# Weak Solution: Convex Integration

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# 1 Introduction

First let's give an example to help define what is the weak solution.

**Example 1.1.**

$$\Delta u = f$$

If $u$ don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \to \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z}\alpha + \frac{\partial u}{\partial \bar{z}}\overline{\alpha} = \frac{\partial u}{\partial x} Re\alpha + \frac{\partial u}{\partial y} Im\alpha$$

i.e. $du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}$.

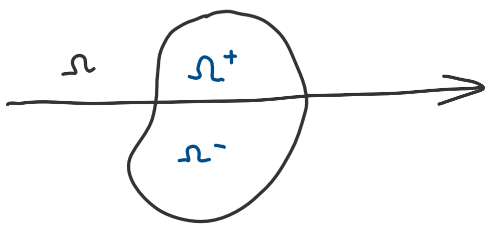
**Theorem 1.1.** *If $\frac{\partial u}{\partial z} = 0$ or $\Delta u = 0$ in the weak sense(against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.*

Useful: $f_k(z)$ holomorphic $u(z) = \sum\limits_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} & - \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ \text{Fubini} \quad &= - \sum_{k=0}^{\infty} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

**Theorem 1.2.** *(Swartz reflection principle.)If $f$ is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If $f$ is continu-*




*ous on $\Omega$ on $\Omega$ including $\Omega \cap \{y = 0\}$. Then $f$ is holomorphic on $\Omega$.*

In $D'$ $f = \lim\limits_{\delta \to 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here $H$ is heaviside function.

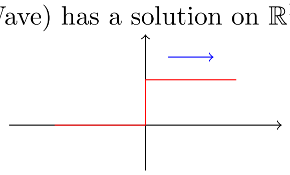
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \to 0} \frac{\partial f}{\partial \bar{z}} + f\left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial z} \delta(\varepsilon - y)\right)$$

Since $f$ is continuous $\lim\limits_{\varepsilon \to 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\Box u = 0 \quad \text{where } \Box := -\partial_t^2 + \partial_1^2 + ... + \partial_n^2 \tag{Wave}$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \qquad \int_{\mathbb{R}}^{d+1} \Box \phi u = 0$$

(Wave) has a solution on $\mathbb{R}^{1+1}$ given by $u(t, x) = u(t - x)$, a traveling wave:



**Example 1.2.**

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on $\mathbb{R}^{1+d}$

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

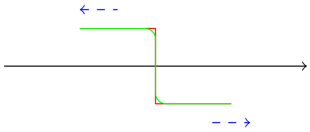


Figure 1: $t=0$

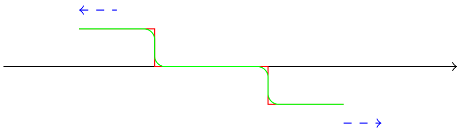


Figure 2: $t=1$

# 2 Fluid Mechanics

## 2.1 Incompressible Euler Equation

Define 2 velocity field: $v: \quad \mathbb{R}\times\mathbb{R}^d \to \mathbb{R} \quad p: \quad \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$

$$\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p = 0$$

$$\nabla_j v^j = 0 \quad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{\text{loc}}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \quad \partial\Omega$

$$\int_{\partial\Omega} v \cdot \overrightarrow{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt}\begin{bmatrix}\text{total momentum}\\ m\cdot v\end{bmatrix} = \begin{bmatrix}\text{Force on } \Omega\end{bmatrix} + \begin{bmatrix}\text{Flux of}\\ \text{momentum}\end{bmatrix} \quad \Rightarrow \frac{d}{dt}\int_\Omega v^\ell dx = \int_{\partial\Omega} p\overrightarrow{n}^\ell dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if $p$ is good enough. If $v, pinC^1$, use $\int_{\partial\Omega} f\overrightarrow{n}_j d\sigma = -\int_\Omega \nabla_j f dx$

$$\frac{d}{dt}\int_\Omega v^\ell = -(\int_\Omega \nabla^\ell p + \nabla_j(v^j v^\ell) dx$$

$$\int_\Omega (\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p) dx = 0 \qquad \forall\Omega, \quad \forall t$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful?
Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I\times\mathbb{R}^d)$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell(t,x) dx = 0$$

If $(1+|x|)v \in L^1_{t,x}(I\times\mathbb{R}^d)$, then linear angular momentum conserved?
Here $\forall K^\ell$ s.t. $\quad \nabla_j K_\ell + \nabla_\ell K_j = 0 \quad on \quad \mathbb{R}^d$

**Example 2.1.** $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_\ell v^\ell dx = \int_{\mathbb{R}^d} v^{(i)} dx$

**Example 2.2.** Rotation $K(a,b) = x^a e_b - x^b e_a$, $1 \le a < b \le d$.

$$\int_{\mathbb{R}^d} K_\ell(\partial_t v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p) dx = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell - \int \nabla : K_\ell(v^j v^\ell) - \int_{\mathbb{R}^d} \nabla^\ell K_\ell p dx = 0$$

where $\operatorname{div} K = \nabla^\ell K_\ell = \delta^{j\ell}\nabla_j K_\ell = \frac{1}{2}\delta^{j\ell}(\nabla_j K_\ell + \nabla_\ell K_j) = 0$ by assumption.

$$\partial v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p)dx = 0$$

Test against a space cut-off function $K^\ell(B) := q(t)\varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here $K$ is rotationally symmetric, so it is divergence-free.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_\ell^{(\beta)} v^\ell dx]dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_\ell^{(\beta)} v^j v^\ell - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$. i.e.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_\ell^{\beta} v^\ell dx]dt - \frac{1}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \left(\nabla_j K_\ell^{(\beta)} + \nabla_\ell K_j^{(\beta)}\right) v^j v^\ell dx dt - \frac{1}{2}\int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi(\frac{|x|}{B}))K_\ell v^j v^\ell dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L^1_{t,x}$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

