Persistence of Geometric Structures in 2-Dimensional Incompressible Fluids

Jean-Yves Chemin

Abstract

In this paper, we study the properties of a solution of the incompressible Euler System for large time. We suppose that the initial vorticity is the characteristic function of a regular bounded domain. Then the vorticity remains, for ail time, the characteristic function of a bounded domain with the same regularity.

Keywords: Vector field(little regular), tangential regularity, flow, vortex(of patches).

Introduction

The principal results shown here are for primary motivation a classic problem from mechanics of the 2 dimensional perfect fluid: the problem of vortex patches. Remember the frame in which we work. The movement of such fluid is described by a vector field in the plane, depended on time, noting $v(t, x)$ and checking

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{cases} \tag{E}$$

where $p(t, x)$ denotes the pressure of fluid at point x and the instant t ant where $v \cdot \nabla = \sum_i v^i \partial_i$. And one may notice that the flow Ψ of the field of the vectors v, that is to say, the sure(checking) application of next differential equation:

$$\partial_t \Psi(t, x) = v(t, \Psi(t, x)) \quad \text{and} \quad \Psi(0, x) = x$$

The fundamental quantity in the study of this equation is the rotational field of speeds, also called vortex. As we are in dimension 2, this antisymmetric matrix is identifying in a real notation $w = \partial_1 v^2 - \partial_2 v^1$. The specific character of the dimension 2 is the conservation of w along the trajectory of the field of vectors v:

$$\partial_t w + v \cdot \nabla w = 0. \tag{0.1}$$

Considering the nullity of the divergence of the field of vectors v, we can, if we stick to the fields of marked vectors, recalculate v, into closed constant vector, from w, by the following well known formula, called the law of Biot-Savart:

$$v = \nabla^{\perp}\Delta^{-1} = \left(-\int \frac{x_2 - y_2}{|x-y|^2} w(y)dy, \int \frac{x_1 - y_1}{|x-y|^2} w(y)dy \right), \tag{0.2}$$

by letting $\nabla^{\perp} f = (-\partial_2 f, \partial_1 f)$.
It is clear that, if $w \in L^{\infty} \cap L^p$ with $p < 2$, above integral defines a field of marked vectors. Furthermore, it's well known (and trivial to verify) that if w satisfies (0.1) with the field of vectors v given by (0.2), so v itself is a solution to (E) with the initial data deducted from w_0 from the relation (0.2). We will always place ourselves on the frame and, in the statement of the theorems, we will not formulate the hypothesis in the vortex.

The problem of vortex patches is the following: suppose that the vortex is, at the initial moment, a open and bounded characteristic function with edge in the Holder class $C^{k+\varepsilon}$, where k is a strictly positive integer and ε a real number in the interval $(0, 1)$. In that case, Yudovitch demonstrated in [11] that there exists a unique vector solutions field of system (E) in $\mathbb{R}^2 \times \mathbb{R}^2$, where the vortex belongs to $L^{\infty}(\mathbb{R}^3)$. This solution is so quasi-Lipschitz, that is to say it's module of continuity is $|x - y| \cdot |\log |x - y||$. Such vector field processes the flow Ψ in exponentially decreasing regularity of function in time, that is to say $\Psi(t, \cdot)$ is a homeomorphism of the Holder class $C^{\exp -\alpha t}$. According to the relation (0.1),the vortex in the moment t is then a open bounded characteristic function in the topology remaining unchanged. On the other hand, it's edge is not better than the class $C^{\exp -\alpha t}$.
2 very natural questions are posed as following:

- Does the open edge stay regular for small time?
- If yes, what happens for big time?

In the case where w_0 is the characteristic function of the interior of a plane curve, closed, simple and in the class $C^{1+\varepsilon}$, the next approach was developed(see example [9]). It is very easy, in this frame, to verify, thanks to Green formula, that if the edge remains in the Holder class $C^{1+\varepsilon}$, (then) there exists a clean setting of the edge satisfying the equation

$$\partial_t \gamma(t, s) = \frac{1}{2\pi} \int_0^{2\pi} \log |\gamma(t, s) - \gamma(t, \sigma)| \partial_\sigma \gamma(t, \sigma) d\sigma. \tag{B}$$

In [9] A. Majda announced a demonstration of existence located in time of a solution of the equation (B) and, it is based on numerical experiment(see [12]), conjecture that the time of existence is usually finished and, in this case, the edge of domain ceases to be rectifiable(?). A less degradation of regularity of edge was also suggested by more recent numerical simulations(see [4] [5]).

One simplified approach is proposed by P. Constantin and Titi(see [8]). In the view of equation (B), we study the small perturbations of circle, which is very sure a solution of (B) and then we don't remember the development in series of the logarithms which are quadratic terms. For this quadratic approximation of equation (B), S. Alinhac has demonstrate in [1] a result of instability which is tend to think that there might not be global existence of regular solution to equation (B) itself.

In regard to the local existence in time, we demonstrated in [6] forgetting equation (B) and demonstrated a local control with Lipschitz norm of the solution of E thanks to the tangential regularity of vortex by relative to a vector field not cancelling in the singular support C^ε of vortex. Furthermore, in [10], P.Serfati demonstrated the existence and the holomorphy regularity , locally in time, for little perturbation of circle, in the frame of (B).

The first motivation of this work is to demonstrate the following theorem:

Theorem 1. ***A*** *Let* ε *belong to the interval* $(0,1)$ *and* γ_0 *a function in the space* $C^{1+\varepsilon}(S^1;\mathbb{R}^2)$ *setted properly a Jordan curve. Then there exist a unique solution* $\gamma(t,s)$ *of the equation* (B) *belonging to the space* $L^\infty_{loc}(\mathbb{R};C^{1+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2))$.

