High-Order Finite-Element Waveguide Computation and Analysis

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***Abstract*—This report shows how to calculate the propagating modes and cutoff frequencies of arbitrary shaped homogeneous waveguides, through the finite element method. The programs using high order interpolation polynomials are described and the outcomes are compared with the analytic solution. Additionally, the convergence rate and relative error are discussed. It is believed that the program applying higher order finite element method produces the waveguide analysis of higher accuracy.**

I. Introduction

THE Finite Element Method (FEM) is a well-established technique widely used as an analysis and design tool in many engineering disciplines like structures and computational fluid mechanics[1]. In past 30 years, the interest in application of finite element method to microwave components such as waveguides and antennas has become stronger. The literature reporting application of FEM to empty or homogeneous waveguides appeared as early as the late 1960s[2]. In the beginning, the order of interpolation polynomials used in FEM was low and a large number of meshes were required. The application of high-order elements was first proposed by P. Silvester[3]. He suggested with high-order interpolation polynomial, the subregions need not be any more numerous than that required to describe the boundary shape. And the approximate solution of the waveguide eigenvalue problem is obtained by finding the polynomial coefficients in each triangle using a variational procedure.

In this project, relatively high-order elements (up to fourth-order) are applied to solve the eigenvalue problem. Thanks to the strong computational capability of MATLAB, the number of elements needs not be too small even though the high-order elements are utilized. The procedures of the implementation of FEM in MATLAB are described in detail and the analytic and numerical solutions are demonstrated. Besides, the benefits brought by high-order elements are presented in different ways.

II. Methods

A. Domain Discretization

The first procedure of the finite element analysis is to fix the dimension of the waveguide and divide the area domain into a number of two-dimensional elements. In this project, all the elements are triangles. For simplicity, the PDE tool in MATLAB is used to draw the cross-section of the waveguide and produce the meshes. An example is shown in Fig. 1.

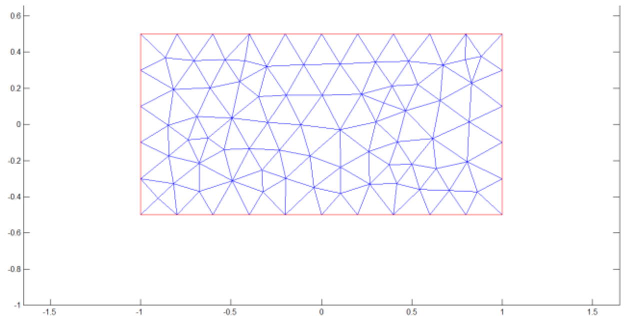

Fig 1: Mesh consists of 93 nodes and 154 triangles.

Because the error in the finite element solution is inversely proportional to the sine of the smallest inner angle, an ideal element is an equilateral triangle[4]. In PDE tool, we can see the quality of each element, which represents how equilateral the triangular element is. If the overall quality of elements is not acceptable, the meshes can be jiggled to get the higher quality.

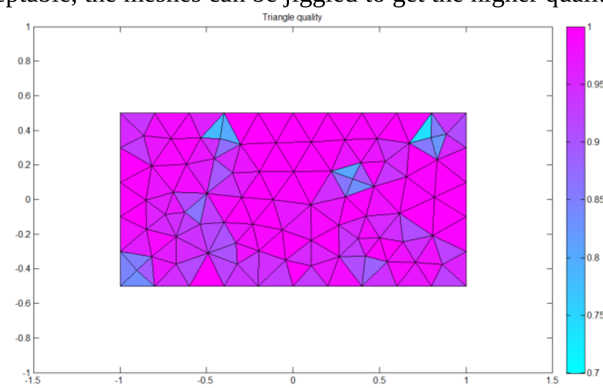

Fig 2: Quality of each element.

After exporting the meshes, three matrixes are obtained, p, e and t. The matrix p contains *x*- and *y*-coordinates of the points in the mesh. The matrix e includes indices of the points on the boundary and the matrix t contains indices to the corner points of each element, given in counter clockwise order.

The problem is that PDE tool is only able to generate the linear triangle element that cannot be used in high order FEM unless some modifications are made. The example of second-order and third-order elements is presented in Fig.3.

