TROPICAL THERMODYNAMIC FORMALISM

ZHIQIANG LI AND YIQING SUN

Abstract. We investigate the correspondence between thermodynamic formalism and ergodic optimization. It has been known that the Bousch operator $\mathcal{L}_A$ is tropical linear and corresponds to the Ruelle operator $\mathcal{R}_A$. In this paper, we present general idempotent analysis (tropical algebra) results, define the tropical adjoint operator $\mathcal{L}_A^{\circledast}$ which corresponds to $\mathcal{R}_A^*$, and study the existence and generic uniqueness of tropical eigen-densities of $\mathcal{L}_A^{\circledast}$. We also investigate the Logarithmic type zero temperature limit of equilibrium states which implies the large deviation principle. It turns out that the rate function is the tropical product of a calibrated sub-action and a tropical eigen-density.

Contents

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1. Introduction

This paper is devoted to explore more connection between ergodic optimization and thermodynamic formalism. The connection of the two areas is related to the zero temperature limit called the Maslov dequantization and consequently the tropical (max-plus) algebra naturally appears. From this viewpoint, the tropical algebra corresponds to the linear algebra on $\mathbb{R}$, the Bousch operator as a tropical linear operator corresponds to the Ruelle operator, the maximal potential energy corresponds to (the exponential of) the topological pressure (as the eigenvalues of the corresponding operators), and the calibrated sub-actions (i.e., the tropical eigenfunctions of the Bousch operator) correspond to the eigenfuction of the Ruelle operator. We add to the existing connection that tropical eigen-densities are just the rate functions for the large deviation principle for the family of eigenmeasures parametrized by the temperature. For relevant background in dynamical systems, see [BLL13] and [LZ23]. For more about the tropical algebra and other applications, see [LMS01] and [Mik06]. Note that [LMS01] focuses on idempotent analysis, i.e., analysis in some abstract max-plus algebra, while [Mik06] is about tropical geometry which is related to algebraic geometry.

Thermodynamic formalism (in ergodic theory) dates back to the works of Ya. G. Sinai, R. Bowen, D. Ruelle, and others around early 1970s [Do68, Sin72, Bow75, Ru78], inspired by statistical mechanics. See also [Bro65, Lyu82] for early works in complex dynamics. Recall the measure-theoretic pressure

$$P_\mu(T,\varphi) = h_\mu(T) + \int \varphi \,\mathrm{d}\mu,$$

where $h_\mu(T)$ is the measure-theoretic entropy and μ is a T-invariant Borel probability measure. We call μ an *equilibrium state* if μ maximizes $P_\mu(T,\varphi)$. In particular, for a constant potential, an equilibrium state reduces to a *measure of maximal entropy*. Equilibrium states are the central focus on thermodynamic formalism. The Ruelle operator $\mathcal{R}_\varphi$, also known as the Ruelle–Perron–Frobenius operator or the transfer operator, was introduced by D. Ruelle to study the equilibrium states.

Our work can be seen as a tropical version of thermodynamic formalism. We first present a relatively complete study in idempotent analysis (tropical algebra) including the definitions and properties of completions of tropical spaces, tropical dual spaces, tropical continuious linear functionals, tropical measures, and densities. Then we investigate the existence and representations of the calibrated sub-actions (i.e., the tropical eigenfucntions of the Bousch operator), define the tropical adjoint operator of the Bousch operator, and investigate the existence and representations of the tropical eigen-densities of the tropical adjoint operator. We remark that generically (when the Hölder potential is uniquely maximizing) the tropical eigenfunction and the tropical eigen-density is unique up to a (tropical multiplicative) constant. Tools from ergodic optimization including the Aubry set and the Mañé potential are used when we give the representations. Moreover, we also investigate the zero temperature limit which links the corresponding objects of thermodynamic formalism to its tropical counterpart. Similar to the fact that the equilibrium state in thermodynamic formalism is the product of the eigenfunction and the eigenmeasure, we see that the zero temperature limit of the equilibrium states is a tropical product of the tropical eigenfuction and the tropical eigen-density.

1.1. **Preliminaries and assumptions.**

1.1.1. *Tropical algebra and uniformly expanding dynamical systems.* We consider $\overline{\mathbb{R}}_{\max} \coloneqq \mathbb{R} \cup \{+\infty, -\infty\}$ equipped with the tropical (max-plus) algebra:

$$x \oplus y \coloneqq \max\{x, y\}, \quad x \otimes y \coloneqq x + y, \quad \text{for } x, y \in \overline{\mathbb{R}}_{\max}.$$

Since $-\infty$ is seen as the tropical zero element, we adopt the convention that $-\infty \otimes +\infty = -\infty$. The basis $\{(a,b), [-\infty, a), (b, +\infty] : a, b \in \mathbb{R}\}$ generates the desired topology on $\overline{\mathbb{R}}_{\max}$. Define $\sup \emptyset \coloneqq -\infty$.

Let $C(X, \mathbb{R})$ be the space consisting of real-valued continuous functions on a topological space X. Let $\overline{\mathbb{R}}_{\max}^X$ denote the space of $\overline{\mathbb{R}}_{\max}$-valued functions on X. The space $\overline{\mathbb{R}}_{\max}^X$ with $(u \oplus v)(x) \coloneqq u(x) \oplus v(x)$ for all $u, v \in \overline{\mathbb{R}}_{\max}^X$ and $(\lambda \otimes u)(x) \coloneqq \lambda \otimes u(x)$ for all $u \in \overline{\mathbb{R}}_{\max}^X$ and $\lambda \in \overline{\mathbb{R}}_{\max}$ is a $\overline{\mathbb{R}}_{\max}$-semimodule and it has the natural order: $u \preccurlyeq v$ if and only if $u(x) \leqslant v(x)$ for all x in X, i.e., $u \preccurlyeq v$ if and only if $u \oplus v = v$. We also consider $(u \otimes v)(x) \coloneqq u(x) \otimes v(x)$ for all $u, v \in \overline{\mathbb{R}}_{\max}^X$.

In the sequel, $\oplus$ always means sup.

Note that $-\infty$ is the zero element (the additive identity) of $\overline{\mathbb{R}}_{\max}$, 0 is the multiplicative identity element of $\overline{\mathbb{R}}_{\max}$, and the constant $-\infty$ function is the zero element of $\overline{\mathbb{R}}_{\max}^X$.

We use $\mathbb{0}_X$ and $\mathbb{1}_X$ to represent the constant zero and one functions on X, respectively.

In this paper, we focus on uniformly expanding systems $T\colon X \to X$ that satisfy the following **Assumptions**:

(i) X is a compact metric space with the metric d.

(ii) T is open, continuous, and topologically transitive (thus surjective).

(iii) T is distance expanding, i.e., there exist constants $\lambda > 1$ and $\eta > 0$ such that $d(x, y) \leqslant 2\eta$ implies $d(Tx, Ty) \geqslant \lambda d(x, y)$ for $x, y \in X$.

1.1.2. *Ruelle operators and Bousch operators.* Let $\operatorname{Lip}(X, d^\alpha)$ denote the space of α-Hölder continuous functions $\varphi\colon X \to \mathbb{R}$ with respect to the metric d where $\alpha \in (0, 1]$. For $A \in C(X, \mathbb{R})$, the operator $\mathcal{R}_A\colon C(X, \mathbb{R}) \to C(X, \mathbb{R})$ given by

$$u \mapsto \mathcal{R}_A(u)(x) \coloneqq \sum_{y \in T^{-1}(x)} u(y) e^{A(y)}$$

is called the *Ruelle operator* with potential A. For a uniformly expanding dyamical system $T\colon X \to X$, it is well known that for a Hölder potential $A \in \operatorname{Lip}(X, d^\alpha)$, $\mathcal{R}_A$ (resp. its adjoint operator $\mathcal{R}_A^*$) has a unique eigenfunction (resp. eigenmeasure) up to a constant with respect to the eigenvalue of maximal modulus, i.e., $e^{P(T,A)}$, where $P(T, A)$ is the topological pressure of T with respect to the potential A. Thus, if m_A is a probability measure that satisfies $m_A(X) = 1$ and $\mathcal{R}_A^*(m_A) = e^{P(T,A)} m_A$, while the function u_A satisfies $\int u_A \, \mathrm{d}m_A = 1$ and $\mathcal{R}_A(u_A) = e^{P(T,A)} u_A$, then $\mu_A \coloneqq u_A \cdot m_A$ is the unique equilibrium state.

Fix $A \in \operatorname{Lip}(X, d^\alpha)$. Let $\widetilde{\mathcal{R}}_A(u) := \frac{1}{e^{P(T,A)}u_A}\mathcal{R}_A(uu_A)$ be the normalized Ruelle operator. Note that $\widetilde{\mathcal{R}}_{\beta A}$ is just the Ruelle operator with potential

$$g_\beta = \beta A + \log u_{\beta A} - \log u_{\beta A} \circ T - P(T, \beta A) \tag{1.1}$$

and $\widetilde{\mathcal{R}}_{\beta A}(\mathbb{1}_X) = \mathbb{1}_X, \widetilde{\mathcal{R}}^*_{\beta A}(\mu_{\beta A}) = \mu_{\beta A}$. See for example, [PU10, Chapters 3 and 5] for more details.

For each $\beta > 0$, denote

$$l^\mu_\beta(f) := \frac{1}{\beta} \log \int e^{\beta f} \,\mathrm{d}\mu_{\beta A} \quad \text{and} \quad l^m_\beta(f) := \frac{1}{\beta} \log \int e^{\beta f} \,\mathrm{d}m_{\beta A}, \tag{1.2}$$

for $f \in C(X, \mathbb{R})$. When we say the "Logarithmic type zero temperature limits", we are considering the accumulation points (in the compact-open topology) of $\{\frac{1}{\beta} \log u_{\beta A}\}_{\beta \in (0,+\infty)}$, $\{l^\mu_\beta(\cdot)\}_{\beta\in(0,+\infty)}$, or $\{l^m_\beta(\cdot)\}_{\beta\in(0,+\infty)}$ as $\beta \to +\infty$.

Definition 1.1. Let $T \colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in C(X, \mathbb{R})$. The Bousch operator $\mathcal{L}_A$ with potential A is defined by

$$\mathcal{L}_A(u)(x) := \bigoplus_{y \in T^{-1}(x)} (u(y) \otimes A(y)) = \sup_{y\in T^{-1}(x)} \{u(y) + A(y)\},$$

for all $u \in C(X, \mathbb{R})$ and $x \in X$. We define its tropical adjoint operator $\mathcal{L}^{\circledast}_A$ by

$$\mathcal{L}^{\circledast}_A(b)(x) := b(T(x)) \otimes A(x) = b(T(x)) + A(x) \tag{1.3}$$

for each $x \in X$ and each density $b \in D_{\max}(X)$.

Remark 1.2. For the Bousch operator $\mathcal{L}_A$, we will recall the fact that $\mathcal{L}_A(u) \in C(X, \mathbb{R})$ for all $u \in C(X, \mathbb{R})$ in Proposition 3.2. It immediately follows that $\mathcal{L}_A$ is a tropical linear map (see also [LZ23, Lemma 6.1]). For the precise definition of tropical linearity, see Definition 2.5.

For the tropical adjoint operator $\mathcal{L}^{\circledast}_A$, as we will see, every tropical linear functional in $C(X, \mathbb{R})^{\circledast}$ corresponds to a unique density in $D_{\max}(X)$ (see Proposition 2.12 and Remark 2.17) and every finite tropical measure corresponds to a unique density in $D_{\max}(X) \smallsetminus \{+\infty\}$ (see Proposition 2.16 and Remark 2.17). For the definition of $D_{\max}(X)$, see Remark 2.17.

We choose to define the tropical adjoint operator $\mathcal{L}^{\circledast}_A$ on the space of densities $D_{\max}(X)$. For a justification for our definition in formula (1.3), see Remark 3.7. For the precise definitions of tropical linear functionals, $C(X, \mathbb{R})^{\circledast}$, tropical measures, and densities, see Definitions 2.5, 2.9, 2.14, and 2.15, respectively.

The constant $Q(T, A)$, which we call the *maximal potential energy*, is defined as

$$Q(T, A) := \sup\left\{ \int A \,\mathrm{d}\mu : \mu \in M(X, T) \right\},$$

where $M(X, T)$ is the set of all T-invariant Borel probability measures on X. The supremum is attained due to the weak*-compactness of $M(X, T)$. We call a measure $\mu \in$

$M(X,T)$ that satisfies $\int A\,\mathrm{d}\mu = Q(T,A)$ a *maximizing measure* for T and A and denote the (nonempty) set of maximizing measures by

$$M_{\max}(T,A) \coloneqq \left\{\mu \in M(X,T) : \int A\,\mathrm{d}\mu = Q(T,A)\right\}.$$

The study of maximizing measures became known as *ergodic optimization* after O. Jenkinson [Je06]. Ergodic optimization originated in 1990s from the works of B. R. Hunt and E. Ott [HO96a, HO96b], with motivation from control theory [OGY90, SGYO93], and the Ph.D. thesis of O. Jenkinson [Je96]. For comprehensive surveys on the subject, see O. Jenkinson [Je06, Je19], J. Bochi [Boc18], and A.T. Baraviera, R. Leplaideur, and A.O. Lopes [BLL13].

Note that $\overline{A} \coloneqq A - Q(T,A)$ is the normalized potential.

We have the following definitions:

(i) $u \in C(X,\mathbb{R})$ is a *tropical eigenfunction* of $\mathcal{L}_A$ (associated with eigenvalue $Q(T,A)$) if $\mathcal{L}_A(u) = u \otimes Q(T,A)$, i.e., $\mathcal{L}_{\overline{A}}(u) = u$.

(ii) $u \in C(X,\mathbb{R})$ is a *sub-action* if $\mathcal{L}_A(u) \preccurlyeq u \otimes Q(T,A)$.

(iii) A density $b \in D_{\max}(X)$ is a *tropical eigen-density* of $\mathcal{L}_A^{\circledast}$ (associated with eigenvalue $Q(T,A)$) if $\mathcal{L}_A^{\circledast}(b) = b \otimes Q(T,A)$, i.e., $\mathcal{L}_{\overline{A}}^{\circledast}(b) = b$.

As we will see in Theorem A the tropical eigenvalue for $\mathcal{L}_A$ (resp. $\mathcal{L}_A^{\circledast}$) is unique.

In ergodic optimization (e.g.[Ga17]), a tropical eigenfunction of $\mathcal{L}_A$ is called a "calibrated sub-action". In order to find a sub-action, T. Bousch [Bou00] proposed to find a calibrated sub-action which is a fixed point of $\mathcal{L}_{\overline{A}}$. The operator $\mathcal{L}_A$ was studied by T. Bousch in [Bou00], which is why we call $\mathcal{L}_A$ the Bousch operator (with potential A). It is also known as the Bousch–Lax operator or the Lax operator in the literature since an analogous construction gives the *Lax–Oleinik semi-groups* in the context of Hamiltonian systems.

We call the potential A *uniquely maximizing* if $M_{\max}(T,A)$ consists of a single measure. It is worth mentioning that for an open and dense subset of A in $\mathrm{Lip}(X,d^{\alpha})$, A is uniquely maximizing. This is proved in G. Contreras, A.O. Lopes, and P. Thieullen [CLT01].

1.2. **Main results.** In [LZ23], it is proposed to study the "dictionary" for the correspondence between ergodic optimization and thermodynamic formalism. A proof of the existence of tropical eigenfunctions following this dictionary is presented in [LZ23]. We investigate along this direction and study the existence of tropical eigen-densities and generic uniqueness of both tropical functions and tropical eigen-densities.

Theorem A (Existence and generic uniqueness of tropical eigenfucntions and tropical eigen-densities.)**.** *Suppose that $T\colon X \to X$ satisfies the Assumptions in Subsection 1.1 and $A \in \mathrm{Lip}(X,d^{\alpha})$. Then the following statements hold:*

(i) *(Existence of tropical eigenfunction) For each $u \in \mathrm{Lip}(X,d^{\alpha})$, define*

$$v_u(x) \coloneqq \limsup_{n\to+\infty} \mathcal{L}_{\overline{A}}^n(u)(x),$$

for each x in X. Then $v_u \in \mathrm{Lip}(X,d^{\alpha})$ and v_u is a tropical eigenfunction of $\mathcal{L}_A$.

(ii) *(Generic uniqueness of tropical eigenfunction) For an open and dense subset of A in $\operatorname{Lip}(X, d^\alpha)$, $\mathcal{L}_A$ has a unique tropical eigenfunction up to a tropical multiplicative constant.*

(iii) *(Existence of tropical eigen-density) There exists a tropical eigen-density of $\mathcal{L}_A^{\circledast}$ different from constant density functions $-\infty$ and $+\infty$.*

(iv) *(Generic uniqueness of tropical eigen-density) For an open and dense subset of A in $\operatorname{Lip}(X, d^\alpha)$, $\mathcal{L}_A^{\circledast}$ has a unique tropical eigen-density up to a tropical multiplicative constant.*

Moreover, $Q(T, A)$ is the unique tropical eigenvalue of $\mathcal{L}_A$ (resp. $\mathcal{L}_A^{\circledast}$).

In order to study the generic uniqueness of tropical eigenfunctions and tropical eigen-densities, we study the general representations which can be described by tools in ergodic optimization.

Theorem B (Representation of tropical eigenfunctions and eigen-densities)**.** *Suppose that $T\colon X \to X$ satisfies the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. Let $\phi_A(\cdot,\cdot)\colon X \times X \to \mathbb{R} \cup \{-\infty\}$ be the Mañé potential associated with A and Ω_A be the Aubry set with respect to A. Then the following statements hold:*

(i) *For every tropical eigenfunction v of $\mathcal{L}_A$, the identity*
$$v(y) = \bigoplus_{x\in\Omega_A} (v(x) \otimes \phi_A(x, y))$$
holds for every y in X.

(ii) *For every tropical eigen-density b of $\mathcal{L}_A^{\circledast}$, we have that $b(\cdot)$ is equivalent to $\bigoplus_{y\in\Omega_A} (\phi_A(\cdot, y) \otimes b(y))$, i.e.,*
$$\bigoplus_{x\in X} (f(x) \otimes b(x)) = \bigoplus_{x\in X, y\in\Omega_A} (f(x) \otimes \phi_A(x, y) \otimes b(y))$$
for every $f \in C(X, \mathbb{R})$.

(iii) *If A is uniquely maximizing, then the entries of $\{\phi_A(x, \cdot)\}_{x\in\Omega_A}$ (resp. $\{\phi_A(\cdot, y)\}_{y\in\Omega_A}$) are the same tropical eigenfunction of $\mathcal{L}_A$ (resp. eigen-density of $\mathcal{L}_A^{\circledast}$) up to a tropical multiplicative constant.*

The Mañé potential and the Aubry set are recalled in Definitions 3.11 and 3.10, respectively. While the parts of Theorem B on eigenfunctions have appeared in [Ga17, Propositions 6.2 and 6.7], the parts on eigen-densities are new.

The zero temperature limits can be seen as a bridge connecting thermodynamic formalism objects and their tropical counterparts. We study the Logarithmic type zero temperature limits below.

Theorem C (Logarithmic type zero temperature limits of eigenfuctions and eigendensities)**.** *Suppose that $T\colon X \to X$ satisfies the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. Then the following two statements hold:*

(i) *The family $\left\{\frac{1}{\beta} \log u_{\beta A}\right\}_{\beta\in(1,+\infty)}$ is normal and the (uniform) limit of every convergent subsequence $\left\{\frac{1}{\beta_n} \log u_{\beta_n A}\right\}_{n\in\mathbb{N}}$ with $\beta_n \to +\infty$ as $n \to +\infty$ is a tropical eigenfunction of $\mathcal{L}_A$.*

(ii) *The family* $\{l^m_\beta(\cdot)\}_{\beta\in(0,+\infty)}$ *is normal and the (pointwise) limit of every convergent subsequence* $\{l^m_{\beta_n}(\cdot)\}_{n\in\mathbb{N}}$ *with* $\beta_n \to +\infty$ *as* $n\to+\infty$ *is a tropical linear functional whose density is a tropical eigen-density of* $\mathcal{L}^{\circledast}_A$.

(iii) *The family* $\{l^\mu_\beta(\cdot)\}_{\beta\in(0,+\infty)}$ *is normal and the (pointwise) limit of every convergent subsequence* $\{l^\mu_{\beta_n}(\cdot)\}_{n\in\mathbb{N}}$ *with* $\beta_n \to +\infty$ *as* $n\to+\infty$ *is a tropical linear functional whose density is the tropical product of a tropical eigenfunction of* $\mathcal{L}_A$ *and a tropical eigen-density of* $\mathcal{L}^{\circledast}_A$.

Here l^m_β and l^μ_β are defined in (1.2) and $u_{\beta A}$ is the eigenfunction of the Ruelle operator $\mathcal{R}_{\beta A}$ defined in Subsection 1.1.

While Theorem C (i) may be a well-known result, Theorem C (ii) and (iii) are new and it follows from Theorem C (ii) and (iii) that the density functions are the corresponding rate functions.

Theorem D (Logarithmic type zero temperature limits of equilibrium states and normalized potentials)**.** *Let* $T\colon X\to X$ *satisfy the Assumptions in Subsection 1.1 and* $A\in\operatorname{Lip}(A,d^\alpha)$. *Then the following statements hold:*

(i) *The two families* $\{\frac{g_\beta}{\beta}\}_{\beta\in(1,+\infty)}$ *and* $\{l^\mu_\beta(\cdot)\}_{\beta\in(0,+\infty)}$ *are normal.*

(ii) *Suppose that* $\{\frac{g_{\beta_k}}{\beta_k}\}_{k\in\mathbb{N}}$ *(uniformly) converges to* $\widehat{A}$ *and* $\{l^\mu_{\beta_k}(\cdot)\}_{k\in\mathbb{N}}$ *(pointwise) converges to* $\widehat{l}(\cdot)$ *with* $\beta_k\to+\infty$ *as* $k\to+\infty$. *Then* $\widehat{l}$ *is a tropical linear functional. Let* $\widehat{b}$ *be the density of* $\widehat{l}$ *in* $D_{\max}(X)$. *Then* $\mathcal{L}^{\circledast}_{\widehat{A}}\big(\widehat{b}\big)=\widehat{b}$, *i.e.,* $\widehat{b}\circ T+\widehat{A}=\widehat{b}$.

Here g_β is the normalized potential defined in (1.1) and $D_{\max}(X)$ is defined in Remark 2.17. Theorem D is a generalization of results in [Me18].

Recall that a family of real-valued continuous functions on X (resp. functionals on $C(X,\mathbb{R})$) is a *normal family* if for every sequence of functions (resp. functionals) of this family, there exists a subsequnce that is uniformlly converging on every compact subsets of X (resp. $C(X,\mathbb{R})$). It immediately follows that if a sequence of real-valued continuous functions on X (resp. functionals on $C(X,\mathbb{R})$) of a normal family pointwise converges, then the pointwise limit of this sequence is the limit of this sequence in the compact-open topology.

In Theorems C and D, all the limits of functions (on X) metioned are uniform limits and all the limits of functionals (on $C(X,\mathbb{R})$) metioned are pointwise limits. Once the normality of a certain family is verified, we do not distinguish between the two kinds of limits.

Theorem A is our main result which corresponds to thermodynamic formalism. We give original proofs of (i) and (iii) following the correspondence of thermodynamic formalism and its tropical couterpart. For (i), recall that the eigenfunction of the Ruelle operator $\mathcal{R}_\varphi$ with respect to a potential $\varphi\in\operatorname{Lip}(X,d^\alpha)$ is the limit of $\frac{1}{n}\sum_{i=0}^n\mathcal{R}^i_{\overline{\varphi}}(u)$ as $n\to+\infty$ for every $u\in\operatorname{Lip}(X,d^\alpha)$ where $\overline{\varphi}:=P(T,\varphi)^{-1}\varphi$. Since $+$ correspnds to $\oplus$ and $\overline{\varphi}$ corresponds to $\overline{A}$, we shall consider some tropical sum $\bigoplus_i\mathcal{L}^i_{\overline{A}}(u)$. It is worth noticing that we use the tail $\bigoplus_{i\geqslant n}\mathcal{L}^i_{\overline{A}}(u)$ instead of $\bigoplus_{i\leqslant n}\mathcal{L}^i_{\overline{A}}(u)$ but one notices that the limit of $\frac{1}{n}\sum_{i=0}^n\mathcal{R}^i_{\overline{\varphi}}(u)$ only depends on the tail of $\{\mathcal{R}^n_{\overline{\varphi}}(u)\}_{n\in\mathbb{N}}$ (see Proposition 3.5 and Corollary 3.6). For (iii), recall that existence of the eigenmeasure of $\mathcal{R}_\varphi^*$ follows from general functional analysis results (the Schauder–Tychonoff fixed point theorem). Thus, we refer to properties of tropical spaces. We use the completeness of the tropical space $\widehat{C(X,\mathbb{R})}$ and apply a version of Perron's method to prove (iii) (see Proposition 3.9).

In Theorem B, we use the Mañé potential to represent tropical eigenfunctions and tropical eigen-densities. Theorem B (i) is well known, see Proposition 3.13 (see also [Ga17, Proposition 6.2]). It is worth mentioning that the proof of Theorem B (ii) interestingly relies on the constructive result Corollary 3.6 (i.e., Theorem A (i)) for tropical eigenfunctions. Theorem A (ii) and (iv) follow from Theorem B (iii). While Theorem A (ii) is a combination of two well-known results (see [Bou00, Lemma C] and [CLT01]), Theorem A is new.

Theorem C is a direct result of our perspective about the Logarithmic type zero temperature limits. Combining with Theorems A and B, we immediately see that if the Hölder potential A is uniquely maximizing, then the accumulation point for the Logarithmic type zero temperature limit in Theorem C (i)(ii) is unique. Recall that the equilibrium state is the product of an eigenfunction of the Ruelle operator $\mathcal{R}_\varphi$ and an eigenmeasure of $\mathcal{R}_\varphi^*$ in our setting (see for example, [PU10, Section 5.2]). Thus, we have the Logarithmic type zero temperature limit for equilibrium states, which implies the large deviation principle for equilibrium states. In this way, we prove that the equilibrium states $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$ satisfy the large deviation principle if the Hölder potential A is uniquely maximizing (Corollary 4.3). To study the equivalence between the large deviation principle of the equilibruim states and the two Logarithmic type zero temperature limits for eigenfunctions and eigenmeasures, we establish Theorem D as a generalization of Corollary 4.4. In the context of subshifts of finite type, Corollary 4.3 has been proved via a "dual shift" technique in [BLT11], and [Me18] proved Corollary 4.4 using methods specific to subshifts of finite type. Our methods in our proofs of Corollaries 4.3 and 4.4 clearly reflect our philosophy in this paper.

