Weak Solution: Convex Integration

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1 Introduction

First let's give an example to help define what is the weak solution.

Example 1.1.

$$\Delta u = f$$

If u don't have to be continuous take a test function: $\forall \phi \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{C}} \phi \Delta u = \int_{\mathbb{C}} \phi f$. If u is holomorphic $u : \mathbb{C} \to \mathbb{C}$

$$\frac{d}{dt} u(z + t_\alpha) = \frac{\partial u}{\partial z}\alpha + \frac{\partial u}{\partial \bar{z}}\bar{\alpha} = \frac{\partial u}{\partial x} Re\alpha + \frac{\partial u}{\partial y} Im\alpha$$

i.e. $du = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}$.

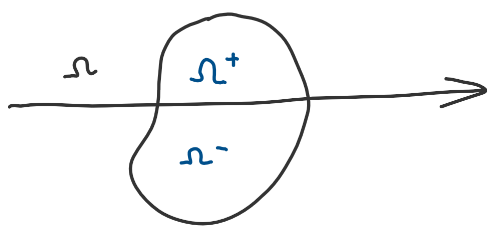
Theorem 1.1. *If $\frac{\partial u}{\partial z} = 0$ or $\Delta u = 0$ in the weak sense(against all test function). Then $u \in C^\infty(\mathbb{R})$ and satisfies the equation in the classical sense.*

Useful: $f_k(z)$ holomorphic $u(z) = \sum\limits_{k=0}^{\infty} f_k(z)$, the series is absolutely convergent.

$$\begin{aligned} & - \textstyle\int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} u(z) \\ \text{Fubini} \quad = & - \sum_{k=0}^{\infty} \textstyle\int_{\mathbb{C}} \frac{\partial \phi}{\partial \bar{z}} f_k(z) \\ = & \sum_{k=0}^{\infty} \textstyle\int_{\mathbb{C}} \phi \frac{\partial f_k}{\partial \bar{z}} = 0 \end{aligned}$$

(???)

Theorem 1.2. *(Swartz reflection principle.)If f is holomorphic on $\Omega \cap \{y > 0\}$ and $\Omega \cap \{y < 0\}$. If f is continuous on Ω on Ω including $\Omega \cap \{y = 0\}$. Then f is holomorphic on Ω.*



In D' $f = \lim\limits_{\delta \to 0} f(H(y - \varepsilon) + H(\varepsilon - y))$, here H is heaviside function.

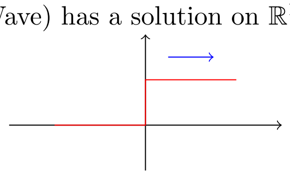
$$\frac{\partial f}{\partial \bar{z}} = \lim_{\varepsilon \to 0} \frac{\partial f}{\partial \bar{z}} + f\left(\frac{\partial y}{\partial \bar{z}} \delta(y - \varepsilon) - \frac{\partial y}{\partial z} \delta(\varepsilon - y)\right)$$

Since f is continuous $\lim\limits_{\varepsilon \to 0} f(\delta(y - \varepsilon) - \delta(\varepsilon - y)) = 0$

$$\Box u = 0 \quad \text{where } \Box := -\partial_t^2 + \partial_1^2 + ... + \partial_n^2 \tag{Wave}$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^{d-1}) \qquad \int_{\mathbb{R}}^{d+1} \Box \phi u = 0$$

(Wave) has a solution on $\mathbb{R}^{1+1}$ given by $u(t, x) = u(t - x)$, a traveling wave:



Example 1.2.

$$u(t, x) = H(t - x) - H(t + x)$$

is the unique solution to (Wave) on $\mathbb{R}^{1+d}$

The green lines are smooth approximation. After some time, it is still good enough to approximate the real world solution.

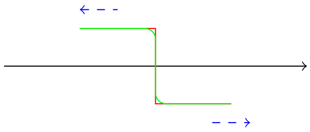


Figure 1: $t = 0$

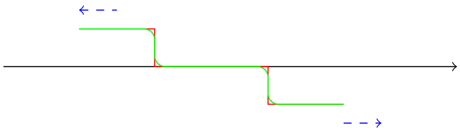


Figure 2: $t = 1$

2 Fluid Mechanics

2.1 Incompressible Euler Equation

Define 2 velocity field: $v: \quad \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \quad p: \quad \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0$$

$$\nabla_j v^j = 0 \quad \text{divergence free}$$

This system obvious make sense for $v \in L^2_{\text{loc}}$. Let's recall the derivation of Euler equation. $\forall \Omega$ with $C^1 \quad \partial\Omega$

$$\int_{\partial\Omega} v \cdot \overrightarrow{n} d\sigma = 0 \quad \forall t$$

meaning water coming in is exactly the same as water going out.

$$\frac{d}{dt}\begin{bmatrix}\text{total momentum}\\ m \cdot v\end{bmatrix} = \begin{bmatrix}\text{Force on } \Omega\end{bmatrix} + \begin{bmatrix}\text{Flux of}\\ \text{momentum}\end{bmatrix} \quad \Rightarrow \frac{d}{dt}\int_\Omega v^\ell dx = \int_{\partial\Omega} p \overrightarrow{n}^\ell dx \quad \forall t$$

These integral gives also the weak form of equation, let's say if p is good enough. If $v, pinC^1$, use $\int_{\partial\Omega} f \overrightarrow{n}_j d\sigma = -\int_\Omega \nabla_j f dx$

$$\frac{d}{dt}\int_\Omega v^\ell = -(\int_\Omega \nabla^\ell p + \nabla_j (v^j v^\ell) dx$$

$$\int_\Omega (\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p) dx = 0 \qquad \forall \Omega, \quad \forall t$$

Here comes a natural question: Are weak solution to the Euler equation physical meaningful?
Some physical properties are required. Take $\Omega = \mathbb{R}^d$ and $v \in L^2_{t,x}(I \times \mathbb{R}^d)$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell (t, x) dx = 0$$

If $(1 + |x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, then linear angular momentum conserved?
Here $\forall K^\ell$ s.t. $\quad \nabla_j K_\ell + \nabla_\ell K_j = 0 \quad on \quad \mathbb{R}^d$

Example 2.1. $K = e_{(i)}$ the basis vector, $\int_{\mathbb{R}^d} K_\ell v^\ell dx = \int_{\mathbb{R}^d} v^{(i)} dx$

Example 2.2. Rotation $K(a, b) = x^a e_b - x^b e_a$, $1 \leq a < b \leq d$.

$$\int_{\mathbb{R}^d} K_\ell (\partial_t v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p) dx = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell - \int \nabla : K_\ell (v^j v^\ell) - \int_{\mathbb{R}^d} \nabla^\ell K_\ell p dx = 0$$

where $\operatorname{div} K = \nabla^\ell K_\ell = \delta^{j\ell}\nabla_j K_\ell = \frac{1}{2}\delta^{j\ell}(\nabla_j K_\ell + \nabla_\ell K_j) = 0$ by assumption.

$$\partial v^\ell + \nabla : (v^j v^\ell) + \nabla^\ell p) dx = 0$$

Test against a space cut-off function $K^\ell(B) := q(t)\varphi(\frac{|x|}{B})(x^b e_m - x^a e_b)$. Here K is rotationally symmetric, so it is divergence-free.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_\ell^{(\beta)} v^\ell dx]dt - \int_{\mathbb{R}} \eta \int_{\mathbb{R}^d} \nabla_j K_\ell^{(\beta)} v^j v^\ell - \int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$$

Here, due to divergence-free, like what we did previously, $\int_{\mathbb{R}^+} \eta(t) \int_{\mathbb{R}^d} \nabla^\ell K_\ell^{(\beta)} p dx dt = 0$. i.e.

$$-\int_{\mathbb{R}^+} \eta'(t)[\int_{\mathbb{R}^d} K_\ell^\beta v^\ell dx]dt - \frac{1}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \left(\nabla_j K_\ell^{(\beta)} + \nabla_\ell K_j^{(\beta)}\right) v^j v^\ell dx dt - \frac{1}{2}\int_{\mathbb{R}} \eta(t) \int_{\mathbb{R}^d} \nabla_j \varphi(\frac{|x|}{B})) K_\ell v^j v^\ell dx dt = 0$$

The 1st term is dominated by $|x| \cdot v \in L^1_{t,x}$ by assumption. $\frac{1}{|x|} \cdot v$ dominated the derivative and integrant.

2.2 Conservation of Energy

If $(1+|x|)v \in L^1_{t,x}(I \times \mathbb{R}^d)$, $v \in L^2_{t,x}(I \times \mathbb{R}^d)$, then $\forall K^\ell$, $\nabla_j K_l + \nabla_l K_j = 0$, then we have the conservation of angular momentum:

$$\frac{d}{dt}\int_{\mathbb{R}^d} K_\ell v^\ell(t,x) dx = 0,$$

where $K \in \operatorname{span}\{x^a e_b = x^b e_a : \ e_i, \ 1 \le i \le d, 1 \le a < b \le d\}$. Here decay assumption is needed but not the regularity assumption. If $f \in \mathcal{D}'(\mathbb{R})$, $\frac{df}{dt} = 0 \Rightarrow f = c$ limit of constant.

$$\delta_j^\ell = \nabla_j w^{j\ell} \quad w^{j\ell} = -w^{lj} \qquad \text{antisymmetric}$$

Approximate by $\nabla(\phi(\frac{|x|}{B} w^{j\ell})$

$$w^{j\ell} = x^2(\delta_2^j \delta_1^\ell - \delta_1^j \delta_2^\ell)$$

$$\begin{aligned} &\nabla_j(x^2(\delta_2^j\delta_1^\ell - \delta_1^j\delta_2^\ell)) \\ &= \delta_j^2(\delta_2^j\delta_1^\ell - \delta_1^j\delta_2^\ell) = \delta_1^\ell \end{aligned}$$