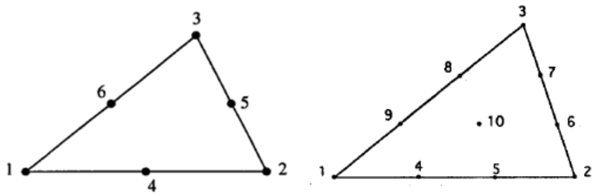

Fig 3: Second-order and third-order elements[4].

To convert the first-order element into high-order element, first, we need to get all the sides of the first-order element which is defined by two vertexes, and then insert the nodes in each side. For the element having order higher than 3, some nodes also need to be insert in the element. The coordinates of the new nodes on sides are determined by the position of the three vertexes while the inside nodes positions are determined by the nodes on sides. A better way to define the boundary nodes positions is putting the nodes just on the boundary instead of the middle of two vertexes as it's shown in Fig. 4, which will produce a more accurate solution for the waveguide having curved boundary. Finally, the local and global node numbers are distributed to these nodes.

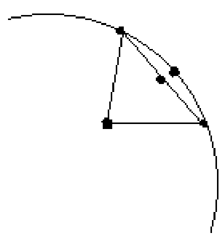

Fig 4: A way to set the nodes on the boundary.

B. *Element Interpolation*

After the domain discretization, the unknown function Φ needs to be approximated. The easiest way is to use the linear polynomial.

$$\Phi^e(x,y) = a^e + b^e x + c^e y \tag{1}$$

For the high-order element, the interpolation function is

$$\Phi^e(x,y) = \sum_{j=1}^{n} N_j^e(x,y)\Phi_j^e \tag{2}$$

where n is the number of the nodes already known in each element and N_j^e are the interpolation functions given by

$$N_i^e = P_I^n(L_1^e)P_J^n(L_2^e)P_K^n(L_3^e) \quad I+J+K=n \tag{3}$$

L_1, L_2 and L_3 are all functions of x and y, thus, the expression of N_i^e in *xy*-plane is complex.

To model the curved boundaries accurately, we should not use the elements with straight sides. Fig.5 demonstrates an isoparametric element with curved sides in *xy*-plane is converted into a right triangle in $\xi\eta -$ plane.

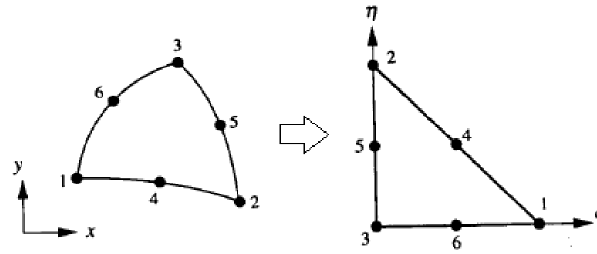

Fig 5: Transformation of a triangular element with curved sides[4].

The inverse transformation is defined as

$$x = \sum_{j=1}^{n} N_j^e(\xi,\eta)\, x_j \tag{4}$$

$$y = \sum_{j=1}^{n} N_j^e(\xi,\eta)\, y_j \tag{5}$$

where $\xi = L_2^e$, $\eta = L_3^e$ and the associated $L_1^e = 1 - \xi - \eta$. The interpolation equation (2) becomes

$$\Phi^e(\xi,\eta) = \sum_{j=1}^{n} N_j^e(\xi,\eta)\Phi_j^e \tag{6}$$

If the interpolation function and the geometrical transformation have the same order, the elements are called isoparametric elements[4]. High-order elements with straight edges are geometrically identical to linear elements, even though they can be used with high-order interpolating functions[5]. In this situation, the elements are called subparametric elements. In fact the subparametric element is a special case of the isoparametric elements. To make sure the program can deal with generous problems, we regard all the elements as isoparametric even though most of them have straight edges.

C. *Derivation of Element Equation*

The functional F in the project is defined as[4]

$$F(\phi) = \frac{1}{2}\iint_{\Omega} \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 - k_t\phi^2\right] d\Omega \tag{7}$$

The elemental matrix $[K^e]$ can be derived from equation (7)[4]

$$K_{ij}^e = \iint_{\Omega^e} \left(\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} - k_t^2 N_i^e N_j^e\right) d\Omega \tag{8}$$

by defining[4]

$$A_{ij}^e = \iint_{\Omega^e} \left(\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\right) d\Omega \tag{9}$$

and

$$B_{ij}^e = \iint_{\Omega^e} N_i^e N_j^e \, d\Omega \tag{10}$$

The problem is simplified to an eigenvalue equation.