The outline of the paper is as follows. In Section 2, we introduce idempotent analysis results in our setting, which include the definitions of completeness, tropical dual spaces, and tropical measures. In Section 3, we focus on the Bousch operator. Subsection 3.1 follows the ideas of [LZ23] in order to reach a constructive proof of Theorem A (i) (i.e., Proposition 3.5 and Corollary 3.6). In Subsection 3.2, we recall the concept of the Mañé potential and discover that tropical eigen-densities of the tropical dual of the Bousch operator can be represented by the Mañé potential. Theorem B is proved there. In Section 4, we investigate the Logarithmic type zero temperature limits and establish Theorems C and D. Moreover, Theorems C and D can simplify the proofs of results in [BLT11] and [Me18] (i.e., Corollaries 4.3 and 4.4) and give a clear picture from our perspective. We keep the proofs that are known to experts in Appendix A.

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2. General analysis results

In this section, we prepare some general idempotent analysis results including the definitions of completeness, tropical dual spaces, and tropical measures.

In this section, we always assume that (X, d) is a metric space. For all $x \in X$ and $r > 0$, denote $B(x, r) \coloneqq \{y \in X : d(x, y) < r\}$. We use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. **Tropical spaces and tropical dual spaces.**

We focus on tropical spaces and tropical dual spaces in this subsection, especially the completion and tropical dual of $C(X, \mathbb{R})$.

We introduce the following notion of completeness and completion, which is different from the "normal completion" in [LMS01, Page 9].

Definition 2.1. Let $S \subseteq \overline{\mathbb{R}}_{\max}^X$. The subset S is said to be *complete* if for each subset $U \subseteq S$, the (pointwise) supremum taken in $\overline{\mathbb{R}}_{\max}^X$, $\sup\{u(\cdot) : u \in U\}$, is in S. We use $\oplus$ to denote sup, and $\bigoplus_{u \in U} u$ means $\sup\{u(\cdot) : u \in U\}$. The completion of S, denoted by $\widehat{S}$, is defined to be the intersection of all complete subsets of $\overline{\mathbb{R}}_{\max}^X$ containing S.

Recall that a function $u \colon X \to \overline{\mathbb{R}}_{\max}$ is upper (resp. lower) semi-continuous if for every $b \in \overline{\mathbb{R}}_{\max}$, the set $\{x \in X : u(x) < b\}$ (resp. $\{x \in X : u(x) > b\}$) is open.

Remark 2.2. The "normal completion" of $C(X, \mathbb{R})$ is $LSC(X) \cup \{+\infty, -\infty\}$ (i.e., real-valued lower semi-continuous functions and the two constant infinity funcitons) while $\widehat{C(X, \mathbb{R})}$ contains more lower semi-continuous functions which take $+\infty$ somewhere but do not equal $+\infty$ everywhere. Indeed, different kinds of completions can be seen as analogs of different norms on (conventional) linear spaces.

Lemma 2.3. *If $g \colon X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, then there exists a family $\{u_i\}_{i \in I}$ in $C(X, \mathbb{R})$ such that $g = \bigoplus_{i \in I} u_i$.*

Proof. Baire's Theorem (see [Ha17]) implies that every lower semi-continuous function $g \colon X \to \mathbb{R} \cup \{+\infty\}$ on a metric space X is the limit of a non-decreasing sequence of continuous functions and if $g \colon X \to \mathbb{R}$, then the continuous functions in the sequence desribed above can be chosen to be real-valued. So if g is real-valued, using Baire's Theorem directly leads to the desired result. If g takes $+\infty$ somewhere, we consider $g_n(x) \coloneqq \min\{g(x), n\}$ for all $n \in \mathbb{N}$ and $x \in X$. Note that for all $n \in \mathbb{N}$, g_n is both lower semi-continuous and real-valued. Then we obtain a non-decreasing sequence $\{f_{k,n}\}_{k \in \mathbb{N}}$ in $C(X, \mathbb{R})$ with

$$\lim_{k \to +\infty} f_{k,n}(x) = g_n(x)$$

for all $x \in X$. Therefore, we conclude that $g = \bigoplus_{k,n \in \mathbb{N}} f_{k,n}$. □

Since our main focus is the space $C(X, \mathbb{R})$, we shall give a description of its completion as follows.

Proposition 2.4. $\widehat{C(X, \mathbb{R})} = \{g \colon X \to \mathbb{R} \cup \{+\infty\} : g$ *lower semi-continuous*$\} \cup \{-\infty\}$.

Proof. Denote $W := \{g\colon X \to \mathbb{R}\cup\{+\infty\} : g \text{ lower semi-continuous}\}\cup\{-\infty\}$, and we need to show that $W = \widehat{C(X,\mathbb{R})}$. By Definition 2.1, it suffices to check that W is complete and each complete subset of $\overline{\mathbb{R}}_{\max}^X$ containing $C(X,\mathbb{R})$ contains W.

We first prove the completeness of W. For each family $\{g_v\}_{v\in V}$ in W, g_v is lower semi-continuous for all v in V. Thus, $\bigoplus_{v\in V} g_v$ is lower semi-continuous. Moreover, if there exists v_0 in V such that g_{v_0} is not constant function $-\infty$, then $\bigoplus_{v\in V} g_v\colon X \to \mathbb{R}\cup\{+\infty\}$ easily follows from $g_{v_0}\colon X \to \mathbb{R}\cup\{+\infty\}$. Otherwise $g_v = -\infty$ for every v in V and $\bigoplus_{v\in V} g_v = -\infty \in W$. We conclude that $\bigoplus_{v\in V} g_v$ is always in W. The completeness of W follows.

Now we show that every complete set $\widetilde{W} \subseteq \overline{\mathbb{R}}_{\max}^X$ containing $C(X,\mathbb{R})$ contains W. Note that $-\infty = \bigoplus_{u\in\emptyset} u \in \widetilde{W}$. By Lemma 2.3, for each $g \in W \setminus \{-\infty\}$, there exists a family $\{u_i\}_{i\in I}$ in $C(X,\mathbb{R}) \subseteq \widetilde{W}$ such that $g = \bigoplus_{i\in I} u_i$. Now $g \in \widetilde{W}$ follows from the completeness of $\widetilde{W}$. We conclude that $W \subseteq \widetilde{W}$. $\square$

Now we define tropical continuous linear maps based on completeness following [LMS01]. Recall that a *R-semimodule* is a set S equipped with a binary operation $S\times S \to S$, which we denote by $\oplus$, and a map from $R\times S \to S$, which we denote by $\otimes$, with the operations for the ring R also denoted by $\oplus$ and $\otimes$, provided that the following axioms are satisfied:

(i) $(a\oplus b)\oplus c = a\oplus(b\oplus c)$ for a,b,c in S.
(ii) $a\oplus b = b\oplus a$ for a,b in S.
(iii) There is an element 0_S in S such that $0_S\oplus a = a$ for every a in S.
(iv) $\lambda\otimes(a\oplus b) = (\lambda\otimes a)\oplus(\lambda\otimes b)$ for $\lambda\in R$ and $a,b\in S$.
(v) $(\lambda_1\oplus\lambda_2)\otimes a = (\lambda_1\otimes a)\oplus(\lambda_2\otimes a)$ for $\lambda_1,\lambda_2\in R$ and $a\in S$.
(vi) $0_R\otimes a = 0_S$ for every $a\in S$.
(vii) $1_R\otimes a = a$ for every $a\in S$.
(viii) $(\lambda_1\otimes\lambda_2)\otimes a = \lambda_1\otimes(\lambda_2\otimes a)$ for $\lambda_1,\lambda_2\in R$ and $a\in S$.

Definition 2.5. Let $V, W \subseteq \overline{\mathbb{R}}_{\max}^X$ be two $\overline{\mathbb{R}}_{\max}$-semimodules. Let $\mathcal{L}$ be a map from V to W.

(i) We call $\mathcal{L}$ *tropical linear* if
$$\mathcal{L}(u\oplus v) = \mathcal{L}(u)\oplus\mathcal{L}(v) \quad\text{and}\quad \mathcal{L}(\lambda\otimes u) = \lambda\otimes\mathcal{L}(u),$$
for all $\lambda\in\overline{\mathbb{R}}_{\max}$ and $u,v\in V$.

(ii) We call $\mathcal{L}$ *tropical continuous* if $\mathcal{L}$ can be extended uniquely to a tropical linear map $\mathcal{L}\colon \widehat{V}\to\widehat{W}$ satisfying
$$\mathcal{L}\Big(\bigoplus_{u\in U} u\Big) = \bigoplus_{u\in U}\mathcal{L}(u)$$
for every subset $U\subseteq\widehat{V}$.

(iii) If $W = \overline{\mathbb{R}}_{\max}$ (seen as the set of constant functions in $\overline{\mathbb{R}}_{\max}^X$), then $\mathcal{L}$ is called a *tropical functional*.

When $\mathcal{L}$ satisfies more than one properties above, we adopt the convention to only use one "tropical", for example, a tropical linear functional.

Let $V, W \subseteq \overline{\mathbb{R}}_{\max}^X$ be two $\overline{\mathbb{R}}_{\max}$-semimodules and $\mathcal{L}$ be a tropical linear map from V to W. Suppose $u, v \in V$ and $u \preccurlyeq v$, i.e., $u \oplus v = v$. Then $\mathcal{L}(v) = \mathcal{L}(u \oplus v) = \mathcal{L}(u) \oplus \mathcal{L}(v)$, i.e., $\mathcal{L}(u) \preccurlyeq \mathcal{L}(v)$. Thus, $\mathcal{L}$ is an order-preserving map. (As subsets of $\overline{\mathbb{R}}_{\max}^X$, V, W inherits the natural order on $\overline{\mathbb{R}}_{\max}^X$.) We record this well-known fact below.

Lemma 2.6. *Let $V, W \subseteq \overline{\mathbb{R}}_{\max}^X$ be two $\overline{\mathbb{R}}_{\max}$-semimodules and $\mathcal{L}$ be a tropical linear map from V to W. Then $\mathcal{L}$ is an order-preserving map.*

We discover the following interesting result which plays a role in the discussion on zero temperature limits.

Proposition 2.7. *If X is compact, then every tropical linear functional $\mathcal{L}\colon C(X, \mathbb{R}) \to \overline{\mathbb{R}}_{\max}$ is tropical continuous.*

Proof. We need to prove that $\mathcal{L}$ can be uniquely extended to some $\widehat{\mathcal{L}}\colon \widehat{C(X, \mathbb{R})} \to \overline{\mathbb{R}}_{\max}$ satisfying $\widehat{\mathcal{L}}\big(\bigoplus_{u \in U} u\big) = \bigoplus_{u \in U} \widehat{\mathcal{L}}(u)$ for each subset $U \subseteq \widehat{C(X, \mathbb{R})}$ and $\widehat{\mathcal{L}}(\lambda \otimes g) = \lambda \otimes \widehat{\mathcal{L}}(g)$ for all $g \in \widehat{C(X, \mathbb{R})}$ and $\lambda \in \overline{\mathbb{R}}_{\max}$. We first construct an extension and then verify the uniqueness.

We consider $\widehat{\mathcal{L}}(g) := \bigoplus_{u \in C(X, \mathbb{R}), u \preccurlyeq g} \mathcal{L}(u)$ for $g \in \widehat{C(X, \mathbb{R})}$. If $g \in C(X, \mathbb{R})$, then the tropical linearity of $\mathcal{L}$ and Lemma 2.6 imply $\mathcal{L}(u) \leqslant \mathcal{L}(g)$ for all $u \in C(X, \mathbb{R})$ with $u \preccurlyeq g$. Note $g \preccurlyeq g$. Thus, $\widehat{\mathcal{L}}(g) = \mathcal{L}(g)$, i.e., $\widehat{\mathcal{L}}$ is an extension of $\mathcal{L}$. Next, we check the linearity and continuity condition for $\widehat{\mathcal{L}}$. Fix $g \in \widehat{C(X, \mathbb{R})}$. Note that when $\lambda \in \mathbb{R}$,

$$\widehat{\mathcal{L}}(\lambda \otimes g) = \bigoplus_{u \in C(X, \mathbb{R}), u \preccurlyeq \lambda \otimes g} \mathcal{L}(u) = \bigoplus_{u \in C(X, \mathbb{R}), u \preccurlyeq g} \mathcal{L}(\lambda \otimes u) = \lambda \otimes \widehat{\mathcal{L}}(g). \tag{2.1}$$

When $\lambda \in \{+\infty, -\infty\}$, the identities in (2.1) obviously hold.

To prove $\widehat{\mathcal{L}}\big(\bigoplus_{u \in U} u\big) = \bigoplus_{u \in U} \widehat{\mathcal{L}}(u)$, denote $g := \bigoplus_{u \in U} u$ for some arbitrary $U \subseteq \widehat{C(X, \mathbb{R})}$. According to our construction of $\widehat{\mathcal{L}}$, $u \preccurlyeq g$ implies $\widehat{\mathcal{L}}(u) \leqslant \widehat{\mathcal{L}}(g)$. We conclude that $\bigoplus_{u \in U} \widehat{\mathcal{L}}(u) \leqslant \widehat{\mathcal{L}}(g)$. Now we need to show $\widehat{\mathcal{L}}(g) \leqslant \bigoplus_{u \in U} \widehat{\mathcal{L}}(u)$. According to our construction of $\widehat{\mathcal{L}}$, it suffices to prove that for every v in $C(X, \mathbb{R})$ satisfying $v \preccurlyeq g$, $\mathcal{L}(v) \leqslant \bigoplus_{u \in U} \widehat{\mathcal{L}}(u)$.

Fix $\epsilon > 0$ and $v \in C(X, \mathbb{R})$ with $v \preccurlyeq g = \bigoplus_{u \in U} u$. For every $x \in X$, there exists $u_x \in U$ such that $u_x(x) > v(x) - \epsilon$. Since $u_x \in \widehat{C(X, \mathbb{R})}$, by Lemma 2.3 there exists $w_x \in C(X, \mathbb{R})$ such that $w_x \preccurlyeq u_x$ and $w_x(x) > v(x) - \epsilon$. Now that w_x and v are both in $C(X, \mathbb{R})$, $w_x > v - 2\epsilon$ holds in some neighbourhood $B(x, r_x)$. Thus, $\bigcup_{x \in X} B(x, r_x)$ forms an open cover of X and compactness of X implies that there is a finite cover $X = \bigcup_{i=1}^{n} B(x_i, r_{x_i})$. We conclude that $v - 2\epsilon \preccurlyeq \bigoplus_{1 \leqslant i \leqslant n} w_{x_i}$.

Note that $\bigoplus_{1\leqslant i\leqslant n} w_{x_i}$ is in $C(X,\mathbb{R})$ since w_{x_i} is in $C(X,\mathbb{R})$. Thus, the linearity of $\mathcal{L}$, $w_x \preccurlyeq u_x$, and $u_x \in U$ implies that

$$\mathcal{L}(v) - 2\epsilon \leqslant \mathcal{L}\big(\bigoplus_{1\leqslant i\leqslant n} w_{x_i}\big) = \bigoplus_{1\leqslant i\leqslant n} \mathcal{L}(w_{x_i}) \leqslant \bigoplus_{1\leqslant i\leqslant n} \widehat{\mathcal{L}}(u_{x_i}) \leqslant \bigoplus_{u\in U} \widehat{\mathcal{L}}(u).$$

Let $\epsilon \to 0^+$ and we conclude that $\mathcal{L}(v) \leqslant \bigoplus_{u\in U} \widehat{\mathcal{L}}(u)$ and the continuity follows.

Finally, we verify the uniqueness of the extension. Let $\widetilde{\mathcal{L}}$ be an extension of $\mathcal{L}$ satisfying $\widetilde{\mathcal{L}}\big(\bigoplus_{u\in U} u\big) = \bigoplus_{u\in U} \widetilde{\mathcal{L}}(u)$ for each subset $U \subseteq \widehat{C(X,\mathbb{R})}$ and $\widetilde{\mathcal{L}}(\lambda \otimes g) = \lambda \otimes \widetilde{\mathcal{L}}(g)$ for all $g \in \widehat{C(X,\mathbb{R})}$ and $\lambda \in \overline{\mathbb{R}}_{\max}$.

Fix $g \in \widehat{C(X,\mathbb{R})}$, consider $U \coloneqq \{u \in C(X,\mathbb{R}) : u \preccurlyeq g\}$. By Lemma 2.3, we have $\bigoplus_{u\in U} u = g$. Now since $\widetilde{\mathcal{L}}$ is a tropical continuous linear extension of $\mathcal{L}$, we see that $\widetilde{\mathcal{L}}(g) = \bigoplus_{u\in U} \widetilde{\mathcal{L}}(u) = \bigoplus_{u\in C(X,\mathbb{R}), u\preccurlyeq g} \mathcal{L}(u) = \widehat{\mathcal{L}}(g)$. Uniqueness follows. $\square$

Remark 2.8. This proposition suggests that when X is compact, there is no difference between tropical linear functionals and tropical continuous linear functionals. Furthermore, we observe that when X is compact, we can replace "every subset $U \subseteq \widehat{V}$" with "every countable subset $U \subseteq \widehat{V}$" in Definition 2.5 (ii).

Definition 2.9. Let $V \subseteq \overline{\mathbb{R}}_{\max}^X$ be a $\overline{\mathbb{R}}_{\max}$-semimodule. The *tropical dual space* $V^{\circledast}$ of V is defined to be the space consisting of all tropical continuous linear functionals from $\widehat{V}$ to $\overline{\mathbb{R}}_{\max}$.

In the sequel, we discuss the connection between the completion and the tropical dual space. The following definitions are important for the representation of the tropical dual space as well as some analysis in Section 4. Similar notions also appear in [CGQ04] and [LMS01].

Definition 2.10. For $u, v \in \overline{\mathbb{R}}_{\max}^X$, we define

$$u \oslash v \coloneqq \sup\big\{\lambda \in \overline{\mathbb{R}}_{\max} : \lambda \otimes v \preccurlyeq u\big\} \in \overline{\mathbb{R}}_{\max} \quad \text{and}$$
$$u \mathbin{\dot{\div}} v \coloneqq u \otimes (-v) \in \overline{\mathbb{R}}_{\max}^X.$$

Recall that $\sup \emptyset = -\infty$.

Definition 2.11. For each $v \in \widehat{C(X,\mathbb{R})}$, we denote

$$l_v(f) \coloneqq -(v \oslash f) = \sup_{x\in X}\{f(x) - v(x)\}$$

for every $f \in \widehat{C(X,\mathbb{R})}$.

Since the sup operation is exchangeable, it immediately follows that l_v is a tropical continuous linear functional, i.e., $l_v \in C(X,\mathbb{R})^{\circledast}$.

Due to the subtlety of our notion of completion in Definition 2.1, the following bijective relation can be verified.

Proposition 2.12. *The functional l_v defined above is a tropical linear functional and the map $l_\bullet\colon \widehat{C(X,\mathbb{R})} \to C(X,\mathbb{R})^{\circledast}$ given by $v \mapsto l_v$ is a bijection.*

Proof. We first prove that $l_\bullet$ is surjective and then verify its injectivity.

Let $L\colon \widehat{C(X,\mathbb{R})} \to \overline{\mathbb{R}}_{\max}$ be a tropical continuous linear functional. First we consider the case where the range of L does not contain any real number, i.e., $L(f) \in \{+\infty, -\infty\}$, for all f in $\widehat{C(X,\mathbb{R})}$. Note again that we adopt $-\infty - (-\infty) = -\infty$ and $+\infty - (+\infty) = -\infty$ following $-\infty \otimes +\infty = -\infty$. If there exists $u \in C(X,\mathbb{R})$ such that $L(u) = -\infty$, then $L(+\infty) = L(+\infty \otimes u) = +\infty \otimes L(u) = +\infty \otimes -\infty = -\infty$. Note that $+\infty$ is the maximal element of $\widehat{C(X,\mathbb{R})}$, it follows that $L(f) = -\infty$ for all $f \in \widehat{C(X,\mathbb{R})}$. Otherwise, $L(u) = +\infty$ for every u in $C(X,\mathbb{R})$. We conclude that the only two tropical linear functionals for the first case are

$$L(f) = -\infty \quad \text{and} \quad L(f) = \begin{cases} -\infty & \text{if } f \equiv -\infty, \\ +\infty & \text{if } f \not\equiv -\infty. \end{cases}$$

The former is $l_{+\infty}$ and the latter is $l_{-\infty}$.

Now we suppose that L takes some real value, then the range of L must contain $\mathbb{R}$ since L is tropical linear. Consider $M \coloneqq \left\{ f \in \widehat{C(X,\mathbb{R})} : L(f) \leqslant 0 \right\}$ and it follows that $L(M) \coloneqq \{L(f) : f \in M\}$ also contains some real number. Since $\widehat{C(X,\mathbb{R})}$ is complete, we consider $v \coloneqq \bigoplus_{f\in M} f$ and v is in $\widehat{C(X,\mathbb{R})}$.

Claim. $L(v) = 0$ and $L = l_v$.

The tropical continuity of L and the definition of M immediately implies that $L(v) = \bigoplus_{f\in M} L(f) \leqslant 0$. Note that $L(M) \cap \mathbb{R} \neq \emptyset$ and this implies $L(v) \in \mathbb{R}$. Thus,

$$L((-L(v)) \otimes v) = (-L(v)) \otimes L(v) = 0,$$

and $(-L(v)) \otimes v$ is in M. Recall that $v = \bigoplus_{f\in M} f$ is the maximal element of M, it follows that $(-L(v)) \otimes v \preccurlyeq v$. Thus,

$$0 = (-L(v)) \otimes L(v) = L((-L(v)) \otimes v) \leqslant L(v)$$

and we conclude that $L(v) = 0$.

Futhermore, for each $f \in \widehat{C(X,\mathbb{R})}$, $L((-L(f)) \otimes f) = (-L(f)) \otimes L(f) \leqslant 0$ implies that $(-L(f)) \otimes f \in M$. Since v is the maximal element of M, we have $(-L(f)) \otimes f \leqslant v$, i.e., $-L(f) \leqslant v \oslash f$ (recall Definition 2.10). Meanwhile, it follows from Definition 2.10 ($(v \oslash f) \otimes f \preccurlyeq v$) and Lemma 2.6 that

$$v \oslash f + L(f) = L((v \oslash f) \otimes f) \leqslant L(v) = 0,$$

i.e., $v \oslash f \leqslant -L(f)$. Combine the two inequalities together and our claim follows.

We conclude that $l_\bullet$ is surjective.

Finally, we need to verify the injectivity of $l_\bullet$, i.e., for each pair of u, v in $\widehat{C(X,\mathbb{R})}$, if

$$\sup_{x\in X}\{f(x) - v(x)\} = \sup_{x\in X}\{f(x) - u(x)\}$$

for all f in $\widehat{C(X,\mathbb{R})}$, then $v = u$. Now take $f = u$ in the condition and we get $u \preccurlyeq v$. Take $f = v$ and we get $v \preccurlyeq u$. We conclude that $u = v$ and injectivity follows. $\square$

The following basic fact is used in the proofs of Proposition 3.8 and Theorem B.

Proposition 2.13. *Assume that X is compact. Then every tropical linear functional $\mathcal{L}\colon C(X,\mathbb{R}) \to \overline{\mathbb{R}}_{\max}$ is continuous with respect to C^0 topology on $C(X,\mathbb{R})$. Moreover, let $\widehat{\mathcal{L}}$ be the tropical continuous extension of $\mathcal{L}$. If $\widehat{\mathcal{L}} \neq l_{-\infty}$ and $\widehat{\mathcal{L}} \neq l_{+\infty}$, then the range of $\mathcal{L}$ is contained in $\mathbb{R}$ and*

$$|\mathcal{L}(u) - \mathcal{L}(v)| \leqslant \|u - v\|_{C^0}$$

for all u, v in $C(X,\mathbb{R})$.

Proof. Fix a tropical linear functional $\mathcal{L}\colon C(X,\mathbb{R}) \to \overline{\mathbb{R}}_{\max}$. If $\widehat{\mathcal{L}} = l_{-\infty}$ (resp. $l_{+\infty}$), then $\mathcal{L}$ is the constant $+\infty$ (resp. $-\infty$) functional. So $\mathcal{L}$ is continuous.

If $\widehat{\mathcal{L}} \neq l_{-\infty}$ and $\widehat{\mathcal{L}} \neq l_{+\infty}$, then by Proposition 2.12, there exsits $v \in \widehat{C(X,\mathbb{R})} \smallsetminus \{+\infty, -\infty\}$ such that $\widehat{\mathcal{L}} = l_v$. Thus, $\mathcal{L}(f) = \sup_{x\in X}\{f(x) - v(x)\}$ for all $f \in C(X,\mathbb{R})$.

By Proposition 2.4, $v \in \widehat{C(X,\mathbb{R})} \smallsetminus \{+\infty, -\infty\}$ implies that $v\colon X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous and there exists $x_0 \in X$ so that $v(x_0) \in \mathbb{R}$. The lower semi-continuity of v and the compactness of X imply that there exists $C \in \mathbb{R}$ so that $v(x) \geqslant C$ for all $x \in X$.

We conclude that

$$f(x_0) - v(x_0) \leqslant \sup_{x\in X}\{f(x) - v(x)\} \leqslant \|f\|_{C^0} - C$$

for all $f \in C(X,\mathbb{R})$. Since X is compact, it follows that $\mathcal{L}(f) \in \mathbb{R}$ for all $f \in C(X,\mathbb{R})$.

Moreover, fix $u_1, u_2 \in C(X,\mathbb{R})$. Since $u_1 \preccurlyeq u_2 \otimes \|u_1 - u_2\|_{C^0}$ and $u_2 \preccurlyeq u_1 \otimes \|u_2 - u_1\|_{C^0}$, it follows from Lemma 2.6 and the tropical linearity of $\mathcal{L}$ that

$$\mathcal{L}(u_1) \leqslant \|u_1 - u_2\|_{C^0} \otimes \mathcal{L}(u_2) \quad \text{and} \quad \mathcal{L}(u_2) \leqslant \|u_2 - u_1\|_{C^0} \otimes \mathcal{L}(u_1).$$

Recall $\mathcal{L}(f) \in \mathbb{R}$ for all $f \in C(X,\mathbb{R})$. We conclude that $|\mathcal{L}(u_1) - \mathcal{L}(u_2)| \leqslant \|u_1 - u_2\|_{C^0}$ and it follows that $\mathcal{L}$ is continuous. $\square$

2.2. **Tropical (cost) measures.**

This subsection is devoted to the connection of abstract tropical measures and tropical linear functionals: the function $-v$ appearing in the representation of tropical linear functionals should be the density b of the corresponding tropical measures.