If $w^{il} = -w^{lj}$

$$\nabla_j\nabla_j w^{j\ell} = -\nabla_\ell\nabla_j w^{lj} = -\nabla_j\nabla_\ell w^{lj} = -\nabla_\ell\nabla_j w^{j\ell}$$

Conservation of energy means that $\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v|}{2}(t,x)dx = 0$. Note that energy is nonlinear.

$$\partial_t(\frac{|v|^2}{2}) + \nabla_j((\frac{|v|}{2} + p)v^j) = 0$$

If $v \in C^1 \cap L^2_{t,x} \cap L^3_{t,x}(I \times \mathbb{R}^d)$ both local and global conservation of energy hold. Note that here B could be ∞. Multiply the local energy by $\eta(t)\varphi(\frac{|x|}{B})$

$$\begin{aligned} &\underset{(1)}{\int \eta \frac{d}{dt}\int \varphi(\tfrac{|x|}{B})\tfrac{|v|^2}{2}(t,x) dx dt} \quad \underset{(2)}{-\int \eta(t)\int \nabla_j[\varphi(\tfrac{|x|}{B})](\tfrac{|v|^2}{2} + p)v^j dx dt} \\ &(1) = -\int \underbrace{\eta'(t)} \int \varphi(\tfrac{|x|}{B}) \underbrace{\frac{|v|}{2}(t,x)} dx dt \quad \text{Intergral by parts} \\ &\quad \text{dominated by } \tfrac{|x|^2}{2}|\eta'| \in L^1_{t,x} \end{aligned} \tag{Local}$$

term (2) converge to 0 pointwisely when $B \to \infty$ and dominated by $|\eta t|(\frac{|v|^3}{2} + |p||v|)$.
Let's recall Euler equation.

$$\begin{cases} \partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p &= 0, \\ \nabla_j v^j &= 0 \end{cases} \tag{Euler}$$

Take divergence over (Euler), $\Rightarrow$

$$\nabla_j\nabla_\ell(v^jv^\ell)+\nabla_\ell\nabla^\ell p=0$$

i.e.

$$\Delta p=-\nabla_\ell\nabla_j(v^jv^\ell)$$

$$p=\underbrace{(-\Delta)^{-1}\nabla_\ell\nabla_j(v^jv^\ell)}_{\text{zero order operator}}\quad\underbrace{(v^jv^\ell)}_{\in L^{3/2}_{t,x}}$$

Thus naturally $p\in L^{3/2}_x$ a.e. $t\in\mathbb{R}^+$

$$\|p\|_{L^{3/2}_x(L^{3/2}_t)}=\|p\|_{L^{3/2}_{t,x}}<\infty$$

$$v_\ell(\partial_t v^\ell+\nabla(v^jv^\ell)+\nabla^\ell p)=0$$

$$\nabla_jv^j=0$$

Thus

$$\Rightarrow\partial_t(\frac{|v|^2}{2})+v_\ell v^j\nabla_jv^\ell+v_\ell\nabla^\ell p=0$$

$$\Rightarrow\partial_t(\frac{|v|^2}{2})+v^j\nabla_j(\frac{|v|^2}{2}+v_jv^jp=0$$

$$\nabla_jv^j=0$$

$$\partial_t(\frac{|v|^2}{2}+\nabla_j((\frac{|v|^2}{2}+p)v^j)=0$$

Using $\nabla_jv^j=0$ and product rule, conservation of energy is proved for sufficient regular solutions. But how sufficient do we need?
In turbulence situation (Navier-Stokes equations) with $\nu<<1$

$$v_\ell(\partial_tv^\ell+\nabla_j(v^jv^\ell)+\nabla^\ell p)=\nu v_\ell\Delta v^\ell$$

$$\frac{d}{dt}\int\frac{|v|^2}{2}(t,x)dx=-\nu\int|\nabla v|^2dx=\nu\int v_\ell\nabla_i\nabla^iv^\ell$$

Taking a formal limit, $\exists$ incompressible Euler flows with

$$\frac{d}{dt}\int\frac{|v|}{2}(t,x)dx<-\varepsilon<0$$

Theorem 2.1. *Onsager's Conjecture*
(+) *If $\alpha>1/3$ and $(v(t,x+\Delta x)-v(t,x))\leq c|\Delta x|$ where $x\in\mathbb{T}^3(v\in L^\infty_tC^\alpha_x)$, then the energy conserved.*
($-$) *(K41)If $\alpha\leq1/3$ $\exists$ incompressible Euler flows with $v\in L^\infty_tL^\alpha_x$ s.t. $\int_{\mathbb{T}^d}\frac{|v|^2}{2}(t,x)dx$ is not constant.*

Now we follow [2] and discuss the (+) part first.

$$\partial_tv^\ell+\nabla_j(v^jv^\ell)+\nabla^\ell p=0$$

In order to get tnto Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series,

$$v(x,t)=\sum_{k\in\mathbb{Z}^3}a_k(t)e^{ik\cdot x}.$$

Obviously $a_{-k}=\overline{a_k}$, because v is real-valued. Moreover the divergence-free con- straint translates into the identity $k\cdot a_k=0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k :

$$\frac{da_k}{dt}=i\sum_\ell a_{k-\ell}\cdot\ell\left[-a_\ell+\frac{(a_\ell\cdot k)\,k}{|k|^2}\right]-\nu|k|^2a_k\tag{1}$$

The total kinetic energy is (up to a constant factors) $\sum_k |a_k|^2$.
(Don't understand)Energy starts at low wave numbers and moves to higher wave numbers in finite number.
$\sum\limits_{\frac{\lambda}{2}\leq|k|\leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ matches (K41), corresponding to exactly 1/3 regularity for solutions.
Low frequency energy will goes to all frequency and when it goes to infinity, it will disappear.
(K 41) $E \lim\limits_{v\to 0} \left\langle v \int |\nabla v|^2 dx \right\rangle$ and v determine all statistic properties of turbulent flows.

$$\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^p |\Delta x|^{1/3}$$

Try to find $|\Delta x| \leq L \sim \varepsilon^a v^b$.
Now $(+)$ is solved by [4] and [1] with the goal $L_t^3 B^{1/3}_{3,C(N)}$, $L_t^3 B^{1/3+\varepsilon}_{3,\infty}$.
$(-)$ is solved $(d \geq 3)$ with $\alpha = \frac{1}{3}$, using convex integration by Phillip Isett [7].
Convex integration originated from the Nash–Kuiper Paradox(50's) for C^1 isometric embedding. Connection to Euler equation discovered by Camillo De Lellis and László Székelyhidi (08,12). First result towards Onsager conjecture is in [8]. And $\alpha < \frac{1}{5}$ by [10]. The non-uniqueness example was first given by [12] and then Shnirelman give a different proof in [13].

2.3 Another way of proving $(+)$

$(+)$ (Eyink, Constantin, E, Titi 94') $L^3(B^\alpha_{3,\infty})$

$$\|v\|_{C^\alpha} = \sup_{h\neq 0} \frac{\|v(x+h) - v(x)\|_{L^\infty}}{|h|^\alpha}$$

$$\|v\|_{B^\alpha_{3,\infty}} = \sup_{h\neq 0} \frac{\|v(x+h) - v(x)\|_{L^3}}{|h|^\alpha}$$

Lemma 2.2. *Commutator Estimate*

$$R^{j\ell}_\varepsilon = \eta_\varepsilon * (v^j v^\ell) - (v^j_\varepsilon v^\ell_\varepsilon)$$
$$\|R_\varepsilon\|_{L^{3/2}} \lesssim \varepsilon^{2\alpha} \|v\|^2_{B^\alpha_{3,\infty}}$$

Let's think $R^{j\ell}_\varepsilon$ as an expectation with the idea:

$$R = \text{"}\mathbb{E}\left[v^2\right] - \left(\mathbb{E}[v]\right)^2\text{"} = \text{"}\mathbb{E}[(v - \mathbb{E}(v))^2]\text{"},$$

which is quadratic.

$$\begin{array}{rl} R^{j\ell}_\varepsilon = & \int v^i(x-h) v^\ell(x-h)\eta_\varepsilon(h)dh - \int v^j(x-h_1)\eta_\varepsilon(h_1)dh_1 \int v^\ell(x-h_2)\eta_\varepsilon(h_2)dh_2 \\ \text{Using} & \int \eta_\varepsilon(h)dh = 1 \\ = & \int (v^j(x-h) - v^j_\varepsilon(x))(v^\ell(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \end{array}$$

By Lemmas in [1], we decompose above equation into $\sum\limits_{i=1}^4 R^{j\ell}_{\varepsilon i}$, where

$$\begin{array}{rl} R_{\varepsilon 1} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^\ell(x-h) - v^\ell_\varepsilon(x-h))\eta_\varepsilon(h)dh \\ R_{\varepsilon 2} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x))(v^\ell(x-h) - v^\ell_\varepsilon(x-h))\eta_\varepsilon(h)dh \\ R_{\varepsilon 3} = & \int (v^j(x-h) - v^j_\varepsilon(x-h))(v^\ell_\varepsilon(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \\ R_{\varepsilon 4} = & \int (v^j_\varepsilon(x-h) - v^j_\varepsilon(x-h))(v^\ell_\varepsilon(x-h) - v^\ell_\varepsilon(x))\eta_\varepsilon(h)dh \end{array}$$

For example,

$$\begin{array}{rl} R_{\varepsilon 2} = & \int_{\mathbb{R}^d} \int_0^1 \frac{d}{d\sigma} v^j_\varepsilon(x-\sigma h)d\sigma(v^\ell(x-h) - v^\ell(x))\eta_\varepsilon(h)dh \\ = & \int_{\mathbb{R}^d} \int_0^1 d\sigma \nabla_i v^i_\varepsilon(x-\sigma h)h^i(v^\ell(x-h) - v^\ell(x)\eta_\varepsilon(h)dh \end{array}$$

$$\left\|R^j_{\varepsilon 2}\right\| \leq_{\mathbb{R}^d} \int_0^1 \|\nabla v_\varepsilon\|_{L^3} |h| \|v(\cdot - h) - v(\cdot)\|_{L^3} |\eta_\varepsilon(h)| dh$$

Modify the equation with modifier η_ε:

$$\begin{aligned} \eta_\varepsilon * (\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p) = & \quad 0 \\ \partial_t v_\varepsilon^\ell + \nabla_j(v_\varepsilon^j v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon = & -\nabla_j R_\varepsilon^{j\ell} \end{aligned}$$

(Thus we need smoothness in time) $\times v_\varepsilon$ then integral by parts:

$$\partial(\frac{|v_\varepsilon|^2}{2}) + v_{\varepsilon l}\nabla_j(v_\varepsilon^j v_\varepsilon^\ell) + v_{\varepsilon l}\nabla^\ell p_\varepsilon = -v_{\varepsilon l}\nabla_j R_\varepsilon^{j\ell} = \int_{\mathbb{R}^d} \nabla_j \left| \frac{v_\varepsilon^2}{2} v_\varepsilon^j \right| \to 0$$

with assumption.