$$[A]\{\phi\} = k_t^2 [B]\{\phi\} \tag{11}$$

As I mentioned in section B, the isoparametric elements algorithm is applied. Therefore, we need to change A_{ij} and B_{ij}

to the integrals in $\xi\eta -$ plane. By applying the chain rule, we get the partial derivative of N_i^e.

$$\frac{\partial N_i^e}{\partial \xi} = \sum_{i=1}^{3} \frac{\partial N_i^e}{\partial L_i} \frac{\partial L_i}{\partial \xi} \tag{12}$$

$$\frac{\partial N_i^e}{\partial \eta} = \sum_{i=1}^{3} \frac{\partial N_i^e}{\partial L_i} \frac{\partial L_i}{\partial \eta} \tag{13}$$

Because[4]

$$\frac{\partial N_i^e}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial N_i^e}{\partial \eta}\right) \tag{14}$$

$$\frac{\partial N_i^e}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial N_i^e}{\partial \eta}\right) \tag{15}$$

A and B can be calculated in $\xi\eta -$ plane.

The final step to get matrix A and B is to calculate the integral of $\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}$ and $N_i^e N_j^e$. We cannot compute these integrals directly and numerical methods have to be implemented. A widely used integral scheme is Gaussian quadrature[4]. For the integration over a triangular element, it can be shown[6]

$$\iint_{\Omega^e} F(L_1^e, L_2^e, L_3^e)dx\, dy = \sum_{i=1}^{m} W_i F(L_{1i}^e, L_{2i}^e, L_{3i}^e)\Delta^e \tag{16}$$

where Δ^e is the area of the triangular element. In normalized $\xi\eta -$ plane, Δ^e is 0.5.

Since high order element is used, we must choose the sampling points and the weight very carefully to ensure the results from Gaussian quadrature are reliable. In this project the highest order of elements is 4. Thus, the highest polynomial degrees of $N_i^e N_j^e$ and $\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}$ are 8 and 6 respectively. The sampling points and the weights employed in this project are shown in Table 1 and 2.

weight	L_{1i}^e	L_{2i}^e	L_{3i}^e
0.22500000	0.33333333	0.33333333	0.33333333
0.13239415	0.05961587	0.47014206	0.47014206
0.12593918	0.79742699	0.10128651	0.10128651

Table 1: Sampling points number m=7 and the highest achieved polynomial degree p=5 [7].

weight	L_{1i}^e	L_{2i}^e	L_{3i}^e
0.14431561	0.33333333	0.33333333	0.333333
0.095091634	0.081414823	0.45929259	0.459293
0.10321737	0.65886138	0.17056931	0.1705693
0.032458498	0.89890554	0.050547228	0.0505472
0.027230314	0.008394777	0.26311283	0.7284924

Table 2: Sampling points number m=16 and the highest achieved polynomial degree p=8 [7].

It is obvious in the tables that more sampling points are required for the higher order elements. Consequently, the computational work becomes heavier.

D. *Incorporation of the Boundary Condition*

In the homogeneously filled waveguide, there are only two distinct set of modes: TE and TM. The boundary conditions at the waveguide wall for TE and TM modes are different, one is Neumann condition, and the other is Dirichlet condition.

The functional F defined in equation (7) already absorbs the homogeneous Neumann boundary condition. Otherwise, F would be defined as[4]

$$F(\phi) = \frac{1}{2}\iint_{\Omega} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 - k_t \phi^2\right] d\Omega + \int_{\Gamma_2} \left(\frac{\Upsilon}{2}\Phi^2 - q\Phi\right) d\Gamma \tag{17}$$

Thus, Neumann boundary condition is satisfied automatically and implicitly.

For TM mode, ϕ is the electrical field whose value is zero on the boundary. To impose Dirichlet boundary condition, we simply delete the columns and rows of K_a and K_b corresponding to the boundary nodes.

E. *Solution of the Eigenvalue Problem*

After these procedures, the resultant system of equation has the form of the generalized eigenvalue equation[4].

$$[A]\{\phi\} - \lambda[B]\{\phi\} = \{0\} \tag{18}$$

According to the Cholesky decomposition, B matrix is factorized as $B = LL^T$ and equation (18) is converted to

$$L^{-1}AL^{-T}y = \lambda y, \qquad y = L^T x \tag{19}$$

Because $L^{-1}AL^{-T}$ is a symmetric matrix, the traditional method can be used to solve this eigenvalue problem.

