Weak Solution: Convex Integration

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1 Introduction

First let's give an example to help define what is the weak solution.

Example 1.1.

$$\Delta u = f$$

If u don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \to \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z}\alpha + \frac{\partial u}{\partial \bar{z}}\bar{\alpha} = \frac{\partial u}{\partial x} Re\alpha + \frac{\partial u}{\partial y} Im\alpha$$

i.e. $du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}$.

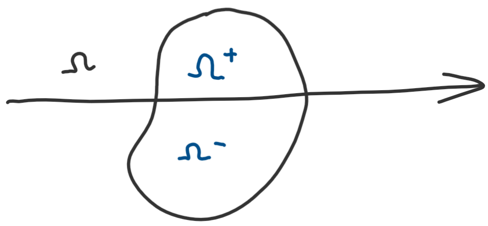
Theorem 1.1. *If $\frac{\partial u}{\partial z} = 0$ or $\Delta u = 0$ in the weak sense(against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.*

Useful: $f_k(z)$ holomorphic $u(z) = \sum_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} & -\textstyle\int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ \text{Fubini} \quad = & -\sum_{k=0}^{\infty} \textstyle\int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ = & \sum_{k=0}^{\infty} \textstyle\int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

Theorem 1.2. *(Swartz reflection principle.)If f is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If f is continu-*



ous on Ω on Ω including $\Omega \cap \{y = 0\}$. Then f is holomorphic on Ω.

In D' $f = \lim_{\delta \to 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here H is heaviside function.

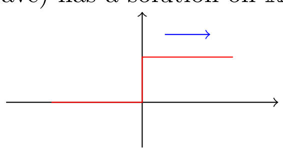
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \to 0} \frac{\partial f}{\partial \bar{z}} + f\left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial z} \delta(\varepsilon - y)\right)$$

Since f is continuous $\lim_{\varepsilon \to 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\Box u = 0 \quad \text{where } \Box := -\partial_t^2 + \partial_1^2 + \ldots + \partial_n^2 \tag{Wave}$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \qquad \int_{\mathbb{R}}^{d+1} \Box \phi u = 0$$

(Wave) has a solution on $\mathbb{R}^{1+1}$ given by $u(t, x) = u(t - x)$, a traveling wave:



Example 1.2.

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on $\mathbb{R}^{1+d}$

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

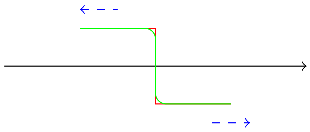


Figure 1: $t = 0$

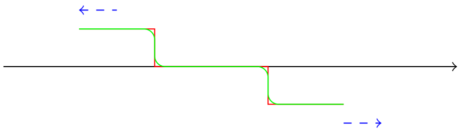


Figure 2: $t = 1$

2 Fluid Mechanics

2.1 Incompressible Euler Equation

Define 2 velocity field: $v: \quad \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \quad p: \quad \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$

$$\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p = 0$$

$$\nabla_j v^j = 0 \qquad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{\text{loc}}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \quad \partial\Omega$

$$\int_{\partial\Omega} v \cdot \overrightarrow{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt}\begin{bmatrix}\text{total momentum} \\ m \cdot v\end{bmatrix} = \begin{bmatrix}\text{Force on } \Omega\end{bmatrix} + \begin{bmatrix}\text{Flux of} \\ \text{momentum}\end{bmatrix} \quad \Rightarrow \frac{d}{dt}\int_{\Omega} v^l dx = \int_{\partial\Omega} p \overrightarrow{n}^l dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if p is good enough. If $v, pinC^1$, use $\int_{\partial\Omega} f \overrightarrow{n}_j d\sigma = -\int_{\Omega} \nabla_j f dx$

$$\frac{d}{dt}\int_{\Omega} v^l = -(\int_{\Omega} \nabla^l p + \nabla_j (v^j v^l) dx$$

$$\int_{\Omega} (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) dx = 0 \qquad \forall \Omega, \quad \forall t$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful?
Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I \times \mathbb{R}^d)$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_l v^l (t, x) dx = 0$$

If $(1 + |x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, then linear angular momentum conserved?
Here $\forall K^l$ s.t. $\quad \nabla_j K_l + \nabla_l K_j = 0 \quad on \quad \mathbb{R}^d$

Example 2.1. $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_l v^l dx = \int_{\mathbb{R}^d} v^{(i)} dx$

Example 2.2. Rotation $K(a, b) = x^a e_b - x^b e_a$, $1 \leq a < b \leq d$.

$$\int_{\mathbb{R}^d} K_l (\partial_t v^l + \nabla : (v^j v^l) + \nabla^l p) dx = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_l v^l - \int \nabla : K_l (v^j v^l) - \int_{\mathbb{R}^d} \nabla^l K_l p dx = 0$$

where $\operatorname{div} K = \nabla^l K_l = \delta^{jl}\nabla_j K_l = \frac{1}{2}\delta^{jl}(\nabla_j K_l + \nabla_l K_j) = 0$ by assumption.

