Persistence of Geometric Structures in 2-Dimensional Incompressible Fluids

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Abstract

In this paper, we study the properties of a solution of the incompressible Euler System for large time. We suppose that the initial vorticity is the characteristic function of a regular bounded domain. Then the vorticity remains, for all time, the characteristic function of a bounded domain with the same regularity.

Keywords: Vector field(little regular), tangential regularity, flow, vortex(of patches).

Introduction

The principal results shown here have as primary motivation of a classic problem from mechanics of the 2 dimensional perfect fluid: the problem of vortex patches. Let us recall the framework where we are going to work. The movement of such fluid is described by a vector field on the plane, depending on time, noted $v(t,x)$ and satisfying

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{cases} \tag{E}$$

where $p(t,x)$ denotes the pressure of the fluid at point x and the instant t and where $v \cdot \nabla = \sum_i v^i \partial_i$. One will notice that the flow Ψ of the field of the vectors v, that is to say, the mapping satisfying the next differential equation:

$$\partial_t \Psi(t,x) = v(t, \Psi(t,x)) \quad \text{and} \quad \Psi(0,x) = x$$

The fundamental quantity in the study of this equation is the curl of the field of speeds, also called vortex. As we are in dimension 2, this antisymmetric matrix is identified with a real notation of $w = \partial_1 v^2 - \partial_2 v^1$. The specific character of the dimension 2 is the conservation of w along the trajectory of the field of vectors v:

$$\partial_t w + v \cdot \nabla w = 0. \tag{0.1}$$

Considering the nullity of the divergence of the field of vectors v, we can, if we stick to the fields of bounded vectors, recalculate v, within closed constant vector, from based on w, by the following well known formula, called the law of Biot-Savart:

$$v = \nabla^{\perp}\Delta^{-1} = \left(-\int \frac{x_2 - y_2}{|x-y|^2} w(y)dy, \int \frac{x_1 - y_1}{|x-y|^2} w(y)dy \right), \tag{0.2}$$

by letting $\nabla^{\perp} f = (-\partial_2 f, \partial_1 f)$.

It is clear that, if $w \in \mathbf{L}^{\infty} \cap \mathbf{L}^{p}$ with $p < 2$, the integral above defines a field of bounded vectors. Furthermore, it is well known (and trivial to verify) that if w satisfies (0.1) with the field of vectors v given by (0.2), then v itself is a solution to (E) with the initial data deducted from w_0 by the relation (0.2). We will always place ourselves in the framework and, in the statement of the theorems, we will formulate the hypothesis only in the vortex.

The problem of vortex patches is the following: suppose that the vortex is, at the initial moment, a open and bounded characteristic function whose edge is in the Holder class $C^{k+\varepsilon}$, where k is a strictly positive integer and ε a real number in the interval $(0,1)$. In that case, Yudovitch proved in [11] that there exists a unique vector solutions field of system (E) in $\mathbb{R}^2 \times \mathbb{R}^2$, whose the vortex belongs to $\mathbf{L}^{\infty}(\mathbb{R}^3)$. This solution is so quasi-Lipschitz, that is to say its module of continuity is $|x-y| \cdot |\log|x-y||$. Such a vector field processes the flow Ψ with exponentially decreasing regularity a function in time, that is to say $\Psi(t,\cdot)$ is a homeomorphism of the Holder class $C^{\exp -\alpha t}$. According to the relation (0.1),the vortex in the moment t is then a open bounded characteristic function in the topology remaining unchanged. On the other hand, its edge is not better than the class $C^{\exp -\alpha t}$.
2 very natural questions are posed as following:

- Does the open edge stay regular for small time?
- If so, what happens for big time?

In the case where w_0 is the characteristic function of the interior of a plane curve, closed, simple and in the class $C^{1+\varepsilon}$, the following approach was developed(see example [9]). It is very easy, in this context, to verify, thanks to Green's formula, that if the edge remains in the Holder class $C^{1+\varepsilon}$, (then) there exists a specific setting of the edge satisfying the equation

$$\partial_t \gamma(t,s) = \frac{1}{2\pi} \int_0^{2\pi} \log|\gamma(t,s) - \gamma(t,\sigma)| \partial_\sigma \gamma(t,\sigma) d\sigma. \tag{B}$$

In [9] A. Majda announced a proof of existence located in time of a solution of the equation (B) and, based on numerical experiment(see [12]), conjecture that the time of existence is usually finished and, in this case, the edge of domain ceases to be rectifiable(?). A less degradation of regularity of edge was also suggested by more recent numerical simulations(see [4] [5]).

One simplified approach has been proposed by P. Constantin and Titi(see [8]). In the view of equation (B), we study the small perturbations of circle, which is very sure a solution of (B) and then we don't remember the development in series of the logarithms which are quadratic terms. For this quadratic approximation of equation (B), S. Alinhac has proved in [1] a result of instability which is tend to think that there might not be global existence of regular solution to equation (B) itself.

In regard to the local existence in time, we proved in [6] forgetting equation (B) and proved a local control with Lipschitz norm of the solution of (E) thanks to the tangential regularity of vortex by relative to a vector field not cancelling in the singular support C^{ε} of vortex. Furthermore, in [10], P.Serfati proved the existence and the holomorphy regularity , locally in time, for little perturbation of circle, in the frame of (B).

The first motivation of this work is to prove the following theorem:

Theorem 1. ***A*** *Let ε belong to the interval $(0,1)$ and γ_0 a function in the space $C^{1+\varepsilon}(S^1;\mathbb{R}^2)$ setted properly a Jordan curve. Then there exist a unique solution $\gamma(t,s)$ of the equation* (B) *belonging to the space $\mathbf{L}^{\infty}_{loc}(\mathbb{R};C^{1+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2))$.*

In [6], we have developed the study of iterated action in the irregular vector fields which allows the deduction in the above theorem in the following corollary:

Corollary. ***B*** *Let ε be in the interval $(0,1)$, k a non-zero positive integer and γ_0 a function in the space $C^{k+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2)$ setted properly as a Jordan curve. Then there exists a unique solution $\gamma(t,s)$ of the equation* (B) *in the space $\mathbf{L}^{\infty}_{loc}(\mathbb{R},C^{k+\varepsilon}(\mathbb{S}^1;\mathbb{R}^2))\cap C^{\infty}(\mathbb{R},C^{k+\varepsilon'}(\mathbb{S}^1;\mathbb{R}^2))$ for all $\varepsilon'<\varepsilon$.*

Our approach will be as follows:

- in the first paragraph, we will explain what concept of regularity which allows to see the previous theorem as an immediate corollary of a much more general theorem, the Theorem 1;
- in the second paragraph, we will prove a estimation on the Lipschitz norm of a vector field;
- in the third paragraph, we will utilize this estimate to prove a priori estimation on the regular solutions of the system (E);
- in the fourth paragraph, we will finish, by regularization of initial data and then transition to the limit, the proof of Theorem 1;

- in the fifth and final paragraph, we will state various global results from the combination of Theorem 1 and the local theorems of regularities proved in [6].

Notations and reminders

In the rest of the article, we will take the following notations and conventions:

- ε denotes a real number strictly between 0 and 1;
- if X is a vector field in plane, we denote $I(A, X)$ the lower bound of $|X(x)|$ for x running through A and we denote ∇X the matrix of general term $\partial_j X^i$;
- if f is a distribution on the plane, we denote $\nabla^\perp f$ the vector field $(-\partial_2 f, \partial_1 f)$ that is sure to be divergence-free.
- if Ω is an open on plane, $C^\varepsilon(\Omega)$[resp. $\mathrm{Lip}(\Omega)$] denotes the set of functions u given on Ω such that we have, for all x and y in Ω, $|u(x) - u(y)| \leqslant C \leqslant |x-y|^\varepsilon$(resp. $|x-y|$) and we will note by $\|\cdot\|_{\varepsilon,\Omega}$(resp. $\|\cdot\|_{Lip(\Omega)}$) the natural norm of $C^\varepsilon(\Omega)$[resp. $\mathrm{Lip}(\Omega)$],
- if $\Omega = \mathbb{R}^2$, one can characterize the space $C^\varepsilon(\Omega)$, then simply noted as C^ε, using a dyadic cutting in the space of frequency. More precisely, it's $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\chi(\xi) = 1 - \sum\limits_{q \geqslant 0} \varphi(2^{-q}\xi) \in C_0^\infty(\mathbb{R})$, we have:

$$u \in C^\varepsilon \quad \Leftrightarrow \quad \chi(D)u \in \mathbf{L}^\infty \quad \text{and} \quad \left\|\varphi(2^{-q}D)u\right\|_{\mathbf{L}^\infty} \leqslant C 2^{-q\varepsilon},$$

  the norm $\|\chi(D)u\|_{\mathbf{L}^\infty} + \sup_{q \geqslant 0} 2^{q\varepsilon} \|\varphi(2^{-q}D)u\|_{\mathbf{L}^\infty}$, noted as $\|u\|_\varepsilon$, being a equivalent norm to the normal norm. This characteristic property serves as the definition in the space C^r when r is some number. Furthermore, if $r = 1$, we don't find the set of Lipschitz function, but the set is traditionally called the class of Zygmund and noted as C_*^1 of functions given as $|u(x+y) + u(x-y) - 2u(x)| \leqslant C|y|$. Finally, if r is a real number (resp. a strict positive real number) and $\mathcal{C}$ (resp. $\mathcal{B}$) an annulus (resp. a ball) of $\mathbb{R}^2$, there exist a constant C such that , for all function sequences, we have:
  if for all integer q, the support of the Fourier transform of u_q is included in $2^q\mathcal{C}$ (resp. $2^q\mathcal{B}$, then

$$\left\|\sum_{q \in \mathcal{N}} u_q\right\|_r \leqslant C \sup_{q \geqslant 0} 2^{qr} \|u_q\|_{\mathbf{L}^\infty}. \tag{1}$$

- We set, for $q \geq 0$, the operator $\psi(2^{-q}D)$ by Δ_q, the operator $\chi(D)$ by Δ_{-1} and finally, agreeing with that $\Delta_p = 0$ when $p \leqslant -2$, the operator $\sum\limits_{p \leqslant q-1} \Delta_p$ by S_q. We set N_0 an integer such that $\operatorname{supp}\chi(2^{N_0}.) + \operatorname{supp}\varepsilon$

doesn't meet the origin. We will use this decomposition of a product introduced by J.M. Bony in [3] very often in this work. We define respectively the operators of paraproduct and the remaining by the formula:

$$T_a = \sum_q S_{q-N_0}(a)\Delta_q, \qquad \text{then } R(a,\cdot) = \sum_{|q-q'|\leqslant N_0} \Delta_q(a)\Delta_{q'}. \tag{2}$$

We will also choose N_0 big enough such that

$$\chi(D)T_a = T_a\chi(D) = 0 \tag{3}$$

It is immediate that we have:

$$ab = T_a b + T_b a + R(a,b) \tag{4}$$

We will also use a very close decomposition:

$$ab = T_a b + \sum_q S_{q+N_0+1}(b)\Delta_q a. \tag{5}$$

Finally, if X is a vector field, we pose $T_X = \sum_i T_{X^i}\partial_i$.

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1 General Theorem of global existence

The object of this paragraph us the reduction of Theorem A to a general theorem of propagation, until any time, of a certain type of regularity of the vortex. To motivate this approach, observed that if ω us the characteristic function of a bounded domain in space $C^{1+\varepsilon}$, then we prove by fairly usual techniques of study of singular integrals that the field of vector associated with ω by the relation (0.2) is Lipschitz. It is very common that the control of the Lipschitz norm of the solution of a hyperbolic quasilinear system is a key point.

