The Perceptron

YuHua Li(李玉华)

Intelligent and Distributed Computing Lab,
Huazhong University of Science & Technology

idcliyuhua@hust.edu.cn

2019.05

Table of Contents

Table of Contents

Assumptions

Basic idea:

In machine learning, the perceptron is an algorithm for supervised learning of binary classifiers. A binary classifier is a function which can decide whether or not an input, represented by a vector of numbers, belongs to some specific class. It is a type of linear classifier, i.e. a classification algorithm that makes its predictions based on a linear predictor function combining a set of weights with the feature vector.

- Binary classification (i.e. $y_i \in \{-1, +1\}$)
- Data is linearly separable

A binary classification example

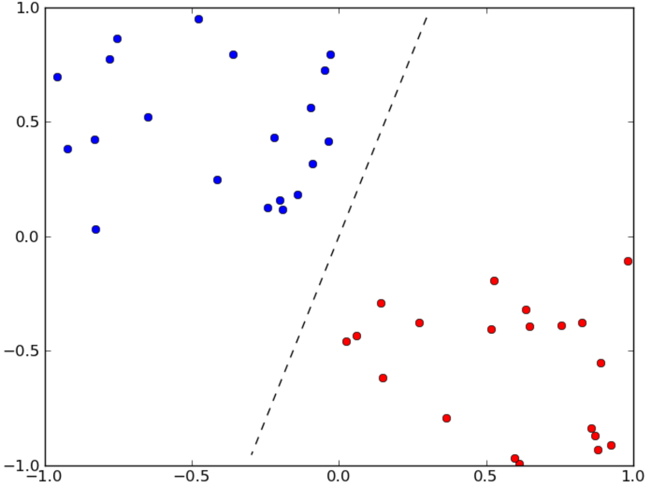
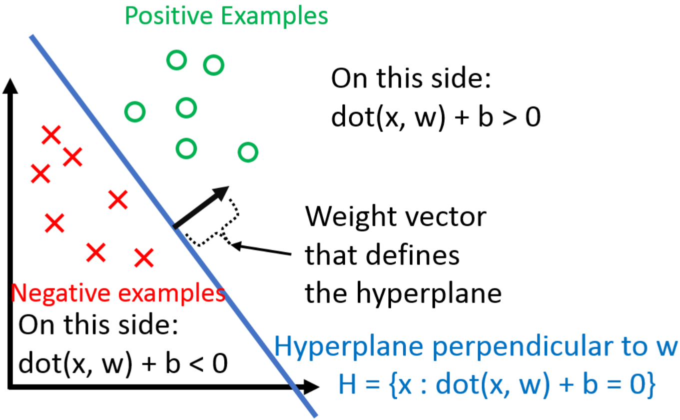


Table of Contents

$$h(x_i) = \mathrm{sign}(\mathbf{w}^\top \mathbf{x}_i + b)$$



Parameter selection

b is the bias term (without the bias term, the hyperplane that $\mathbf{w}$ defines would always have to go through the origin). Dealing with b can be a pain, so we 'absorb' it into the feature vector $\mathbf{w}$ by adding one additional constant dimension. Under this convention,

$$\mathbf{x}_i \quad \text{becomes} \quad \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix}$$

$$\mathbf{w} \quad \text{becomes} \quad \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix}$$

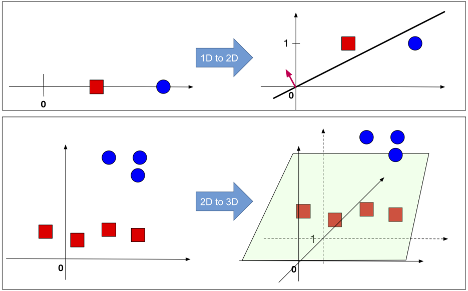
We can verify that

$$\begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix} = \mathbf{w}^\top \mathbf{x}_i + b$$

Hyperplane

Using this, we can simplify the above formulation of $h(\mathbf{x}_i)$ to

$$h(\mathbf{x}_i) = \mathrm{sign}(\mathbf{w}^\top \mathbf{x})$$



(Left:) The original data is 1-dimensional (top row) or 2-dimensional (bottom row). There is no hyper-plane that passes through the origin and separates the red and blue points.

(Right:) After a constant dimension was added to all data points such a hyperplane exists.

Observation

Note that

$$y_i(\mathbf{w}^\top \mathbf{x}_i) > 0 \Longleftrightarrow \mathbf{x}_i \quad \text{is classified correctly}$$

where 'classified correctly' means that x_i is on the correct side of the hyperplane defined by $\mathbf{w}$. Also, note that the left side depends on $y_i \in \{-1, +1\}$(it wouldn't work if, for example $y_i \in \{0, +1\}$).