$$\partial v^l + \nabla : (v^j v^l) + \nabla^l p)dx = 0$$

Test against a space cut-off function $K^l(B) := q(t)\varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here K is rotationally symmetric, so it is divergence-free.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_l^{(\beta)} v^l dx]dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_l^{(\beta)} v^j v^l - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$. i.e.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_l^{\beta} v^l dx]dt - \frac{1}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \left(\nabla_j K_l^{(\beta)} + \nabla_l K_j^{(\beta)}\right) v^j v^l dx dt - \frac{1}{2}\int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi(\frac{|x|}{B})) K_l v^j v^l dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L^1_{t,x}$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

2.2 Conservation of Energy

If $(1+|x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, $v \in L^2_{t,x}(I \times \mathbb{R}^d)$, then $\forall K^l$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_l v^l(t,x)dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : \ e_i, \ 1 \le i \le d, 1 \le a < b \le d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^l = \nabla_j w^{jl} \quad w^{jl} = -w^{lj} \qquad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B} w^{jl})$

$$w^{jl} = x^2(\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)$$

$$\begin{aligned} &\nabla_j (x^2(\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)) \\ &= \delta_j^2(\delta_2^j \delta_1^l - \delta_1^j \delta_2^l) = \delta_1^l \end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j \nabla_j w^{jl} = -\nabla_l \nabla_j w^{lj} = -\nabla_j \nabla_l w^{lj} = -\nabla_l \nabla_j w^{jl}$$

Conservation of energy means that $\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v|}{2}(t,x)dx = 0$. Note that energy is nonlinear.

$$\partial_t(\frac{|v|^2}{2}) + \nabla_j((\frac{|v|}{2} + p)v^j) = 0$$

If $v \in C^1 \cap L^2_{t,x} \cap L^3_{t,x}(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here B could be ∞. Multiply the local energy by $\eta(t)\varphi(\frac{|x|}{B})$

$$\begin{aligned} &\underset{(1)}{\int \eta \tfrac{d}{dt} \int \varphi(\tfrac{|x|}{B}) \tfrac{|v|^2}{2}(t,x) dx dt} \quad \underset{(2)}{- \int \eta(t) \int \nabla_j [\varphi(\tfrac{|x|}{B})](\tfrac{|v|^2}{2} + p) v^j dx dt} \\ &(1) = -\int \underbrace{\eta'(t)} \int \varphi(\tfrac{|x|}{B}) \underbrace{\frac{|v|}{2}(t,x)} dx dt \quad \text{Intergral by parts} \\ &\quad \text{dominated by } \tfrac{|x|^2}{2}|\eta'| \in L^1_{t,x} \end{aligned} \tag{Local}$$

term (2) converge to 0 pointwisely when $B \to \infty$ and dominated by $|\eta t|(\frac{|v|^3}{2} + |p||v|)$.
Let's recall Euler equation.

$$\begin{cases} \partial_t v^l + \nabla_j(v^j v^l) + \nabla^l p &= 0, \\ \nabla_j v^j &= 0 \end{cases} \tag{Euler}$$

Take divergence over (Euler), $\Rightarrow$

$$\nabla_j\nabla_l(v^jv^l) + \nabla_l\nabla^l p = 0$$

i.e.

$$\Delta p = -\nabla_l\nabla_j(v^jv^l)$$

$$p = \underbrace{(-\Delta)^{-1}\nabla_l\nabla_j(v^jv^l)}_{\text{zero order operator}} \quad \underbrace{(v^jv^l)}_{\in L^{3/2}_{t,x}}$$

Thus naturally $p \in L^{3/2}_x$ a.e. $t \in \mathbb{R}^+$

$$\|p\|_{L^{3/2}_x(L^{3/2}_t)} = \|p\|_{L^{3/2}_{t,x}} < \infty$$

$$v_l(\partial_t v^l + \nabla(v^jv^l) + \nabla^l p) = 0$$

$$\nabla_j v^j = 0$$

Thus

$$\Rightarrow \partial_t(\frac{|v|^2}{2}) + v_lv^j\nabla_jv^l + v_l\nabla^l p = 0$$

$$\Rightarrow \partial_t(\frac{|v|^2}{2}) + v^j\nabla_j(\frac{|v|^2}{2} + v_jv^jp = 0$$

$$\nabla_j v^j = 0$$

$$\partial_t(\frac{|v|^2}{2} + \nabla_j((\frac{|v|^2}{2} + p)v^j) = 0$$

Using $\nabla_j v^j = 0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?
In turbulence situation (Navier-Stokes equations) with $\nu << 1$

$$v_l(\partial_t v^l + \nabla_j(v^jv^l) + \nabla^l p) = \nu v_l \Delta v^l$$

$$\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx = -\nu\int |\nabla v|^2 dx = \nu\int v_l\nabla_i\nabla^i v^l$$

Taking a formal limit, $\exists$ incompressible Euler flows with

$$\frac{d}{dt}\int \frac{|v|}{2}(t,x)dx < -\varepsilon < 0$$

Theorem 2.1. *Onsager's Conjecture*
(+) If $\alpha > 1/3$ and $(v(t, x + \Delta x) - v(t, x)) \leq c|\Delta x|$ where $x \in \mathbb{T}^3 (v \in L^\infty_t C^\alpha_x)$, then the energy conserved.
(−) (K41)If $\alpha \leq 1/3$ $\exists$ incompressible Euler flows with $v \in L^\infty_t L^\alpha_x$ s.t. $\int_{\mathbb{T}^d} \frac{|v|^2}{2}(t,x)dx$ is not constant.

