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Positive Mathematical Programming

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A method for calibrating models of agricultural production and resource use using nonlinear yield or cost functions is developed. The nonlinear parameters are shown to be implicit in the observed land allocation decisions at a regional or farm level. The method is implemented in three stages and initiated by a constrained linear program. The procedure automatically calibrates the model in terms of output, input use, objective function values and dual values on model constraints. The resulting nonlinear models show smooth responses to parameterization and satisfy the Hicksian conditions for competitive firms.

Key words: calibration, mathematical programming, nonlinear optimization, production model, sectoral model.

This paper is a methodological paper for practitioners rather than theorists. Instead of a new method that requires additional data, I take a different perspective on mathematical programming using a more flexible specification than traditional linear constraints. Sometimes new methodologies are published, but not implemented. Positive mathematical programming (PMP) is a methodology that has been implemented but not published. Over the past eight years the PMP approach has been used on several policy models at the sectoral, regional and farm level. National sectoral models using PMP for the U.S., Canada, and Turkey include House; Ribaudo, Osborn, and Konyar; Horner et al.; and Kasnakoglu and Bauer. Regional models include Hatchett, Horner, and Howitt; Oamek and Johnson; and Quinby and Leuck. Rosen and Sexton apply PMP to individual farms. The PMP approach uses the farmer's crop allocation in the base year to generate self-calibrating models of agricultural production and resource use, consistent with microeconomic theory, that accomodate heterogeneous quality of land and livestock.

Mathematical programming models are widely used for agricultural economic policy analysis, despite few methodological developments in the past decade. Their popularity stems from several sources. First, they can be constructed from a minimal data set. In many cases, analysts are required to construct models for systems where time-series data are absent or are inapplicable due to structural changes in a developing or shifting economy. Second, the constraint structure inherent in programming models is well suited to characterizing resource, environmental, or policy constraints. In some cases, a set of inequality constraints, such as those found in farm commodity programs, strongly influences crop and resource allocation. Third, the Leontief production technology inherent in most programming models has an intrinsic appeal of input determinism when modeling farm production (Just, Zilberman, and Hochman). In addition, linear programming models are consistent with the Von Liebig production specification, which is preferable for several inputs (Paris and Knapp).

While the PMP approach is unconventional in that it employs both programming constraints and "positive" inferences from the base-year crop allocations, it has one strong attraction for applied analysis: it works. That is to say, the PMP approach automatically calibrates models using minimal data, and without using "flexibility" constraints. The resulting models are more flexible in their response to policy changes, and priors on yield variation or supply elasticities can be specified. With modern algorithms and microcomputers, the resulting quadratic programming problems can be readily solved.

Following a brief overview of past approaches to calibrating programming models of farm production and problems associated with these models, the equivalency of the Kuhn

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Tucker conditions for the constrained and cali-
brated models are shown, and three proposi-
tions that justify the nonlinearity and dimension
of the calibration specification are presented.
Formal statement and proofs of the propositions
are in the appendix. This is followed by presen-
tation of an empirical calibration method with a
simplified graphical and numerical example.
The final section of the paper addresses some
common empirical policy modeling problems.
The ability of PMP models to yield smooth
parametric functions and nest LP problems
within them is briefly discussed.

Calibration Problems in Programming Models

Programming models should calibrate against a
base year or an average over several years.
Policy analysis based on normative models that
show a wide divergence between base period
model outcomes and actual production patterns
is generally unacceptable. However, models
that are tightly constrained can only produce
that subset of normative results that the calibra-
tion constraints dictate. The policy conclusions
are thus bounded by a set of constraints that are
expedient for the base year, but often inappro-
priate under policy changes. This problem is
exacerbated when the model is on a regional
basis with very few empirical constraints, but
with a wide diversity of crop production.

Brevity only permits a brief overview of
some of the past calibration methods in math-
ematical programming models. A more compre-
hensive discussion can be found in Hazell and
Norton or Bauer and Kasnakoglu. It is worth
noting that no one approach has proved satis-
factory enough to dominate the applied litera-
ture.

Previous researchers (e.g., Day) attempt to
provide more realism by imposing upper and
lower bounds to production levels as con-
straints. McCarl advocates a decomposition
methodology to reconcile sectoral equilibria
and farm-level plans. Both of these approaches
require additional micro-level data, and result
in calibration constraints influencing policy re-
sponse.

Meister, Chen, and Heady, in their national
quadratic programming model, specify 103 pro-
ducing regions and aggregate the results to ten
market regions. Despite this structure, they note
the problem of overspecialization and suggest
the use of rotational constraints to curtail the

overspecialization. However, it is compara-
tively rare that agronomic practices are fixed at
the margin; more commonly they reflect net
revenue maximizing trade-offs between yields,
costs of production, and externalities between
crops. In the latter case, rotations are functions
of relative resource scarcity, output prices, and
input costs.

Hazell and Norton suggest six tests to vali-
date a sectoral model. The first is a capacity
test for overconstrained models; the second is a
marginal cost test to ensure that marginal costs
of production, including the implicit opportu-
nity costs of fixed inputs, are equal to the out-
put price; and the third is a comparison of the
dual value on land with actual rental values.
They also advocate three additional compari-
sons of input use, production level and product
price tests. Hazell and Norton show that the
percentage of absolute deviation for production
and acreage over five sectoral models ranges
from 7% to 14%. The constraint structures
needed for this validation are not defined.

In contrast, the PMP approach aims to
achieve exact calibration in acreage, produc-
tion, and price. Bauer and Kanakoglu subse-
quently applied the PMP approach to one of the
sectoral models cited by Hazell and Norton.
The results for the Turkish Agricultural Sector
model (TASM) showed consistent calibration
over seven years.

The calibration problem in farm-level, re-
gional, and sectoral models can be mathemati-
cally defined by the common situation in which
the number of binding constraints in the opti-
mal solution are less than the number of non-
zero activities observed in the base solution. If
the modeler has enough data to specify a con-
straint set to reproduce the optimal base-year
solution, then additional model calibration will
be redundant. The PMP approach is developed
for the majority of modelers who, for lack of an
empirical justification, data availability, or cost,
find that the empirical constraint set does not
reproduce the base-year results. The LP solu-
tion is an extreme point of the binding con-
straints. In contrast, the PMP approach views
the optimal farm production as a boundary
point, which is a combination of binding con-
straints and first-order conditions.

Relevant constraints should be based on ei-
ther economic logic or the technical environ-
ment under which the agricultural production is
operating. Calibration problems are especially
prevalent where the constraints represent allo-
catable inputs, actual rotational limits, and

policy constraints. When the basis matrix has a rank less than the number of observed base-year activities, the resulting optimal solution will suffer from overspecialization of production activities compared to the base year.

A source of these problems is that linear programming was originally used as a normative farm planning method assuming full knowledge of the production technology. Under these conditions, any production technology can be represented as a Leontief technology, subject to resource and stepwise constraints. For aggregate policy models, this normative approach produces a production and cost technology that is too simplified due to inadequate knowledge. In most cases, the only regional production data are average or "representative" values for crop yields and inputs.[1] This common situation means that the analyst is attempting to estimate marginal behavioral reactions to policy changes based on average data observations. The average conditions can be assumed to be equal to the marginal conditions only where the policy range is small enough to admit linear technologies.

