

Weak Solution: Convex Integration

Yiran Hu

March 2, 2020

1 Introduction

First let's give an example to help define what is the weak solution.

Example 1.1.

$$\Delta u = f$$

If u don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \rightarrow \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z} \alpha + \frac{\partial u}{\partial \bar{z}} \bar{\alpha} = \frac{\partial u}{\partial x} \operatorname{Re} \alpha + \frac{\partial u}{\partial y} \operatorname{Im} \alpha$$

$$\text{i.e. } du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}.$$

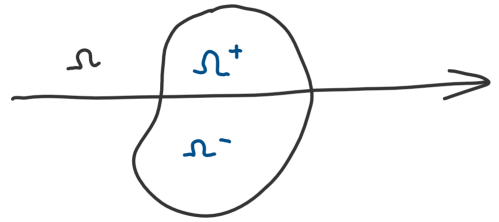
Theorem 1.1. If $\frac{\partial u}{\partial \bar{z}} = 0$ or $\Delta u = 0$ in the weak sense (against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.

Useful: $f_k(z)$ holomorphic $u(z) = \sum_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} \text{Fubini} &= - \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ &= - \sum_{k=0}^{\infty} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

Theorem 1.2. (Swartz reflection principle.) If f is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If f is continu-



ous on Ω on Ω including $\Omega \cap \{y = 0\}$. Then f is holomorphic on Ω .

In $D' f = \lim_{\delta \rightarrow 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here H is heaviside function.

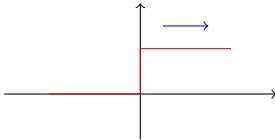
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \rightarrow 0} \frac{\partial f}{\partial \bar{z}} + f \left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial \bar{z}} \delta(\varepsilon - y) \right)$$

Since f is continuous $\lim_{\varepsilon \rightarrow 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\square u = 0 \quad \text{where } \square := -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2 \quad (\text{Wave})$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \quad \int_{\mathbb{R}} \square \phi u = 0$$

(Wave) has a solution on \mathbb{R}^{1+1} given by $u(t, x) = u(t - x)$, a traveling wave:



Example 1.2.

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on \mathbb{R}^{1+d}

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

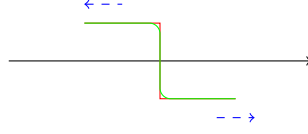


Figure 1: $t = 0$

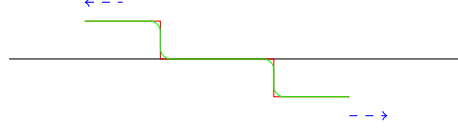


Figure 2: $t = 1$

2 Fluid Mechanics

2.1 Incompressible Euler Equation

Define 2 velocity field: $v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ $p : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0$$

$$\nabla_j v^j = 0 \quad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{loc}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \partial\Omega$

$$\int_{\partial\Omega} v \cdot \vec{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt} \left[\begin{array}{c} \text{total momentum} \\ m \cdot v \end{array} \right] = [\text{Force on } \Omega] + \left[\begin{array}{c} \text{Flux of} \\ \text{momentum} \end{array} \right] \Rightarrow \frac{d}{dt} \int_{\Omega} v^\ell dx = \int_{\partial\Omega} p \vec{n}^\ell dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if p is good enough. If $v, p \in C^1$, use $\int_{\partial\Omega} f \vec{n}_j d\sigma = - \int_{\Omega} \nabla_j f dx$

$$\frac{d}{dt} \int_{\Omega} v^\ell = - \left(\int_{\Omega} \nabla^\ell p + \nabla_j (v^j v^\ell) dx \right)$$

$$\int_{\Omega} (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) dx = 0 \quad \forall \Omega, \quad \forall t$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful?

Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I \times \mathbb{R}^d)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_\ell v^\ell(t, x) dx = 0$$

If $(1 + |x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, then linear angular momentum conserved?

Here $\forall K^\ell$ s.t. $\nabla_j K_\ell + \nabla_\ell K_j = 0$ on \mathbb{R}^d

Example 2.1. $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_\ell v^\ell dx = \int_{\mathbb{R}^d} v^{(i)} dx$

Example 2.2. Rotation $K(a, b) = x^a e_b - x^b e_a$, $1 \leq a < b \leq d$.

$$\int_{\mathbb{R}^d} K_\ell (\partial_t v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p) dx = 0$$

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_\ell v^\ell - \int_{\mathbb{R}^d} \nabla : K_\ell (v^j v^\ell) - \int_{\mathbb{R}^d} \nabla^\ell K_\ell p dx = 0$$

where $\operatorname{div} K = \nabla^\ell K_\ell = \delta^{j\ell} \nabla_j K_\ell = \frac{1}{2} \delta^{j\ell} (\nabla_j K_\ell + \nabla_\ell K_j) = 0$ by assumption.

$$\partial v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p dx = 0$$

Test against a space cut-off function $K^\ell(B) := q(t) \varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here K is rotationally symmetric, so it is divergence-free.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_\ell^{(\beta)} v^\ell dx \right] dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_\ell^{(\beta)} v^j v^\ell - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$. i.e.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_\ell^\beta v^\ell dx \right] dt - \frac{1}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left(\nabla_j K_\ell^{(\beta)} + \nabla_\ell K_j^{(\beta)} \right) v^j v^\ell dx dt - \frac{1}{2} \int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi\left(\frac{|x|}{B}\right) K_\ell v^j v^\ell dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L_{t,x}^1$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

2.2 Conservation of Energy

If $(1 + |x|)v \in L_{t,x}^1(I \times \mathbb{R}^d)$, $v \in L_{t,x}^2(I \times \mathbb{R}^d)$, then $\forall K^\ell$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_\ell v^\ell(t, x) dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : e_i, 1 \leq i \leq d, 1 \leq a < b \leq d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^\ell = \nabla_j w^{j\ell} \quad w^{j\ell} = -w^{lj} \quad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B}) w^{j\ell})$

$$w^{j\ell} = x^2 (\delta_2^j \delta_1^\ell - \delta_1^j \delta_2^\ell)$$

$$\begin{aligned} & \nabla_j (x^2 (\delta_2^j \delta_1^\ell - \delta_1^j \delta_2^\ell)) \\ &= \delta_j^2 (\delta_2^j \delta_1^\ell - \delta_1^j \delta_2^\ell) = \delta_1^\ell \end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j \nabla_j w^{j\ell} = -\nabla_\ell \nabla_j w^{lj} = -\nabla_j \nabla_\ell w^{lj} = -\nabla_\ell \nabla_j w^{j\ell}$$

Conservation of energy means that $\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|v|^2}{2}(t, x) dx = 0$. Note that energy is nonlinear.

