

Weak Solution: Convex Integration

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1 Introduction

First let's give an example to help define what is the weak solution.

Example 1.1.

$$\Delta u = f$$

If u don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \rightarrow \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z} \alpha + \frac{\partial u}{\partial \bar{z}} \bar{\alpha} = \frac{\partial u}{\partial x} \operatorname{Re} \alpha + \frac{\partial u}{\partial y} \operatorname{Im} \alpha$$

$$\text{i.e. } du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}.$$

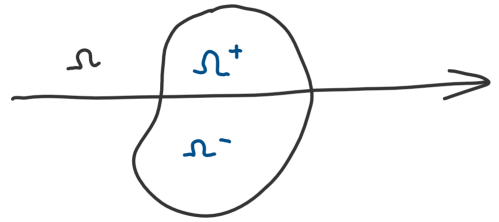
Theorem 1.1. If $\frac{\partial u}{\partial \bar{z}} = 0$ or $\Delta u = 0$ in the weak sense (against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.

Useful: $f_k(z)$ holomorphic $u(z) = \sum_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} \text{Fubini} &= - \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ &= - \sum_{k=0}^{\infty} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

Theorem 1.2. (Swartz reflection principle.) If f is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If f is continu-



ous on Ω on Ω including $\Omega \cap \{y = 0\}$. Then f is holomorphic on Ω .

In $D' f = \lim_{\delta \rightarrow 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here H is heaviside function.

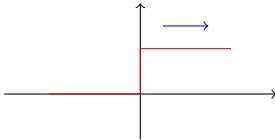
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \rightarrow 0} \frac{\partial f}{\partial \bar{z}} + f \left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial \bar{z}} \delta(\varepsilon - y) \right)$$

Since f is continuous $\lim_{\varepsilon \rightarrow 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\square u = 0 \quad \text{where } \square := -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2 \quad (\text{Wave})$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \quad \int_{\mathbb{R}} \square \phi u = 0$$

(Wave) has a solution on \mathbb{R}^{1+1} given by $u(t, x) = u(t - x)$, a traveling wave:



Example 1.2.

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on \mathbb{R}^{1+d}

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

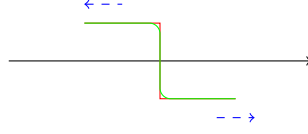


Figure 1: $t = 0$

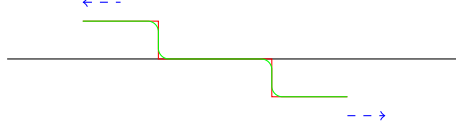


Figure 2: $t = 1$

2 Fluid Mechanics

2.1 Incompressible Euler Equation

Define 2 velocity field: $v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ $p : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p = 0$$

$$\nabla_j v^j = 0 \quad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{loc}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \partial\Omega$

$$\int_{\partial\Omega} v \cdot \vec{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt} \left[\begin{array}{c} \text{total momentum} \\ m \cdot v \end{array} \right] = [\text{Force on } \Omega] + \left[\begin{array}{c} \text{Flux of} \\ \text{momentum} \end{array} \right] \Rightarrow \frac{d}{dt} \int_{\Omega} v^l dx = \int_{\partial\Omega} p \vec{n}^l dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if p is good enough. If $v, p \in C^1$, use $\int_{\partial\Omega} f \vec{n}_j d\sigma = - \int_{\Omega} \nabla_j f dx$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} v^l &= - \left(\int_{\Omega} \nabla^l p + \nabla_j (v^j v^l) dx \right) \\ \int_{\Omega} (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) dx &= 0 \quad \forall \Omega, \quad \forall t \end{aligned}$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful? Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I \times \mathbb{R}^d)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l(t, x) dx = 0$$

If $(1 + |x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, then linear angular momentum conserved?