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{|v_\varepsilon|^2}{2}(t,x)dx + \int_{\mathbb{R}^d} v_\varepsilon^j \nabla_j v_\varepsilon^\ell v_{\varepsilon l} + \underline{\nabla^\ell v_{\varepsilon l} p_\varepsilon} = \int_{\mathbb{R}^d} \nabla v_{\varepsilon l} R_\varepsilon^{j\ell}$$

$\underline{\nabla^\ell v_{\varepsilon l} p_\varepsilon} = 0$ for divergence-free.
LHS converges to $\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx$ in $\mathcal{D}'(\mathbb{R})$ since $v_\varepsilon \to v$ in $L^2_{t,x}$.

$$\begin{aligned} &\left\| \tfrac{d}{dt}\int \tfrac{|v|^2}{2}(t,x)dx \right\|_{L^1_t} \leq \limsup_{\varepsilon\to 0} \int\int \left|\nabla_j v(t,x) R_\varepsilon^{j\ell}\right| dxdt \\ &\leq \limsup_{\varepsilon\to 0} \int \|\nabla v_\varepsilon(t,\cdot)\|_{L^3_x} \|R_\varepsilon\|_{L^{3/2}} dt \\ &\leq \limsup_{\varepsilon\to 0} \int \varepsilon^{-1+\alpha}\|v(t)\|_{B^\alpha_{3,\infty}} \|v(t,\cdot)\|^2_{B^\alpha_{3,\infty}})dt \\ &< \limsup_{\varepsilon\to 0} \varepsilon^{-1+3\alpha} \int \|v(t,\cdot)\|^3_{B^\alpha_{3,\infty}} dt \quad \to 0 \quad \text{with } \alpha > \tfrac{1}{3} \end{aligned}$$

If $\alpha = \frac{1}{3}$ and v bounded in $L^1_t(I)$ for some finite time period.

$$\frac{d}{dt}\int \frac{|v|^2}{2}(t,x)dx = \lim_{\varepsilon\to 0}\frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dx$$

$v\phi \in C_c^\infty(I)$

$$\frac{d}{dt} < \int \frac{|v|^2}{2}(t,x)dx, \phi > \leq \|\phi\|_{L^\infty(I)}$$

LHS is of finite measure. $e(t) = \int \frac{|v|^2}{2}(t,x)$ is of bounded variation. IN fact $\frac{d}{dt}e(t)$ is finite.
(???)If $v \in L^r B^{1/3}_{3,\infty}$, consider $\left\|\frac{d}{dt}e(t)\right\|_{L^{r/t}_t}$ using duality. $u \in L^\infty_t B^{1/3}_{3,\infty}$ uniformly $\left\|\frac{d}{dt}e(t)\right\|_{L^inftyt_t} \leq C$ and also $\frac{d}{dt}e(t) \leq -\varepsilon < 0$ is stable under perturbation. If not, the dissipation $\int_{\mathbb{R}^d} \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx$ can be really big.
Remark. The singular support of a generalized function u is the complement of the largest open set on which u is smooth. Roughly speaking, it is the closed set where the distribution does not correspond to a smooth function.

2.4 Local energy conservation

$$\partial_t v_\varepsilon^\ell + \nabla_\ell(v_\varepsilon^\ell v_\varepsilon^\ell) + \nabla^\ell p_\varepsilon = -\nabla_j R_\varepsilon^{j\ell}$$

where $R_\varepsilon^{j\ell} = \eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell$

$$\|R_\varepsilon(t,.)\|_{L^{3/2}_\ell} \leq \varepsilon^{2\alpha}\|v(t)\|^2_{B^\alpha_{3,\infty}}$$

$$\frac{1}{2}\int \frac{|v_\varepsilon|}{2}(t,x)dx = \lim_{\varepsilon\to 0}\int \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} dx$$

Here to clarify the space:

$$B^{1/3}_{3,c(N)} = (\overline{C^\infty})^{B^{1/3}_{3,\infty}} = B^{1/3}_{3,\infty} \cap \{\lim_{h\to 0} \frac{|v(x+h)-v(x)|}{|h|^{1/3}} = 0\}$$

The "Holder Continuity" is the reason for smooth approximation. Define

$$c^{1/3} = (\overline{C^\infty})^{C^{1/3}}$$

Note that, here $c^{1/3}$ is not dense in $C^{1/3}$. Let $\varphi(x)$ be a smooth cut off function, then, $|x|^{1/3} \in C^{1/3}\backslash c^{1/3}$, but $\varphi(x)|x|^{1/3} \notin C^{1/3}\backslash c^{1/3}$

Lemma 2.3. $\|\nabla v_\varepsilon\|_{L^3} = o(\varepsilon^{-1+\alpha})$ *if* $v \in B^\alpha_{3,c(N)}$

Proof. Claim: $\varepsilon^{1-\alpha}\nabla(\eta_\varepsilon * \cdot) : B^\alpha_{3,\infty} \to L^3$ is uniformly bounded.

$$\|\nabla v_\varepsilon\|_{L^3_x} \lesssim \varepsilon^{-1+\alpha}\|v\|_{B^\alpha_{3,\infty}}$$

Let $\delta > 0$ be given, choose $\tilde{v} \in C^\infty$ s.t. $\|v - \tilde{v}\|_{B^\alpha_{3,\infty}} < \frac{\delta}{2C_2}$.

$$\begin{aligned}
\left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} &\le \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * (v-\tilde{v})\right\|_{L^3_x} + \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * \tilde{v}\right\|_{L^3_x} \\
&\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\|\nabla\eta_\varepsilon * \tilde{v}\|_{L^3_x} \\
&\le \tfrac{\delta}{2} + \varepsilon^{1-\alpha}\tilde{c} \quad \text{for} \varepsilon^{1-\alpha} < \tfrac{\delta}{2\tilde{c}} \text{ and } \left\|\varepsilon^{1-\alpha}\nabla\eta_\varepsilon * v\right\|_{L^3_x} < \delta
\end{aligned}$$

$$\begin{aligned}
&\textstyle\int_I \frac{d}{dt}\int \frac{|v_\varepsilon|^2}{2}(t,x)dxdt \le \limsup_{\varepsilon\to 0}\int_I\int \left|\nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon\right| dxdt \\
&\textstyle\le \limsup_{\varepsilon\to 0}\int_I \|\nabla v_\varepsilon(t)\|_{L^3_x}\varepsilon^{2\alpha}\|v\|^2_{B^\alpha_{3,\infty}}dt
\end{aligned}$$

For a.e. t, $v \in B^{1/3}_{3,C(N)}$, the integrant is bounded by $o(\varepsilon^{-1+2/3})\varepsilon^{3/2} = o(1)$. Thus above integral is dominated by:

$$\int_I \varepsilon^{-1\times 1/3+2/3}\|v(t)\|^3_{B^{1/3}_{3,\infty}}dt \le \int_I \|v(t)\|^3_{B^{1/3}_{3,\infty}}dt$$

By assumption and DCT, bounded. □

Theorem 2.4. *(Isett 18') An energy dissipating solution whose singularities have 0 Lebesgue measure in* $\mathbb{R}^4$ *cannot be of class* $L^r_t B^{1/3}_{\zeta,\infty}$ *if* $r > 3$.

Compared with Meneveau-Sreenivasan [11],

$$< |v(x+\Delta x) - v(x)|^r >= |\Delta x|^{\xi_r}$$

singular support in $L^3_t B^{1/3}_{3,C(N)}$. (K41) implies $\xi_r \sim \frac{r}{3}$ (only correct when $r = 3$).

Lemma 2.5. *(Local energy conservation Duchon-Robert[3] formula* $D[v,p] = \partial_t(\frac{|v|^2}{2}) + \nabla(\frac{|v|^2}{2} + p)v^j) = \lim_{\varepsilon\to 0}\nabla_j v_{\varepsilon\to 0}\nabla_j v_{\varepsilon l}R^{j\ell}_\varepsilon$ *dissipation distribution* $v \in L^3_{t,x}$. *If* $D[v,p] = 0$ *and* $v \in L^2_{t,x} \cap L^3_{t,x}$, *then* $\int \frac{|v|^2}{2}(t,x)dx$ *is constant and* $D[v,p]$ *if* $v, p \in C^1$.