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|}{2} + p \right) v^j \right) = 0$$

If $v \in C^1 \cap L_{t,x}^2 \cap L_{t,x}^3(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here B could be ∞ . Multiply the local energy by $\eta(t) \varphi(\frac{|x|}{B})$

$$\int \eta \frac{d}{dt} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|^2}{2}(t, x) dx dt \quad - \int \eta(t) \int \nabla_j \left[\varphi\left(\frac{|x|}{B}\right) \right] \left(\frac{|v|^2}{2} + p \right) v^j dx dt$$

(1)
(2)

$$(1) = - \int \underbrace{\eta'(t)} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|}{2}(t, x) dx dt \quad \text{Integral by parts} \quad (\text{Local})$$

$$\text{dominated by } \frac{|x|^2}{2} |\eta'| \in L_{t,x}^1$$

term (2) converge to 0 pointwisely when $B \rightarrow \infty$ and dominated by $|\eta t| \left(\frac{|v|^3}{2} + |p||v| \right)$.
Let's recall Euler equation.

$$\begin{cases} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= 0, \\ \nabla_j v^j &= 0 \end{cases} \quad (\text{Euler})$$

Take divergence over (Euler), \Rightarrow

$$\nabla_j \nabla_\ell (v^j v^\ell) + \nabla_\ell \nabla^\ell p = 0$$

i.e.

$$\begin{aligned} \Delta p &= -\nabla_\ell \nabla_j (v^j v^\ell) \\ p &= \underbrace{(-\Delta)^{-1} \nabla_\ell \nabla_j (v^j v^\ell)}_{\text{zero order operator}} \underbrace{(v^j v^\ell)}_{\in L_{t,x}^{3/2}} \end{aligned}$$

Thus naturally $p \in L_x^{3/2}$ a.e. $t \in \mathbb{R}^+$

$$\|p\|_{L_x^{3/2}(L_t^{3/2})} = \|p\|_{L_{t,x}^{3/2}} < \infty$$

$$v_\ell (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) = 0$$

$$\nabla_j v^j = 0$$

Thus

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v_\ell v^j \nabla_j v^\ell + v_\ell \nabla^\ell p = 0$$

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v^j \nabla_j \left(\frac{|v|^2}{2} + v_j v^j p \right) = 0$$

$$\nabla_j v^j = 0$$

$$\partial_t \left(\frac{|v|^2}{2} + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \right) = 0$$

Using $\nabla_j v^j = 0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?

In turbulence situation (Navier-Stokes equations) with $\nu \ll 1$

$$v_\ell (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) = \nu v_\ell \Delta v^\ell$$

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx = -\nu \int |\nabla v|^2 dx = \nu \int v_\ell \nabla_i \nabla^i v^\ell$$

Taking a formal limit, \exists incompressible Euler flows with

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx < -\varepsilon < 0$$

Theorem 2.1. *Onsager's Conjecture*

(+) If $\alpha > 1/3$ and $(v(t, x + \Delta x) - v(t, x)) \leq c|\Delta x|$ where $x \in \mathbb{T}^3 (v \in L_t^\infty C_x^\alpha)$, then the energy conserved.

(-) (K41) If $\alpha \leq 1/3 \exists$ incompressible Euler flows with $v \in L_t^\infty L_x^\alpha$ s.t. $\int_{\mathbb{T}^d} \frac{|v|^2}{2}(t, x) dx$ is not constant.

Now we follow [2] and discuss the (+) part first.

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0$$

In order to get into Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series,

$$v(x, t) = \sum_{k \in \mathbb{Z}^3} a_k(t) e^{ik \cdot x}.$$

Obviously $a_{-k} = \overline{a_k}$, because v is real-valued. Moreover the divergence-free constraint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k :

$$\frac{da_k}{dt} = i \sum_\ell a_{k-\ell} \cdot \ell \left[-a_\ell + \frac{(a_\ell \cdot k) k}{|k|^2} \right] - \nu |k|^2 a_k \quad (1)$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.

(Don't understand) Energy starts at low wave numbers and moves to higher wave numbers in finite number.

$\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.

Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.

(K 41) $E \lim_{v \rightarrow 0} \langle v \int |\nabla v|^2 dx \rangle$ and v determine all statistic properties of turbulent flows.

$$\langle |v(x + \Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.

Now (+) is solved by [4] and [1] with the goal $L_t^3 B_{3,C(N)}^{1/3}, L_t^3 B_{3,\infty}^{1/3+\varepsilon}$.

(-) is solved ($d \geq 3$) with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [7].

Convex integration originated from the Nash–Kuiper Paradox(50's) for C^1 isometric embedding. Connection to Euler equation discovered by Camillo De Lellis and László Székelyhidi (08,12). First result towards Onsager conjecture is in [8]. And $\alpha < \frac{1}{5}$ by [10]. The non-uniqueness example was first given by [12] and then Shnirelman give a different proof in [13].

2.3 Another way of proving (+)

(+) (Eyink, Constantin, E, Titi 94') $L^3(B_{3,\infty}^\alpha)$

$$\|v\|_{C^\alpha} = \sup_{h \neq 0} \frac{\|v(x+h) - v(x)\|_{L^\infty}}{|h|^\alpha}$$

$$\|v\|_{B_{3,\infty}^\alpha} = \sup_{h \neq 0} \frac{\|v(x+h) - v(x)\|_{L^3}}{|h|^\alpha}$$

Lemma 2.2. *Commutator Estimate*

$$R_\varepsilon^{j\ell} = \eta_\varepsilon * (v^j v^\ell) - (v_\varepsilon^j v_\varepsilon^\ell)$$

$$\|R_\varepsilon\|_{L^{3/2}} \lesssim \varepsilon^{2\alpha} \|v\|_{B_{3,\infty}^\alpha}^2$$

Let's think $R_\varepsilon^{j\ell}$ as an expectation with the idea:

$$R = \mathbb{E}[v^2] - (\mathbb{E}[v])^2 = \mathbb{E}[(v - \mathbb{E}(v))^2],$$

which is quadratic.

$$R_\varepsilon^{j\ell} = \int v^i(x-h) v^\ell(x-h) \eta_\varepsilon(h) dh - \int v^j(x-h_1) \eta_\varepsilon(h_1) dh_1 \int v^\ell(x-h_2) \eta_\varepsilon(h_2) dh_2$$

$$\text{Using } \int \eta_\varepsilon(h) dh = 1$$

$$= \int (v^j(x-h) - v_\varepsilon^j(x)) (v^\ell(x-h) - v_\varepsilon^\ell(x)) \eta_\varepsilon(h) dh$$