It is then appeared to us that a real comprehension of problem was going on by answering to the following question: if we regularize initial data of vortex patch , do we have, by the solution associated with given regularities, a uniform estimation of their Lipschitz norm in a fixed timer interval? More precisely, we consider θ a indefinitely differentiable function with compact support, positive and with integral 1, and $(w_n)_{n\in N}$ is defined by $\omega_n = (1+n)^{-2}\theta((1+n)^{-1}.)*\omega$, ω is always a characteristic function of a regular bounded domain.

Do we have a uniform estimation over n of the Lipschitz norm of the vector field v_n associated with w_n by the relation (0.2)? The solution of the problem imposes the construction of a space of adapted functions, hence the following definition:

Definition 1.1. Let A be a closed plan and X a divergence-free vector field in coefficients C^{ε}, not cancelling on A, we set by $C_{\varepsilon}(A, X)$ together some given functions on plane such that:

- $u \in C^{\varepsilon}(\mathbb{R}^2 \setminus A)$,
- $X(x, D)u \in C^{\varepsilon-1}$.

The 2 important points are a part belonging to ω in $C_{\varepsilon}(A, X)$ ensure that the v in space of Lipschitz vector field and that, modulo a slightly enlargement of A without any consequence, the family $(\omega_n)_{n\in N}$ is given in $C_{\varepsilon}(A, X)$ in an appropriate sense.

Let's now state the principal theorem of this work.

Theorem 2. *Let X_0 be a divergence-free vector field and A^0 closed on the plan such that X_0 is in the class C^{ε} and $I(A^0, X_0)$ strictly positive. If $w_0 \in C_{\varepsilon}(A^0, X_0) \cap \mathbf{L}^p$. with $p < 2$, there exists a unique solution of* (E) *in $C(\mathbb{R}; \mathbf{L}^{\infty}(\mathbb{R}^2)) \cap \mathbf{L}^{\infty}_{loc}(\mathbb{R}; Lip(\mathbb{R}^2))$ which also checks:*

1. *$X_0(x, D)\psi \in \mathbf{L}^{\infty}_{loc}(\mathbb{R}; C^{\varepsilon})$,*
2. *if $X_t = (\psi_t)_* X_0$ and $A^t = \psi(t, A^0)$, then $w(t, \cdot) \in C_{\varepsilon}(A^t, X_t)$.*

Proof of theorem 1 is part of theorem 2. - Let f_0 be a $C^{1+\varepsilon}$ equation of a curve Γ_0, image of circle S^1 by γ_0, we set the $X_0 = \nabla^{\perp} f_0$. Then we consider the closed A^0 of of points on the plane in sufficient small distance of Γ_0 such that X_0 didn't cancel it on A^0. As X_0 is a tangent vector in γ_0, not vanish on Γ_0, there exists a function f belonging to $C^{\varepsilon}(S^2; \mathbb{R}$ such that $\partial_s \gamma_0(s) = f(s) X_0(\gamma(s))$. In the context of the equation of the equation(B) we have$\gamma(t, s) = \psi(t, \gamma(s))$. Thus $\partial_s \gamma(t, s) = f(s)(X_0(x, D)\psi)(t, \gamma_0(s))$. In theorem 2 1., we deduce the existence of a solution γ of the equation (B) in the space $\mathbf{L}^{\infty}_{loc}(\mathbb{R}; C^{1+\varepsilon}(\mathbb{S}^1; \mathbb{R}^2))$.

Remark. The statement of theorem 2 contains a particular fact that the field of vector field v_0 is Lipschitz.

The point 1. of theorem 2, to know the tangential regularity of the flow by link to X_0, is sufficient to establish the theorem 1. The point 2. must be understood as a theorem of persistence, that is to say the propagation until infinity, with the regularity $C_{\varepsilon}(A, X)$. This adopted step is the same as that one in [6]. In [6], the lack of sufficient precise estimation, we have proved only that one local version of theorem 1. The major difficulty of proof remains in the control of Lipschitz norm of the solution vector fields of (E) over time, then the regularity given by the relations (0.1) and (0.2) is less(C^1_* or better, in derivatives **BMO**). In [6], we have proved a upper bound of the Lipschitz norm of the field of the vector v, which ensures the Lipschitz character of v when its curl ω belongs to $C_\varepsilon(A, X)$. The key point to obtain the global existence consist in establish a logarithmic version of this inequality, that is to say a version or the geometric data only defines the additional regularity will appear that through a logarithm. This is the subject of the following paragraph.

2 Control of the gradient of the divergence free vector field in part of its curl

This paragraph is consecrated in the proof of the following theorem:

Theorem 3. *Let ε be a real number in the interval $(0,1)$ and p a real number in the interval $[1,2)$, there exists a constant C such that, if A is any closed set on the plane and X a vector field of class C^ε, divergence free and not vanishing on A, we have:*

$$\|v\|_{Lip} \leqslant C\left\{\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}(1+\log N_{\varepsilon,p}(A,X,\omega))\right\},$$

with

$$N_{\varepsilon,p}(A,A,\omega) = \frac{\|X\|_\varepsilon \|\omega\|_{\varepsilon,\mathbb{R}^2\setminus A}}{I(A,X)\|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2\setminus A)}}\left(1+\frac{\|X(\chi,D)\omega\|_{\varepsilon-1}}{I(A,X)\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}}\right).$$

Remark. As $\|X\|_\varepsilon \geqslant \|X\|_{\mathbf{L}^\infty} \geqslant I(A,X)$ and $\|\omega\|_{\varepsilon,\mathbb{R}^2\setminus A)}$, it is clear that $N_{\varepsilon,p}(A,X,\omega)\geqslant 1$.

Proof: Let us, for the first time, prove this theorem in a very particular where appear, with overload techniques, the 2 essential ideas of the proof. Suppose that the vector field X is ∂_1 and accidentally the support of the Fourier transform of ω doesn't contain the origin. In this case, the set A is any set and we can take $A=\mathbb{R}^2$ and agree with that $\|\omega\|_{\varepsilon,\mathbb{R}^2\setminus A}/\|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2\setminus A)}$ values 1. It is clear that $\|X\|_\varepsilon = I(A,X)=1$. The first well-known idea, is that the space C^0_* including in $\mathbf{L}^\infty$ is true, almost at a logarithm of the norm C^ε. More precisely, we have the following lemma:

Lemma 4. *A strictly positive real number ε is given. There exist a constant C such that, for all function f and couple (α,β) of real number satisfies $\|f\|_0 \leqslant \alpha$, $\|f\|_\varepsilon \leqslant \beta$ and $\alpha\leqslant\beta$, we have:*

$$\|f\|_{\mathbf{L}^\infty} \leqslant C\alpha(1+\log\frac{\alpha}{\beta}).$$

Proof, we write $f = S_n f + (Id - S_n)f$, the characterization of Holder driven space such that we have $\|f\|_{\mathbf{L}^\infty} \leqslant (N+1)\alpha + 4\varepsilon^{-1}w^{-N\varepsilon}\beta$; thus we obtain the lemma from choosing by example:

$$N = \left[\frac{1}{\varepsilon}\log\frac{\beta}{\alpha}\right]+1$$

where log denotes the logarithm with base 2. The second idea is the following: If $X = \partial_1$, we have, according to the lemma above, applying with $\alpha = C\|\omega\|_{\mathbf{L}^\infty}$ and $\beta = C(\|\omega\|_{\mathbf{L}^\infty} + \|\partial_1\omega\|_{\varepsilon-1})$,

$$\left\|\partial_1\partial_j\Delta^{-1}\omega\right\|_{\mathbf{L}^\infty} \leqslant C_\varepsilon\|\omega\|_{\mathbf{L}^\infty}\left(1+\log\left(1+\frac{\|\partial_1\omega\|_{\varepsilon-1}}{\|\omega\|_{\mathbf{L}^\infty}}\right)\right).$$

Or, $\left\|\partial_2^2\Delta^{-1}\omega\right\|_{\mathbf{L}^\infty} \leqslant \|\omega\|_{\mathbf{L}^\infty} + \left\|\partial_1^2\Delta^{-1}\omega\right\|_{\mathbf{L}^\infty}$, hence the theorem 3 in this very particular case.

As for a general case, there presents 2 difficulties. First, a serious one comes from the weak regularity of the vector field X, ans its possible cancellation during the evolution. The second one comes from the necessary space truncation and its easily solving by analysing with the precaution pseudo-locality of multiplier of Fourier. One of the crucial points us that all these tendencies to disturb the inequality do appear, it also, mitigated by a logarithm.

To obtain the result, we need to proceed gradually. So, we are going to start by demonstrating the following Lemma:

Lemma 5. *Let ε be a real number in interval $(0,1)$ and p a real number in interval $[1,2)$, there exist a constant C such that if A is a closed set on the plane and Y a vector field of class C^ε not vanishing on A, we have*

$$\|v\|_{Lip} \leqslant C\|w\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}\frac{\|Y\|_{\mathbf{L}^\infty}^2}{I(A,Y)^2}\times\left(1+\log\frac{\|\mathbf{Y}\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}+\|\mathbf{Y}(x,\mathbf{D})v\|_\varepsilon}{\mathbf{I}(\mathbf{A},\mathbf{Y})\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}}+\log\frac{\|\omega\|_{\varepsilon,\mathbf{R}^2\backslash\mathbf{A}}}{\|\omega\|_{\mathbf{L}^\infty(\mathbf{R}^2\backslash\mathbf{A})}}\right)$$

Proof.–Then we, firstly, focus on the high frequencies. We will pose $a = (Id - \chi(D))\Delta^{-1}\omega$. It must increase $\|\partial_j\partial_i a\|_{\mathbf{L}^\infty}$. The decisive step is the increasing of $\|\mathbf{Y}(x,\mathbf{D})\partial_i a\|_{\mathbf{L}^\infty}$.

Before we go further, we need some properties of paradifferential calculus of Bony (see [3]) which is summarized in the following lemma:

Lemma 6. *Let r, s and m be 3 real numbers and σ an indefinitely derivable function such that, for all ξ of norm larger than 1 and all real λ greater than 1, $\sigma(\lambda\xi) = \lambda^m\sigma(\xi)$. There exists a constant C such that*

1. $\|\mathrm{T}_a b\|_s \leqq \mathrm{C}\|a\|_{\mathrm{L}}\infty\|b\|_s$ *and if* $r<0, \|\mathrm{T}_a b\|_{r+s} \leqq \mathrm{C}\|a\|_r\|b\|_s$
2. $\mathrm{sir} + s > 0, \|\mathbf{R}(a,b)\|_{r+s} \leqq \mathrm{C}\|a\|_r\|b\|_s et\|\mathbf{R}(a,b)\|_{r+s} \leqq \mathrm{C}\|a\|_{\mathbf{L}^\infty}\|b\|_{r+s}$
3. *si* $\quad r>0, \|ab\|_r \leqq \mathrm{C}\left(\|a\|_{\mathrm{L}^\infty}\|b\|_r + \|a\|_r\|b\|_{\mathrm{L}^\infty}\right) \quad$ *et* $\quad$ *si* $\quad r+s>0 \quad$ *et* $\quad r<0\ \|ab\|_{r+s} \leqq \mathrm{C}\|a\|_r\|b\|_s$
4. $\|[\mathrm{T}_a,\sigma(\mathrm{D})]\,b\|_{s-m+1} \leqq \mathrm{C}\|\nabla a\|_{\mathrm{L}}\infty\|b\|_s$ *and if* $r<1$ $\|[\mathrm{T}_a,\sigma(\mathrm{D})]\,b\|_{s-m+r} \leqq \mathrm{C}\|\nabla a\|_{r-1}\|b\|_s$
5. *Let X be a vector field with divergence-free coefficients C^ε and A a linear operator mapping from C^r to C^{r-m} for all real numberr, so, for all real number r such that r and $r - m$are strictly larger than $-\varepsilon$ and*

for all u in C^r, we have

$$[\mathrm{X}(x,\mathrm{D}),\mathrm{A}]u = [\mathrm{T}_{\mathrm{x}},\mathrm{A}]\,u + \sum_j \left\{\mathrm{T}_{\partial_j \mathrm{Au}}\mathrm{X}^j + \partial_j \mathrm{R}\left(\mathrm{A}u,\mathrm{X}^j\right)\right\} - \mathrm{A}\sum_j \left\{\mathrm{T}_{\partial j}\mathrm{X}^j + \partial_j \mathrm{R}\left(u,\mathrm{X}^j\right)\right\}$$

Proof. The points 1 and 2 result in a clear way of (1) and the definition (2) of operators to remain R and the paraproduct T. The point 3 is an immediate consequence of the decomposition (4) of paraproduct and the rest.

