Conclusion of chapter 1 & 2: Probability Reasoning & Basic Graph Concepts

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1 Prob

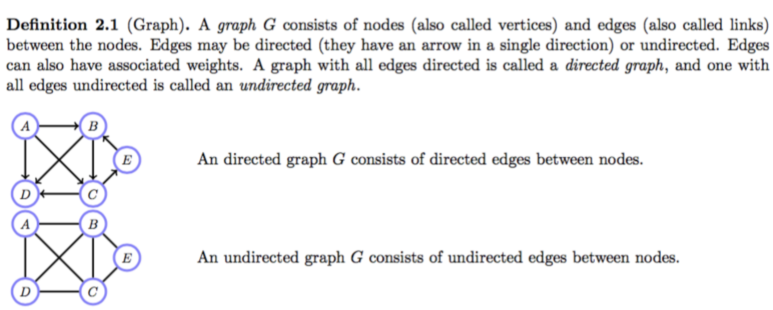
1. Definition 1.4 (Conditional Probability / Bayes Rule).

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} \tag{1}$$

2. Definition 1.7 (Conditional Independence). $X \perp\!\!\!\perp Y | Z$, if $p(X, Y|Z) = p(X|Z)p(Y|Z)$.

2 Graphic

1. Definition 2.1(Directed and Undirected Graph).



Definition 2.1 (Graph)**.** A *graph* G consists of nodes (also called vertices) and edges (also called links) between the nodes. Edges may be directed (they have an arrow in a single direction) or undirected. Edges can also have associated weights. A graph with all edges directed is called a ***directed graph***, and one with all edges undirected is called an ***undirected graph***.

An directed graph G consists of directed edges between nodes.

An undirected graph G consists of undirected edges between nodes.

2. Definition 2.2(Ancestors/ Parents and Descendants/ Children).A path $A \mapsto B$ from node A to node B is a sequence of nodes that connects

A to B. That is, a path is of the form $A_0, A_1, ..., A_{n-1}, A_n$, with $A_0 = A$ and $A_n = B$ and each edge $(A_{k-1}, A_k), k = 1, ..., n$ being in the graph. A directed path is a sequence of nodes which when we follow the direction of the arrows leads us from A to B. In directed graphs, the nodes A such that $A \mapsto B$ and $B \not\mapsto A$ are the ancestors of B. The nodes B such that $A \mapsto B$ and $B \not\mapsto A$ are the descendants of A.

3. Definition 2.3 (Cycle, loop and chord). A cycle is a directed path that starts and returns to the same node $a \to b \to ... \to z \to a$. A loop is a path containing more than two nodes, irrespective of edge direction, that starts and returns to the same node. For example in fig(2.2b) $1-2-4-3-1$ forms a loop, but the graph is acyclic (contains no cycles). A chord is an edge that connects two non-adjacent nodes in a loop for example, the $2-3$ edge is a chord in the $1-2-4-3-1$ loop of fig(2.2a).

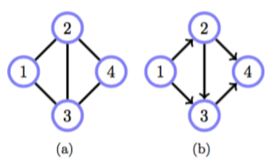


Figure 2.2: **(a)**: An undirected graph can be represented as a symmetric adjacency matrix. **(b)**: A directed acyclic graph with nodes labelled in ancestral order corresponds to a triangular adjacency matrix.

4. Definition 2.4 (Directed Acyclic Graph (DAG)). A DAG is a graph G with directed edges (arrows on each link) between the nodes such that by following a path of nodes from one node to another along the direction of each edge no path will revisit a node. In a DAG the ancestors of B are those nodes who have a directed path ending at B. Conversely, the descendants of A are those nodes who have a directed path starting at A.

5. Definition 2.5 (Relationships in a DAG).

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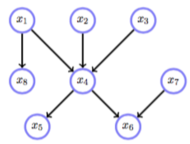

The *parents* of x_4 are $\text{pa}(x_4) = \{x_1, x_2, x_3\}$. The *children* of x_4 are $\text{ch}(x_4) = \{x_5, x_6\}$. The *family* of a node is itself and its parents. The *Markov blanket* of a node is its parents, children and the parents of its children (excluding itself). In this case, the Markov blanket of x_4 is $x_1, x_2, x_3, x_5, x_6, x_7$.

6. Definition 2.6 (Neighbour). For an undirected graph G the neighbours of x, ne (x) are those nodes directly connected to x.

7. Definition 2.7 (Clique).

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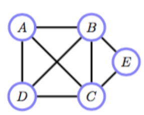

Given an undirected graph, a clique is a fully connected subset of nodes. All the members of the clique are neighbours; for a maximal clique there is no larger clique that contains the clique. For example this graph has two maximal cliques, $\mathcal{C}_1 = \{A, B, C, D\}$ and $\mathcal{C}_2 = \{B, C, E\}$. Whilst A, B, C are fully connected, this is a non-maximal clique since there is a larger fully connected set, A, B, C, D that contains this. A non-maximal clique is sometimes called a *cliquo*.

8. Adjacency matrix

An alternative is to use an adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

where $A_{ij} = 1$ if there is an edge from node i to node j in the graph, and 0 otherwise. Some authors include self-connections and place 1s on the diagonal in this definition. An undirected graph has a symmetric adjacency matrix.

Provided that the nodes are labelled in ancestral order (parents always come before children) a directed graph fig(2.2b) can be represented as a triangular adjacency matrix:

$$T = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

9. Clique matrix

For an undirected graph with N nodes and maximal cliques $C_1, ..., C_K$ a clique matrix is an $N \times K$ matrix in which each column c_k has zeros except for ones on entries describing the clique. For example:

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a clique matrix for fig(2.2)(a). A cliquo matrix relaxes the constraint that cliques are required to be maximal1. A cliquo matrix containing only two-node cliques is called an incidence matrix. For example:

$$C_{int} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is an incidence matrix for fig(2.2a). It is straightforward to show that $C_{inc}C_{inc}^T$ is equal to the adjacency matrix except that the diagonals now contain the degree of each node (the number of edges it touches). Similarly, for any cliquo matrix the diagonal entry of $[CC^T]_{ii}$ expresses the number of cliquos (columns) that node i occurs in. Off diagonal elements $[CC^T]_{ij}$ contain the number of cliquos that nodes i and j jointly inhabit.