## 2.2 Conservation of Energy

If $(1+|x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, $v \in L^2_{t,x}(I \times \mathbb{R}^d)$, then $\forall K^\ell$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell(t,x)dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : \; e_i, \; 1 \le i \le d, 1 \le a < b \le d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^\ell = \nabla_j w^{j\ell} \quad w^{j\ell} = -w^{lj} \qquad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B} w^{j\ell})$

$$w^{j\ell} = x^2(\delta_2^j\delta_1^\ell - \delta_1^j\delta_2^\ell)$$

$$\begin{aligned}&\nabla_j(x^2(\delta_2^j\delta_1^\ell - \delta_1^j\delta_2^\ell))\\ &= \delta_j^2(\delta_2^j\delta_1^\ell - \delta_1^j\delta_2^\ell) = \delta_1^\ell\end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j\nabla_j w^{j\ell} = -\nabla_\ell\nabla_j w^{lj} = -\nabla_j\nabla_\ell w^{lj} = -\nabla_\ell\nabla_j w^{j\ell}$$

Conservation of energy means that $\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v|}{2}(t,x)dx = 0$. Note that energy is nonlinear.

$$\partial_t(\frac{|v|^2}{2}) + \nabla_j((\frac{|v|}{2} + p)v^j) = 0$$

If $v \in C^1 \cap L^2_{t,x} \cap L^3_{t,x}(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here $B$ could be $\infty$. Multiply the local energy by $\eta(t)\varphi(\frac{|x|}{B})$

$$\begin{aligned}&\underset{(1)}{\int \eta \tfrac{d}{dt}\int \varphi(\tfrac{|x|}{B})\tfrac{|v|^2}{2}(t,x)dxdt} \quad \underset{(2)}{-\int \eta(t)\int \nabla_j[\varphi(\tfrac{|x|}{B})](\tfrac{|v|^2}{2}+p)v^j dxdt}\\ &(1) = -\int \underbrace{\eta'(t)}\int \varphi(\tfrac{|x|}{B})\underbrace{\frac{|v|}{2}(t,x)}dxdt \quad \text{Intergral by parts}\\ &\quad\text{dominated by } \tfrac{|x|^2}{2}|\eta'| \in L^1_{t,x}\end{aligned} \tag{Local}$$

term (2) converge to 0 pointwisely when $B \to \infty$ and dominated by $|\eta t|(\frac{|v|^3}{2} + |p||v|)$.
Let's recall Euler equation.

$$\begin{cases} \partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p &= 0,\\ \nabla_j v^j &= 0\end{cases} \tag{Euler}$$

Take divergence over (Euler), $\Rightarrow$

$$\nabla_j \nabla_\ell (v^j v^\ell) + \nabla_\ell \nabla^\ell p = 0$$

i.e.

$$\Delta p = -\nabla_\ell \nabla_j (v^j v^\ell)$$

$$p = \underbrace{(-\Delta)^{-1} \nabla_\ell \nabla_j (v^j v^\ell)}_{\text{zero order operator}} \quad \underbrace{(v^j v^\ell)}_{\in L^{3/2}_{t,x}}$$

Thus naturally $p \in L^{3/2}_x$ a.e. $t \in \mathbb{R}^+$

$$\|p\|_{L^{3/2}_x (L^{3/2}_t)} = \|p\|_{L^{3/2}_{t,x}} < \infty$$

$$v_\ell (\partial_t v^\ell + \nabla (v^j v^\ell) + \nabla^\ell p) = 0$$

$$\nabla_j v^j = 0$$

Thus

$$\Rightarrow \partial_t (\frac{|v|^2}{2}) + v_\ell v^j \nabla_j v^\ell + v_\ell \nabla^\ell p = 0$$

$$\Rightarrow \partial_t (\frac{|v|^2}{2}) + v^j \nabla_j (\frac{|v|^2}{2} + v_j v^j p = 0$$

$$\nabla_j v^j = 0$$

$$\partial_t (\frac{|v|^2}{2} + \nabla_j ((\frac{|v|^2}{2} + p) v^j) = 0$$

Using $\nabla_j v^j = 0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?
In turbulence situation (Navier-Stokes equations) with $\nu << 1$

$$v_\ell (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) = \nu v_\ell \Delta v^\ell$$

$$\frac{d}{dt} \int \frac{|v|^2}{2} (t, x) dx = -\nu \int |\nabla v|^2 dx = \nu \int v_\ell \nabla_i \nabla^i v^\ell$$

Taking a formal limit, $\exists$ incompressible Euler flows with

$$\frac{d}{dt} \int \frac{|v|}{2} (t, x) dx < -\varepsilon < 0$$

**Theorem 2.1.** *Onsager's Conjecture*
(+) *If* $\alpha > 1/3$ *and* $(v(t, x + \Delta x) - v(t, x)) \leq c|\Delta x|$ *where* $x \in \mathbb{T}^3 (v \in L^\infty_t C^\alpha_x)$*, then the energy conserved.*
($-$) *(K41)If* $\alpha \leq 1/3$ $\exists$ *incompressible Euler flows with* $v \in L^\infty_t L^\alpha_x$ *s.t.* $\int_{\mathbb{T}^d} \frac{|v|^2}{2} (t, x) dx$ *is not constant.*

Now we follow [2] and discuss the (+) part first.