By Lemmas in [1], we decompose above equation into $\sum_{i=1}^4 R_{\varepsilon i}^{j\ell}$, where

$$R_{\varepsilon 1} = \int (v^j(x-h) - v_\varepsilon^j(x-h)) (v^\ell(x-h) - v_\varepsilon^\ell(x-h)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 2} = \int (v_\varepsilon^j(x-h) - v_\varepsilon^j(x)) (v^\ell(x-h) - v_\varepsilon^\ell(x-h)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 3} = \int (v^j(x-h) - v_\varepsilon^j(x-h)) (v_\varepsilon^\ell(x-h) - v_\varepsilon^\ell(x)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 4} = \int (v_\varepsilon^j(x-h) - v_\varepsilon^j(x)) (v_\varepsilon^\ell(x-h) - v_\varepsilon^\ell(x)) \eta_\varepsilon(h) dh$$

For example,

$$R_{\varepsilon 2} = \int_{\mathbb{R}^d} \int_0^1 \frac{d}{d\sigma} v_\varepsilon^j(x - \sigma h) d\sigma (v^\ell(x-h) - v_\varepsilon^\ell(x)) \eta_\varepsilon(h) dh$$

$$= \int_{\mathbb{R}^d} \int_0^1 d\sigma \nabla_i v_\varepsilon^j(x - \sigma h) h^i (v^\ell(x-h) - v_\varepsilon^\ell(x)) \eta_\varepsilon(h) dh$$

$$\|R_{\varepsilon 2}^j\| \leq_{\mathbb{R}^d} \int_0^1 \|\nabla v_\varepsilon\|_{L^3} |h| \|v(\cdot - h) - v(\cdot)\|_{L^3} |\eta_\varepsilon(h)| dh$$

Modify the equation with modifier η_ε :

$$\begin{aligned} \eta_\varepsilon * (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) &= 0 \\ \partial_t v_\varepsilon^\ell + \nabla_j (v_\varepsilon^j v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon &= -\nabla_j R_\varepsilon^{j\ell} \end{aligned}$$

(Thus we need smoothness in time) $\times v_\varepsilon$ then integral by parts:

$$\partial_t \left(\frac{|v_\varepsilon|^2}{2} \right) + v_{\varepsilon l} \nabla_j (v_\varepsilon^j v_\varepsilon^\ell) + v_{\varepsilon l} \nabla^\ell p_\varepsilon = -v_{\varepsilon l} \nabla_j R_\varepsilon^{j\ell} = \int_{\mathbb{R}^d} \nabla_j \left| \frac{v_\varepsilon^2}{2} v_\varepsilon^j \right| \rightarrow 0$$

with assumption.

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|v_\varepsilon|^2}{2} (t, x) dx + \int_{\mathbb{R}^d} v_\varepsilon^j \nabla_j v_\varepsilon^\ell v_{\varepsilon l} + \nabla^\ell v_{\varepsilon l} p_\varepsilon = \int_{\mathbb{R}^d} \nabla v_{\varepsilon l} R_\varepsilon^{j\ell}$$

$\nabla^\ell v_{\varepsilon l} p_\varepsilon = 0$ for divergence-free.

LHS converges to $\frac{d}{dt} \int \frac{|v|^2}{2} (t, x) dx$ in $\mathcal{D}'(\mathbb{R})$ since $v_\varepsilon \rightarrow v$ in $L_{t,x}^2$.

$$\begin{aligned} \left\| \frac{d}{dt} \int \frac{|v|^2}{2} (t, x) dx \right\|_{L_t^1} &\leq \limsup_{\varepsilon \rightarrow 0} \int \int |\nabla_j v(t, x) R_\varepsilon^{j\ell}| dx dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int \|\nabla v_\varepsilon(t, \cdot)\|_{L_x^3} \|R_\varepsilon\|_{L^{3/2}} dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int \varepsilon^{-1+\alpha} \|v(t)\|_{B_{3,\infty}^\alpha} \|v(t, \cdot)\|_{B_{3,\infty}^\alpha}^2 dt \\ &< \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1+3\alpha} \int \|v(t, \cdot)\|_{B_{3,\infty}^\alpha}^3 dt \rightarrow 0 \quad \text{with } \alpha > \frac{1}{3} \end{aligned}$$

If $\alpha = \frac{1}{3}$ and v bounded in $L_t^1(I)$ for some finite time period.

$$\frac{d}{dt} \int \frac{|v|^2}{2} (t, x) dx = \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \int \frac{|v_\varepsilon|^2}{2} (t, x) dx$$

$v\phi \in C_c^\infty(I)$

$$\frac{d}{dt} < \int \frac{|v|^2}{2} (t, x) dx, \phi > \leq \|\phi\|_{L^\infty(I)}$$

LHS is of finite measure. $e(t) = \int \frac{|v|^2}{2} (t, x)$ is of bounded variation. IN fact $\frac{d}{dt} e(t)$ is finite.

(???) If $v \in L^r B_{3,\infty}^{1/3}$, consider $\left\| \frac{d}{dt} e(t) \right\|_{L_t^{r'/t}}$ using duality. $u \in L_t^\infty B_{3,\infty}^{1/3}$ uniformly $\left\| \frac{d}{dt} e(t) \right\|_{L^{i_nfty} t} \leq C$ and also $\frac{d}{dt} e(t) \leq -\varepsilon < 0$ is stable under perturbation. If not, the dissipation $\int_{\mathbb{R}^d} \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx$ can be really big.

Remark. The singular support of a generalized function u is the complement of the largest open set on which u is smooth. Roughly speaking, it is the closed set where the distribution does not correspond to a smooth function.

2.4 Local energy conservation

$$\partial_t v_\varepsilon^\ell + \nabla_\ell (v_\varepsilon^\ell v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon = -\nabla_j R_\varepsilon^{j\ell}$$

where $R_\varepsilon^{j\ell} = \eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell$

$$\begin{aligned} \|R_\varepsilon(t, \cdot)\|_{L_t^{3/2}} &\leq \varepsilon^{2\alpha} \|v(t)\|_{B_{3,\infty}^\alpha}^2 \\ \frac{1}{2} \int \frac{|v_\varepsilon|^2}{2} (t, x) dx &= \lim_{\varepsilon \rightarrow 0} \int \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx \end{aligned}$$

Here to clarify the space:

$$B_{3,c(N)}^{1/3} = (\overline{C^\infty})^{B_{3,\infty}^{1/3}} = B_{3,\infty}^{1/3} \cap \left\{ \lim_{h \rightarrow 0} \frac{|v(x+h) - v(x)|}{|h|^{1/3}} = 0 \right\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3} \setminus c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3} \setminus c^{1/3}$

Lemma 2.3. $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ if $v \in B_{3,c(N)}^\alpha$

Proof. Claim: $\varepsilon^{1-\alpha}\nabla(\eta_\varepsilon * \cdot) : B_{3,\infty}^\alpha \rightarrow L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L_x^3} \lesssim \varepsilon^{-1+\alpha} \|v\|_{B_{3,\infty}^\alpha}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B_{3,\infty}^\alpha} < \frac{\delta}{2C_2}$.

