

Weak Solution: Convex Integration

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1 Introduction

First let's give an example to help define what is the weak solution.

Example 1.1.

$$\Delta u = f$$

If u don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \rightarrow \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z} \alpha + \frac{\partial u}{\partial \bar{z}} \bar{\alpha} = \frac{\partial u}{\partial x} \operatorname{Re} \alpha + \frac{\partial u}{\partial y} \operatorname{Im} \alpha$$

$$\text{i.e. } du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}.$$

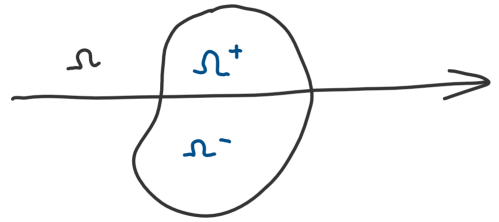
Theorem 1.1. If $\frac{\partial u}{\partial \bar{z}} = 0$ or $\Delta u = 0$ in the weak sense (against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.

Useful: $f_k(z)$ holomorphic $u(z) = \sum_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} \text{Fubini} &= - \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ &= - \sum_{k=0}^{\infty} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

Theorem 1.2. (Swartz reflection principle.) If f is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If f is continu-



ous on Ω on Ω including $\Omega \cap \{y = 0\}$. Then f is holomorphic on Ω .

In $D' f = \lim_{\delta \rightarrow 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here H is heaviside function.

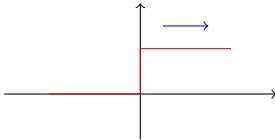
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \rightarrow 0} \frac{\partial f}{\partial \bar{z}} + f \left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial \bar{z}} \delta(\varepsilon - y) \right)$$

Since f is continuous $\lim_{\varepsilon \rightarrow 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\square u = 0 \quad \text{where } \square := -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2 \quad (\text{Wave})$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \quad \int_{\mathbb{R}} \square \phi u = 0$$

(Wave) has a solution on \mathbb{R}^{1+1} given by $u(t, x) = u(t - x)$, a traveling wave:



Example 1.2.

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on \mathbb{R}^{1+d}

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

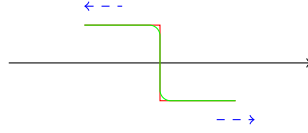


Figure 1: $t = 0$

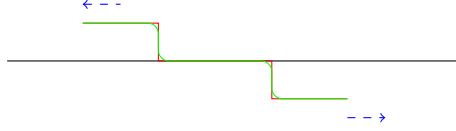


Figure 2: $t = 1$

2 Fluid Mechanics

2.1 Incompressible Euler Equation

Define 2 velocity field: $v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ $p : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p = 0$$

$$\nabla_j v^j = 0 \quad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{loc}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \partial\Omega$

$$\int_{\partial\Omega} v \cdot \vec{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt} \left[\begin{array}{c} \text{total momentum} \\ m \cdot v \end{array} \right] = [\text{Force on } \Omega] + \left[\begin{array}{c} \text{Flux of} \\ \text{momentum} \end{array} \right] \Rightarrow \frac{d}{dt} \int_{\Omega} v^l dx = \int_{\partial\Omega} p \vec{n}^l dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if p is good enough. If $v, p \in C^1$, use $\int_{\partial\Omega} f \vec{n}_j d\sigma = - \int_{\Omega} \nabla_j f dx$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} v^l &= - \left(\int_{\Omega} \nabla^l p + \nabla_j (v^j v^l) dx \right) \\ \int_{\Omega} (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) dx &= 0 \quad \forall \Omega, \quad \forall t \end{aligned}$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful? Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I \times \mathbb{R}^d)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l(t, x) dx = 0$$

If $(1 + |x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, then linear angular momentum conserved?

Here $\forall K^l$ s.t. $\nabla_j K_l + \nabla_l K_j = 0$ on \mathbb{R}^d

Example 2.1. $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_l v^l dx = \int_{\mathbb{R}^d} v^{(i)} dx$

Example 2.2. Rotation $K(a, b) = x^a e_b - x^b e_a$, $1 \leq a < b \leq d$.

$$\begin{aligned} \int_{\mathbb{R}^d} K_l (\partial_t v^l + \nabla : (v^j v^l) + \nabla^l p) dx &= 0 \\ \frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l - \int_{\mathbb{R}^d} \nabla : K_l (v^j v^l) - \int_{\mathbb{R}^d} \nabla^l K_l p dx &= 0 \end{aligned}$$

where $\operatorname{div} K = \nabla^l K_l = \delta^{jl} \nabla_j K_l = \frac{1}{2} \delta^{jl} (\nabla_j K_l + \nabla_l K_j) = 0$ by assumption.