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0$$

In order to get tnto Onsager's explanation of how this might be possible, we expand the velocity $v$ in Fourier series,

$$v(x, t) = \sum_{k \in \mathbb{Z}^3} a_k(t) e^{ik \cdot x}.$$

Obviously $a_{-k} = \overline{a_k}$, because $v$ is real-valued. Moreover the divergence-free con- straint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of ( 2.2) as an infinite-dimensional system of ODEs for the $a_k$ :

$$\frac{da_k}{dt} = i \sum_\ell a_{k-\ell} \cdot \ell \left[ -a_\ell + \frac{(a_\ell \cdot k)\, k}{|k|^2} \right] - \nu |k|^2 a_k \tag{1}$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.
(Don't understand)Energy starts at low wave numbers and moves to higher wave numbers in finite number.
$\sum\limits_{\frac{\lambda}{2}\leq|k|\leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.
Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.
(K 41) $E \lim\limits_{v\to 0} \left\langle v \int |\nabla v|^2 dx \right\rangle$ and $v$ determine all statistic properties of turbulent flows.

$$\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.
Now $(+)$ is solved by [4] and [1] with the goal $L^3_t B^{1/3}_{3,C(N)}$, $L^3_t B^{1/3+\varepsilon}_{3,\infty}$.
$(-)$ is solved $(d \geq 3)$ with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [7].
Convex integration originated from the Nash–Kuiper Paradox(50's) for $C^1$ isometric embedding. Connection to Euler equation discovered by Camillo De Lellis and László Székelyhidi (08,12). First result towards Onsager conjecture is in [8]. And $\alpha < \frac{1}{5}$ by [10]. The non-uniqueness example was first given by [12] and then Shnirelman give a different proof in [13].

## 2.3 Another way of proving $(+)$

$(+)$ (Eyink, Constantin, E, Titi 94') $L^3(B^\alpha_{3,\infty})$

$$\|v\|_{C^\alpha} = \sup_{h\neq 0} \frac{\|v(x+h)-v(x)\|_{L^\infty}}{|h|^\alpha}$$

$$\|v\|_{B^\alpha_{3,\infty}} = \sup_{h\neq 0} \frac{\|v(x+h)-v(x)\|_{L^3}}{|h|^\alpha}$$

**Lemma 2.2.** *Commutator Estimate*

$$R^{j\ell}_\varepsilon = \eta_\varepsilon * (v^j v^\ell) - (v^j_\varepsilon v^\ell_\varepsilon)$$
$$\|R_\varepsilon\|_{L^{3/2}} \lesssim \varepsilon^{2\alpha} \|v\|^2_{B^\alpha_{3,\infty}}$$

Let's think $R^{j\ell}_\varepsilon$ as an expectation with the idea:

$$R = \text{"}\mathbb{E}\left[v^2\right] - \left(\mathbb{E}[v]\right)^2\text{"} = \text{"}\mathbb{E}[(v-\mathbb{E}(v))^2]\text{"},$$

which is quadratic.

$$\begin{array}{rl} R^{j\ell}_\varepsilon = & \int v^i(x-h)v^\ell(x-h)\eta_\varepsilon(h)dh - \int v^j(x-h_1)\eta_\varepsilon(h_1)dh_1 \int v^\ell(x-h_2)\eta_\varepsilon(h_2)dh_2 \\ \text{Using} & \int \eta_\varepsilon(h)dh = 1 \\ = & \int (v^j(x-h) - v^j_\varepsilon(x))(v^\ell(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \end{array}$$

By Lemmas in [1], we decompose above equation into $\sum\limits_{i=1}^{4} R^{j\ell}_{\varepsilon i}$, where

$$\begin{array}{rl} R_{\varepsilon 1} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^\ell(x-h) - v^\ell_\varepsilon(x-h))\eta_\varepsilon(h)dh \\ R_{\varepsilon 2} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x))(v^\ell(x-h) - v^\ell_\varepsilon(x-h))\eta_\varepsilon(h)dh \\ R_{\varepsilon 3} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^\ell_\varepsilon(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \\ R_{\varepsilon 4} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x-h))(v^\ell_\varepsilon(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \end{array}$$

For example,

$$\begin{array}{rl} R_{\varepsilon 2} = & \int_{\mathbb{R}^d} \int_0^1 \frac{d}{d\sigma} v^j_\varepsilon(x-\sigma h)d\sigma(v^\ell(x-h) - v^\ell(x))\eta_\varepsilon(h)dh \\ = & \int_{\mathbb{R}^d} \int_0^1 d\sigma \nabla_i v^i_\varepsilon(x-\sigma h)h^i(v^\ell(x-h) - v^\ell(x)\eta_\varepsilon(h)dh \end{array}$$

$$\left\|R^j_{\varepsilon 2}\right\| \leq_{\mathbb{R}^d} \int_0^1 \|\nabla v_\varepsilon\|_{L^3}|h|\|v(\cdot - h) - v(\cdot)\|_{L^3}|\eta_\varepsilon(h)|dh$$

Modify the equation with modifier $\eta_\varepsilon$:

$$\begin{aligned}\eta_\varepsilon * (\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p) = & \quad 0\\ \partial_t v_\varepsilon^\ell + \nabla_j(v_\varepsilon^j v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon = & -\nabla_j R_\varepsilon^{j\ell}\end{aligned}$$

(Thus we need smoothness in time) $\times v_\varepsilon$ then integral by parts:

$$\partial(\frac{|v_\varepsilon|^2}{2}) + v_{\varepsilon l}\nabla_j(v_\varepsilon^j v_\varepsilon^\ell) + v_{\varepsilon l}\nabla^\ell p_\varepsilon = -v_{\varepsilon l}\nabla_j R_\varepsilon^{j\ell} = \int_{\mathbb{R}^d} \nabla_j \left| \frac{v_\varepsilon^2}{2} v_\varepsilon^j \right| \to 0$$

with assumption.