Two broad approaches have been used to reduce the specialization errors in optimizing models. The demand-based methods use a range of methods to add risk or endogenize prices. These help resolve the problem, but substantial calibration problems remain in many models (Just).

The other common approach is to constrain the crop supply activities by rotational (or flexibility) constraints, or step functions, over multiple activities (Meister, Chen, and Heady). In regional and sectoral models of farm production, there are few empirically justifiable constraints. Land area and soil type are clearly constraints, as is water in some irrigated regions. Crop contracts and quotas, breeding stock, and perennial crops are others. However, it is harder to justify other constraints such as labor, machinery, or crop rotations on short-run marginal production decisions. These inputs are limiting, but only in the sense that once the normal availability is exceeded, the cost-per-unit output increases due to overtime, increased probability of machinery failure, or disease. If the assumption of linear production (cost) tech-

nology is retained, the observed output levels imply that additional binding constraints on the optimal solution should be specified. Comprehensive rotational constraints are a common example of this approach.

An alternative explanation to linear technologies with constraints is that the profit function is nonlinear in land for most crops, and that the observed crop allocations are a result of a mix of unconstrained and constrained optima. The most common reasons for a decreasing gross margin per acre are declining yields due to heterogeneous land quality, risk aversion, or increasing costs due to restricted management or machinery capacity.

Given the exhaustive literature on the addition of risk to LP models, I concentrate on calibrating the supply side by introducing a nonlinear yield (or cost) specification for each production activity. While risk is clearly an important determinant of cropping patterns, as shown below, risk alone usually provides insufficient nonlinear calibration terms to completely calibrate a model.

Behavioral Calibration Theory

Calibrating models to observed outcomes is an integral part of constructing physical and engineering models, but it is rarely formally analyzed for optimization models in agricultural economics. In this section I show that observed behavioral reactions provide a basis for model calibration in a formal manner that is consistent with microeconomic theory. By analogy to econometrics, the calibration approach draws a distinction between the two modeling phases of calibration (estimation) and policy prediction.

On a regional level, information on the output levels produced and the land allocations by farmers is usually more accurate than the estimates of crop marginal production costs. This is particularly true with micro data on land class variability, technology, and risk. This information often features in the farmers' decisions, but is absent in the aggregate cost data available to the model builder. Accordingly, the PMP approach uses the observed acreage allocations and outputs to infer marginal cost conditions for each observed regional crop allocation. This inference is based on those parameters that are accurately observed, and the usual profit-maximizing and concavity assumptions.

Proposition 1 (see appendix A) shows that if the model does not calibrate to observed production activities with the full set of general

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[1] The paper is written using cropping activities as examples, but the same procedure can be directly applied to livestock fattening and other activities where the key input is not land but a livestock unit, such as a breeding cow. For an example of PMP applied to a wide range of livestock activities in a national model see Bauer and Kasnakoglu.

linear constraints that are empirically justified by the model, a necessary condition for profit maximization is that the objective function be nonlinear in at least some of the activities.

Many regional models have some nonlinear terms in the objective function reflecting endogenous price formation or risk specifications. Although it is well known that the addition of nonlinear terms improves the diversity of the optimal solution, there are usually an insufficient number of independent nonlinear terms to accurately calibrate the model.

Proposition 2 (appendix A) shows that the ability to calibrate the model with complete accuracy depends on the number of nonlinear terms that can be independently calibrated.

The ability to adjust some nonlinear parameters in the objective function, typically the risk aversion coefficient, can improve model calibration. However, with insufficient independent nonlinear terms the model cannot be calibrated precisely. In technical terms, the number of instruments available for model calibration may not span the set of activities that need to be calibrated.

Consider the following problem where the objective function is specified in a general linear or nonlinear form, $f(\mathbf{x})$. For simplicity, and without loss of generality, activities not observed in the base data are removed from the specification.

$$(1) \quad \max_x f(\mathbf{x})$$

subject to

$$\mathbf{Ax} \leq \mathbf{b}$$
$$(\bar{\mathbf{x}} - \boldsymbol{\varepsilon}_1) \leq \mathbf{x} \leq (\bar{\mathbf{x}} - \boldsymbol{\varepsilon}_2) \quad \mathbf{x} \geq 0, \bar{\mathbf{x}} > 0$$
$$\mathbf{x} \text{ is } k \times 1, \mathbf{A} \text{ is } m \times k, m < k$$

where the $\boldsymbol{\varepsilon}_i$ perturbations are defined in appendix B.

Let $\bar{\boldsymbol{\lambda}}_1$ be the $m \times 1$ dual solution vector to problem (1) associated with the set of general constraints. The dual values associated with the set of calibration constraints can be ignored in the analysis of the general constraint duals ($\boldsymbol{\lambda}_1$), since proposition 3 (appendix B) shows that the optimal values for $\boldsymbol{\lambda}_1$ are not changed by the addition of the calibration constraints. Define the $k \times 1$ vector $\bar{\boldsymbol{\gamma}}$ as

$$(2) \quad \bar{\boldsymbol{\gamma}} = \nabla f(\bar{\mathbf{x}})' - \mathbf{A}'\bar{\boldsymbol{\lambda}}_1$$

where $\nabla f(\mathbf{x})$ is the $1 \times k$ gradient vector of first

derivatives of $\mathbf{f(x)}$. Let $\boldsymbol{\alpha}$ be a $k \times 1$ set of constants such that

$$(3) \quad (\bar{\boldsymbol{\gamma}}_i - \boldsymbol{\alpha}_i) \geq 0 .$$

Define the $k \times k$ diagonal matrix $\boldsymbol{\Gamma}$ as

$$(4) \quad \boldsymbol{\Gamma} = diag\,[(\bar{\gamma}_1 - \alpha_1)/\bar{x}_1, \,...,\, (\bar{\gamma}_k - \alpha_k)/\bar{x}_k].$$

The matrix $\boldsymbol{\Gamma}$ is positive definite by construction.

Consider the following problem:

$$(5) \quad \max_x \mathbf{f(x)} - \frac{1}{2} \mathbf{x}'\boldsymbol{\Gamma}\mathbf{x} - \boldsymbol{\alpha}'\mathbf{x}$$

subject to

$$\mathbf{Ax} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0.$$

The first-order Kuhn-Tucker conditions for this problem are

$$(6) \quad \nabla\mathbf{f(x)}' - \boldsymbol{\Gamma}\mathbf{x} - \boldsymbol{\alpha} - \mathbf{A}'\boldsymbol{\lambda} = 0.$$

From equation (4) we see that $\boldsymbol{\Gamma}x = (\bar{\boldsymbol{\gamma}} - \boldsymbol{\alpha})$; therefore, substituting for $\boldsymbol{\Gamma}x$ in (6), we get

$$(7) \quad \nabla\mathbf{f(x)} - \mathbf{A}'\boldsymbol{\lambda} = \bar{\boldsymbol{\gamma}}.$$

From equation (2) we see that the Kuhn-Tucker condition (6) holds exactly when $\mathbf{x} = \bar{\mathbf{x}}$ and $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}}_1$. That is, the calibrated problem (5) will optimize at the values $\bar{\mathbf{x}}$ and $\bar{\boldsymbol{\lambda}}_1$ if the values $\boldsymbol{\Gamma}$ and α are defined by equations (3) and (4).