$$\begin{aligned} \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\|_{L_x^3} &\leq \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * (v - \tilde{v})\|_{L_x^3} + \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \|\nabla\eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \tilde{C} \quad \text{for } \varepsilon^{1-\alpha} < \frac{\delta}{2\tilde{C}} \text{ and } \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\|_{L_x^3} < \delta \end{aligned}$$

$$\begin{aligned} \int_I \frac{d}{dt} \int \frac{|v_\varepsilon|^2}{2}(t,x) dx dt &\leq \limsup_{\varepsilon \rightarrow 0} \int_I \int |\nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell}| dx dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_I \|\nabla v_\varepsilon(t)\|_{L_x^3} \varepsilon^{2\alpha} \|v\|_{B_{3,\infty}^\alpha}^2 dt \end{aligned}$$

For a.e. t , $v \in B_{3,C(N)}^{1/3}$, the integrand is bounded by $o(\varepsilon^{-1+2/3})\varepsilon^{3/2} = o(1)$. Thus above integral is dominated by:

$$\int_I \varepsilon^{-1 \times 1/3 + 2/3} \|v(t)\|_{B_{3,\infty}^{1/3}}^3 dt \leq \int_I \|v(t)\|_{B_{3,\infty}^{1/3}}^3 dt$$

By assumption and DCT, bounded. \square

Theorem 2.4. (Isett 18') An energy dissipating solution whose singularities have 0 Lebesgue measure in \mathbb{R}^4 cannot be of class $L_t^r B_{\zeta,\infty}^{1/3}$ if $r > 3$.

Compared with Meneveau-Sreenivasan [11],

$$< |v(x + \Delta x) - v(x)|^r > = |\Delta x|^{\xi_r}$$

singular support in $L_t^3 B_{3,C(N)}^{1/3}$. (K41) implies $\xi_r \sim \frac{r}{3}$ (only correct when $r = 3$).

Lemma 2.5. (Local energy conservation Duchon-Robert[3] formula $D[v,p] = \partial_t(\frac{|v|^2}{2}) + \nabla(\frac{|v|^2}{2} + p)v^j = \lim_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell}$ dissipation distribution $v \in L_{t,x}^3$. If $D[v,p] = 0$ and $v \in L_{t,x}^2 \cap L_{t,x}^3$, then $\int \frac{|v|^2}{2}(t,x) dx$ is constant and $D[v,p]$ if $v,p \in C^1$.

If $v \in L_t^r B_{r,\infty}^{1/3}$ with $r > 3$ is energy dissipating, since $v \in L_{t,x}^2 \cap L_{t,x}^3$, $D[v,p] \neq 0$. Since $\frac{d}{dt} \int \frac{|v|^2}{2} = \int D[v,p] dx$ But we claim that

$$\|D[v,p]\|_{L_{t,x}^{r/3}} < \infty$$

using Duchon-Robert formula: $\|\nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell}\|_{L_{t,x}^{r/3}}$ is bounded uniformly in $\varepsilon > 0$.

$$\begin{aligned} \|\nabla_j v_\varepsilon\|_{L_x^r} &\lesssim \varepsilon^{-1+1/3} \|v(t)\|_{B_{r,\infty}^{1/3}} \\ \|R_\varepsilon^{j\ell}\| &\lesssim \varepsilon^{2/3} \|v(t)\|_{B_{r,\infty}^{1/3}}^2 \\ \Rightarrow \quad \forall \phi \in C^i nfty_{ty_c}(I \times \mathbb{R}^d) \quad < D < [v,p], p > \lesssim C \|p\|_{L_{t,x}^s} \end{aligned}$$

Then $D[v,p]$ is in the dual of $L_{t,x}^s$ which is $L_{t,x}^{r/3}$ provided $r > 3$. Let $\frac{1}{s} + \frac{3}{r} = 1$. $\text{supp } D[v,p]$ has positive Lebesgue measure, but $\text{supp } D[u,p] \subset \text{sing}(\text{supp } U)$ also has positive measure.

There is an open problem to find a function $f(r)$ s.t. the condition $\frac{\zeta_r}{r} < \frac{1}{3} - f(r)$ works.

Proof. (Proof of Duchon-Robert formula) Considering Euler equation(Euler)

$$\eta_{\varepsilon\delta} * u := J_\zeta *_t \eta_\varepsilon *_x u$$

Let's test against $w_\varepsilon \delta = \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v)$.

$$0 = - \int_{I \times \mathbb{R}^d} v^\ell \partial_t \eta_{\varepsilon\delta} + (\phi \eta * v_\ell) + v^j v^\ell \nabla_j \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v) + p \nabla^\ell \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v_\ell) dx dt,$$

where $\phi \in C_c^\infty(I \times \mathbb{R}^d)$. Use the definition of self adjointness solution $\eta_{\varepsilon\delta}*$ and divergence-free properties of $\eta_{\varepsilon\delta} * v_\ell$. Then Euler equation becomes

$$0 = - \int \partial_t \phi \frac{|\eta_{\varepsilon\delta} * v|^2}{2} + v^j v^\ell \eta_{\varepsilon\delta} \nabla_j \phi \eta_{\varepsilon\delta} * v_\ell + p \eta_{\varepsilon\delta} * (\nabla^\ell \phi \eta_{\varepsilon\delta} * v_\ell) dx dt$$

Let $\varepsilon \rightarrow 0$ using uniform boundedness of $\eta_\varepsilon*$ and $\nabla_j \eta_\varepsilon*$. As $\delta \rightarrow 0$, thanks to $\nabla_j \eta_\varepsilon*$, the time derivative naturally goes away. Then

$$0 = - \int \partial_t \phi \frac{v_\varepsilon^2}{2} + \nabla_j \phi \left(\frac{|v_\varepsilon|^2}{2} v^j + \eta_\varepsilon * p v_\varepsilon^j \right) dx dt \quad (2)$$

$$+ \int \phi \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} + Z_\varepsilon, \quad (3)$$

where $Z_\varepsilon = \int \nabla_j \phi R_\varepsilon^{j\ell} v_{\varepsilon l}$. Take both time and space derivative of ϕ . Using $v \in L_{t,x}^2 \cap L_{t,x}^2$ and $p = (-\Delta)^{-1} \nabla_j \nabla_\ell (v^j v^\ell) \in L_{t,x}^{3/2}$
(2) $\Rightarrow \langle D[u, p], \phi \rangle$ as $\varepsilon \rightarrow 0$.
(3) $\Rightarrow \lim_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} + Z_\varepsilon$

$$Z_\varepsilon = \int \nabla_j \phi (\eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell) v_{\varepsilon l} dx dt =: B_\varepsilon[v, v]$$

Here we define the commutator $B_\varepsilon[\cdot, \cdot]$.

$$\|B_\varepsilon[u, w]\|_{L_{t,x}^{3/2}} \leq C \|u\|_{L_{t,x}^3} \|w\|_{L_{t,x}^3}$$

which is independent of t .