$$\partial v^l + \nabla : (v^j v^l) + \nabla^l p) dx = 0$$

Test against a space cut-off function $K^l(B) := q(t) \varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here K is rotationally symmetric, so it is divergence-free.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_l^{(\beta)} v^l dx \right] dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_l^{(\beta)} v^j v^l - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$. i.e.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_l^{(\beta)} v^l dx \right] dt - \frac{1}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left(\nabla_j K_l^{(\beta)} + \nabla_l K_j^{(\beta)} \right) v^j v^l dx dt - \frac{1}{2} \int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi\left(\frac{|x|}{B}\right) K_l v^j v^l dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L_{t,x}^1$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

2.2 Conservation of Energy

If $(1 + |x|)v \in L_{t,x}^1(I \times \mathbb{R}^d)$, $v \in L_{t,x}^2(I \times \mathbb{R}^d)$, then $\forall K^l$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l(t, x) dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : e_i, 1 \leq i \leq d, 1 \leq a < b \leq d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^l = \nabla_j w^{jl} \quad w^{jl} = -w^{lj} \quad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B}) w^{jl})$

$$w^{jl} = x^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)$$

$$\begin{aligned} & \nabla_j (x^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)) \\ &= \delta_j^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l) = \delta_1^l \end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j \nabla_j w^{jl} = -\nabla_l \nabla_j w^{lj} = -\nabla_j \nabla_l w^{lj} = -\nabla_l \nabla_j w^{jl}$$

Conservation of energy means that $\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|v|^2}{2}(t, x) dx = 0$. Note that energy is nonlinear.

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) = 0$$

If $v \in C^1 \cap L_{t,x}^2 \cap L_{t,x}^3(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here B could be ∞ . Multiply the local energy by $\eta(t) \varphi(\frac{|x|}{B})$

$$\begin{aligned} & \int \eta \frac{d}{dt} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|^2}{2}(t, x) dx dt \quad - \int \eta(t) \int \nabla_j \left[\varphi\left(\frac{|x|}{B}\right) \right] \left(\frac{|v|^2}{2} + p \right) v^j dx dt \\ & \quad (1) \quad (2) \\ (1) &= - \int \underbrace{\eta'(t)} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|^2}{2}(t, x) dx dt \quad \text{Integral by parts} \quad (\text{Local}) \\ & \quad \text{dominated by } \frac{|x|^2}{2} |\eta'| \in L_{t,x}^1 \end{aligned}$$

term (2) converge to 0 pointwisely when $B \rightarrow \infty$ and dominated by $|\eta t| \left(\frac{|v|^3}{2} + |p||v| \right)$. Let's recall Euler equation.

$$\begin{cases} \partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p &= 0, \\ \nabla_j v^j &= 0 \end{cases} \quad (\text{Euler})$$

Take divergence over (Euler), \Rightarrow

$$\nabla_j \nabla_l (v^j v^l) + \nabla_l \nabla^l p = 0$$

i.e.

$$\begin{aligned} \Delta p &= -\nabla_l \nabla_j (v^j v^l) \\ p &= \underbrace{(-\Delta)^{-1} \nabla_l \nabla_j (v^j v^l)}_{\text{zero order operator}} \underbrace{(v^j v^l)}_{\in L_{t,x}^{3/2}} \end{aligned}$$

Thus naturally $p \in L_x^{3/2}$ a.e. $t \in \mathbb{R}^+$

$$\|p\|_{L_x^{3/2}(L_t^{3/2})} = \|p\|_{L_{t,x}^{3/2}} < \infty$$

$$v_l (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) = 0$$

$$\nabla_j v^j = 0$$

Thus

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v_l v^j \nabla_j v^l + v_l \nabla^l p = 0$$

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v^j \nabla_j \left(\frac{|v|^2}{2} + v_j v^j p \right) = 0$$

$$\nabla_j v^j = 0$$

$$\partial_t \left(\frac{|v|^2}{2} + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \right) = 0$$

Using $\nabla_j v^j = 0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?