We recall the following definitions in [ACG94, Definition 18].

Definition 2.14. Let $\mathcal{U}$ be the collection of all open subsets of X. A map $m\colon \mathcal{U} \to \overline{\mathbb{R}}_{\max}$ is a *tropical measure* if it satisfies the following conditions:

(i) $m(\emptyset) = -\infty$.
(ii) $m\big(\bigcup_{i\in I} A_i\big) = \bigoplus_{i\in I} m(A_i)$, where I is countable and $A_i \in \mathcal{U}$ for each $i \in I$.

When $m(X) < +\infty$, m is *finite*; and m is a *tropical probability measure* if $m(X) = 0$.

Remark. In [ACG94, Definition 18], tropical probability measures are called *cost measures*.

Definition 2.15. A function $b\colon X \to \overline{\mathbb{R}}_{\max}$ is a *density* of a tropical measure m if

$$m(U) = \bigoplus_{u \in U} b(u)$$

for every open subset U of X.

For a tropical measure m with a density b, we define the tropical integral with respect to m by

$$\int_V^{\oplus} f(x)\,\mathrm{d}m \coloneqq \bigoplus_{x \in V} (f(x) \otimes b(x))$$

for each open subset V of X and each $f \in C(X, \mathbb{R})$. For the well-definedness of the tropical integral, see Remark 2.17 below.

A function $b\colon X \to \overline{\mathbb{R}}_{\max}$ is a *density* of a tropical linear functional l on $C(X, \mathbb{R})$ if

$$l(f) = \bigoplus_{x \in X} (f(x) \otimes b(x))$$

for all $f \in C(X, \mathbb{R})$.

Proposition 2.16. *For each finite tropical measure m on a compact space X, there exists a unique upper semi-continuous function $b\colon X \to \mathbb{R} \cup \{-\infty\}$ such that b is a density of m.*

Remark. Here we present a direct proof in the case where X is compact, which is the only case we need. For more general discussions, see [ACG94, Theorem 19], [Ak99, Proposition 3.15], and [Ak99, Corollary 3.22].

Proof. We first verify the existence. By the definition of tropical measures, $m(A \cup B) = m(A) \oplus m(B)$. Thus, $A \subseteq B$ implies $m(A) \leqslant m(B)$. Then for every $x \in X$, we define $b(x)$ as

$$b(x) \coloneqq \lim_{\epsilon \to 0^+} m(B(x, \epsilon)).$$

We need to show that $b(\cdot)$ is upper semi-continuous and

$$m(U) = \bigoplus_{u \in U} b(u)$$

for every open subset U of X.

Consider a sequence $\{x_k\}$ in X that converges to x in X as $k \to +\infty$. Note that

$$b(x_k) \leqslant m(B(x_k, d(x, x_k))) \leqslant m(B(x, 2d(x, x_k))).$$

Combining $x_k \to x$ and the definition of $b(x)$, we see that

$$\limsup_{k \to +\infty} b(x_k) \leqslant b(x).$$

This establishs the upper semi-continuity.

Now suppose that U is open, we get $b(u) \leqslant m(U)$ for every u in U from the definition of b. Thus,

$$m(U) \geqslant \bigoplus_{u \in U} b(u).$$

To prove the inverse inequality, we need to use the fact that X is a second-countable topological space, so that for every open cover of U, we can find a countable subcover of U. Fix $\epsilon > 0$. By the definition of b, there exists a neighbourhood $B(u, r_u)$ such that $b(u) \otimes \epsilon \geqslant m(B(u, r_u))$ for every u in U. Now that $\{B(u, r_u)\}_{u \in U}$ forms an open cover of

U, we then have a countable subcover $\{B(u_k, r_{u_k})\}_{k\in\mathbb{N}}$. According to the definition of m, we get

$$m(U) \leqslant \bigoplus_{k\in\mathbb{N}} m(B(u_k, r_{u_k})) \leqslant \Big(\bigoplus_{u\in U} b(u)\Big) \otimes \epsilon.$$

As ϵ tends to 0 from above, we get

$$m(U) \leqslant \bigoplus_{u\in U} b(u).$$

We conclude that b is a density of m and the finiteness of m implies that $b\colon X \to \mathbb{R}\cup\{-\infty\}$.

Finally, it is straightforward to check that if $b\colon X \to \mathbb{R} \cup \{-\infty\}$ is an upper semi-continuous density of the measure m, then $b(x)$ must be equal to the limit of $m(B(x,\epsilon))$ as ϵ tends to 0 from above. The uniqueness follows. $\square$

Remark 2.17. Denote

$$D_{\max}(X) \coloneqq \{b\colon X \to \mathbb{R} \cup \{-\infty\} : b \text{ upper semi-continuous}\} \cup \{+\infty\}. \tag{2.2}$$

Assume that X is compact. By Proposition 2.16, $D_{\max}(X) \smallsetminus \{+\infty\}$ consists of upper semi-continuous densities of finite tropical measures. Moreover, $v \in \widehat{C(X,\mathbb{R})}$ is equivalent to $-v \in D_{\max}(X)$. Thus, by Proposition 2.12, $D_{\max}(X)$ consists of densities of tropical continuous linear functionals and $b \mapsto l_{-b}$ gives a bijection from $D_{\max}(X)$ to $C(X,\mathbb{R})^{\circledast}$. Recall Definition 2.15. We conclude that Proposition 2.16 together with Proposition 2.12 forms a tropical analog of the Riesz representation theorem for $C(X,\mathbb{R})$ when X is compact.

Existence of the density provides convenience to study tropical measures and tropical linear functionals. But one needs to be aware that a tropical measure (resp. linear functional) can have different densities that may not be upper semi-continuous. So in the sequel, when we say 2 densities b_1 and b_2 are *equivalent*, we mean that they induce the same tropical linear functional, i.e., $\bigoplus_{x\in X}(f(x)\otimes b_1(x)) = \bigoplus_{x\in X}(f(x)\otimes b_2(x))$ for all $f \in C(X,\mathbb{R})$.

Note that there is a slight difference between $\bigoplus_{x\in X}(f(x)\otimes b_1(x)) = \bigoplus_{x\in X}(f(x)\otimes b_2(x))$ for all $f \in C(X,\mathbb{R})$ and $\bigoplus_{u\in U} b_1(u) = \bigoplus_{u\in U} b_2(u)$ for every open subset U of X. If $\bigoplus_{x\in X} b_1(x) < +\infty$ and $\bigoplus_{x\in X} b_2(x) < +\infty$, then the two conditions are equivalent.

Generally, one can prove that $\bigoplus_{u\in U} b_1(u) = \bigoplus_{u\in U} b_2(u)$ for every open subset U of X implies $\bigoplus_{x\in X}(f(x)\otimes b_1(x)) = \bigoplus_{x\in X}(f(x)\otimes b_2(x))$ for all $f \in C(X,\mathbb{R})$ and consequently for all $f \in \widehat{C(X,\mathbb{R})}$. Fix an open subset V of X and a function $h \in C(X,\mathbb{R})$. Note that the function h_V which equals h at points in V and takes $-\infty$ at points in $X \smallsetminus V$ is lower semi-continuous. We conclude that $h_V \in \widehat{C(X,\mathbb{R})}$ and consequently $\bigoplus_{x\in X}(h_V(x)\otimes b_1(x)) = \bigoplus_{x\in X}(h_V(x)\otimes b_2(x))$, i.e., $\bigoplus_{x\in V}(h(x)\otimes b_1(x)) = \bigoplus_{x\in V}(h(x)\otimes b_2(x))$. This justifies our definition of tropical integral in Definition 2.15.

For every b_1 with $\bigoplus_{x\in X} b_1(x) = +\infty$, b_1 and the constant function $+\infty$ induce the same tropical linear functional but they may not induce the same tropical measure.

We can also define the counterparts of invariant measures and ergodic measures for future reference. Consider the dynamical system (X,T) where $T\colon X \to X$ is a continuous map. We focus on the case where X is compact so that density functions are available.

Definition 2.18. Let $T\colon X \to X$ be a continuous transformation on a compact topological space X. Let m be a finite tropical measure on X with the upper semi-continuous density $b\colon X \to \mathbb{R} \cup \{-\infty\}$.

(i) m is *T-invariant* if for every point x in X, $b(x) = \bigoplus_{y \in T^{-1}(x)} b(y)$.

(ii) m is *ergodic* if m is *T-invariant* and for every point x in X,

$$\lim_{k \to +\infty} b\bigl(T^k(x)\bigr) = \begin{cases} \bigoplus_{y \in X} b(y) & \text{if } b(x) \in \mathbb{R}, \\ -\infty & \text{if } b(x) = -\infty. \end{cases}$$

3. Bousch operator and its tropical dual

In parallel to the classical Ruelle operator theory, we establish in this section the representation of tropical eigenfunctions and tropical eigen-densities with respect to a Hölder potential.

Tropical eigenfunctions will be systematically studied. We first present analysis similar to that in thermodynamic formalism which gives a specific constructive result for the tropical eigenfunctions (Corollary 3.6) and reach the general representation using the Mañé potential.

However, analysis becomes different for tropical eigen-densities. It turns out that the Mañé potential can also give a representation of tropical eigen-densities with the help of the constructive result Corollary 3.6.

Subsection 3.1 presents analysis similar to that in thermodynamic formalism and Subsection 3.2 presents analysis about the Mañé potential. To be consistent with the previous context, we adopt the notation of the tropical algebra. We always assume that $T\colon X \to X$ satisfies the Assumptions in Subsection 1.1.

3.1. **Analysis in the tropical thermodynamic approach.** This subsection can be seen as a tropical version of a part of the thermodynamic formalism. In the following, many ingredients are similar to propositions in the thermodynamic formalism, but there is much difference when we deal with the tropical adjoint operator. We present an original approach to the existence of tropical eigen-densities.

The following lemma is well known (see for example, [PU10, Lemmas 4.1.2 and 4.1.3]). We omit its proof.

Lemma 3.1. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1. Then there exists a constant $\xi > 0$ such that for each x in X, $B(T(x), \xi) \subseteq T(B(x, \eta))$. Here η is the constant from the Assumptions in Subsection 1.1.. Moreover, the restriction $T|_{B(x,\eta)}$ is injective and the inverse map $T_x^{-1}\colon B(T(x), \xi) \to B(x, \eta)$ has the property that $d\bigl(T_x^{-1}(y), T_x^{-1}(z)\bigr) \leqslant \lambda^{-1} d(y, z)$. Futhermore,*

$$\sup_{x \in X} \operatorname{card} T^{-1}(x) =: N < +\infty.$$

Remark. For each $n \in \mathbb{N}$, we denote $T_x^{-n}\colon B(T^n(x), \xi) \to B(x, \lambda^{-n}\xi)$ as the composition of inverse maps $T_{T^i(x)}^{-1}$ for $0 \leqslant i \leqslant n-1$.

We first deal with the construction of tropical eigenfucntions. The idea of this process follows [LZ23] but the proof is slightly easier in the present setting. For the convenience of the reader, we consequently leave the proofs of Proposition 3.2, Lemma 3.4, and Proposition 3.5 below to the Appendix A. As is already mentioned, the construction in Corollary 3.6 turns out to be useful in the general representation of tropical eigen-densities.

Recall that we use $\mathrm{Lip}(X, d^\alpha)$ to denote the space of α-Hölder continuous functions $\varphi\colon X \to \mathbb{R}$ with respect to the metric d where $\alpha \in (0, 1]$. For each $\varphi \in \mathrm{Lip}(X, d^\alpha)$, denote

$$|\varphi|_{d^\alpha} := \sup\biggl\{ \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha} : x, y \in X,\ 0 < d(x, y) \biggr\} \text{ and}$$
$$|\varphi|_{d^\alpha,\epsilon} := \sup\biggl\{ \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha} : x, y \in X,\ 0 < d(x, y) < \epsilon \biggr\}.$$

Recall that $\mathbb{0}_X$ is used to represent the constant zero function on X. For a potential A in $C(X, \mathbb{R})$, denote

$$S_nA(x) := A(x) + A(T(x)) + \cdots + A\bigl(T^{n-1}(x)\bigr).$$

Proposition 3.2. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1. The operator $\mathcal{L}_A$ with potential $A \in C(X, \mathbb{R})$ (resp. $\mathrm{Lip}(X, d^\alpha)$) maps a real-valued continuous (resp. α-Hölder) function to a real-valued continuous (resp. α-Hölder) function.*

Lemma 3.3. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \mathrm{Lip}(X, d^\alpha)$. Then the Bousch operator $\mathcal{L}_A\colon C(X, \mathbb{R}) \to C(X, \mathbb{R})$ satisfies*

$$\|\mathcal{L}_A(u_1) - \mathcal{L}_A(u_2)\|_{C^0} \leqslant \|u_1 - u_2\|_{C^0}$$

for all u_1, u_2 in $C(X, \mathbb{R})$.

Proof. Note that $u_1 \preccurlyeq u_2 \otimes \|u_1 - u_2\|_{C^0}$ and $u_2 \preccurlyeq u_1 \otimes \|u_2 - u_1\|_{C^0}$. Thus, by Lemma 2.6 and the tropical linearity of $\mathcal{L}_A$ (directly from definition), we have

$$\mathcal{L}_A(u_1) \preccurlyeq \mathcal{L}_A(u_2) \otimes \|u_1 - u_2\|_{C^0} \quad \text{and} \quad \mathcal{L}_A(u_2) \preccurlyeq \mathcal{L}_A(u_1) \otimes \|u_1 - u_2\|_{C^0}.$$

It follows that $\|\mathcal{L}_A(u_1) - \mathcal{L}_A(u_2)\|_{C^0} \leqslant \|u_1 - u_2\|_{C^0}$. $\square$

Lemma 3.4. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1. For every $A \in \mathrm{Lip}(X, d^\alpha)$, there exists a constant $C_1(A) > 0$ such that for all $n \in \mathbb{N}$ and $x, y \in X$, we have*

$$\Bigl| \bigoplus_{\overline{x} \in T^{-n}(x)} S_nA(\overline{x}) - \bigoplus_{\overline{y} \in T^{-n}(y)} S_nA(\overline{y}) \Bigr| \leqslant C_1(A).$$

Moreover, there exists a constant $C_2 > 0$ such that for all $A, u \in \mathrm{Lip}(X, d^\alpha)$ and $n \in \mathbb{N}$,

$$|\mathcal{L}_A^n(u)|_{d^\alpha} \leqslant C_2(|A|_{d^\alpha} + |u|_{d^\alpha}).$$

Proposition 3.5 (Construction of a tropical eigenfunction)**.** *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \mathrm{Lip}(X, d^\alpha)$. Consider $\overline{A} := A - Q(T, A)$ and $\mathcal{L}_{\overline{A}}$. Define*

$$v_{\mathbb{0}_X}(x) := \limsup_{n \to +\infty} \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(x)$$

for all x in X. Then $v_{\mathbb{0}_X}$ is a function that satisfies the following properties:

(i) $\|v_{\mathbb{0}_X}\|_{C^0} \leqslant C_2|A|_{d^\alpha}(\operatorname{diam} X)^\alpha$.

(ii) $v_{\mathbb{0}_X} \in \operatorname{Lip}(X, d^\alpha)$ *with* $|v_{\mathbb{0}_X}|_{d^\alpha} \leqslant C_2 |A|_{d^\alpha}$, $v_{\mathbb{0}_X}$ *is the uniform limit of*

$$\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(\mathbb{0}_X)$$

as $n \to +\infty$.

(iii) $\mathcal{L}_{\overline{A}}(v_{\mathbb{0}_X}) = v_{\mathbb{0}_X}$, *i.e.,* $v_{\mathbb{0}_X}$ *is a tropical eigenfunction of* $\mathcal{L}_A$ *with tropical eigenvalue* $Q(T, A)$.

Here $C_2 > 0$ *is the constant from Lemma 3.4.*

We generalize Proposition 3.5 in the following corollary.

Corollary 3.6. *Let* $T \colon X \to X$ *satisfy the Assumptions in Subsection 1.1 and* $A \in \operatorname{Lip}(X, d^\alpha)$. *For each* $u \in \operatorname{Lip}(X, d^\alpha)$, *denote*

$$v_u(x) \coloneqq \limsup_{n \to +\infty} \mathcal{L}_{\overline{A}}^n(u)(x),$$

for all x *in* X. *Let* $C_2 > 0$ *be the constant from Proposition 3.4. Then* v_u *satisfies the following properties:*

(i) $\|v_u\|_{C^0} \leqslant C_2 |A|_{d^\alpha} (\operatorname{diam} X)^\alpha + \|u\|_{C^0}$.

(ii) $v_u \in \operatorname{Lip}(X, d^\alpha)$ *with* $|v_u|_{d^\alpha} \leqslant C_2(|A|_{d^\alpha} + |u|_{d^\alpha})$, v_u *is the uniform limit of*

$$\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)$$

as $n \to +\infty$.

(iii) $\mathcal{L}_{\overline{A}}(v_u) = v_u$, *i.e.,* v_u *is a tropical eigenfunction of* $\mathcal{L}_A$ *with tropical eigenvalue* $Q(T, A)$.

Proof. By Lemma 3.3, we have $\bigl\|\mathcal{L}_{\overline{A}}^{n+1}(u) - \mathcal{L}_{\overline{A}}^{n+1}(\mathbb{0}_X)\bigr\|_{C^0} \leqslant \bigl\|\mathcal{L}_{\overline{A}}^n(u) - \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)\bigr\|_{C^0}$ for every $n \in \mathbb{N}_0$. Thus, $\bigl\|\mathcal{L}_{\overline{A}}^n(u) - \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)\bigr\|_{C^0} \leqslant \|u - \mathbb{0}_X\|_{C^0}$ for every $n \in \mathbb{N}$. We conclude that $\|v_u - v_{\mathbb{0}_X}\|_{C^0} \leqslant \|u - \mathbb{0}_X\|_{C^0}$. Now (i) follows from Proposition 3.5 (i).

By Lemma 3.4, for each $n \in \mathbb{N}$, we have $\bigl|\mathcal{L}_{\overline{A}}^n(u)\bigr|_{d^\alpha} \leqslant C_2(|A|_{d^\alpha} + |u|_{d^\alpha})$ and it follows that $\bigl|\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)\bigr|_{d^\alpha} \leqslant C_2(|A|_{d^\alpha} + |u|_{d^\alpha})$. Since v_u is the pointwise decreasing limit of $\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)$ as $n \to +\infty$, (i) implies a uniform lower bound of $\bigl\{\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)\bigr\}_{n \in \mathbb{N}}$. A uniform upper bound now follows from the above uniform Lipschitz constant estimate and the compactness of X.

We conclude that $\bigl\{\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)\bigr\}_{n \in \mathbb{N}}$ forms a normal family since it is equicontinuous and uniformly bounded. Thus, v_u is its uniform limit and it follows that $|v_u|_{d^\alpha} \leqslant C_2(|A|_{d^\alpha} + |u|_{d^\alpha})$.

Note that for each $n \in \mathbb{N}$, $\mathcal{L}_{\overline{A}}\bigl(\sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^k(u)\bigr) = \sup_{k \geqslant n+1} \mathcal{L}_{\overline{A}}^k(u)$ (see [LZ23, Lemma 6.1 (iii)]). By Lemma 3.3 and (ii), as $n \to +\infty$, the left-hand side of the above identity uniformly converges to $\mathcal{L}_{\overline{A}}(v_u)$ and the right-hand side of the above identity uniformly converges to v_u. Thus, $\mathcal{L}_{\overline{A}}(v_u) = v_u$ and (iii) is verified. $\square$

Remark. The map $u \mapsto v_u$ is a projection into the semimodule of tropical eigenfunctions of $\mathcal{L}_A$, which plays a role in the representation of tropical eigen-densities of $\mathcal{L}_{A}^{\circledast}$.

Now we look into the tropical adjoint operator $\mathcal{L}_A^{\circledast}$. Next remark explains the reason of the definition of $\mathcal{L}_A^{\circledast}$ in (1.3).

Remark 3.7. According to Remark 2.17, we define the tropical adjoint operators on the space $D_{\max}(X)$. It is straightforward to check that $b \in D_{\max}(X)$ implies $\mathcal{L}_A^{\circledast}(b) \in D_{\max}(X)$.

Recall $b \mapsto l_{-b}$ gives a bijection from $D_{\max}(X)$ to $C(X,\mathbb{R})^{\circledast}$. We denote $c \coloneqq \mathcal{L}_A^{\circledast}(b)$. Then

$$\begin{aligned} l_{-c}(u) &= \bigoplus_{x\in X}\big(u(x) + \mathcal{L}_A^{\circledast}(b)(x)\big) = \bigoplus_{x\in X}(u(x) + b(T(x)) + A(x)) \\ &= \bigoplus_{x\in X}\Big(\bigoplus_{y\in T^{-1}(x)}(u(y)+A(y)) + b(x)\Big) = \bigoplus_{x\in X}(\mathcal{L}_A(u)(x) + b(x)) = l_{-b}(\mathcal{L}_A(u)) \end{aligned}$$

for all $u \in C(X,\mathbb{R})$. The fact that T is surjective is used here. By identifying b with l_{-b}, $\mathcal{L}_A^{\circledast}$ can be seen as a map from $C(X,\mathbb{R})^{\circledast}$ to $C(X,\mathbb{R})^{\circledast}$, i.e., $\mathcal{L}_A^{\circledast}(l_{-b}) = l_{-c}$. Now the above identities imply that

$$\mathcal{L}_A^{\circledast}(l)(u) = l(\mathcal{L}_A(u))$$

for all $l \in C(X,\mathbb{R})^{\circledast}$ and $u \in C(X,\mathbb{R})$. To avoid confusions on notations, we will not use this identification in this paper.

For $b \in D_{max}(X) \smallsetminus \{+\infty\}$, let m_b be the finite tropical measure satifying $m_b(U) = \bigoplus_{x\in U} b(x)$ for every open subset U of X. Recall $b \mapsto m_b$ gives a bijection from $D_{\max}(X) \smallsetminus \{+\infty\}$ to the set of finite tropical measures, $c = \mathcal{L}_A^{\circledast}(b)$, and Definition 2.15. Then for every open subset U of X,

$$\begin{aligned} m_c(U) &= \bigoplus_{x\in U}\mathcal{L}_A^{\circledast}(b)(x) = \bigoplus_{x\in U}(b(T(x)) + A(x)) \\ &= \bigoplus_{x\in T(U)}\Big(b(x) + \bigoplus_{y\in T^{-1}(x)}A(y)\Big) = \int_{T(U)}^{\oplus}\bigoplus_{y\in T^{-1}(x)}A(y)\,\mathrm{d}m_b \end{aligned}$$

By identifying b with m_b, $\mathcal{L}_A^{\circledast}$ can be seen as a map on the set of finite tropical measures, i.e.,

$$\mathcal{L}_A^{\circledast}(m)(U) = \int_{T(U)}^{\oplus}\bigoplus_{y\in T^{-1}(x)}A(y)\,\mathrm{d}m$$

for every open subset U of X. Note that the above identity can serve as the defining identity for the tropical adjoint operator on other spaces of tropical measures.

Proposition 3.8 (Uniqueness of tropical eigenvalue)**.** *Let $T\colon X\to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X,d^\alpha)$. Then the following statements hold:*

(i) *If there exists $u \in C(X,\mathbb{R})$ and $\lambda \in \overline{\mathbb{R}}_{\max}$ such that $\mathcal{L}_A(u) = \lambda \otimes u$, then $\lambda = Q(T,A)$.*

(ii) *If there exists a density $b \in D_{\max}(X) \smallsetminus \{+\infty, -\infty\}$ and $\lambda \in \overline{\mathbb{R}}_{\max}$ such that $\mathcal{L}_A^{\circledast}(b) = \lambda \otimes b$, then $\lambda = Q(T,A)$.*

Proof. (i). Suppose $\mathcal{L}_A(u) = \lambda \otimes u$ for some $u \in C(X,\mathbb{R})$ and some $\lambda \in \overline{\mathbb{R}}_{\max}$. By Proposition 3.2, $\mathcal{L}_A(u) \in C(X,\mathbb{R})$, so $\lambda \in \mathbb{R}$. Recall that we have constructed $v_{\mathbb{0}_X} \in \operatorname{Lip}(X,d^\alpha)$ in Proposition 3.5 such that $\mathcal{L}_A(v_{\mathbb{0}_X}) = Q(T,A) \otimes v_{\mathbb{0}_X}$.

By Lemma 3.3, we have $\|\mathcal{L}_A^{n+1}(u) - \mathcal{L}_A^{n+1}(v_{\mathbb{0}_X})\|_{C^0} \leqslant \|\mathcal{L}_A^{n}(u) - \mathcal{L}_A^{n}(v_{\mathbb{0}_X})\|_{C^0}$ for every $n \in \mathbb{N}_0$. Thus, $\|\mathcal{L}_A^{n}(u) - \mathcal{L}_A^{n}(v_{\mathbb{0}_X})\|_{C^0} \leqslant \|u - v_{\mathbb{0}_X}\|_{C^0}$ for every $n \in \mathbb{N}$. Since $\mathcal{L}_A(u) = \lambda \otimes u$ and $\mathcal{L}_A(v_{\mathbb{0}_X}) = Q(T, A) \otimes v_{\mathbb{0}_X}$, it follows that

$$\|n(\lambda - Q(T, A)) + u - v_{\mathbb{0}_X}\|_{C^0} \leqslant \|u - v_{\mathbb{0}_X}\|_{C^0}$$

for every $n \in \mathbb{N}$. This implies $\lambda = Q(T, A)$.

(ii). Suppose $\mathcal{L}_A^{\circledast}(b) = \lambda \otimes b$ for some density $b \in D_{\max}(X) \setminus \{+\infty, -\infty\}$ and some $\lambda \in \overline{\mathbb{R}}_{\max}$. By Remark 3.7, l_{-b} is a tropical linear functional different from $l_{+\infty}$ and $l_{-\infty}$ satisfying

$$l_{-b}(\mathcal{L}_A(v)) = \lambda \otimes l_{-b}(v)$$

for all $v \in C(X, \mathbb{R})$. Thus, $l_{-b}(\mathcal{L}_A(v_{\mathbb{0}_X})) = \lambda \otimes l_{-b}(v_{\mathbb{0}_X})$. It follows from the tropical linearity of l_{-b} that

$$Q(T, A) \otimes l_{-b}(v_{\mathbb{0}_X}) = \lambda \otimes l_{-b}(v_{\mathbb{0}_X}).$$

By Proposition 2.13, it follows from $l_{-b} \neq l_{-\infty}$ and $l_{-b} \neq l_{+\infty}$ that $l_{-b}(v) \in \mathbb{R}$ for all $v \in C(X, \mathbb{R})$. Thus, $l_{-b}(v_{\mathbb{0}_X}) \in \mathbb{R}$ and $\lambda = Q(T, A)$. □

Here we demonstrate the existence of a tropical eigen-density.