If u or $w \in C_c^\infty$ $\|B_\varepsilon[u, w]\|_{L_{t,x}^{3/2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. By approximation $\|B_\varepsilon[v, v]\|_{L_{t,x}^{3/2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

By Holder inequality,

$$|Z_\varepsilon| \leq \|\nabla \phi\|_{L^\infty} \|B_\varepsilon[v, v]\|_{L_{t,x}^{3/2}} \|v_\varepsilon\|_{L^2} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

□

Remark. Improvement:

- Calderon-Zygmund Theorem.
- More regularity in time.

Proof. ($\text{supp } D[u, p] \subset \text{sing}(\text{supp } v)$) In fact $\text{supp } D[u, p] \subset \text{sing}(\text{supp } L_\varepsilon^3 B_{3,C(N)}^{1/3} v)$.

What's good for not using Littlewood-Paley definition of Besov space? the solution above can be defined locally.

$\phi \in C_c^\infty(I \times B_q)$ and $B'_q \subset B_q$ a smaller ball with same center q . Let $q \notin \text{sing}(\text{supp } B_{3,C(N)}^{1/3} v)$

$$\langle \phi, D[v, p] \rangle = \lim_{\varepsilon \rightarrow 0} \int_I \int_{B'_q} \phi(t, x) \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} \lesssim \|\phi\|_{L^\infty} \int_I \|\nabla v_\varepsilon(t)\|_{L(B'_q)} \|R_\varepsilon\|_{L_x^{2/3}}$$

dominated by $\|\phi\|_{L^\infty} \int_I \|v(t, \cdot)\|_{B_{3,\infty}^{1/3}(B_\varepsilon)}^3$. For a.e. t , we have $\|\nabla v_\varepsilon\|_{L^3} \cdot \|R_\varepsilon\|_{L^{3/2}} = o(1)$ and $D[u, p] \rightarrow 0$ by dominate convergence theorem. □

Remark. 1. Heat flow approach can also be applied to this problem. The proof is quite different[9] and is on a compact Riemann manifold(no convolution can be used).

2. Compressible Euler Case. The problem lies when dealing with commutator estimation.

3 Holder Continuity

The following lecture are basic on [9].

Note that $B_{3,C(\mathbb{N})}^{1/3} \subsetneq B_{3,\infty}^{1/3}$ and we can find a function in $B_{3,\infty}^{1/3} \setminus B_{3,C(N)}^{1/3}$.

$\phi(x)\chi_{\{x'>0\}} \in B_{p,\infty}^{1/p} \quad \forall 1 < p < \infty$ Let's consider

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0 \quad L_t^\infty B_{3,\infty}^{1/3}$$

Energy dissipation at time $t = 0$

$$\frac{d}{dt} e(t) = \int_{\mathbb{T}^d} \nabla_j v_{\varepsilon l} R_{\varepsilon}^{j\ell}(0, x) dx$$

Eyink proved that there exists a divergence-free vector field in the space $C^{1/3} B_{3,C(N)}^{1/3}$, s.t. $\frac{d}{dt}|_{t=0} e(t) < 0$. We have a useful counter example:

$$v(x) = \sum_q 2^{2q\alpha} \sin(2^{2q}x) \in \dot{B}_{3,C(N)}^{1/3} \setminus \dot{B}_{3,\infty}^{1/3}$$

Now we consider this problem on compact Riemannian manifold for the conclusion $L_t^3 B_{3,C(N)}^{1/3}$. Consider (Euler), instead of $\eta_\varepsilon * v^\ell$, we consider

- Estimates(Commutator)
- Keeping divergence-free property

Define the opetator $-\Delta_H = d_\delta + \delta_d$, which looks like a 1-form. In Hodge heat flow equation,

$$\partial_s v^\ell = \Delta_H v^\ell = \nabla_j \nabla^j v^\ell - \text{Ric}_\ell^k v^k$$

Since we know what the solution exactly is,

$$\eta_\varepsilon * v^\ell \rightarrow e^{s\Delta_H} v^\ell$$

The square root of heat time $s^{1/2} \sim \varepsilon$ and solution at time s $S_{[s]}v = e^{s\Delta_H}v$.

To estimate

$$\eta_\varepsilon * \nabla_j (v^j v^\ell) - \nabla_j (\eta_\varepsilon v^j \eta_\varepsilon * v^\ell),$$

we would need the commutator

$$w^\ell(s) = S_{[s]} \nabla_j (v^j v^\ell) - \nabla_j (S_{[s]} v^j S_{[s]} v^\ell)$$

and Riemannian manifold M will be always assumed to be smooth.

$$s \in (0, 1] \quad \int_{I \times M} \eta(t) [S_{[s]} \nabla_j (v^j v^\ell) - \nabla_j (S_{[s]} v^j S_{[s]} v^\ell) S_{[s]} v_\ell] d^{1+d} \text{vol},$$

here volume is in time \times space.

Let's calculate

$$\begin{aligned} (\partial_s - \Delta_H) w^\ell &= N^\ell(t, s) \\ w^\ell(s) &= \int_0^s e^{(s-s')\Delta_H} N^\ell(t, s') ds', \end{aligned}$$

by d'Alembert's formula.

$$\begin{aligned} w^\ell(t, s) &= S_{[s]} \nabla_j (v^j v^\ell) - \nabla_j (S_{[s]} v^j S_{[s]} v^\ell) \\ (\partial_s - \Delta_H) w^\ell &= (\partial_s - \Delta_H) \nabla_j (S_{[s]} v^j S_{[s]} v^\ell) \\ &= (\partial_s - \nabla_i \nabla^i) \nabla_j (S_{[s]} v^j S_{[s]} v^\ell) + \text{curvature terms} \\ &= -2 \nabla_j (\nabla_i S_{[s]} v^j \nabla^i S_{[s]} v^\ell) + \text{low order terms} \end{aligned}$$

$$\begin{aligned}
\text{Commutator} &= -2 \int_{I \times M} \eta(t) \int_0^s e^{(s-s')\Delta} \nabla_j (\nabla_i S_{[s]} v^j \nabla^i S_{[s]} v^\ell) ds' S_{[s]} v_\ell d^{1+d} \text{vol} \\
\text{Integral by parts} &= 2 \int_{I \times M} \eta(t) \int_0^s \nabla_i S_{[s]} v^j \nabla^i S_{[s]} \cdot \underbrace{S_{[2s-s']} v_\ell ds'}_{\text{very low frequency}} d^{1+d} \text{vol}
\end{aligned}$$

Assume that $v \in L_t^3 B_{3,\infty}^\alpha$, claim that $\|\nabla S_{[s]} v\|_{L^3} \lesssim s^{-\frac{1+\alpha}{2}} \|v\|_{L_t^3 B_{3,\infty}^\alpha}$ with $\alpha > 1/3$. First we can try $v \in L_t^3 C^\alpha$ or $v \in L_t^3 W^{\alpha,3}$.

$$\begin{aligned}
|\text{Commutator}| &\lesssim \|\eta\|_{L^\infty} \int_0^s (2s-s')^{-\frac{1+\alpha}{2}} (s')^{-2\frac{1+\alpha}{2}} ds' \\
&\lesssim s^{-1/2+3\alpha/2} \int_0^1 (2-\sigma)^{-\frac{1+\alpha}{2}} \sigma^{-1+\alpha} d\sigma \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ with } \alpha < 1/3
\end{aligned}$$

No derivatives that we can take over the heat flow. In order to prove the claim: $\|\nabla S_{[s]} v\|_{L^3} \lesssim s^{-\frac{1+\alpha}{2}} \|v\|_{W^{\alpha,3}}$ with $\alpha > 1/3$.

