

## 1 5005

1.  $\sup E|S_n| < +\infty$ ,  $S_n \xrightarrow{a.s.} S$ . Prove or give a counter example on  $E|S_n - S| \rightarrow 0$
2.  $\{X_n\}$  i.i.d.,  $p_n = P(X = a_n) = 1 - P(X = -a_n)$ , characterize the sequence  $\{(a_n, p_n)\}$  such that  $\sum X_n$  converges almost surely.
3. Simple random walk recurrent, prove that the probability to reach any given point  $b$  is 1.
4. Lindeberg randomized method.  $E[X_i] = E[Y_i] = 0$ ,  $\sum E[X_i^2] = \sum E[Y_i^2] = n$ ,  $E[X_i^3], E[Y_i^3]$  bounded by  $\gamma$ ,  $W_n = \sqrt{n}\bar{X}_n, V_n = \sqrt{n}\bar{Y}_n$ , prove that  $E[g(W_n) - g(V_n)] \leq \frac{C\gamma\|g'''\|_\infty}{\text{something}}$ . [Hint: write LHS as  $(1/n!) \sum_{\sigma(n)} (g(W_{\sigma(n)}) - g(V_{\sigma(n)}))$ ], where  $\sigma(n)$  is the permutation of  $n$  variables.](Don't remember the exact right hand side. NOT sure about the hint.)

## 2 5010

1.  $Poisson(\lambda)$ ,  $\{X_n\}$  i.i.d.,
  - Give a complete and sufficient statistic.
  - Prove that  $E[B^{n\bar{X}_n}] = e^{n(B-1)\lambda}$ .
  - Give a UMVUE on  $e^{-\lambda}$ .
  - Is  $e^{-\bar{X}_n}$  a consistent estimator of  $e^{-\lambda}$ ?
  - Asymptotic normality of  $e^{-\bar{X}_n}$ .
2.  $\theta \in [-C, C]$ ,  $X \sim N(\theta, 1)$ . ( $C = \frac{1}{2}$ )
  - $C \tanh(CX)$  minimax (NOT sure).
  - MLE?
  - MLE's risk function?
  - Sketch two risk functions.
3.  $f(x) = a \exp(-a(x-b))$ ,  $\{X_n\}$  i.i.d. Both parameters unknown.  $a > 0$ .
  - UMPT of  $H_0 : b_0$  versus  $H_1 : b \neq b_0$
  - UMPT of  $H_0 : a_0, b_0$  versus  $H_1 : a > a_0, b < b_0$ . UMPT for  $H_0 : a_0, b_0$  versus  $H_1 : a \neq a_0, b \neq b_0$  exists?
4. (Bonus) Maximize  $\prod_{i=1}^n p_i$  subject to  $p_i > 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i \mathbf{u}_i = 0$ . Prove that solution is:  $\hat{p}_i = \frac{1}{n(1+\lambda^T \mathbf{u}_i)}$ , where  $\lambda$  satisfies  $\sum_{i=1}^n \frac{\mathbf{u}_i}{1+\lambda^T \mathbf{u}_i} = 0$ . Here,  $\mathbf{u}_i$  and  $\lambda \in R^s$  are vectors. (Not covered in the course, see Shao Jun exercise & solution book for *Mathematical Statistics* Exercise 5.7)

### 3 5020

1. Consider model

$$p(y_{ik}|\boldsymbol{\omega}_i) = \exp[y_{ik}\vartheta - b(\vartheta_{ik})]/\psi_{\epsilon k} + c_k(y_{ik}, \phi_{\epsilon k}),$$

$$E(y_{ik}|\boldsymbol{\omega}_i) = \dot{b}(\vartheta_{ik}), \text{Var}(y_{ik}|\boldsymbol{\omega}_i) = \psi_{\epsilon k} \ddot{b}(\vartheta_{ik}),$$

$y_{ik}$  is binary.

- Specify  $\vartheta_{ik}, b(\vartheta_{ik}), \psi_{\epsilon k}, C_k(y_{ik}, \psi_{\epsilon k}), \dot{b}(\vartheta_{ik})$ .
- Prior and derive posterior distribution.
- Why should we use  $p(\psi_{\epsilon k})$  and  $p(\boldsymbol{\Lambda}_k|\psi_{\epsilon k})$  instead of  $p(\psi_{\epsilon k})$  and  $p(\boldsymbol{\Lambda}_k)$ ? Is there any conjugate prior for  $p(\psi_{\epsilon k})$  and  $p(\boldsymbol{\Lambda}_k|\psi_{\epsilon k})$ ?
- When use MH algorithm, the proposal distribution for  $p(\psi_{\epsilon k}|\cdot)$  and  $p(\boldsymbol{\Lambda}_k|\cdot)$  are  $N(\cdot, \sigma_{\psi}^2 \Omega_{\psi k})$  and  $N(\cdot, \sigma_{\lambda}^2 \Omega_{\lambda k})$ . Show that

$$\Omega_{\psi k}^{-1} = 1 - n/2 - \alpha_{0\epsilon k} - 2 \sum_{i=1}^n [y_{ik}\vartheta_{ik} - b(\vartheta_{ik})] - \ddot{c}_k(y_{ik}, \psi_{\epsilon k}) + 2\beta_{0\epsilon k},$$

$$\Omega_{\lambda k}^{-1} = \sum_{i=1}^n \ddot{b}(\vartheta_{ik}) \boldsymbol{\omega}_i \boldsymbol{\omega}_i^T + \psi_{\epsilon k}^{-1} + \psi_{\epsilon k}^{-1} \mathbf{H}_{0yk}^{-1}.$$

- How could we derive MLE( $\boldsymbol{\theta}$ ) through CSA?
- Explain why classical CSA approach cannot be applied to the analyses of advanced SEMs, such as nonlinear.

2. A path diagram for nonlinear SEM, all are continuous variable.

- Write down the explicit form.
- Assumption and identifiability.
- How could we extend it to mixture, multi-level and multisample model? What is the difference in in identifiability? How could we solve these?
- Derive posterior distribution for each model.

### 4 5030

1. Design matrix

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$

,  $Y = X\boldsymbol{\theta} + \epsilon$ ,  $\epsilon$  i.i.d. with mean 0 and variance  $\sigma^2$

- OLS.
- Unbiased estimator for  $\sigma^2$
- Test  $H_0 : \theta_1 = 2\theta_2$

2. Consider a regression model,  $Y = X\beta + \epsilon$ , where  $X_{n \times p}$  is full column rank.  $Y = (Y_1, \dots, Y_n)'$ . Further, assume that  $\text{Var}(\epsilon) = \sigma^2 I$ . Let  $e_i$  be the  $i$ -th residual,  $h_{ii}$  be the  $i$ -th diagonal element of the hat matrix. Let  $\hat{\beta}$  and  $\hat{\beta}_{(i)}$  be the LSE of  $\beta$  with and without the  $i$ -th case in the data respectively.
- Prove  $X'X = X'_{(i)}X_{(i)} + x_i x_i'$ , where denote the regression matrix with the  $i$ -th row  $x_i$  deleted (should be a column vector).
  - Show that  $(X'_{(i)}X_{(i)})^{-1} = (X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1-h_{ii}}$
  - Prove that  $\hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1}x_i e_i}{1-h_{ii}}$ .