When considering point 4, it should explain the commutator. According to (3), we can suppose the function σ homogeneous. We have that the following formula, N_1 is a integer depending on ϕ and N_0

$$[\mathrm{T}_a,\sigma(\mathrm{D})]\,b = \sum_q \sum_{k=-\mathrm{N}_1}^{\mathrm{N}_1} 2^{(q+k)m} c_{q,k} \quad \text{avec} \quad c_{q,k} = \left[\mathrm{S}_{q-\mathrm{N}_0}(a), \varphi\sigma\left(2^{-q-k}\mathrm{D}\right)\right]\Delta_q b$$

Taylor formula ensures in order that we have

$$\begin{gathered} c_{q,k} = 2^{-(q+k)} \sum_j 2^{2(q+k)} \int dy 2^{(q+k)}(x-y)_j h\left(2^{q+k}(x-y)\right)\Delta_q b(y) \\ \times \int_0^1 dt S_{q-N_0}\left(\partial_j a\right)(x+t(y-x)) \quad \text{with } \hat{h} = \varphi\sigma \end{gathered}$$

It is clear that the support of Fourier transform of $c_{q,k}$ is included in a annulus of type $2^1\mathcal{C}'$, $\mathcal{C}'$ is a fixed annulus. The increasing of $c_{q,k}$ clearly result that the characterization of the Holder space and h and $x_j h_k$ belonging to $\mathbf{L}^1$, from the point 4.

For proof of point 5, we use (2), hence

$$\mathrm{X}(x,\mathrm{D})u = \mathrm{T}_{\mathrm{x}}u + \sum_j \left\{\mathrm{T}_{\partial_j \mu}\mathrm{X}^j + \partial_j \mathrm{R}\left(u,\mathrm{X}^j\right)\right\}$$

X being divergence free ensures that we have

$$\mathrm{X}(x,\mathrm{D})u = \mathrm{T}_{\mathrm{x}}u + \sum_j \left\{\mathrm{T}_{\partial j \mu}\mathrm{X}^j + \partial_j \mathrm{R}\left(u,\mathrm{X}^j\right)\right\}$$

The commutation with a fixed operator A is linear, we obtain the desired formula and then the lemma 6.

Let's go back to the proof of Lemma 5. We are going to utilize the identity (5). We have

$$\mathrm{Y}(x,\mathrm{D})\partial_i a = \Phi_1 + \Phi_2 \tag{6}$$

with

$$\Phi_1 = \mathrm{T_Y}\partial_i a \quad \text{and} \quad \Phi_2 = \sum_{k=1}\sum_q \mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_i\partial_k a\right)\Delta_q \mathrm{Y}^k$$

For the increasing of $\|\Phi_1\|_{\mathbf{L}^\infty}$ we use the lemma 4. It is clear that

$$\|\Phi_1\|_0 \leqq \mathrm{C}\|\mathrm{Y}\|_{\mathrm{L}^\infty}\|\omega\|_{\mathrm{L}^\infty}$$

Moreover, we prove that $\Phi_1 = \sum_{j=1}^4 \Phi_{1,j}$ with:

$$\Phi_{1,1} = \left[\mathrm{T_Y}, \partial_i(\mathrm{Id} - \chi(\mathrm{D}))\Delta^{-1}\right]\omega,$$

$$\Phi_{1,2} = \partial_i(\mathrm{Id} - \chi(\mathrm{D}))\Delta^{-1}\textstyle\sum_{k=1}^2\left\{\mathrm{T}_{\partial_2\mathrm{Y}^k}\partial_k v^1 - \mathrm{T}_{\partial_1\mathrm{Y}^k}\partial_k v^2\right\}$$
$$\Phi_{1,3} = \partial_i(\mathrm{Id} - \chi(\mathrm{D}))\Delta^{-1}\left\{\partial_1\mathrm{Y}(x,\mathrm{D})v^2 - \partial_2\mathrm{Y}(x,\mathrm{D})v^1\right\}$$

and

$$\begin{aligned}\Phi_{1,4} = \partial_i(\mathrm{Id} - \chi(\mathrm{D}))\Delta^{-1}\Big\{&\partial_2\textstyle\sum_{k=1}^2\sum_q \mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_k v^1\right)\Delta_q\left(\mathrm{Y}^k\right)\\ &-\partial_1\textstyle\sum_{k=1}^2\sum_q \mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_k v^2\right)\Delta_q\left(\mathrm{Y}^k\right)\Big\}\end{aligned}$$

Indeed, it's clear that $\Phi_1 = \Phi_{1,1} + \partial_i(Id - \chi(D))\Delta^{-1}T_\gamma\omega$. By definition of ω, we have

$$\begin{aligned}\mathrm{T_Y}\omega &= \sum_{k=1}^2 \mathrm{T}_{\mathrm{Y}^k}\partial_k\left(\partial_1 v^2 - \partial_2 v^1\right)\\ &= \sum_{k=1}^2\left\{\partial^1\left(\mathrm{T}_{\mathrm{Y}^k}\partial_k v^2\right) - \partial_2\left(\mathrm{T}_{\mathrm{Y}^k}\partial_k v^1\right)\right\} - \sum_{k=1}^2\left\{\mathrm{T}_{\partial_1\mathrm{Y}^k\partial_k v^2 - \mathrm{T}_{\partial_2\mathrm{Y}^k}\partial_k v^1}\right\}\end{aligned}$$

Hence,

$$\Phi_1 = \Phi_{1,1} + \Phi_{1,2} + \partial_i(\mathrm{Id} - \chi(\mathrm{D}))\Delta^{-1}\sum_{k=1}^2\left\{\partial_1\left(\mathrm{T}_{\mathrm{Y}^k}\partial_k v^2\right) - \partial_2\left(\mathrm{T}_{\mathrm{Y}^k}\partial_k v^1\right)\right\}$$

Or, according to (5), we have

$$\mathrm{T}_{\mathrm{Y}^k}\partial_k v^i = \mathrm{Y}(x,\mathrm{D})v^i - \sum_q \mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_k v^i\right)\Delta_q\mathrm{Y}^k$$

Hence we have the desired formula.

The Lemma 6 ensures that we have, for all $\varepsilon \in (0,1)$:

$$\|\Phi_{1,3}\|_\varepsilon \leqq C_\varepsilon\|Y(x,D)v\|_\varepsilon \quad \text{et} \quad \|\Phi_{1,i}\|_\varepsilon \leqq C_\varepsilon\|Y\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty}, \quad \text{pour} \quad i \in \{1,2\}$$

Moreover, seeing that $v=(\mathrm{Id}-\chi(\mathrm{D}))\nabla^{\perp}\Delta^{-1}\omega+\chi(\mathrm{D})v$, it is clear that

$$\|v\|_1 \leqq \mathbf{C}\left(\|\omega\|_{\mathbf{L}^\infty}+\|v\|_{\mathbf{L}^\infty}\right)$$

Or, according to (0.2), we have

$$|v(x)| \leqq \int \frac{\chi(x-y)}{|x-y|}|\omega(y)|dy+\int \frac{1-\chi(x-y)}{|x-y|}|\omega(y)|dy$$

As $\chi(x)|x|^{-1}$ belongs to $\mathbf{L}^1$ and $(1-\chi(x))|x|^{-1}$ belongs to $\mathbf{L}^{p'}$, p' representing the conjugate exponent of p, we have

$$\|v\|_{\mathbf{L}^\infty} \leqq \mathbf{C}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p} \text{ and so } \|v\|_1 \leqq \mathbf{C}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p} \tag{7}$$

As we have $\left\|\mathbf{S}_{a+\mathbf{N}_0+1}\left(\partial_k v^i\right)\right\|_{\mathbf{L}^\infty} \leqq \sum_{q'\leq q+\mathbf{N}_0}\|\Delta_{q'}\left(\partial_k v^i\right)\|_{\mathbf{L}^\infty}$, it results from (7) that

$$\left\|\mathbf{S}_{q+\mathbf{N}_0+1}\left(\partial_k v^i\right)\right\|_{\mathbf{L}}\infty \leqq \mathbf{C}\left(q+\mathbf{N}_0+2\right)\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}.$$

In particular, we have $\left\|\mathbf{S}_{\boldsymbol{q}+\mathbf{N_0}+1}\left(\partial_{\boldsymbol{k}} v^i\right)\right\|_{\mathbf{L}}^{\infty} \leqq \mathbf{C}_{\boldsymbol{\epsilon}'}2^{\boldsymbol{q}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}'\right)}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}$ for all $0<\varepsilon'<\varepsilon$.

According to the definition of the Holder space using the dyadic annulus, we have that $\|\Delta_q(\mathbf{Y})\|_{\mathbf{L}^\infty} \leqq 2^{-q\varepsilon}\|\mathbf{Y}\|_{\varepsilon}$. Hence, according to (1),

$$\|\Phi_{1,4}\|_{e'} \leqq \mathrm{C}_{\varepsilon'}\|\mathrm{Y}\|_{\epsilon}\|\omega\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}, \quad \text{for all } \varepsilon'\in(0,\varepsilon)$$

and a real number in interval $(0,\varepsilon)$, for example $\varepsilon/2$, then ensures the following inequality:

$$\|\Phi_1\|_{\mathbf{L}^\infty} \leqq \mathbf{C}_{\varepsilon}\|\mathbf{Y}\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}\left(1+\log\frac{\|\mathbf{Y}\|_{\epsilon}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}}+\|\mathbf{Y}(x,\mathbf{D})v\|_{\varepsilon}}{\|\mathbf{Y}\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}}\right) \tag{8}$$

The term Φ_2 is treated differently. Let N be a integer, we set $\Phi_2=\Phi_{3,N}+\Phi_{4,N}$ with:

$$\Phi_{3,\mathrm{N}}=\sum_{k=1}^{2}\sum_{q\leq\mathrm{N}-1}\mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_k\partial_i a\right)\Delta_q\left(\mathrm{Y}^k\right)$$
$$\Phi_{4,\mathrm{N}}=\sum_{k=1}^{2}\sum_{q\geq\mathrm{N}}\mathrm{S}_{q+\mathrm{N}_0+1}\left(\partial_k\partial_i a\right)\Delta_q\left(\mathrm{Y}^k\right)$$

The increasing of $\Phi_{4,N}$ is done using the C^{ε} regularity of the vector field Y.