III. Results and Discussions

In this project, programs are written to generate the high-order mesh from the first-order mesh and to calculate TE and TM modes in a rectangular waveguide. The resultant high-order mesh and electromagnetic field are presented separately.

A. *High Order Meshes*

The second to fourth order meshes are illustrated in Fig.6, Fig.7 and Fig.8. The triangles having black edges are the first-order meshes that are identical to the meshes shown in Fig.1. Each second-order element has total 6 nodes, the third-order element has 10 and the fourth-order element has 15. When the order of elements increases, the number of unknowns becomes larger and larger, which means the computational work becomes heavier and heavier.

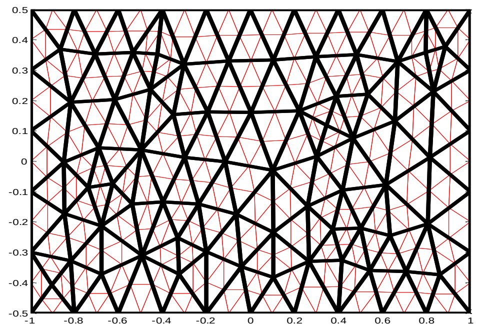

Fig 6: Second-order meshes, number of nodes is 339.

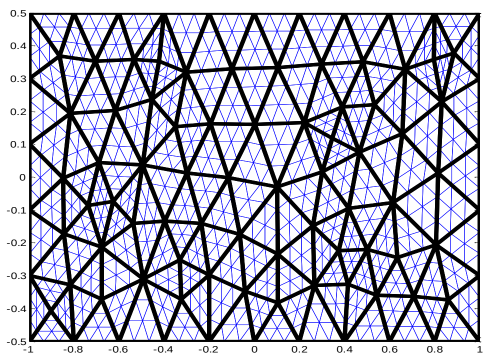

Fig 7: Third-order meshes, number of nodes is 739.

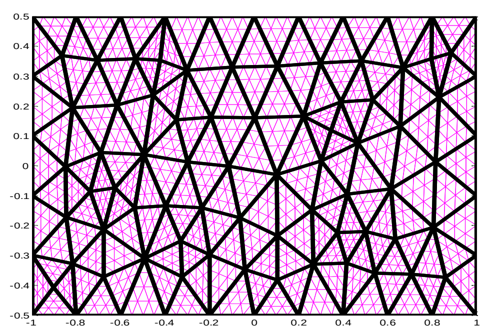

Fig 8: Fourth-order meshes, number of nodes is 1293.

B. *TE and TM Modes*

First the analytic solutions of the rectangular waveguide are presented. The EM wave is transmitted in z axis.

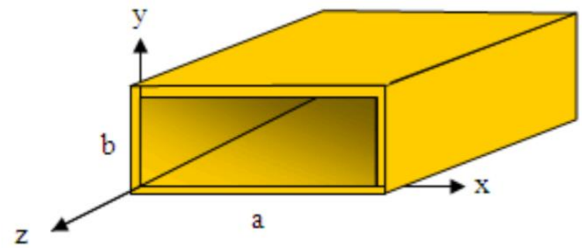

Fig 9: A rectangular waveguide with perfectly conducting walls[8].

In TM modes, the electric field in z direction is[9]

$$E_{zs} = E_o sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{20}$$

and the magnetic field in TE modes is[9]

$$H_{zs} = H_o cos\left(\frac{m\pi x}{a}\right) cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \tag{21}$$

The cut-off frequency is[9]

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2} \tag{22}$$

The eigenvalues calculated through FEM are $\mathrm{k_t} = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$. The analytic solution is compared with numerical solution to present how accurate the finite element analysis is when the cut-off frequency becomes higher.

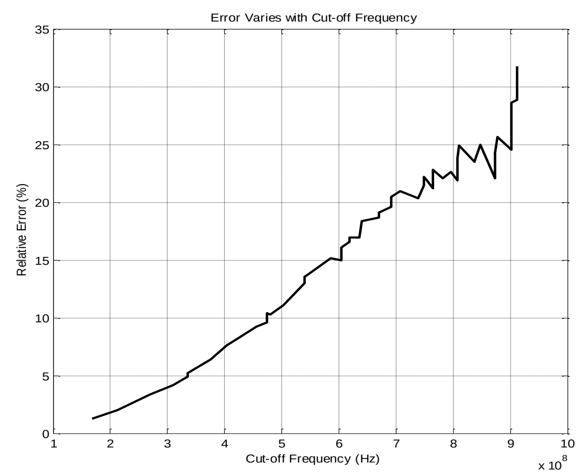

Fig 10: The relative error of first-order FEM when 63 unknowns are calculated.