If $v \in L^r_t B^{1/3}_{r,\infty}$ with $r > 3$ is energy dissipating, since $v \in L^2_{t,x} \cap L^3_{t,x}$, $D[v,p] \ne 0$. Since $\frac{d}{dt}\int\frac{|v|^2}{2} = \int D[v,p]dx$ But we claim that

$$\|D[v,p]\|_{L^{r/3}_{t,x}} < \infty$$

using *Duchon-Robert* formula: $\left\|\nabla_j v_{\varepsilon l}R^{j\ell}_\varepsilon\right\|_{L^{r/3}_{t,x}}$ is bounded uniformly in $\varepsilon > 0$.

$$\begin{aligned}
&& \|\nabla_j v_\varepsilon\|_{L^r_x} &\lesssim \varepsilon^{-1+1/3}\|v(t)\|_{B^{1/3}_{r,\infty}} \\
&& \left\|R^{j\ell}_\varepsilon\right\| &\lesssim \varepsilon^{2/3}\|v(t)\|^2_{B^{1/3}_{r,\infty}} \\
\Rightarrow \quad & \forall\phi \in C^inifty_c(I\times\mathbb{R}^d) \quad & < D < [v,p], p &>\lesssim C\|p\|_{L^s_{t,x}}
\end{aligned}$$

Then $D[v,p]$ is in the dual of $L^s_{t,x}$ which is $L^{r/3}_{t,x}$ provided $r > 3$. Let $\frac{1}{s} + \frac{3}{r} = 1$. $\operatorname{supp} D[v,p]$ has positive Lebesgue measure, but $\operatorname{supp} D[u,p] \subset \operatorname{sing}(\operatorname{supp} U)$ also has positive measure.
There is an open problem to find a function $f(r)$ s.t. the condition $\frac{\zeta_r}{r} < \frac{1}{3} - f(r)$ works.

Proof. (Proof of *Duchon-Robert* formula) Considering Euler equation(Euler)

$$\eta_{\varepsilon\delta} * u := J_\zeta *_t \eta_\varepsilon *_x u$$

Let's test against $w_\varepsilon\delta = \eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v)$.

$$0 = -\int_{I\times\mathbb{R}^d} v^\ell\partial_t\eta_{\varepsilon\delta} + (\phi\eta * v_\ell) + v^j v^\ell\nabla_j\eta_{\varepsilon\delta} * (\phi\eta_{\varepsilon\delta} * v) + p\nabla^\ell\eta_{\varepsilon_\delta} * (\phi\eta_{\varepsilon\delta} * v_\ell)dxdt,$$

where $\phi \in C_c^\infty(I \times \mathbb{R}^d)$. Use the definition of self adjointness solution $\eta_{\varepsilon_\delta}*$ and divergence-free properties of $\eta_{\varepsilon_\delta} * v_\ell$. Then Euler equation becomes

$$0 = -\int \partial_t \phi \frac{|\eta_{\varepsilon\delta} * v|^2}{2} + v^j v^\ell \eta_{\varepsilon\delta} \nabla) j \phi \eta_{\varepsilon\delta} * v_\ell + p\eta_{\varepsilon\delta} * (\nabla^\ell \phi \eta_{\varepsilon\delta} * v_\ell) dxdt$$

Let $\varepsilon \to 0$ using uniform bounded ness of $\eta_\varepsilon*$ and $\nabla_j \eta_\varepsilon *$. As $\delta \to 0$, thanks to $\nabla_j \eta_\varepsilon *$, the time derivative naturally goes away. Then

$$0 = -\int \partial_t \phi \frac{{v_\varepsilon}^2}{2} + \nabla_j \phi (\frac{|v_\varepsilon|^2}{2} v^j + \eta_\varepsilon * p v_\varepsilon^j) dxdt \tag{2}$$

$$+ \int \phi \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} + Z_\varepsilon, \tag{3}$$

where $Z_\varepsilon = \int \nabla_j \phi R_\varepsilon^{j\ell} v_{\varepsilon l}$. Take both time and space derivative of ϕ. Using $v \in L^2_{t,x} \cap L^2_{t,x}$ and $p = (-\Delta)^{-1} \nabla_j \nabla_\ell (v^j v^\ell) \in L^{3/2}_{t,x}$
(2) $\Rightarrow < D[u,p], \phi >$ as $\varepsilon \to 0$.
(3) $\Rightarrow \lim_{\varepsilon \to 0} \nabla_j v_{\varepsilon j} R_\varepsilon^{j\ell} + Z_\varepsilon$

$$Z_\varepsilon = \int \nabla_j \phi (\eta_\varepsilon * (v^j v^\ell) - v_\varepsilon^j v_\varepsilon^\ell) v_{\varepsilon l} dxdt =: B_\varepsilon[v,v]$$

Here we define the commutator $B_\varepsilon[\cdot,\cdot]$.

$$\|B_\varepsilon[u,w]\|_{L^{3/2}_{t,x}} \leqslant C \|u\|_{L^3_{t,x}} \|w\|_{L^3_{t,x}}$$

which is independent of t.
If u or $w \in C_c^\infty$ $\|B_\varepsilon[u,w]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$. By approximation $\|B_\varepsilon[v,v]\|_{L^{3/2}_{t,x}} \to 0$ as $\varepsilon \to 0$.
By Holder inequality,

$$|Z_\varepsilon| \leqslant \|\nabla \phi\|_{L^\infty} \|B_\varepsilon[v,v]\|_{L^{3/2}_{t,x}} \|v_\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0$$

□

Remark. Improvement:

- Calderon-Zygmund Theorem.
- More regularity in time.

Proof. ($\operatorname{supp} D[u,p] \subset \operatorname{sing}(\operatorname{supp} v)$) In fact $\operatorname{supp} D[u,p] \subset \operatorname{sing}(\operatorname{supp} L^3_\varepsilon B^{1/3}_{3,C(N)} v)$.
What's good for not using Littlewood-Paley definition of Besov space? the solution above can be defined locally.
$\phi \in C_c^\infty(I \times B_q)$ and $B_q' \subset B_q$ a smaller ball with same center q. Let $q \notin \operatorname{sing}(\operatorname{supp} B^{1/3}_{3,C(N)} v)$

$$< \phi, D[v,p] >= \lim_{\varepsilon \to 0} \int_I \int_{B_q'} \phi(t,x) \nabla_j v_{\varepsilon l} R_\varepsilon^{j\ell} \lesssim \|\phi\|_{L^\infty} \int_I \|\nabla v_\varepsilon(t)\|_{L(B_q')} \|R_\varepsilon\|_{L_x^{2/3}}$$

dominated by $\|\phi\|_{L^\infty} \int_I \|v(t,\cdot)\|^3_{B^{1/3}_{3,\infty}(B_\varepsilon)}$. For a.e. t, we have $\|\nabla v_\varepsilon\|_{L^3} \cdot \|R_\varepsilon\|_{L^{3/2}} = o(1)$ and $D[u,p] \to 0$ by dominate convergence theorem. □

Remark.

1. Heat flow approach can also be applied to this problem. The proof is quite different[9] and is on a compact Riemann manifold(no convolution can be used).
2. Compressible Euler Case. The problem lies when dealing with commutator estimation.

3 Holder Continuity

The following lecture are basic on [9].
Note that $B^{1/3}_{3,C(\mathbb{N}} \subsetneqq B^{1/3}_{3,\infty}$ and we can find a function in $B^{1/3}_{3,\infty} \setminus B^{1/3}_{3,C(N)}$.
$\phi(x)\chi_{\{x'>0\}} \subset B^{1/p}_{p,\infty} \quad \forall 1 < p < \infty$ Let's consider

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0 \quad L^\infty_t B^{1/3}_{3,\infty}$$

Energy dissipation at time $t = 0$

$$\frac{d}{dt}e(t) = \int_{\mathbb{T}^d} \nabla_j v_{\varepsilon l} R^{j\ell}_\varepsilon(0,x)dx$$

Eyink proved that there exists a divergence-free vector field in the space $C^{1/3}B^{1/3}_{3,C(N)}$, s.t. $\frac{d}{dt}|_{t=0}e(t) < 0$. We have a useful counter example:

$$v(x) = \sum_q 2^{2q\alpha} \sin\bigl(2^{2q}x\bigr) \in \dot{B}^{1/3}_{3,C(N)} \setminus \dot{B}^{1/3}_{3,\infty}$$

Now we consider this problem on compact Riemannian manifold for the conclusion $L^3_t B^{1/3}_{3,C(N)}$. Consider (Euler), instead of $\eta_\varepsilon * v^\ell$, we consider

- Estimates(Commutator)
- Keeping divergence-free property

Define the opetator $-\Delta_H = d_\delta + \delta_d$, which looks like a 1-form. In Hodge heat flow equation,

$$\partial_s v^\ell = \Delta_H v^\ell = \nabla_j \nabla^j v^\ell - \mathrm{Ric}^k_\ell\, v^k$$

Since we know what the solution exactly is,

$$\eta_\varepsilon * v^\ell \to e^{s\Delta_H} v^\ell$$

The square root of heat time $s^{1/2} \sim \varepsilon$ and solution at time s $S_{[s]}v = e^{s\Delta_h}v$.
To estimate

$$\eta_\varepsilon * \nabla_j(v^j v^\ell) - \nabla_j(\eta_\varepsilon v^j \eta_\varepsilon * v^\ell),$$

we would need the commutator

$$w^\ell(s) = S_{[s]}\nabla_j(v^j v^\ell) - \nabla_j(S_{[s]}v^j S_{[s]}v^\ell)$$

and Riemannian manifold M will be always assumed to be smooth.

$$s \in (0,1] \qquad \int_{I\times M} \eta(t)[S_{[s]}\nabla_j(v^j v^\ell) - \nabla_j \left(S_{[s]}v^j S_{[s]}v^\ell\right) S_{[s]}v_\ell d^{1+d}\mathrm{vol},$$

here volume is in time$\times$ space.
Let's calculate

$$(\partial_s - \Delta_H)w^\ell = N^\ell(t,s)$$
$$w^\ell(s) = \int_0^s e^{(s-s')\Delta_H} N^\ell(t,s')ds',$$

by d'Alembert's formula.