Now we follow [2] and discuss the (+) part first.

$$\partial_t v^l + \nabla_j(v^jv^l) + \nabla^l p = 0$$

In order to get tnto Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series,

$$v(x,t) = \sum_{k\in\mathbb{Z}^3} a_k(t)e^{ik\cdot x}.$$

Obviously $a_{-k} = \overline{a_k}$, because v is real-valued. Moreover the divergence-free con- straint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k :

$$\frac{da_k}{dt} = i\sum_\ell a_{k-\ell}\cdot\ell\left[-a_\ell + \frac{(a_\ell\cdot k)\,k}{|k|^2}\right] - \nu|k|^2 a_k \tag{1}$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.
(Don't understand)Energy starts at low wave numbers and moves to higher wave numbers in finite number.
$\sum\limits_{\frac{\lambda}{2}\leq|k|\leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.
Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.
(K 41) $E \lim\limits_{v\to 0} \left\langle v \int |\nabla v|^2 dx \right\rangle$ and v determine all statistic properties of turbulent flows.

$$\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.
Now (+) is solved by [4] and [1] with the goal $L_t^3 B^{1/3}_{3,C(N)}$, $L_t^3 B^{1/3+\varepsilon}_{3,\infty}$.
(−) is solved ($d \geq 3$) with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [5].
Convex integration originated from the Nash–Kuiper Paradox(50's) for C^1 isometric embedding. Connection to Euler equation discovered by Camillo De Lellis and László Székelyhidi (08,12). First result towards Onsager conjecture is in [6]. And $\alpha < \frac{1}{5}$ by [8]. The non-uniqueness example was first given by [10] and then Shnirelman give a different proof in [11].

2.3 Another way of proving (+)

(+) (Eyink, Constantin, E, Titi 94') $L^3(B^\alpha_{3,\infty})$

$$\|v\|_{C^\alpha} = \sup_{h\neq 0} \frac{\|v(x+h) - v(x)\|_{L^\infty}}{|h|^\alpha}$$

$$\|v\|_{B^\alpha_{3,\infty}} = \sup_{h\neq 0} \frac{\|v(x+h) - v(x)\|_{L^3}}{|h|^\alpha}$$

Lemma 2.2. *Commutator Estimate*

$$R^{jl}_\varepsilon = \eta_\varepsilon * (v^j v^l) - (v^j_\varepsilon v^l_\varepsilon)$$
$$\|R_\varepsilon\|_{L^{3/2}} \lesssim \varepsilon^{2\alpha} \|v\|^2_{B^\alpha_{3,\infty}}$$

Let's think R^{jl}_ε as an expectation with the idea:

$$R = \text{"}\mathbb{E}\left[v^2\right] - \left(\mathbb{E}[v]\right)^2\text{"} = \text{"}\mathbb{E}[(v - \mathbb{E}(v))^2]\text{"},$$

which is quadratic.

$$\begin{array}{rl} R^{jl}_\varepsilon = & \int v^i(x-h) v^l(x-h) \eta_\varepsilon(h) dh - \int v^j(x-h_1)\eta_\varepsilon(h_1) dh_1 \int v^l(x-h_2)\eta_\varepsilon(h_2) dh_2 \\ \text{Using} & \int \eta_\varepsilon(h) dh = 1 \\ = & \int (v^j(x-h) - v^j_\varepsilon(x))(v^l(x-h) - v^l_\varepsilon(x))\eta_\varepsilon(h) dh \end{array}$$

By Lemmas in [1], we decompose above equation into $\sum\limits_{i=1}^{4} R^{jl}_{\varepsilon i}$, where

$$\begin{array}{rl} R_{\varepsilon 1} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^l(x-h) - v^l_\varepsilon(x-h))\eta_\varepsilon(h) dh \\ R_{\varepsilon 2} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x))(v^l(x-h) - v^l_\varepsilon(x-h))\eta_\varepsilon(h) dh \\ R_{\varepsilon 3} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^l_\varepsilon(x-h) - v^l_\varepsilon(x))\eta_\varepsilon(h) dh \\ R_{\varepsilon 4} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x-h))(v^l_\varepsilon(x-h) - v^l_\varepsilon(x))\eta_\varepsilon(h) dh \end{array}$$