Proposition 3.9. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. There exists a density $b \in D_{\max}(X) \setminus \{+\infty, -\infty\}$ such that*

$$\mathcal{L}_A^{\circledast}(b) = Q(T, A) \otimes b.$$

Proof. Since $\widehat{C(X, \mathbb{R})}$ is complete, we switch to $v := -b$ according to Remark 2.17 and it suffices to find a solution of $v \circ T = v + \overline{A}$ in $\widehat{C(X, \mathbb{R})} \setminus \{+\infty, -\infty\}$.

Recall that we have constructed a tropical eigenfunction $v_{\mathbb{0}_X} \in C(X, \mathbb{R})$ of $\mathcal{L}_A$ in Proposition 3.5, so

$$v_{\mathbb{0}_X} + \overline{A} \preccurlyeq v_{\mathbb{0}_X} \circ T. \tag{3.1}$$

Consider $w := v - v_{\mathbb{0}_X}$ and $\varphi := v_{\mathbb{0}_X} + \overline{A} - v_{\mathbb{0}_X} \circ T \in C(X, \mathbb{R})$. It suffices to find a nontrivial solution (not constant function $-\infty$ or $+\infty$) $w \in \widehat{C(X, \mathbb{R})}$ such that $w \circ T = w + \varphi$.

Note that $\varphi \preccurlyeq \mathbb{0}_X$ (see 3.1) and

$$\begin{aligned} Q(T, \varphi) &= \sup\biggl\{\int \varphi \,\mathrm{d}\mu : \mu \in M(X, T)\biggr\} = \max\biggl\{\int \varphi \,\mathrm{d}\mu : \mu \in M(X, T)\biggr\} \\ &= \max\biggl\{\int \overline{A} \,\mathrm{d}\mu : \mu \in M(X, T)\biggr\} = Q(T, \overline{A}) = 0. \end{aligned}$$

The supremum is attained due to the weak*-compactness of $M(X, T)$ and we fix a maximizing meaure $\mu \in M(X, T)$ for φ.

Note that $\operatorname{supp}(\mu) \subseteq \varphi^{-1}(0)$ follows from $C(X, \mathbb{R}) \ni \varphi \preccurlyeq \mathbb{0}_X$ and $\int \varphi \,\mathrm{d}\mu = 0$. Moreover, since μ is T-invariant, $T(\operatorname{supp}(\mu)) \subseteq \operatorname{supp}(\mu)$. We use these two properties to construct a solution $w = w_0$.

Consider $S := \bigl\{w \in \widehat{C(X, \mathbb{R})} : w + \varphi \preccurlyeq w \circ T,\, w|_{\operatorname{supp}(\mu)} = 0,\, \mathbb{0}_X \preccurlyeq w\bigr\}$ and $w_0 := \bigoplus_{w \in S} w \in \widehat{C(X, \mathbb{R})}$ (since $\widehat{C(X, \mathbb{R})}$ is complete). Note that $\varphi \preccurlyeq \mathbb{0}_X$ implies $\mathbb{0}_X \in S$. Thus, S is not empty and $w_0 \not\equiv -\infty$.

It is straightforward to check that $w_0 \in S$. Thus, $w_0 \not\equiv +\infty$ follows from $w_0|_{\mathrm{supp}(\mu)} = 0$. Now consider $w_1 := w_0 \circ T - \varphi$.

We claim that $w_1 \in S$.

Since $w_0 \in S$, we have $\mathbb{0}_X \preccurlyeq w_0 \preccurlyeq w_1$. So $w_0 \circ T \preccurlyeq w_1 \circ T$, i.e., $w_1 + \varphi \preccurlyeq w_1 \circ T$. Recall $\mathrm{supp}(\mu) \subseteq \varphi^{-1}(0)$ and $T(\mathrm{supp}(\mu)) \subseteq \mathrm{supp}(\mu)$. It follows that $w_1|_{\mathrm{supp}(\mu)} = 0$. We conclude that $w_1 \in S$.

It follows that $w_1 \preccurlyeq w_0$ since w_0 is the maximal element of S. We conclude that $w_1 = w_0$, $w_0 \circ T = w_0 + \varphi$, and $w_0 \in \widehat{C(X, \mathbb{R})} \smallsetminus \{+\infty, -\infty\}$. $\square$

3.2. **Mañé potential and representation.** In this subsection, we recall properties of the Mañé potential and the representation of tropical eigenfunctions (Proposition 3.13) and prepare for the representation of tropical eigen-densities (Theorem B).

As in Proposition 3.5, we write $\overline{A} := A - Q(T, A)$ in the following discussion. For the discussion below, whether the system is a subshift of finite type or a general uniformly expanding system makes no essential difference. We work in the more general setting while [Ga17] presents the theories for subshifts of finite type.

Definition 3.10 (Aubry set)**.** Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1. For a continuous potential A, we call $x \in X$ an *Aubry point* if for every $\epsilon > 0$, there exists $y \in X$ and $n \in \mathbb{N}$ such that

$$d(x, y) \leqslant \epsilon, \quad d(T^n(y), x) \leqslant \epsilon, \quad \text{and} \quad |S_n\overline{A}(y)| \leqslant \epsilon.$$

The collection of all Aubry points is called the *Aubry set* and denoted by Ω_A.

Some basic knowledge about the Aubry set is contained in Chapter 4 of [Ga17], for example the nonemptiness of the Aubry set and the fact that the Aubry set is closed and T-invariant. The mechanism of the nonemptiness will show up in the proof of Proposition 3.13.

Definition 3.11 (Mañé potential)**.** Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1. For a potential $A \in \mathrm{Lip}(X, d^\alpha)$, the *Mañé potential* associated with A is the function ϕ_A defined on $X \times X$ given by

$$\phi_A(x, y) := \lim_{\epsilon \to 0^+} \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{d(z,x) \leqslant \epsilon \\ d(T^n(z), y) \leqslant \epsilon}} S_n\overline{A}(z) = \lim_{\epsilon \to 0^+} \sup_{n \in \mathbb{N}} \sup_{\substack{d(z,x) \leqslant \epsilon \\ d(T^n(z), y) \leqslant \epsilon}} S_n\overline{A}(z)$$

Remark. In order to match the tropical (max-plus) algebra, our definition is a slightly different from that in [Ga17, Proposition 5.A], where the Mañé potential is defined as

$$\lim_{\epsilon \to 0^+} \inf_{n \in \mathbb{N}} \inf_{\substack{d(z,x) \leqslant \epsilon \\ d(T^n(z), y) \leqslant \epsilon}} (-S_n\overline{A}(z)).$$

It is straightforward to check that our $\phi_A(\cdot, \cdot)$ is upper semi-continuous. Moreover, it immediately follows from Proposition 3.5 (i) that there exists $D \in \mathbb{R}$ such that

$$S_n(\overline{A})(x) \leqslant D \tag{3.2}$$

for all $n \in \mathbb{N}$ and $x \in X$. Thus, $\phi_A(\cdot, \cdot)\colon X \times X \to \mathbb{R} \cup \{-\infty\}$.

Proposition 3.12 (Main properties of the Mañé potential)**.** *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \mathrm{Lip}(X, d^\alpha)$. Then the following statements hold for all $x, y, z \in X$:*

(i) *For every tropical eigenfunction u of $\mathcal{L}_A$,*

$$u(x) \otimes \phi_A(x, y) \leqslant u(y).$$

(ii) *For every tropical eigen-density b of $\mathcal{L}_A^{\circledast}$,*

$$\phi_A(x, y) \otimes b(y) \leqslant b(x).$$

(iii) $\phi_A(x, z) \geqslant \phi_A(x, y) \otimes \phi_A(y, z)$.

(iv) $x \in \Omega_A$ *if and only if* $\phi_A(x, x) = 0$.

(v) *If $x \in \Omega_A$, then $\phi_A(x, \cdot)$ is a tropical eigenfunction of $\mathcal{L}_A$.*

Statements (i), (iii), (iv), and (v) already appear in [Ga17, Proposition 5.2]. Statement (ii) is a new. Statement (v) is the counterpart of Lemma 3.14. We only prove (ii) here. For a sketch of the proof of others, see Appendix A.

Proof. (ii) Since b is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$, $b(T(x)) \otimes \overline{A}(x) = b(x)$ for every $x \in X$ and b is upper semi-continuous. Since $A \in \mathrm{Lip}(X, d^\alpha)$ and b takes values in $\overline{\mathbb{R}}_{\max}$, it follows that $b(x) \in \mathbb{R}$ if and only if $b(T(x)) \in \mathbb{R}$ and that $b(x) = -\infty$ if and only if $b(T(x)) = -\infty$. Thus,

$$b(T^n(x)) \otimes S_n(\overline{A})(x) = b(x) \tag{3.3}$$

for all $x \in X$.

Denote $\widetilde{\phi_A}(x, y) := \lim\limits_{\epsilon \to 0^+} \bigoplus\limits_{n \in \mathbb{N}} \bigoplus\limits_{\substack{d(y_0, x) \leqslant \epsilon \\ T^n(y_0) = y}} S_n(\overline{A})(y_0)$.

Claim. $\widetilde{\phi_A} \equiv \phi_A$.

To prove the claim, by Definition 3.11, it immediately follows that

$$\phi_A(x, y) = \lim_{\epsilon \to 0^+} \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{d(z, x) \leqslant \epsilon \\ d(T^n(z), y) \leqslant \epsilon}} S_n\overline{A}(z) \geqslant \lim_{\epsilon \to 0^+} \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{d(z, x) \leqslant \epsilon \\ T^n(z) = y}} S_n\overline{A}(z) = \widetilde{\phi_A}(x, y)$$

for all $x, y \in X$. So it suffices to prove $\widetilde{\phi_A}(x, y) \geqslant \phi_A(x, y)$ for all $x, y \in X$.

Fix $x, y \in X$, $n \in \mathbb{N}$, and $\epsilon \in (0, \xi)$ where ξ is the constant in Lemma 3.1. We show that for every $z \in X$ satisfying $d(z, x) \leqslant \frac{\epsilon}{2}$ and $d(T^n(z), y) \leqslant \frac{\epsilon}{2}$, there exists $y_0 \in X$ satisfying $d(y_0, x) \leqslant \epsilon$ and $T^n(y_0) = y$ such that

$$\big|S_n(\overline{A})(z) - S_n(\overline{A})(y_0)\big| \leqslant \frac{|A|_{d^\alpha} \epsilon^\alpha}{2^\alpha(\lambda^\alpha - 1)}.$$

According to Lemma 3.1, if $d(z, x) \leqslant \frac{\epsilon}{2}$ and $d(T^n(z), y) \leqslant \frac{\epsilon}{2}$, then there exists $y_0 \in X$ such that $T^n(y_0) = y$ and $d\big(T^i(z), T^i(y_0)\big) \leqslant \lambda^{-n+i} \frac{\epsilon}{2}$ for all $0 \leqslant i \leqslant n$. Since A is in

$\operatorname{Lip}(X, d^\alpha)$,

$$
\begin{aligned}
|S_n(\overline{A})(z) - S_n(\overline{A})(y_0)| &\leqslant \sum_{i=0}^{n-1} |A|_{d^\alpha} d\bigl(T^i(z), T^i(y_0)\bigr)^\alpha \\
&\leqslant |A|_{d^\alpha} \sum_{i=0}^{n-1} \lambda^{(-n+i)\alpha} \Bigl(\frac{\epsilon}{2}\Bigr)^\alpha = \frac{|A|_{d^\alpha} \epsilon^\alpha}{2^\alpha(\lambda^\alpha - 1)}.
\end{aligned}
$$

Note that $d(y_0, x) \leqslant d(z, x) + d(z, y_0) \leqslant \epsilon$. We conclude that

$$
\bigoplus_{\substack{d(z,x)\leqslant \frac{\epsilon}{2} \\ d(T^n(z),y)\leqslant\frac{\epsilon}{2}}} S_n\overline{A}(z) \leqslant \Bigl(\bigoplus_{\substack{d(y_0,x)\leqslant\epsilon \\ T^n(y_0)}} S_n(\overline{A})(y_0) \Bigr) \otimes \frac{|A|_{d^\alpha} \epsilon^\alpha}{2^\alpha(\lambda^\alpha - 1)}
$$

for all $n \in \mathbb{N}$ and $\epsilon \in (0, \xi)$.

Thus,

$$
\begin{aligned}
\phi_A(x, y) &= \lim_{\epsilon\to 0^+} \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant \frac{\epsilon}{2} \\ d(T^n(z),y)\leqslant\frac{\epsilon}{2}}} S_n(\overline{A})(z) \\
&\leqslant \lim_{\epsilon\to 0^+} \Bigl(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(y_0,x)\leqslant\epsilon \\ T^n(y_0)=y}} S_n(\overline{A})(y_0) \Bigr) \otimes \frac{|A|_{d^\alpha} \epsilon^\alpha}{2^\alpha(\lambda^\alpha - 1)} = \widetilde{\phi_A}(x, y)
\end{aligned}
$$

for all $x, y \in X$ and the claim follows.

By the claim and (3.3),

$$
\begin{aligned}
\phi_A(x, y) \otimes b(y) &= \lim_{\epsilon\to 0^+} \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\epsilon \\ T^n(z)=y}} (S_n(\overline{A})(z) \otimes b(y)) \\
&= \lim_{\epsilon\to 0^+} \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\epsilon \\ T^n(z)=y}} b(z) \leqslant \limsup_{z\to x} b(z) \leqslant b(x),
\end{aligned}
$$

where the last inequality follows from the upper semi-continuity of b. □

Proposition 3.13 (Representation of eigenfuctions)**.** *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. Then every tropical eigenfunction u of $\mathcal{L}_A$ satisfies $u(\cdot) = \bigoplus_{x\in\Omega_A} (u(x) \otimes \phi_A(x, \cdot))$.*

This proposition is [Ga17, Proposition 6.2 (iii)] and we present a sketch of the proof in Appendix A.

Lemma 3.14 ($\phi_A(\cdot, y)$ is an eigen-density for each $y \in \Omega_A$)**.** *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. The Mañé potential satisfies*

$$
\phi_A(T(x), y) = \phi_A(x, y) - \overline{A}(x) \tag{3.4}
$$

for all $x, y \in X$ with $T(x) \neq y$. In particular, for every $y \in \Omega_A$, (3.4) holds for every $x \in X$, that is to say, $\phi_A(\cdot, y)$ is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$.

Recall $\mathcal{L}_A^{\circledast}$ from Definition 1.1. This lemma is similar to [Ga17, Proposition 5.3]. We present a different and concise proof.

Proof. The condition $T(x) \neq y$ ensures that the sum $S_n(\overline{A})(z)$ that approximates $\phi_A(x,y)$ must have length $n > 1$ when $0 < \epsilon < d\big(x, T^{-1}(y)\big)$. Thus, (3.4) follows immediately from the definition of ϕ_A and the fact that $\lim\limits_{z\to x} d(T(x),T(z)) = 0$ since T is continuous. This proves the first part of the lemma.

Recall $\phi_A(\cdot,\cdot)\colon X \times X \to \mathbb{R} \cup \{-\infty\}$ and is upper semi-continuous. It suffices to show that $\phi_A(T(x),y) = \phi_A(x,y) - \overline{A}(x)$ for all $y \in \Omega_A$ and $x \in T^{-1}(y)$. Fix $y \in \Omega_A$ and $x \in T^{-1}(y)$. By Proposition 3.12 (iv), we have $\phi_A(T(x),y) = \phi_A(y,y) = 0$. Thus, it suffices to prove $\phi_A(x,T(x)) = \overline{A}(x)$.

Claim. $\phi_A(x,T(x)) = \overline{A}(x)$ for all $x \in X$.

Fix a point $x \in X$. On the one hand, $S_n(\overline{A})(z) = \overline{A}(x)$ for $n = 1, z = x$ and it follows that $\phi_A(x,T(x)) \geqslant \overline{A}(x)$.

On the other hand, since T is continuous and X is compact, for every $\epsilon > 0$, there exists $\eta(\epsilon) \in (0,\epsilon)$ such that $d(y_1,y_2) \leqslant \eta(\epsilon)$ implies $d(T(y_1),T(y_2)) \leqslant \epsilon$ for all $y_1, y_2 \in X$.

Thus,

$$\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\eta(\epsilon)\\ d(T^n(z),T(x))\leqslant\eta(\epsilon)}} S_n(\overline{A})(z) \leqslant \Big(\bigoplus_{d(z,x)\leqslant\eta(\epsilon)} \overline{A}(z)\Big) \otimes \Big(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\eta(\epsilon)\\ d(T^n(z),T^n(x))\leqslant\eta(\epsilon)}} S_{n-1}(\overline{A})(T(z))\Big)$$

$$\leqslant \Big(\bigoplus_{d(z,x)\leqslant\eta(\epsilon)} \overline{A}(z)\Big) \otimes \Big(\bigoplus_{m\in\mathbb{N}_0} \bigoplus_{\substack{d(\widetilde{z},T(x))\leqslant\epsilon\\ d(T^m(\widetilde{z}),T(x))}} S_m(\overline{A})(\widetilde{z})\Big).$$

In the above inequalities, we use the observation that every trajectory $\big\{z, \cdots, T^{n-1}(z)\big\}$ satisfying $d(z,x) \leqslant \epsilon$ and $d(T^n(z),T(x)) \leqslant \epsilon$ can be decomposed into one single step from z to $T(z)$ and steps from $T(z)$ to $T^{n-1}(z)$.

Note that $S_m(\overline{A})(\widetilde{z}) = 0$ for all $\widetilde{z} \in X$ if $m = 0$. As ϵ tends to zero from above, we get

$$\phi_A(x,T(x)) \leqslant \overline{A}(x) \otimes (0 \oplus \phi_A(T(x),T(x))).$$

Recall the tropical eigenfunction $v_{\mathbb{0}_X}$ of $\mathcal{L}_A$ constructed in Proposition 3.5. It follows from Proposition 3.12 (i) that $\phi_A(x,x) \leqslant v_{\mathbb{0}_X}(x) - v_{\mathbb{0}_X}(x) = 0$ for all x in X.

Combining the last two inequalities above, we conclude that $\phi_A(x,T(x)) \leqslant \overline{A}(x)$. Hence, the claim is now verified.

This finishes the proof. $\square$

The following characterizations will be useful in the proof of Theorem B (iii).

Lemma 3.15. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \mathrm{Lip}(X,d^\alpha)$. If $\phi_A(x,y) \otimes \phi_A(y,x) = 0$ for all $x,y \in \Omega_A$, then $\phi_A(x,\cdot) = \phi_A(y,\cdot) \otimes \phi_A(x,y)$ and $\phi_A(\cdot,x) = \phi_A(\cdot,y) \otimes \phi_A(y,x)$ for all $x,y \in \Omega_A$.*

Proof. Fix $x,y \in \Omega_A$.

By Proposition 3.12 (iii), we have

$$\phi_A(x,\cdot) \otimes \phi_A(y,x) \preccurlyeq \phi_A(y,\cdot) \quad \text{and} \quad \phi_A(y,\cdot) \otimes \phi_A(x,y) \preccurlyeq \phi_A(x,\cdot).$$

The above two inequalities and $\phi_A(x,y) \otimes \phi_A(y,x) = 0$ imply

$$\phi_A(x,\cdot) = \phi_A(y,\cdot) \otimes \phi_A(x,y).$$

By Proposition 3.12 (iii), we have

$$\phi_A(\cdot,x)\otimes\phi_A(x,y)\preccurlyeq\phi_A(\cdot,y)\quad\text{and}\quad\phi_A(\cdot,y)\otimes\phi_A(y,x)\leqslant\phi_A(\cdot,x).$$

Therefore, $\phi_A(\cdot,x)=\phi_A(\cdot,y)\otimes\phi_A(y,x)$ follows from the above two inequalities and $\phi_A(x,y)\otimes\phi_A(y,x)=0$. □

Proposition 3.16. *Let $T\colon X\to X$ satisfy the Assumptions in Subsection 1.1 and $A\in\operatorname{Lip}(X,d^\alpha)$. Then the following statements are equivalent:*

(i) *The entries of $\{\phi_A(x,\cdot)\}_{x\in\Omega_A}$ are the same up to a tropical multiplicative constant.*

(ii) *The entries of $\{\phi_A(\cdot,y)\}_{y\in\Omega_A}$ are the same up to a tropical multiplicative constant.*

(iii) *For all $x,y\in\Omega_A$, $\phi_A(x,y)\otimes\phi_A(y,x)=0$.*

Proof. Fix $x,y\in\Omega_A$.

To see that (i) implies (iii), suppose $\phi_A(x,\cdot)\otimes c=\phi_A(y,\cdot)$. It follows that $\phi_A(x,x)\otimes c=\phi_A(y,x)$ and $\phi_A(x,y)\otimes c=\phi_A(y,y)$. By Proposition 3.12 (iv), we get $c=\phi_A(y,x)$ and $\phi_A(x,y)\otimes c=0$. Thus, $\phi_A(x,y)\otimes\phi_A(y,x)=0$.

To see that (ii) implies (iii), suppose $\phi_A(\cdot,x)\otimes d=\phi_A(\cdot,y)$. It follows that $\phi_A(x,x)\otimes d=\phi_A(x,y)$ and $\phi_A(y,x)\otimes d=\phi_A(y,y)$. By Proposition 3.12 (iv), we get $d=\phi_A(x,y)$ and $\phi_A(y,x)\otimes d=0$. Thus, $\phi_A(y,x)\otimes\phi_A(x,y)=0$.

That (iii) implies (i) and (ii) follows from Lemma 3.15. □

Proposition 3.17. *Let $T\colon X\to X$ satisfy the Assumptions in Subsection 1.1 and $A\in\operatorname{Lip}(X,d^\alpha)$. If the restriction $T|_{\Omega_A}$ is transitive, then $\phi_A(x,y)\otimes\phi_A(y,x)=0$ for all $x,y\in\Omega_A$.*

Proof. Recall that $x\in\Omega_A$ implies $T(x)\in\Omega_A$. So by Lemma 3.14 and Proposition 3.12 (iv), we have

$$\phi_A(x,T^n(x))=S_n(\overline{A})(x)\otimes\phi_A(T^n(x),T^n(x))=S_n(\overline{A})(x)\quad\text{and}$$
$$0=\phi_A(x,x)=\phi_A(T^n(x),x)+S_n(\overline{A})(x)$$

for all $x\in\Omega_A$ and $n\in\mathbb{N}_0$.

We conclude that $\phi_A(x,T^n(x))+\phi_A(T^n(x),x)=0$ for all x in Ω_A and $n\in\mathbb{N}_0$. Now that $T|_{\Omega_A}$ is transitive, we fix a point w in Ω_A such that its orbit is dense in Ω_A. Suppose $y\in\Omega_A$ is the limit of a subsequence $\{T^{n_k}(w)\}_{k\in\mathbb{N}}$ with $\lim\limits_{k\to+\infty}n_k=+\infty$. The upper semi-continuity of ϕ_A implies

$$\phi_A(w,y)\otimes\phi_A(y,w)\geqslant\limsup_{k\to+\infty}(\phi_A(w,T^{n_k}(w))\otimes\phi_A(T^{n_k}(w),w))=0.$$

Thus, $\phi_A(w,y)\otimes\phi_A(y,w)=0$ since $\phi_A(w,y)\otimes\phi_A(y,w)\leqslant\phi_A(w,w)=0$ (Proposition 3.12 (iii)(iv)).

Note that replacing w with $T^m(w)$ does not affect the above analysis. Therefore, $\phi_A(T^m(w),y)\otimes\phi_A(y,T^m(w))=0$ for all y in Ω_A and $m\in\mathbb{N}_0$. Again using the upper semi-continuity of ϕ_A and that the orbit of w is dense in Ω_A, we conclude from Proposition 3.12 (iii)(iv) that $\phi_A(x,y)\otimes\phi_A(y,x)=0$ for all x,y in Ω_A. □

3.3. **Proof of Theorem B.** We discover the representation of tropical eigen-densities in light of the duality of $\phi_A(\cdot,\cdot)$. We are now ready to give the proof of Theorem B.

Proof of Theorem B. For (i), see Proposition 3.13.

For (ii), it suffices to prove that if b is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$, then

$$\bigoplus_{x\in X}(u(x)\otimes b(x)) = \bigoplus_{x\in X, y\in\Omega_A}(u(x)\otimes\phi_A(x,y)\otimes b(y)) \tag{3.5}$$

for all $u\in C(X,\mathbb{R})$.

Fix a tropical eigen-density b of $\mathcal{L}_A^{\circledast}$. We first reduce the proof to tropical eigenfunctions v of $\mathcal{L}_A$ using Corollary 3.6 and then apply the representation of tropical eigenfunctions (Proposition 3.13).

By Proposition 2.13, it suffices to prove (3.5) for a dense subset of $C(X,\mathbb{R})$, for example, the set $\mathrm{Lip}(X,d^\alpha)$ (by the Stone–Weierstrass theorem). Now fix $u\in\mathrm{Lip}(X,d^\alpha)$.

For the left-hand side of (3.5), by Remark 3.7 and that b is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$, we have

$$\bigoplus_{x\in X}(\mathcal{L}_A(u)(x)\otimes b(x)) = \bigoplus_{y\in X}\big(u(y)\otimes\mathcal{L}_A^{\circledast}(b)(y)\big) = \bigoplus_{y\in X}(u(y)\otimes b(y)\otimes Q(T,A)). \tag{3.6}$$

By repeated use of (3.6), we get

$$\bigoplus_{x\in X}(u(x)\otimes b(x)) = \bigoplus_{x\in X}\big(\mathcal{L}_A^n(u)(x)\otimes b(x)\big) = \bigoplus_{x\in X}\Big(\Big(\bigoplus_{m\geqslant n}\mathcal{L}_A^m(u)(x)\Big)\otimes b(x)\Big)$$

for all $n\in\mathbb{N}$. Recall that $\bigoplus_{m\geqslant n}\mathcal{L}_A^m(u)$ uniformly converges to the tropical function v_u of $\mathcal{L}_A$ as $n\to+\infty$ (Corollary 3.6 (ii)(iii)). Thus, by Proposition 2.13, we conclude that

$$\bigoplus_{x\in X}(u(x)\otimes b(x)) = \bigoplus_{x\in X}(v_u(x)\otimes b(x)). \tag{3.7}$$

For the right-hand side of (3.5), according to Lemma 3.14, $\phi_A(\cdot,y)$ is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$ for each $y\in\Omega_A$. Thus, we can subtitute $b(\cdot)$ in (3.7) with $\phi_A(\cdot,y)$ and it follows that for each $y\in\Omega_A$

$$\bigoplus_{x\in X}(u(x)\otimes\phi_A(x,y)) = \bigoplus_{x\in X}(v_u(x)\otimes\phi_A(x,y)).$$

Hence,

$$\begin{aligned}\bigoplus_{x\in X,y\in\Omega_A}(u(x)\otimes\phi_A(x,y)\otimes b(y)) &= \bigoplus_{y\in\Omega_A}\Big(\bigoplus_{x\in X}(\phi_A(x,y)\otimes u(x))\otimes b(y)\Big)\\ &= \bigoplus_{y\in\Omega_A}\Big(\bigoplus_{x\in X}(\phi_A(x,y)\otimes v_u(x))\otimes b(y)\Big)\\ &= \bigoplus_{x\in X,y\in\Omega_A}(v_u(x)\otimes\phi_A(x,y)\otimes b(y)).\end{aligned}$$

We have achieved the first step of reduction. Now it suffices to show

$$\bigoplus_{x\in X}(v(x)\otimes b(x)) = \bigoplus_{x\in X,y\in\Omega_A}(v(x)\otimes\phi_A(x,y)\otimes b(y))$$

for all tropical eigenfunctions v of $\mathcal{L}_A$.