$$\begin{aligned} w^\ell(t,s) &= S_{[s]}\nabla_j(v^j v^\ell) - \nabla(S_{[s]}v^j S_{[s]}v^\ell) \\ (\partial_s - \Delta_H)w^\ell &= (\partial_s - \Delta_H)\nabla_j(S_{[s]}v^j S_{[s]}v^\ell) \\ &= (\partial_s - \nabla_i\nabla^i)\nabla_j(S_{[s]}v^j S_{[s]}v^\ell) + \text{curvature terms} \\ &= -2\nabla_j(\nabla_i S_{[s]}v^j \nabla^i S_{[s]}v^\ell) + \text{low order terms} \end{aligned}$$

$$\begin{aligned} \text{Commutator} &= -2\int_{I\times M}\eta(t)\int_0^s e^{(s-s')\Delta}\nabla_j(\nabla_i S_{[s]}v^j\nabla^i S_{[s]}v^\ell)ds' S_{[s]}v_\ell d^{1+d}\mathrm{vol} \\ \text{Integral by parts} &= 2\int_{I\times M}\eta(t)\int_0^s \nabla_i S_{[s]}v^j\nabla^i S_{[s]}\cdot \underbrace{S_{[2s-s']}}_{\text{very low frequency}} v_\ell ds' d^{1+d}\mathrm{vol} \end{aligned}$$

Assume that $v\in L_t^3 B^\alpha_{3,\infty}$, claim that $\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{L_t^3 B^\alpha_{3,\infty}}$ with $\alpha>1/3$. First we can try $v\in L_t^3 C^\alpha$ or $v\in L_t^3 W^{\alpha,3}$.

$$\begin{aligned} |Commutator| &\lesssim \|\eta\|_{L^\infty}\int_0^s(2s-s')^{\frac{-1+\alpha}{2}}(s')^{-2\frac{1+\alpha}{2}}ds' \\ &\lesssim s^{-1/2+3\alpha/2}\int_0^1(2-\sigma)^{\frac{-1+\alpha}{2}}\sigma^{-1+\alpha}d\sigma\to 0 \text{ as } \varepsilon\to 0 \text{ with } \alpha<1/3 \end{aligned}$$

No derivatives that we can take over the heat flow. In order to prove the claim:$\left\|\nabla S_{[s]}v\right\|_{L^3}\lesssim s^{\frac{-1+\alpha}{2}}\|v\|_{W^{\alpha,3}}$ with $\alpha>1/3$.

Proof.

$$\left\|\nabla S_{[s]}v\right\|_{L^3}\leqslant s^{-\frac{1}{2}}\|v\|_{L^3}$$
$$\left\|\nabla S_{[s]}v\right\|_{L^3}\leqslant \|v\|_{W^{1,3}}$$

Since $u(s)=e^{s\Delta}u$, $\|u\|_{L^\infty}\lesssim\|u\|_{L^p}\quad s\in(0,1]$ On compact manifold we have that

$$\|\nabla u\|_{L^p}\lesssim\|\nabla u\|_{L^r}+\|u\|_{L^p}$$

Here let $p=2z$ where $z\geqslant 2$ is an integer. □

$$\partial_s|u|^2-\Delta|u|+|\nabla u|^2=0(\text{or }-2\,\mathrm{Ric}_{jk}\,u^ju^k) \tag{4}$$
$$\Rightarrow\int_M|u|^2d\mathrm{vol}\searrow\text{ and }\int_M u^{2z}d\mathrm{vol}\searrow$$
$$\begin{aligned} \tfrac{1}{z}\partial_s\textstyle\int|u|^{2z}d\mathrm{vol} &= \textstyle\int_M(\Delta|u|^2-2|\nabla u|^2)|u|^{2(z-1)}d\mathrm{vol} \\ &= -\textstyle\int_M\nabla^j|u|^2\nabla_j|u||u|^{2(z-2)}-2|\nabla u|^2|u|^{2(z-1)}\leqslant 0 \end{aligned}$$

For curvature terms, they can be bounded by $\|\mathrm{Ric}\|_{L^\infty}\int_M|u|^{2z}d\mathrm{vol}$ remains bounded. So $\int_M u^{2z}d\mathrm{vol}\leqslant\int_M u^{2z}(t=0)d\mathrm{vol}$.

$$\partial_s|\nabla u|^2-\Delta|\nabla u|^2+|\nabla\nabla u|^2(\mathrm{BAD})=0(\text{ or }\mathrm{Riem}(\nabla u\nabla u)+\nabla\mathrm{Riem}u\nabla u) \tag{5}$$

Multiply by $|\nabla u|^{2(z-1)}$,

$$\frac{1}{z}\partial_j\int_M|\nabla u|^{2z}d\mathrm{vol}\quad\searrow$$

or

$$\begin{aligned} &\textstyle\int_M|\nabla u|^{2(z-1)}\nabla_{\mathrm{Riem}}u\nabla ud\mathrm{vol}\quad\text{by integration by parts} \\ &\leqslant 2\underbrace{\int_M|\nabla\nabla u|^2|\nabla u|^{2(z-1)}}_{BAD}+\underbrace{Cz\int_M\|R\|_{L^\infty}\|\nabla u\|^{2z}+\|\mathrm{Riem}\|^{z+1}\|u\|^{2z}}_{GOOD} \end{aligned}$$

which can cancel bad terms from (5).

$$\partial_s(s|\nabla u|^2)-\Delta(s|\nabla u|^2)+2s|\nabla\nabla u|^2-|\nabla u|^2(\mathrm{BAD})$$

But here the bad term can cancel with (4). Let $\Phi(s)=s|\nabla u|^2+\frac{1}{2}|u|^2$, then we have

$$\partial_z\Phi_s-\Delta\Phi_s=2s|\nabla\nabla u|^2$$

Thus

$$\frac{1}{z}\partial_s\int_M|\Phi_s|^z(\searrow)\leqslant\frac{1}{z}\int_M|\Phi_s|^zd\mathrm{vol}\leqslant\frac{1}{z}\int_M|\Phi_0|^z=\frac{1}{z}\int_M|u|^{2z}$$
$$\frac{1}{z}(\int_M|\nabla u|^{2z})^{1/2z}\leqslant s^{-1/2}(\int_M|u|^{2z})^{1/2z}$$

Hodge Laplacian commute with derivative$d$ and divergenceδ. $(\partial_s-\Delta_H)\delta w_\ell=\delta(\partial_s-\Delta_H)w_\ell=0$ with 0 initial condition.

3.1 Isentropic Compressible Euler

$$\begin{array}{ll} \text{Mass} & \partial_t \rho + \nabla_j(\rho v^\ell) = 0 \\ \text{Momentum} & \partial_t(\rho v^\ell) + \nabla_j(\rho v^j v^\ell) + \nabla^l(P\rho) = 0 \end{array} \tag{6}$$

We made some assumption $p(\rho) = \rho^2\gamma$ and $\rho \in C^2$ and away from $\rho \equiv 0$. Here exists a problem: modifier doesn't commute with nonlinearity.

$$\partial_t(\frac{1}{2}\rho|v| + p(\rho)) + \nabla_j((\frac{1}{2}\rho|v| + p(\rho) + p(\rho)v^j)p(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}dr$$

If conservation holds in $B^\alpha_{3,\infty}$ in both (t,x), we need to estimate the commutator:$= \eta_\varepsilon * (p(\rho)) - p\eta * \rho)$.

Remark.
- Heat flow also works.
- [5] gives another method

$$\begin{aligned} \text{Commutator} &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p(\int \rho(x-h)\eta_\varepsilon(h)dh \\ &= \int p(\rho(x-h))\eta_\varepsilon(h)dh - p(\int(\rho(x-h)\eta_\varepsilon(h))dh = \int(1-\sigma)\int p"((1-\eta)\rho_\varepsilon(x) + \sigma\rho(x-h))(\rho(x-h) - \rho_\varepsilon(x \end{aligned}$$

Therefore bounded.

$$\overline{f(X)} - f(\overline{X}) = \mathbb{E}[f(X)] - f(\mathbb{E}[X]) \tag{7}$$

Since

$$\begin{aligned} \phi(1) &= \phi(0) + \textstyle\int_0^1 \frac{d}{d\sigma}\phi(\sigma)d\sigma \\ &= \phi(0) + \frac{d}{d\sigma}|_{\sigma=0}\phi(\sigma) + \textstyle\int_0^1(1-\sigma)\frac{d^2}{d\sigma^2}\phi(\sigma)d\sigma \end{aligned}$$

RHS of (7) becomes

$$\begin{aligned} &\textstyle\int +0^1 \frac{d}{d\sigma}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X))] \\ &= \frac{d}{d\sigma}|_{\sigma=0}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma(X)] + \textstyle\int_0^1(1-\sigma)\frac{d^2}{d\sigma^2}\mathbb{E}[f(1-\sigma)\overline{X} + \sigma X)d\sigma \\ &= \mathbb{E}[\nabla_i f(\overline{X})(X^i - \overline{X}^i)] + \textstyle\int_0^1(1-\sigma)\mathbb{E}[\nabla_a\nabla_b f((1-\sigma)\overline{X} + \sigma X)(X^a - \overline{X}^b)(X^b - \overline{X}^a)d\sigma \end{aligned}$$

Here last term is a quadratic form.

4 Convex Integration

Here "convex" refers to convex linear combination.