Here $\forall K^l$ s.t. $\nabla_j K_l + \nabla_l K_j = 0$ on \mathbb{R}^d

Example 2.1. $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_l v^l dx = \int_{\mathbb{R}^d} v^{(i)} dx$

Example 2.2. Rotation $K(a, b) = x^a e_b - x^b e_a$, $1 \leq a < b \leq d$.

$$\begin{aligned} \int_{\mathbb{R}^d} K_l (\partial_t v^l + \nabla : (v^j v^l) + \nabla^l p) dx &= 0 \\ \frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l - \int_{\mathbb{R}^d} \nabla : K_l (v^j v^l) - \int_{\mathbb{R}^d} \nabla^l K_l p dx &= 0 \end{aligned}$$

where $\operatorname{div} K = \nabla^l K_l = \delta^{jl} \nabla_j K_l = \frac{1}{2} \delta^{jl} (\nabla_j K_l + \nabla_l K_j) = 0$ by assumption.

$$\partial v^l + \nabla : (v^j v^l) + \nabla^l p) dx = 0$$

Test against a space cut-off function $K^l(B) := q(t) \varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here K is rotationally symmetric, so it is divergence-free.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_l^{(\beta)} v^l dx \right] dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_l^{(\beta)} v^j v^l - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^l K_l^{(\beta)} p dx dt = 0$. i.e.

$$- \int_{\mathbb{R}^+} \eta'(t) \left[\int_{\mathbb{R}^d} K_l^{(\beta)} v^l dx \right] dt - \frac{1}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \left(\nabla_j K_l^{(\beta)} + \nabla_l K_j^{(\beta)} \right) v^j v^l dx dt - \frac{1}{2} \int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi\left(\frac{|x|}{B}\right) K_l v^j v^l dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L_{t,x}^1$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

2.2 Conservation of Energy

If $(1 + |x|)v \in L_{t,x}^1(I \times \mathbb{R}^d)$, $v \in L_{t,x}^2(I \times \mathbb{R}^d)$, then $\forall K^l$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt} \int_{\mathbb{R}^d} K_l v^l(t, x) dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : e_i, 1 \leq i \leq d, 1 \leq a < b \leq d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^l = \nabla_j w^{jl} \quad w^{jl} = -w^{lj} \quad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B}) w^{jl})$

$$w^{jl} = x^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)$$

$$\begin{aligned} & \nabla_j (x^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l)) \\ &= \delta_j^2 (\delta_2^j \delta_1^l - \delta_1^j \delta_2^l) = \delta_1^l \end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j \nabla_j w^{jl} = -\nabla_l \nabla_j w^{lj} = -\nabla_j \nabla_l w^{lj} = -\nabla_l \nabla_j w^{jl}$$

Conservation of energy means that $\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|v|^2}{2}(t, x) dx = 0$. Note that energy is nonlinear.

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) = 0$$

If $v \in C^1 \cap L_{t,x}^2 \cap L_{t,x}^3(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here B could be ∞ . Multiply the local energy by $\eta(t) \varphi(\frac{|x|}{B})$

$$\int \eta \frac{d}{dt} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|^2}{2}(t, x) dx dt \quad (1) \quad - \int \eta(t) \int \nabla_j \left[\varphi\left(\frac{|x|}{B}\right) \right] \left(\frac{|v|^2}{2} + p \right) v^j dx dt \quad (2)$$

$$(1) = - \int \underbrace{\eta'(t)} \int \varphi\left(\frac{|x|}{B}\right) \frac{|v|^2}{2}(t, x) dx dt \quad \text{Integral by parts} \quad (\text{Local})$$

$$\text{dominated by } \frac{|x|^2}{2} |\eta'| \in L_{t,x}^1$$

term (2) converge to 0 pointwisely when $B \rightarrow \infty$ and dominated by $|\eta t| \left(\frac{|v|^3}{2} + |p||v| \right)$.

Let's recall Euler equation.

$$\begin{cases} \partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p &= 0, \\ \nabla_j v^j &= 0 \end{cases} \quad (\text{Euler})$$

Take divergence over (Euler), \Rightarrow

$$\nabla_j \nabla_l (v^j v^l) + \nabla_l \nabla^l p = 0$$

i.e.

$$\begin{aligned} \Delta p &= -\nabla_l \nabla_j (v^j v^l) \\ p &= \underbrace{(-\Delta)^{-1} \nabla_l \nabla_j (v^j v^l)}_{\text{zero order operator}} \underbrace{(v^j v^l)}_{\in L_{t,x}^{3/2}} \end{aligned}$$

Thus naturally $p \in L_x^{3/2}$ a.e. $t \in \mathbb{R}^+$

$$\|p\|_{L_x^{3/2}(L_t^{3/2})} = \|p\|_{L_{t,x}^{3/2}} < \infty$$

$$v_l (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) = 0$$

$$\nabla_j v^j = 0$$

Thus

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v_l v^j \nabla_j v^l + v_l \nabla^l p = 0$$

$$\Rightarrow \partial_t \left(\frac{|v|^2}{2} \right) + v^j \nabla_j \left(\frac{|v|^2}{2} + v_j v^j p \right) = 0$$

$$\nabla_j v^j = 0$$

$$\partial_t \left(\frac{|v|^2}{2} + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \right) = 0$$

Using $\nabla_j v^j = 0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?