Fig.10 illustrates how the relative error varies with the cut-off frequency. The ratio between the length *a* and width *b* of the empty waveguide is 2. The numerical solution is obtained from first-order FEM. In general, the relative error increases with the cut-off frequency because the high frequency electromagnetic field is very hard to model. To achieve higher accuracy in finite element solution, we have to resort to smaller elements and higher-order interpolation function.

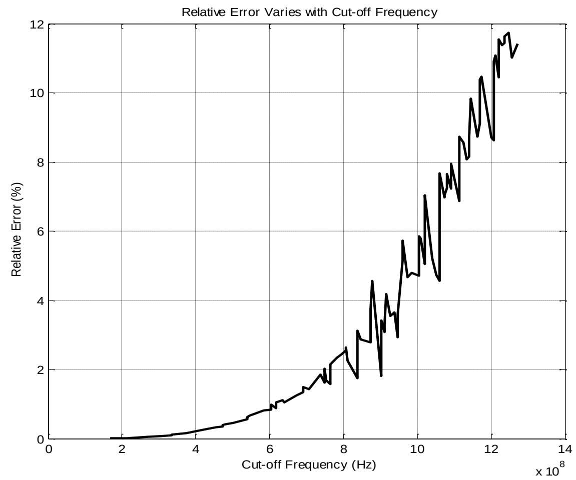

Fig 11: The relative error of second-order FEM when 279 unknowns are calculated.

Compared with Fig.10, Fig.11 presents a much better result where the relative error is less than 6% when the cut-off frequency is up to 1 GHz.

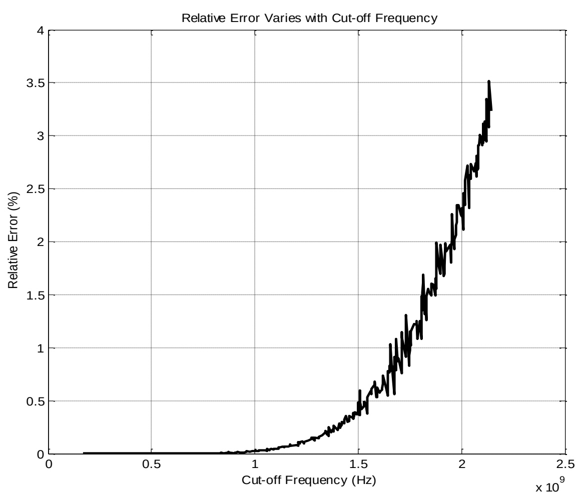

Fig 12: The relative error of fourth-order FEM when 1173 unknowns are calculated.

If we continue to increase the order of the element, we can get more accurate result that is shown in Fig.12. Even if the cut-off frequency is as high as 2 GHz, the relative error is still less than 2.5%. Especially at frequency lower than 1 GHz, the relative error is negligible.

Until now the accuracy of the results produced by the high-order FEM is displayed. The disadvantage of high-order FEM is more unknowns have to be calculated, which means the matrixes A and B in equation (18) have large dimensions and the eigenvalue problem is difficult to solve.

We may wonder whether the accuracy of the numerical solution improves if the number of unknowns used in low-order FEM is the same as that used in the high-order FEM. The relative errors versus cut-off frequency are shown in Fig.13 and Fig.14, where the number of unknowns calculated is 1173.

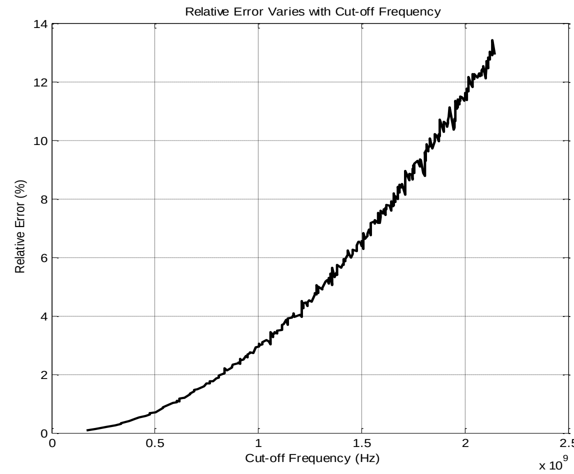

Fig 13: The relative error of first-order FEM when 1173 unknowns are calculated.

