5005 Advanced Probability Theory

Q1

(a)

Suppose F_n, F are all distribution functions, $F_n(x)$ converge weakly to F, and F is a continuous function. Show that

$$sup|F_n(x) - F(x)| \to 0$$

(b)

Show that X_n converge to X in probability if and only if

$$E[\frac{|X_n - X|}{1 + |X_n - X|}] \to 0$$

(c)

Let $\mathcal{A}_1$,..., $\mathcal{A}_n$ are collections of sets that are independent to each other. Suppose each $\mathcal{A}_k$ is a π-system, prove that $\sigma(\mathcal{A}_1)$,...,$\sigma(\mathcal{A}_n)$ are independent to each other.

(d)

Let X_1,...X_n be any random variables,$S_n = \sum_{k=1}^{n} X_k$, Prove that $\frac{S_n}{n}$ converge to 0 in probability implies that $\frac{X_n}{n}$ converge to 0 in probability.

Q2

Let $X_1, ..., X_n$ be independent variables such that $0 \leq X_i \leq 1$ and $S_n = \sum_{k=1}^{n} X_k$. Show that

$$\frac{S_n}{ES_n} \to 1$$

almost surely.

Q3

Let $X_1, ..., X_n$ be i.i.d. random variables such that $Ee^{tX_1} = 1$ for some t. Let $S_n = \sum_{k=1}^{n} X_k$.

(a)

Show that $T_n = e^{tS_n}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, ..., X_n)$

(b)

Show that $P(X_k > x$ for some $k) \leq e^{-tx}$

5030 Linear Model

Q1

Consider the ridge regression model with loss function

$$L(\beta) = (Y - X\beta)^T(Y - X\beta) + \lambda\beta^T\beta$$

(a)

Derive the estimation of ridge regression $\beta(k)$ (Show the detailed step properly).

(b)

Use the singular value decomposition of X to calculate the ridge estimator $\beta(k)$.

(c)

Let $\hat{\beta}$ be OLS estimator. Show that there exists a λ such that

$$MSE(\beta(k)) \leq MSE(\hat{\beta})$$

Q2

$\beta_1, \beta_2, \beta_3$ be interior angles of a triangle such that $\beta_1 + \beta_2 + \beta_3 = 180.$

Let Y_1, Y_2, Y_3 be observations of β_1, β_2 and β_3 such that Y_i follows $N(\beta_i, \sigma^2)$(σ^2 unknown).

Derive a test statistic to test $\beta_0 = \beta_1 = \beta_3 = 60$ and specify your reject region under level α.