In turbulence situation (Navier-Stokes equations with $\nu \ll 1$)

$$v_l (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) = \nu v_l \Delta v^l$$

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx = -\nu \int |\nabla v|^2 dx = \nu \int v_l \nabla_i \nabla^i v^l$$

Taking a formal limit, \exists incompressible Euler flows with

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx < -\varepsilon < 0$$

Theorem 2.1. *Onsager's Conjecture*

(+) If $\alpha > 1/3$ and $(v(t, x + \Delta x) - v(t, x)) \leq c|\Delta x|$ where $x \in \mathbb{T}^3 (v \in L_t^\infty C_x^\alpha)$, then the energy conserved.

(-) (K41) If $\alpha \leq 1/3 \exists$ incompressible Euler flows with $v \in L_t^\infty L_x^\alpha$ s.t. $\int_{\mathbb{T}^d} \frac{|v|^2}{2}(t, x) dx$ is not constant.

Now we follow [2] and discuss the (+) part first.

$$\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p = 0$$

In order to get into Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series,

$$v(x, t) = \sum_{k \in \mathbb{Z}^3} a_k(t) e^{ik \cdot x}.$$

Obviously $a_{-k} = \overline{a_k}$, because v is real-valued. Moreover the divergence-free constraint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k :

$$\frac{da_k}{dt} = i \sum_{\ell} a_{k-\ell} \cdot \ell \left[-a_\ell + \frac{(a_\ell \cdot k) k}{|k|^2} \right] - \nu |k|^2 a_k \quad (1)$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.

(Don't understand) Energy starts at low wave numbers and moves to higher wave numbers in finite number.

$\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.

Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.

(K 41) $E \lim_{v \rightarrow 0} \left\langle v \int |\nabla v|^2 dx \right\rangle$ and v determine all statistic properties of turbulent flows.

$$\langle |v(x + \Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.

Now (+) is solved by [3] and [1] with the goal $L_t^3 B_{3,C(N)}^{1/3}, L_t^3 B_{3,\infty}^{1/3+\varepsilon}$.

(-) is solved ($d \geq 3$) with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [4].

2.3 Local energy conservation

$$\partial_t v_\varepsilon^l + \nabla_l (v_\varepsilon^l v_\varepsilon^l) + \nabla^l p_\varepsilon = -\nabla_j R_\varepsilon^{jl}$$

where $R_\varepsilon^{jl} = \eta_\varepsilon * (v^j v^l) - v_\varepsilon^j v_\varepsilon^l$

$$\|R_\varepsilon(t, \cdot)\|_{L_t^{3/2}} \leq \varepsilon^{2\alpha} \|v(t)\|_{B_{3,\infty}^\alpha}^2$$

$$\frac{1}{2} \int \frac{|v_\varepsilon|^2}{2}(t, x) dx = \lim_{\varepsilon \rightarrow 0} \int \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} dx$$

Here to clarify the space:

$$B_{3,c(N)}^{1/3} = (\overline{C^\infty})^{B_{3,\infty}^{1/3}} = B_{3,\infty}^{1/3} \cap \left\{ \lim_{h \rightarrow 0} \frac{|v(x+h) - v(x)|}{|h|^{1/3}} = 0 \right\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3} \setminus c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3} \setminus c^{1/3}$

Lemma 2.2. $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ if $v \in B_{3,c(N)}^\alpha$

Proof. Claim: $\varepsilon^{1-\alpha} \nabla(\eta_\varepsilon * \cdot) : B_{3,\infty}^\alpha \rightarrow L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L_x^3} \lesssim \varepsilon^{-1+\alpha} \|v\|_{B_{3,\infty}^\alpha}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B_{3,\infty}^\alpha} < \frac{\delta}{2C_2}$.

$$\begin{aligned} \|\varepsilon^{1-\alpha} \nabla \eta_\varepsilon * v\|_{L_x^3} &\leq \|\varepsilon^{1-\alpha} \nabla \eta_\varepsilon * (v - \tilde{v})\|_{L_x^3} + \|\varepsilon^{1-\alpha} \nabla \eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \|\nabla \eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \tilde{C} \quad \text{for } \varepsilon^{1-\alpha} < \frac{\delta}{2\tilde{C}} \text{ and } \|\varepsilon^{1-\alpha} \nabla \eta_\varepsilon * v\|_{L_x^3} < \delta \end{aligned}$$

$$\int_I \frac{d}{dt} \int \frac{|v_\varepsilon|^2}{2}(t, x) dx dt \leq$$

□

References

- [1] Peter Constantin, Edriss S Titi, and F Weinan. Onsager's conjecture on the energy conservation for solutions of euler's equation. *Communications in Mathematical Physics*, 165(1):207, 1994.
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- [4] Philip Isett. On the endpoint regularity in onsager's conjecture. *arXiv preprint arXiv:1706.01549*, 2017.