

STAT 5010 Tutorial 1*

Sept. 2022

Definitions

- For $X, \{X_n\}$ random variables, $F, \{F_n\}$ the corresponding distribution functions,
 - $X_n \xrightarrow{a.s.} X: \mathbf{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$
 - $X_n \xrightarrow{L^r} X: \lim_{n \rightarrow \infty} \mathbf{E}|X_n - X|^r = 0$
 - $X_n \xrightarrow{p} X: \forall \delta > 0, \lim_{n \rightarrow \infty} \mathbf{P}(|X_n - X| > \delta) = 0$
 - $F_n \xrightarrow{w} F: \forall x \text{ continuity point of } F, \lim_{n \rightarrow \infty} F_n(x) = F(x)$
 - $X_n \xrightarrow{d} X: F_n \xrightarrow{w} F$
- Let X_1, X_2, \dots be random vectors and Y_1, Y_2, \dots be random variables defined on a common probability space.
 - $X_n = O_p(Y_n)$ if and only if, $\forall \epsilon > 0, \exists C_\epsilon > 0$ such that $\sup_n \mathbf{P}(\|X_n\| \geq C_\epsilon |Y_n|) < \epsilon$.
 - $X_n = o_p(Y_n)$ if and only if $X_n/Y_n \xrightarrow{p} 0$

Propositions and Theorems

- Chebyshev's Inequality: $\forall t > 0, \mathbf{P}(|X - EX| \geq t) \leq \frac{Var(X)}{t^2}$

Proof. $t^2 \mathbf{P}(|X - EX| \geq t) = t^2 \mathbf{P}(|X - EX|^2 \geq t^2) = t^2 \mathbf{E}[I_{|X-EX|^2 \geq t^2}] \leq E[|X - EX|^2] = Var(X)$ ■

- $X_n \xrightarrow{a.s.} X \Leftrightarrow \forall \epsilon > 0, \lim_{k \rightarrow \infty} \mathbf{P}(\bigcup_{n=k}^{\infty} \{|X_n - X| > \epsilon\}) = 0$

Proof. $\mathbf{P}(\lim_{n \rightarrow \infty} X_n = X) = \mathbf{P}(\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = \lim_{k \rightarrow \infty} \mathbf{P}(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = 1$.

Note that $A_k \triangleq \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}$ are nonincreasing in k . The equations above hold if and only if $\mathbf{P}(A_k) = 1$ for all k , which is equivalent to the statement in the right hand side. ■

- $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{p} X$

Proof. Similar to the proof of Chebyshev's Inequality. ■

- $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$

Proof. The result follows directly from the proposition 2 above. ■

- $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

Proof. Let x be a continuity point of F , $\epsilon > 0$ given,

$$\begin{aligned} F(x - \epsilon) &= \mathbf{P}(X \in (-\infty, x - \epsilon)) \\ &\leq \mathbf{P}(X_n \in (-\infty, x - \epsilon), X_n \notin (-\infty, x]) + \mathbf{P}(X_n \in (-\infty, x]) \\ &\leq F_n(x) + \mathbf{P}(|X_n - X| > \epsilon) \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain that $F(x - \epsilon) \leq \liminf_n F_n(x)$.

Similarly, we have $F(x + \epsilon) \geq \limsup_n F_n(x)$. Since ϵ is arbitrary and F is continuous at x ,

$$F(x) = \lim_{n \rightarrow \infty} F_n(x)$$

■

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6. The corresponding converses of propositions 3-5 are all incorrect.

Proof. Construction of counter examples. ■

7. $X_n \xrightarrow{p} c \Leftrightarrow X_n \xrightarrow{d} c$, here c is a constant

8. Slutsky's Theorem: $X_n \xrightarrow{d} c$, $Y_n \xrightarrow{d} Y$, then $X_n + Y_n \xrightarrow{d} c + Y$.

Question 1

1. If random variables $X_n = O_p(1)$, $Y_n = o_p(1)$, prove that $X_n Y_n = o_p(1)$.

Solution:

Proof. $\forall \epsilon, \delta > 0$, by definition, one can find M s.t. $\sup_n \mathbf{P}(|X_n| > M) < \frac{\epsilon}{2}$ and $N_0 \in \mathbf{N}^+$ s.t. $\mathbf{P}(|Y_n| > \frac{\delta}{M}) < \frac{\epsilon}{2}$ holds for all $n > N_0$. Then, for all $n > N_0$,

$$\begin{aligned} \mathbf{P}(|X_n Y_n| > \delta) &= \mathbf{P}(|X_n| > M, |X_n Y_n| > \delta) + \mathbf{P}(|X_n| \leq M, |X_n Y_n| > \delta) \\ &\leq \mathbf{P}(|X_n| > M) + \mathbf{P}(|Y_n| > \frac{\delta}{M}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore, $X_n Y_n = o_p(1)$. ■

Question 2

1. $X_1, X_2, \dots, X_n, \dots$ are a series of i.i.d. random variables following $Uniform[0, 1]$ and defined on the same probability space. Give the asymptotic distribution of $n(1 - X_{(n)})$.

Solution: For $\forall t > 0$ fixed, $n > t$ eventually,

$$\begin{aligned} \mathbf{P}(n(1 - X_{(n)}) \leq t) &= \mathbf{P}(X_{(n)} \geq 1 - \frac{t}{n}) = 1 - \mathbf{P}(X_{(n)} \leq 1 - \frac{t}{n}) \\ &= 1 - (1 - \frac{t}{n})^n \xrightarrow{n \rightarrow \infty} 1 - e^{-t}. \end{aligned}$$

Therefore, $n(1 - X_{(n)})$ is asymptotically distributed as $exp(1)$.

2. Is it possible to have a random variable Y s.t. $n(1 - X_{(n)}) \xrightarrow{a.s.} Y$?

Solution: NO! For simplicity, use notation: $P_n \triangleq \max_{1 \leq i \leq n} X_i$, $Q_n \triangleq \max_{1 \leq i \leq 2n} X_i$. Suppose there exists such Y , we naturally have $n(1 - X_{(n)}) \xrightarrow{p} Y$ and $Y \sim exp(1)$. Hence,

$$2n(1 - P_n) \xrightarrow{p} 2Y, \quad 2n(1 - Q_n) \xrightarrow{p} Y.$$

Thus, take difference and we get

$$2n(P_n - Q_n) \xrightarrow{p} Y.$$

Note that $\mathbf{P}(P_n = Q_n) \geq \frac{1}{2}$, we have $\mathbf{P}(Y = 0) \geq \frac{1}{2}$ and we have contradiction here given that $Y \sim exp(1)$.