# STAT5030 Linear Models (Final Exam 2021-2022)

25 April 2022

1. (**15 marks)** Consider the one-way ANOVA model

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

for $i = 1, 2, 3$ and $j = 1, 2$, where $E(\epsilon_{ij}) = 0$. Then,

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{31} \\ Y_{32} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

(a) Is $\mu$ estimable? Is $\alpha_1$ estimable?

(b) Is $\mu + \alpha_1$ estimable? Is $\alpha_1 - (\alpha_2 + \alpha_3)/2$ estimable?

(c) Assume that $\epsilon_{ij}$ are independently distributed as $N(0, \sigma^2)$. Consider a hypothesis $H_{01} : \alpha_1 = \alpha_2 = \alpha_3$. Is $H_0$ testable? If yes, derive a test for testing $H_0$.

2. (**20 marks)** Suppose that $\boldsymbol{Y} = (Y_1, Y_2, Y_3)^\top$ follows $N(0, \sigma^2 I_3)$.

(a) Let

$$Q = \frac{(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + (Y_3 - Y_1)^2}{3}.$$

Write $Q$ as $\boldsymbol{Y}^\top \boldsymbol{A} \boldsymbol{Y}$ where $\boldsymbol{A}$ is symmetric. Is $\boldsymbol{A}$ idempotent? what is the distribution of $Q/\sigma^2$? Find $E(Q)$. Provide detailed arguments to your answers.

(b) What is the distribution of $L = Y_1 + Y_2 + Y_3$? Find $E(L)$ and $Var(L)$.

(c) Are $Q$ and $L$ independent? Provide detailed proofs.

(d) Find the distribution of $Q/L^2$.

3. **(20 marks)** Consider a linear model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1}$$

where $\boldsymbol{Y}$ is $n \times 1$, $\boldsymbol{X}$ is an $n \times (p+1)$ fixed design matrix, $\boldsymbol{\beta}$ is a $(p+1)$-vector of regression coefficient, $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $\mathrm{cov}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$.

(a) When $\boldsymbol{X}$ is of full rank, find the best linear unbiased estimates (BLUE) of $\boldsymbol{p}^\top \boldsymbol{\beta}$, where $\boldsymbol{p} \in \mathbb{R}^{p+1}$ is a constant vector. (Students are required to show the detailed proof of the Gauss-Markov theorem.)

(b) When $\boldsymbol{X}$ is not of full rank, find a sufficient and necessary condition for $\boldsymbol{c}^\top \boldsymbol{\beta}$ to be estimable, where $\boldsymbol{c} \in \mathbb{R}^{p+1}$.

(c) Suppose that $\boldsymbol{\Lambda}^\top \boldsymbol{\beta}$ is any $k$-dimensional estimable vector and that $\boldsymbol{c} + \boldsymbol{A}^\top \boldsymbol{Y}$ is any vector of linear unbiased estimators of the elements of $\boldsymbol{\Lambda}^\top \boldsymbol{\beta}$. Let $\hat{\boldsymbol{\beta}}$ denote any solution to the normal equations. Prove that the matrix $\mathrm{cov}(\boldsymbol{c} + \boldsymbol{A}^\top \boldsymbol{Y}) - \mathrm{cov}(\boldsymbol{\Lambda}^\top \hat{\boldsymbol{\beta}})$ is nonnegative definite.

4. **(15 marks)** Let $\beta_1, \beta_2, \beta_3$ be the interior angles of a triangle, so that $\beta_1 + \beta_2 + \beta_3 = 180$ degrees. Suppose we have available estimates $Y_1, Y_2, Y_3$ of $\beta_1, \beta_2, \beta_3$, respectively. We assume that $Y_i \sim N(\beta_i, \sigma^2)$, $i = 1, 2, 3$ ($\sigma$ is unknown) and that the $Y_i$'s are independent. Derive a test for testing the null hypothesis that the triangle is equilateral, that is $\beta_1 = \beta_2 = \beta_3 = 60$ degrees. Please state clearly the test statistics and the rejection rule when the level of significance is $\alpha$.

5. **(30 marks)** Consider a linear regression model

$$y_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \ldots x_{ip}\beta_p + \epsilon_i = \sum_{j=1}^{p} x_{ij}\beta_j + \epsilon_i, \quad i = 1, \ldots, n. \tag{2}$$

By convention, the response and covariates are centered and standardized. The ridge regression is to apply squared penalty on the least square estimate by minimizing

$$\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} x_{ij}\beta_j)^2 + \lambda \sum_{j=1}^{p} \beta_j^2,$$

where $\lambda \geq 0$ is a tuning parameter, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^\top$. The resulting estimate is denoted by $\hat{\boldsymbol{\beta}}^{\text{ridge}}$.

(a) **(10 marks)** Show that $\|\hat{\boldsymbol{\beta}}^{\text{ridge}}\|_2$ increases as the tuning parameter $\lambda \to 0$. Here $\|\cdot\|_2$ is the $L_2$ norm.

(b) **(10 marks)** When the design matrix $\boldsymbol{X}_{n\times p} = (x_1, \dots, x_p)$ is orthogonal design, that is $\boldsymbol{X}^\top \boldsymbol{X} = \boldsymbol{I}_p$, find the explicit expression of $\hat{\boldsymbol{\beta}}^{\text{ridge}}$.

(c) **(10 marks)** Consider model (2) in matrix form

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where $\boldsymbol{X}_{n\times p} = (x_1, \dots, x_p)$ is orthogonal design and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$. Instead of fitting ridge regression, one may consider the *non-negative lasso*, which is to minimize

$$\frac{1}{2}\|\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_1 \text{ subject to } \beta_j \geq 0, j = 1, \dots, p, \tag{3}$$

over $\boldsymbol{\beta}$. Here $\|\cdot\|_2$ and $\|\cdot\|_1$ are the $L_2$ norm and the $L_1$ norm, respectively. Find the solution to the non-negative lasso problem. Explain the difference between lasso and the non-negative lasso by comparing the solutions. (Students are required to provide detailed steps.)