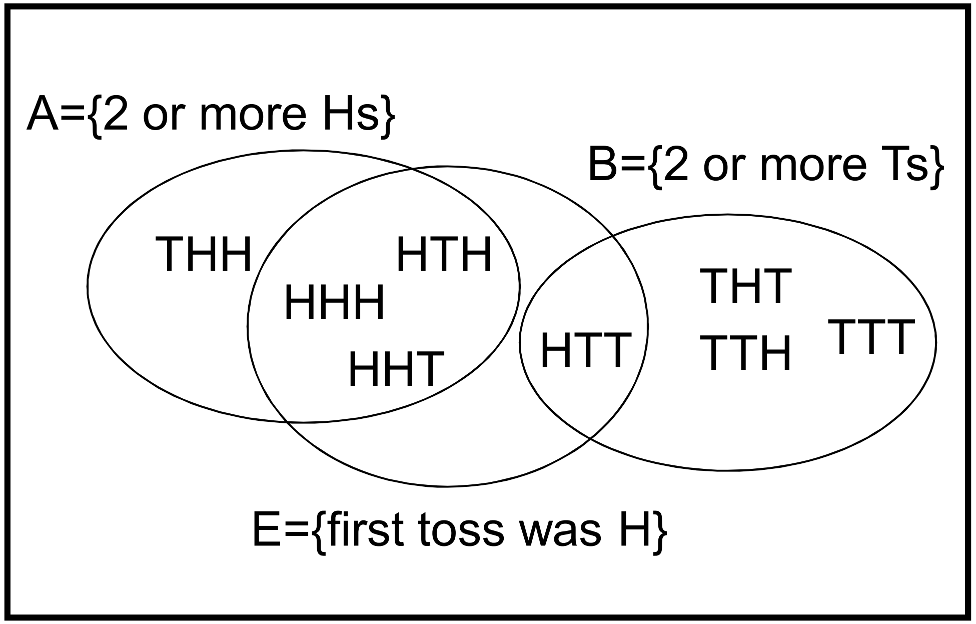
$$P(A \mid E) = \frac{\text{number of elemets in } A \cap E}{\text{number of elements in E}}$$

Example

- We toss a fair coin 3 times. I win (**event A**) if 2 or more Hs come up, you win (**event B**) if 2 or more Ts come up.
  - What is the probability that I win given that the first toss came up as H?
  - What is the probability that you win given that the first toss came up as H?

A graphical illustration

S={HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}



$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} =$$

(because the space is equiprobable) =

$$\frac{|A \cap E|}{|E|} = 3/4$$

$$P(B \mid E) = \frac{P(B \cap E)}{P(E)} =$$

(because the space is equiprobable) =

$$\frac{|B \cap E|}{|E|} = 1/4$$

The general multiplication rule

- We saw in week 1 the **multiplication rule for independent events**: $P(A \cap B) = P(A)P(B)$
- What if the events are not independent?

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} \Rightarrow P(B \cap A) = P(A \mid B)P(B)$$

  - $P(B)$ must not be zero
- We can use this formula regardless of whether events A and B are independent
  - we will see later why
- The multiplication rule can be extended to include 3 or more events (not covered here)

The 2nd test for examining event independence

- If events A and B are independent, by definition, the knowledge that one event has happened does not affect the probability of the other event happening
- **So, if A and B are independent:**
  - **$P(A|B) = P(A)$ (equally $P(B|A)= P(B)$)**
- This is our 2nd test for independence between two events

Using the new test for independence

- Rolling a single die. Events A={die comes up odd}, B={die comes up 1}, C={die comes up 1 or 2}
  - A and B independent?
  - A and C independent?
  - B and C independent?

Sample answer

- **Events A and B:** P(A)=3/6 P(A|B)=1, P(A) and P(A|B) are not equal, so the events are NOT independent
- **Events A and C:** P(A)=3/6, P(A|C)=1/2, the two probabilities are the same so the events are independent
- **Events B and C:** P(B)=1/6, P(B|C)=1/2, P(B) and P(B|C) are not equal, so the events are NOT independent

NOTE: You can always try to change the order of events, e.g. try to see if P(B|A)=P(B), this should give you the same answer as P(A|B)=P(A) with regards to the independence of the events.So, if A and B are independent so are B and A.