KNN vs. Perceptron

KNN	Perceptron
non-linear	linear
Binary/Multi-class/Regression	Binary Class
$n \to \infty$, accurate $\uparrow$	no performance guarantees as data increase
$O(nd)$	not rely on n or d

Table: Difference between KNN and Perceptron

Table of Contents

Algorithm

Now that we know what the **w** is supposed to do (defining a hyperplane the separates the data), let's look at how we can get such **w**.

Initialize $\vec{w} = \vec{0}$	// Initialize $\vec{w}$. $\vec{w} = \vec{0}$ misclassifies everything.
while TRUE **do**	// Keep looping
$m = 0$	// Count the number of misclassifications, m
for $(x_i, y_i) \in D$ **do**	// Loop over each (data, label) pair in the dataset, D
if $y_i(\vec{w}^T \cdot \vec{x}_i) \leq 0$ **then**	// If the pair $(\vec{x}_i, y_i)$ is misclassified
$\vec{w} \leftarrow \vec{w} + y\vec{x}$	// Update the weight vector $\vec{w}$
$m \leftarrow m + 1$	// Counter the number of misclassification
end if	
end for	
if $m = 0$ **then**	// If the most recent $\vec{w}$ gave 0 misclassifications
break	// Break out of the while-loop
end if	
end while	// Otherwise, keep looping!

Geometric Intuition

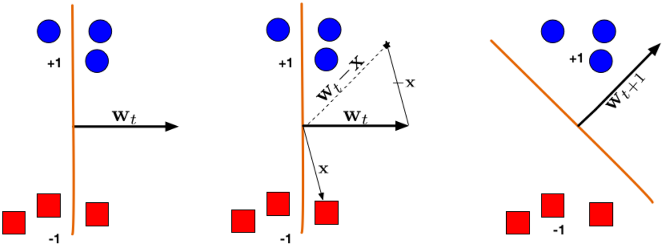


Illustration of a Perceptron update.(Left:) The hyperplane defined by $\mathbf{w}_t$ misclassifies one red (-1) and one blue (+1) point. (Middle:) The red point $\mathbf{x}$ is chosen and used for an update. Because its label is -1 we need to **subtractx** from $\mathbf{w}_t$. (Right:) The udpated hyperplane $\mathbf{w}_{t+1} = \mathbf{w}_t - \mathbf{x}$ separates the two classes and the Perceptron algorithm has converged.

Quiz

Assume a data set consists only of a single data point $\{(\mathbf{x}, +1)\}$. How often can a Perceptron misclassify this point $\mathbf{x}$ repeatedly? What if the initial weight vector $\mathbf{w}$ was initialized randomly and not as the all-zero vector?

Table of Contents

Perceptron Convergence

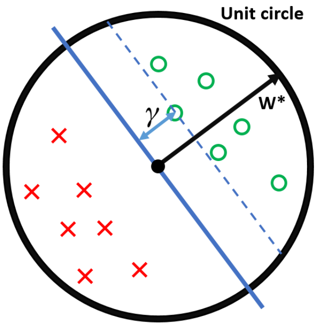
The Perceptron was arguably the first algorithm with a strong formal guarantee. If a data set is linearly separable, the Perceptron will find a separating hyperplane in a finite number of updates. (If the data is not linearly separable, it will loop forever.)

The argument goes as follows: Suppose $\exists \mathbf{w}^*$ such that $y_i(\mathbf{x}^\top \mathbf{w}^*) > 0 \; \forall (\mathbf{x}_i, y_i) \in D$

Now, suppose that we rescale each data point and the $\mathbf{w}^*$ such that

$$||\mathbf{w}^*|| = 1 \qquad \text{and} \qquad ||\mathbf{x}_i|| \leq 1 \quad \forall \mathbf{x}_i \in D$$

Let us define the Margin γ of the hyperplane $\mathbf{w}^*$ as $\gamma = \min_{(\mathbf{x}_i, y_i) \in D} |\mathbf{x}_i^\top \mathbf{w}^*|$.



To summarize our setup:

- All inputs $\mathbf{x}_i$ live within the unit sphere
- There exists a separating hyperplane defined by $\mathbf{w}^*$, with $\|\mathbf{w}\|^* = 1$ (i.e. $\mathbf{w}^*$ lies exactly on the unit sphere).
- γ is the distance from this hyperplane (blue) to the closest data point.

Theorem and Proof

Theorem: If all of the above holds, then the perceptron algorithm makes at most $1/\gamma^2$ mistakes.