For example,

$$\begin{array}{rl} R_{\varepsilon 2} = & \int_{\mathbb{R}^d} \int_0^1 \frac{d}{d\sigma} v^j_\varepsilon(x-\sigma h) d\sigma (v^l(x-h) - v^l(x))\eta_\varepsilon(h) dh \\ = & \int_{\mathbb{R}^d} \int_0^1 d\sigma \nabla_i v^i_\varepsilon(x-\sigma h) h^i (v^l(x-h) - v^l(x)\eta_\varepsilon(h) dh \end{array}$$

$$\left\| R^j_{\varepsilon 2} \right\| \leq_{\mathbb{R}^d} \int_0^1 \|\nabla v_\varepsilon\|_{L^3} |h| \|v(\cdot - h) - v(\cdot)\|_{L^3} |\eta_\varepsilon(h)| dh$$

Modify the equation with modifier η_ε:

$$\begin{aligned} \eta_\varepsilon * (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) = &\quad 0 \\ \partial_t v_\varepsilon^l + \nabla_j (v_\varepsilon^j v_\varepsilon^l) + \nabla^l p_\varepsilon = & -\nabla_j R_\varepsilon^{jl} \end{aligned}$$

(Thus we need smoothness in time) $\times v_\varepsilon$ then integral by parts:

$$\partial(\frac{|v_\varepsilon|^2}{2}) + v_{\varepsilon l}\nabla_j (v_\varepsilon^j v_\varepsilon^l) + v_{\varepsilon l}\nabla^l p_\varepsilon = -v_{\varepsilon l}\nabla_j R_\varepsilon^{jl} = \int_{\mathbb{R}^d} \nabla_j \left| \frac{v_\varepsilon^2}{2} v_\varepsilon^j \right| \to 0$$

with assumption.

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v_\varepsilon|^2}{2}(t,x)dx + \int_{\mathbb{R}^d} v_\varepsilon^j \nabla_j v_\varepsilon^l v_{\varepsilon l} + \underline{\nabla^l v_{\varepsilon l} p_\varepsilon} = \int_{\mathbb{R}^d} \nabla v_{\varepsilon l} R_\varepsilon^{jl}$$

$\underline{\nabla^l v_{\varepsilon l} p_\varepsilon} = 0$ for divergence-free.
LHS converges to $\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx$ in $\mathcal{D}'(\mathbb{R})$ since $v_\varepsilon \to v$ in $L^2_{t,x}$.

$$\begin{aligned} &\left\| \tfrac{d}{dt}\textstyle\int \tfrac{|v|^2}{2}(t,x)dx \right\|_{L^1_t} \le \limsup_{\varepsilon\to 0} \textstyle\int\int \left|\nabla_j v(t,x) R_\varepsilon^{jl}\right| dxdt \\ &\le \limsup_{\varepsilon\to 0} \textstyle\int \|\nabla v_\varepsilon(t,\cdot)\|_{L^3_x} \|R_\varepsilon\|_{L^{3/2}} dt \\ &\le \limsup_{\varepsilon\to 0} \textstyle\int \varepsilon^{-1+\alpha} \|v(t)\|_{B^\alpha_{3,\infty}} \|v(t,\cdot)\|^2_{B^\alpha_{3,\infty}})dt \\ &< \limsup_{\varepsilon\to 0} \varepsilon^{-1+3\alpha} \textstyle\int \|v(t,\cdot)\|^3_{B^\alpha_{3,\infty}} dt \quad \to 0 \quad \text{with } \alpha > \tfrac{1}{3} \end{aligned}$$

If $\alpha = \frac{1}{3}$ and v bounded in $L^1_t(I)$ for some finite time period.

$$\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx = \lim_{\varepsilon\to 0} \frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dx$$

$v\phi \in C_c^\infty(I)$

$$\frac{d}{dt} < \int \frac{|v|^2}{2}(t,x)dx, \phi > \le \|\phi\|_{L^\infty(I)}$$

LHS is of finite measure. $e(t) = \int \frac{|v|^2}{2}(t,x)$ is of bounded variation. IN fact $\frac{d}{dt}e(t)$ is finite.
(???)If $v \in L^r B^{1/3}_{3,\infty}$, consider $\left\|\frac{d}{dt}e(t)\right\|_{L^{r/t}_t}$ using duality. $u \in L^\infty_t B^{1/3}_{3,\infty}$ uniformly $\left\|\frac{d}{dt}e(t)\right\|_{L^infty_t} \le C$ and also $\frac{d}{dt}e(t) \le -\varepsilon < 0$ is stable under perturbation. If not, the dissipation $\int_{\mathbb{R}^d} \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} dx$ can be really big.
Remark. The singular support of a generalized function u is the complement of the largest open set on which u is smooth. Roughly speaking, it is the closed set where the distribution does not correspond to a smooth function.