Apply Propositions 3.12 and 3.13 and it follows from discussions below that

$$\begin{aligned}\bigoplus_{x\in X, y\in\Omega_A}(v(x)\otimes\phi_A(x,y)\otimes b(y)) &= \bigoplus_{x\in X, y,z\in\Omega_A}(v(z)\otimes\phi_A(z,x)\otimes\phi_A(x,y)\otimes b(y))\\ &= \bigoplus_{y,z\in\Omega_A}(v(z)\otimes\phi_A(z,y)\otimes b(y))\\ &= \bigoplus_{y\in X, z\in\Omega_A}(v(z)\otimes\phi_A(z,y)\otimes b(y))\\ &= \bigoplus_{y\in X}(v(y)\otimes b(y))\end{aligned}$$

for all tropical eigenfunctions v of $\mathcal{L}_A$. Here the first and fourth identity follow from the representations of tropical eigenfunctions (Proposition 3.13), and the second identity immediately follows from properties of the Mañé potential (Propostion 3.12 (iii)(iv)). For the third identity, we remark that $\phi_A(z,y)\otimes b(y)\leqslant b(z)$ for all $y,z\in X$ (Proposition 3.12 (ii)) and if $z\in\Omega_A$, then the equality is achieved at $y=z\in\Omega_A$ as $\phi_A(z,z)=0$ (Proposition 3.12 (iv)). Thus, for all $z\in\Omega_A$,

$$\bigoplus_{y\in\Omega_A}(\phi_A(z,y)\otimes b(y)) = \bigoplus_{y\in X}(\phi_A(z,y)\otimes b(y)) = b(z).$$

Now (ii) is verified.

For the proof of (iii), recall for each $x\in\Omega_A$, $\phi_A(x,\cdot)$ is a tropical eigen function of $\mathcal{L}_A$ (Proposition 3.12 (v)) and $\phi_A(\cdot,x)$ is a tropical eigen-density of $\mathcal{L}_A$ (Lemma 3.14).

By Propositions 3.16 and 3.17, it suffices to show that if A is uniquely maximizing, then $T|_{\Omega_A}$ is transitive. It is well known that a T-invariant measure μ is in $M_{\max}(T,A)$ if and only if $\mathrm{supp}(\mu)\subseteq\Omega_A$, see [Ga17, Theorem 7.1]. If $T|_{\Omega_A}$ is not transitive, then it has at least two transitive pieces. Thus, the weak* limit of $\big\{\frac{\delta_x+\cdots+\delta_{T^{n-1}(x)}}{n}\big\}_{n\in\mathbb{N}}$ as $n\to+\infty$ for x in the two transitive pieces will give two maximizing measures singular to each other. We conclude that the uniquely maximizing condition implies the transitivity of $T|_{\Omega_A}$ and (iii) is established. □

3.4. Uniqueness of eigenfunction and eigen-density.

Proposition 3.18 (Sufficient condition of uniqueness)**.** *Let $T\colon X\to X$ satisfy the Assumptions in Subsection 1.1 and $A\in\mathrm{Lip}(X,d^\alpha)$. If A is uniquely maximizing, then up to a tropical multiplicative constant, there exists a unique tropical eigenfunction of $\mathcal{L}_A$ and a unique tropical eigen-density of $\mathcal{L}_A^{\circledast}$.*

The part of Proposition 3.18 on eigenfunctions is a reformulation of [Bou00, Lemma C]. We give a different proof using the relationship $0=\phi_A(x,y)\otimes\phi_A(y,x)$ (Proposition 3.16). The part of Proposition 3.18 on eigen-densities is new.

Proof. By Theorem B (iii), Proposition 3.16, and Lemma 3.15, we conclude that the entries of $\{\phi_A(x,\cdot)\}_{x\in\Omega_A}$ (resp. $\{\phi_A(\cdot,x)\}_{x\in\Omega_A}$) are the same tropical eigenfunction of $\mathcal{L}_A$ (resp. eigen-density of $\mathcal{L}_A^{\circledast}$) up to a tropical multiplicative constant and these constants are given by

$$\phi_A(x,\cdot)=\phi_A(x,y)\otimes\phi_A(y,\cdot)\quad\text{and}\quad\phi_A(\cdot,x)=\phi_A(\cdot,y)\otimes\phi_A(y,x)\tag{3.8}$$

for all $x,y\in\Omega_A$.

We then apply Theorem B (i) (resp. (ii)) to prove the uniqueness of tropical eigenfunction of $\mathcal{L}_A$ (resp. eigen-density of $\mathcal{L}_A^{\circledast}$). Fix $x_0 \in \Omega_A$. For every tropical eigenfunction v of $\mathcal{L}_A$, it follows from Theorem B (i) and (3.8) that

$$\begin{aligned} v(\cdot) = \bigoplus_{x\in\Omega_A}(\phi_A(x,\cdot)\otimes v(x)) &= \bigoplus_{x\in\Omega_A}(\phi_A(x_0,\cdot)\otimes\phi_A(x,x_0)\otimes v(x)) \\ &= \phi_A(x_0,\cdot)\otimes\Big(\bigoplus_{x\in\Omega_A}(\phi_A(x_0,x)\otimes v(x))\Big), \end{aligned}$$

where $\bigoplus_{x\in\Omega_A}(\phi_A(x_0,x)\otimes v(x))$ is a constant in $\overline{\mathbb{R}}_{\max}$. We conclude that up to a (tropical multiplicative) constant, $v(\cdot)$ is the same as $\phi_A(x_0,\cdot)$.

For every tropical eigen-density b of $\mathcal{L}_A^{\circledast}$, it follows from Theorem B (ii) and (3.8) that

$$\begin{aligned} \bigoplus_{x\in X}(f(x)\otimes b(x)) &= \bigoplus_{x\in X, y\in\Omega_A}(f(x)\otimes\phi_A(x,y)\otimes b(y)) \\ &= \bigoplus_{x\in X, y\in\Omega_A}(f(x)\otimes\phi_A(x,x_0)\otimes\phi_A(x_0,y)\otimes b(y)) \\ &= \bigoplus_{x\in X}\Big(f(x)\otimes\phi_A(x,x_0)\otimes\Big(\bigoplus_{y\in\Omega_A}(\phi_A(x_0,y)\otimes b(y))\Big)\Big) \end{aligned} \tag{3.9}$$

for all $f \in C(X,\mathbb{R})$. Denote $c := \bigoplus_{y\in\Omega_A}(\phi_A(x_0,y)\otimes b(y))$ and c is constant in $\overline{\mathbb{R}}_{\max}$. Recall $\phi_A(\cdot,x_0)$ is a tropical eigen-density (Lemma 3.14). Thus, $b(\cdot)$ and $\phi_A(\cdot,x_0)\otimes c$ are both in $D_{\max}(X)$. Then by (3.9) and Remark 2.17, we conclude that $b(\cdot) = \phi_A(\cdot,x_0)\otimes c$, i.e., $b(\cdot)$ is the same as $\phi_A(\cdot,x_0)$ up to a (tropical multiplicative) constant. □

3.5. **Proof of Theorem A.**

Proof of Theorem A. (i) follows from Corollary 3.6 and (iii) follows from Proposition 3.9. For (ii) and (iv), we recall the well-known fact in [CLT01] that for an open and dense subset of A in $\mathrm{Lip}(X,d^\alpha)$, A is uniquely maximizing. Thus, (ii) and (iv) follow from Proposition 3.18. The uniqueness of tropical eigenvalue of $\mathcal{L}_A$ (resp. $\mathcal{L}_A^{\circledast}$) follows from Proposition 3.8. □

4. Zero temperature limit

In this section, we always assume that $T\colon X\to X$ satisfies the Assumptions in Subsection 1.1 and $A\in\mathrm{Lip}(X,d^\alpha)$.

There have been two kinds of zero temperature limits. One is to study the weak* limits of the equilibrium states $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$ as the inverse temperature $\beta\to+\infty$. The other is to study the accumulation points (in C^0 topology) of $\big\{\frac{1}{\beta}\log u_{\beta A}\big\}_{\beta\in(0,+\infty)}$ as $\beta\to+\infty$ and the accumulation points turn out to be tropical eigenfunctions of $\mathcal{L}_A$. This process is also called *selection of sub-actions*.

Thus, it is also natural to consider the Logarithmic type zero temperature limits of the equilibrium states $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$ which gives the rate function of $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$. With the knowledge of the tropical ajoint operator and tropical eigen-densities, we see that the rate function of $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$ should be the tropical product of a tropical eigen-density of $\mathcal{L}_A^{\circledast}$ and a tropical eigenfunction of $\mathcal{L}_A$.

In this section, we investigate along the above philosophy and consequently give new proofs for results in [BLT11] and [Me18].

Recall that for each $f \in C(X,\mathbb{R})$ and each $\beta > 0$,

$$l^{\mu}_{\beta}(f) = \frac{1}{\beta} \log \int e^{\beta f} \,\mathrm{d}\mu_{\beta A} \quad \text{and} \quad l^{m}_{\beta}(f) = \frac{1}{\beta} \log \int e^{\beta f} \,\mathrm{d}m_{\beta A}.$$

In the sequel, we do not distinguish between the pointwise limit and the limit in the compact-open topology when considering the accumulation points of normal families of real-valued continuous functions on X (resp. functionals on $C(X,\mathbb{R})$) since they coincide. We remark that all limits of functionals on $C(X,\mathbb{R})$ mentioned below are pointwise limits and all limits of fucntions on X metioned below are uniform limits (limits in the C^0 topology).

Since X is compact due to our assumptions, X and $C(X,\mathbb{R})$ are both separable. Thus, by Arzelà–Ascoli theorem, an equicontinuous family of real-valued continuous functions on X (resp. functionals on $C(X,\mathbb{R})$) that is uniformly bounded on every compact subsets of X (resp. $C(X,\mathbb{R})$) is a normal family. (citation need)

Thus, we only verify the equicontinuity and the (locally) uniform bound when showing the normality of a certain family below. Recall that $\mathbb{0}_X$ and $\mathbb{1}_X$ are used to represent the constant zero and one functions on X, respectively.

4.1. **Proofs of Theorems C and D.**

Proof of Theorem C. (i). We first show the normality with estimates in [PU10] and then verify that every accumulation point of $\left\{\frac{1}{\beta}\log u_{\beta A}\right\}$ as $\beta \to +\infty$ is a tropical eigenfunction of $\mathcal{L}_A$.

By [PU10, Proposition 4.4.3], there exists a constant $D > 0$ only depending on T such that for all $x, y \in X$ with $y \in B(x,\xi)$, $n \in \mathbb{N}$, and $\beta > 0$,

$$\frac{\mathcal{R}^n_{\beta A}(\mathbb{1}_X)(x)}{\mathcal{R}^n_{\beta A}(\mathbb{1}_X)(y)} \leqslant \exp(D|\beta A|_{d^\alpha} d(x,y)^\alpha) \tag{4.1}$$

and for all $x, y \in X$, $n \in \mathbb{N}$, and $\beta > 0$,

$$\frac{\mathcal{R}^n_{\beta A}(\mathbb{1}_X)(x)}{\mathcal{R}^n_{\beta A}(\mathbb{1}_X)(y)} \leqslant \exp(D(\|\beta A\|_{C^0} + |\beta A|_{d^\alpha} + 1)). \tag{4.2}$$

Recall $m_{\beta A}$ is the probability measure satisfying $\mathcal{R}^{\star}_{\beta A}(m_{\beta A}) = e^{P(T,\beta A)} m_{\beta A}$. It follows from the definition of $\mathcal{R}_{\beta A}$ that $\mathbb{0}_X \preccurlyeq \mathcal{R}^n_{\beta A}(\mathbb{1}_X)$ for all $n \in \mathbb{N}$. Thus, multiplying (4.2) by $\mathcal{R}^n_{\beta A}(\mathbb{1}_X)(y)$ and integrating with $m_{\beta A}$ with respect to variables x and y respectively, we get for all $n \in \mathbb{N}$ and $\beta > 0$,

$$\exp(-D(\|\beta A\|_{C^0} + |\beta A|_{d^\alpha} + 1)) \preccurlyeq e^{-nP(T,\beta A)} \mathcal{R}^n_{\beta A}(\mathbb{1}_X) \preccurlyeq \exp(D(\|\beta A\|_{C^0} + |\beta A|_{d^\alpha} + 1)). \tag{4.3}$$

By [PU10, Propositions 5.3.1 and 5.4.5], $\frac{1}{n}\sum_{i=0}^{n-1} e^{-iP(T,\beta A)} \mathcal{R}^i_{\beta A}(\mathbb{1}_X)$ uniformly converges to $u_{\beta A}$ as $n \to +\infty$.

It follows from (4.3) that for all $\beta > 0$,

$$\exp(-D(\|\beta A\|_{C^0} + |\beta A|_{d^\alpha} + 1)) \preccurlyeq u_{\beta A} \preccurlyeq \exp(D(\|\beta A\|_{C^0} + |\beta A|_{d^\alpha} + 1))$$

and consequently for all $\beta > 1$,

$$\left\|\beta^{-1}\log u_{\beta A}\right\|_{C^0} \leqslant D(\|A\|_{C^0} + |A|_{d^\alpha} + 1).$$

Thus, $\left\{\frac{1}{\beta}\log u_{\beta A}\right\}_{\beta\in(1,+\infty)}$ is uniformly bounded.

It follows from (4.1) that for all $x, y \in X$ with $y \in B(x,\xi)$ and $\beta > 0$,

$$\frac{u_{\beta A}(x)}{u_{\beta A}(y)} \leqslant \exp(D|\beta A|_{d^\alpha} d(x,y)^\alpha)$$

and consequently for all $\beta > 0$,

$$\left|\beta^{-1}\log u_{\beta A}\right|_{d^\alpha,\xi} \leqslant D|A|_{d^\alpha}.$$

Thus, $\left\{\frac{1}{\beta}\log u_{\beta A}\right\}_{\beta\in(0,+\infty)}$ is equicontinuous.

We conclude that $\left\{\frac{1}{\beta}\log u_{\beta A}\right\}_{\beta\in(1,+\infty)}$ is normal.

Now suppose v is the uniform limit of a convergent subsequence $\left\{\frac{1}{\beta_k}\log u_{\beta_k A}\right\}_{k\in\mathbb{N}}$ with $\beta_k \to +\infty$ as $k \to +\infty$. To show that v is a tropical eigenfunction of $\mathcal{L}_A$, we compare the Ruelle operator $\mathcal{R}_{\beta A}$ with the Bousch operator $\mathcal{L}_A$. It follows from definitions of the two operators that for all $\beta > 0$ and $f \in C(X,\mathbb{R})$,

$$\mathcal{L}_A(f) \preccurlyeq \frac{1}{\beta}\log\mathcal{R}_{\beta A}\big(e^{\beta f}\big) \preccurlyeq \mathcal{L}_A(f) \otimes \frac{\log N}{\beta}, \tag{4.4}$$

where $N = \sup_{x\in X}\operatorname{card} T^{-1}(x) < +\infty$ is the constant in Lemma 3.1. Recall $\mathcal{R}_{\beta A}(u_{\beta A}) = e^{P(T,\beta A)}u_{\beta A}$. Thus,

$$\begin{aligned}
\|\mathcal{L}_A(v) - Q(T,A)\otimes v\|_{C^0} &\leqslant \left\|\mathcal{L}_A(v) - \mathcal{L}_A\big(\beta_k^{-1}\log u_{\beta_k A}\big)\right\|_{C^0} \\
&\quad + \left\|\mathcal{L}_A\big(\beta_k^{-1}\log u_{\beta_k A}\big) - \beta_k^{-1}\log\mathcal{R}_{\beta_k A}(u_{\beta_k A})\right\|_{C^0} \\
&\quad + \left\|\beta_k^{-1}\log u_{\beta_k A} + \beta_k^{-1}P(T,\beta_k A) - v - Q(T,A)\right\|_{C^0} \\
&\leqslant 2\left\|v - \beta_k^{-1}\log u_{\beta_k A}\right\|_{C^0} + \frac{\log N}{\beta_k} + \left|\beta_k^{-1}P(T,\beta_k A) - Q(T,A)\right|,
\end{aligned}$$

where the second inequality follows from Lemma 3.3 and (4.4). As $k \to +\infty$, it follows that $\mathcal{L}_A(v) = Q(T,A)\otimes v$, i.e., v is a tropical eigenfunction of $\mathcal{L}_A$.

(ii). It immediately follows from the definition of l_β^m that

$$\left|l_\beta^m(f)\right| \leqslant \|f\|_{C^0} \quad\text{and}\quad \left|l_\beta^m(f) - l_\beta^m(g)\right| \leqslant \|f-g\|_{C^0} \tag{4.5}$$

for all $f, g \in C(X,\mathbb{R})$. We conclude that $\left\{l_\beta^m(\cdot)\right\}_{\beta\in(0,+\infty)}$ is locally uniformly bounded, equicontinuous, and consequently normal.

Now suppose $l(\cdot)$ is the limit of a convergent subsequence $\left\{l_{\beta_k}^m(\cdot)\right\}_{k\in\mathbb{N}}$ with $\beta_k \to +\infty$ as $k \to +\infty$.

Claim. l is a tropical linear functional.

It follows from the definition of l_β^m that $l_\beta^m(a\otimes f) = a\otimes l_\beta^m(f)$ for all $a \in \overline{\mathbb{R}}_{\max}$, $f \in C(X,\mathbb{R})$, and $\beta > 0$. Thus, $l(a\otimes f) = a\otimes l(f)$ for all $a \in \overline{\mathbb{R}}_{\max}$ and $f \in C(X,\mathbb{R})$.

Fix $f, g \in C(X, \mathbb{R})$. It suffices to prove $l(f \oplus g) = l(f) \oplus l(g)$. For this reason, we introduce the plus operation at inverse temperature $\beta > 0$:

$$h_1 \underset{\beta}{\oplus} h_2 := \frac{1}{\beta} \log\bigl(e^{\beta h_1} + e^{\beta h_2}\bigr),$$

for all $h_1, h_2 \in C(X, \mathbb{R})$.

It immediately follows that $h_1 \oplus h_2 \leqslant h_1 \underset{\beta}{\oplus} h_2 \leqslant (h_1 \oplus h_2) \otimes \frac{\log 2}{\beta}$ and $l_\beta^m\bigl(h_1 \underset{\beta}{\oplus} h_2\bigr) = l_\beta^m(h_1) \underset{\beta}{\oplus} l_\beta^m(h_2)$ for all $h_1, h_2 \in C(X, \mathbb{R})$ and $\beta > 0$. Thus, for all $k \in \mathbb{N}$,

$$\begin{aligned}
l_{\beta_k}^m(f \oplus g) \leqslant l_{\beta_k}^m\bigl(f \underset{\beta_k}{\oplus} g\bigr) &\leqslant l_{\beta_k}^m(f \oplus g) \otimes \frac{\log 2}{\beta_k}, \\
l_{\beta_k}^m\bigl(f \underset{\beta_k}{\oplus} g\bigr) &= l_{\beta_k}^m(f) \underset{\beta_k}{\oplus} l_{\beta_k}^m(g), \\
l_{\beta_k}^m(f) \oplus l_{\beta_k}^m(g) \leqslant l_{\beta_k}^m(f) \underset{\beta_k}{\oplus} l_{\beta_k}^m(g) &\leqslant \bigl(l_{\beta_k}^m(f) \oplus l_{\beta_k}^m(g)\bigr) \otimes \frac{\log 2}{\beta_k}.
\end{aligned}$$

We conclude that $l_{\beta_k}^m(f \oplus g) - \frac{\log 2}{\beta_k} \leqslant l_{\beta_k}^m(f) \oplus l_{\beta_k}^m(g) \leqslant l_{\beta_k}^m(f \oplus g) + \frac{\log 2}{\beta_k}$. Recall $\lim\limits_{k \to +\infty} \beta_k = +\infty$ and $\lim\limits_{k \to +\infty} l_{\beta_k}^m(\cdot) = l(\cdot)$. As $k \to +\infty$, it follows that $l(f \oplus g) = l(f) \oplus l(g)$. Now the claim is verified.

Now we need to show that the density of l in $D_{\max}(X)$ is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$. By Remark 3.7, it suffices to prove

$$l(\mathcal{L}_A(u)) = l(u) \otimes Q(T, A)$$

for all $u \in C(X, \mathbb{R})$. Fix $f \in C(X, \mathbb{R})$.

It follows from (4.4) and (4.5) that for all $k \in \mathbb{N}$,

$$\left| l_{\beta_k}^m(\mathcal{L}_A(f)) - l_{\beta_k}^m\left(\frac{1}{\beta_k} \log \mathcal{R}_{\beta_k A}\bigl(e^{\beta_k f}\bigr)\right)\right| \leqslant \left\| \mathcal{L}_A(f) - \frac{1}{\beta_k} \log \mathcal{R}_{\beta_k A}\bigl(e^{\beta_k f}\bigr)\right\|_{C^0} \leqslant \frac{\log N}{\beta_k}. \tag{4.6}$$

Note that

$$\begin{aligned}
l_{\beta_k}^m\left(\frac{1}{\beta_k} \log \mathcal{R}_{\beta_k A}\bigl(e^{\beta_k f}\bigr)\right) &= \frac{1}{\beta_k} \log \int \mathcal{R}_{\beta_k A}\bigl(e^{\beta_k f}\bigr) \, \mathrm{d}m_{\beta_k A} \\
&= \frac{1}{\beta_k} \log \int e^{P(T, \beta_k A)} \cdot e^{\beta_k f} \, \mathrm{d}m_{\beta_k A} \\
&= \frac{P(T, \beta_k A)}{\beta_k} \otimes \frac{1}{\beta_k} \log \int e^{\beta_k f} \, \mathrm{d}m_{\beta_k A} \\
&= \frac{P(T, \beta_k A)}{\beta_k} \otimes l_{\beta_k}^m(f),
\end{aligned} \tag{4.7}$$

where the second equality holds since $m_{\beta A}$ is the eigenmeasure of $\mathcal{R}_{\beta A}^*$ with eigenvalue $e^{P(T, \beta A)}$. Recall $\lim\limits_{k \to +\infty} \beta_k = +\infty$, $\lim\limits_{k \to +\infty} l_{\beta_k}^m(\cdot) = l(\cdot)$, and $\lim\limits_{\beta \to +\infty} \frac{P(T, \beta A)}{\beta} = Q(T, A)$.

Combine (4.6) and (4.7) and let $k \to +\infty$. We conclude that $l(\mathcal{L}_A(f)) = l(f) \otimes Q(T, A)$.

(iii). It follows from the definition of l^{μ}_{β} that

$$\left|l^{\mu}_{\beta}(f)\right| \leqslant \|f\|_{C^0} \quad \text{and} \quad \left|l^{\mu}_{\beta}(f) - l^{\mu}_{\beta}(g)\right| \leqslant \|f-g\|_{C^0} \tag{4.8}$$

for all $f, g \in C(X,\mathbb{R})$. This implies that $\left\{l^{\mu}_{\beta}(\cdot)\right\}_{\beta\in(0,+\infty)}$ is normal.

Now suppose $\widehat{l}(\cdot)$ is the limit of a convergent subsequence $\left\{l^{\mu}_{\beta_k}(\cdot)\right\}_{k\in\mathbb{N}}$ with $\beta_k \to +\infty$ as $k\to+\infty$. By (i) and (ii), we take a subsequence $\left\{\widetilde{\beta}_k\right\}_{k\in\mathbb{N}}$ from $\{\beta_k\}_{k\in\mathbb{N}}$ with $\widetilde{\beta}_k \to +\infty$ as $k\to+\infty$ such that $\left\{\widetilde{\beta}_k^{-1}\log u_{\widetilde{\beta}_k A}\right\}_{k\in\mathbb{N}}$ uniformly converges to v and $\left\{l^{m}_{\widetilde{\beta}_k}(\cdot)\right\}_{k\in\mathbb{N}}$ converges to $l(\cdot)$.

Note that for all $\beta > 0$ and $f \in C(X,\mathbb{R})$, $l^{\mu}_{\beta}(f) = l^{m}_{\beta}\left(f + \beta^{-1}\log u_{\beta A}\right)$ since $\mu_{\beta A} = u_{\beta A}\cdot m_{\beta A}$. Thus, for all $k\in\mathbb{N}$ and $f\in C(X,\mathbb{R})$,

$$\begin{aligned}
\left|l(v\otimes f) - l^{\mu}_{\widetilde{\beta}_k}(f)\right| &= \left|l(v+f) - l^{m}_{\widetilde{\beta}_k}\left(f + \widetilde{\beta}_k^{-1}\log u_{\widetilde{\beta}_k A}\right)\right| \\
&\leqslant \left|l(v+f) - l^{m}_{\widetilde{\beta}_k}(v+f)\right| + \left|l^{m}_{\widetilde{\beta}_k}(v+f) - l^{m}_{\widetilde{\beta}_k}\left(f+\widetilde{\beta}_k^{-1}\log u_{\widetilde{\beta}_k A}\right)\right| \\
&\leqslant \left|l(v+f) - l^{m}_{\widetilde{\beta}_k}(v+f)\right| + \left\|v - \widetilde{\beta}_k^{-1}\log u_{\widetilde{\beta}_k A}\right\|_{C^0},
\end{aligned}$$

where the last inequality follows from (4.5). According to our definition of $\left\{\widetilde{\beta}_k\right\}_{k\in\mathbb{N}}$, let $k\to+\infty$ and it follows that $l(v\otimes f) = \widehat{l}(f)$ for all $f\in C(X,\mathbb{R})$. Let b be the density of l in $D_{\max}(X)$. Thus, $\widehat{l}(f) = \bigoplus_{x\in X}(f(x)\otimes v(x)\otimes b(x))$ for all $f\in C(X,\mathbb{R})$, i.e., $\widehat{l}$ is tropical linear and $v\otimes b$ is the density of $\widetilde{l}$ in $D_{\max}(X)$. $\square$

Recall that u_A is the unique eigenfuction of the Ruelle operator $\mathcal{R}_A$ satisfying $\int u_A \,\mathrm{d}m_A = 1$ with eigenvalue $e^{P(T,A)}$. Let $\widetilde{\mathcal{R}}_A(u) := \frac{1}{e^{P(T,A)}u_A}\mathcal{R}_A(uu_A)$ be the normalized Ruelle operator. Note that $\widetilde{\mathcal{R}}_{\beta A}$ is just the Ruelle operator with potential

$$g_\beta = \beta A + \log u_{\beta A} - \log u_{\beta A}\circ T - P(T,\beta A)$$

and $\widetilde{\mathcal{R}}_{\beta A}(\mathbb{1}_X) = \mathbb{1}_X, \widetilde{\mathcal{R}}^*_{\beta A}(\mu_{\beta A}) = \mu_{\beta A}$ (see for example, [PU10, Section 5.4]).