Proof: Keeping what we defined above, consider the effect of an update ($\mathbf{w}$ becomes $\mathbf{w} + y\mathbf{x}$) on the two terms $\mathbf{w}^\top \mathbf{w}^*$ and $\mathbf{w}^\top \mathbf{w}$. We will use two facts:

- $y(\mathbf{x}^\top \mathbf{w}) \leq 0$: This holds because $\mathbf{x}$ is misclassified by $\mathbf{w}$ - otherwise we wouldn't make the update.
- $y(\mathbf{x}^\top \mathbf{w}^*) > 0$: This holds because $\mathbf{w}^*$ is a separating hyper-plane and classifies all points correctly.

Theorem and Proof

1.Consider the effect of an update on $\mathbf{w}^\top \mathbf{w}^*$:

$$(\mathbf{w} + y\mathbf{x})^\top \mathbf{w}^* = \mathbf{w}^\top \mathbf{w}^* + y(\mathbf{x}^\top \mathbf{w}^*) \geq \mathbf{w}^\top \mathbf{w}^* + \gamma$$

The inequality follows from the fact that, for $\mathbf{w}^*$, the distance from the hyperplane defined by $\mathbf{w}^*$ to $\mathbf{x}$ must be at least γ (i.e. $y(\mathbf{x}^\top \mathbf{w}^*) = |\mathbf{x}^\top \mathbf{w}^*| \geq \gamma$).

This means that for each update, $\mathbf{w}^\top \mathbf{w}^*$ grows by **at least** γ.

2.Consider the effect of an update on $\mathbf{w}^\top \mathbf{w}$:

$$(\mathbf{w} + y\mathbf{x})^\top (\mathbf{w} + y\mathbf{x}) = \mathbf{w}^\top \mathbf{w} + \underbrace{2y(\mathbf{w}^\top \mathbf{x})}_{<0} + \underbrace{y^2(\mathbf{x}^\top \mathbf{x})}_{0\leq \quad \leq 1} \leq \mathbf{w}^\top \mathbf{w} + 1$$

The inequality follows from the fact that

- $2y(\mathbf{w}^\top \mathbf{x}) < 0$ as we had to make an update, meaning $\mathbf{x}$ was misclassified
- $0 \leq y^2(\mathbf{x}^\top \mathbf{x}) \leq 1$as $y^2 = 1$ and all $\mathbf{x}^\top \mathbf{x} \leq 1$ (because $\|\mathbf{x}\| \leq 1$).

This means that for each update, $\mathbf{w}^\top \mathbf{w}$ grows by **at most 1**.

3.Now we can put together the above findings. Suppose we had M updates.

$$
\begin{align}
M\gamma &\leq \mathbf{w}^\top \mathbf{w}^* && \text{By first point} \tag{1}\\
&= |\mathbf{w}^\top \mathbf{w}^*| && \text{Simply because } M\gamma \geq 0 \tag{2}\\
&\leq ||\mathbf{w}||\ ||\mathbf{w}^*|| && \text{By Cauchy-Schwartz inequality}^* \tag{3}\\
&= ||\mathbf{w}|| && \text{As } ||\mathbf{w}^*|| = 1 \tag{4}\\
&= \sqrt{\mathbf{w}^\top \mathbf{w}} && \text{by definition of } \|\mathbf{w}\| \tag{5}\\
&\leq \sqrt{M} && \text{By second point} \tag{6}\\
& \tag{7}\\
&\Rightarrow M\gamma \leq \sqrt{M} \tag{8}\\
&\Rightarrow M^2\gamma^2 \leq M \tag{9}\\
&\Rightarrow M \leq \frac{1}{\gamma^2} \tag{10}
\end{align}
$$

And hence, the number of updates M is bounded from above by a constant.

*Alternative explanation: $|\mathbf{w}^\top \mathbf{w}^*| = \|\mathbf{w}\|\|\mathbf{w}^*\||\cos(\alpha)|$, but $|\cos(\alpha)| \leq 1$

Quiz

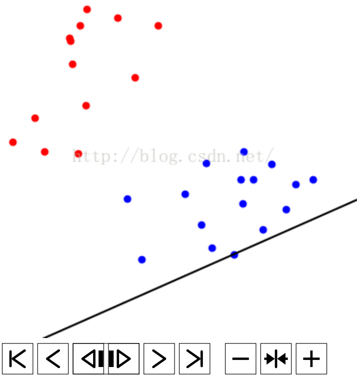
Given the theorem above, what can you say about the margin of a classifier (what is more desirable, a large margin or a small margin?) Can you characterize data sets for which the perceptron algorithm will converge quickly? Draw an example.

Table of Contents

Perceptron example

Click here: Perceptron example python code

Sample Animation



References:

1) CS4780 course of Cornell University, taught by Prof. Kilian Weinberger.

2) Prof. Kun He's teaching of machine learning at HUST basing on CS4780 of Cornell.

The End