We use $\|\mathbf{S}_{q+\mathbf{N}_0+1}\left(\partial_k\partial_i a\right)\|_{\mathbf{L}^\infty} \leqq \mathbf{C}\left(q+\mathbf{N}_0+1\right)\|\omega\|_{\mathbf{L}^\infty} \leqq \mathbf{C}_{\mathfrak{e}}2^{q\varepsilon/2}\|\omega\|_{\mathbf{L}^\infty}$. From where comes that

$$\|\Phi_{4,\mathrm{N}}\|_{\mathrm{L}^\infty} \leqq \mathrm{C}_{\mathrm{e}}2^{-\mathrm{N}\epsilon/2}\|\omega\|_{\mathrm{L}^\infty}\|\mathrm{Y}\|_{\mathrm{e}}$$

The term $\Phi_{3,N}$ must be dealt with a little more fineness. An Abel group means that

$$\Phi_{3,\mathrm{N}} = \sum_{k=1}^{2}\left\{\mathrm{S_N}\left(\mathrm{Y}^k\right)\mathrm{S_{N+N_0}}\left(\partial_k\partial_i a\right) - \sum_{q\leqq \mathrm{N}-1}\mathrm{S}_q\left(\mathrm{Y}^k\right)\Delta_{q+\mathrm{N}_0}\left(\partial_k\partial_i a\right)\right\}$$

Or, for all integer q and all integer N, we have, by definition of the Holder space,

$$\left\|\mathbf{S}_q\left(\mathbf{Y}^k\right)\right\|_{\mathbf{L}^\infty} \leqq \|\mathbf{Y}\|_{\mathbf{L}^\infty} \quad \text{and} \quad \|\mathbf{S_{N+N_0}}\left(\partial_k\partial_i a\right)\|_{\mathbf{L}^\infty} \leqq \mathbf{C}\left(\mathbf{N}+\mathbf{N}_0+1\right)\|\omega\|_{\mathbf{L}^\infty}.$$

It follows that

$$\|\Phi_{3,\mathrm{N}}\|_{\mathrm{L}^\infty} \leqq \mathrm{C}\left(\mathrm{N}+\mathrm{N_o}+1\right)\|\mathrm{Y}\|_{\mathrm{L}^\infty}\|\omega\|_{\mathrm{L}^\infty}$$

Then we optimize the choice of N taking by example $\frac{\mathrm{N}}{2} = \left[\frac{1}{\varepsilon}\log\frac{\|\mathrm{Y}\|_{\mathrm{e}}}{\|\mathrm{Y}\|_{\mathrm{L}^\infty}}\right]+1$, hence it comes that

$$\|\Phi_2\|_{\mathbf{L}}\ \propto\leqq \mathbf{C}_\varepsilon\|\mathbf{Y}\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^\infty}\left(1+\log\frac{\|\mathbf{Y}\|_{\mathbf{e}}}{\|\mathbf{Y}\|_{\mathbf{L}^\infty}}\right)$$

Applying (6) (8) and this inequality, we obtain, for all $i\in\{1,2\}$:

$$\begin{aligned}&\|\mathrm{Y}(x,\mathrm{D})\partial_i a\|_{\mathrm{L}}\infty\\ &\quad\leqq \mathrm{C}_\varepsilon\|\mathrm{Y}\|_{\mathrm{L}^\infty}\|\omega\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}\left(1+\log\frac{\|\mathrm{Y}\|_\epsilon\|\omega\|_{\mathrm{L}^\infty\cap\mathbf{L}^p}+\|\mathrm{Y}(x,\mathrm{D})v\|_\epsilon}{\|\mathrm{Y}\|_{\mathrm{L}^\infty}\|\omega\|_{\mathrm{L}^\infty\cap\mathbf{L}^p}}\right).\end{aligned} \tag{9}$$

Then it's enough to observe, that we have

$$|Y(x)|^2\partial_1^2 = Y^1(x)Y(x,D)\partial_1 - Y^2(x)Y(x,D)\partial_2 + \left(Y^2(x)\right)^2\Delta,$$

$$|Y(x)|^2\partial_2^2 = Y^2(x)Y\left(x,\ \mathrm{D}\right)\partial_2 - Y^1(x)Y(x,D)\partial_1 + \left(Y^1(x)\right)^2\Delta$$

and

$$|Y(x)|^2\partial_1\partial_2 = Y^1(x)Y(x,D)\partial_2 + Y^2(x)Y(x,D)\partial_1 - Y^1(x)Y^2(x)\Delta,$$

for conclusion, according to (9), the following inequality, for all i and j valued 1 or 2:

$$\begin{aligned}&\left\||Y(x)|^2\partial_i\partial_j a\right\|_{\mathbf{L}^\infty}\\ &\quad\leqq C_\varepsilon\left(\|Y\|_{\mathbf{L}^\infty}\right)^2\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}\left(1+\log\frac{\|Y\|_\epsilon\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}+\|Y(x,D)v\|_\epsilon}{\|Y\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}}\right)\end{aligned} \tag{10}$$

Once this equality is obtained, it remains to be truncated outside the place of possible cancellation of the vector field Y. To do this, we set $\delta = \left(\mathrm{I}(\mathrm{A},\mathrm{Y})/2\|\mathrm{Y}\|_e\right)^{1/\varepsilon}$ and we consider a function g(resp.g) belonging to $C^\varepsilon(\mathbb{R}^2;[0,1])$, value identically 1 near $A_{\delta/4}$(resp. $A_{\delta/2}$) and supported in $A_{3\delta/4}$(resp. $A_{5\delta/6}$) such that $\|f\|_\varepsilon \leqslant C\delta^{-\varepsilon}$($A_\delta$ denoting the set of point whose distance from A is less than δ). By definition of δ, we have

$2\mathrm{I}(\mathrm{A}_\delta, \mathrm{Y}) \geqq \mathrm{I}(\mathrm{A}, \mathrm{Y})$, hence:

$$\|g\partial_i\partial_j a\|_{\mathbf{L}^\infty} \leqq C_\epsilon \frac{\|Y\|^2_{\mathbf{L}^\infty}}{I(A,Y)^2}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}\left(1+\log\frac{\|Y\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty\cap L}p+\|Y(x,D)v\|_\varepsilon}{\|Y\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}}\right) \tag{11}$$

We will now increase $\|(1-g)\partial_i\partial_j a\|_{\mathbf{L}^\infty}$, the following lemma is necessary.

Lemma 7. *Let σ be a function of class C^∞ such that $|\partial^\alpha\sigma(\xi)| \leqq C_\alpha(1+|\xi|)^{-|\alpha|}$, there exists a constant C such that, if P and Q are 2 closed disjoint sets such that $d(\mathrm{P},\mathrm{Q}) = \mathrm{Inf}\{|x-y| \mid (x,y)\in \mathrm{P}\times\mathrm{Q}\}$ is not a null set, then for all bounded function h supported in Q, we have:*

$$\|\sigma(\mathrm{D})h\|_{\mathrm{L}^\infty(\mathrm{P})} \leqq \mathrm{C}\left(1-\log^- d(\mathrm{P},\mathrm{Q})\right)\|h\|_{\mathrm{L}^\infty}, \qquad \textit{en posant } \mathrm{Log}^- = \mathrm{Min}(0,\log)$$

Remark. We will find the fact that the Fourier multiplier of Fourier of order 0 certainly do not operate in $\mathbf{L}^\infty$, but this non-operation never appears except through a logarithm.

Proof Denoted by K the kernel of the operator $\sigma(D)$. Since K is the inverse Fourier transform of σ, for all integer index α, $x^\alpha K(x)$ is the inverse Fourier transform of $(-D_\xi)^\alpha\sigma(\xi)$. By hypothesis, when $|\alpha|$ value $d-1$, $(-D_\xi)^\alpha\,\sigma(\xi)$ is integrable and so $x^\alpha K(x)$ is bounded. Moreover, $|\alpha|$ always value $d-1$ we can write

$$x^\alpha \mathrm{K}(x) = \int \chi(|x|\xi)e^{i(x|\xi)}\left(-\mathrm{D}_\xi\right)^\alpha\sigma(\xi)d\xi + \int(1-\chi(|x|\xi))e^{i(x|\xi)}\left(-\mathrm{D}_\xi\right)^\alpha\sigma(\xi)d\xi$$

After integration by parts, the first term becomes

$$\sum_{\beta\leq\alpha}\int C^\beta_\alpha\left(\mathrm{D}_\xi\right)^{\alpha-\beta}(\chi(|x|\xi))x^\beta e^{i(x|\xi)}\sigma(\xi)d\xi$$

It results that, when x is small enough, we have $|x^\alpha\mathrm{K}(x)| \leqq \mathrm{C}|x|$ for all integer index α at length $d+1$. Thus, we have

$$|\mathrm{K}(x)| \leqq \frac{\mathrm{C}}{|x|^d+|x|^{d+1}}$$

For all x in P, and for all bounded function h supported in Q, we have, since $d(P,Q)$ is strictly positive

$$|\sigma(\mathrm{D})h(x)| \leqq \mathrm{C}\int_{|x-y|\geq d(\mathrm{P},\mathrm{Q})}\frac{1}{|x-y|^d+|x-y|^{d+1}}|h(y)|dy.$$

Hence the lemma 7.

Recall the proof of Lemma 5. The application of lemma 7 above, with $P=\mathrm{supp}(1-g)$ and $Q=\mathrm{supp}\,f$, it results, as $d(P,Q) \geqslant \delta/4$, that

$$\left\|(1-g)\partial_i\partial_j(\mathrm{Id}-\chi(\mathrm{D}))\Delta^{-1}f\omega\right\|_{\mathrm{L}^\infty} \leqq \mathrm{C}\|\omega\|_{\mathrm{L}^\infty}(1-\log\delta) \tag{12}$$

$\left\|(1-g)\partial_i\partial_j(\mathrm{Id}-\chi(\mathrm{D}))\Delta^{-1}(1-f)\omega\right\|_{\mathrm{L}^\infty}$ remains to increase. To do this, we are going to use the lemma 4. It is clear that

$$\left\|\partial_i\partial_j(\mathrm{Id}-\chi(\mathrm{D}))\Delta^{-1}(1-f)\omega\right\|_0 \leqq \mathrm{C}\|(1-f)\omega\|_{\mathrm{L}^\infty}.$$

As the support of $1-f$ is included in $\mathbb{R}^2\setminus A$, we have

$$\left\|\partial_i\partial_j(\mathrm{Id}-\chi(\mathrm{D}))\Delta^{-1}(1-f)\omega\right\|_0 \leqq \mathrm{C}\|\omega\|_{\mathrm{L}^\infty(\mathbf{R}^2\setminus\mathrm{A})}.$$

Otherwise, the operator $\partial_i\partial_j(\mathrm{Id}-\chi(\mathrm{D}))\Delta^{-1}$ maps continuously from C^ε to C^ε; so $\|(1-f)\omega\|_\varepsilon$ is increasing. Assuming for example that $x\in\mathrm{supp}(1-f)$, we have

$$|(1-f(x))\omega(x)-(1-f(y))\omega(y)| \leqq |f(x)-f(y)|\|\omega\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A})} + |1-f(y)||\omega(x)-\omega(y)|$$

As $\|f\|_\varepsilon \leqslant C\delta^{-\varepsilon}$, we have $\|(1-f)\omega\|_\varepsilon \leqq \mathrm{C}\delta^{-\varepsilon}\|\omega\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}}$.
The lemma 4, applied with $\alpha=\mathrm{C}\|\omega\|_{\mathrm{L}^\infty(\mathbf{R}^2\setminus\mathbf{A})}$ et $\beta=\mathrm{C}\delta^{-\varepsilon}\|\omega\|_{\varepsilon,\mathbf{R}^2\setminus A}$, the ensures that

$$\left\|(\mathrm{Id}-\chi(\mathrm{D}))\partial_i\partial_j\Delta^{-1}(1-f)\omega\right\|_{\mathbf{L}^\infty} \leqq \mathrm{C}_\varepsilon\|\omega\|_{\mathbf{L}^\infty}\left(1+\log\frac{\|\omega\|_{\epsilon,\mathbf{R}^2\setminus\mathbf{A}}}{\|\omega\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A})}}-\log\delta\right)$$

Then it results from (12) that

$$\|(1-g)\partial_i\partial_j a\|_{\mathbf{L}^\infty} \leqq C\|\omega\|_{\mathbf{L}^\infty}\left(1+\log\frac{\|\omega\|_{\varepsilon,\mathbf{R}^2\setminus A}}{\|\omega\|_{\mathbf{L}^\infty(R^2\setminus A)}}-\log\delta\right) \tag{13}$$

By definition of δ, we have

$$1-\log\delta = 1+\frac{1}{\varepsilon}\log 2+\frac{1}{\varepsilon}\log\frac{\|\mathrm{Y}\|_\varepsilon}{\mathrm{I}(\mathrm{A},\mathrm{Y})}$$

Hence, as $I(A,Y)\leqslant\|Y\|_\varepsilon$, we have, according to (11),

$$\begin{gathered}\|(\mathrm{Id}-\chi(\mathrm{D}))\nabla v\|_{\mathrm{L}^\infty} \leqq \mathrm{C}_\epsilon\tfrac{\|\mathrm{Y}\|^2_{\mathrm{L}^\infty}}{\mathrm{I}(\mathrm{A},\mathrm{Y})^2}\|\omega\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}\\ \times\left(1+\log\tfrac{\|\mathrm{Y}\|_\epsilon\|\omega\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}+\|\mathrm{Y}(x,\mathrm{D})v\|_\varepsilon}{\mathrm{I}(\mathrm{A},\mathrm{Y})\|\omega\|_{\mathrm{L}^\infty\cap\mathbf{L}^p}}+\log\tfrac{\|\omega\|_{\epsilon,\mathrm{R}^2\setminus A}}{\|\omega\|_{\mathrm{L}^\infty(\mathrm{R}^2\setminus\mathrm{A})}}\right)\end{gathered} \tag{14}$$

Now remains the case of low frequencies. It's sufficient to observe that $\|\chi(\mathbf{D})\nabla v\|_{\mathbf{L}^\infty}\leqq\mathbf{C}\|v\|_{\mathbf{L}^\infty}$ and to use (7) to obtain that $\|\chi(\mathbf{D})\nabla v\|_{\mathbf{L}^\infty}\leqq\mathbf{C}\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}$. Decomposing ∇v as $\nabla v=(\mathrm{Id}-\chi(\mathbf{D}))\nabla v+\chi(\mathbf{D})\nabla v$, lemma 5 results in (14).