Proof.

$$\begin{aligned}
\|\nabla \mathbb{B} v\|_{L^3} &\leq s^{-\frac{1}{2}} \|v\|_{L^3} \\
\|\nabla \mathbb{B} v\|_{L^3} &\leq \|v\|_{W^{1,3}}
\end{aligned}$$

Since $u(s) = e^{s\Delta} u$, $\|u\|_{L^\infty} \lesssim \|u\|_{L^p}$ $s \in (0, 1]$ On compact manifold we have that

$$\|\nabla u\|_{L^p} \lesssim \|\nabla u\|_{L^r} + \|u\|_{L^p}$$

Here let $p = 2z$ where $z \geq 2$ is an integer. □

$$\begin{aligned}
\partial_s |u|^2 - \Delta |u| + |\nabla u|^2 &= 0 \text{ (or } -2 \text{Ric}_{jk} u^j u^k) \\
&\Rightarrow \int_M |u|^2 d\text{vol} \searrow \text{ and } \int_M u^{2z} d\text{vol} \searrow \\
\frac{1}{z} \partial_s \int |u|^{2z} d\text{vol} &= \int_M (\Delta |u|^2 - 2|\nabla u|^2) |u|^{2(z-1)} d\text{vol} \\
&= - \int_M \nabla^j |u|^2 \nabla_j |u| |u|^{2(z-2)} - 2|\nabla u|^2 |u|^{2(z-1)} \leq 0
\end{aligned} \tag{4}$$

For curvature terms, they can be bounded by $\|\text{Ric}\|_{L^\infty} \int_M |u|^{2z} d\text{vol}$ remains bounded. So $\int_M u^{2z} d\text{vol} \leq \int_M u^{2z}(t=0) d\text{vol}$.

$$\partial_s |\nabla u|^2 - \Delta |\nabla u|^2 + |\nabla \nabla u|^2 (\text{BAD}) = 0 \text{ (or } \text{Riem}(\nabla u \nabla u) + \nabla \text{Riem} u \nabla u) \tag{5}$$

Multiply by $|\nabla u|^{2(z-1)}$,

$$\frac{1}{z} \partial_j \int_M |\nabla u|^{2z} d\text{vol} \searrow$$

or

$$\begin{aligned}
&\int_M |\nabla u|^{2(z-1)} \nabla_{\text{Riem}} u \nabla u d\text{vol} \text{ by integration by parts} \\
&\leq 2 \underbrace{\int_M |\nabla \nabla u|^2 |\nabla u|^{2(z-1)} + C z \int_M \|R\|_{L^\infty} \|\nabla u\|^{2z} + \|\text{Riem}\|^{z+1} \|u\|^{2z}}_{\text{GOOD}}
\end{aligned}$$

which can cancel bad terms from (5).

$$\partial_s (s |\nabla u|^2) - \Delta (s |\nabla u|^2) + 2s |\nabla \nabla u|^2 - |\nabla u|^2 (\text{BAD})$$

But here the bad term can cancel with (4). Let $\Phi(s) = s |\nabla u|^2 + \frac{1}{2} |u|^2$, then we have

$$\partial_z \Phi_s - \Delta \Phi_s = 2s |\nabla \nabla u|^2$$

Thus

$$\begin{aligned}
\frac{1}{z} \partial_s \int_M |\Phi_s|^z (\searrow) &\leq \frac{1}{z} \int_M |\Phi_s|^z d\text{vol} \leq \frac{1}{z} \int_M |\Phi_0|^z = \frac{1}{z} \int_M |u|^{2z} \\
\frac{1}{z} \left(\int_M |\nabla u|^{2z} \right)^{1/2z} &\leq s^{-1/2} \left(\int_M |u|^{2z} \right)^{1/2z}
\end{aligned}$$

Hodge Laplacian commute with derivatived and divergenced. $(\partial_s - \Delta_H) \delta w_\ell = \delta(\partial_s - \Delta_H) w_\ell = 0$ with 0 initial condition.

3.1 Isentropic Compressible Euler

$$\begin{array}{ll} \text{Mass} & \partial_t \rho + \nabla_j(\rho v^j) = 0 \\ \text{Momentum} & \partial_t(\rho v^\ell) + \nabla_j(\rho v^j v^\ell) + \nabla^\ell(P\rho) = 0 \end{array} \quad (6)$$

We made some assumption $p(\rho) = \rho^2 \gamma$ and $\rho \in C^2$ and away from $\rho \equiv 0$. Here exists a problem: modifier doesn't commute with nonlinearity.

$$\partial_t(\frac{1}{2}\rho|v| + p(\rho)) + \nabla_j((\frac{1}{2}\rho|v| + p(\rho) + p(\rho)v^j)p(\rho)) = \rho \int_1^\rho \frac{p(r)}{r^2} dr$$

If conservation holds in $B_{3,\infty}^\alpha$ in both (t, x) , we need to estimate the commutator: $\eta_\varepsilon * (p(\rho)) - p\eta * \rho$.

Remark. • Heat flow also works.

- [5] gives another method

$$\begin{aligned} \text{Commutator} &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p\left(\int \rho(x-h)\eta_\varepsilon(h)dh\right) \\ &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p\left(\int (\rho(x-h)\eta_\varepsilon(h))dh\right) = \int (1-\sigma) \int p''((1-\eta)\rho_\varepsilon(x) + \sigma\rho(x-h))(\rho(x-h) - \rho_\varepsilon(x)) \end{aligned}$$

Therefore bounded.

$$\overline{f(X)} - f(\overline{X}) = \mathbb{E}[f(X)] - f(\mathbb{E}[X]) \quad (7)$$

Since

$$\begin{aligned} \phi(1) &= \phi(0) + \int_0^1 \frac{d}{d\sigma} \phi(\sigma) d\sigma \\ &= \phi(0) + \frac{d}{d\sigma}|_{\sigma=0} \phi(\sigma) + \int_0^1 (1-\sigma) \frac{d^2}{d\sigma^2} \phi(\sigma) d\sigma \end{aligned}$$

RHS of (7) becomes

$$\begin{aligned} &\int +0^1 \frac{d}{d\sigma} \mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X)] \\ &= \frac{d}{d\sigma}|_{\sigma=0} \mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X)] + \int_0^1 (1-\sigma) \frac{d^2}{d\sigma^2} \mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X)] d\sigma \\ &= \mathbb{E}[\nabla_i f(\overline{X})(X^i - \overline{X}^i)] + \int_0^1 (1-\sigma) \mathbb{E}[\nabla_a \nabla_b f((1-\sigma)\overline{X} + \sigma X)(X^a - \overline{X}^b)(X^b - \overline{X}^a)] d\sigma \end{aligned}$$

Here last term is a quadratic form.