So considering the Logarithmic type zero temperature limit, we predict that if $\widehat{A}$ is the limit of $\frac{g_\beta}{\beta}$ and $\widehat{b}$ is the density in $D_{\max}(X)$ of the limit of $l^{\mu}_{\beta}(\cdot)$, then $\mathcal{L}_{\widehat{A}}(\mathbb{0}_X) = \mathbb{0}_X$ and $\mathcal{L}^{\circledast}_{\widehat{A}}(\widehat{b}) = \widehat{b}$.

Proof of Theorem D. (i). Since $g_\beta = \beta A + \log u_{\beta A} - \log u_{\beta A}\circ T - P(T,\beta A)$, $Q(T,A) = \lim\limits_{\beta\to+\infty}\frac{P(T,\beta A)}{\beta}$, and $\left\{\frac{1}{\beta}\log u_{\beta A}\right\}_{\beta\in(0,+\infty)}$ is a normal family (Theorem C (i)), it immediately follows that $\left\{\frac{g_\beta}{\beta}\right\}_{\beta\in(0,+\infty)}$ is a normal family. It has been verified in Theorem C (iii) that $\left\{l^{\mu}_{\beta}(\cdot)\right\}_{\beta\in(0,+\infty)}$ is normal.

(ii). By Theorem C (iii), $\widehat{l}$ is a tropical linear functional. Thus, by Remark 3.7, it suffices to show that $\widehat{l}\left(\mathcal{L}_{\widehat{A}}(f)\right) = \widehat{l}(f)$ for all $f\in C(X,\mathbb{R})$. Fix $f\in C(X,\mathbb{R})$.

Recall $\widetilde{\mathcal{R}}^*_{\beta A}(\mu_{\beta A}) = \mu_{\beta A}$ and $\widetilde{\mathcal{R}}_{\beta A} = \mathcal{R}_{g_\beta}$. It follows that for all $\beta > 0$,

$$
\begin{aligned}
l^\mu_\beta\biggl(\frac{1}{\beta}\log \mathcal{R}_{g_\beta}\bigl(e^{\beta f}\bigr)\biggr) &= \frac{1}{\beta}\log\int \mathcal{R}_{g_\beta}(e^{\beta f})\,\mathrm{d}\mu_{\beta A}\\
&= \frac{1}{\beta}\log\int e^{\beta f}\,\mathrm{d}\mathcal{R}^*_{g_\beta}(\mu_{\beta A})\\
&= \frac{1}{\beta}\log\int e^{\beta f}\,\mathrm{d}\mu_{\beta A}\\
&= l^\mu_\beta(f).
\end{aligned} \tag{4.9}
$$

It turns out that we need to compare $\mathcal{L}_{\widehat{A}}(f)$ with $\frac{1}{\beta}\log \mathcal{R}_{g_\beta}\bigl(e^{\beta f}\bigr)$. More precisely, it directly follows from definitions of the operators that for all $\beta > 0$,

$$
\begin{aligned}
\mathcal{L}_{\widehat{A}}(f) \preccurlyeq \beta^{-1}\log R_{\beta\widehat{A}}\bigl(e^{\beta f}\bigr) &\preccurlyeq \mathcal{L}_{\widehat{A}}(f) + \beta^{-1}\log N,\\
\bigl\|\widehat{A} - \beta^{-1}g_\beta\bigr\|_{C^0} \geqslant \bigl\|\beta^{-1}\log R_{\beta\widehat{A}}\bigl(e^{\beta f}\bigr) &- \beta^{-1}\log R_{g_\beta}\bigl(e^{\beta f}\bigr)\bigr\|_{C^0},
\end{aligned}
$$

where N is the constant in Lemma 3.1. We conclude that for all $\beta > 0$.

$$
\bigl\|\mathcal{L}_{\widehat{A}}(f) - \beta^{-1}\log R_{g_\beta}\bigl(e^{\beta f}\bigr)\bigr\|_{C^0} \leqslant \beta^{-1}\log N + \bigl\|\widehat{A} - \beta^{-1}g_\beta\bigr\|_{C^0} \tag{4.10}
$$

Thus, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
&\bigl|\widehat{l}(\mathcal{L}_{\widehat{A}}(f)) - \widehat{l}(f)\bigr|\\
&\quad\leqslant \bigl|\widehat{l}(f) - l^\mu_{\beta_k}(f)\bigr| + \bigl|l^\mu_{\beta_k}(f) - l^\mu_{\beta_k}\bigl(\beta_k^{-1}\log \mathcal{R}_{g_{\beta_k}}\bigl(e^{\beta_k f}\bigr)\bigr)\bigr|\\
&\qquad\quad + \bigl|l^\mu_{\beta_k}\bigl(\beta_k^{-1}\log R_{g_{\beta_k}}\bigl(e^{\beta_k f}\bigr)\bigr) - l^\mu_{\beta_k}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr)\bigr| + \bigl|l^\mu_{\beta_k}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr) - \widehat{l}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr)\bigr|\\
&\quad\leqslant \bigl|\widehat{l}(f) - l^\mu_{\beta_k}(f)\bigr| + 0 + \bigl\|\mathcal{L}_{\widehat{A}}(f) - \beta_k^{-1}\log R_{g_{\beta_k}}\bigl(e^{\beta_k f}\bigr)\bigr\|_{C^0} + \bigl|l^\mu_{\beta_k}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr) - \widehat{l}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr)\bigr|\\
&\quad\leqslant \bigl|\widehat{l}(f) - l^\mu_{\beta_k}(f)\bigr| + \beta_k^{-1}\log N + \bigl\|\widehat{A} - \beta_k^{-1}g_{\beta_k}\bigr\|_{C^0} + \bigl|l^\mu_{\beta_k}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr) - \widehat{l}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr)\bigr|,
\end{aligned} \tag{4.11}
$$

where the second inequality follows from (4.9) and (4.8), and the third inequality follows from (4.10).

Recall $\lim\limits_{k\to+\infty} l^\mu_{\beta_k}(\cdot) = \widehat{l}(\cdot)$, $\lim\limits_{k\to+\infty}\frac{g_{\beta_k}}{\beta_k} = \widehat{A}$, and $\lim\limits_{k\to+\infty}\beta_k = +\infty$. As $k \to +\infty$ in (4.11), we conclude that $\widehat{l}(f) = \widehat{l}\bigl(\mathcal{L}_{\widehat{A}}(f)\bigr)$. □

4.2. **Applications.** From the point of view that we adopt in this paper, we are able to derive in Corollary 4.3 the main result of [BLT11] without using the "dual shift" technique used there and prove in Corollary 4.4 the main result of [Me18] using its generalization Theorem D. Note that we prove these results in a more general setting while [BLT11] and [Me18] present the results for subshifts of finite type.

Definition 4.1. A famliy of probability measures $\{\mu_\beta\}_{\beta\in(0,+\infty)}$ satisfies the *large deviation principle as* $\beta \to +\infty$ if there exists a lower semi-continuous function $I\colon X \to [0, +\infty]$

(called the *rate function*) so that the two inequalities holds:

$$\liminf_{\beta\to+\infty}\frac{1}{\beta}\log\mu_\beta(\mathcal{G})\geqslant-\inf_{x\in\mathcal{G}}I(x),\quad\text{for every open set }\mathcal{G}\subseteq X,$$
$$\limsup_{\beta\to+\infty}\frac{1}{\beta}\log\mu_\beta(\mathcal{K})\leqslant-\inf_{x\in\mathcal{K}}I(x),\quad\text{for every closed set }\mathcal{K}\subseteq X.$$

Remark 4.2. Note that $I(x)\in[0,+\infty]$ for all $x\in X$ follows from the fact that μ_β is a probability measure. By [DZ09, Theorems 4.3.1 and 4.4.2], when X is a compact metric space, the above definition is equivalent to the following condition

$$\lim_{\beta\to+\infty}\frac{1}{\beta}\log\int e^{\beta f}\,\mathrm{d}\mu_\beta=\sup_{x\in X}(f(x)-I(x))\tag{4.12}$$

for all $f\in C(X,\mathbb{R})$.

Corollary 4.3. *Let $T\colon X\to X$ satisfy the Assumptions in Subsection 1.1. Suppose that the potential A in $\mathrm{Lip}(X,d^\alpha)$ is uniquely maximizing. Then the family of equilibrium states $\{\mu_{\beta A}\}_{\beta\in(0,+\infty)}$ satisfies the large deviation principle as $\beta\to+\infty$ with rate function $-(b\otimes v)$, where b is the unique tropical eigen-density of $\mathcal{L}_A^{\circledast}$ satisfying $\bigoplus_{x\in X}b(x)=0$ and v is the unique tropical eigenfunction of $\mathcal{L}_A$ satisfying $\bigoplus_{x\in X}(v(x)\otimes b(x))=0$.*

Proof. Since A is uniquely maximizing, Proposition 3.18 implies the uniqueness of tropical eigen-density of $\mathcal{L}_A^{\circledast}$ up to a constant. Note that for all $\beta>0$, $l_\beta^m(\mathbb{0}_X)=\frac{1}{\beta}\log 1=0$ since $m_{\beta A}$ is a probability measure. By Theorem C (ii), it follows that as $\beta\to+\infty$, $l_\beta^m(\cdot)$ must converge to the unique tropical linear functional l whose density b in $D_{\max}(X)$ is a tropical eigen-density of $\mathcal{L}_A^{\circledast}$ and $l(\mathbb{0}_X)=0$, i.e., b is the unique tropical eigen-density of $\mathcal{L}_A^{\circledast}$ satisfying $\bigoplus_{x\in X}b(x)=0$.

Claim. As $\beta\to+\infty$, $\frac{1}{\beta}\log u_{\beta A}$ must converge to the unique tropical eigenfunction v of $\mathcal{L}_A$ satisfying $l(v)=0$.

Recall that $\int u_{\beta A}\,\mathrm{d}m_{\beta A}=1$ for all $\beta>0$. It follows that for all $\beta>0$,

$$l_\beta^m\bigl(\beta^{-1}\log u_{\beta A}\bigr)=\frac{1}{\beta}\log\int u_{\beta A}\,\mathrm{d}m_{\beta A}=0.$$

Suppose $\widehat{v}$ is the uniform limit of a convergent subsequence $\bigl\{\frac{1}{\beta_k}\log u_{\beta_k A}\bigr\}_{k\in\mathbb{N}}$ with $\beta_k\to+\infty$ as $k\to+\infty$ according to Theorem C (i). Recall (4.5) and it follows that for all $k\in\mathbb{N}$,

$$\bigl|l_{\beta_k}^m(\widehat{v})-l_{\beta_k}^m\bigl(\beta_k^{-1}\log u_{\beta_k A}\bigr)\bigr|\leqslant\bigl\|\widehat{v}-\beta_k^{-1}\log u_{\beta_k A}\bigr\|_{C^0}.$$

We conclude that

$$\begin{aligned}|l(\widehat{v})|&\leqslant\bigl|l(\widehat{v})-l_{\beta_k}^m(\widehat{v})\bigr|+\bigl|l_{\beta_k}^m(\widehat{v})\bigr|\\&=\bigl|l(\widehat{v})-l_{\beta_k}^m(\widehat{v})\bigr|+\bigl|l_{\beta_k}^m(\widehat{v})-l_{\beta_k}^m\bigl(\beta_k^{-1}\log u_{\beta_k A}\bigr)\bigr|\\&\leqslant\bigl|l(\widehat{v})-l_{\beta_k}^m(\widehat{v})\bigr|+\bigl\|\widehat{v}-\beta_k^{-1}\log u_{\beta_k A}\bigr\|_{C^0}\end{aligned}\tag{4.13}$$

for all $k \in \mathbb{N}$. Since $\lim_{\beta \to +\infty} l_\beta^m(\cdot) = l(\cdot)$ and $\lim_{k \to +\infty} \beta_k^{-1} \log u_{\beta_k A} = \widehat{v}$, we get $l(\widehat{v}) = 0$ as $k \to +\infty$ in (4.13). Now our claim follows from the uniqueness of the tropical eigenfunction of $\mathcal{L}_A$ up to a tropical multiplicative constant since A is uniquely maximizing (see Proposition 3.18).

Finally, note that for all $\beta > 0$ and $f \in C(X, \mathbb{R})$, $l_\beta^\mu(f) = l_\beta^m\bigl(f + \beta^{-1} \log u_{\beta A}\bigr)$ since $\mu_{\beta A} = u_{\beta A} \cdot m_{\beta A}$. Thus, for all $\beta > 0$ and $f \in C(X, \mathbb{R})$,

$$\begin{aligned} \bigl|l(v+f) - l_\beta^\mu(f)\bigr| &= \bigl|l(v+f) - l_\beta^m\bigl(f + \beta^{-1} \log u_{\beta A}\bigr)\bigr| \\ &\leqslant \bigl|l(v+f) - l_\beta^m(v+f)\bigr| + \bigl|l_\beta^m(v+f) - l_\beta^m\bigl(f + \beta^{-1}\log u_{\beta A}\bigr)\bigr| \\ &\leqslant \bigl|l(v+f) - l_\beta^m(v+f)\bigr| + \bigl\|v - \beta^{-1} \log u_{\beta_A}\bigr\|_{C^0}, \end{aligned} \tag{4.14}$$

where the last inequality follows from (4.5).

Since we have proved $\lim_{\beta \to +\infty} l_\beta^m(\cdot) = l(\cdot)$ and $\lim_{\beta \to +\infty} \frac{1}{\beta} \log u_{\beta A} = v$, it follows from (4.14) that $\lim_{\beta \to +\infty} l_\beta^\mu(f) = l(v \otimes f)$ for all $f \in C(X, \mathbb{R})$. Recall b is the density of l in $D_{\max}(X)$. Thus, $l(v+f) = \sup_{x \in X}(f(x) + (b(x) + v(x)))$ for all $f \in C(X, \mathbb{R})$. Therefore, by Remark 4.2, $-(b+v)$ is the rate function I in (4.12). $\square$

Corollary 4.4. *Let $T\colon X \to X$ satisfy the Assumptions in Subsection 1.1 and $A \in \operatorname{Lip}(X, d^\alpha)$. Suppose that the family of equilibrium states $\{\mu_{\beta A}\}_{\beta \in (0,+\infty)}$ satisifies the large deviation principle as $\beta \to +\infty$, then the following statements hold:*

(i) $\bigl\{\frac{g_\beta}{\beta}\bigr\}_{\beta \in (0,+\infty)}$ *uniformly converges as $\beta \to +\infty$.*

(ii) $\bigl\{\frac{1}{\beta} \log u_{\beta A}\bigr\}_{\beta \in (0,+\infty)}$ *uniformly converges as $\beta \to +\infty$.*

(iii) $\bigl\{l_\beta^m(\cdot)\bigr\}_{\beta \in (0,+\infty)}$ *locally uniformly coverges as $\beta \to +\infty$.*

Proof. We have shown that the three families $\{\frac{g_\beta}{\beta}\}_{\beta \in (0,+\infty)}$, $\{\frac{1}{\beta} \log u_{\beta A}\}_{\beta \in (0,+\infty)}$, and $\bigl\{l_\beta^m(\cdot)\bigr\}_{\beta \in (0,+\infty)}$ are normal in Theorem D (i) and Theorem C (i)(ii). It suffices to show that the limit of every convergent subsequence must be the same fuction or functional. Since $\{\mu_{\beta A}\}_{\beta \in (0,+\infty)}$ satisfies the large deviation principle as $\beta \to +\infty$, it follows from Remark 4.2 that $\bigl\{l_\beta^\mu(\cdot)\bigr\}_{\beta \in (0,+\infty)}$ locally uniformly converges a tropical linear functional as $\beta \to +\infty$. We denote the functional by $\widehat{l}(\cdot)$ with its density $\widehat{b} \in D_{\max}(X)$.

(i). Suppose that the subsequence $\bigl\{\beta_k^{-1} g_{\beta_k}\bigr\}_{k \in \mathbb{N}}$ uniformly converges to $\widehat{A} \in C(X, \mathbb{R})$ with $\beta_k \to +\infty$ as $k \to +\infty$. Then it follows from Theorem D (ii) that

$$\widehat{b}(T(x)) + \widehat{A}(x) = \widehat{b}(x) \tag{4.15}$$

for all $x \in X$. If $\widehat{b}(x_0) \in \mathbb{R}$ for some $x_0 \in X$, then it follows from $\widehat{A} \in C(X, \mathbb{R})$ that $\widehat{b}(T(x_0)) \in \mathbb{R}$ and $\widehat{A}(x_0) = \widehat{b}(x_0) - \widehat{b}(T(x_0))$. We conclude that the values of $\widehat{A}$ at points in $\{y \in X : \widehat{b}(y) \in \mathbb{R}\}$ are determined by (4.15).

Recall that $\mu_{\beta A}$ is a probability measure for all $\beta > 0$. Note that

$$\bigoplus_{x \in X} \bigl(0 \otimes \widehat{b}(x)\bigr) = \lim_{\beta \to +\infty} l_\beta^\mu(\mathbb{0}_X) = \lim_{\beta \to +\infty} \frac{1}{\beta} \log \mu_{\beta A}(X) = 0$$

and it follows that $\widehat{b} : X \to \mathbb{R} \cup \{-\infty\}$.

We claim that $\{y \in X : \widehat{b}(y) \in \mathbb{R}\}$ is dense in X. If the claim holds, then the values of $\widehat{A}$ on X are all determined since $\widehat{A} \in C(X, \mathbb{R})$ and $\{y \in X : \widehat{b}(y) \in \mathbb{R}\}$ is dense. Thus, every convergent subsequence $\{\beta_k^{-1} g_{\beta_k}\}_{k\in\mathbb{N}}$ converges to the same funtion $\widehat{A} \in C(X, \mathbb{R})$ and (i) is verified.

Now we prove the claim by contradiction. Suppose that $\{y \in X : \widehat{b}(y) \in \mathbb{R}\}$ is not dense, i.e., there is an open set $U \subsetneq X$ so that $\widehat{b}(y) = -\infty$ for all $y \in U$. Moreover, (4.15) implies that $\widehat{b}(T(y)) = -\infty$ if $\widehat{b}(y) = -\infty$. It follows that for all $y \in U$ and $n \in \mathbb{N}$, $\widehat{b}(T^n(y)) = -\infty$. Since T is open distance-expanding and transitive, there exists a positive integer M so that $X = \bigcup_{i=0}^{M} T^i(U)$ (see [PU10, Theorem 4.3.12]). Thus,

$$0 = \bigoplus_{x\in X} \widehat{b}(x) = \bigoplus_{0\leqslant i\leqslant M} \bigoplus_{y\in T^i(U)} \widehat{b}(y) = \bigoplus_{0\leqslant i\leqslant M} (-\infty) = -\infty.$$

This is a contradiction and our claim follows.

(ii). Recall $g_\beta = \beta A + \log u_{\beta A} - \log u_{\beta A} \circ T - P(T, \beta A)$ and $\lim_{\beta\to+\infty} \frac{P(T,\beta A)}{\beta} = Q(T, A)$.

We have proved in (i) that $\frac{g_\beta}{\beta}$ uniformly converges to $\widehat{A}$ as $\beta \to +\infty$. Now suppose that the subsequence $\big\{\frac{1}{\beta_k} \log u_{\beta_k A}\big\}_{k\in\mathbb{N}}$ uniformly comverges to $v \in C(X, \mathbb{R})$ with $\beta_k \to +\infty$ as $k \to +\infty$, then

$$\begin{aligned}\widehat{A} = \lim_{k\to+\infty} \frac{g_{\beta_k}}{\beta_k} &= A - Q(T, A) + \lim_{k\to+\infty} \frac{\log u_{\beta_k A} - \log u_{\beta_k A} \circ T}{\beta_k} \\ &= A - Q(T, A) + v - v \circ T.\end{aligned}$$

This implies that $v - v \circ T$ is uniquely determined.

Recall $\int u_{\beta A} \,\mathrm{d}m_{\beta A} = 1$ for all $\beta > 0$. Suppose a subsequence $\{\widehat{\beta}_k\}_{k\in\mathbb{N}}$ of the sequence $\{\beta_k\}_{k\in\mathbb{N}}$ satisfies

$$\lim_{k\to+\infty} \widehat{\beta}_k = +\infty \quad \text{and} \quad \lim_{k\to+\infty} l^m_{\widehat{\beta}_k}(\cdot) =: \check{l}(\cdot),$$

where $\check{l}$ is the pointwise limit of $l^m_{\widehat{\beta}_k}$ as $k \to +\infty$.

Similar to the argument for the claim in the proof of Corollary 4.3 (see (4.13)), we have

$$\begin{aligned}\big|\check{l}(v)\big| &\leqslant \big|\check{l}(v) - l^m_{\widehat{\beta}_k}(v)\big| + \big|l^m_{\widehat{\beta}_k}(v)\big| \\ &= \big|\check{l}(v) - l^m_{\widehat{\beta}_k}(v)\big| + \big|l^m_{\widehat{\beta}_k}(v) - l^m_{\widehat{\beta}_k}\big(\widehat{\beta}_k^{-1} \log u_{\widehat{\beta}_k A}\big)\big| \\ &\leqslant \big|\check{l}(v) - l^m_{\widehat{\beta}_k}(v)\big| + \big\|v - \widehat{\beta}_k^{-1} \log u_{\widehat{\beta}_k A}\big\|,\end{aligned}$$

where the equality follows from $\int u_{\beta A} \,\mathrm{d}m_{\beta A} = 1$ and the second inequality follows from (4.5). As $k \to +\infty$ in the above inequalities, we have $\check{l}(v) = 0$. Moreover, $\check{l}$ is tropical linear by Theorem C (ii).

We claim that the uniqueness of $v - v \circ T$ and that $\check{l}(v) = 0$ implies the uniqueness of v. If there exists $v_1, v_2 \in C(X, \mathbb{R})$ such that $v_1 - v_1 \circ T = v_2 - v_2 \circ T$ and $\check{l}(v_1) = \check{l}(v_2) = 0$, then $v_1 - v_2 = (v_1 - v_2) \circ T$ and $v_1 - v_2 \in C(X, \mathbb{R})$. The transitivity of T immediately implies that $v_1 - v_2$ must be a constant function c. Thus, it follows that

$$0 = \check{l}(v_1) = \check{l}(v_2 \otimes c) = \check{l}(v_2) \otimes c = 0 \otimes c = c,$$

i.e., $v_1 = v_2$ and the claim follows. Now (ii) is verified.

(iii). Recall $l^{\mu}_{\beta}(f) = l^{m}_{\beta}\big(f + \frac{1}{\beta}\log u_{\beta A}\big)$ for all $\beta > 0$ and $f \in C(X,\mathbb{R})$ since $\mu_{\beta A} = u_{\beta A} \cdot m_{\beta A}$. By (ii), $\frac{1}{\beta}\log u_{\beta A}$ uniformly converges to some $v \in C(X,\mathbb{R})$ as $\beta \to +\infty$. Recall that $l^{\mu}_{\beta}(\cdot)$ pointwise converges to $\widehat{l}(\cdot)$ as $\beta \to +\infty$.

Now suppose that $\check{l}(\cdot)$ is the pointwise limit of $l^{m}_{\beta_k}(\cdot)$ with $\beta_k \to +\infty$ as $k \to +\infty$. Similar to the argument for the rate function in the proof of Corollary 4.3 (see (4.14)), we have for all $f \in C(X,\mathbb{R})$,

$$\begin{aligned}
\big|l^{m}_{\beta_k}(v+f) - \widehat{l}(f)\big| &\leqslant \big|l^{m}_{\beta_k}(v+f) - l^{\mu}_{\beta_k}(f)\big| + \big|l^{\mu}_{\beta_k}(f) - \widehat{l}(f)\big| \\
&= \big|l^{m}_{\beta_k}(v+f) - l^{m}_{\beta_k}(\beta_k^{-1}\log u_{\beta_k A} + f)\big| + \big|l^{\mu}_{\beta_k}(f) - \widehat{l}(f)\big| \\
&\leqslant \big\|v - \beta_k^{-1}\log u_{\beta_k A}\big\|_{C^0} + \big|l^{\mu}_{\beta_k}(f) - \widehat{l}(f)\big|,
\end{aligned}$$

where the second inequality follows from (4.5). As $k \to +\infty$ in the above inequalities, it follows that

$$\widehat{l}(f) = \check{l}(f+v)$$

for all $f \in C(X,\mathbb{R})$, i.e., $\check{l}(g) = \widehat{l}(g-v)$ for all $g \in C(X,\mathbb{R})$. We conclude that $\widehat{l}$ is uniquely determined and (iii) is verified. $\square$

Appendix A. Additional proofs

We include the proofs of a few results known to experts in this appendix for the convenience of the reader.

Proof of Proposition 3.2. Fix $x \in X$ and denote $T^{-1}(x) \coloneqq \{x_1, \cdots, x_n\}$. Let $\xi > 0$ be the constant in Lemma 3.1. For all $y \in B(x,\xi)$, denote $y_i \coloneqq T^{-1}_{x_i}(y)$ for all $1 \leqslant i \leqslant n$ and consequently $T^{-1}(y) = \{y_1, \cdots, y_n\}$ according to Lemma 3.1. Thus, $\mathcal{L}_A(u)(x) - \mathcal{L}_A(u)(y) = \max\limits_{1 \leqslant i \leqslant n}\{u(x_i) + A(x_i)\} - \max\limits_{1 \leqslant i \leqslant n}\{u(y_i) + A(y_i)\}$. It follows that

$$|\mathcal{L}_A(u)(x) - \mathcal{L}_A(u)(y)| \leqslant \bigoplus_{1 \leqslant i \leqslant k} |u(x_i) - u(y_i) + A(x_i) - A(y_i)|. \tag{A.1}$$

If A and u are in $C(X,\mathbb{R})$, the compactness of X implies that A and u are uniformly continuous. Thus, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|u(z_1) - u(z_2) + A(z_1) - A(z_2)| < \epsilon$ for all $z_1, z_2 \in X$ with $d(z_1,z_2) < \delta$.