Theorem 4.1. *(Old conclution)* $\forall E(t) \in C^\infty$, $E(t) \geq C$, $\forall\alpha < \frac{1}{10}$, $\exists v \in C^\alpha_{t,x}(I \times \mathbb{T}^3)$ *s.t.* $\int \frac{|v|^2}{2}(t,x)dx = E(t)$ *conserved.*

Theorem 4.2. *(Isset [6])*$\forall\alpha < 1/5$, $\exists v \in C^\alpha_{t,x}(I \times \mathbb{T}^3$, $p \in C^{2,\alpha}_{t,x}$. *A non trivial solution with compact support in time. (0 is not the only solution stays 0 implies non-uniqueness).*

- Question: How to construct continuous solution?
- Idea: Euler-Reynolds flows

For R a symmetric tensor (v,p,R) that solves

$$\begin{aligned} \partial_t v^l + \nabla_j(v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell} \\ \nabla_j v^j = 0 \end{aligned}$$

Here $R^{j\ell} = R^{lj}$. If $R = 0$, we have the Euler equation(Euler). If $R \neq 0$, we have a smooth approximation of Euler Equation.
Construction (v_q, p_q, R_q) the Euler-Reynolds flow:

$$R_q \to 0 \text{ as } q \to \infty$$

and

$$(v_q, p_q)$$

converge uniformly.

$$-\int \partial_t \phi_\ell v_q^l + \nabla_j \phi_\ell + v_q^j v_q^\ell + \nabla^\ell \phi \cdot p_q = \nabla_j \phi R_q^{j\ell} \to 0$$

The idea would make sense if every continuous solution (v, p) is a uniform limit of $(v_\varepsilon, p_\varepsilon, R_\varepsilon)$ as $\varepsilon \to 0$. I.e. every Euler flow is a limit of Euler-Reynolds flow. Proof will also be given by modification. To check weather it is a E-R flow, we need to check weather it conserves energy.

Lemma 4.3. *(Main Lemma) Given (v, p, R) $\exists$ new*

$$(\overset{*}{\mathrm{v}}, \overset{*}{\mathrm{p}}, \overset{*}{\mathrm{R}})$$

with

$$\|\overset{*}{\mathrm{R}}\|_0 << \|R\|_0,$$

where

$$\overset{*}{\mathrm{v}} = v + V \qquad \overset{*}{\mathrm{p}} = p + P. \tag{8}$$

We expect that $\|V\|_0 \leqslant \|R\|_0^{1/2} \leftarrow [\frac{m}{s}]$ and $\|P\|_0 \leq \|R\|_0 \leftarrow [\frac{m}{s}]$ by dimension analysis.

Apply 4.3 over and over again generate (v_q, p_q, R_q) with $\|R_q\|_0 \to 0$ rapidly. Set $v = v(0) + \sum_q v_q$ and $p = p(0) + \sum_q P_q$. Plug in new 8

$$\partial_t \overset{*}{\mathrm{v}}^\ell + \nabla_j (\overset{*}{\mathrm{v}}^i \overset{*}{\mathrm{v}}^l) + \nabla^\ell \overset{*}{\mathrm{p}} = \nabla_j R^{j\ell} + \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla_j (v^j v^\ell) + \nabla^\ell p + \nabla_j (v^j v^\ell)$$

i.e.

$$\mathrm{RHS} = \nabla v^\ell + \nabla_j (v^j V^l \ell) + \nabla_j (V^j V^l + P\delta^{j\ell} + R^{jl}) + \nabla_j (v^l V^j)$$

where R^{jl} is the old error. We want $\nabla_j \mathrm{R}^{*j\ell}$ =RHS with R^* small. [1]
The rest is not small. Let them equals to $\nabla_j Q^{jl}$ with $Q^{jl} = Q^{lj}$. Either $\nabla_j (v^{jl} + p\delta^{jl} + R^{jl}$ or $\nabla_j (v^j V^l)$ or $\nabla_j (v^l V^j)$ is not small.

4.1 Non-stationary Phase Lemma

When you can find a small solution $Q^{j\ell} = Q^{\ell j}$ to the symmetric divergence equation

$$\nabla_j Q^{j\ell} = u^l$$

Necessary for u^ℓ to be high-frequency $n \geq 1$ and $\int_{\mathbb{T}^3} u^\ell dx = 0$.
What does v^ℓ look like? [2] Beltrami flows.

$$v^\ell = \sum_{|k|=R} a_k B_k^\ell e^{i\lambda k \cdot x}, \quad k \in \mathbb{R}^d$$

$\nabla \times v = R\lambda v$ if $(ik) \times B_k = |k| B_k$ and $a_k B_k$ are constant.

$$v^l = \sum_{|k|=R} a_k(R, E, v, \lambda t) B_l^\ell e^{i\lambda kx}$$

Construction of frequency:

$$v^\ell = \sum_I v_I^l, \quad v_I^\ell = \bar{v}_I^l, \quad \nabla_j v_I^j = 0$$

$$v_I^l = \nabla_j w_I^{jl}, \quad w_I^{jl} = -w_I^{lj} \Leftrightarrow$$

[1] Here the position of $*$ is decided by whether it is a equation or just math script. The different position of $*$ doesn't have different meaning

$w^j l$ is a curl. For high frequency wave

$$v_I^\ell + \delta v_I^\ell \qquad \text{with } \lambda >> 1$$

$$v_I \cdot \nabla \xi_I = 0 \qquad \text{for divergence free}$$

Here ξ_I is always tangent to level phase function.

$$w_I^{jl} = \frac{e^{i\lambda\xi_I}}{i\lambda} w_I^{j\ell} = \frac{e^{i\lambda\xi}}{i\lambda}(\nabla^j \xi_I v_I^\ell - v_I^j \nabla^l \xi_I)$$

then it is anti-symmetric. Thus

$$\nabla_j w_I^{j\ell} = e^{i\lambda\xi_I}(\nabla_j \xi_I w_I^{il} + \frac{\nabla_j w^{j\ell}}{i\lambda}) = e^{i\lambda\xi_I}(v_I^\ell + \delta v_I^l),$$

where $\delta_{V_I} = \mathcal{O}(\frac{1}{\lambda})$, thus small. The transport term:

$$\nabla_j R_T^{jl} = \partial_t v^\ell + \nabla_J(v^j V^\ell) + \nabla_j(V^j v^l)$$

want to solve

$$\nabla_j Q^{jl} = u^l = e^{i\lambda\xi_I} v_I^\ell$$

with small $Q^{j\ell} = Q^{\ell j}$ on $\mathbb{T}^3$. The equations above are under-determined. So the solution is not unique and we are try to find a small one.

Lemma 4.4. *Non-stationary Phase Lemma If $\left\| |\nabla \xi_I|^{-1} \right\|_0 \leqslant A$ (no zero frequency) and $\int_{\mathbb{T}^3} u^\ell(x)dx = 0$. We note that if $|\nabla\xi_I| \to 0$, then we just rotating a constant but there is no waves. Then $\exists Q^{j\ell} = Q^{\ell j}$ of size $\|Q\| \leqslant \mathcal{O}(\lambda^{-1})\|u\|_0 + \mathbb{O}_{(u,\nabla q)}(\lambda^{-2})$. Since $\int \partial_t V^l dx = 0 \Rightarrow \frac{d}{dt}\int v^\ell dx = 0$ conserve the momentum, consistent with solution of Euler.*

Proof. d=1.

$$\frac{dQ}{dx} = e^{i\lambda\xi(x)}u(x) \qquad \text{on } \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

$\Rightarrow Q(x) = \int_0^x e^{i\lambda\xi(x)}u(s)ds$ is periodic because $\int_{\mathbb{T}} u(x)dx = 0$. By integration by parts,

$$= \frac{1}{i\lambda\xi(s)} e^{i\lambda\xi(s)}u(s)|_{s=0}^x - \frac{1}{i\lambda}\int_0^x e^{i\lambda\xi(s)}\frac{d}{ds}(\frac{u(s)}{\xi(s)}ds = \mathcal{O}(\frac{1}{\lambda} + \frac{1}{\lambda}\mathcal{O}(\lambda^{-2})$$

Let's look at the transport term

$$\begin{aligned}\nabla R_T^{j\ell} &= \partial_t v^l + \nabla_j(v^j v^\ell) + \nabla_j(v^j v^\ell) = \partial_t v^l + v^j\nabla_j v^l + v^j \nabla_j v^\ell \\ &= (i\lambda)\sum_I e^{i\lambda\xi_I}(\partial_t \xi_I)v_q^\ell + (v^j\nabla_j)\xi_I v^\ell + \sum_I e^{i\lambda\xi_q}\end{aligned}$$

□

Apply the Non-stationary Phase Lemma 4.4

$$\|R_T\|_0 \lesssim \lambda^{-1}\|RHS\|_0 \lesssim \lambda^{-1}(\lambda\left\|(\partial_t + v^j\nabla_j)\xi_I\right\|_0 \|V_I\|_0) + \mathcal{O}(\lambda^{-2})$$

Impose the phase function:

$$\begin{aligned}(\partial_t + V\cdot\nabla)\xi_I &= 0 \\ D_t\xi_I &= 0\end{aligned} \tag{9}$$

Taylor hypothesis: High frequency flows are actually carried by low frequency flows. So it make sense with our assumption. If $\left\| |\nabla \xi_I|^{-1} \right\|_0 \leqslant A$ fails to control the amplitude, we include time cut-offs in v_I. Look at $\nabla_j(v^j v^l + p\delta^{jl} + R^{jl})$, want it to be div of something small. Want $R_{small}^{j\ell} = \sum_I v_I^j \bar{v}_I^\ell + P_0\delta + R^{j\ell}$

$$\sum_I v_I^j \bar{v}_I^\ell + P_0\delta^{j\ell} + \sum_{j\neq\bar{I}} v_I^j v_j^\ell + P_{Ij}\delta^{j\ell} = \sum_I v_I^j\bar{v}_I^\ell + P_0\delta^{j\ell} + R^{j\ell} + \sum_I \delta v_I^j v^\ell \left(= 0 + \mathcal{O}(\frac{1}{\lambda})\right)$$

Let's pick a t to make this $= 0$ pointwisely.

$$\sum_I v_I^j \bar{v}_j^\ell = -P_0 \delta^{i\ell} - R^{j\ell} = e(t)\delta^{j\ell} - E^{j\ell}$$

Choose $e(t) \geq 10^4 \|R\|_0$ on $\operatorname{supp} R$, here $e(t)$ is the lifting function.