Remark. The lemma 5 stated above uses nothing but the divergence-free of vector field Y, which didn't involve any restriction in its application. Let X be a divergence-free vector field, the striated structure that it defines is only faithfully described by a vector field collinear with X, unique in the region which we're interested in. The vector field of course don't need to be divergence-free. This is the key to lemma 5, the linked geometric data in striated structure appears "squared", in theorem 3 the data doesn't present the modification by logarithm.

End of the proof of theorem 3 - Then we apply the lemma 5 in a judiciously constructed vector field Y from X. To do this, set $\eta = (I(A,X)/2\|X\|_\varepsilon)^{1/\varepsilon}$, and if ρ denotes a regular function supported in the unite ball of plane and with integral 1 and if A_α denotes the set of points in distance α from A, we set:

$$\tau = \frac{16}{\eta^2}\rho\left(\frac{4}{\eta}\cdot\right) * 1_{\mathrm{A}_{\mathrm{v}/2}} \quad \text{and} \quad \mathrm{Y}(x) = \tau(x)\frac{\mathrm{X}(x)}{|\mathrm{X}(x)|}$$

Immediately noticed that $\|\mathbf{Y}\|_{\mathbf{L}^\infty} = 1(\mathbf{A},\mathbf{Y}) = 1$. Apply with a ε' strictly between 0 and ε(for example $\varepsilon' = \varepsilon/2$), the lemma 5 ensures:

$$\|v\|_{\mathrm{Lip}} \leqq \mathrm{C}\|\omega\|_{\mathrm{L}^\infty} \cap L^p \left\{1 + \log\left(\|\mathrm{Y}\|_\epsilon + \frac{\|\mathrm{Y}(x,\mathrm{D})v\|_{\varepsilon'}}{\|\omega\|_{\mathrm{L}^\infty \cap \mathrm{L}^p}}\right) + \log\frac{\|\omega\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}}}{\|\omega\|_{\mathrm{L}^\infty(\mathbf{R}^2\setminus\mathbf{A})}}\right\}. \tag{15}$$

$\|Y\|_\varepsilon$ and $\|Y(x,\mathbf{D})v\|_{\varepsilon'}$ remains increasing. The increasing of $\|Y\|_\varepsilon$ is easy. We study $Y(x) - Y(y)$ assuming for example that x belongs to the support of τ. A very element calculation ensures that

$$|Y(x) - Y(y)| \leqq |\tau(x) - \tau(y)| + 2|\tau(y)|\frac{|X(x) - X(y)|}{|X(x)|}.$$

As $\mathrm{Inf}\{|\mathbf{X}(x)|, x \in \operatorname{supp}\tau\} \geqq (1/2)\mathrm{I}(\mathrm{A},\mathrm{X})$ and that $\|\tau\|_{\mathbf{L}^\infty} \leqq 1$, it follows:

$$|Y(x) - Y(y)| \leqq \|\tau\|_\varepsilon|x-y|^\varepsilon + 2|\tau(y)|\frac{|X(x) - X(y)|}{|X(x)|}$$

It results from the fact that $\|\tau\|_\varepsilon \leqq \mathrm{C}\eta^{-\varepsilon}$ and the definition of η that

$$\|\mathrm{Y}\|_\varepsilon \leqq \mathrm{C}\frac{\|\mathrm{X}\|_\varepsilon}{\mathrm{I}(\mathrm{A},\mathrm{X})}. \tag{16}$$

Notice that it is possible to increase $\|Y(x,\mathbf{D})v\|_\varepsilon$ by $\|\mathbf{X}\|_\varepsilon$, $\|\mathbf{X}(x,\mathbf{D})\omega\|_{\varepsilon-1}$ and $\|\omega\|_{L^\infty}$. On the other hand, we can, for all $\varepsilon' < \varepsilon$, increase $\|Y(x,\mathbf{D})v\|_\varepsilon$ by $\|\mathbf{X}\|_\varepsilon$, $\|\mathbf{X}(x,\mathbf{D})\omega\|_{\varepsilon-1}$ and $\|\omega\|_{L^\infty}$ as will be shown in the lemma 8.

Observing that $\mathbf{Y}(x, \mathbf{D})v = \frac{\tau(x)}{|\mathbf{X}(x)|}\mathbf{X}(x, \mathbf{D})v$, it follows that

$$\|\mathrm{Y}(x, \mathrm{D})v\|_{\varepsilon'} \leqq \mathrm{C}_{\varepsilon'} \frac{\|\mathrm{X}\|_{\varepsilon}\|\mathrm{X}(x, \mathrm{D})v\|_{\varepsilon'}}{\mathrm{I}(\mathrm{A}, \mathrm{X})^2}, \quad \text{et ce pour tout } \varepsilon' < \varepsilon \tag{17}$$

The increasing of $\|X(x, \mathbf{D})v\|_{\varepsilon'}$ is done using the lemma below.

Lemma 8. *There exist 2 operators W_1 and W_2 operating on a pairs of divergence-free vector field, with values in the vector field such that, for all pairs (ε, p) in $(0,1) \times (1,2)$, there exists a constant C such that given v and X 2 divergence-free vector field, if ω is the vortex of v, we have $X(x, \mathbf{D})v = W_1(X, v) + W_2(X, v)$ with:*

1. $\|\mathbf{W}_1(\mathbf{X}, v)\|_{\epsilon} \leqq \mathbf{C}\|\mathbf{X}(x, \mathbf{D})\omega\|_{\epsilon-1}$,
2. *for all* $\varepsilon' < \varepsilon, \|\mathbf{W}_2(\mathbf{X}, v)\|_{e'} \leqq \mathbf{C}_{\boldsymbol{\varepsilon}'}\|\mathbf{X}\|_{\boldsymbol{e}}\|\omega\|_{\mathbf{L}^\infty \cap \mathbf{L}^p}$,
3. $\|\mathbf{W}_2(\mathbf{X}, v)\|_{\varepsilon} \leqq \mathbf{C}\|\mathbf{X}\|_{\varepsilon}\|v\|_{\mathbf{Lip}}$.

Proof - We will use the dyadic cutting of the frequency space to prove the following formula:

$$\mathbf{X}(x, \mathbf{D})v = \sum_{i=1}^{4} \mathbf{V}_i,$$

with

$$\mathrm{V}_1 = (\mathrm{Id} - \chi(\mathrm{D}))\nabla^{\perp}\Delta^{-1}\mathrm{X}(x, \mathrm{D})\omega$$
$$\mathrm{V}_2 = \left[\mathrm{T}_{\mathbf{x}}, \nabla^{\perp}\Delta^{-1}\right]\omega$$
$$\mathrm{v}_3 = -(\mathrm{Id} - \chi(\mathrm{D}))\nabla^{\perp}\Delta^{-1}\textstyle\sum_{j=1}^{2}\left\{\mathrm{T}\partial_j\omega\mathrm{X}^j + \partial_j\mathrm{R}\left(\omega, \mathrm{X}^j\right)\right\}$$

and

$$\mathrm{V}_4 = \sum_{j=1}^{2}\left\{\mathrm{T}\partial_j v\mathrm{X}^j + \partial_j\mathrm{R}\left(v, \mathrm{X}^j\right)\right\}$$

The lemma 6 point 5 applied with $A = (Id - \chi(D))\nabla^{\perp}\Delta^{-1}$ and $u = \omega$ shows that we have

$$\mathrm{X}(x, \mathrm{D})(\mathrm{Id} - \chi(\mathrm{D}))v = \sum_{i=1}^{3}\mathrm{V}_i + \mathrm{V}_4'$$

with

$$\mathrm{V}_4' = \sum_{j=1}^{2}\left\{\mathrm{T}_{\partial_j(\mathrm{d}-\chi(\mathrm{D}))}\mathrm{X}^j + \partial_j\mathrm{R}\left((\mathrm{Id} - \chi(\mathrm{D}))v, \mathrm{X}^j\right)\right\}.$$

According to (3), $T_x\chi(\mathbf{D}) = 0$; moreover, X is divergence-free, so

$$\mathbf{X}(x, \mathbf{D})\chi(\mathbf{D})v = \sum_{j=1}^{2}\left\{\mathrm{T}\partial_j\chi(\mathbf{D})v\mathbf{X}^j + \partial_j\mathbf{R}\left(\chi(\mathbf{D})v, \mathbf{X}^j\right)\right\}.$$

Hence the desired formula; then we set $\mathbf{W}_1(\mathbf{X}, v) = \mathbf{V}_1$ and $\mathbf{W}_2(\mathbf{X}, v) = \mathbf{X}(x, \mathbf{D})v - \mathbf{W}_1$.

It is immediate that $\|V_1\|_\varepsilon \leqq C\|X(x,D)\omega\|_{\varepsilon-1}$. Moreover, the lemma 6 ensures that we have $\|\mathbf{V}_i\|_\varepsilon \leqq \mathbf{C}\|\mathbf{X}\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty}$ for i valued 2 or 3 and that $\|V_4\|_\varepsilon \leqq C\|X\|_\varepsilon\|v\|_{\mathrm{Lip}}$. Moreover, $\|\partial_j v\|_{-(\varepsilon-\varepsilon')} \leqq C\|v\|_1$. Thus, according to (7) and the lemma 6, we have $\|\mathbf{V}_4\|_{\varepsilon'} \leqq \mathbf{C}\|\mathbf{X}\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p}$, hence the lemma 8.

We deduce from the lemma that

$$\|\mathbf{X}(x,\mathbf{D})v\|_{\varepsilon'} \leqq \mathbf{C}\|\mathbf{X}(x,\mathbf{D})\omega\|_{\varepsilon-1} + \|\mathbf{X}\|_\varepsilon\|\omega\|_{\mathbf{L}^\infty\cap\mathbf{L}^p} \tag{18}$$

Then the theorem 3 results from (15)-(18).