In turbulence situation (Navier-Stokes equations) with $\nu \ll 1$

$$v_l (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) = \nu v_l \Delta v^l$$

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx = -\nu \int |\nabla v|^2 dx = \nu \int v_l \nabla_i \nabla^i v^l$$

Taking a formal limit, \exists incompressible Euler flows with

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx < -\varepsilon < 0$$

Theorem 2.1. *Onsager's Conjecture*

(+) If $\alpha > 1/3$ and $(v(t, x + \Delta x) - v(t, x)) \leq c|\Delta x|$ where $x \in \mathbb{T}^3 (v \in L_t^\infty C_x^\alpha)$, then the energy conserved.

(-) (K41) If $\alpha \leq 1/3 \exists$ incompressible Euler flows with $v \in L_t^\infty L_x^\alpha$ s.t. $\int_{\mathbb{T}^d} \frac{|v|^2}{2}(t, x) dx$ is not constant.

Now we follow [2] and discuss the (+) part first.

$$\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p = 0$$

In order to get into Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series,

$$v(x, t) = \sum_{k \in \mathbb{Z}^3} a_k(t) e^{ik \cdot x}.$$

Obviously $a_{-k} = \overline{a_k}$, because v is real-valued. Moreover the divergence-free constraint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k :

$$\frac{da_k}{dt} = i \sum_{\ell} a_{k-\ell} \cdot \ell \left[-a_\ell + \frac{(a_\ell \cdot k) k}{|k|^2} \right] - \nu |k|^2 a_k \quad (1)$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.

(Don't understand) Energy starts at low wave numbers and moves to higher wave numbers in finite number.

$\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.

Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.

(K 41) $E \lim_{v \rightarrow 0} \langle v \int |\nabla v|^2 dx \rangle$ and v determine all statistic properties of turbulent flows.

$$\langle |v(x + \Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.

Now (+) is solved by [4] and [1] with the goal $L_t^3 B_{3,C(N)}^{1/3}, L_t^3 B_{3,\infty}^{1/3+\varepsilon}$.

(-) is solved ($d \geq 3$) with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [5].

Convex integration originated from the Nash–Kuiper Paradox(50's) for C^1 isometric embedding. Connection to Euler equation discovered by Camillo De Lellis and László Székelyhidi (08,12). First result towards Onsager conjecture is in [6]. And $\alpha < \frac{1}{5}$ by [8]. The non-uniqueness example was first given by [10] and then Shnirelman give a different proof in [11].

2.3 Another way of proving (+)

(+) (Eyink, Constantin, E, Titi 94') $L^3(B_{3,\infty}^\alpha)$

$$\|v\|_{C^\alpha} = \sup_{h \neq 0} \frac{\|v(x+h) - v(x)\|_{L^\infty}}{|h|^\alpha}$$

$$\|v\|_{B_{3,\infty}^\alpha} = \sup_{h \neq 0} \frac{\|v(x+h) - v(x)\|_{L^3}}{|h|^\alpha}$$

Lemma 2.2. *Commutator Estimate*

$$R_\varepsilon^{jl} = \eta_\varepsilon * (v^j v^l) - (v_\varepsilon^j v_\varepsilon^l)$$

$$\|R_\varepsilon\|_{L^{3/2}} \lesssim \varepsilon^{2\alpha} \|v\|_{B_{3,\infty}^\alpha}^2$$

Let's think R_ε^{jl} as an expectation with the idea:

$$R = \mathbb{E}[v^2] - (\mathbb{E}[v])^2 = \mathbb{E}[(v - \mathbb{E}(v))^2],$$

which is quadratic.