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v_\varepsilon|^2}{2}(t,x)dx + \int_{\mathbb{R}^d} v_\varepsilon^j \nabla_j v_\varepsilon^\ell v_{\varepsilon l} + \underline{\nabla^\ell v_{\varepsilon l} p_\varepsilon} = \int_{\mathbb{R}^d} \nabla v_{\varepsilon l} R_\varepsilon^{j\ell}$$

$\underline{\nabla^\ell v_{\varepsilon l} p_\varepsilon} = 0$ for divergence-free.
LHS converges to $\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx$ in $\mathcal{D}'(\mathbb{R})$ since $v_\varepsilon \to v$ in $L^2_{t,x}$.

$$\begin{aligned}&\left\| \tfrac{d}{dt}\int \tfrac{|v|^2}{2}(t,x)dx \right\|_{L^1_t} \le \limsup_{\varepsilon\to 0} \int\int \left|\nabla_j v(t,x) R_\varepsilon^{j\ell}\right| dxdt\\ &\le \limsup_{\varepsilon\to 0} \int \|\nabla v_\varepsilon(t,\cdot)\|_{L^3_x} \|R_\varepsilon\|_{L^{3/2}} dt\\ &\le \limsup_{\varepsilon\to 0} \int \varepsilon^{-1+\alpha} \|v(t)\|_{B^\alpha_{3,\infty}} \|v(t,\cdot)\|^2_{B^\alpha_{3,\infty}})dt\\ &< \limsup_{\varepsilon\to 0} \varepsilon^{-1+3\alpha} \int \|v(t,\cdot)\|^3_{B^\alpha_{3,\infty}} dt \quad \to 0 \quad \text{with } \alpha > \tfrac{1}{3}\end{aligned}$$

If $\alpha = \frac{1}{3}$ and $v$ bounded in $L^1_t(I)$ for some finite time period.

$$\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx = \lim_{\varepsilon\to 0} \frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dx$$

$v\phi \in C_c^\infty(I)$

$$\frac{d}{dt} < \int \frac{|v|^2}{2}(t,x)dx, \phi > \le \|\phi\|_{L^\infty(I)}$$

LHS is of finite measure. $e(t) = \int \frac{|v|^2}{2}(t,x)$ is of bounded variation. IN fact $\frac{d}{dt}e(t)$ is finite.
(???)If $v \in L^r B^{1/3}_{3,\infty}$, consider $\left\|\frac{d}{dt}e(t)\right\|_{L_t^{r/t}}$ using duality. $u \in L_t^\infty B^{1/3}_{3,\infty}$ uniformly $\left\|\frac{d}{dt}e(t)\right\|_{L^infty_t} \le C$ and also $\frac{d}{dt}e(t) \le -\varepsilon < 0$ is stable under perturbation. If not, the dissipation $\int_{\mathbb{R}^d} \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx$ can be really big.
*Remark.* The singular support of a generalized function u is the complement of the largest open set on which u is smooth. Roughly speaking, it is the closed set where the distribution does not correspond to a smooth function.

## 2.4 Local energy conservation

$$\partial_t v_\varepsilon^\ell + \nabla_\ell(v_\varepsilon^\ell v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon = -\nabla_j R_\varepsilon^{j\ell}$$

where $R_\varepsilon^{j\ell} = \eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell$

$$\|R_\varepsilon(t,.)\|_{L_\ell^{3/2}} \le \varepsilon^{2\alpha}\|v(t)\|^2_{B^\alpha_{3,\infty}}$$

$$\frac{1}{2}\int \frac{|v_\varepsilon|}{2}(t,x)dx = \lim_{\varepsilon\to 0}\int \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx$$

Here to clarify the space:

$$B^{1/3}_{3,c(N)} = (\overline{C^\infty})^{B^{1/3}_{3,\infty}} = B^{1/3}_{3,\infty} \cap \{\lim_{h\to 0} \frac{|v(x+h)-v(x)|}{|h|^{1/3}} = 0\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3}\backslash c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3}\backslash c^{1/3}$

**Lemma 2.3.** $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ *if* $v \in B^\alpha_{3,c(N)}$

*Proof.* Claim: $\varepsilon^{1-\alpha}\nabla(\eta_\varepsilon * \cdot) : B^\alpha_{3,\infty} \to L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L^3_x} \lesssim \varepsilon^{-1+\alpha}\|v\|_{B^\alpha_{3,\infty}}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B^\alpha_{3,\infty}} < \frac{\delta}{2C_2}$.

$$\begin{aligned}\left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} &\le \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * (v-\tilde{v})\right\|_{L^3_x} + \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * \tilde{v}\right\|_{L^3_x} \\ &\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\|\nabla\eta_\varepsilon * \tilde{v}\|_{L^3_x} \\ &\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\tilde{c} \quad \text{for}\varepsilon^{1-\alpha} < \tfrac{\delta}{2\tilde{c}} \text{ and } \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} < \delta\end{aligned}$$

$$\begin{aligned}&\textstyle\int_I \frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dxdt \le \limsup_{\varepsilon\to 0}\int_I\int \left|\nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon\right| dxdt \\ &\le \limsup_{\varepsilon\to 0}\textstyle\int_I \|\nabla v_\varepsilon(t)\|_{L^3_x}\varepsilon^{2\alpha}\|v\|^2_{B^\alpha_{3,\infty}}\,dt\end{aligned}$$

For a.e. t, $v \in B^{1/3}_{3,C(N)}$, the integrant is bounded by $o(\varepsilon^{-1+2/3})\varepsilon^{3/2} = o(1)$. Thus above integral is dominated by:

$$\int_I \varepsilon^{-1\times 1/3+2/3}\|v(t)\|^3_{B^{1/3}_{3,\infty}}\,dt \le \int_I \|v(t)\|^3_{B^{1/3}_{3,\infty}}\,dt$$

By assumption and DCT, bounded. ☐

**Theorem 2.4.** *(Isett 18') An energy dissipating solution whose singularities have 0 Lebesgue measure in* $\mathbb{R}^4$ *cannot be of class* $L^r_t B^{1/3}_{\zeta,\infty}$ *if* $r > 3$.

Compared with Meneveau-Sreenivasan [11],

$$< |v(x+\Delta x) - v(x)|^r >= |\Delta x|^{\xi_r}$$

singular support in $L^3_t B^{1/3}_{3,C(N)}$. (K41) implies $\xi_r \sim \frac{r}{3}$ (only correct when $r = 3$).