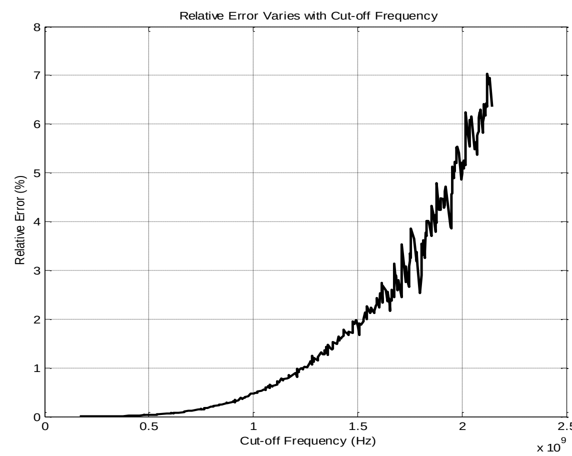

Fig 14: The relative error of second-order FEM when 1173 unknowns are calculated.

Comparing Fig.10 and Fig.13, we find the relative error decreases because more elements are utilized and more unknowns are calculated for Fig.13. The same conclusion can be drawn by comparing Fig.11 and Fig.14. But when we compare Fig.13 and Fig.14 with Fig.12, where the numbers of unknowns are the same, we find the relative errors in the first two figures are much larger than that in Fig.12. Therefore, we reach the conclusion that high order finite element analysis brings more accurate result even though the number of unknowns computed in the eignvalue problem does not change.

Additionally, ‘tic’ and ‘toc’ commands in MATLAB are used to record CPU time of running programs. The result shows that the running time for second-order and fourth-order program is 146.30s and 148.52s respectively. This is because the heaviest computational work in the program is solving the eigenvalue program. The numbers of unknowns are the same in second-order and fourth-order programs. Therefore, the CPU time used by these two programs is almost the same. Since the high-order analysis produces more accurate result without taking up too much extra CPU time, we always prefer to use the high order elements.

C. *TM32 Mode Analysis*

Now the high-order program is applied to solve the cut-off frequency of TM32 mode for a 2:1 rectangular waveguide. Besides, the 'accuracy versus cost' results are explained in detail. First, the analytic solution is presented in Fig.15.

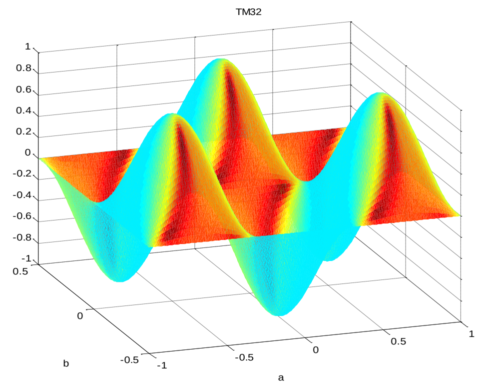

Fig 15: Electric field of TM32 mode.

Fig.16 and 17 are obtained by the program based on fourth-order FEM, which illustrate clearly that more unknowns are calculated, nearer the numerical solution approaches to the analytic one.

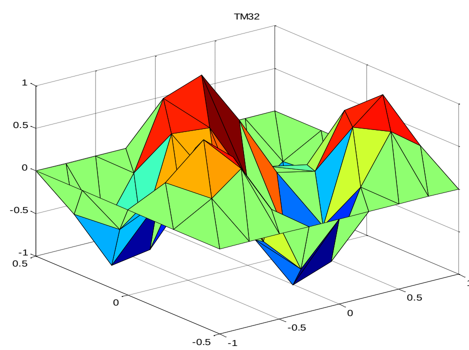

Fig 16: Electric field obtained by solving 53 unknowns.

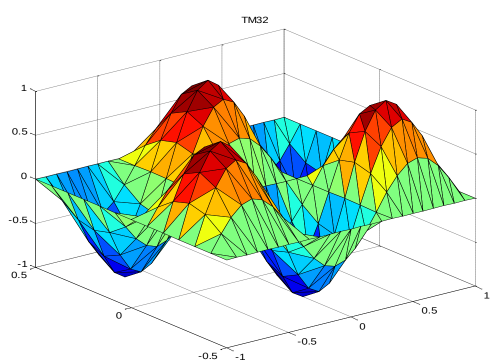

Fig 17: Electric field obtained by solving 253 unknowns.