2.4 Local energy conservation

$$\partial_t v_\varepsilon^l + \nabla_l (v_\varepsilon^l v_\varepsilon^l) + \nabla^l p_\varepsilon = -\nabla_j R_\varepsilon^{jl}$$

where $R_\varepsilon^{jl} = \eta_\varepsilon * (v^j v^l) - v_\varepsilon^j v_\varepsilon^l$

$$\|R_\varepsilon(t,.)\|_{L^{3/2}_l} \le \varepsilon^{2\alpha} \|v(t)\|^2_{B^\alpha_{3,\infty}}$$

$$\frac{1}{2}\int \frac{|v_\varepsilon|}{2}(t,x)dx = \lim_{\varepsilon\to 0}\int \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} dx$$

Here to clarify the space:

$$B^{1/3}_{3,c(N)} = (\overline{C^\infty})^{B^{1/3}_{3,\infty}} = B^{1/3}_{3,\infty} \cap \{\lim_{h\to 0} \frac{|v(x+h)-v(x)|}{|h|^{1/3}} = 0\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3} \backslash c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3} \backslash c^{1/3}$

Lemma 2.3. $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ *if* $v \in B^\alpha_{3,c(N)}$

Proof. Claim: $\varepsilon^{1-\alpha}\nabla(\eta_\varepsilon * \cdot) : B^\alpha_{3,\infty} \to L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L^3_x} \lesssim \varepsilon^{-1+\alpha}\|v\|_{B^\alpha_{3,\infty}}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B^\alpha_{3,\infty}} < \frac{\delta}{2C_2}$.

$$\begin{aligned}\left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} &\le \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * (v-\tilde{v})\right\|_{L^3_x} + \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * \tilde{v}\right\|_{L^3_x} \\ &\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\|\nabla\eta_\varepsilon * \tilde{v}\|_{L^3_x} \\ &\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\tilde{c} \quad \text{for}\varepsilon^{1-\alpha} < \tfrac{\delta}{2\tilde{c}} \text{ and } \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} < \delta\end{aligned}$$

$$\begin{aligned}&\textstyle\int_I \frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dxdt \le \limsup\limits_{\varepsilon\to 0}\int_I\int \left|\nabla_j v_{\varepsilon l}R^{jl}_\varepsilon\right|dxdt \\ &\le \limsup\limits_{\varepsilon\to 0}\textstyle\int_I \|\nabla v_\varepsilon(t)\|_{L^3_x}\varepsilon^{2\alpha}\|v\|^2_{B^\alpha_{3,\infty}}dt\end{aligned}$$

For a.e. t, $v \in B^{1/3}_{3,C(N)}$, the integrant is bounded by $o(\varepsilon^{-1+2/3})\varepsilon^{3/2} = o(1)$. Thus above integral is dominated by:

$$\int_I \varepsilon^{-1\times 1/3+2/3}\|v(t)\|^3_{B^{1/3}_{3,\infty}}dt \le \int_I \|v(t)\|^3_{B^{1/3}_{3,\infty}}dt$$

By assumption and DCT, bounded. □

Theorem 2.4. *(Isett 18') An energy dissipating solution whose singularities have 0 Lebesgue measure in* $\mathbb{R}^4$ *cannot be of class* $L^r_t B^{1/3}_{\zeta,\infty}$ *if* $r > 3$.

Compared with Meneveau-Sreenivasan [9],

$$< |v(x+\Delta x) - v(x)|^r >= |\Delta x|^{\xi_r}$$

singular support in $L^3_t B^{1/3}_{3,C(N)}$. (K41) implies $\xi_r \sim \frac{r}{3}$ (only correct when $r = 3$).

Lemma 2.5. *(Local energy conservation Duchon-Robert[3] formula* $D[v,p] = \partial_t(\frac{|v|^2}{2}) + \nabla(\frac{|v|^2}{2} + p)v^j) = \lim\limits_{\varepsilon\to 0}\nabla_j v_{\varepsilon\to 0}\nabla_j v_{\varepsilon l}R^{jl}_\varepsilon$ *dissipation distribution* $v \in L^3_{t,x}$. *If* $D[v,p] = 0$ *and* $v \in L^2_{t,x}\cap L^3_{t,x}$, *then* $\int \frac{|v|^2}{2}(t,x)dx$ *is constant and* $D[v,p]$ *if* $v, p \in C^1$.

If $v \in L^r_t B^{1/3}_{r,\infty}$ with $r > 3$ is energy dissipating, since $v \in L^2_{t,x}\cap L^3_{t,x}$, $D[v,p] \ne 0$. Since $\frac{d}{dt}\int\frac{|v|^2}{2} = \int D[v,p]dx$
But we claim that

$$\|D[v,p]\|_{L^{r/3}_{t,x}} < \infty$$

using *Duchon-Robert* formula: $\left\|\nabla_j v_{\varepsilon l}R^{jl}_\varepsilon\right\|_{L^{r/3}_{t,x}}$ is bounded uniformly in $\varepsilon > 0$.