**Lemma 2.5.** *(Local energy conservation Duchon-Robert[3] formula* $D[v,p] = \partial_t(\frac{|v|^2}{2}) + \nabla(\frac{|v|^2}{2} + p)v^j) = \lim_{\varepsilon\to 0}\nabla_j v_{\varepsilon\to 0}\nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon$ *dissipation distribution* $v \in L^3_{t,x}$. *If* $D[v,p] = 0$ *and* $v \in L^2_{t,x}\cap L^3_{t,x}$*, then* $\int \frac{|v|^2}{2}(t,x)dx$ *is constant and* $D[v,p]$ *if* $v, p \in C^1$.

If $v \in L^r_t B^{1/3}_{r,\infty}$ with $r > 3$ is energy dissipating, since $v \in L^2_{t,x}\cap L^3_{t,x}$, $D[v,p] \neq 0$. Since $\frac{d}{dt}\int\frac{|v|^2}{2} = \int D[v,p]dx$ But we claim that

$$\|D[v,p]\|_{L^{r/3}_{t,x}} < \infty$$

using *Duchon-Robert* formula: $\left\|\nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon\right\|_{L^{r/3}_{t,x}}$ is bounded uniformly in $\varepsilon > 0$.

$$\begin{aligned}&& \|\nabla_j v_\varepsilon\|_{L^r_x} &\lesssim \varepsilon^{-1+1/3}\|v(t)\|_{B^{1/3}_{r,\infty}} \\ && \left\|R^{j\ell}_\varepsilon\right\| &\lesssim \varepsilon^{2/3}\|v(t)\|^2_{B^{1/3}_{r,\infty}} \\ \Rightarrow \quad \forall\phi \in C^i nfty_c(I\times\mathbb{R}^d) && < D < [v,p], p &>\lesssim C\|p\|_{L^s_{t,x}}\end{aligned}$$

Then $D[v,p]$ is in the dual of $L^s_{t,x}$ which is $L^{r/3}_{t,x}$ provided $r > 3$. Let $\frac{1}{s} + \frac{3}{r} = 1$. $\operatorname{supp} D[v,p]$ has positive Lebesgue measure, but $\operatorname{supp} D[u,p] \subset \operatorname{sing}(\operatorname{supp} U)$ also has positive measure.
There is an open problem to find a function $f(r)$ s.t. the condition $\frac{\zeta_r}{r} < \frac{1}{3} - f(r)$ works.

*Proof.* (Proof of *Duchon-Robert* formula) Considering Euler equation(Euler)

$$\eta_{\varepsilon\delta} * u := J_\zeta *_t \eta_\varepsilon *_x u$$

Let's test against $w_\varepsilon\delta = \eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v)$.

$$0 = -\int_{I\times\mathbb{R}^d} v^\ell\partial_t\eta_{\varepsilon\delta} + (\phi\eta * v_\ell) + v^j v^\ell\nabla_j\eta_{\varepsilon\delta} * (\phi\eta_{\varepsilon\delta} * v) + p\nabla^\ell\eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v_\ell)dxdt,$$

where $\phi \in C_c^\infty(I \times \mathbb{R}^d)$. Use the definition of self adjointness solution $\eta_{\varepsilon_\delta} *$ and divergence-free properties of $\eta_{\varepsilon_\delta} * v_\ell$. Then Euler equation becomes

$$0 = -\int \partial_t \phi \frac{|\eta_{\varepsilon\delta} * v|^2}{2} + v^j v^\ell \eta_{\varepsilon\delta} \nabla) j \phi \eta_{\varepsilon\delta} * v_\ell + p \eta_{\varepsilon\delta} * (\nabla^\ell \phi \eta_{\varepsilon\delta} * v_\ell) dx dt$$

Let $\varepsilon \to 0$ using uniform bounded ness of $\eta_\varepsilon *$ and $\nabla_j \eta_\varepsilon *$. As $\delta \to 0$, thanks to $\nabla_j \eta_\varepsilon *$, the time derivative naturally goes away. Then

$$0 = -\int \partial_t \phi \frac{{v_\varepsilon}^2}{2} + \nabla_j \phi (\frac{|v_\varepsilon|^2}{2} v^j + \eta_\varepsilon * p v_\varepsilon^j) dx dt \tag{2}$$

$$+ \int \phi \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} + Z_\varepsilon, \tag{3}$$

where $Z_\varepsilon = \int \nabla_j \phi R_\varepsilon^{j\ell} v_{\varepsilon l}$. Take both time and space derivative of $\phi$. Using $v \in L^2_{t,x} \cap L^2_{t,x}$ and $p = (-\Delta)^{-1} \nabla_j \nabla_\ell (v^j v^\ell) \in L^{3/2}_{t,x}$
(2) $\Rightarrow < D[u,p], \phi >$ as $\varepsilon \to 0$.
(3) $\Rightarrow \lim_{\varepsilon \to 0} \nabla_j v_{\varepsilon j} R_\varepsilon^{j\ell} + Z_\varepsilon$

$$Z_\varepsilon = \int \nabla_j \phi (\eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell) v_{\varepsilon l} dx dt =: B_\varepsilon[v, v]$$

Here we define the commutator $B_\varepsilon[\cdot, \cdot]$.

$$\|B_\varepsilon[u, w]\|_{L^{3/2}_{t,x}} \leqslant C \|u\|_{L^3_{t,x}} \|w\|_{L^3_{t,x}}$$

which is independent of $t$.
If $u$ or $w \in C_c^\infty$ $\|B_\varepsilon[u, w]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$. By approximation $\|B_\varepsilon[v, v]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$.
By Holder inequality,

$$|Z_\varepsilon| \leqslant \|\nabla \phi\|_{L^\infty} \|B_\varepsilon[v, v]\|_{L^{3/2}_{t,x}} \|v_\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0$$

□

*Remark.* Improvement:

- Calderon-Zygmund Theorem.
- More regularity in time.