$$R_\varepsilon^{jl} = \int v^j(x-h) v^l(x-h) \eta_\varepsilon(h) dh - \int v^j(x-h_1) \eta_\varepsilon(h_1) dh_1 \int v^l(x-h_2) \eta_\varepsilon(h_2) dh_2$$

$$\text{Using } \int \eta_\varepsilon(h) dh = 1$$

$$= \int (v^j(x-h) - v_\varepsilon^j(x)) (v^l(x-h) - v_\varepsilon^l(x)) \eta_\varepsilon(h) dh$$

By Lemmas in [1], we decompose above equation into $\sum_{i=1}^4 R_{\varepsilon i}^{jl}$, where

$$R_{\varepsilon 1} = \int (v^j(x-h) - v_\varepsilon^j(x-h)) (v^l(x-h) - v_\varepsilon^l(x-h)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 2} = \int (v_\varepsilon^j(x-h) - v_\varepsilon^j(x)) (v^l(x-h) - v_\varepsilon^l(x-h)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 3} = \int (v^j(x-h) - v_\varepsilon^j(x-h)) (v_\varepsilon^l(x-h) - v_\varepsilon^l(x)) \eta_\varepsilon(h) dh$$

$$R_{\varepsilon 4} = \int (v_\varepsilon^j(x-h) - v_\varepsilon^j(x)) (v_\varepsilon^l(x-h) - v_\varepsilon^l(x)) \eta_\varepsilon(h) dh$$

For example,

$$R_{\varepsilon 2} = \int_{\mathbb{R}^d} \int_0^1 \frac{d}{d\sigma} v_\varepsilon^j(x - \sigma h) d\sigma (v^l(x-h) - v_\varepsilon^l(x)) \eta_\varepsilon(h) dh$$

$$= \int_{\mathbb{R}^d} \int_0^1 d\sigma \nabla_i v_\varepsilon^j(x - \sigma h) h^i (v^l(x-h) - v_\varepsilon^l(x)) \eta_\varepsilon(h) dh$$

$$\|R_{\varepsilon 2}^j\| \leq_{\mathbb{R}^d} \int_0^1 \|\nabla v_\varepsilon\|_{L^3} |h| \|v(\cdot - h) - v(\cdot)\|_{L^3} |\eta_\varepsilon(h)| dh$$

Modify the equation with modifier η_ε :

$$\begin{aligned} \eta_\varepsilon * (\partial_t v^l + \nabla_j (v^j v^l) + \nabla^l p) &= 0 \\ \partial_t v_\varepsilon^l + \nabla_j (v_\varepsilon^j v_\varepsilon^l) + \nabla^l p_\varepsilon &= -\nabla_j R_\varepsilon^{jl} \end{aligned}$$

(Thus we need smoothness in time) $\times v_\varepsilon$ then integral by parts:

$$\partial(\frac{|v_\varepsilon|^2}{2}) + v_{\varepsilon l} \nabla_j (v_\varepsilon^j v_\varepsilon^l) + v_{\varepsilon l} \nabla^l p_\varepsilon = -v_{\varepsilon l} \nabla_j R_\varepsilon^{jl} = \int_{\mathbb{R}^d} \nabla_j \left| \frac{v_\varepsilon^2}{2} v_\varepsilon^j \right| \rightarrow 0$$

with assumption.

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|v_\varepsilon|^2}{2}(t, x) dx + \int_{\mathbb{R}^d} v_\varepsilon^j \nabla_j v_\varepsilon^l v_{\varepsilon l} + \nabla^l v_{\varepsilon l} p_\varepsilon = \int_{\mathbb{R}^d} \nabla v_{\varepsilon l} R_\varepsilon^{jl}$$

$\nabla^l v_{\varepsilon l} p_\varepsilon = 0$ for divergence-free.

LHS converges to $\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx$ in $\mathcal{D}'(\mathbb{R})$ since $v_\varepsilon \rightarrow v$ in $L^2_{t,x}$.

$$\begin{aligned} \left\| \frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx \right\|_{L^1_t} &\leq \limsup_{\varepsilon \rightarrow 0} \int \int |\nabla_j v(t, x) R_\varepsilon^{jl}| dx dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int \|\nabla v_\varepsilon(t, \cdot)\|_{L^3_x} \|R_\varepsilon\|_{L^{3/2}} dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int \varepsilon^{-1+\alpha} \|v(t)\|_{B^{\alpha}_{3,\infty}} \|v(t, \cdot)\|_{B^{\alpha}_{3,\infty}}^2 dt \\ &< \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1+3\alpha} \int \|v(t, \cdot)\|_{B^{\alpha}_{3,\infty}}^3 dt \rightarrow 0 \quad \text{with } \alpha > \frac{1}{3} \end{aligned}$$