4 Convex Integration

Here "convex" refers to convex linear combination.

Theorem 4.1. (Old conclusion) $\forall E(t) \in C^\infty, E(t) \geq C, \forall \alpha < \frac{1}{10}, \exists v \in C_{t,x}^\alpha(I \times \mathbb{T}^3)$ s.t. $\int \frac{|v|^2}{2}(t, x) dx = E(t)$ conserved.

Theorem 4.2. (Isset [6]) $\forall \alpha < 1/5, \exists v \in C_{t,x}^\alpha(I \times \mathbb{T}^3, p \in C_{t,x}^{2,\alpha})$. A non trivial solution with compact support in time. (0 is not the only solution stays 0 implies non-uniqueness).

- Question: How to construct continuous solution?
- Idea: Euler-Reynolds flows

For R a symmetric tensor (v, p, R) that solves

$$\begin{aligned} \partial_t v^l + \nabla_j(v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned}$$

Here $R^{j\ell} = R^{\ell j}$. If $R = 0$, we have the Euler equation(Euler). If $R \neq 0$, we have a smooth approximation of Euler Equation.

Construction (v_q, p_q, R_q) the Euler-Reynolds flow:

$$R_q \rightarrow 0 \text{ as } q \rightarrow \infty$$

and

$$(v_q, p_q)$$

converge uniformly.

$$-\int \partial_t \phi_\ell v_q^l + \nabla_j \phi_\ell + v_q^j v_q^\ell + \nabla^\ell \phi \cdot p_q = \nabla_j \phi R_q^{j\ell} \rightarrow 0$$

The idea would make sense if every continuous solution (v, p) is a uniform limit of $(v_\varepsilon, p_\varepsilon, R_\varepsilon)$ as $\varepsilon \rightarrow 0$. I.e. every Euler flow is a limit of Euler-Reynolds flow. Proof will also be given by modification. To check weather it is a E-R flow, we need to check weather it conserves energy.

Lemma 4.3. (Main Lemma) Given $(v, p, R) \ni$ new

$$(\mathring{v}, \mathring{p}, \mathring{R})$$

with

$$\|\mathring{R}\|_0 < \|R\|_0,$$

where

$$\mathring{v} = v + V \quad \mathring{p} = p + P. \quad (8)$$

We expect that $\|V\|_0 \leq \|R\|_0^{1/2} \leftarrow [\frac{m}{s}]$ and $\|P\|_0 \leq \|R\|_0 \leftarrow [\frac{m}{s}]$ by dimension analysis.

Apply 4.3 over and over again generate (v_q, p_q, R_q) with $\|R_q\|_0 \rightarrow 0$ rapidly. Set $v = v(0) + \sum_q v_q$ and $p = p(0) + \sum_q p_q$. Plug in new 8

$$\partial_t \mathring{v}^\ell + \nabla_j (\mathring{v}^i \mathring{v}^{*l}) + \nabla^\ell \mathring{p} = \nabla_j R^{j\ell} + \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla_j (v^j v^\ell) + \nabla^\ell p + \nabla_j (v^j v^\ell)$$

i.e.

$$\text{RHS} = \nabla v^\ell + \nabla_j (v^j V^l) + \nabla_j (V^j V^l + P \delta^{jl} + R^{jl}) + \nabla_j (v^l V^j)$$

where R^{jl} is the old error. We want $\nabla_j R^{*jl} = \text{RHS}$ with R^* small. ¹

The rest is not small. Let them equals to $\nabla_j Q^{jl}$ with $Q^{jl} = Q^{lj}$. Either $\nabla_j (v^{jl} + p \delta^{jl} + R^{jl})$ or $\nabla_j (v^l V^j)$ is not small.

4.1 Non-stationary Phase Lemma

When you can find a small solution $Q^{j\ell} = Q^{\ell j}$ to the symmetric divergence equation

$$\nabla_j Q^{j\ell} = u^\ell$$

Necessary for u^ℓ to be high-frequency $n \geq 1$ and $\int_{\mathbb{T}^3} u^\ell dx = 0$.

What does v^ℓ look like? [2] Beltrami flows.

$$v^\ell = \sum_{|k|=R} a_k B_k^\ell e^{i\lambda k \cdot x}, \quad k \in \mathbb{R}^d$$

$\nabla \times v = R\lambda v$ if $(ik) \times B_k = |k|B_k$ and $a_k B_k$ are constant.

$$v^l = \sum_{|k|=R} a_k(R, E, v, \lambda t) B_k^\ell e^{i\lambda k x}$$

Construction of frequency:

$$v^\ell = \sum_I v_I^l, \quad v_I^\ell = \bar{v}_I^l, \quad \nabla_j v_I^j = 0$$

$$v_I^l = \nabla_j w_I^{jl}, \quad w_I^{jl} = -w_I^{lj} \Leftrightarrow$$

¹Here the position of $*$ is decided by whether it is a equation or just math script. The different position of $*$ doesn't have different meaning

w^{jl} is a curl. For high frequency wave

$$\begin{aligned} v_I^\ell + \delta v_I^\ell & \quad \text{with } \lambda \gg 1 \\ v_I \cdot \nabla \xi_I &= 0 \quad \text{for divergence free} \end{aligned}$$

Here ξ_I is always tangent to level phase function.

$$w_I^{jl} = \frac{e^{i\lambda\xi_I}}{i\lambda} w_I^{j\ell} = \frac{e^{i\lambda\xi}}{i\lambda} (\nabla^j \xi_I v_I^\ell - v_I^j \nabla^l \xi_I)$$

then it is anti-symmetric. Thus

$$\nabla_j w_I^{j\ell} = e^{i\lambda\xi_I} (\nabla_j \xi_I w_I^{il} + \frac{\nabla_j w_I^{j\ell}}{i\lambda}) = e^{i\lambda\xi_I} (v_I^\ell + \delta v_I^\ell),$$

where $\delta v_I = \mathcal{O}(\frac{1}{\lambda})$, thus small. The transport term:

$$\nabla_j R_T^{jl} = \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla_j (V^j v^\ell)$$

want to solve

$$\nabla_j Q^{jl} = u^\ell = e^{i\lambda\xi_I} v_I^\ell$$

with small $Q^{j\ell} = Q^{\ell j}$ on \mathbb{T}^3 . The equations above are under-determined. So the solution is not unique and we are try to find a small one.