Conditional probabilities in practice

- Conditional probability, and especially **Bayes theorem (next week)**, are extensively applied to various computer science areas
  - Information retrieval (e.g. retrieve web pages in response to some query words)
  - Automatic Classification (e.g. is this email spam or not? is this shape a human head?)
  - Automatic translation of one language to another
  - etc. etc. etc.

Counting: Permutations and combinations

- Remember the very first formula we saw for the classical method last week?

$$P(A) = \frac{\text{number of ways A can occur}}{\text{number of ways the experiment can proceed}}$$

- When the probability problems become more complex, it becomes difficult to list all elements of event sets
- We need to find more general methods for "enumerating" the number of ways that a certain outcome can occur

Counting: Permutations and combinations

- **Permutation**: an arrangement of objects in a specific order (**order matters**)
  - Example: the number of ways that from a group of 10 students, we can select 3 to place in a specific seating order
  - Example: selecting characters for a password, or digits for a pin (pins 2345 and 2435 are different)
- **Combination:** an arrangement of objects without regard to order **(order does not matter)**
  - Example: the number of ways that from a group of 10 students, we can make committees of 3 students
  - Example: winning the lottery (the number of ways to choose 6 numbers from 49 numbers)

Permutations: with / without repetitions

- In general we can have 2 types of permutations
  - **With repetition:** e.g. passwords, pins, etc. allow the same characters/digits to be repeated
  - **Without repetition:** e.g. take the 20 football teams in the premier league and calculate all possible orderings of these teams – which team will be 1st, which 2nd, etc. - a team can not appear twice in such ordering
- Depending on whether repetitions are allowed or not, we use different formulas for counting

Permutations with repetitions

- When we have ***n*** things to choose from, we have ***n*** choices every time
- When we choose ***r*** of these, the total number of permutations is: **n x n … (r times) =** $\mathbf{n^r}$
- **Example:** For a 4-digit pin there are ***n***=10 numbers to choose from (0…9) and we choose ***r***=4 of them
  - So the total number of permutations (i.e. of 4-digit pins) is $\mathbf{n^r = 10^4 =}$ 10,000 (as you would expect, all numbers from 0000 to 9999)

Permutations without repetitions

- In this case, we have to reduce the number of available choices every time (once an item is used, it can not be used again)
- We are asked to arrange **r items chosen from n items (with no repetitions),** ***r* ≤ *n***
  - Notation: **P(n,r)** or ${}_nP_r$ or P^n_r

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutations without repetitions

- In how many ways can you arrange ***n*** people to sit in ***r*** chairs?
  - In how many ways can the first chair be filled? In n ways
  - Once the first chair is filled, in how many ways can the 2nd be filled? In $(n-1)$ ways (remember, one person already is sitting in chair 1, **and no repetitions allowed**)
  - Once the 1st and 2nd chairs are filled, there are $(n-2)$ ways in which the 3rd chair can be filled . . .
  - There will be $(n-(r-1)) = n-r+1$ ways to fill the rth (last) chair

$$\mathbf{P(n,r) = n \times n-1 \times n-2 \times \ldots \times n-r+1 = n!/(n-r)!}$$

Chair 1

Chair 2

Chair 3

Chair r

Counting permutations: factorials

- ***n* factorial**

$$n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$$

The quantity n! is called in mathematics
n factorial.

  - **by definition, 0! = 1 and also n! = n(n-1)!**
  - so 0! = 1, 1! = 1, 2!=2, 3!=6, 4!=24, 5!=120, 6!=720
    . . . .10! = 3,628,800
  - numbers increase **VERY** steeply

Permutations of *n* items chosen from *n* items (without repetitions)

- In how many ways can you arrange 5 people to sit in 5 chairs?
- By applying the general formula as:

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

P(n,n)= n x n-1 x n-2 x . . . x 1 = n!

P(5,5)= 5 x 4 x 3 x 2 x 1 = 5! = 120 ways

Exercises

- Suppose you have 6 people in a group going to the theatre, but you can select only 4 to sit in a row together. How many ways do you have to select the 4 people, and arrange them in one row in the theatre?

- In how many ways can you choose 5 distinct numbers from 0-9 to put them on a license plate?