To summarize, given the three propositions in the appendices, linear and nonlinear optimization problems can be calibrated by the addition of a specific number of nonlinear terms. We use a simple quadratic specification to show that if the quadratic parameters satisfy equations (2), (3), and (4), then the resulting quadratic problem will calibrate exactly in the primal and dual values of the original problem, but without inequality calibration constraints.

In the next section I show how the calibration procedure can be simply implemented in a two-stage process that is initiated with a linear program.

An Empirical Calibration Method

The previous section showed that if the correct

nonlinear parameters are calculated for the $(k - m)$ unconstrained (independent) activities, the model will exactly calibrate to the base-year values without additional constraints. The problem addressed in this section is to show how the calibrating parameters can be simply and automatically calculated using the minimal data set for a base-year LP.

Because nonlinear terms in the supply side of the profit function are needed to calibrate a production model, the task is to define the simplest specification which is consistent with the technological basis of agriculture, microeconomic theory, and the data base available to the modeler.

A highly probable source of nonlinearity on the primal side is heterogeneous land quality, and declining marginal yields as the proportion of a crop in a specific area is increased. This phenomenon, first formalized by Ricardo (Peach), is widely noted by farmers, agronomists, and soil scientists, but often omitted from quantitative production models.

I use a "Primal" PMP approach which keeps the variable cost/acre constant and has a yield function that decreases the marginal crop yield per acre as a linear function of the acreage planted.[2] This specification is consistent with the large body of evidence from soil science and agronomy that shows variability in soil suitability and consequent crop yield in most agricultural areas, whether on the farm or regional scale. The production function in this paper is Leontief with heterogenous and restricted land inputs.

Obviously this is a considerable simplification of the complete production process. Given the applied goal of this "positive" modeling method, the calibration criteria used is not whether the simple production specification is true, but whether it captures the essential behavioral response of farmers, and can be made to work with available restricted data bases and model structures.[3]

The output from a given cropping activity i under the primal PMP specification with land x_i and two other inputs is

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[2] Past working papers on PMP, and most of the applications, have specified the nonlinear part of the profit function as originating from an increase in variable cost per acre with constant yields. Both yield and cost changes are probably present; however, data on yield variability are more easily obtained by an empirical modeler than cost variation.

[3] If more complex specifications of the production function are required, Howitt shows how the calibration principles can be extended to include Cobb-Douglas and nested Constant Elasticity of Substitution (CES) production functions.

$$(8) \quad y_i = (\beta_i - \delta_i x_i) \min(x_i, a_{i2} x_i, a_{i3} x_i)$$

where β_i and δ_i are, respectively, the intercept and slope of the marginal yield function for crop i.

The calibrated optimization problem equivalent to equation (5), therefore, becomes

$$(9) \quad \max \sum_i P_i (\beta_i - \delta_i x_i) x_i - \sum_{j=1}^{3} \omega_j a_{ij} x_i$$

subject to

$$\mathbf{Ax} \leq \mathbf{b} \text{ and } \mathbf{x} \geq 0$$

where $a_{i1 = 1}$, $\mathbf{A} = (m \times n)$ with elements a_{ij}, x_i is the acreage of land allocated to crop i, and ω_j is the cost per unit of the jth input.

The PMP calibration approach uses three stages. In the first stage a constrained LP model is used to generate particular dual values. In the second stage, the dual values are used, along with the data based average yield function, to uniquely derive the calibrating yield function parameters. In the third stage, the yield parameters (β and δ) are used with the base-year data to specify the PMP model in equation (9). The resulting model calibrates exactly to the base-year solution and original constraint structure.

Figure 1 shows problem (1) in a diagrammatic form for two activities, with $\mathbf{f(x)}$ simplified to $\mathbf{c'x}$, one resource constraint and two upper-bound calibration constraints. Note that at the optimum, the calibration constraint will be binding for wheat, the activity with the higher average gross margin, while the resource constraint will restrict the acreage of oats.

Two equations are solved for the two unknown yield parameters (β and δ). Defining $\mathbf{f(x)}$ as the quadratic total output function specified in (9), the first equation is the average yield for crop i, $\bar{y}_i$

$$(10) \quad \bar{y}_i = \beta_i - \delta_i x_i.$$

The second equation uses the value of the dual on the LP calibration constraint (λ_2) which is shown below to be the difference between the value average product (VAP) of the crop and the value marginal product (VMP).

The derivation of the two types of dual value λ_1 and λ_2, can be shown for the general case using appendix B. The $\mathbf{A}$ matrix in (1) is partitioned by the optimal solution of (1) into an $m \times m$ matrix $\mathbf{B}$ associated with the variables $\mathbf{x_B}$,
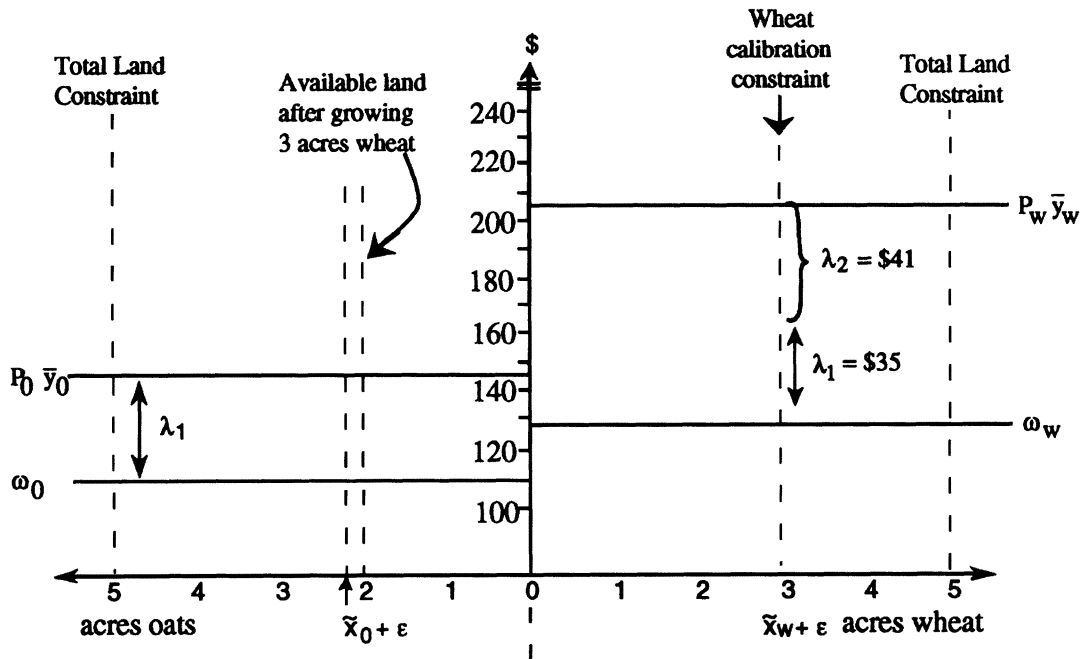
Figure 1. L.P. problem with calibration constraints—two activity/one resource constraint

an $m \times 1$ subset of $\mathbf{x}$ with inactive calibration constraints. The second partition of $\mathbf{A}$ is into an $m \times (k - m)$ matrix $\mathbf{N}$ associated with a $(k - m) \times 1$ partition of $\mathbf{x}$, $\mathbf{x_N}$ of nonzero activities constrained by the calibration constraints. The first partition of equation (B13) in appendix B for λ_1 is

$$(11) \quad \lambda_1^* = \mathbf{B}'^{-1}\mathbf{V}_{\mathbf{x_B}} \mathbf{f}(\mathbf{x}^*)$$

where $\mathbf{V}_{\mathbf{x_B}} \mathbf{f}(\mathbf{x}^*)$ is the gradient of value marginal products (VMPs) of the vector $\mathbf{x_B}$ at the optimum value.