In [6], we have developed the study of iterated action in the irregular vector fields which allows the deduction in the above theorem in the following corollary:

Corollary. ***B*** *Let* ε *be in the interval* $(0,1)$, k *a non-zero positive integer and* γ_0 *a function in the space* $C^{k+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2)$ *setted properly as a Jordan curve. Then there exists a unique solution* $\gamma(t,s)$ *of the equation* (B) *in the space* $L^\infty_{loc}(\mathbb{R},C^{k+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2))\cap C^\infty(\mathbb{R},C^{k+\varepsilon'}(\mathbb{S}^1;\mathbb{R}^2))$ *for all* $\varepsilon'<\varepsilon$.

Our approach will be as follows:

- in the first paragraph, we will explain what concept of regularity which allows to see the previous theorem as an immediate corollary of a much more general theorem, the Theorem 1;
- in the second paragraph, we will demonstrate a estimation on the Lipschitz norm of a vector field;
- in the third paragraph, we will utilize this estimate to demonstrate a priori estimation on the regular solutions of the system (E);
- in the fourth paragraph, we will finish, by regularization of initial data and then transition to the limit, demonstrating Theorem 1;
- in the fifth and final paragraph, we will state various global results from the combination of Theorem 1 and the local theorems of regularities demonstrated in [6].

Notations and reminders

In the rest of the article, we will take the following notations and conventions:

- ε denotes a real number strictly between 0 and 1;
- if X is a vector field in plane, we denote $I(A, X)$ the lower bound of $|X(x)|$ for x running through A and we denote ∇X the matrix of general term $\partial_j X^i$;
- if f is a distribution on the plane, we denote $\nabla^\perp f$ the vector field $(-\partial_2 f, \partial_1 f)$ that is sure to be divergence-free.
- if Ω is an open on plane, $C^\varepsilon(\Omega)$[resp. $\mathrm{Lip}(\Omega)$] denotes the set of functions u given on Ω such that we have, for all x and y in Ω, $|u(x) - u(y)| \leqslant C \leqslant |x - y|^\varepsilon$(resp. $|x - y|$) and we will note by $\|\cdot\|_{\varepsilon,\Omega}$(resp. $\|\cdot\|_{Lip(\Omega)}$) the natural norm of $C^\varepsilon(\Omega)$[resp. $\mathrm{Lip}(\Omega)$],
- if $\Omega = \mathbb{R}^2$, one can characterize the space $C^\varepsilon(\Omega)$, then simply noted as C^ε, using a dyadic cutting in the space of frequency. More precisely, it's $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\chi(\xi) = 1 - \sum_{q \geqslant 0} \varphi(2^{-q}\xi) \in C_0^\infty(\mathbb{R})$, we have:

$$u \in C^\varepsilon \quad \Leftrightarrow \quad \chi(D)u \in L^\infty \quad \text{and} \quad \left\|\varphi(2^{-q}D)u\right\|_{L^\infty} \leqslant C 2^{-q\varepsilon},$$

  the norm $\|\chi(D)u\|_{L^\infty} + \sup_{q \geqslant 0} 2^{q\varepsilon} \|\varphi(2^{-q}D)u\|_{L^\infty}$, noted as $\|u\|_\varepsilon$, being a equivalent norm to the normal norm. This characteristic property serves as the definition in the space C^r when r is some number. Furthermore, if $r = 1$, we don't find the set of Lipschitz function, but the set is traditionally called the class of Zygmund and noted as C^1_* of functions given as $|u(x + y) + u(x - y) - 2u(x)| \leqslant C|y|$. Finally, if r is a real number (resp. a strict positive real number) and $\mathcal{C}$ (resp. $\mathcal{B}$) an annulus (resp. a ball) of $\mathbb{R}^2$, there exist a constant C such that , for all function sequences, we have:
  if for all integer q, the support of the Fourier transform of u_q is included in $2^q\mathcal{C}$ (resp. $2^q\mathcal{B}$, then

$$\left\| \sum_{q \in \mathcal{N}} u_q \right\|_r \leqslant C \sup_{q \geqslant 0} 2^{qr} \|u_q\|_{L^\infty}. \tag{1}$$

- We set, for $q \geq 0$, the operator $\psi(2^{-q}D)$ by Δ_q, the operator $\chi(D)$ by Δ_{-1} and finally, agreeing with that $\Delta_p = 0$ when $p \leqslant -2$, the operator $\sum_{p \leqslant q-1} \Delta_p$ by S_q. We set N_0 an integer such that $\operatorname{supp} \chi(2^{N_0}.) + \operatorname{supp} \varepsilon$ doesn't meet the origin. We will use this decomposition of a product introduced by J.M. Bony in [3] very often in this work. We define respectively the operators of paraproduct and the remaining by the formula:

$$T_a = \sum_q S_{q-N_0}(a)\Delta_q, \qquad \text{then } R(a, \cdot) = \sum_{|q-q'| \leqslant N_0} \Delta_q(a)\Delta_{q'}. \tag{2}$$

We will also choose N_0 big enough such that

$$\chi(D)T_a = T_a\chi(D) = 0 \tag{3}$$

It is immediate that we have:

$$ab = T_a b + T_b a + R(a,b) \tag{4}$$

We will also use a very close decomposition:

$$ab = T_a b + \sum_q S_{q+N_0+1}(b)\Delta_q a. \tag{5}$$

Finally, if X is a vector field, we pose $T_X = \sum_i T_{X^i}\partial_i$.

I want to thank S.Alinhac, H.-M Bony, P. Gerard and G. Lebeau. These discussions that I could have with them or their remarks that they have made were a precious help during the production on this work.
L. Hormander have reread with great care of the most all previous version of this work and have made me many suggestions. I sincerely thank him.

1 General Theorem of global existence