Sample answers

- We are asked to calculate the possible permutations of r = 4 items chosen from a total of n = 6 items, P(6,4)=6!/(6-4)!=6!/2!= (6x5x4x3x2x1)/(2x1)=6x5x4x3=360
- This is essentially asking us in how many ways can we arrange 5 numbers chosen from 10 to put in the plate, n=10 and r=5, P(10,5)=10!/(10-5)! = (10x9x8x7x6x5x4x3x2x1)/(5x4x3x2x1)= (10x9x8x7x6)= 30,240

When some items are the same

- In how many ways can you arrange the letters of the word “Mississippi”? (note that the arrangements do not have to make sense!!!)
- In general, when we have ***n* items being permuted, and n_1, n_2, …, n_k the number of each of the k types of objects being involved** (e.g. the sets of letters in the example), then the number of distinguishable arrangements is given by:

$$\frac{n!}{n_1!n_2!\ldots n_k!}, \text{ where } n = n_1 + n_2 + \ldots + n_k$$

Probability problems involving permutations

- If a problem asks you to calculate the probability of an event, and not just the number of permutations:
  - you will need to calculate a numerator and a denominator
  - **Numerator:** number of ways (permutations) in which the event of interest can happen
  - **Denominator:** total number of ways (permutations) to arrange all items

Exercise

- You choose 4 different letters of the alphabet and re-arrange them in a 4-letter word. What is the probability of making the word with letters only from *a*-*m*?

  Sample answer

  The alphabet has 26 letters, so n=26. The problem asks us for a probability which we will need to calculate as: (number of ways in which we can select 4 letters from 13 letters (a-m))/(number of ways we can select 4 letters from the entire alphabet of 26 letters)

  Numerator = P(13,4) and denominator = P(26,4)

  So p= P(13,4) / P(26,4)=0.048

Combinations (without repetition)

- Similar to permutations, but here we are **not interested in the order** in which we can arrange the items
- Intuitively, the number of possible ways should be less than for permutations
- For example, the combinations of the 4 letters *a*, *b*, *c*, *d*, taken 3 at a time, are:
  - {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}
- We will use ***C*(*n*,*r*)** to mean combinations of *r* objects taken from *n* objects

Counting combinations

- The general formula is:

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

- Notice the relationship with P(*n*,*r*)?
  - to calculate **C(*n*,*r*)** we take **P(*n*,*r*)** and we divide it with the possible ways to rearrange the *r* objects (*r*!)
- We can also count combinations with repetitions, but we will not cover this here

Exercise

- A supplier ships 20 memory chips to a purchaser, of which 5 have some flaw. The purchaser will select 3 chips at random and test them, and he will accept the lot if no flaws are found. In how many ways can the supplier be saved (i.e. in how many ways can the purchaser select the 3 chips from among the working chips)?

  Sample answer

  In this problem the order in which the chips are selected does not matter, so this is a combination problem. In how many ways can the purchaser select the 3 chips from among the working chips - the working chips are 15 (20 minus the 5 problematic ones), so we are asked to calculate C(15,3) = 455

Probability problems involving combinations

- If a problem asks you to calculate the probability of an event, and not just the number of combinations:
  - you will need to calculate a numerator and a denominator
  - **Numerator:** number of ways (combinations) in which the event of interest can happen
  - **Denominator:** total number of ways (combinations) to arrange all items

Exercises

- Two cards are drawn at random from a standard 52-card deck. Find the probability *p* that:
  - Both are hearts
  - One is a heart and one is a spade
- A supplier ships 20 memory chips to a purchaser, of which 5 have some flaw. The purchaser will select 3 chips at random and test them, and he will accept the lot if no flaws are found. What is the probability *p* that the lot will be accepted?

Sample answers

- There are C(52,2)=1326 ways to draw 2 cards from the deck of 52 cards.
  - To draw 2 hearts from the 13 hearts in the deck, there are C(13,2) = 78 ways, so p=78/1326
  - To draw one heart and one spade there are C(13,1)C(13,1) =169 ways, so p=169/1326 = 0.127
- We calculated in previous slides the number of ways in which the purchaser will select the 3 chips from the 15 working chips: C(15,3)=455

  The total umber of ways in which 3 chips can be chosen from 20 chips is C(20,3)=1140, so p=455/1140 = 0.399

Summary of lecture

- In Week 2 we covered:
  - Conditional probability
  - Event independence and conditional probability
  - CS applications for conditional probability
  - Probability problems using permutations
  - Probability problems using combinations
- Don’t forget:
  - Work on the exercises for Friday **BEFORE** you come to the class