The true value of cut-off frequency of TE32 is 374.75 MHz. Table.3 shows how the numerical solution approaches the true value

Unknowns	21	53	105	253
Cut-off Frequency (MHz)	400.69	377.29	375.39	374.78
CPU time (s)	0.1524	0.3003	0.6788	2.3427

Table 3: Cut-off frequency varies with the number of unknowns based on fourth-order FEM.

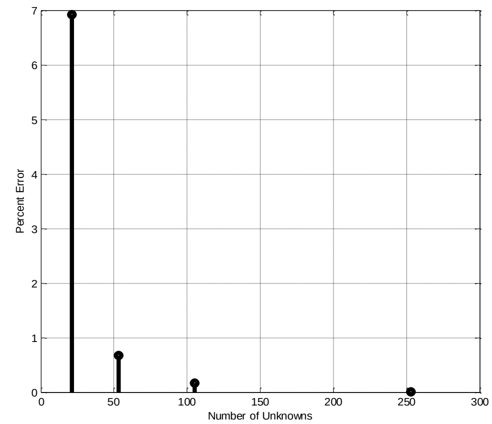

Fig 18: Relative error versus number of unknowns based on fourth-order FEM.

Now different meshes are inputted to the second-order program and Fig. 19 and Table 4 are obtained.

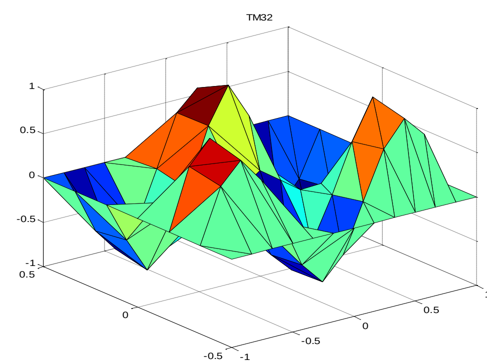

Fig 19: Electric field obtained by solving 55 unknowns.

Unknowns	11	21	55	127	253
Cut-off Frequency (MHz)	499.18	430.62	387.03	377.28	375.85
CPU time (s)	0.0254	0.0407	0.0814	0.2015	1.1299

Table 4: Cut-off frequency varies with the number of unknowns based on second-order FEM.

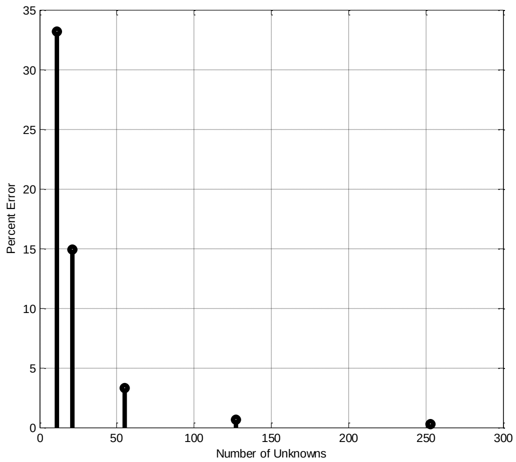

Fig 20: Relative error versus number of unknowns based on second-order FEM.

Comparing Table 3 and Table 4, we find the accuracy of the numerical solution improves when the computational cost becomes larger, but the numerical solution approaches to the analytic solution slower and slower. Therefore, we can only calculate a certain number of unknowns to get the acceptable result. If the relative error is already very small, lots of computational work needs to be done to further enhance the accuracy.

IV. Conclusion

In this project, the high-order FEM is deeply studied and the programs are written in MATLAB to solve the electromagnetic field in a rectangular homogenous waveguide. Besides, the solutions from the programs applying different order elements show that higher elemental order brings the better solution accuracy without adding too much computational cost. The relation between the accuracy of numerical solution and the computational cost is also presented based on second-order and fourth-order FEM. The numerical and analytic solutions of TE32 mode are demonstrated. Furthermore, the programs written for this project are usable for any waveguide shape whose boundary is made up of straight-line segments. With some modifications, the program is capable of analyzing inhomogeneous waveguides.

References

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