3 A priori non-linear estimation

This part is the dynamic intervenes. We consider in this paragraph a regular global solution of the system (E). More precisely, we know that, if the vector field at the initial moment belongs to C^r for all $r>0$ and has its curl in L^p, we there exists a global solution locally bounded in time to value in C^r for all $r>0$ (see [2]). The goal of this paragraph is the following proof of a priori estimation:

Theorem 9. *Let $\varepsilon>0$ and $p\in[1,2)$, there exists C such that for all vector field v as a $\mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbf{R};\mathrm{C}^\infty_b\left(\mathbf{R}^2\right)\right)$ solution to* (E)*, all divergence-free vector field X_0 and class C^ε and all bounded set A^0 on plan such that $I(A^0,X_0)$ is strictly positive, we have*

$$\|v(t,.)\|_{\mathrm{Lip}} \leqq \mathrm{C}\,\|\omega_0\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}\left(1+\log N_{\varepsilon,p}\left(\mathrm{A}^0,\mathrm{X}_0,\omega_0\right)\right)\exp\left(\mathrm{C}t\,\|\omega_0\|_{\mathrm{L}^\infty}\right),$$

with, as in theorem 3

$$\mathrm{N}_{\varepsilon,p}(\mathrm{A},\mathrm{X},\omega) = \frac{\|\mathrm{X}\|_\varepsilon\|\omega\|_{\varepsilon,\mathbf{R}^2\setminus\mathrm{A}^0}}{\mathrm{I}(\mathrm{A},\mathrm{X})\|\omega\|_{\mathrm{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)}}\left(1+\frac{\|\mathrm{X}(x,\mathrm{D})\omega\|_{\varepsilon-1}}{\mathrm{I}(\mathrm{A},\mathrm{X})\|\omega\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}}\right)$$

Proof. - Let us define, for each time, a vector field X, and a bounded A^t which we can apply, at each moment, the theorem 3 with profit.

So we defines X_t and A^t by

$$\mathbf{A}^t = \psi\left(t,\mathbf{A}^0\right) \tag{19}$$

$$\mathbf{X}_t = (\psi_t)_*\mathbf{X}_0 \quad \text{i.e. } (\mathbf{X}_t)^i = (\mathbf{X}_0(x,\mathbf{D})\psi)^i\left(t,\psi^{-1}(t,x)\right) \tag{20}$$

It is necessary to control the quantities appearing on the RHS of inequality of theorem 3. The key point of demonstrating is:

$$\mathbf{N}_{\varepsilon,p}\left(\mathbf{A}^t,\mathbf{X}_t,\omega_t\right) \leqq \mathbf{N}_{\varepsilon,p}\left(\mathbf{A}^0,\mathbf{X}_0,\omega_0\right)\exp\left(\mathbf{C}\int_0^t\|v(s,.)\|_{\mathbf{Lip}}ds\right) \tag{21}$$

Increasing of $\|\omega(t,.)\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^t}/\|\omega(t,.)\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^t)}$. - According to the conservation of vortex along the fluid lines (0.1), we have

$$\omega(t,x)-\omega(t,y)=\omega_0\left(\psi^{-1}(t,x)\right)-\omega_0\left(\psi^{-1}(t,y)\right).$$

thus

$$\|\omega(t,.)\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^t}\leqslant\|\omega_0\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^0}\exp\left(\varepsilon\int_0^t\|v(s,.)\|_{\mathbf{Lip}}ds\right).$$

However by the definition of A^t, it's clear that

$$\|\omega(t,.)\|_{\mathrm{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^t)}=\|\omega_0\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)}$$

Then it follows that

$$\frac{\|\omega(t,.)\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^t}}{\|\omega(t,.)\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^t)}}\leqq\frac{\|\omega_0\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^0}}{\|\omega_0\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)}}\exp\left(\varepsilon\int_0^t\|v(s,.)\|_{\mathbf{Lip}}ds\right)\tag{22}$$

Decreasing of $I(A^t,X_t)$. - By derivation of following flow X_0, we obtain:

$$\partial_t\mathbf{X}_0(x,\mathbf{D})\psi(t,x)=\nabla v(t,\psi(t,x))\cdot\mathbf{X}_0(x,\mathbf{D})\psi(t,x)$$

Integrating above equation between t and 0, it becomes:

$$|\mathbf{X}_0(x)|\leqq|\mathbf{X}_0(x,\mathbf{D})\psi(t,x)|\exp\int_0^t\|v(s,.)\|_{\mathrm{Lip}}\,ds$$

Increasing of $\|X_t\|_\varepsilon$ and $\|X_t(x,D)\omega\|_{\varepsilon-1}$. It is described by the following proposition:

Proposition 10. *For all ε in the interval $(0,1)$, there exists a constant C, such that*

1.
$$\|\mathbf{X}_t(x,\mathbf{D})\omega(t,.)\|_{\varepsilon-1}\leqq\mathbf{C}\,\|\mathbf{X}_0(x,\mathbf{D})\omega_0\|_{\varepsilon-1}\exp\left(\mathbf{C}\int_0^t\|v(s,.)\|_{\mathbf{Lip}}ds\right)$$

2.
$$\|\mathbf{X}_t\|_\varepsilon\leqq\mathbf{C}\left(\|\mathbf{X}_0\|_\varepsilon+\frac{\|\mathbf{X}_0(x,\mathbf{D})\omega_0\|_{\varepsilon-1}}{\|\omega_0\|_{\mathbf{L}^\infty}}\right)\exp\left(\mathbf{C}\int_0^t\|v(s,.)\|_{\mathbf{Lip}}ds\right)$$

Proof - the key point is the following lemma of propagation of the holder regularity, lemma which carefully describes the necessary regularity on the vector field propagating the regularity:

Lemma 11. *Let r be a non-zero in the interval $(-1,1)$ and F a bilinear map defined on $Lip(\mathbb{R}^2;\mathbb{R}^2)\times C^r$ such that $\|F(v,f)\|_r\leqslant C\|f\|_r\|v\|_{\mathbf{Lip}}$. There exists a constant C_1 such that for all divergence-free Lipschitz vector field v such that $V(s)=\|v(s,\cdot)\|_{\mathbf{Lip}}\geqslant V>0$ and for all pair (f,h) of functions of $L^\infty_{\mathbf{loc}}(\mathbb{R};C^r)$ satisfies*

$\partial_t f + v \cdot \nabla f = \mathrm{F}(v, f) + h$, *we have:*

$$\|f(t,.)\|_r \leqq \mathrm{C}_1 \left(\|f(0,.)\|_r + \frac{1}{\mathrm{V}} \sup_{s\in[0,t]} \|h(s,.)\|_r \right) \exp\left(\mathrm{C}_1 \int_0^t \mathrm{V}(s) ds \right)$$

Remark. The hypothesis $V(s) \geqslant V > 0$ is useless if $h = 0$. The reason of this hypothesis (hypothesis of course verified in the case of solutions of the system of incompressible Euler where we have $2V(s) \geqslant \|w_0\|_{L^\infty}$) remains in our commitment to obtain suitable homogeneity in the relative term of the initial data in the point 2. of the proposition 10.

Proof - The first thing to do is to come down to the easy case where r is strictly included between 0 and 1. We define the operator $\Lambda = \chi(\mathrm{D}) + (\mathrm{Id} - \chi(\mathrm{D}))|\mathrm{D}|^{-1}$ if $r < 0$ and by $\Lambda = Id$ if $r > 0$. Thus we have $\partial_t \Lambda f + v.\nabla \Lambda f = \Lambda f(v, f) + \Lambda h + [v \cdot \nabla, \Lambda] f$. According to the lemma 6 5., applied with $X = v \cdot \nabla$, and $A = \Lambda$, we have

$$\begin{aligned} &\partial_t \Lambda f + v \cdot \nabla \Lambda f \\ &\quad = \Lambda \mathrm{F}(v, f) + \Lambda h + [\mathrm{T}_v, \Lambda] f + \textstyle\sum_j \left\{ \mathrm{T}_{\partial j \Lambda f} v^j + \partial_j \mathrm{R}\left(\Lambda f, v^j\right) \right\} - \Lambda \sum_j \left\{ \mathrm{T}_{\partial j} v^j + \partial_j \mathrm{R}\left(f, v^j\right) \right\} \end{aligned}$$

Hence, by setting $g(t) = \Lambda f(t)$ and $l(t) = \Lambda h(t)$, it follows:

$$\partial_t g + v \cdot \nabla g = l + G(v, g) \tag{23}$$

with

$$\begin{aligned} \mathrm{G}(v, g) &= \Lambda \mathrm{F}\left(v, \Lambda^{-1} g\right) + [\mathrm{T}_v, \Lambda]\Lambda^{-1} g \\ &\qquad + \sum_j \left\{ \mathrm{T}_{\partial_j g} v^j + \partial_j \mathrm{R}\left(g, v^j\right) \right\} - \Lambda \sum_j \left\{ \mathrm{T}_{\partial_j \Lambda^{-1} g} v^j + \partial_j \mathrm{R}\left(\Lambda^{-1} g, v^j\right) \right\} \end{aligned}$$

Let's set $r' = r$ if $r = 0$ and $r' = r + 1$ if $r < 0$. According to the lemma 6, it is clear that $\|\mathbf{G}(v, g)\|_{r'} \leqq \mathbf{C}\|v\|_{\mathbf{Lip}} \|g\|_{r'}$. Moreover, by simple integration of the equation along the characteristics, it follows:

$$g(t,x) = g_0\left(\psi^{-1}(t,x)\right) + \int_0^t \left\{ G(v,g)\left(s, \psi\left(s, \psi^{-1}(t,x)\right)\right) ds + l\left(s, \psi\left(s, \psi^{-1}(t,x)\right)\right) \right\} ds.$$

Moreover it is clear that $\left\|\nabla \left\{ \psi\left(s, \psi^{-1}(t,.)\right) \right\}\right\|_{\mathbf{L}^\infty} \leqq \exp \int_s^t \mathbf{V}(s) ds$, thus, according to the relation (23), we have

$$\begin{aligned} \|g(t,.)\|_{r'} &\leqq \|g(0,.)\|_{r'} \exp\left(r' \int_0^t \mathbf{V}(s) ds \right) \\ &\quad + \mathrm{C} \int_0^t \|g(s,.)\|_{r'} \mathbf{V}(s) \exp\left(r' \int_s^t \mathbf{V}\left(s'\right) ds' \right) ds + \int_0^t \|l(s)\|_{r'} \exp\left(r' \int_s^t \mathbf{V}\left(s'\right) ds' \right) ds \end{aligned}$$

We will multiply above integral by $-\lambda \int_0^t V(s)ds$, λ is a real number strictly larger than r'.

Let's

$$G_{r',\lambda}(t) = \mathrm{Sup}_{s\in[0,t]} \left\{ \|g(s,.)\|_{r'} \exp\left(-\lambda \int_0^s V\left(s'\right) ds'\right)\right\}$$

and

$$\mathbf{H}_{r'\lambda}(t) = \mathrm{Sup}_{s\in[0,t]} \left\{ \|h(s,.)\|_{r'} \exp\left(-\lambda \int_0^s \mathbf{V}\left(s'\right) ds'\right)\right\}.$$

By using that $V(s) > V$, it follows that

$$\mathrm{G}_{r',\lambda}(t) \leqq \|g(0,.)\|_{r^r} + \left(\mathrm{C}'\mathrm{G}_{r',\lambda}(t) + \frac{\mathrm{H}_{r',\lambda}(t)}{\mathrm{V}}\right) \int_0^t \mathrm{V}(s) \exp\left((\lambda - r') \int_t^s \mathrm{V}\left(s'\right) ds'\right) ds.$$

However, a immediate integral ensures that

$$\int_0^t \mathrm{V}(s) \exp\left((\lambda - r') \int_t^s \mathrm{V}\left(s'\right) ds'\right) ds \leqslant \frac{1}{\lambda - r'}.$$

It results that, for all $\lambda > C' + r'$.

$$G_{r',\lambda}(t) \leqq \frac{\lambda - r'}{\lambda - r' - \mathrm{C}'} \|g(0,.)\|_{r'} + \frac{\mathrm{H}_{r',\lambda}(t)}{\mathrm{V}\left(\lambda - r' - \mathrm{C}'\right)} \tag{24}$$

As there exists a real number $\alpha > 1$ such that, for all t, $\alpha^{-1}\|g(t)\|_{r'} \leqslant \|f(t)\|_t \leqslant \alpha\|g(t)\|_{r'}$, the lemma 11 results from (24) with the choice by example $\lambda = C' + 3$.