If $\alpha = \frac{1}{3}$ and v bounded in $L^1_t(I)$ for some finite time period.

$$\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx = \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \int \frac{|v_\varepsilon|^2}{2}(t, x) dx$$

$v\phi \in C_c^\infty(I)$

$$\frac{d}{dt} < \int \frac{|v|^2}{2}(t, x) dx, \phi > \leq \|\phi\|_{L^\infty(I)}$$

LHS is of finite measure. $e(t) = \int \frac{|v|^2}{2}(t, x)$ is of bounded variation. IN fact $\frac{d}{dt} e(t)$ is finite.

(???)If $v \in L^r B^1/3_{3,\infty}$, consider $\|\frac{d}{dt} e(t)\|_{L^{r/t}_t}$ using duality. $u \in L^\infty_t B^1/3_{3,\infty}$ uniformly $\|\frac{d}{dt} e(t)\|_{L^{inf t y_t}} \leq C$ and also $\frac{d}{dt} e(t) \leq -\varepsilon < 0$ is stable under perturbation. If not, the dissipation $\int_{\mathbb{R}^d} \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} dx$ can be really big.

Remark. The singular support of a generalized function u is the complement of the largest open set on which u is smooth. Roughly speaking, it is the closed set where the distribution does not correspond to a smooth function.

2.4 Local energy conservation

$$\partial_t v_\varepsilon^l + \nabla_l (v_\varepsilon^l v_\varepsilon^l) + \nabla^l p_\varepsilon = -\nabla_j R_\varepsilon^{jl}$$

where $R_\varepsilon^{jl} = \eta_\varepsilon * (v^j v^l) - v_\varepsilon^j v_\varepsilon^l$

$$\begin{aligned} \|R_\varepsilon(t, \cdot)\|_{L^{3/2}_t} &\leq \varepsilon^{2\alpha} \|v(t)\|_{B^{\alpha}_{3,\infty}}^2 \\ \frac{1}{2} \int \frac{|v_\varepsilon|^2}{2}(t, x) dx &= \lim_{\varepsilon \rightarrow 0} \int \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} dx \end{aligned}$$

Here to clarify the space:

$$B^{1/3}_{3,c(N)} = (\overline{C^\infty})^{B^{1/3}_{3,\infty}} = B^{1/3}_{3,\infty} \cap \left\{ \lim_{h \rightarrow 0} \frac{|v(x+h) - v(x)|}{|h|^{1/3}} = 0 \right\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3} \setminus c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3} \setminus c^{1/3}$

Lemma 2.3. $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ if $v \in B_{3,c(N)}^\alpha$

Proof. Claim: $\varepsilon^{1-\alpha}\nabla(\eta_\varepsilon * \cdot) : B_{3,\infty}^\alpha \rightarrow L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L_x^3} \lesssim \varepsilon^{-1+\alpha} \|v\|_{B_{3,\infty}^\alpha}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B_{3,\infty}^\alpha} < \frac{\delta}{2C_2}$.

$$\begin{aligned} \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\|_{L_x^3} &\leq \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * (v - \tilde{v})\|_{L_x^3} + \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \|\nabla\eta_\varepsilon * \tilde{v}\|_{L_x^3} \\ &\leq \frac{\delta}{2} + \varepsilon^{1-\alpha} \tilde{c} \quad \text{for } \varepsilon^{1-\alpha} < \frac{\delta}{2\tilde{c}} \text{ and } \|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\|_{L_x^3} < \delta \end{aligned}$$

$$\begin{aligned} \int_I \frac{d}{dt} \int \frac{|v_\varepsilon|^2}{2}(t, x) dx dt &\leq \limsup_{\varepsilon \rightarrow 0} \int_I \int |\nabla_j v_{\varepsilon l} R_\varepsilon^{jl}| dx dt \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_I \|\nabla v_\varepsilon(t)\|_{L_x^3} \varepsilon^{2\alpha} \|v\|_{B_{3,\infty}^\alpha}^2 dt \end{aligned}$$