$$\begin{aligned}\|\nabla_j v_\varepsilon\|_{L^r_x} &\lesssim \varepsilon^{-1+1/3}\|v(t)\|_{B^{1/3}_{r,\infty}} \\ \left\|R^{jl}_\varepsilon\right\| &\lesssim \varepsilon^{2/3}\|v(t)\|^2_{B^{1/3}_{r,\infty}} \\ \Rightarrow \quad \forall\phi \in C^inifty_c(I\times\mathbb{R}^d) \quad < D < [v,p], p &> \lesssim C\|p\|_{L^s_{t,x}}\end{aligned}$$

Then $D[v,p]$ is in the dual of $L^s_{t,x}$ which is $L^{r/3}_{t,x}$ provided $r > 3$. Let $\frac{1}{s} + \frac{3}{r} = 1$. $\operatorname{supp} D[v,p]$ has positive Lebesgue measure, but $\operatorname{supp} D[u,p] \subset \operatorname{sing}(\operatorname{supp} U)$ also has positive measure.
There is an open problem to find a function $f(r)$ s.t. the condition $\frac{\zeta_r}{r} < \frac{1}{3} - f(r)$ works.

Proof. (Proof of *Duchon-Robert* formula) Considering Euler equation(Euler)

$$\eta_{\varepsilon\delta} * u := J_\zeta *_t \eta_\varepsilon *_x u$$

Let's test against $w_\varepsilon\delta = \eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v)$.

$$0 = -\int_{I\times\mathbb{R}^d} v^l\partial_t\eta_{\varepsilon\delta} + (\phi\eta * v_l) + v^j v^l\nabla_j\eta_{\varepsilon\delta} * (\phi\eta_{\varepsilon\delta} * v) + p\nabla^l\eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v_l)dxdt,$$

where $\phi \in C_c^\infty(I \times \mathbb{R}^d)$. Use the definition of self adjointness solution $\eta_{\varepsilon_\delta} *$ and divergence-free properties of $\eta_{\varepsilon_\delta} * v_l$. Then Euler equation becomes

$$0 = -\int \partial_t \phi \frac{|\eta_{\varepsilon\delta} * v|^2}{2} + v^j v^l \eta_{\varepsilon\delta} \nabla) j \phi \eta_{\varepsilon\delta} * v_l + p\eta_{\varepsilon\delta} * (\nabla^l \phi \eta_{\varepsilon\delta} * v_l) dxdt$$

Let $\varepsilon \to 0$ using uniform bounded ness of $\eta_\varepsilon *$ and $\nabla_j \eta_\varepsilon *$. As $\delta \to 0$, thanks to $\nabla_j \eta_\varepsilon *$, the time derivative naturally goes away. Then

$$0 = -\int \partial_t \phi \frac{{v_\varepsilon}^2}{2} + \nabla_j \phi (\frac{|v_\varepsilon|^2}{2} v^j + \eta_\varepsilon * p v_\varepsilon^j) dxdt \tag{2}$$

$$+ \int \phi \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} + Z_\varepsilon, \tag{3}$$

where $Z_\varepsilon = \int \nabla_j \phi R_\varepsilon^{jl} v_{\varepsilon l}$. Take both time and space derivative of ϕ. Using $v \in L^2_{t,x} \cap L^2_{t,x}$ and $p = (-\Delta)^{-1} \nabla_j \nabla_l (v^j v^l) \in L^{3/2}_{t,x}$
(2) $\Rightarrow < D[u,p], \phi >$ as $\varepsilon \to 0$.
(3) $\Rightarrow \lim_{\varepsilon \to 0} \nabla_j v_{\varepsilon j} R_\varepsilon^{jl} + Z_\varepsilon$

$$Z_\varepsilon = \int \nabla_j \phi (\eta_\varepsilon * (v^j v^l) - v_\varepsilon^j v_\varepsilon^l) v_{\varepsilon l} dxdt =: B_\varepsilon[v,v]$$

Here we define the commutator $B_\varepsilon[\cdot,\cdot]$.

$$\|B_\varepsilon[u,w]\|_{L^{3/2}_{t,x}} \leqslant C \|u\|_{L^3_{t,x}} \|w\|_{L^3_{t,x}}$$

which is independent of t.
If u or $w \in C_c^\infty$ $\|B_\varepsilon[u,w]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$. By approximation $\|B_\varepsilon[v,v]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$.
By Holder inequality,

$$|Z_\varepsilon| \leqslant \|\nabla \phi\|_{L^\infty} \|B_\varepsilon[v,v]\|_{L^{3/2}_{t,x}} \|v_\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0$$

□

Remark. Improvement:

- Calderon-Zygmund Theorem.
- More regularity in time.