The elements of vector x_B are the acreages produced in the crop group limited by the general constraints, and λ_1 are the dual values associated with the set of $m \times 1$ binding general constraints. Equation (11) states that the value of marginal product of the constraining resources is a function of the revenues from the constrained crops. The more profitable crops $(\mathbf{x_N})$ do not influence the dual value of the resources (proposition 3, appendix B). This is consistent with the principle of opportunity cost in which the marginal net return from a unit increase in the constrained resource determines its opportunity cost. Since the more profitable crops x_N are constrained by the calibration constraints, the less profitable crop group x_B are those that could use the increased resources and, hence, determine the opportunity cost.

The second partition of appendix equation B13 determines the dual values on the upper-bound calibration constraints on the crops

$$(12) \quad \lambda_2 = -\mathbf{N}'\mathbf{B}'^{-1}\mathbf{V}_{\mathbf{x_B}} \mathbf{f}(\mathbf{x}^*) + \mathbf{IV}_{\mathbf{x_N}} \mathbf{f}(\mathbf{x}^*)$$

[and substituting equation (11)]

$$\lambda_2 = \mathbf{V}_{\mathbf{x_N}} \mathbf{f}(\mathbf{x}^*) - \mathbf{N}'\lambda_1^*$$

Note that the right-hand side of (12) is a $(k - m)$ partition of the right-hand side of (2).

The dual values for the binding calibration constraints are equal to the difference between the marginal revenues for the calibrated crops $(\mathbf{x_N})$ and the marginal opportunity cost of resources used in production of the constrained crops $(\mathbf{x_B})$. Since the stage I problem in figure 1 has a linear objective function, the first term in (12) is the crop average value product of land in activities $\mathbf{x_N}$. The second term in (12) is the marginal value product of land from equation (11). In this PMP specification, the difference between the average and marginal value product of land is attributed to changing land quality. Thus the PMP dual value (λ_2) is a hedonic measure of the difference between the average and marginal products of land, for the calibrated crops. By analogy to revealed prefer-
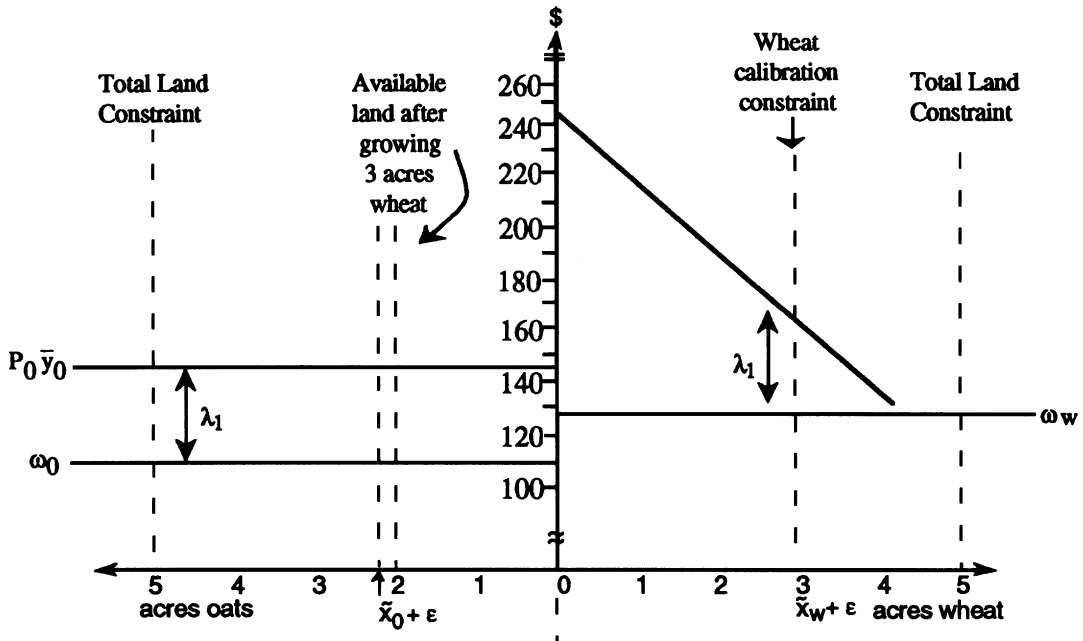
Figure 2. PMP yield function on wheat

ence, PMP can be thought of as revealed effi-ciency based on observed land allocations.

Equation (12) substantiates the dual values shown in figure 1, where the duals for the cali-bration constraint set (λ_2) in the stage I problem are equal to the divergence between the LP av-erage net value product per acre and opportu-nity cost per acre. Since the value λ_2 represents the difference between VAP and VMP for the more profitable crops, and given the linear yield function in (8), a single element of λ_2 can be expressed as

$$(13) \quad \lambda_{2i} = P_i(\beta_i - \delta_i x_i) - P_i(\beta_i - 2\delta_i x_i)$$
$$= P_i \delta_i x.$$

Using (13) the yield slope coefficient can be solved as

$$(14) \quad \delta_i = \frac{\lambda_{2i}}{P_i x_i}.$$

Using equation (10) the intercept coefficient (β_i) for crop i can be solved in terms of δ_i and $\bar{y}_i$.

Despite all the notation, the basic concept of PMP is numerically simple and easy to solve automatically, even on desktop computers. A numerical example applied to the problem in

equation (1) and figures 1 and 2 demonstrates this simplicity.