For a.e. t , $v \in B^{1/3}_{3,C(N)}$, the integrand is bounded by $o(\varepsilon^{-1+2/3})\varepsilon^{3/2} = o(1)$. Thus above integral is dominated by:

$$\int_I \varepsilon^{-1 \times 1/3 + 2/3} \|v(t)\|_{B^{1/3}_{3,\infty}}^3 dt \leq \int_I \|v(t)\|_{B^{1/3}_{3,\infty}}^3 dt$$

By assumption and DCT, bounded. \square

Theorem 2.4. (Isett 18') An energy dissipating solution whose singularities have 0 Lebesgue measure in \mathbb{R}^4 cannot be of class $L_t^1 B^{1/3}_{\zeta,\infty}$ if $r > 3$.

Compared with Meneveau-Sreenivasan [9],

$$\langle |v(x + \Delta x) - v(x)|^r \rangle = |\Delta x|^{\xi_r}$$

singular support in $L_t^3 B^{1/3}_{3,C(N)}$. (K41) implies $\xi_r \sim \frac{r}{3}$ (only correct when $r = 3$).

Lemma 2.5. (Local energy conservation Duchon-Robert[3] formula $D[v, p] = \partial_t(\frac{|v|^2}{2}) + \nabla(\frac{|v|^2}{2} + p)v^j = \lim_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon l} R_\varepsilon^{jl}$ dissipation distribution $v \in L_{t,x}^3$. If $D[v, p] = 0$ and $v \in L_{t,x}^2 \cap L_{t,x}^3$, then $\int \frac{|v|^2}{2}(t, x) dx$ is constant and $D[v, p]$ if $v, p \in C^1$.

If $v \in L_t^r B_{r,\infty}^{1/3}$ with $r > 3$ is energy dissipating, since $v \in L_{t,x}^2 \cap L_{t,x}^3$, $D[v, p] \neq 0$. Since $\frac{d}{dt} \int \frac{|v|^2}{2} = \int D[v, p] dx$ But we claim that

$$\|D[v, p]\|_{L_{t,x}^{r/3}} < \infty$$

using Duchon-Robert formula: $\|\nabla_j v_{\varepsilon l} R_\varepsilon^{jl}\|_{L_{t,x}^{r/3}}$ is bounded uniformly in $\varepsilon > 0$.

$$\begin{aligned} \|\nabla_j v_\varepsilon\|_{L_x^r} &\lesssim \varepsilon^{-1+1/3} \|v(t)\|_{B_{r,\infty}^{1/3}} \\ \|R_\varepsilon^{jl}\| &\lesssim \varepsilon^{2/3} \|v(t)\|_{B_{r,\infty}^{1/3}}^2 \\ \Rightarrow \quad \forall \phi \in C^n \text{ nfty}_c(I \times \mathbb{R}^d) \quad &< D < [v, p], p > \lesssim C \|p\|_{L_{t,x}^s} \end{aligned}$$

Then $D[v, p]$ is in the dual of $L_{t,x}^s$ which is $L_{t,x}^{r/3}$ provided $r > 3$. Let $\frac{1}{s} + \frac{3}{r} = 1$. $\text{supp } D[v, p]$ has positive Lebesgue measure, but $\text{supp } D[u, p] \subset \text{sing}(\text{supp } U)$ also has positive measure.

There is an open problem to find a function $f(r)$ s.t. the condition $\frac{\zeta_r}{r} < \frac{1}{3} - f(r)$ works.

Proof. (Proof of Duchon-Robert formula) Considering Euler equation(Euler)

$$\eta_{\varepsilon\delta} * u := J_\zeta * \eta_\varepsilon * u$$

Let's test against $w_\varepsilon \delta = \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v)$.

$$0 = - \int_{I \times \mathbb{R}^d} v^l \partial_t \eta_{\varepsilon\delta} + (\phi \eta * v_l) + v^j v^l \nabla_j \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v) + p \nabla^l \eta_{\varepsilon\delta} * (\phi \eta_{\varepsilon\delta} * v_l) dx dt,$$

where $\phi \in C_c^\infty(I \times \mathbb{R}^d)$. Use the definition of self adjointness solution $\eta_{\varepsilon\delta}*$ and divergence-free properties of $\eta_{\varepsilon\delta} * v_l$. Then Euler equation becomes

$$0 = - \int \partial_t \phi \frac{|\eta_{\varepsilon\delta} * v|^2}{2} + v^j v^l \eta_{\varepsilon\delta} \nabla_j \phi \eta_{\varepsilon\delta} * v_l + p \eta_{\varepsilon\delta} * (\nabla^l \phi \eta_{\varepsilon\delta} * v_l) dx dt$$