Proof. (supp $D[u,p] \subset$ sing(supp v)) In fact supp $D[u,p] \subset$ sing(supp $L^3_\varepsilon B^{1/3}_{3,C(N)} v$).
What's good for not using Littlewood-Paley definition of Besov space? the solution above can be defined locally.
$\phi \in C_c^\infty(I \times B_q)$ and $B'_q \subset B_q$ a smaller ball with same center q. Let $q \notin$ sing(supp $B^{1/3}_{3,C(N)} v$)

$$< \phi, D[v,p] >= \lim_{\varepsilon \to 0} \int_I \int_{B'_q} \phi(t,x) \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} \lesssim \|\phi\|_{L^\infty} \int_I \|\nabla v_\varepsilon(t)\|_{L(B'_q)} \|R_\varepsilon\|_{L_x^{2/3}}$$

dominated by $\|\phi\|_{L^\infty} \int_I \|v(t,\cdot)\|^3_{B^{1/3}_{3,\infty}(B_\varepsilon)}$. For a.e. t, we have $\|\nabla v_\varepsilon\|_{L^3} \cdot \|R_\varepsilon\|_{L^{3/2}} = o(1)$ and $D[u,p] \to 0$ by dominate convergence theorem. □

Remark.
1. Heat flow approach can also be applied to this problem. The proof is quite different[7] and is on a compact Riemann manifold(no convolution can be used).
2. Compressible Euler Case. The problem lies when dealing with commutator estimation.

3 Holder Continuity

The following lecture are basic on [7].
Note that $B^{1/3}_{3,C(\mathbb{N}} \subsetneq B^{1/3}_{3,\infty}$ and we can find a function in $B^{1/3}_{3,\infty} \setminus B^{1/3}_{3,C(N)}$.
$\phi(x)\chi_{\{x'>0\}} \subset B^{1/p}_{p,\infty} \quad \forall 1 < p < \infty$ Let's consider

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0 \quad L^\infty_t B^{1/3}_{3,\infty}$$

Energy dissipation at time $t = 0$

$$\frac{d}{dt}e(t) = \int_{\mathbb{T}^d} \nabla_j v_{\varepsilon l} R^{jl}_\varepsilon(0,x)dx$$

Eyink proved that there exists a divergence-free vector field in the space $C^{1/3}B^{1/3}_{3,C(N)}$, s.t. $\frac{d}{dt}|_{t=0}e(t) < 0$. We have a useful counter example:

$$v(x) = \sum_q 2^{2q\alpha} \sin\bigl(2^{2q}x\bigr) \in \dot{B}^{1/3}_{3,C(N)} \setminus \dot{B}^{1/3}_{3,\infty}$$

Now we consider this problem on compact Riemannian manifold for the conclusion $L^3_t B^{1/3}_{3,C(N)}$. Consider (Euler), instead of $\eta_\varepsilon * v^l$, we consider

- Estimates(Commutator)
- Keeping divergence-free property

Define the opetator $-\Delta_H = d_\delta + \delta_d$, which looks like a 1-form. In Hodge heat flow equation,

$$\partial_s v^l = \Delta_H v^l = \nabla_j \nabla^j v^\ell - \mathrm{Ric}^k_\ell \, v^k$$

Since we know what the solution exactly is,

$$\eta_\varepsilon * v^l \to e^{s\Delta_H} v^l$$

The square root of heat time $s^{1/2} \sim \varepsilon$ and solution at time s $S_{[s]}v = e^{s\Delta_h}v$.
To estimate

$$\eta_\varepsilon * \nabla_j(v^j v^l) - \nabla_j(\eta_\varepsilon v^j \eta_\varepsilon * v^l),$$

we would need the commutator

$$w^l(s) = S_{[s]}\nabla_j(v^j v^l) - \nabla_j(S_{[s]}v^j S_{[s]}v^l)$$

and Riemannian manifold M will be always assumed to be smooth.

$$s \in (0,1] \qquad \int_{I\times M} \eta(t)[S_{[s]}\nabla_j(v^j v^l) - \nabla_j\left(S_{[s]}v^j S_{[s]}v^l\right) S_{[s]}v_l d^{1+d}\mathrm{vol},$$

here volume is in time$\times$ space.
Let's calculate

$$(\partial_s - \Delta_H)w^l = N^l(t,s)$$

$$w^l(s) = \int_0^s e^{(s-s')\Delta_H} N^l(t,s')ds',$$

by d'Alembert's formula.

$$\begin{aligned} w^l(t,s) \quad &= S_{[s]}\nabla_j(v^j v^l) - \nabla(S_{[s]}v^j S_{[s]}v^l) \\ (\partial_s - \Delta_H)w^l \quad &= (\partial_s - \Delta_H)\nabla_j(S_{[s]}v^j S_{[s]}v^l) \\ &= (\partial_s - \nabla_i\nabla^i)\nabla_j(S_{[s]}v^j S_{[s]}v^l) + \text{curvature terms} \\ &= -2\nabla_j(\nabla_i S_{[s]}v^j \nabla^i S_{[s]}v^l) + \text{low order terms} \end{aligned}$$

$$
\begin{aligned}
\text{Commutator} &= -2\int_{I\times M}\eta(t)\int_0^s e^{(s-s')\Delta}\nabla_j(\nabla_i S_{[s]}v^j\nabla^i S_{[s]}v^l)ds' S_{[s]}v_l d^{1+d}\text{vol}\\
\text{Integral by parts} &= 2\int_{I\times M}\eta(t)\int_0^s \nabla_i S_{[s]}v^j\nabla^i S_{[s]}\cdot \underbrace{S_{[2s-s']}}_{\text{very low frequency}} v_l ds' d^{1+d}\text{vol}
\end{aligned}
$$