$$\Rightarrow \|V_I\|_0 \lesssim e^{1/2} \lesssim \|R\|_0^{1/2}$$

We need to choose time depending on amplitude. Note that $\operatorname{supp}_t e \subset B(\operatorname{supp} R, \varepsilon)$,

$$\Rightarrow \operatorname{supp}_t(\text{New Error}) \subset \operatorname{supp}_t e$$

For (8), we rewrite our (Euler) as:

$$\begin{aligned} \partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p &= \nabla_j R^j l \\ \nabla_j v^j &= 0 \end{aligned} \tag{10}$$

where we decompose $v^\ell = \sum\limits_I v_I^l$ and $\nabla_j V_I^j = 0$ by $v^l = \nabla_j w_I^{jl}$ since w_I^{jl} is anti-symmetric tensor. $V_I^l = e^{i\lambda\xi_I}(v_I^\ell + \delta v_I^\ell)$, where $\delta v_I^\ell \sim \mathbb{O}(\lambda^{-1})$. As for phase $\xi_I \in \mathbb{C}$, we have $\left\| |\nabla\xi_I|^{-1} \right\|_0 \leq A$, $V_I \cdot \nabla\xi_I = 0$. Let's impose $(\partial_t + v\cdot\nabla)\xi_I = 0 \Rightarrow$ We need time cut offs inside v_I to maintain the non-stationary phase. Let $\|R\|_0 \leqslant e_R$

$$\begin{array}{ccccc} \mathrm{R}^* = & R_T^{j\ell} + & R_L^{j\ell} + & R_S^{j\ell} + & R_H^{j\ell} \\ & \text{Transport} & \text{Low Frequency} & \text{Stress} & \text{High Frequency} \end{array}$$

$$\begin{aligned} \nabla_j R_T^{j\ell} &= \partial_t v^l + \nabla_j(v_j V^\ell) \\ \nabla_j R_L^{j\ell} &= \nabla_j(v^j V^L) \\ \nabla_j R_S^{j\ell} &= \nabla_j\left(\textstyle\sum_I v_I^j \bar{v}_I^\ell + p\delta^{j\ell} + R^{j\ell}\right) \\ \nabla_j R_H^{j\ell} &= \sum_{j\neq \bar{I}} \nabla_j \end{aligned}$$

$$R_S^{j\ell} = \sum_I V_I^j \bar{v}_I^\ell + \rho\delta^{j\ell} + R^{j\ell} + \mathbb{O}(\delta v_I)$$

(Phase cancel thus high frequency disappear)

$$\sum_I V_I^j V_I^{-\ell} = -P_0\delta^{j\ell} - R^{j\ell}$$

Let's set $p_0 = -e(t)$, $e(t) \geqslant Ke_R(\text{error})$. $\sum\limits_I v_I^j v_I^{-\ell} = e(t)\delta^{i\ell} - R^{j\ell}$. $e(t) \geqslant Ke_\ell$ on $\operatorname{supp} R$.

What about the high frequency?

$$\begin{aligned} \nabla_j R_H^{j\ell} &= \sum_{J\neq\bar{I}} \nabla_j(V_I^j \bar{V}_I^l) + \nabla^l P_{IJ} \\ &= \tfrac{1}{2}\sum_{J\neq I} V_I^j \nabla_j V_J^\ell + V_J^j \nabla_j V_I^\ell + \nabla^\ell P_{IJ} \\ &= (i\lambda)\sum_{J\neq J} e^{i\lambda(\xi_I+\xi_J)} \nabla_J \xi_J V_J^\ell + \text{L.O.T} \\ \|R_H\|_0 &\lesssim \lambda^{-1}\|\text{RHS}\|_0 + \mathbb{O}(\lambda^{-2}\|\text{RHS}\|) \\ &= \lambda^{-1}\lambda\|V_I\|_0 \|\nabla\xi_I\| \|V_{\bar{I}}\|_0 \lesssim e_R^{1/2} 1 e_R^{1/2} \end{aligned} \tag{11}$$

Better way "write V_I as a steady state solution".

Idea: If $\nabla\cdot V_I = \lambda V_I$ and $\nabla\cdot V_J = \lambda V_J$ then $V_I + V_J$ is steady state for E-R flow with appropriate pressure P_{Ij} s.t.

$$V_I^j \nabla_j V_J^i + V_J^j \nabla_j V_I^i + \nabla^i P_{IJ} = 0$$

To make $\nabla\times V_I \sim \lambda V_I$

$$\nabla\times(e^{1\lambda\xi_I} V_I) = \lambda e^{i\lambda\xi_I}((i\nabla\xi_I)\times V_I)$$

$$(i\nabla\xi_I)\times V_I = |\nabla\xi_I| V_I \sim \lambda e^{i\lambda\xi_I}|\nabla\xi_I|\times V_I$$

Here $|\nabla\xi_I|$ is eigenvalue. If $\||\nabla\xi_I|-1\|_0 << 1$, (by sharp time cut-off)

$$\begin{aligned}&(i\nabla\xi_I)\times V_I = -(\nabla\xi)\times b_I + i(\nabla\xi_I)\times a_I\\ &|\nabla\xi_I|\cdot V_I = |\nabla\xi_I|a_I + i|\nabla\xi_I|b_I\end{aligned}$$

choose $b_I \in <\nabla\xi_I>^+$ and set $a_I = \frac{-(\nabla\xi_I)\times b_I}{|\nabla\xi_I|}$, thus

$$(\nabla\xi_I)\times a_I = -\nabla\xi_I \times \frac{(\nabla\xi_I\times b_I)}{|\nabla\xi_I|} = \frac{-(-|\nabla\xi|^2 b_I}{|\nabla\xi_I|} = |\nabla\xi_2|b_I$$

$$e^{i\lambda\xi_I}V_I^l = e^{i\lambda\xi_I}(a_J^l + ib^l + I) = \cos\lambda\xi_I a_I^\ell - \sin\lambda\xi_I b_I^\ell$$

$$a_I = \frac{-\nabla\xi_I\times b_I}{|\xi_I|}$$

By (11)

$$\frac{1}{2}\sum_{J+I}(V_I)_j(\nabla^j V_J^j) + (V_J)_j(\nabla^j V_I^\ell - \nabla^\ell v_I^j) + \nabla^\ell\frac{V_iV_J}{2} + \nabla^\ell P_{IJ},$$

$\nabla^\ell\frac{V_iV_J}{2} + \nabla^\ell P_{IJ} = 0$ by our choice of P_{IJ}.

$$P_{IJ} = -\frac{1}{2}V_IV_J = -\frac{1}{2}\sum_{J\neq\bar{I}}(V_I\times(\nabla\times V_J) + V_J\times(\nabla\times V_I)) = 0$$

If $V_j,\ V_I$ are eigen-function of $\nabla\times$.

$$\begin{aligned}P_{IJ} = -\frac{1}{2}V_IV_J &= \lambda e^{\lambda(\xi_I\xi_J)}(V_I\times((i\nabla\xi_J)\times V_j) + V_J\times(i\nabla\xi_I)V_I)) + \text{L.O.T}\\ &= \lambda e^{\lambda(\xi_I\xi_J)}(V_I\times(|\nabla\xi_J|-1)V_J) - V_J\times(|\nabla\xi_I|-1)V_I) + \text{L.O.T}\end{aligned}$$

for $V_I\times V_j + V_J\times V_I = 0$. Since non-stationary phase $\left\||\nabla\xi_I+\nabla\xi_J|^{-3}\right\| \leqslant A$.

$$\begin{aligned}&\|R_H\|_0 \lesssim \lambda\cdot\lambda\max_I\|V_I\|_0^2\cdot\max_I\||\nabla\xi_I|-1\|_0\\ &\lesssim e_R\max_I\||\nabla\xi_I|-1\|_0 + \text{L.O.T}\end{aligned}$$

Can be made small with a sharp tine cut off in each V_I: $\tau\sim b|\nabla V|^{-1}$. Then we need to pay the price for time cut-off.

$$\begin{aligned}\nabla_j R_T^{j\ell} &= \partial_t v^\ell + \nabla_j(v^jV^\ell) = \partial_t v^\ell + v^j\nabla_j V^\ell\\ &= \sum_I \exp(i\lambda\xi_I)(\partial_t + v^j\nabla_j)V_I^\ell\\ \|R_T\|_0 &\lesssim \lambda^{-1}\text{RHS} + \mathcal{O}(\lambda^{-2}) \lesssim \lambda^{-1}\tau^{-1}\end{aligned}$$

Choose $\lambda >> \tau^{-1} \Rightarrow \|R_\tau\|_0$ is arbitrarily small.

Remark.
- Convex linear combination problem $\sum v^j\bar{v}_I^\ell = e(t)\delta^{jl} - R^{jl}$
- Integration means using $\nabla R^{j\ell}$ to approximate $R^j\ell$.

Lemma 4.5. *(Main Lemma) $\exists\, K$, $\exists\, C>1$ s.t. $\forall\varepsilon>0\ \forall(v,p,R)$ uniformly C^3 solution to Euler-Reynolds with $\operatorname{supp} R\subset I\times\mathbb{T}^3$, and $\|R\|_0\leqslant e_R$. Let $e(t):\mathbb{R}\to\mathbb{R}^*$, s.t.*

$$\frac{d}{dt}e^{\frac{1}{2}}(t)\in C^\infty$$

and $e(t)\geqslant Ke_R$ on a neighbourhood of $\operatorname{supp} R$. $\exists(\mathrm{v}^*,\mathrm{p}^*,\mathrm{R}^*)$, $\mathrm{v}^* = v+V$, $\mathrm{p}^* = p+P$. $\operatorname{supp}(V,P,R)\subset\operatorname{supp} I\times\mathbb{T}^3$, with $\|V\|_0\lesssim e_R^{1/2}$, $\|P\|_0\lesssim e_R$ and $\|\mathrm{R}^*\|_0<\varepsilon\ \forall\varepsilon>0$ by choosing arbitrarily big λ. $\left\|\int_{\mathbb{T}^3}|\mathrm{v}^*|^2 - \int_{\mathbb{T}^3}|v|^2 + e(t)dx\right\|_0\leqslant\varepsilon$ (i.e. nice low frequency).