Back to the proposition 10. By deriving the equation of the flow along X_0, it follows, according to the relation (20) of definition of X_t,

$$\partial_t \mathbf{X}_t + v\nabla \mathbf{X}_t = \mathbf{X}_t(x, \mathbf{D})v \tag{25}$$

The above relation simply means that the vector fields $\partial_t + v\cdot\nabla$ and X_t commutes. It follows that:

$$\partial_t \mathbf{X}_t(x, \mathbf{D})\omega + v.\nabla \mathbf{X}_t(x, \mathbf{D})\omega = 0. \tag{26}$$

The increasing of $\|X_t(x,D)\omega\|_{\varepsilon-1}$ is then only a direct application of the Lemma 11 with $f(t,x) = X_t(xD)\omega(t,x)$, $F = 0$ and $h = 0$.

To increase $\|X_t\|_\varepsilon$, we need to return to the lemma 8. With the notation in that lemma, we have $X_t(xD)v = W_1 + W_2$. According to the point 1. that we have just proved, the application of lemma 11 with $h = W_1$ and $F = W_2$ (reasonable because $2\|v(s,\cdot)\|_{\mathbf{Lip}} \geqslant 2V = \|\omega_0\|_{L^\infty}$) ensures that point 3 and also the proposition.

In the proposition 10 and the inequality (**??**) and (**??**), we deducted the inequality (21). The theorem 3 ensures that

$$\begin{aligned}&\|v(t,.)\|_{\mathrm{Lip}}\\&\leqq C\left\|\omega_0\right\|_{L^\infty\cap L^p}\left(1+\log N_{\varepsilon,p}\left(A^0,X_0,\omega_0\right)\right)+C\left\|\omega_0\right\|_{L^\infty}\int_0^t\|v(s,.)\|_{\mathrm{Lip}}\,ds\end{aligned}\tag{27}$$

The theorem 9 follows the integration of the inequality (27) above.

Remark. If $X_0(xD)\omega_0$ belongs to L^∞, the proposition 10 1. is then useless. Indeed, according to the relation (26), we have in this case $\left\|X_0(x,D)\omega_0\right\|_{L^\infty}=\left\|X_t(x,D)\omega_t\right\|_{L^\infty}$.

4 Proof of the Theorem 2

We proceed in a manner analogous to that of [6]. The 2 steps of the demonstration is to maintain the regularization of the initial data and the passage at the limit of the solutions associated with the regularized initial data.

1. Regularization of initial data. - Let v_0 be a given initial data verified the hypotheses of theorem 2, we consider a function $\theta\in C_0^\infty(B(0,\alpha))$, positive and with integral 1, α is to be chosen. Then we set $\theta_n=(1+n)^{-2}\theta((1+n))$ and $v_{0,n}(\text{resp}.\omega_{0,n})=\theta_n * v_0(\text{resp}.w_0)$. Since v_0 is Lipschitz, the sequence $(v_{0,n})_{n\in N}$ converge to L^∞. Moreover, it is clear that

$$\left\|\omega_{0,n}\right\|_{L^p}\leqq\left\|\omega_0\right\|_{L^p}\ \text{ and that }\ \left\|\omega_{0,n}\right\|_{L^\infty}\leqq\left\|\omega_0\right\|_{\mathrm{L}^\infty}\tag{28}$$

We choose α sufficiently small such that, on the set denotes as A_α^0 of points with distance less than α from A^0, we have

$$2\mathrm{I}\left(\mathrm{A}_\alpha^0,\mathrm{X}_0\right)\geqq\mathrm{I}\left(\mathrm{A}^0,\mathrm{X}_0\right).\tag{29}$$

In addition, if x and x' are 2 points of $\mathbb{R}\setminus A_\alpha^0$,then for all y in $B(0,\alpha)$, $x-y$ and $x'-y$ is in $\mathbb{R}^2\setminus A^0$;so it is clear that

$$\left\|\omega_{0,n}\right\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}_\alpha^0)}\leqq\left\|\omega_0\right\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)}\qquad\text{and}\qquad\left\|\omega_{0,n}\right\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}_\alpha^0}\leqq\left\|\omega_0\right\|_{\varepsilon,\mathbf{R}^2\setminus\mathbf{A}^0.}\tag{30}$$

Moreover, it must be shown that the sequence $(X_0(xD)\omega_{0,n})_{n\in\mathbb{N}}$ is a bounded sequence in $C^{\varepsilon-1}$. By setting $\Theta_n a=\theta_n * a$, it follows, according to the lemma 6 5. applied with $X=X_0$, $u=\omega$ and $A=\Theta_n$,

$$\left[\mathbf{X}_0(x,\mathbf{D}),\mathbf{\Theta}_n\right]\omega_0=\left[\mathbf{T}_{\mathbf{X}_0},\mathbf{\Theta}_n\right]\omega_0+\mathbf{R}_0\left(\mathbf{X}_0,\omega_0\right)$$

with

$$\mathbf{R}_0\left(\mathbf{X}_0,\omega_0\right)=\sum_j \mathbf{T}_{\partial j\theta_n \mathbf{w}_0}\mathbf{X}_0^j+\partial_j\mathbf{R}\left(\mathbf{X}_0^j,\mathbf{\Theta}_n\omega_0\right)-\Theta_n\left(\mathbf{T}_{\partial_j\omega_0}\mathbf{X}_0^j+\partial_j\mathbf{R}\left(\mathbf{X}_0^j,\omega_0\right)\right)$$

The it follows from the lemma (6) 2. and 4. that we have

$$\|\mathbf{X}_0(x,\mathbf{D})\omega_{0,n}\|_{\varepsilon-1}\leqq \mathbf{C}\left(\|\mathbf{X}_0(x,\mathbf{D})\omega_0\|_{\varepsilon-1}+\|\mathbf{X}_0\|_{\varepsilon}\|\omega_0\|_{\mathbf{L}^\infty}\right) \tag{31}$$

Finally, since the quantities $\|w_0\|_{L^\infty}$ and $\|w_0\|_{L^\infty(\mathbb{R}^2\setminus A^0)}$ appear in the denominator in the inequality of the theorem 9, it is interesting to reduce them. The sequence $(\omega_{0,n})_{n\in\mathbb{N}}$ converge weakly to ω_0 in L^∞. The estimations (28) and (30) combined with the weak compactness of the L^∞ ensures that $\lim\|\omega_{0,n}\|_{L^\infty(\mathbb{R}^2\setminus A^0)}=\|\omega_0\|_{L^\infty(\mathbb{R}^2\setminus A^0)}$ and $\lim\|\omega_{0,n}\|_{L^\infty}=\|\omega_0\|_{L^\infty}$. Thus, since elsewhere $\omega_{0,n}$ tend strongly towards ω_0 in L^p, we can, taking n large enough, suppose

$$\begin{gathered}\|\omega_{0,n}\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)}\geqslant(1/2)\|\omega_0\|_{\mathbf{L}^\infty(\mathbf{R}^2\setminus\mathbf{A}^0)},\quad \|\omega_{0,n}\|_{\mathbf{L}^\infty}\geqq(1/2)\|\omega_0\|_{\mathbf{L}^\infty}\\ \text{and}\\ \|\omega_{0,n}\|_{L^p}\geqq(1/2)\|\omega_0\|_{L^p.}\end{gathered} \tag{32}$$

Then it follows some estimations (28)-(32) that there exists a constant C such that

$$\mathbf{N}_{\varepsilon,p}\left(\mathbf{A}^0,\mathbf{X}_{0,n},\omega_{0,n}\right)\leqq \mathbf{C}\mathbf{N}_{\varepsilon,p}\left(\mathbf{A}^0,\mathbf{X}_0,\omega_0\right)$$

The theorem 9, applied with A^0_α, X_0 and $\omega_{0,n}$, then immediately ensures that

$$\|v_n(t,.)\|_{\mathrm{Lip}}\leqq \mathrm{C}\|\omega_0\|_{\mathrm{L}^\infty\cap\mathrm{L}^p}\left(1+\log\mathrm{N}_{\varepsilon,p}\left(\mathrm{A}^0,\mathrm{X}_0,\omega_0\right)\right)\exp\left(\mathrm{C}t\|\omega_0\|_{\mathrm{L}^\infty}\right) \tag{33}$$

So we have a uniform and global control of the Lipschitz norm of the vector field v_n. We can move to the second step.

2. Pass to the limit. - It is a question of demonstrating that the sequence $(v_n)_{n\in\mathbb{N}}$ is Cauchy in a suitable space. Here we follow the process in [7] by considering the operator $\pi(v,w)=\nabla\Delta^{-1}(\mathrm{tr}(dvdw))$, where Δ^{-1} denotes the convolution with $\log|x|$. It is trivial to verify that a solution of the system (E) is a solution of the system (E') where ∇p is replaced by $\pi(v,v)$. The following lemma ensures the definition of all operator π independently to decrease to infinity of vector fields v and w.

Lemma 12. *For all $r\in(-1,1)$, there exists a constant C such that, for all divergence-free vector field v and w, we have*

$$\|\pi(v,w)\|_r\leqq \mathrm{C}\|v\|_{Lip}\|w\|_r$$

Proof. - It is of course important to use the divergence free of the vector field. Thanks to the nullity, we can write $\pi(v,w)=\sum_i \pi_i(v,w)$ with

$$\begin{gathered}
\pi_1(v,w)=\sum_{i,j}(\mathrm{Id}-\chi(\mathrm{D}))\nabla\Delta^{-1}\partial_j \mathrm{T}_{\partial_i v^j} w^i \\
\pi_2(v,w)=\sum_{i,j}(\mathrm{Id}-\chi(\mathrm{D}))\nabla\Delta^{-1}\partial_i \mathrm{T}_{\partial_j w^i} v^j \\
\pi_3(v,w)=\sum_{i,j}(\mathrm{Id}-\chi(\mathrm{D}))\nabla\Delta^{-1}\partial_i\partial_j \mathrm{R}\left(v^j,w^i\right) \\
\pi_4(v,w)=\sum_{i,j}\chi(\mathrm{D})\nabla\partial_i\partial_j \int \chi(x-y)\log|x-y|\mathrm{R}\left(v^j,w^i\right)(y)dy
\end{gathered}$$

and

$$\pi_5(v,w)=\sum_{i,j}\chi(\mathrm{D})\int \nabla\partial_i\partial_j(1-\chi(x-y))\log|x-y|\mathrm{R}\left(v^j,w^i\right)(y)dy$$

The high frequencies part does not pose any problems. It is sufficient to apply lemma 6 to obtain that $\|\pi_i(v,w)\|_r \leqq \mathrm{C}\|v\|_{\mathrm{Lip}}\|w\|_r$ for $i=1,2$ or 3. Since the Fourier transform of π_4 and π_5 is supported in a fixed compact set, it is sufficient, to increase $\|\pi_4(v,w)\|_r$ and $\|\pi_5(v,w)\|_r$ to increase their L^∞ norm. However, the functions $\chi(x)\log|x|$ and $\nabla\partial_i\partial_j(1-\chi(x))\log|x|$ are integrable functions, thus, for $i=4,5$, we have $\|\pi_i(v,w)\|_r \leqq \mathrm{C}\mathrm{Sup}_{i,j}\left\|\mathrm{R}\left(v^i,w^j\right)\right\|_{\mathrm{L}^\infty}$. Then, according to lemma 6

$$\left\|\mathbf{R}\left(v^i,w^j\right)\right\|_{\mathbf{L}^\infty} \leqq \mathbf{C}\|v\|_{\mathbf{Lip}}\|w\|_r;$$

hence the lemma 12.