The problems shown in figures 1 and 2 have a single land constraint (500 acres) and two crops, wheat and oats. The following param-eters are used:

	Wheat (*w*)	(Oats) (*o*)
Crop prices	$P_w = \$2.98/\text{bu}$	$P_0 = \$2.20/\text{bu}$
Variable cost/acre	$\omega_w = \$129.62$	$\omega_0 = \$109.98$
Average yield/acre	$\bar{y}_w = 69 \text{ bu}$	$\bar{y}_0 = 65.9 \text{ bu}$

The observed acreage allocation in the base year is 300 acres of wheat and 200 acres of oats. The problem in figure 1 is

$$(15) \quad \max \ (2.98*69 - 130)x_w$$
$$+ (2.20*65.9 - 110)x_0$$

subject to

(*i*)	x_w	$+$	$x_0 \leq$	500
(*ii*)	x_w		$\leq$	300.01
(*iii*)			$x_0 \leq$	200.01

Note the addition of the ε perturbation term (0.01) on the right-hand side of the calibration constraints. The average gross margin from
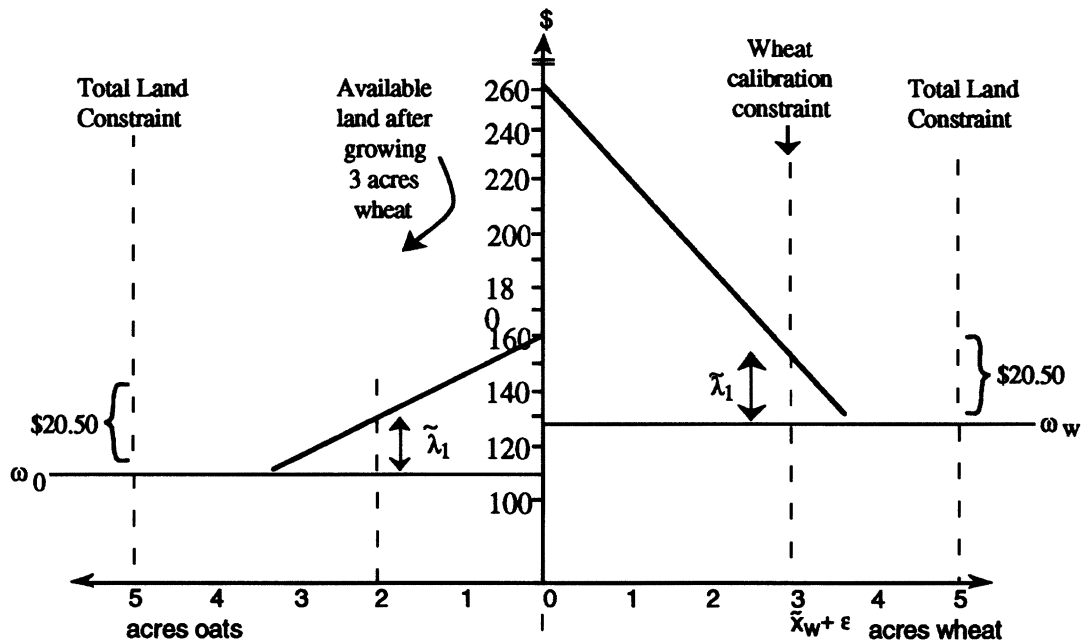
Figure 3. PMP model—quadratic yields on all crops

wheat is $76/acre and for oats is $35/acre. The optimal solution to the stage 1 problem (1) is when the wheat calibration constraint is binding at a value of 300.01 and constraint (*i*) is binding when the oat acreage equals 199.99. The oat calibration constraint is slack.

The dual value on land (λ_1) is $35 and on the two calibration constraints (λ_2) = [41 and 0]. Using equation (14), the λ_2 value for wheat and the base-year data, the yield function slope for wheat is calculated as

$$(16) \quad \delta_w = 41/(2.98*300.01) = 0.04586.$$

Quantity δ_w is now substituted into equation (10) to calculate the yield slope intercept β_w

$$(17) \quad \beta_w = 69 + (0.04586*300.01) = 82.76.$$

Using the yield function parameters, the Stage II primal PMP problem becomes (see figure 2)

$$(18) \quad \max [2.98(82.76 - 0.04586*x_w) - 130]x_w$$
$$+ (2.20*65.9 - 110)x_0$$

subject to

$$x_w + x_0 \le 500.$$

A quick empirical check of the calibration of

problem (18) to the base values can be performed by calculating the VMP of wheat at 300 acres. If it is close to the VMP (VAP) of oats and convergent, the model will calibrate without additional calibration constraints.

The marginal yield per acre of wheat is

$$y_{|300} = 82.76 - 2*0.04586*300 = 55.25$$
$$VMP_{w,300} = 2.98*55.25 - 130 = 34.65$$

The VMP for wheat at 300 acres of $34.65 is marginally below the VMP for oats ($35). Thus, the unconstrained PMP model will calibrate within the rounding error of this example.

This numerical example shows that PMP models can be calibrated using simple methods. The three-stage process and calculation of the parameters is easily programmable as a single process using GAMS/MINOS.[4] Thus, given the initial data and specifications, the PMP model is automatically calibrated in the time it takes to solve an LP and QP solution for the model.

The PMP model specified in (18) calibrates in all aspects. That is, the optimal solution, binding constraints, objective function value

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[4] A PMP program written for the GAMS/MINOS optimization package is available from the author by e-mail (rehowitt@ucdavis.edu). The program can be used to automatically calibrate and run a range of agricultural production problems by PMP.

and dual values will all be within rounding error of the original LP in (15) that is constrained by the calibration constraints.

A valid objection to the simple PMP specification in (15) is that we assume a decreasing yield/acre function for the more profitable unconstrained crops x_N, but the crop set x_B that is constrained by resources is assumed to have constant yields.

Calibrating the marginal crops (x_B) with decreasing yield functions requires additional empirical information. The independent variables, as x_N are termed, use both the constrained resource opportunity cost (λ_1) and their own calibration dual (λ_2) (figure 1) to solve for the yield function parameters implied by the observed crop allocations. However, the marginal crops (x_B) have no binding calibration constraint, and thus cannot empirically differentiate marginal and average yield of the observed calibration acreage using the minimal LP data set specified.

Clearly some additional data are needed. The simplest source of additional data are measurements on the expected yield variation of the marginal crops (x_B) within a given region and year. Regional acreage response elasticities would supply the equivalent information, but it would seem that yield variation is an easier empirical value to obtain from farmers, particularly if it is simplified into percentage deviations above and below the mean yields in the region.

Returning to the simple pedagogical example in equation (15) and figure 2, the stage 1 calibrated problem is run exactly as before. One of the important pieces of information from the optimal solution of the stage 1 problem is the activities which are in the x_N and x_B groups. This information is unlikely to be known beforehand.

In the example, assume that the a priori information on oats is that expected yield variation is plus or minus 10% of the mean. The reduced marginal yield information now causes a recalculation of the opportunity cost of land. Given an average yield ($\bar{y}_0$) for oats of 65.9 bu/acre and a price of $2.20, the marginal return given 10% yield reduction will now be based on a yield of $59.31 bu/acre; therefore, the dual value on land (l1) is reduced by $14.50 to $20.50. The PMP dual ($\lambda_2$) must also be increased by this same amount to ensure the first-order conditions (12) hold. The new value for $\lambda_2 = \$55.50$.

The calculations for the yield coefficients in (16) and (17) are now applied to all activities, both marginal (x_B) and independent (x_N). Note

that the adjusted λ_2 values are used for the independent activities, and the MVP based on the prior data is used for the marginal crops.

The PMP problem, given the information on marginal yields for the oat crop, is

(19)   max $[2.98(87.63 - 0.0621 * x_w) - 130]x_w$
         $+ [2.20(72.49 - 0.0329 * x_0) - 110]x_0$

subject to

$$x_w + x_0 \leq 500.$$

The problem is shown in figure 3. The calibration acreage can be checked by calculating the VMP for each crop at the calibration acreages of $\tilde{x}_w = 300$ and $\tilde{x}_0 = 200$.