*Proof.* (supp $D[u,p] \subset$ sing(supp $v$)) In fact supp $D[u,p] \subset$ sing(supp $L^3_\varepsilon B^{1/3}_{3,C(N)} v$).
What's good for not using Littlewood-Paley definition of Besov space? the solution above can be defined locally.
$\phi \in C_c^\infty(I \times B_q)$ and $B'_q \subset B_q$ a smaller ball with same center $q$. Let $q \notin$ sing(supp $B^{1/3}_{3,C(N)} v$)

$$< \phi, D[v,p] >= \lim_{\varepsilon \to 0} \int_I \int_{B'_q} \phi(t,x) \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} \lesssim \|\phi\|_{L^\infty} \int_I \|\nabla v_\varepsilon(t)\|_{L^(B'_q)} \|R_\varepsilon\|_{L_x^{2/3}}$$

dominated by $\|\phi\|_{L^\infty} \int_I \|v(t,\cdot)\|^3_{B^{1/3}_{3,\infty}(B_\varepsilon)}$. For a.e. $t$, we have $\|\nabla v_\varepsilon\|_{L^3} \cdot \|R_\varepsilon\|_{L^{3/2}} = o(1)$ and $D[u,p] \to 0$ by dominate convergence theorem. □

*Remark.*
1. Heat flow approach can also be applied to this problem. The proof is quite different[9] and is on a compact Riemann manifold(no convolution can be used).
2. Compressible Euler Case. The problem lies when dealing with commutator estimation.

# 3 Holder Continuity

The following lecture are basic on [9].
Note that $B^{1/3}_{3,C(\mathbb{N}} \subsetneq B^{1/3}_{3,\infty}$ and we can find a function in $B^{1/3}_{3,\infty} \setminus B^{1/3}_{3,C(N)}$.
$\phi(x)\chi_{\{x'>0\}} \subset B^{1/p}_{p,\infty} \quad \forall 1 < p < \infty$ Let's consider

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0 \quad L^\infty_t B^{1/3}_{3,\infty}$$

Energy dissipation at time $t = 0$

$$\frac{d}{dt}e(t) = \int_{\mathbb{T}^d} \nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon(0,x)dx$$

Eyink proved that there exists a divergence-free vector field in the space $C^{1/3}B^{1/3}_{3,C(N)}$, s.t. $\frac{d}{dt}|_{t=0}e(t) < 0$. We have a useful counter example:

$$v(x) = \sum_q 2^{2q\alpha}\sin\bigl(2^{2q}x\bigr) \in \dot{B}^{1/3}_{3,C(N)} \setminus \dot{B}^{1/3}_{3,\infty}$$

Now we consider this problem on compact Riemannian manifold for the conclusion $L^3_t B^{1/3}_{3,C(N)}$. Consider (Euler), instead of $\eta_\varepsilon * v^\ell$, we consider

- Estimates(Commutator)
- Keeping divergence-free property

Define the opetator $-\Delta_H = d_\delta + \delta_d$, which looks like a 1-form. In Hodge heat flow equation,

$$\partial_s v^\ell = \Delta_H v^\ell = \nabla_j \nabla^j v^\ell - \mathrm{Ric}^k_\ell\, v^k$$

Since we know what the solution exactly is,

$$\eta_\varepsilon * v^\ell \to e^{s\Delta_H} v^\ell$$

The square root of heat time $s^{1/2} \sim \varepsilon$ and solution at time $s$ $S_{[s]}v = e^{s\Delta_h}v$.
To estimate

$$\eta_\varepsilon * \nabla_j(v^j v^\ell) - \nabla_j(\eta_\varepsilon v^j \eta_\varepsilon * v^\ell),$$

we would need the commutator

$$w^\ell(s) = S_{[s]}\nabla_j(v^j v^\ell) - \nabla_j(S_{[s]}v^j S_{[s]}v^\ell)$$

and Riemannian manifold $M$ will be always assumed to be smooth.

$$s \in (0,1] \qquad \int_{I\times M} \eta(t)[S_{[s]}\nabla_j(v^j v^\ell) - \nabla_j\left(S_{[s]}v^j S_{[s]}v^\ell\right) S_{[s]}v_\ell d^{1+d}\mathrm{vol},$$

here volume is in time$\times$ space.
Let's calculate

$$(\partial_s - \Delta_H)w^\ell = N^\ell(t,s)$$
$$w^\ell(s) = \int_0^s e^{(s-s')\Delta_H} N^\ell(t,s')ds',$$

by d'Alembert's formula.

$$\begin{aligned} w^\ell(t,s) &= S_{[s]}\nabla_j(v^j v^\ell) - \nabla(S_{[s]}v^j S_{[s]}v^\ell) \\ (\partial_s - \Delta_H)w^\ell &= (\partial_s - \Delta_H)\nabla_j(S_{[s]}v^j S_{[s]}v^\ell) \\ &= (\partial_s - \nabla_i\nabla^i)\nabla_j(S_{[s]}v^j S_{[s]}v^\ell) + \text{curvature terms} \\ &= -2\nabla_j(\nabla_i S_{[s]}v^j \nabla^i S_{[s]}v^\ell) + \text{low order terms} \end{aligned}$$

$$\begin{aligned}\text{Commutator} &= -2\int_{I\times M}\eta(t)\int_0^s e^{(s-s')\Delta}\nabla_j(\nabla_i S_{[s]}v^j\nabla^i S_{[s]}v^\ell)ds' S_{[s]}v_\ell d^{1+d}\text{vol}\\ \text{Integral by parts} &= 2\int_{I\times M}\eta(t)\int_0^s \nabla_i S_{[s]}v^j\nabla^i S_{[s]}\cdot \underbrace{S_{[2s-s']}}_{\text{very low frequency}} v_\ell ds' d^{1+d}\text{vol}\end{aligned}$$