Let $\varepsilon \rightarrow 0$ using uniform boundedness of $\eta_\varepsilon*$ and $\nabla_j \eta_\varepsilon*$. As $\delta \rightarrow 0$, thanks to $\nabla_j \eta_\varepsilon*$, the time derivative naturally goes away. Then

$$0 = - \int \partial_t \phi \frac{v_\varepsilon^2}{2} + \nabla_j \phi \left(\frac{|v_\varepsilon|^2}{2} v^j + \eta_\varepsilon * p v_\varepsilon^j \right) dx dt \quad (2)$$

$$+ \int \phi \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} + Z_\varepsilon, \quad (3)$$

where $Z_\varepsilon = \int \nabla_j \phi R_\varepsilon^{jl} v_{\varepsilon l}$. Take both time and space derivative of ϕ . Using $v \in L_{t,x}^2 \cap L_{t,x}^2$ and $p = (-\Delta)^{-1} \nabla_j \nabla_l (v^j v^l) \in L_{t,x}^{3/2}$
(2) $\Rightarrow \langle D[u, p], \phi \rangle$ as $\varepsilon \rightarrow 0$.
(3) $\Rightarrow \lim_{\varepsilon \rightarrow 0} \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} + Z_\varepsilon$

$$Z_\varepsilon = \int \nabla_j \phi (\eta_\varepsilon * (v^j v^l) - v_\varepsilon^j v_\varepsilon^l) v_{\varepsilon l} dx dt =: B_\varepsilon[v, v]$$

Here we define the commutator $B_\varepsilon[\cdot, \cdot]$.

$$\|B_\varepsilon[u, w]\|_{L_{t,x}^{3/2}} \leq C \|u\|_{L_{t,x}^3} \|w\|_{L_{t,x}^3}$$

which is independent of t .

If u or $w \in C_c^\infty$ $\|B_\varepsilon[u, w]\|_{L_{t,x}^{3/2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. By approximation $\|B_\varepsilon[v, v]\|_{L_{t,x}^{3/2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

By Holder inequality,

$$|Z_\varepsilon| \leq \|\nabla \phi\|_{L^\infty} \|B_\varepsilon[v, v]\|_{L_{t,x}^{3/2}} \|v_\varepsilon\|_{L^2} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

□

Remark. Improvement:

- Calderon-Zygmund Theorem.
- More regularity in time.

Proof. ($\text{supp } D[u, p] \subset \text{sing}(\text{supp } v)$) In fact $\text{supp } D[u, p] \subset \text{sing}(\text{supp } L_\varepsilon^3 B_{3,C(N)}^{1/3} v)$.

What's good for not using Littlewood-Paley definition of Besov space? the solution above can be defined locally.

$\phi \in C_c^\infty(I \times B_q)$ and $B'_q \subset B_q$ a smaller ball with same center q . Let $q \notin \text{sing}(\text{supp } B_{3,C(N)}^{1/3} v)$

$$\langle \phi, D[v, p] \rangle = \lim_{\varepsilon \rightarrow 0} \int_I \int_{B'_q} \phi(t, x) \nabla_j v_{\varepsilon l} R_\varepsilon^{jl} \lesssim \|\phi\|_{L^\infty} \int_I \|\nabla v_\varepsilon(t)\|_{L(B'_q)} \|R_\varepsilon\|_{L_x^{2/3}}$$

dominated by $\|\phi\|_{L^\infty} \int_I \|v(t, \cdot)\|_{B_{3,\infty}^{1/3}(B_\varepsilon)}^3$. For a.e. t , we have $\|\nabla v_\varepsilon\|_{L^3} \cdot \|R_\varepsilon\|_{L^{3/2}} = o(1)$ and $D[u, p] \rightarrow 0$ by dominate convergence theorem. □

Remark. 1. Heat flow approach can also be applied to this problem. The proof is quite different[7] and is on a compact Riemann manifold(no convolution can be used).

2. Compressible Euler Case. The problem lies when dealing with commutator estimation.

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