(20) (i) $VMP_w\big|_{\tilde{x}_{w=300}} = 2.98 * 50.37 - 130 = 20.10$

     (ii) $VMP_0\big|_{\tilde{x}_{0=200}} = 2.20 * 59.33 - 110 = 20.53$

With the VMP's equal, aside from rounding error, the PMP with endogenous yield functions will calibrate arbitrarily close to the base-year acreages.

The resulting model will calibrate acreage allocation and input use, and the objective function value precisely. However, the dual value on resources will be lower reflecting the additional, and presumably more accurate, data on the yield variation among the marginal crops.

Policy Modeling with PMP

The purpose of most programming models is to analyze the impact of quantitative policy scenarios which take the form of changes in prices, technology, or constraints on the system. The policy response of the model can be characterized by its response to sensitivity analysis and changes in constraints.

Advantages of the PMP specification are not only the automatic calibration feature, but also its ability to respond smoothly to policy scenarios. Paris shows that input demand functions and output supply functions obtained by parameterizing a PMP problem satisfy the Hicksian conditions for the competitive firm. In addition, the input demand and supply functions are continuous and differentiable with respect to prices, costs, and right-hand side quantities. At the point of a change in basis, the supply and demand functions are not differentiable. This is in contrast to LP or stepwise problems, where

the dual values, and sometimes the optimal solution, are unchanged by parameterization until there is a discrete change in basis, when they jump discontinuously to a new level.

The ability to represent policies by constraint structures is important. The PMP formulation has the property that the nonlinear calibration can take place at any level of aggregation. That is, one can nest an LP subcomponent within the quadratic objective function and obtain the optimum solution to the full problem. An example of this is used in technology selection where a specification that causes discrete choices may be appropriate. Suppose a given regional commodity can be produced by a combination of five alternative linear technologies, whose aggregate output has a common supply function. The PMP can calibrate the supply function while a nested LP problem selects the optimal set of linear technology levels that make up the aggregate supply (Hatchett, Horner, and Howitt).

Since the intersection of the convex sets of constraints for the main problem and the convex nested subproblem is itself convex, then the optimal solution to the nested LP subproblem will be unchanged when the main problem is calibrated by replacing the calibration constraints with quadratic PMP cost functions. The calibrating functions can thus be introduced at any level of the linear model. In some cases, the available data on base-year values will dictate the calibration level. Ideally, the level of calibration would be determined by the properties of the production functions, as in the example of linear irrigation technology selection. The PMP approach does not replace all linear cost functions with equivalent quadratic specifications, but only replaces those that data or theory suggest are best modeled as nonlinear.

If one has prior information on the nature of yield externalities and rotational effects between crops, they can be explicitly incorporated by specifying cross-crop yield interaction coefficients in equations (13) and (14). The PMP yield slope coefficient matrix is positive definite, $k \times k$, and has rank k. Without the cross-crop effects the matrix is diagonal.

Resource-using activities such as fodder crops consumed on the farm may be specified with zero valued objective function coefficients. Where an activity is not resource-using, but merely acts as a transfer between other activities, there is no empirical basis or need to modify the objective function coefficients.

Conclusions

Programming models have a strong role to play in agricultural policy analysis, particularly where reliable time-series data are absent, or shifts in market institutions or constraints have changed substantially over time. The problem addressed in this paper is one of calibrating programming models without adding constraints that cannot be justified by economic theory or agricultural technology. The solution proposed by the PMP approach is based on the derivation of nonlinear yield functions from the base-year data and prior crop yield data. The derivation is achieved by a simple three-step procedure.

Calibration of a model to the base-year data set and constraints is a necessary, but not sufficient, condition for a meaningful policy model. The ultimate test of a policy model is its ability to predict behavioral responses out of the sample base-year. If the yield response functions calibrated in the PMP method have a basis in regional soil variation and farmer behavior, then they should be relatively stable over time and can provide additional structural information for policy response. Empirical tests of the stability of the PMP values are required to evaluate the stability of the calibrated models. Initial tests in Kasnakoglu and Bauer are encouraging.

The PMP approach is shown to satisfy the main criteria for calibrating sectoral and regional models. Using PMP, the model calibrates precisely to output and input quantities, the objective function value, dual constraint values, and output prices. In addition, the PMP approach can incorporate priors on yield variability or supply elasticities.

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Appendix A

PROPOSITION 1. *Given an agent maximizing multi-output profit subject to linear constraints on some inputs or outputs, if the number of nonzero nondegenerate production activity levels observed (k) exceeds the number of binding constraints (m), then a necessary and sufficient condition for profit maximization at the observed levels is that the profit function be nonlinear (in output) in some of the (k) production activities.*

Proof. Define the profit function in general as a function of input allocation $\mathbf{x}$, $\mathbf{f(x)}$.

(a1)   problem is max $\mathbf{f(\bar{x})}$

subject to

$$\bar{\mathbf{A}}\bar{\mathbf{x}} \leq \mathbf{b} \quad \bar{\mathbf{x}} = n \times 1$$
$$\bar{\mathbf{A}} = m \times n \quad m < n$$

At the observed optimal solution (nondegenerate in primal and dual specifications) there are k non-zero values of $\bar{\mathbf{x}}$. Drop the zero values of $\bar{\mathbf{x}}$ and define the $m \times m$ basic partition of $\mathbf{A}$ as the $(m \times m)$ optimal solution basis matrix $\mathbf{B}$ and the remaining partition of $\mathbf{A}$ as $\mathbf{N}$ $(m \times k - m)$. Partitioning the $k \times 1$ vector x into the $m \times 1$ vector $\mathbf{x_B}$ and $(k - m)$ x1 vector $\mathbf{x_N}$, the problem (a1) is written as

(a2)   max $\mathbf{f(x)}$ subject to $[\mathbf{B} : \mathbf{N}]\begin{bmatrix}\mathbf{x_B}\\\mathbf{x_N}\end{bmatrix} = \mathbf{b}$

or

(a3)   max $\mathbf{f(x_B, x_N)}$ subject to $\mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{b}$

Given the constraint set in (a3), $\mathbf{x_B}$ can be written

(a4) $\mathbf{x_B} = \mathbf{B^{-1}b} - \mathbf{B^{-1}Nx_N}.$

Since the binding constraints are implicit in (a4), substituting (a4) into the (a3) objective function gives

(a5)   max $f(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N, \mathbf{x}_N)$.