Assume that $v\in L_t^3 B_{3,\infty}^\alpha$, claim that $\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{L_t^3 B_{3,\infty}^\alpha}$ with $\alpha>1/3$. First we can try $v\in L_t^3 C^\alpha$ or $v\in L_t^3 W^{\alpha,3}$.

$$\begin{aligned}|Commutator| &\lesssim \|\eta\|_{L^\infty}\int_0^s(2s-s')^{\frac{-1+\alpha}{2}}(s')^{-2\frac{1+\alpha}{2}}ds'\\ &\lesssim s^{-1/2+3\alpha/2}\int_0^1(2-\sigma)^{\frac{-1+\alpha}{2}}\sigma^{-1+\alpha}d\sigma\to 0 \text{ as } \varepsilon\to 0 \text{ with } \alpha<1/3\end{aligned}$$

No derivatives that we can take over the heat flow. In order to prove the claim: $\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{W^{\alpha,3}}$ with $\alpha>1/3$.

*Proof.*

$$\begin{aligned}\|\nabla ßv\|_{L^3}&\leqslant s^{-\frac{1}{2}}\|v\|_{L^3}\\ \|\nabla ßv\|_{L^3}&\leqslant \|v\|_{W^{1,3}}\end{aligned}$$

Since $u(s)=e^{s\Delta}u$, $\|u\|_{L^\infty}\lesssim\|u\|_{L^p}\quad s\in(0,1]$ On compact manifold we have that

$$\|\nabla u\|_{L^p}\lesssim\|\nabla u\|_{L^r}+\|u\|_{L^p}$$

Here let $p=2z$ where $z\geqslant 2$ is an integer. □

$$\partial_s|u|^2-\Delta|u|+|\nabla u|^2=0(\text{or }\ -2\,\mathrm{Ric}_{jk}\,u^j u^k) \tag{4}$$

$$\Rightarrow\int_M|u|^2 d\text{vol}\searrow\ \text{ and }\int_M u^{2z}d\text{vol}\searrow$$

$$\begin{aligned}\tfrac{1}{z}\partial_s\int|u|^{2z}d\text{vol}&=\int_M(\Delta|u|^2-2|\nabla u|^2)|u|^{2(z-1)}d\text{vol}\\ &=-\int_M\nabla^j|u|^2\nabla_j|u||u|^{2(z-2)}-2|\nabla u|^2|u|^{2(z-1)}\leqslant 0\end{aligned}$$

For curvature terms, they can be bounded by $\|\mathrm{Ric}\|_{L^\infty}\int_M|u|^{2z}d\text{vol}$ remains bounded. So $\int_M u^{2z}d\text{vol}\leqslant\int_M u^{2z}(t=0)d\text{vol}$.

$$\partial_s|\nabla u|^2-\Delta|\nabla u|^2+|\nabla\nabla u|^2(\text{BAD})=0(\text{ or }\mathrm{Riem}(\nabla u\nabla u)+\nabla\mathrm{Riem}u\nabla u) \tag{5}$$

Multiply by $|\nabla u|^{2(z-1)}$,

$$\frac{1}{z}\partial_j\int_M|\nabla u|^{2z}d\text{vol}\quad\searrow$$

or

$$\begin{aligned}&\int_M|\nabla u|^{2(z-1)}\nabla_{\mathrm{Riem}}u\nabla u d\text{vol}\quad\text{by integration by parts}\\ &\leqslant 2\underbrace{\int_M|\nabla\nabla u|^2|\nabla u|^{2(z-1)}}_{BAD}+\underbrace{Cz\int_M\|R\|_{L^\infty}\|\nabla u\|^{2z}+\|\mathrm{Riem}\|^{z+1}\|u\|^{2z}}_{GOOD}\end{aligned}$$

which can cancel bad terms from (5).

$$\partial_s(s|\nabla u|^2)-\Delta(s|\nabla u|^2)+2s|\nabla\nabla u|^2-|\nabla u|^2(\text{BAD})$$

But here the bad term can cancel with (4). Let $\Phi(s)=s|\nabla u|^2+\frac{1}{2}|u|^2$, then we have

$$\partial_z\Phi_s-\Delta\Phi_s=2s|\nabla\nabla u|^2$$

Thus

$$\begin{aligned}\frac{1}{z}\partial_s\int_M|\Phi_s|^z(\searrow)\leqslant\frac{1}{z}\int_M|\Phi_s|^z d\text{vol}\leqslant\frac{1}{z}\int_M|\Phi_0|^z=\frac{1}{z}\int_M|u|^{2z}\\ \frac{1}{z}(\int_M|\nabla u|^{2z})^{1/2z}\leqslant s^{-1/2}(\int_M|u|^{2z})^{1/2z}\end{aligned}$$

Hodge Laplacian commute with derivative$d$ and divergence$\delta$. $(\partial_s-\Delta_H)\delta w_\ell=\delta(\partial_s-\Delta_H)w_\ell=0$ with 0 initial condition.

### 3.1 Isentropic Compressible Euler

$$\begin{array}{ll} \text{Mass} & \partial_t \rho + \nabla_j(\rho v^\ell) = 0 \\ \text{Momentum} & \partial_t(\rho v^\ell) + \nabla_j(\rho v^j v^\ell) + \nabla^l(P\rho) = 0 \end{array} \tag{6}$$

We made some assumption $p(\rho) = \rho^2\gamma$ and $\rho \in C^2$ and away from $\rho \equiv 0$. Here exists a problem: modifier doesn't commute with nonlinearity.

$$\partial_t(\frac{1}{2}\rho|v| + p(\rho)) + \nabla_j((\frac{1}{2}\rho|v| + p(\rho) + p(\rho)v^j)p(\rho) = \rho\int_1^\rho \frac{p(r)}{r^2}dr$$

If conservation holds in $B^\alpha_{3,\infty}$ in both $(t,x)$, we need to estimate the commutator:$= \eta_\varepsilon * (p(\rho)) - p\eta * \rho)$.

