Department of Statistics, The Chinese University of Hong Kong
STAT5010 Advanced Statistical Inference | Term 1, 2020–21

Take-home Examination

Instruction to the candidates: Please attempt all of the questions. Each problem carries an equal weight of 8 points. Your final score will be capped by 40, which is also the defined full mark of this exam. Good luck!

1. Let $X_1, \dots X_n$ be a random sample from a $N(\theta, \sigma^2)$ population with σ^2 known. Consider estimating θ using the squared error loss. Let $\pi(\theta)$ be a $N(\mu, \tau^2)$ prior distribution on θ and let δ^π be the Bayes estimator of θ. Verify the following formulas for the risk function and Bayes risk.

   (a) For any constants a and b, the estimator $\delta(\boldsymbol{X}) = a\bar{\boldsymbol{X}} + b$ has risk function

   $$R(\theta, \delta) = a^2 \frac{\sigma^2}{n} + \{b - (1-a)\theta\}^2.$$

   (b) Let $\eta = \sigma^2/(n\tau^2 + \sigma^2)$. The risk function for the Bayes estimator is

   $$R(\theta, \delta^\pi) = (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2(\theta - \mu)^2.$$

   (c) The Bayes risk for the Bayes estimator is

   $$B(\pi, \delta^\pi) = \tau^2 \eta.$$

2. Let X be an observation from the pdf

   $$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x \in \{-1, 0, 1\}; \theta \in [0, 1].$$

   (a) Find the MLE of θ.

   (b) Define an estimator $T(X)$ by

   $$T(X) = \begin{cases} 2 & \text{, if } x = 1 \\ 0 & \text{, otherwise} \end{cases}.$$

   Show that $T(X)$ is an unbiased estimator of θ, meaning that $E(T(X)) = \theta$.

   (c) Find a better estimator than $T(X)$ and prove that it is better.

3. Consider a Bayesian model in which the prior distribution for Θ is standard exponential and the density for X given Θ is

   $$f(x \mid \theta) = e^{\theta - x} I(x > \theta).$$

   (a) Find the marginal density for X and $E(X)$ in the Bayesian model.

   (b) Find the Bayes estimator for Θ under squared error loss. (Assume $X > 0$.)

4. Let F be a cumulative distribution function that is continuous and strictly increasing on $[0, \infty)$ with $F(0) = 0$, and let q_α denote the upper αth quantile for F, i.e. $F(q_\alpha) = 1 - \alpha$. Suppose we have a single observation X with

$$P_\theta(X \le x) = F(x/\theta), \quad x \in \mathbb{R}, \theta > 0.$$

   (a) Consider testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$. Find the significance level for the test $\phi(X) = I(X > c)$. What choice for c will give a specified level α?

   (b) Let ϕ_α denote the test with level α in part (a). Show that the tests $\phi_\alpha, \alpha \in (0, 1)$, are nested in the sense described in Problem 2 of Assignment 6. Give a formula to compute the p-value $P(X)$.

5. Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, 1)$ and let $U_1, \ldots, U_n$ be i.i.d. from a uniform distribution on $(0, 1)$, with all $2n$ variables independent. Define $Y_i = X_i U_i$, $i = 1, \ldots, n$. If the X_i and U_i are both observed, then $\bar{X}$ would be a natural estimator for θ. If only the products $Y_1, \ldots, Y_n$ are observed, then $2\bar{Y}$ may be a more responsible estimator. Determine the asymptotic relative efficiency (ARE) of $2\bar{Y}$ with respect to $\bar{X}$, where ARE of $\hat{\theta}_n$ with respect to $\tilde{\theta}_n$ is defined as the ratio $\sigma^2_{\tilde{\theta}}/\sigma^2_{\hat{\theta}}$ if

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma^2_{\hat{\theta}}) \quad \text{and} \quad \sqrt{n}(\tilde{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma^2_{\tilde{\theta}}),$$

   respectively.

6. (a) Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, 1)$. For the hypotheses $H_0 : \mu = 0$ versus $H_1 : \mu \ne 0$, show that the test that rejects H_0 when $\sqrt{n}|\bar{X}_n| > z_{\alpha/2}$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, is not uniformly most powerful (UMP) at level α. Show also that the test $\sqrt{n}\bar{X}_n > z_\alpha$ is UMP at level α for testing $H_0 : \mu \le 0$ versus $\mu > 0$.

   (b) Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Uniform$(\theta, \theta + 1)$. Construct a UMP test procedure for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ at level α.

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