Taking the gradient of (a5) with respect to $\mathbf{x}_N$ yields the reduced gradient $(\mathbf{r}_{\mathbf{x}_N})$

(a6) $\mathbf{r}_{\mathbf{x}_N} = \nabla\mathbf{f}_{\mathbf{x}_N} - \nabla\mathbf{f}_{\mathbf{x}_B}\mathbf{B}^{-1}\mathbf{N}.$

A zero reduced gradient is a necessary condition for optimality (Luenberger). Without loss of generality we define the basic part of the objective function as linear with coefficients $\mathbf{c}_B$, which yields the optimality condition

(a7) $\mathbf{r}_{\mathbf{x}_N} = \nabla\mathbf{f}_{\mathbf{x}_N} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} = 0.$

The objective function associated with the independent $(\mathbf{x}_N)$ variables has either zero coefficients, linear coefficients, or a nonlinear specification. If $f(\mathbf{x}_N)$ had zero coefficients, $\mathbf{x}_N$ would have to be zero at the optimum given the positive opportunity cost of resources. If $f(\mathbf{x}_N)$ was linear, say $\mathbf{c}_N$, then (a7) would be the reduced cost of the activity. A zero reduced cost of a nonbasic activity implies degeneracy when coupled with a zero activity level $\mathbf{x}_N$. Since $\mathbf{x}_N > 0$ at the optimum, $f(\mathbf{x}_N)$ cannot be linear and hence must be nonlinear for (a7) to hold.

PROPOSITION 2. *A necessary condition for the exact calibration of a $k \times 1$ vector $\mathbf{x}$ is that the objective function associated with the $(k - m) \times 1$ vector of independent variables $\mathbf{x}_N$ contain at least $(k - m)$ linearly independent instruments that change the first derivatives of $f(\mathbf{x}_N)$.*

Proof. By proposition 1 $f(\mathbf{x}_N)$ is nonlinear in $\mathbf{x}_N$. Each element of the gradient $\nabla\mathbf{f}(\mathbf{x}_N)$ has a component that is a function of $\mathbf{x}_N$, and probably also a constant term. The optimality conditions in equation (a7) are modified by subtracting the constant components in the gradient $(\overline{\mathbf{k}})$ from both sides to give

(a8) $\nabla\overline{\mathbf{f}}_{\mathbf{x}_N} = \mathbf{c}*$

where

$\nabla\overline{\mathbf{f}}_{\mathbf{x}_N} = \nabla\mathbf{f}_{\mathbf{x}_N} - \overline{\mathbf{k}}'$

and

$\mathbf{c}* = \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{N} - \overline{\mathbf{k}}'$

The $1 \times (k - m)$ vector $\nabla\overline{\mathbf{f}}_{\mathbf{x}_N}$ can be written as the product of $\mathbf{x}_N$ and a $(k - m) \times (k - m)$ matrix $\mathbf{F}$, where the ith column of $\mathbf{F}$ has elements

$$\frac{\partial f(x_N)}{\partial x_i} \frac{1}{x_i}$$

as in equation (4).

Using this decomposition

(a9) $\nabla\overline{\mathbf{f}}_{\mathbf{x}_N} \equiv \mathbf{x}_N'\mathbf{F}$

the necessary reduced gradient condition (a8) can now be rewritten as

(a10) $\mathbf{x}_N'\mathbf{F} = \mathbf{c}*$

Calibration of an optimization model requires that the observed solution vector $\tilde{\mathbf{x}}$ results from the optimal solution of the calibrated model. From equation (a4) the independent values $\tilde{\mathbf{x}}_N$ imply the dependent values $\tilde{\mathbf{x}}_B$. Since from (a8), $\mathbf{c}*$ is a vector of fixed parameters, the necessary condition (a10) can only hold at $\tilde{\mathbf{x}}_i$ if the values of $\mathbf{F}^{-1}$ can be calibrated to map $\mathbf{c}*$ into $\tilde{\mathbf{x}}_N$. Thus the matrix of calibrating gradients $\mathbf{F}^{-1}$ must span $\tilde{\mathbf{x}}$ such that

(a11) $\tilde{\mathbf{x}}_N' = \mathbf{c}*\mathbf{F}^{-1}$

It follows that the rank of $\mathbf{F}$ must be $(k - m)$ and there have to be $(k - m)$ linearly independent instruments which change the values of $\mathbf{F}$ to exactly calibrate $\tilde{\mathbf{x}}$.

Example. Let x_n be a 2×1 vector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

(a12) $\mathbf{f}(\mathbf{x}_N) = \alpha'\mathbf{x}_N - \mathbf{x}_N'\mathbf{Q}\mathbf{x}_N$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

and symmetric. Writing (a7) as

(a13) $[\alpha_1 - 2x_1 q_{11} - 2x_2 q_{12}, \alpha_2 - 2x_2 q_{22} - 2x_1 q_{21}]$
$$- \mathbf{c}_B'\mathbf{B}^{-1}N = 0$$

defining the $1 \times (k - m)$ row vector $\mathbf{c}*$ as in equation (a8) results in

(a14) $[2x_1 q_{11} + 2x_2 q_{12}, 2x_1 q_{21} + 2x_2 q_{22}] = \mathbf{c}*$

By definition, the left-hand side of equation (a14) can be written as the product of $\mathbf{x}_N'$ and a matrix $\mathbf{F}$ where

(a15) $\mathbf{F} = \begin{bmatrix} 2q_{11} & 2q_{21} \\ 2q_{12} & 2q_{22} \end{bmatrix}.$

Therefore the optimality condition that the reduced gradient equals 0 requires that $x_N F = c^*$. If particular values of x_N, say $\tilde{x}_N$, are required by changing the coefficients of F, then $\tilde{x}_N = c^* F^{-1}$.

Note from equation (a8) that $-c^*$ is the difference between the constant linear term in the objective function $\overline{k}$ and the opportunity cost of the resources. Thus $-c^*$ is equal to the vector of PMP dual values λ_2. Solving for the parameters of F, given c^* and $\tilde{x}_N$ is computationally identical to solving for the vector of δ_i parameters which requires the necessary condition that F is linearly independent and of rank $(k - m)$.

CorolLARY. *The number of calibration terms in the objective function must be equal to or greater than the number of independent variables to be calibrated.*

Appendix B

Perturbation of the calibration constraints is shown to preserve the primal and dual values.

Constraint Decoupling

Constraint decoupling is shown given the degenerate problem where the binding and slack resource constraints under values $\tilde{x}$ are separated into groups I and II.

Problem P1.

(b1)   maximize $f(x)$
      subject to $Ax = b$ (I)
                 $\hat{A}x < \hat{b}$ (II)
                 $Ix = \tilde{x}$ (III)

$x = k \times 1, A = m \times k \quad \hat{A} = (l - m) \times k$

$\tilde{x} = k \times 1 \quad k > m \quad b = m \times 1 \quad \hat{b} = (1 - m) \times 1.$

$\tilde{x}$ is a $k \times 1$ vector of activities that are observed to be nonzero in the base-year data; $k > m$ implies that there are more nonzero activities to calibrate than the number of binding resource constraints (I).

We assume that $f(x)$ is monotonically increasing in x with first and second derivatives at all points, and that problem P1 is not primal or dual degenerate.

PROPOSITION 3. *There exists a $k \times 1$ vector of perturbations ε ($\varepsilon > 0$) of the values $\tilde{x}$ such that*
(a) The constraint set (I) in equation (b1) is decoupled from the constraint set (III), in the sense that the dual values associated with constraint set I do not depend on constraint set III;
(b) The number of binding constraints in constraint set III is reduced so that the problem is no longer degenerate; and

(c) The binding constraint set I remains unchanged.

Proof. Define the perturbed problem with the calibration constraints defined as upper bounds without loss of generality.

Problem P2.