Then for all $y \in B(x, \min\{\xi, \lambda\delta\})$, Lemma 3.1 implies that $d(x_i,y_i) \leqslant \lambda^{-1}d(x,y) < \delta$ for all $1 \leqslant i \leqslant n$. It follows that $|u(x_i) - u(y_i) + A(x_i) - A(y_i)| < \epsilon$ for all $1 \leqslant i \leqslant n$. Thus, it follows from (A.1) that $|\mathcal{L}_A(u)(x) - \mathcal{L}_A(u)(y)| < \epsilon$ for all $y \in B(x, \min\{\xi, \lambda\delta\})$. Now the continuity of $\mathcal{L}_A(u)$ is verified.

If A and u are in $\mathrm{Lip}(X,d^{\alpha})$, we immediately have $\mathcal{L}_A(u) \in C(X,\mathbb{R})$ and consequently $\|\mathcal{L}_A(u)\|_{C^0} < +\infty$ since X is compact. Moreover, it follows from (A.1) and Lemma 3.1 that

$$|\mathcal{L}_A(u)|_{d^{\alpha},\xi} \leqslant \lambda^{-\alpha}(|u|_{d^{\alpha}} + |A|_{d^{\alpha}}).$$

Thus, $|\mathcal{L}_A(u)|_{d^\alpha} \leqslant \max\bigl\{2\|\mathcal{L}_A(u)\|_{C^0}/\xi^\alpha,\ \lambda^{-\alpha}(|u|_{d^\alpha} + |A|_{d^\alpha})\bigr\} < +\infty$ and we conclude that $\mathcal{L}_A(u) \in \operatorname{Lip}(X, d^\alpha)$. $\square$

Proof of Lemma 3.4. Let $\xi > 0$ be the constant in Lemma 3.1. Fix $x, y \in X$ with $d(x, y) < \xi$ and $n \in \mathbb{N}$. Denote $T^{-n}(x) := \{x_1, \cdots, x_k\}$ and $y_i := T^{-n}_{x_i}(y)$ for all $1 \leqslant i \leqslant k$. Then Lemma 3.1 implies that $T^{-n}(y) = \{y_1, \cdots, y_k\}$ and $d(T^l(x_i), T^l(y_i)) \leqslant \lambda^{l-n} d(x, y)$ for all $0 \leqslant l \leqslant n$ and $1 \leqslant i \leqslant k$. It follows that

$$\biggl|\bigoplus_{\overline{x}\in T^{-n}(x)} S_nA(\overline{x}) - \bigoplus_{\overline{y}\in T^{-n}(y)} S_nA(\overline{y})\biggr| \leqslant \bigoplus_{1\leqslant i\leqslant k} |S_nA(x_i) - S_nA(y_i)|.$$

Since $A \in \operatorname{Lip}(X, d^\alpha)$, it follows that for all $1 \leqslant i \leqslant k$,

$$\begin{aligned}|S_nA(x_i) - S_nA(y_i)| &\leqslant |A|_{d^\alpha}(d(x_i, y_i)^\alpha + \dots + d(T^{n-1}(x_i), T^{n-1}(y_i))^\alpha)\\ &\leqslant |A|_{d^\alpha} d(x, y)^\alpha(\lambda^{-n\alpha} + \dots + \lambda^{-\alpha})\\ &< |A|_{d^\alpha} d(x, y)^\alpha \lambda^{-\alpha}/(1 - \lambda^{-\alpha}).\end{aligned} \tag{A.2}$$

We take $C_0(A) := |A|_{d^\alpha}\xi^\alpha \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}$ and conclude that for all $x, y \in X$ with $d(x, y) < \xi$ and $n \in \mathbb{N}$,

$$\biggl|\bigoplus_{\overline{x}\in T^{-n}(x)} S_nA(\overline{x}) - \bigoplus_{\overline{y}\in T^{-n}(y)} S_nA(\overline{y})\biggr| \leqslant C_0(A). \tag{A.3}$$

Recall that X is compact and T is topologically transitive

Claim. There exists $N_\xi \in \mathbb{N}$ such that for all $x, y \in X$, there exists an integer m satisfying $0 \leqslant m \leqslant N_\xi$ and $T^m(B(x, \xi)) \cap B(y, \xi) \neq \emptyset$.

Since X is compact, there exists a finite set $\{z_1, \cdots, z_s\} \subseteq X$ such that $\bigcup_{i=1}^{s} B\bigl(z_i, \frac{\xi}{2}\bigr) = X$. For all $x, y \in X$, there exists $i, j \in \{1, \cdots, s\}$ such that $d(x, z_i) < \frac{\xi}{2}$ and $d(y, z_j) < \frac{\xi}{2}$. Thus, $B\bigl(z_i, \frac{\xi}{2}\bigr) \subseteq B(x, \xi)$ and $B\bigl(z_j, \frac{\xi}{2}\bigr) \subseteq B(y, \xi)$. We conclude that if for some $m \in \mathbb{N}_0$, $T^m\bigl(B\bigl(z_i, \frac{\xi}{2}\bigr)\bigr) \cap B\bigl(z_j, \frac{\xi}{2}\bigr) \neq \emptyset$, then $T^m(B(x, \xi)) \cap B(y, \xi) \neq \emptyset$.

Since T is topologically transitive, there exists $m_{ij} \in \mathbb{N}_0$ such that $T^{m_{ij}}\bigl(B\bigl(z_i, \frac{\xi}{2}\bigr)\bigr) \cap B\bigl(z_j, \frac{\xi}{2}\bigr) \neq \emptyset$ for all $i, j \in \{1, \cdots, s\}$. Thus, denote $N_\xi := \max\limits_{1\leqslant i,j\leqslant s} m_{ij}$ and $s < +\infty$ implies $N_\xi < +\infty$. Now the claim is verified.

Fix $x, y \in X$ and $n \in \mathbb{N}$. Then the claim implies that there exists $y' \in T^{-m}(B(y,\xi)) \cap B(x,\xi)$ for some integer m satisfying $0 \leqslant m \leqslant N_\xi$. It follows from (A.3) that

$$\begin{aligned}
\bigoplus_{\overline{x} \in T^{-n}(x)} S_n A(\overline{x}) &\leqslant C_0(A) + \bigoplus_{\overline{y'} \in T^{-n}(y')} S_n A(\overline{y'}) \\
&= C_0(A) - S_m A(y') + \bigoplus_{\overline{y'} \in T^{-n}(y')} S_{n+m} A(\overline{y'}) \\
&\leqslant C_0(A) - m \inf A + \bigoplus_{\overline{y'} \in T^{-n-m}(T^m(y'))} S_{n+m} A(\overline{y'}) \\
&= C_0(A) - m \inf A + \bigoplus_{\overline{y'} \in T^{-n-m}(T^m(y'))} \left(S_m A(\overline{y'}) + S_n A(T^m(\overline{y'})) \right) \\
&\leqslant C_0(A) - m \inf A + m \sup A + \bigoplus_{\overline{y'} \in T^{-n-m}(T^m(y'))} S_n A(T^m(\overline{y'})) \\
&= C_0(A) - m \inf A + m \sup A + \bigoplus_{z \in T^{-n}(T^m(y'))} S_n A(z) \\
&\leqslant 2C_0(A) + N_\xi(\sup A - \inf A) + \bigoplus_{\overline{y} \in T^{-n}(y)} S_n A(\overline{y}).
\end{aligned}$$

Thus, we can take $C_1(A) \coloneqq 2C_0(A) + N_\xi(\sup A - \inf A)$ and

$$\left| \bigoplus_{\overline{x} \in T^{-n}(x)} S_n A(\overline{x}) - \bigoplus_{\overline{y} \in T^{-n}(y)} S_n A(\overline{y}) \right| \leqslant C_1(A) \tag{A.4}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now it suffices to give the estimate for $\left| \mathcal{L}_A^n(u) \right|_{d^\alpha}$ when $A, u \in \operatorname{Lip}(X, d^\alpha)$. Fix $A, u \in \operatorname{Lip}(X, d^\alpha)$. Note that for all $x \in X$,

$$\mathcal{L}_A^n(u)(x) = \bigoplus_{\overline{x} \in T^{-n}(x)} (S_n A(\bar{x}) + u(\overline{x})).$$

For $x, y \in X$ with $d(x,y) < \xi$, (A.2) implies that for all $n \in \mathbb{N}$,

$$\left| \mathcal{L}_A^n(u) \right|_{d^\alpha, \xi} \leqslant \frac{|A|_{d^\alpha} \lambda^{-\alpha}}{1 - \lambda^{-\alpha}} + |u|_{d^\alpha} \lambda^{-\alpha} \leqslant \frac{\lambda^{-\alpha}}{1 - \lambda^{-\alpha}} (|A|_{d^\alpha} + |u|_{d^\alpha}).$$

For $x, y \in X$ with $d(x,y) \geqslant \xi$, (A.4) implies that for all $n \in \mathbb{N}$,

$$\frac{|\mathcal{L}_A^n(u)(x) - \mathcal{L}_A^n(u)(y)|}{d(x,y)^\alpha} \leqslant \frac{C_1(A) + \sup u - \inf u}{d(x,y)^\alpha} \leqslant \frac{C_1(A) + \sup u - \inf u}{\xi^\alpha}.$$

We conclude that for all $n \in \mathbb{N}$,

$$|\mathcal{L}_A^n(u)|_{d^\alpha} \leqslant \max \left\{ \frac{\lambda^{-\alpha}}{1 - \lambda^{-\alpha}} (|A|_{d^\alpha} + |u|_{d^\alpha}), \ \frac{C_1(A) + \sup u - \inf u}{\xi^\alpha} \right\}. \tag{A.5}$$

Now let $C_2(A,u)$ denote a positive constant satifying $|\mathcal{L}_A^n(u)|_{d^\alpha} \leqslant C_2(A,u)(|A|_{d^\alpha} + |u|_{d^\alpha})$ for a specific pair $A, u \in \operatorname{Lip}(X, d^\alpha)$ and all $n \in \mathbb{N}$.

If $A, u \in \operatorname{Lip}(X, d^\alpha)$ are two constant functions, then for each $n \in \mathbb{N}$, $\mathcal{L}_A^n(u)$ is a constant function. Thus, $0 = |A|_{d^\alpha} = |u|_{d^\alpha} = \left| \mathcal{L}_A^n(u) \right|_{d^\alpha}$ for all $n \in \mathbb{N}$ and consequently $C_2(A,u)$ can be arbitrary positive number.

Now suppose that there is a non-constant function between A and u, i.e., $|A|_{d^\alpha} + |u|_{d^\alpha} > 0$. By (A.5), we can take $C_2(A,u) \coloneqq \max\left\{ \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}, \ \frac{C_1(A) + \sup u - \inf u}{\xi^\alpha (|A|_{d^\alpha} + |u|_{d^\alpha})} \right\}$.

Moreover, $A, u \in \operatorname{Lip}(X, d^\alpha)$ implies that

$$\sup A - \inf A \leqslant |A|_{d^\alpha} (\operatorname{diam} X)^\alpha \quad \text{and} \quad \sup u - \inf u \leqslant |u|_{d^\alpha} (\operatorname{diam} X)^\alpha.$$

Recall that $C_1(A) = 2C_0(A) + N_\xi(\sup A - \inf A)$ and $C_0(A) = |A|_{d^\alpha} \xi^\alpha \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}$. We conclude that

$$\begin{aligned}
C_2(A,u) &\leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} + \frac{C_1(A) + \sup u - \inf u}{\xi^\alpha (|A|_{d^\alpha} + |u|_{d^\alpha})} \\
&\leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} + \frac{2C_0(A) + |u|_{d^\alpha} (\operatorname{diam} X)^\alpha + N_\xi |A|_{d^\alpha} (\operatorname{diam} X)^\alpha}{\xi^\alpha (|A|_{d^\alpha} + |u|_{d^\alpha})} \\
&\leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} + \frac{N_\xi (\operatorname{diam} X)^\alpha}{\xi^\alpha} + \frac{2C_0(A)}{\xi^\alpha (|A|_{d^\alpha} + |u|_{d^\alpha})} \\
&\leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} + \frac{N_\xi (\operatorname{diam} X)^\alpha}{\xi^\alpha} + \frac{2\lambda^{-\alpha}}{1-\lambda^{-\alpha}}.
\end{aligned}$$

Thus, we take $C_2 \coloneqq \frac{3\lambda^{-\alpha}}{1-\lambda^{-\alpha}} + \frac{N_\xi (\operatorname{diam} X)^\alpha}{\xi^\alpha}$ and conclude that

$$|\mathcal{L}_A^n(u)|_{d^\alpha} \leqslant C_2 (|A|_{d^\alpha} + |u|_{d^\alpha})$$

for all $A, u \in \operatorname{Lip}(X, d^\alpha)$ and $n \in \mathbb{N}$. The proof is now complete. $\square$

Proof of Proposition 3.5. The proof of this proposition is very similar to that of [LZ23, Proposition 6.4]. For the convenience of readers, we keep it completely as follows.

Denote $D \coloneqq C_2 |A|_{d^\alpha} (\operatorname{diam} X)^\alpha$ in this proof, where C_2 is the constant in Lemma 3.4. To establish (i), we fix a maximizing measure $\mu \in M(X,T)$ for A (i.e., μ is T-invariant Borel probability measure satisfying $\int A \, \mathrm{d}\mu = Q(T,A)$).

We choose for each $n \in \mathbb{N}$ a point $y_n \in X$ such that $S_n \overline{A}$ attains its maximum at y_n (due to compactness of X). Recall that $\mathcal{L}_A^n(\mathbb{0}_X)(x) = \bigoplus_{\overline{x} \in T^{-n}(x)} S_n A(\overline{x})$ for all $x \in X$ and $n \in \mathbb{N}$. Then we have the following estimate that for all $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(x) &\geqslant \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(T^n(y_n)) - \left|\mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)\right|_{d^\alpha} d(x, T^n(y_n))^\alpha \\
&\geqslant \bigoplus_{\overline{y} \in T^{-n}(T^n(y_n))} S_n \overline{A}(\overline{y}) - C_2 |\overline{A}|_{d^\alpha} (\operatorname{diam} X)^\alpha \\
&\geqslant S_n \overline{A}(y_n) - D \\
&\geqslant \int S_n \overline{A} \, \mathrm{d}\mu - D \\
&= -D,
\end{aligned} \tag{A.6}$$

where the second inequality follows from Lemma 3.4, the third inequality follows from the definition of y_n, and the last two lines of the above inequalities follow from the definition of μ. We conclude that for all $x \in X$,

$$v_{\mathbb{0}_X}(x) = \limsup_{n \to +\infty} \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(x) \geqslant -D.$$

In order to prove $v_{\mathbb{0}_X} \preccurlyeq D$, we claim that there exists a point $z \in X$ such that $v_{\mathbb{0}_X}(z) \leqslant 0$.

If the claim holds, then it follows from Lemma 3.4 that

$$\mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(x) \leqslant \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(z) + \left|\mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)\right|_{d^\alpha} d(x,z)^\alpha \leqslant \mathcal{L}_{\overline{A}}^n(\mathbb{0}_X)(z) + D \tag{A.7}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus, $v_{\mathbb{0}_X}(x) \leqslant v_{\mathbb{0}_X}(z) + D \leqslant D$ for all x in X and (i) is verified.

To establish the claim above, we argue by contradiction. Suppose that $v_{\mathbb{0}_X}(z) > 0$ for all $z \in X$. By the definition of $v_{\mathbb{0}_X}$, for every $z \in X$, there exists $n_z \in \mathbb{N}$ such that $\mathcal{L}_{\overline{A}}^{n_z}(\mathbb{0}_X)(z) > \frac{1}{2} v_{\mathbb{0}_X}(z)$. By the continuity of $\mathcal{L}_{\overline{A}}^{n_z}(\mathbb{0}_X)$, there exists $\delta_z > 0$ such that $\mathcal{L}_{\overline{A}}^{n_z}(\mathbb{0}_X)(y) > \frac{1}{3} v_{\mathbb{0}_X}(z)$ for all $y \in B(z, \delta_z)$. By the compactness of X, there exists finite points $x_1, \cdots, x_k$ such that $X = \bigcup_{i=1}^{k} B(x_i, \delta_{x_i})$.

For each $y \in X$, we choose an i such that $y \in B(x_i, \delta_{x_i})$ and set $\overline{n}_y \coloneqq n_{x_i}$. Let $c \coloneqq \min\bigl\{ v_{\mathbb{0}_X}(x_i)/(3n_{x_i}) : 1 \leqslant i \leqslant k \bigr\} > 0$. Then for all $y \in X$, $\mathcal{L}_{\overline{A}}^{\overline{n}_y}(\mathbb{0}_X)(y) \geqslant \overline{n}_y c$, i.e.,

$$\bigoplus_{\overline{y} \in T^{-\overline{n}_y}(y)} S_n \overline{A}(\overline{y}) \geqslant \overline{n}_y c. \tag{A.8}$$

Note that $T^{-\overline{n}_y}(y)$ is finite for all $y \in X$.

Now fix a point $z_0 \in X$. For every $i \in \mathbb{N}$, we can recursively choose $z_i \in X$ satisfying $T^{\overline{n}_{i-1}}(z_i) = z_{i-1}$ and $S_{\overline{n}_{z_{i-1}}} \overline{A}(z_i) \geqslant \overline{n}_{z_{i-1}} c$. Then consider the sequence of Borel probability measures (on X) $\{\mu_i\}_{i \in \mathbb{N}}$ given by

$$\mu_i \coloneqq \frac{1}{m_i} \sum_{j=0}^{m_i - 1} \delta_{T^j(z_i)}$$

where $m_i \coloneqq \overline{n}_{z_0} + \overline{n}_{z_1} + \cdots + \overline{n}_{z_{i-1}}$ and $\delta_{T^j(z_i)}$ is the dirac measure at point $T^j(z_i)$. Due to the weak*-compactness of the unit ball of the space consisting of all finite signed Borel measures on X, we conclude that there is a subsequence $\{\mu_{i_k}\}_{k \in \mathbb{N}}$ of $\{\mu_i\}_{i \in \mathbb{N}}$ that converges to a Borel probability measure $\widetilde{\mu}$ in the weak* topology. Moreover, it directly follows from the definitions of μ_i that $\bigl| \int f \,\mathrm{d}\mu_i - \int f \,\mathrm{d}\mu_i \circ T^{-1} \bigr| \leqslant \frac{2}{m_i} \|f\|_{C^0}$. Note that $m_i \geqslant i$. Thus, $\widetilde{\mu}$ should be a T-invariant Borel probability measure, i.e., $\widetilde{\mu} \in M(X,T)$. Recall that $Q(T,A) = \max\{\int A \,\mathrm{d}\mu : \mu \in M(X,T)\}$ and $\overline{A} = A - Q(T,A)$. It follows that

$$\begin{aligned} 0 \geqslant \int \overline{A} \,\mathrm{d}\tilde{\mu} = \lim_{k \to +\infty} \frac{1}{m_{i_k}} \sum_{j=0}^{m_{i_k}-1} \overline{A}\bigl(T^j(z_{i_k})\bigr) &= \lim_{k \to +\infty} \frac{1}{m_{i_k}} \sum_{j=0}^{i_k - 1} S_{\overline{n}_{z_j}} \overline{A}(z_{j+1}) \\ &\geqslant \lim_{k \to +\infty} \frac{1}{m_{i_k}} \sum_{j=0}^{i_k-1} \overline{n}_{z_j} c = c > 0, \end{aligned}$$

where the last inequality follows from (A.8). This is a contradiction and consequently our claim is verified.

Next, we prove (ii). It follows from Lemma 3.4 that $\bigl| \mathcal{L}_{\overline{A}}^{n}(\mathbb{0}_X) \bigr|_{d^\alpha} \leqslant C_2 \|A\|_{d^\alpha}$ for all $n \in \mathbb{N}$. Thus, $\bigl\{ \mathcal{L}_{\overline{A}}^{n}(\mathbb{0}_X) \bigr\}_{n \in \mathbb{N}}$ is an equicontinuous family. It follows from (A.6) that $\bigl\{ \mathcal{L}_{\overline{A}}^{n}(\mathbb{0}_X) \bigr\}_{n \in \mathbb{N}}$ has uniform lower bound $-D$. For a uniform upper bound, recall that there exists $z \in X$ such that $v_{\mathbb{0}_X}(z) \leqslant 0$. Fix $\epsilon > 0$. It follows that $\mathcal{L}_{\overline{A}}^{m}(\mathbb{0}_X)(z) \leqslant \epsilon$ for all m big enough. Thus, it follows from (A.7) that $\bigl\{ \mathcal{L}_{\overline{A}}^{n}(\mathbb{0}_X) \bigr\}_{n \geqslant m}$ has uniform upper bound $D + \epsilon$.

In conclusion,

$$\bigl\{ \mathcal{L}_{\overline{A}}^{n}(\mathbb{0}_X) \bigr\}_{n \geqslant m} \quad \text{and} \quad \Bigl\{ \sup_{k \geqslant n} \mathcal{L}_{\overline{A}}^{k}(\mathbb{0}_X) \Bigr\}_{n \geqslant m}$$

are equicontinuous, uniformly bounded, and consequently normal. Thus, $v_{\mathbb{0}_X}$, as the pointwise decreasing limit of $\sup_{k\geqslant n}\mathcal{L}_{\overline{A}}^k(\mathbb{0}_X)$ as $n\to+\infty$, is the uniform limit (limit in the C^0 topology) of $\sup_{k\geqslant n}\mathcal{L}_{\overline{A}}^k(\mathbb{0}_X)$. Now it follows clearly that $|v_{\mathbb{0}_X}|_{d^\alpha}\leqslant C_2|A|_{d^\alpha}$ and (ii) is verified.

For (iii), it is straightforward to check that (see [LZ23, Lemma 6.1 (iii)])

$$\mathcal{L}_{\overline{A}}\Bigl(\sup_{k\geqslant n}\mathcal{L}_{\overline{A}}^k(\mathbb{0}_X)\Bigr)=\sup_{k\geqslant n+1}\mathcal{L}_{\overline{A}}^k(\mathbb{0}_X).$$

By (ii) and Lemma 3.3, we conclude that the left-hand side converges to $\mathcal{L}_{\overline{A}}(v_{\mathbb{0}_X})$ and the right-hand side converges to v_0 as $k\to+\infty$, i.e., $\mathcal{L}_{\overline{A}}(v_{\mathbb{0}_X})=v_{\mathbb{0}_X}$. $\square$

The next lemma is useful in the proof of Proposition 3.12 (v).

Lemma A.1. *Let $T\colon X\to X$ satisfy the Assumptions in Subsections 1.1 and $A\in\operatorname{Lip}(X,d^\alpha)$. Let $\xi>0$ be the constant in Lemma 3.1. If $x_0\in\Omega_A$ is an Aubry point, then for all $\epsilon\in(0,\xi)$ and $l\in\mathbb{N}$, there exists a trajectory from x_1 to $T^n(x_1)$ satisfying $n>l$, $d(x_1,x_0)\leqslant\epsilon$, $T^n(x_1)=x_0$ and $|S_n(\overline{A})(x_1)|\leqslant\epsilon$.*

Remark. This lemma is slightly different from [Ga17, Corollary 4.5]. In this lemma, we assume $A\in\operatorname{Lip}(X,d^\alpha)$ and require $T^n(x_1)=x_0$ while [Ga17, Corollary 4.5] assumes $A\in C(X,\mathbb{R})$ and only requires $d(T^n(x_1),x_0)\leqslant\epsilon$. Although [Ga17, Corollary 4.5] is stated for subshifts of finite type, its proof is applicable to our setting. Thus, we directly use its result in the following proof.

Proof. Fix $\epsilon\in(0,\xi)$, $l\in\mathbb{N}$, and $x_0\in\Omega_A$. Then fix $\delta>0$ satisfying

$$\delta+\frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\delta^\alpha<\epsilon.$$

It follows from [Ga17, Corollary 4.5] that there exists a trajectory from x_2 to $T^n(x_2)$ satisfying $n>l$, $d(x_2,x_0)\leqslant\delta$, $d(T^n(x_2),x_0)\leqslant\delta$ and $|S_n(\overline{A})(x_2)|\leqslant\delta$.

Since $d(x_2,x_0)\leqslant\delta<\xi$, Lemma 3.1 implies that we have the trajectory from $T_{x_2}^{-n}(x_0)$ to x_0 with

$$\begin{aligned}d\bigl(T_{x_2}^{-n}(x_0),x_0\bigr)&\leqslant d\bigl(T_{x_2}^{-n}(x_0),x_2\bigr)+d(x_2,x_0)\\&\leqslant\lambda^{-n}d(x_0,T^n(x_2))+d(x_2,x_0)\leqslant 2\delta<\epsilon.\end{aligned}$$

Denote $x_1:=T_{x_2}^{-n}(x_0)$. It follows from (A.2) that $|S_n(\overline{A})(x_1)-S_n(\overline{A})(x_2)|\leqslant\frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\delta^\alpha$. We conclude that $d(x_1,x_0)<\epsilon$, $T^n(x_1)=x_0$, and

$$|S_n(\overline{A})(x_1)|\leqslant\delta+\frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\delta^\alpha<\epsilon,$$

where the last inequality follows from the definition of δ. $\square$

Now we provide a sketch of the proof of Proposition 3.12 below.