*Remark.*
- Heat flow also works.
- [5] gives another method

$$\begin{aligned} \text{Commutator} &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p(\int \rho(x-h)\eta_\varepsilon(h)dh \\ &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p(\int(\rho(x-h)\eta_\varepsilon(h))dh = \int(1-\sigma)\int p"((1-\eta)\rho_\varepsilon(x) + \sigma\rho(x-h))(\rho(x-h) - \rho_\varepsilon(x \end{aligned}$$

Therefore bounded.

$$\overline{f(X)} - f(\overline{X}) = \mathbb{E}[f(X)] - f(\mathbb{E}[X]) \tag{7}$$

Since

$$\begin{aligned} \phi(1) &= \phi(0) + \textstyle\int_0^1 \frac{d}{d\sigma}\phi(\sigma)d\sigma \\ &= \phi(0) + \frac{d}{d\sigma}|_{\sigma=0}\phi(\sigma) + \textstyle\int_0^1(1-\sigma)\frac{d^2}{d\sigma^2}\phi(\sigma)d\sigma \end{aligned}$$

RHS of (7) becomes

$$\begin{aligned} &\textstyle\int +0^1 \frac{d}{d\sigma}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X))] \\ &= \frac{d}{d\sigma}|_{\sigma=0}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X)] + \textstyle\int_0^1(1-\sigma)\frac{d^2}{d\sigma^2}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma X)d\sigma \\ &= \mathbb{E}[\nabla_i f(\overline{X})(X^i - \overline{X}^i)] + \textstyle\int_0^1(1-\sigma)\mathbb{E}[\nabla_a\nabla_b f((1-\sigma)\overline{X} + \sigma X)(X^a - \overline{X}^b)(X^b - \overline{X}^a)d\sigma \end{aligned}$$

Here last term is a quadratic form.

# 4 Convex Integration

Here "convex" refers to convex linear combination.

**Theorem 4.1.** *(Old conclution)* $\forall E(t) \in C^\infty$, $E(t) \geq C$, $\forall \alpha < \frac{1}{10}$, $\exists v \in C^\alpha_{t,x}(I \times \mathbb{T}^3)$ *s.t.* $\int \frac{|v|^2}{2}(t,x)dx = E(t)$ *conserved.*

**Theorem 4.2.** *(Isset [6])*$\forall \alpha < 1/5$, $\exists v \in C^\alpha_{t,x}(I \times \mathbb{T}^3$, $p \in C^{2,\alpha}_{t,x}$. *A non trivial solution with compact support in time. (0 is not the only solution stays 0 implies non-uniqueness).*

- Question: How to construct continuous solution?
- Idea: Euler-Reynolds flows

For $R$ a symmetric tensor $(v,p,R)$ that solves

$$\begin{aligned} \partial_t v^l + \nabla_j(v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned}$$

Here $R^{j\ell} = R^{lj}$. If $R = 0$, we have the Euler equation(Euler). If $R \neq 0$, we have a smooth approximation of Euler Equation.
Construction $(v_q, p_q, R_q)$ the Euler-Reynolds flow:

$$R_q \to 0 \text{ as } q \to \infty$$

and

$$(v_q, p_q)$$

converge uniformly.

$$-\int \partial_t \phi_\ell v_q^l + \nabla_j \phi_\ell + v_q^j v_q^\ell + \nabla^\ell \phi \cdot p_q = \nabla_j \phi R_q^{j\ell} \to 0$$

The idea would make sense if every continuous solution $(v,p)$ is a uniform limit of $(v_\varepsilon, p_\varepsilon, R_\varepsilon)$ as $\varepsilon \to 0$. I.e. every Euler flow is a limit of Euler-Reynolds flow. Proof will also be given by modification. To check weather it is a E-R flow, we need to check weather it conserves energy.

**Lemma 4.3.** *(Main Lemma) Given $(v,p,R)$ $\exists$ new*

$$(\overset{*}{\mathrm{v}}, \overset{*}{\mathrm{p}}, \overset{*}{\mathrm{R}})$$

*with*

$$\|\overset{*}{\mathrm{R}}\|_0 << \|R\|_0,$$

*where*

$$\overset{*}{\mathrm{v}} = v + V \qquad \overset{*}{\mathrm{p}} = p + P. \tag{8}$$

*We expect that $\|V\|_0 \leqslant \|R\|_0^{1/2} \leftarrow [\frac{m}{s}]$ and $\|P\|_0 \leq \|R\|_0 \leftarrow [\frac{m}{s}]$ by dimension analysis.*

Apply 4.3 over and over again generate $(v_q, p_q, R_q)$ with $\|R_q\|_0 \to 0$ rapidly. Set $v = v(0) + \sum\limits_q v_q$ and $p = p(0) + \sum\limits_q P_q$. Plug in new 8

$$\partial_t \overset{*}{\mathrm{v}}^\ell + \nabla_j (\overset{*}{\mathrm{v}}^i \overset{*}{\mathrm{v}}^l) + \nabla^\ell \overset{*}{\mathrm{p}} = \nabla_j R^{j\ell} + \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla_j (v^j v^\ell) + \nabla^\ell p + \nabla_j (v^j v^\ell)$$

i.e.

$$\mathrm{RHS} = \nabla v^\ell + \nabla_j (v^j V^l \ell) + \nabla_j (V^j V^l + P\delta^{j\ell} + R^{jl}) + \nabla_j (v^l V^j)$$

where $R^{jl}$ is the old error. We want $\nabla_j \mathrm{R}^{*j\ell}$ =RHS with $\mathrm{R}^*$ small. [1]

[1] Here the position of $*$ is decided by whether it is a equation or just math script. The different position of $*$ doesn't have different meaning

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