We will now be able to increase the difference v_n-v_m. WE have the following equation

$$\partial_t\left(v_n-v_m\right)+v_n\cdot\nabla\left(v_n-v_m\right)=\pi\left(v_n-v_m,v_n+v_m\right)+\left(v_n-v_m\right)\cdot\nabla v_m \tag{34}$$

According to (32), we can apply, for n big enough, the inequality (24) with $v=v_n$, $\mathrm{V}=(1/4)\|\omega_0\|_{\mathrm{L}^\infty}$ and $h=\pi\left(v_n-v_m,v_n+v_m\right)+\left(v_n-v_m\right).\nabla v_m$. Thus, if $\lambda>r+1+C$,

$$M_{n,m}(t,\lambda)\leqq C\left\|v_{0,n}-v_{0,m}\right\|_r+\frac{C}{V(\lambda-r-1-C)}M_{n,m}(t,\lambda)$$

with

$$\mathbf{M}_{n,m}(t,\lambda)=\mathrm{Sup}_{0\leqq s\leq t}\left\|v_n(s,.)-v_m(s,.)\right\|_r\exp\left(-\lambda\int_0^t\left\|v_n(s,.)\right\|_{\mathrm{Lip}}ds\right)$$

By choosing λ big enough, we deduce the existence of a real number C such that, for all t, we have

$$\left\|v_n(t,.)-v_m(t,.)\right\|_r\leqq \mathrm{C}\left\|v_{0,n}-v_{0,m}\right\|_r\exp\left(\mathrm{C}\int_0^t\left\|v_n(s,.)\right\|_{\mathrm{Lip}}ds\right)$$

It follows the (33) that the sequence $(v_n)_{n\in N}$ is a Cauchy sequence in $L^\infty_{loc}(\mathbb{R};C^r)$. So, by interpolation with the inequality (33), the sequence $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_{loc}(\mathbb{R};C^r)$ for all real number strictly positive and smaller than 1.

To conclude the proof of the theorem 2, it remains to prove the properties of the regularity of the solution built by passing the limit. We will recall without proof of the very easy lemma 2.4 of [6].

Lemma 13. *Let $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_{1oc}\left(\mathbf{R};\mathrm{Lip}\left(\mathbf{R}^2\right)\right)$ converge to v in$L^\infty_{loc}(\mathbb{R};C^r)$ for r strictly inferior to 1. If ψ(resp. ψ_n) denotes the flow associated with v_n(resp. v), then for all r strictly smaller than 1, $\lim(\psi_n)_{n\in\mathbb{N}}=\psi$ and $\lim\left(\left(\psi_n\right)^{-1}\right)_{n\in\mathbb{N}}=\psi^{-1}$ in $\mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbf{R};\mathrm{Id}+\mathrm{C}'\right)$.*

Since the field X_0 is divergence-free, $\mathbf{X}_0(x,\mathbf{D})\psi_n=\Sigma_j\partial_j\left(\mathbf{X}_0^j\psi_n\right)$; according to lemma **??** above and lemma 6 3., it is clear that $\lim(X_0^j\psi_n)_{n\in\mathbb{N}}=X_0^j\psi$ in $L^\infty_{loc}(\mathbb{R};C^\varepsilon)$. Thus $\lim\left(\mathrm{X}_0(x,\mathrm{D})\psi_n\right)_{n\in\mathbb{N}}=\mathrm{X}_0(x,\mathrm{D})\psi$ in $\lim(X_0^j\psi_n)_{n\in\mathbb{N}}=X_0^j\psi$ in $L^\infty_{loc}(\mathbb{R};C^{\varepsilon-1})$. Then, according to the theorem 9 and the proposition 10 2., the sequence $(X_0(x,D)\Psi)n)_{n\in\mathbb{N}}$ is a bounded sequence in $\lim(X_0^j\psi_n)_{n\in\mathbb{N}}=X_0^j\psi$ in $L^\infty_{loc}(\mathbb{R};C^\varepsilon)$. It follows that

$$\left\{\begin{array}{c}\lim\left(\mathrm{X}_0(x,\mathrm{D})\psi_n\right)_{n\in\mathbb{N}}=\mathrm{X}_0(x,\mathrm{D})\psi \text{ dans } \mathrm{L}^\infty_{\mathrm{loc}}\ (\mathbf{R};\mathrm{C}^r)\qquad\text{for all } r<\varepsilon\\ \text{so } \mathrm{X}_0(x,\mathrm{D})\psi\in\mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbf{R};\mathrm{C}^\varepsilon\right)\qquad\text{and}\qquad \mathrm{X}_t\in\mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbf{R};\mathrm{C}^\varepsilon\right)\end{array}\right.\tag{35}$$

The point 1 of theorem 1 is thus proved. Let us now show that

$$\lim\left(X_{n,t}\right)_{n\in\mathbb{N}}=X_t \text{ dans l'espace } L^\infty_{1oc}\left(\mathbf{R};C^{\varepsilon'}\right)\qquad\text{for all } \varepsilon'<\varepsilon\tag{36}$$

By definition of vector field $X_{n,t}$ and X_t, it follows

$$\begin{aligned}\mathbf{X}_{n,t}(x)-\mathbf{X}_t(x)=\mathbf{X}_0(x,\mathbf{D})\psi_n\left(t,\left(\psi_n\right)^{-1}(t,x)\right)-\mathbf{X}_0(x,\mathbf{D})\psi_n\left(t,\psi^{-1}(t,x)\right)\\ +\left(\mathbf{X}_0(x,\mathbf{D})\psi_n-\mathbf{X}_0(x,\mathbf{D})\psi\right)\left(t,\psi^{-1}(t,x)\right)\end{aligned}.$$

Hence the following inequality:

$$\begin{aligned}\|\mathbf{X}_{n,t}-\mathbf{X}_t\|_{\mathbf{L}^\infty}\leqq\|\mathbf{X}_0(x,\mathbf{D})\psi_n(t,\cdot)\|_\varepsilon\left(\left\|\psi_n^{-1}(t,.)-\psi^{-1}(t,\cdot)\right\|_{\mathbf{L}^\infty}\right)^\varepsilon\\ +\|\mathbf{X}_0(x,\mathbf{D})\psi_n(t,.)-\mathbf{X}_0(x,\mathbf{D})\psi(t,.)\|_{\mathbf{L}^\infty}\end{aligned}$$

Assertion (35) says that the sequence $(X_0(x,D)\psi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty_{loc}(\mathbb{R};C^\varepsilon)$ converge to $X_0(x,D)\psi$ in the space $L^\infty_{loc}(\mathbb{R};C^{\varepsilon'})$ for all $\varepsilon'<\varepsilon$. The lemma **??** and the fact that the sequence $(X_{n,t})_{n\in\mathbb{N}}$ is bounded in the space $L^\infty_{loc}(\mathbb{R};C^\varepsilon)$ then allow to immediately deduce the property (36) from the inequality above.

We must now study the convergence of the sequence $(X_{n,t}(x,D)\omega_n)_{n\in\mathbb{N}}$. Let's remember that, since $\lim(v_n)_{n\in\mathbb{N}} = v$ in $L^\infty_{loc}(\mathbb{R};C^r)$, for all $r<1$, we have

$$\lim\left(\omega_n\right)_{n\in\mathrm{N}} = \omega \text{ in } \mathrm{L}^\infty_{1\text{oc}}\left(\mathbf{R};\mathrm{C}'\right), \qquad \text{for all } r<0 \tag{37}$$

It results the lemma 6 3. that $\lim(\omega_n X_n)_{n\in\mathbb{N}} = \omega X_t$ in $L^\infty_{loc}(\mathbb{R};C^r)$, for all $r<0$ and thus $\lim(\operatorname{div}(\omega_n X_{n,t}))_{n\in\mathbb{N}} = \operatorname{div}(X_t\omega)$ in $L^\infty_{loc}(\mathbb{R};C^r)$, for all $r<-1$. According to the inequality (33) and the proposition 10 1., the sequence $(X_{n,t}(x,D)\omega_n)_{n\in\mathbb{N}}$ is a bounded sequence of the space $L^\infty_{loc}(\mathbb{R};C^{\varepsilon-1})$. It follows that

$$\mathrm{X}_t(x,\mathrm{D})\omega \text{ belongs to the space } \mathrm{L}^\infty_{1\text{oc}}\left(\mathbf{R};\mathrm{C}^{\varepsilon-1}\right) \tag{38}$$

Finally, the relation of conservation of the vortex, together with the fact that the solution v belongs to $L^\infty_{loc}(\mathbb{R};\mathrm{Lip}(\mathbb{R}^2))$, ensures, according to relation (0.1), that $\omega(t,\cdot)\in C^\varepsilon(\mathbb{R}\setminus A_t)$ finally becomes $\omega(t,\cdot)\in C_\varepsilon(A^t,X_t)$. The theorem 2 is then completely proved.

5 Some conclusive remarks

In the paragraph, we gave the global version of some local theorems in [6]. If S is a sub-manifold of C^∞, we denotes by $C^\rho(\mathbb{S},\infty)$ the set of u belongs to C^ρ such that , for all family $z_1,...,z_j$ of vector field C^∞ tangent to S, $z_1(x,D)...z_j(x,D)u$ remains in C^ρ. Then we have the following theorem, so the proof is only the juxtaposition of the theorem F in [6] and theorem 2:

Theorem 14. *Let Γ_0 be a bounded curve of class C^∞. If ω_0 belongs to $C^{\varepsilon-1}(\Gamma_0,\infty)\cap L^p$ with $p<2$, then there exists a unique solution v in the space $L^\infty_{loc}(\mathbb{R};Lip(\mathbb{R}^2))$ which moreover checks:*

1. *the surface $\Gamma=\{\psi(t,\Gamma_0),t\in\mathbb{R}\}$ is of class C^∞;*
2. *$v\in C^{\varepsilon'}(\Gamma,\infty)$ for all $\varepsilon'<\varepsilon$ locally in time.*

With this theorem, we would like to insist on the following point: in the problem of vortex patch, the fact that the vortex is exactly 1 on one side of the curve and exactly 0 on the other, appears, from the point of view of evolution of the regularity of the curve, anecdotal information.

As a conclusion, we would like to give a global version of theorem more general than in [6], which, unfortunately, is stated in the framework of the families of vector fields called $1-\varepsilon,k$ regularity. It is out of the question to redefine this notion here and we invite the interested reader to refer to the third paragraph of [3] for a precise definition. Let us simply say that it is a family of vector field so the coefficients can be derived by the elements of the family itself without significant loss of regularity. The theorem, whose proof is again nothing, just the juxtaposition of the theorem 4.1 of [6] and the theorem 2, is the following:

Theorem 15. *Let k be a strictly positive integer and Z_0 a family of divergence-free vector fields, $1-\varepsilon, k$ regularity. We consider given initial v_0 in the space $\mathrm{C}^{\varepsilon-1}_{1-\varepsilon}(\mathrm{Z}_0, k) \cap \mathrm{L}^p$ with $p < 2$, we further suppose the existence of a strictly positive real number β such that the family Z_0 does not vanish identically at any point of the C^β singular support of ω_0. Then there exists a unique solution v of* (E) *in the space $L^\infty_{loc}(\mathbb{R}; Lip(\mathbb{R}^2))$ which moreover checks:*

1. *the flow $\psi \in Id + C^1_{(1-\varepsilon,0)}(Z_0 \cup \{\partial_t\}, (k, \infty))$;*
2. *if $Z = \psi * Z_0$, then the family $\underline{Z} = Z \cup \{\partial_t + v \cdot \nabla\}$ is $(1-\varepsilon, 0)$, (k, ∞) regularity and v belongs to the space $C^1_{(1-\varepsilon,0)}(\underline{Z}, (k, \infty))$ locally in time.*