(b2)   maximize $f(x)$
      subject to $Ax = b$ (I)
                 $\hat{A}x < \hat{b}$ (II)
                 $Ix \leq \tilde{x} + e$ (III)

Any row of the nonbinding resource constraints (II) $\hat{A}x < \hat{b}$ in problem P1 can be written

(b3) $\displaystyle\sum_{j=1}^{k} |\hat{a}_{ij} x_j| < \hat{b}_i \qquad i = 1, ..., (1 - m)$

Select the constraint $i = 1, ..., (l - m)$ such that

$$b_i - \sum_{j=1}^{k} \hat{a}_{ij} \tilde{x}_j$$

is minimized. If $\varepsilon_j > 0, j = 1, ..., k$ are selected such that

(b4) $\displaystyle\sum_{j=1}^{k} |\hat{a}_{ij} \varepsilon_j| < \left[b_i - \sum_{j=1}^{k} \hat{a}_{ij} \tilde{x}_j \right].$

By rearranging (b4), an inequality holds for the constraint when $x = \tilde{x} + \varepsilon$, but x cannot exceed $\tilde{x} + \varepsilon$ from constraint set (III); therefore, those constraints in $Ax \leq b$ that are inactive under the values $\tilde{x}$ will remain inactive after the perturbation to $\tilde{x} + \varepsilon$.

The invariance of the binding resource constraints for (I) under the perturbation ε can be shown using the reduced gradient approach (Luenberger). Using (b4) we can write problem P2 using only constraint sets I and III.

(b5)   maximize $f(x)$
      subject to $Ax = b$
                 $Ix \leq \tilde{x} + \varepsilon$

where $A(m \times k)$, and $I = k \times k$. Invoking the nondegeneracy assumption for A and starting with the solution for problem P1 $\tilde{x}$, the constraints can be partitioned

(b6) $\begin{bmatrix} B & N \\ I_1 & \\ & I_2 \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} \begin{matrix} = \\ \leq \\ \leq \end{matrix} \begin{bmatrix} b \\ \tilde{x}_B + \varepsilon_B \\ \tilde{x}_N + \varepsilon_N \end{bmatrix}$

For brevity, the partition of $\mathbf{A}$ has been made so that the $(k - m)$ activities associated with $\mathbf{N}$ have the highest value of marginal products for the constraining resources. The reduced gradient for changes in $\tilde{\mathbf{x}}_N$ is therefore

$$\text{(b7)} \quad \mathbf{r}_{\mathbf{x}_N} = \nabla \mathbf{f}_{\tilde{\mathbf{x}}_N} - \nabla \mathbf{f}_{\tilde{\mathbf{x}}_B} \mathbf{B}^{-1} \mathbf{N}.$$

Since $\mathbf{f}(\bullet)$ is monotonically increasing in $\mathbf{x}_N$ and $\mathbf{x}_B$, the resource constraints will continue to be binding since the optimization criterion will maximize those activities with a nonnegative reduced gradient until the reduced gradient is zero or the upper-bound calibration constraint $\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}$ is encountered. Since $m < n$, the model overspecializes in the more profitable crops when subject only to constraint sets I and II. Under the specification in problem P2 the most profitable activities will not have a zero-reduced gradient before being constrained by the calibration set II at values of $\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}$. Thus, the binding constraint set I remains binding under the $\boldsymbol{\varepsilon}$ perturbation.

The resource vector for the resource constrained crop activities $(\mathbf{x}_B)$ now is

$$\text{(b8)} \quad \mathbf{b} - \mathbf{N}(\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon})$$

and from (b6)

$$\mathbf{x}_B = \mathbf{B}^{-1}[\mathbf{b} - \mathbf{N}(\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon})].$$

Since $\mathbf{B}$ is of full rank m, exactly m values of $\mathbf{x}_B$ are determined by the binding resource constraints, which depend on the input requirements for the subset of calibrated crop acre values $\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}$.

The slackness in the m calibration constraints associated with the m resource constrained output levels $\mathbf{x}_B$, follows from the monoticity of the production function in the rational stage of production. Since the production function is monotonic, the input requirement functions are also monotonic, and expansion of the output level of the subset of crop acreage to $\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}$ will have a nonpositive effect on the resource vector remaining for the vector of crop acreages constrained by the right-hand side, $\mathbf{x}_B$. That is

$$\text{(b9)} \quad \mathbf{b} - \mathbf{N}(\tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}_N) \le \mathbf{b} - \mathbf{N}\tilde{\mathbf{x}}_N \quad \text{for } \boldsymbol{\varepsilon}_N > 0.$$

But since the input requirement functions for the $\mathbf{x}_B$ subset are also monotonic, (b9) and (b6) imply that

$$\text{(b10)} \quad \mathbf{x}_B \le \tilde{\mathbf{x}}_B \quad \text{or} \quad \mathbf{x}_B < \tilde{\mathbf{x}}_B + \boldsymbol{\varepsilon}_B \quad \text{for } \boldsymbol{\varepsilon}_B > 0.$$

From (b10) it follows that the m perturbed upper bound calibration constraints associated with $\mathbf{x}_B$ will be slack at the optimum solution. Given (b4) and (b10), the constraints at the optimal solution to the perturbed problem P2 are

$$\text{(b11)} \quad \begin{bmatrix} \mathbf{B} & \mathbf{N} \\ \hat{\mathbf{A}}_1 & \hat{\mathbf{A}}_2 \\ \mathbf{I}_1 & \\ & \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_B \\ \tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}_N \end{bmatrix} \begin{matrix} = \mathbf{b} \\ < \hat{\mathbf{b}} \\ < \tilde{\mathbf{x}}_B + \boldsymbol{\varepsilon}_B \\ = \tilde{\mathbf{x}}_N + \boldsymbol{\varepsilon}_N \end{matrix}$$

Thus, there are k binding constraints, $\mathbf{b}(m \times 1)$ and $\mathbf{x}_n + \boldsymbol{\varepsilon}_N [(k - m) \times 1]$.

The dual constraints to this solution are

$$\text{(b12)} \quad \begin{bmatrix} \mathbf{B}' & \mathbf{0} \\ \mathbf{N}' & \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1^* \\ \overline{\boldsymbol{\lambda}_2^*} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}_B} \mathbf{f}(\mathbf{x}^*) \\ \nabla_{\mathbf{x}_N} \mathbf{f}(\mathbf{x}^*) \end{bmatrix}$$

using the partitioned inverse,

$$\text{(b13)} \quad \begin{bmatrix} \boldsymbol{\lambda}_1^* \\ \boldsymbol{\lambda}_2^* \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{Q} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{x}_B} \mathbf{f}(\mathbf{x}^*) \\ \nabla_{\mathbf{x}_N} \mathbf{f}(\mathbf{x}^*) \end{bmatrix}$$

where $\mathbf{P} = \mathbf{B}'^{-1}$ and $\mathbf{Q} = -\mathbf{N}'\mathbf{B}'^{-1}$.

Thus, the $\boldsymbol{\varepsilon}$ perturbation on the upper-bound constraint set II decouples the dual values of constraint set I from constraint set II. This ensures that k constraints are binding and the partitioning of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{N}$ is the unique outcome of the optimal solution to problem P2 in the first stage of PMP.