Proof of Proposition 3.12. (i). Since u is a tropical eigenfunction of $\mathcal{L}_A$, $u(T(x))\geqslant u(x)\otimes\overline{A}(x)$ for all $x\in X$. Thus, $u(z)\otimes S_n(\overline{A})(z)\leqslant u(T^n(z))$ for all $z\in X$ and $n\in\mathbb{N}$. Since X

is compact and $u \in C(X,\mathbb{R})$, u is uniformly continuous. It follows that for every $\epsilon > 0$, there exists $\eta(\epsilon) \in (0,\epsilon)$ such that

$$|u(y) - u(x)| \leqslant \epsilon$$

for all $x, y \in X$ with $d(y,x) \leqslant \eta(\epsilon)$. By Definition 3.11, we see that for all $x, y \in X$,

$$\begin{aligned}
u(x) \otimes \phi_A(x,y) &= \lim_{\epsilon\to 0^+} u(x) \otimes \Big(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\eta(\epsilon)\\ d(T^n(z),y)\leqslant\eta(\epsilon)}} S_n\overline{A}(z)\Big) \\
&\leqslant \limsup_{\epsilon\to 0^+}\Big(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\eta(\epsilon)\\ d(T^n(z),y)\leqslant\eta(\epsilon)}} S_n\overline{A}(z) \otimes u(z) \otimes \epsilon\Big) \\
&\leqslant \limsup_{\epsilon\to 0^+}\Big(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x)\leqslant\eta(\epsilon)\\ d(T^n(z),y)\leqslant\eta(\epsilon)}} u(T^n(z)) \otimes \epsilon\Big) \\
&\leqslant \limsup_{\epsilon\to 0^+}(u(y) \otimes 2\epsilon) \\
&= u(y).
\end{aligned}$$

(ii). Statement (ii) is already established in Subsection 3.2.

(iii). In the analysis below, we use Lemma 3.1 to connect two trajectories when the end of one trajectory is close to the beginning of the other.

For a trajectory from w_1 to $T^n(w_1)$ satisfying $d(w_1,x) \leqslant \epsilon$ and $d(T^n(w_1),y) \leqslant \epsilon$ and a trajectory from w_2 to $T^m(w_2)$ satisfying $d(w_2,y) \leqslant \epsilon$ and $d(T^m(w_2),z) \leqslant \epsilon$, Lemma 3.1 implies that for all $0 < \epsilon < \frac{\xi}{2}$, we have the trajectory from $T^{-n}_{w_1}(w_2)$ to $T^m(w_2)$ satisfying $d\big(T^{-n}_{w_1}(w_2), x\big) \leqslant 3\epsilon$ and $d(T^m(w_2),z) \leqslant \epsilon$.

Note that $d(w_2, T^n(w_1)) \leqslant 2\epsilon < \xi$. Thus, it follows from (A.2) that

$$\big|S_n(\overline{A})(w_1) - S_n(\overline{A})\big(T^{-n}_{w_1}(w_2)\big)\big| \leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}(2\epsilon)^\alpha.$$

We conclude that

$$S_{n+m}(\overline{A})\big(T^{-n}_{w_1}(w_2)\big) + \frac{\lambda^{-\alpha}|A|_{d^\alpha}(2\epsilon)^\alpha}{1-\lambda^{-\alpha}} \geqslant S_n(\overline{A})(w_1) + S_m(\overline{A})(w_2).$$

Now (iii) follows from Definition 3.11.

(iv). This is a direct consequence of the definition of the Aubry set and the Mañé potential. See Definitions 3.10 and 3.11.

(v). We remark that the following idea, which is also a generalization of Lemma A.1, is important for the proof below. When $x_0 \in \Omega_A$, for all $l \in \mathbb{N}$, $z \in X$, and $\epsilon > 0$, there exists a trajectory from x_1 to $T^n(x_1)$ satisfies $n > l$, $d(x_1,x_0) \leqslant \epsilon$, $d(T^n(x_1),z) \leqslant \epsilon$, and $|S_n(\overline{A})(x_1) - \phi_A(x_0,z)| \leqslant \epsilon$. This idea is best captured by the concept of the Peierls boundary which we do not elaborate on.

We first show $\phi_A(x_0,z) = \bigoplus_{y\in T^{-1}(z)} (\phi_A(x_0,y) \otimes \overline{A}(y))$ for all $x_0 \in \Omega_A$ and $z \in X$. We have already proved

$$\phi_A(x,y) = \lim_{\epsilon\to 0^+} \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(y_0,x)\leqslant\epsilon\\ T^n(y_0)=y}} S_n(\overline{A})(y_0) \tag{A.9}$$

for all $x, y \in X$ in the claim in the proof of Proposition 3.12 (ii).

Thus, for all $z, x \in X$,

$$\begin{aligned}
\bigoplus_{y\in T^{-1}(z)}(\phi_A(x,y)\otimes \overline{A}(y)) &= \bigoplus_{y\in T^{-1}(z)}\Bigl(\lim_{\epsilon\to0^+}\bigoplus_{n\in\mathbb{N}}\bigoplus_{\substack{d(y_0,x)\leqslant\epsilon\\ T^n(y_0)=y}} S_n(\overline{A})(y_0)\otimes\overline{A}(y)\Bigr)\\
&= \lim_{\epsilon\to0^+}\bigoplus_{y\in T^{-1}(z)}\bigoplus_{n\in\mathbb{N}}\bigoplus_{\substack{d(y_0,x)\leqslant\epsilon\\ T^n(y_0)=y}}(S_n(\overline{A})(y_0)\otimes\overline{A}(y))\\
&= \lim_{\epsilon\to0^+}\bigoplus_{n\in\mathbb{N}}\bigoplus_{\substack{d(y_0,x)\leqslant\epsilon\\ T^{n+1}(y_0)=z}} S_{n+1}(\overline{A})(y_0)\\
&= \lim_{\epsilon\to0^+}\bigoplus_{m\geqslant2}\bigoplus_{\substack{d(y_0,x)\leqslant\epsilon\\ T^m(y_0)=z}} S_m(\overline{A})(y_0),
\end{aligned}$$

where the second equality follows from the fact that $T^{-1}(z)$ is finite.

Fix $x_0 \in \Omega_A$ and $z \in X$ and it suffices to show

$$\phi_A(x_0,z) = \lim_{\epsilon\to0^+}\bigoplus_{m\geqslant2}\bigoplus_{\substack{d(y_0,x_0)\leqslant\epsilon\\ T^n(y_0)=z}} S_m(\overline{A})(y_0). \tag{A.10}$$

Let $\xi > 0$ be the constant in Lemma 3.1. Fix $\epsilon \in \bigl(0, \frac{\xi}{2}\bigr)$. Lemma A.1 implies that there exists a trajectory from x_1 to $T^{n_0}(x_1)$ satisfying $n_0 \in \mathbb{N}$, $d(x_1, x_0) \leqslant \epsilon$, $T^{n_0}(x_1) = x_0$, and $|S_{n_0}(\overline{A})(x_1)| \leqslant \epsilon$.

Thus, for each trajectory from y_0 to $T^n(y_0)$ satisfying $n \in \mathbb{N}$, $T^n(y_0) = z$, and $d(y_0, x_0) \leqslant \epsilon$, Lemma 3.1 implies that we have the trajectory from $T^{-n_0}_{x_1}(y_0)$ to $T^n(y_0)$ with $d\bigl(T^{-n_0}_{x_1}(y_0), x_0\bigr) \leqslant d\bigl(T^{-n_0}_{x_1}(y_0), x_1\bigr) + d(x_1, x_0) \leqslant 2\epsilon$.

Since $d(y_0, x_0) \leqslant \epsilon < \xi$, it follows from (A.2) that

$$\bigl|S_{n_0}(\overline{A})(x_1) - S_{n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr)\bigr| \leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\epsilon^\alpha.$$

Thus,

$$\begin{aligned}
S_n(\overline{A})(y_0) &= S_{n+n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr) - S_{n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr)\\
&\leqslant S_{n+n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr) - S_{n_0}(\overline{A})(x_1) + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\epsilon^\alpha\\
&\leqslant S_{n+n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr) + \epsilon + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\epsilon^\alpha.
\end{aligned}$$

We denote $E(\epsilon) := \epsilon + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}\epsilon^\alpha$ and conclude that

$$\begin{aligned}
\bigoplus_{n\in\mathbb{N}}\bigoplus_{\substack{d(y_0,x_0)\leqslant\epsilon\\ T^n(y_0)=z}} S_n(\overline{A})(y_0) &\leqslant E(\epsilon) + \bigoplus_{n\in\mathbb{N}}\bigoplus_{\substack{d(y_0,x_0)\leqslant\epsilon\\ T^n(y_0)=z}} S_{n+n_0}(\overline{A})\bigl(T^{-n_0}_{x_1}(y_0)\bigr)\\
&\leqslant E(\epsilon) + \bigoplus_{m\geqslant2}\bigoplus_{\substack{d(y_1,x_0)\leqslant2\epsilon\\ T^m(y_1)=z}} S_m(\overline{A})(y_1).
\end{aligned}$$

As $\epsilon \to 0^+$ in the above inequalities, we get $\phi_A(x_0, z) \leqslant \lim_{\epsilon\to 0^+} \bigoplus_{m\geqslant 2} \bigoplus_{\substack{d(y_0,x_0)\leqslant\epsilon\\ T^n(y_0)=z}} S_m(\overline{A})(y_0)$. It directly follows from Definition 3.11 that $\phi_A(x_0, z) \geqslant \lim_{\epsilon\to 0^+} \bigoplus_{m\geqslant 2} \bigoplus_{\substack{d(y_0,x_0)\leqslant\epsilon\\ T^n(y_0)=z}} S_m(\overline{A})(y_0)$ and consequently (A.10) is verified.

Next we follow a similar process to prove $\phi_A(x_0, z) \in \mathbb{R}$ for all $x_0 \in \Omega_A$ and $z \in X$. More precisely, we have the following claim. Fix $x_0 \in \Omega_A$ and $z \in X$.

Claim 1. The inequality

$$\phi_A(x_0, z) \geqslant S_n(\overline{A})(y_0) - \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} |A|_{d^\alpha} \xi^\alpha \tag{A.11}$$

holds for every $y_0 \in B(x_0, \xi)$ satisfying $T^n(y_0) = z$ for some $n \in \mathbb{N}$.

Now fix a point $y_0 \in B(x_0, \xi)$ with $T^n(y_0) = z$ for some $n \in \mathbb{N}$.

Fix $\epsilon > 0$ and $l \in \mathbb{N}$ satisfying $\lambda^{-l}\xi < \frac{\epsilon}{2}$. By Lemma A.1, there exists a trajectory from x_1 to $T^{n_0}(x_1)$ satisfying $n_0 > l$, $d(x_1, x_0) \leqslant \frac{\epsilon}{2}$, $T^{n_0}(x_1) = x_0$, and $|S_{n_0}(\overline{A})(x_1)| \leqslant \epsilon$.

By Lemma 3.1, we have the trajectory from $T_{x_1}^{-n_0}(y_0)$ to $T^n(y_0)$ with

$$\begin{aligned} d\big(T_{x_1}^{-n_0}(y_0), x_0\big) &\leqslant d\big(T_{x_1}^{-n_0}(y_0), x_1\big) + d(x_1, x_0) \\ &\leqslant \lambda^{-n_0} d(y_0, x_0) + d(x_1, x_0) \leqslant \lambda^{-l}\xi + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Since $y_0 \in B(x_0, \xi)$, it follows from (A.2) that

$$\big|S_{n_0}(\overline{A})(x_1) - S_{n_0}(\overline{A})\big(T_{x_1}^{-n_0}(y_0)\big)\big| \leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} |A|_{d^\alpha} \xi^\alpha.$$

We conclude that

$$\begin{aligned} \bigoplus_{m\in\mathbb{N}} \bigoplus_{\substack{d(y_1,x_0)\leqslant\epsilon\\ T^m(y_1)=z}} S_m(\overline{A})(y_1) &\geqslant S_{n_0+n}(\overline{A})\big(T_{x_1}^{-n_0}(y_0)\big) \\ &\geqslant S_{n_0}(\overline{A})(x_1) - \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} |A|_{d^\alpha} \xi^\alpha + S_n(\overline{A})(y_0) \\ &\geqslant -\epsilon + S_n(\overline{A})(y_0) - \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} |A|_{d^\alpha} \xi^\alpha. \end{aligned}$$

As $\epsilon \to 0^+$ in the above identities, A.11 follows and Claim 1 is verified.

Recall that $\phi_A(\cdot,\cdot)\colon X \times X \to \mathbb{R} \cup \{-\infty\}$ and note that the existence of $y_0 \in B(x_0, \xi)$ satisfying $T^n(y_0) = z$ follows from the transitivity of T. We conclude that $\phi_A(x_0, z) \in \mathbb{R}$ for all $x_0 \in \Omega_A$ and $z \in X$.

Finally, we verify the continuity of $\phi_A(x_0, \cdot)$ for all $x_0 \in \Omega_A$.

Claim 2. For all $z_1, z_2 \in X$ satisfying $d(z_1, z_2) < \xi$ and $x_0 \in \Omega_A$,

$$|\phi_A(x_0, z_1) - \phi_A(x_0, z_2)| \leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}} |A|_{d^\alpha} d(z_1, z_2)^\alpha. \tag{A.12}$$

Now fix $x_0 \in \Omega_A$ and $z_1, z_2 \in X$ with $d(z_1, z_2) < \xi$.

Fix $\epsilon \in \big(0, \frac{\xi}{2}\big)$ and $l \in \mathbb{N}$ satisfying $\lambda^{-l}\xi < \frac{\epsilon}{2}$. By Lemma A.1, there exists a trajectory from x_1 to $T^{n_1}(x_1)$ satisfying $n_1 > l$, $d(x_1, x_0) \leqslant \frac{\epsilon}{2}$, $T^{n_1}(x_1) = x_0$ and $|S_{n_1}(\overline{A})(x_1)| \leqslant \epsilon$.

For each trajectory from y_1 to $T^n(y_1)$ satisfying $n \in \mathbb{N}$, $T^n(y_1) = z_1$, and $d(y_1, x_0) \leqslant \frac{\epsilon}{2} < \xi$, Lemma 3.1 implies that we have the trajectory from $T_{x_1}^{-n_1}(y_1)$ to $T^n(y_1)$ with

$$\begin{aligned} d\big(T_{x_1}^{-n_1}(y_1), x_0\big) &\leqslant d\big(T_{x_1}^{-n_1}(y_1), x_1\big) + d(x_1, x_0) \\ &\leqslant \lambda^{-n_1} d(y_1, x_0) + d(x_1, x_0) \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{A.13}$$

Since $d(y_1, x_0) \leqslant \frac{\epsilon}{2} < \xi$, it follows from (A.2) that

$$\Big|S_{n_1}(\overline{A})\big(T_{x_1}^{-n_1}(y_1)\big)\Big| \leqslant |S_{n_1}(\overline{A})(x_1)| + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}(\epsilon/2)^\alpha \leqslant \epsilon + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}(\epsilon/2)^\alpha. \tag{A.14}$$

Denote $y_2 := T_{x_1}^{-n_1}(y_1)$ and $n_2 := n + n_1$. Note that $T^{n_2}(y_2) = T^n(y_1) = z_1$. Since $d(z_1, z_2) < \xi$, Lemma 3.1 implies that we have the trajectory from $T_{y_2}^{-n_2}(z_2)$ to z_2 with

$$\begin{aligned} d\big(T_{y_2}^{-n_2}(z_2), x_0\big) &\leqslant d\big(T_{y_2}^{-n_2}(z_2), y_2\big) + d(y_2, x_0) \\ &\leqslant \lambda^{-n_2} d(z_2, z_1) + \epsilon < \lambda^{-l}\xi + \epsilon < 2\epsilon, \end{aligned} \tag{A.15}$$

where the second inequality follows from Lemma 3.1 and (A.13), and the third inequality follows from $n_1 > l$. Moreover, it directly follows from (A.2) that

$$\Big|S_{n_2}(\overline{A})\big(T_{y_2}^{-n_2}(z_2)\big) - S_{n_2}(\overline{A})(y_2)\Big| \leqslant \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha} d(z_1, z_2)^\alpha. \tag{A.16}$$

Denote $F(\epsilon) := \epsilon + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}(\epsilon/2)^\alpha$. We conclude that for each trajectory from y_1 to $T^n(y_1)$ satisfying $n \in \mathbb{N}$, $T^n(y_1) = z_1$, and $d(y_1, x_0) \leqslant \frac{\epsilon}{2}$,

$$\begin{aligned} S_n(\overline{A})(y_1) &= S_{n_2}(\overline{A})(y_2) - S_{n_1}(\overline{A})(y_2) \\ &\leqslant S_{n_2}(\overline{A})\big(T_{y_2}^{-n_2}(z_2)\big) + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha} d(z_1, z_2)^\alpha + \epsilon + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha}(\epsilon/2)^\alpha \\ &\leqslant \bigoplus_{m\in\mathbb{N}} \bigoplus_{\substack{d(z,x_0)\leqslant 2\epsilon \\ T^m(z)=z_2}} S_m(\overline{A})(z) + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha} d(z_1, z_2)^\alpha + F(\epsilon), \end{aligned}$$

where the inequality in the second line follows from (A.16) and (A.14), and the inequality in the third line follows from (A.15). Thus,

$$\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(y_1,x_0)\leqslant \epsilon/2 \\ T^n(y_1)=z_1}} S_n(\overline{A})(y_1) \leqslant \bigoplus_{m\in\mathbb{N}} \bigoplus_{\substack{d(z,x_0)\leqslant 2\epsilon \\ T^m(z)=z_2}} S_m(\overline{A})(z) + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha} d(z_1, z_2)^\alpha + F(\epsilon).$$

As $\epsilon \to 0^+$ in the above identity, we get

$$\phi_A(x_0, z_1) \leqslant \phi_A(x_0, z_2) + \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}|A|_{d^\alpha} d(z_1, z_2)^\alpha.$$

Thus, Claim 2 follows and consequently $\phi_A(x_0, \cdot) \in C(X, \mathbb{R})$.

We conclude that $\phi_A(x_0, \cdot)$ is a tropical eigenfunction and (v) is verified. $\square$

Proof of Proposition 3.13. It follows from Proposition 3.12 (i) that $u(\cdot) \geqslant \bigoplus_{x\in\Omega_A} (u(x) \otimes \phi_A(x, \cdot))$. It suffices to find an Aubry point x_y for each y in X such that $u(x_y) \otimes \phi_A(x_y, y) \geqslant u(y)$. Fix $y \in X$.

Since u is a tropical eigenfunction of $\mathcal{L}_A$, $u(x) = \bigoplus_{z\in T^{-1}(x)} (u(z) + \overline{A}(z))$ for all $x \in X$. By Lemma 3.1, $T^{-1}(x)$ is finite for all $x \in X$. Consider $u(y) = \bigoplus_{z\in T^{-1}(y)} (u(z) + \overline{A}(z))$. Then there exists $y_1 \in T^{-1}(y)$ such that $u(y) = u(y_1) + \overline{A}(y_1)$. Then consider $u(y_1) = \bigoplus_{z\in T^{-1}(y_1)} (u(z) + \overline{A}(z))$ and there exists $y_2 \in T^{-1}(y_1)$ such that $u(y_1) = u(y_2) + \overline{A}(y_2)$. Repeating this process reductively, we get a sequence $\{y_k\}_{k\in\mathbb{N}} \subseteq X$ satisfying $T(y_{k+1}) = y_k$ and $u(y_k) = u(y_{k+1}) + \overline{A}(y_{k+1})$ for all $k \in \mathbb{N}$.

Claim. Every accumulation point of $\{y_k\}_{k\in\mathbb{N}}$ as $k \to +\infty$ is an Aubry point and these Aubry points satisfy $u(\cdot) + \phi_A(\cdot, y) \geqslant u(y)$.

Suppose that the subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ coverges to x_y with $n_k \to +\infty$ as $k \to +\infty$. Without loss of generality, we assume that $n_{k+1} > n_k + k$ for all $k \in \mathbb{N}$.

Fix $\epsilon > 0$, the continuity and consequently the uniformly-continuity of u (due to the compactness of X) implies that there exists $\eta \in (0, 2\epsilon)$ such that $|u(x) - u(z)| \leqslant \epsilon$ for all $x, z \in X$ with $d(x, z) \leqslant \eta$. Thus, when k is big enough, $d(y_{n_k}, x_y) \leqslant \frac{\eta}{2} < \epsilon$ and $d(y_{n_{k+1}}, x_y) \leqslant \frac{\eta}{2} < \epsilon$. Let $n_{k+1} - n_k =: m_k \in \mathbb{N}$ and consequently for all k big enough,

$$|S_{m_k}(\overline{A})(y_{n_k})| = |u(y_{n_k}) - u(y_{n_{k+1}})| \leqslant \epsilon$$

since $d(y_{n_k}, y_{n_{k+1}}) \leqslant d(y_{n_k}, x_y) + d(x_y, y_{n_{k+1}}) \leqslant \eta$. According to Definition 3.10, we conclude that x_y is an Aubry point. Moreover, $d(y_{n_k}, x_y) \leqslant \frac{\eta}{2} < \eta$ implie that for all k big enough,

$$u(y) = u(y_{n_k}) \otimes S_{n_k}(\overline{A})(y_{n_k}) \leqslant (u(x_y) + \epsilon) \otimes \Bigg(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{d(z,x_y)\leqslant\epsilon \\ d(T^n(z),y)\leqslant\epsilon}} S_n(\overline{A})(z) \Bigg).$$

As $\epsilon \to 0^+$ in the above inequality, we get $u(y) \leqslant u(x_y) \otimes \phi_A(x_y, y)$. Now the claim is verified and it follows that $u(\cdot) = \bigoplus_{x\in\Omega_A} (u(x) \otimes \phi_A(x, \cdot))$. □

References

[ACG94] Akian, M., Cohen, G., Gaubert, S., Quadrat, J.P., and Viot, M., Max-Plus Algebra and Applications to System Theory and Optimal Control. In *Proc. Internat. Congr. Math. (Zürich 1994)*, vol.2, Birkhäuser Verlag, 1995, 1511–1522.

[Ak99] Akian, M., Densities of idempotent measures and large deviations. *Trans. Amer. Math. Soc.* 351 (1999), 4515–4543.

[BLL13] Baraviera, A.T., Leplaideur, R., and Lopes, A.O., Ergodic Optimization, Zero Temperature Limits and the Max-Plus Algebra, IMPA, Rio de Janeiro, 2013, the 29th Coloquio Brasilerio de Matematica.

[BLT11] Baraviera, A.T., Lopes, A.O., and Thieullen, P., A large deviation principle for the equilibrium states of Hölder potentials: the zero temperature case. *Stoch. Dyn.* 2011.

[Boc18] Bochi, J., Ergodic optimization of Birkhoff averages and Lyapunov exponents. In *Proc. Internat. Congr. Math. (Rio de Janeiro 2018)*, Volume III, World Sci. Publ., Singapore, 2018, pp. 1843–1866.

[Bou00] Bousch, T., Le poisson n'a pas d'arêtes. *Ann. Inst. Henri Poincaré Probab. Stat.* 36 (2000), 489–508.

[Bow75] Bowen, R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Math.*, Springer, Berlin, 1975.

[Bro65] Brolin, H., Invariant sets under iteration of rational functions. *Ark. Mat.* 6 (1965), 103–144.

[CGQ04] Cohen, G., Gaubert, S., and Quadrat, J.P., Duality and separation theorem in idempotent semimodules. *Linear Algebra Appl.* 379 (2004), 395–422.

[CLT01] Contreras, G., Lopes, A.O., and Thieullen, P., Lyapunov minimizing measures for expanding maps of the circle. *Ergodic Theory Dynam. Systems* 21 (2001), 1379–1409.

[Do68] Dobrušin, R.L., Description of a random field by means of conditional probabilities and conditions for its regularity. *Theor. Verojatnost. i Primenen* 13 (1968), 201–229.

[DZ09] Dembo, A. and Zeitouni, O., *Large Deviations Techniques and Applications*, volume 38 of Stochastic Modelling and Applied Probability, Springer-Verlag Berlin Heidelberg, corrected printing 2009.

[Ga17] Garibaldi, E., *Ergodic Optimization in the Expanding Case-Concepts, Tools and Applications*, SpringerBriefs in Mathematics, Springer, 2017.

[Ha17] Hahn, H., Über halbstetige und unstetige Funktionen. *Sitzungsberichte Akad. Wiss. Wien Abt.* IIa 126 (1917), 91–110.

[HO96a] Hunt, B.R. and Ott, E., Optimal periodic orbits of chaotic systems. *Phys. Rev. Lett.* 76 (1996), 2254–2257.

[HO96b] Hunt, B.R. and Ott, E., Optimal periodic orbits of chaotic systems occur at low period. *Phys. Rev. E (3)* 54 (1996), 328–337.

[Je96] Jenkinson, O., *Conjugacy rigidity, cohomological triviality, and barycentres of invariant measures*, Ph.D. thesis, Warwick University, 1996.

[Je06] Jenkinson, O., Ergodic optimization. *Discrete Contin. Dyn. Syst.* 15 (2006), 197–224.

[Je19] Jenkinson, O., Ergodic optimization in dynamical systems. *Ergodic Theory Dynam. Systems* 39 (2019), 2593–2618.

[Kol92] Kolokoltsov, V.N., On linear, additive and homogeneous operators in idempotent analysis. In [MS92], 87–101.

[LMS01] Litvinov, G.L., Maslov, V.P., and Shpiz, G.B., Idempotent Functional Analysis: An Algebraic Approach. *Math. Notes* vol.69, (2001), 696–729.

[Lyu82] Lyubich, M.Yu., The maximum-entropy measure of a rational endomorphism of the Riemann sphere. *Funct. Anal. Appl.* 16 (1982), 309–311.

[LZ23] Zhiqiang Li and Yiwei Zhang, Ground states and periodic orbits for expanding Thurston maps. Preprint, (arXiv:2303.00514), 2023.

[Me18] Mengue, J.K., Large deviations for equilibrium measures and selection of subaction. *Bull. Braz. Math. Soc.* 49 (2018), 17–42.

[Mik06] Mikhalkin, G., Tropical geometry and its applications. Preprint, arXiv:0601041v2, 2006.

[MS92] Maslov, V.P. and Samborskii, S.N., *Idempotent analysis*, volume 13 of Advances in Soviet Mathematics, American Mathematical Society, 1992.

[OGY90] Ott, E., Grebogi, C., and Yorke, J.A., Controlling chaos. *Phys. Rev. Lett.* 64 (1990), 1196–1199.

[PU10] Przytycki, F. and Urbański, M., *Conformal fractals: ergodic theory methods*, Cambridge Univ. Press, Cambridge, 2010.

[Ru78] Ruelle, D., *Thermodynamic formalism*, Addison-Wesley, Reading, MA, 1978.

[SGYO93] Shinbrot, T., Grebogi, C., Yorke, J.A., and Ott, E., Using small perturbations to control chaos. *Nature* 363 (1993), 411–417.

[Sin72] Sinai, Ya.G., Gibbs measures in ergodic theory. *Russian Math. Surveys* 27 (1972), 21–69.

Zhiqiang Li, School of Mathematical Sciences & Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China.
Email address: zli@math.pku.edu.cn

Yiqing Sun, School of Mathematical Sciences, Peking University, Beijing 100871, China.
Email address: 2100010781@stu.pku.edu.cn