Assume that $v\in L_t^3B_{3,\infty}^{\alpha}$, claim that $\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{L_t^3B_{3,\infty}^{\alpha}}$ with $\alpha>1/3$. First we can try $v\in L_t^3C^{\alpha}$ or $v\in L_t^3W^{\alpha,3}$.

$$
\begin{aligned}
|Commutator| &\lesssim \|\eta\|_{L^\infty}\int_0^s(2s-s')^{\frac{-1+\alpha}{2}}(s')^{-2\frac{1+\alpha}{2}}ds'\\
&\lesssim s^{-1/2+3\alpha/2}\int_0^1(2-\sigma)^{\frac{-1+\alpha}{2}}\sigma^{-1+\alpha}d\sigma\to 0 \text{ as } \varepsilon\to 0 \text{ with } \alpha<1/3
\end{aligned}
$$

No derivatives that we can take over the heat flow. In order to prove the claim: $\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{W^{\alpha,3}}$ with $\alpha>1/3$.

Proof.

$$
\begin{aligned}
\|\nabla ßv\|_{L^3}&\leqslant s^{-\frac{1}{2}}\|v\|_{L^3}\\
\|\nabla ßv\|_{L^3}&\leqslant \|v\|_{W^{1,3}}
\end{aligned}
$$

Since $u(s)=e^{s\Delta}u$, $\|u\|_{L^\infty}\lesssim\|u\|_{L^p}\quad s\in(0,1]$ On compact manifold we have that

$$
\|\nabla u\|_{L^p}\lesssim\|\nabla u\|_{L^r}+\|u\|_{L^p}
$$

Here let $p=2z$ where $z\geqslant 2$ is an integer. □

$$
\partial_s|u|^2-\Delta|u|+|\nabla u|^2=0(\text{or } -2\,\mathrm{Ric}_{jk}\,u^ju^k) \tag{4}
$$

$$
\Rightarrow\int_M|u|^2d\text{vol}\searrow \text{ and } \int_M u^{2z}d\text{vol}\searrow
$$

$$
\begin{aligned}
\tfrac{1}{z}\partial_s\int|u|^{2z}d\text{vol}&=\int_M(\Delta|u|^2-2|\nabla u|^2)|u|^{2(z-1)}d\text{vol}\\
&=-\int_M\nabla^j|u|^2\nabla_j|u||u|^{2(z-2)}-2|\nabla u|^2|u|^{2(z-1)}\leqslant 0
\end{aligned}
$$

For curvature terms, they can be bounded by $\|\mathrm{Ric}\|_{L^\infty}\int_M|u|^{2z}d\text{vol}$ remains bounded. So $\int_M u^{2z}d\text{vol}\leqslant\int_M u^{2z}(t=0)d\text{vol}$.

$$
\partial_s|\nabla u|^2-\Delta|\nabla u|^2+|\nabla\nabla u|^2(\text{BAD})=0(\text{ or Riem}(\nabla u\nabla u)+\nabla\text{Riem}u\nabla u \tag{5}
$$

Multiply by $|\nabla u|^{2(z-1)}$

References

[1] Peter Constantin, Edriss S Titi, and F Weinan. Onsager's conjecture on the energy conservation for solutions of euler's equation. *Communications in Mathematical Physics*, 165(1):207, 1994.

[2] Camillo De Lellis and László Székelyhidi Jr. Continuous dissipative euler flows and a conjecture of onsager. In *European Congress of Mathematics*, pages 13–29. Eur. Math. Soc. Zürich, 2013.

[3] Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible euler and navier-stokes equations. *Nonlinearity*, 13(1):249, 2000.

[4] Gregory L Eyink and Katepalli R Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Reviews of modern physics*, 78(1):87, 2006.

[5] Philip Isett. On the endpoint regularity in onsager's conjecture. *arXiv preprint arXiv:1706.01549*, 2017.

[6] Philip Isett. A proof of onsager's conjecture. *Annals of Mathematics*, 188(3):871–963, 2018.

[7] Philip Isett and Sung-Jin Oh. A heat flow approach to onsager's conjecture for the euler equations on manifolds. *Transactions of the American Mathematical Society*, 368(9):6519–6537, 2016.

[8] Philip James Isett et al. Hölder continuous euler flows with compact support in time. 2013.

[9] Charles Meneveau, KR Sreenivasan, P Kailasnath, and MS Fan. Joint multifractal measures: Theory and applications to turbulence. *Physical Review A*, 41(2):894, 1990.

[10] Vladimir Scheffer. An inviscid flow with compact support in space-time. *The Journal of Geometric Analysis*, 3(4):343–401, 1993.

[11] Alexander Shnirelman. On the nonuniqueness of weak solution of the euler equation. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 50(12):1261–1286, 1997.