Lemma 4.4. *Non-stationary Phase Lemma* If $\left\| |\nabla \xi_I|^{-1} \right\|_0 \leq A$ (no zero frequency) and $\int_{\mathbb{T}^3} u^\ell(x) dx = 0$. We note that if $|\nabla \xi_I| \rightarrow 0$, then we just rotating a constant but there is no waves. Then $\exists Q^{j\ell} = Q^{\ell j}$ of size $\|Q\| \leq \mathcal{O}(\lambda^{-1}) \|u\|_0 + \mathcal{O}_{(u, \nabla q)}(\lambda^{-2})$. Since $\int \partial_t V^l dx = 0 \Rightarrow \frac{d}{dt} \int v^\ell dx = 0$ conserve the momentum, consistent with solution of Euler.

Proof. d=1.

$$\frac{dQ}{dx} = e^{i\lambda\xi(x)} u(x) \quad \text{on } \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

$\Rightarrow Q(x) = \int_0^x e^{i\lambda\xi(s)} u(s) ds$ is periodic because $\int_{\mathbb{T}} u(x) dx = 0$. By integration by parts,

$$= \frac{1}{i\lambda\xi(s)} e^{i\lambda\xi(s)} u(s) \Big|_{s=0}^x - \frac{1}{i\lambda} \int_0^x e^{i\lambda\xi(s)} \frac{d}{ds} \left(\frac{u(s)}{\xi(s)} \right) ds = \mathcal{O}\left(\frac{1}{\lambda} + \frac{1}{\lambda} \mathcal{O}(\lambda^{-2})\right)$$

Let's look at the transport term

$$\begin{aligned} \nabla R_T^{j\ell} &= \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla_j (v^j v^\ell) = \partial_t v^\ell + v^j \nabla_j v^\ell + v^j \nabla_j v^\ell \\ &= (i\lambda) \sum_I e^{i\lambda\xi_I} (\partial_t \xi_I) v_q^\ell + (v^j \nabla_j) \xi_I v^\ell + \sum_I e^{i\lambda\xi_q} \end{aligned}$$

□

Apply the Non-stationary Phase Lemma 4.4

$$\|R_T\|_0 \lesssim \lambda^{-1} \|RHS\|_0 \lesssim \lambda^{-1} (\lambda \|(\partial_t + v^j \nabla_j) \xi_I\|_0 \|V_I\|_0) + \mathcal{O}(\lambda^{-2})$$

Impose the phase function:

$$\begin{aligned} (\partial_t + V \cdot \nabla) \xi_I &= 0 \\ D_t \xi_I &= 0 \end{aligned} \tag{9}$$

Taylor hypothesis: High frequency flows are actually carried by low frequency flows. So it make sense with our assumption. If $\left\| |\nabla \xi_I|^{-1} \right\|_0 \leq A$ fails to control the amplitude, we include time cut-offs in v_I . Look at $\nabla_j (v^j v^\ell + p \delta^{jl} + R^{jl})$, want it to be div of something small. Want $R_{small}^{j\ell} = \sum_I v_I^j \bar{v}_I^\ell + P_0 \delta + R^{j\ell}$

$$\sum_I v_I^j \bar{v}_I^\ell + P_0 \delta^{j\ell} + \sum_{j \neq \bar{I}} v_I^j v_j^\ell + P_{Ij} \delta^{j\ell} = \sum_I v_I^j \bar{v}_I^\ell + P_0 \delta^{j\ell} + R^{j\ell} + \sum_I \delta v_I^j v^\ell \left(= 0 + \mathcal{O}\left(\frac{1}{\lambda}\right) \right)$$

Let's pick a t to make this = 0 pointwisely.

$$\sum_I v_I^j \bar{v}_j^\ell = -P_0 \delta^{j\ell} - R^{j\ell} = e(t) \delta^{j\ell} - E^{j\ell}$$

Choose $e(t) \geq 10^4 \|R\|_0$ on $\text{supp } R$, here $e(t)$ is the lifting function.

$$\Rightarrow \|V_I\|_0 \lesssim e^{1/2} \lesssim \|R\|_0^{1/2}$$

We need to choose time depending on amplitude. Note that $\text{supp}_t e \subset B(\text{supp } R, \varepsilon)$,

$$\Rightarrow \text{supp}_t(\text{New Error}) \subset \text{supp}_t e$$

References

- [1] Peter Constantin, Edriss S Titi, and F Weinan. Onsager’s conjecture on the energy conservation for solutions of euler’s equation. *Communications in Mathematical Physics*, 165(1):207, 1994.
- [2] Camillo De Lellis and László Székelyhidi Jr. Continuous dissipative euler flows and a conjecture of onsager. In *European Congress of Mathematics*, pages 13–29. Eur. Math. Soc. Zürich, 2013.
- [3] Jean Duchon and Raoul Robert. Inertial energy dissipation for weak solutions of incompressible euler and navier-stokes equations. *Nonlinearity*, 13(1):249, 2000.
- [4] Gregory L Eyink and Katepalli R Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Reviews of modern physics*, 78(1):87, 2006.
- [5] Eduard Feireisl, Piotr Gwiazda, Agnieszka Świerczewska-Gwiazda, and Emil Wiedemann. Dissipative measure-valued solutions to the compressible navier–stokes system. *Calculus of Variations and Partial Differential Equations*, 55(6):141, 2016.
- [6] Philip Isett. *Hölder continuous Euler flows in three dimensions with compact support in time*, volume 196. Princeton University Press, 2017.
- [7] Philip Isett. On the endpoint regularity in onsager’s conjecture. *arXiv preprint arXiv:1706.01549*, 2017.
- [8] Philip Isett. A proof of onsager’s conjecture. *Annals of Mathematics*, 188(3):871–963, 2018.
- [9] Philip Isett and Sung-Jin Oh. A heat flow approach to onsager’s conjecture for the euler equations on manifolds. *Transactions of the American Mathematical Society*, 368(9):6519–6537, 2016.
- [10] Philip James Isett et al. Hölder continuous euler flows with compact support in time. 2013.
- [11] Charles Meneveau, KR Sreenivasan, P Kailasnath, and MS Fan. Joint multifractal measures: Theory and applications to turbulence. *Physical Review A*, 41(2):894, 1990.
- [12] Vladimir Scheffer. An inviscid flow with compact support in space-time. *The Journal of Geometric Analysis*, 3(4):343–401, 1993.
- [13] Alexander Shnirelman. On the nonuniqueness of weak solution of the euler equation. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 50(12):1261–1286, 1997.