Proof.

$$\int_{\mathbb{T}^3} \left|\overset{*}{\mathrm{v}}\right|^2 - \left|\overset{*}{\mathrm{v}}\right|^2 = \int_{\mathbb{T}^3} |v+V|^2 - |v|^2 dx = \int_{\mathbb{T}^3} 2vV + V^2 dx$$

The cross term is the correction of high frequency and is small because v, V are almost orthogonal. Claim: $\int_{\mathbb{T}^3} |V|^2 2dx + \mathcal{O}(\lambda^{-1})$. Let $V^\ell = \nabla_j W^{j\ell}$, where $W^{j\ell} - \frac{1}{\lambda} e^{i\lambda\xi_I}(w_I^{j\ell})$.

$$\begin{aligned}
&\int vV = -\int \nabla_j v_\ell W_I^{i\ell} dx = \mathcal{O}(\lambda^{-1}) \\
&\int |V|^2 dx = \sum_I \int V_I \bar{V}_I + \sum_{J\neq \bar{I}} \int V_I \cdot V_J
\end{aligned}$$

$\sum_{J\neq\bar{I}} \int V_I \cdot V_J$ is small because for $\sum_{J\neq\bar{I}} \int e^{i\lambda_I + \lambda_J)} V_I V_J dx$, since $\left\| |\nabla\xi_I + \nabla\xi_J|^{-1} \right\|_0 \leqslant A$. If $|\nabla\xi|^{-1} \leqslant A$ failed.

$$\begin{aligned}
e^{i\lambda\xi} &= \frac{\nabla^a \xi_I \nabla_a \xi_J}{(i\lambda)|\nabla\xi|^2} \qquad \xi = \xi_I + \xi_J \\
&= \sum_{J\neq\bar{I}} \int e^{i\lambda_I + \lambda_J)} \frac{\nabla^a\xi}{|\nabla\xi|^2} V_I V_J dx \\
&= \mathcal{O}(\lambda^{-1}) \qquad \text{by IBP}
\end{aligned}$$

$$\begin{aligned}
&\sum_I \int V^I \bar{V}_I, \qquad V_I = e^{i\lambda\xi_I}(V_I^l + \delta V_I^\ell) \\
&\int \delta_{jl} \sum_I V_I^j \bar{V}_I^\ell dx \quad \text{since } \sum_I V_I^j \bar{V}_I^\ell = -P_0 \lambda^{j\ell} - R^{j\ell} \\
&= \delta_{jl} \int (-P_0 \delta^{j\ell} - R^{j\ell}) \\
&= \delta_{jl} \int (e(t)\frac{\delta^{j\ell}}{d} - \mathring{R}^{j\ell}) dx \quad \text{by choose } p_0 = -\frac{e(t)}{d} - \frac{\delta_{ij}R^{j\ell}}{d}
\end{aligned}$$

□

Since $V_I^J \nabla_j V_J^\ell = (V_I)_j (\nabla^j V_j^I - \nabla_J^j) = -V_I \times (\nabla \times V_J)$. Define $(\nabla \times V_J)^c = \varepsilon^{cdf} \nabla_d V_{Jf}$. we notice that here:

$$\varepsilon^{abs} = -\varepsilon^{bac} = -\varepsilon^{cbs}$$

etc. And $\varepsilon^{123} = 1$,

$$\varepsilon^{\ell jc}\varepsilon^{cdf} = \delta_d^\ell \delta_f^j - \delta_f^\ell \delta_d^j$$

We find it anti-symmetric in ℓj and df. Proof:

$$\varepsilon^{12c}\varepsilon^{cdf} = \begin{cases} 0 & \text{if} df \notin \{(12) or (21)\} \\ 1 & \text{if} df = (12) \\ -1 & \text{if} df = (21) \end{cases} = \delta_d^\ell \delta_f^j - \delta_f^\ell \delta_d^j$$

$$\begin{aligned}
V_I \times \nabla \times V_J &= \varepsilon^{\ell jc} (V_I)_j (\nabla \times V_J)_c \\
&= \varepsilon^{\ell jc} (V_I)_j \varepsilon^{cdf} \nabla^d V_j^f \\
&= (\delta_d^\ell \delta_f^j - \delta_f^\ell \delta_d^j)(V_I)_j \nabla^d V_J^\ell \\
&= (V_I)_j \nabla^\ell V_J^j - (V_I)_j \nabla^j V_J^\ell \\
&= (V_I)_j (\nabla^\ell V_J^j 0 \nabla^j V_J^\ell)
\end{aligned}$$

Lemma 4.6. *(Main Lemma) $\exists k \geq 1$ $\exists C \geq 1$ s.t. $\forall \varepsilon > 0$, $\forall (v, p, R)$ uniform C^1 Euler-Reynolds flow with $\operatorname{supp} R \subset I \times \mathbb{T}^3$, $\|R\|_0 \leqslant e_R$. For any $e(t)$ $\mathbb{R}^\star \to \mathbb{R}^+$*

- *$e^{1/2}(t)$ is C_c^∞*
- *$e(t) \geqslant Ke_R$ on a neighbourhood of I.*

$\exists(\mathrm{v}^*, \mathrm{p}^*, \mathrm{R}^*)$, where $\mathrm{v}^* = v + V$, $\mathrm{p}^* = p + P$, $\|R\|_0 < \varepsilon$ $supp(v, p, R) \subset supp\, I \times \mathbb{T}^d$

Proof.

$$\left\| \int \left|\overset{*}{\mathrm{v}}\right| dx - \int (|v|^2 + e(t)) dx \right\|_0 \leq \varepsilon$$

$$\int |v|^2 dx = \int \sum_I V_I \bar{V}_I + \sum_{J \neq \bar{I}} V_J \cdot V_I$$

Since $V_I = e^{i\lambda \xi_I}(V_I + \delta V_I) = e^{i\lambda \xi_I}\tilde{V}_I$,

$$\sum_{J \neq \bar{I}} \int e^{i\lambda(\xi_I + \xi_J)} \tilde{V}_I \cdot \tilde{V}_J dx \quad \text{not exactly } \tilde{V}_I \cdot \tilde{V}_J)$$

$$\int \nabla_a e^{i\lambda(\xi_a + \xi_J)} \frac{\nabla^a \xi_I + \nabla^a \xi_J}{|\nabla^a \xi_I + \nabla^a \xi_J|^2} \tilde{V}_I \tilde{V}_J dx$$

They are oscillations in orthogonal direction, thus integral by parts gives cancellation. Set $(v, p, R) = (0, 0, 0)$ Let $e^{1/2}(t)$ be a function smooth and supported in a simple connected area, and apply lemma4.6 with $\varepsilon_{(1)} = e_R(1) = \frac{1}{10k}\left\|e^{1/2}\right\|_0^2$.

$$... \leqslant \varepsilon_{(2)} \leqslant \varepsilon_{(1)} \leqslant \varepsilon_{(0)}.$$

By choosing the lifting function to have larger L^∞ norm and smaller support. Then $\left\|R_{(k)}\right\| \leqslant e_{R,(k)} \leqslant \varepsilon_{(k)} \searrow 0$ rapidly. $\left\|v_{(k)}\right\|_0 \leqslant Ce_{R,(k)}^{1/2} \searrow 0$ rapidly. $\left\|P_{(k)}\right\| \leqslant Ce_{R,(k)} \searrow 0$ rapidly. Thus $\sum_k v_{(k)}^\ell$ converges to some solution v^ℓ and $\sum_k P_{(k)}$ converge to some P. And since $\left\|R_{(k)}\right\|_0 \searrow 0$ (v, p) is a weak solution to Euler

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = &\quad \nabla_j R^{j\ell} \\ \nabla_j v^j &\quad = 0 \end{aligned} \tag{12}$$

$$\int |v_k|^2 - \left|v_{(k-1)}\right|^2 (t, x) dx \geqslant \int e_(k)(t) dx - e(0)$$

Choose $e_{(k)} < \frac{1}{2}\int e_k(0) dx$

$$\int |v_k|^2 - \left|v_{(k-1)}\right|^2 (0, x) dx \geqslant \frac{1}{2}\int e_k(0) dx \geq 0,$$

by assumption. Thus the continuous solution we found was nontrivial and compactly supported. $\square$

We have a To-Do list: For non-stationary phase

$$\nabla_j Q^{j\ell} = u_j^\ell = e^{i\lambda\xi} u_j^\ell \quad \left\||\nabla \xi|^{-1}\right\|_0 \leq A, \quad \int u^\ell dx = 0$$

$$\exists Q^{j\ell} = Q^{\ell j}, \quad \|Q\|_0 \leqslant \lambda^{-1}\|u\|_0 + \mathcal{O}_{a,\nabla\xi}(\lambda^{-2})$$

1. How to solve $\sum\limits_I V_I^j \bar{V}_I^\ell = e(t)\delta^{j\ell}/d - \mathrm{R}^{oj\ell}$
2. How to ensure $\left\||\nabla \xi|^{-1}\right\|_0$, $\left\|(\nabla \xi_I \cdot \nabla \xi_J)^{-1}\right\|_0 \leqslant A$ with $J \neq \bar{I}$ that interact and $\||\nabla \xi_I| - 1\|_0 \lesssim b$ small.

Suppose $u^\ell = e^{i\lambda\xi} u^\ell$ $\left\||\nabla \xi|^{-1}\right\|_0 \leq A$ $\int u^\ell dx = 0$.

(Pair symmetric) From a solution of the form

$$\mathcal{Q}^{j\ell} = \frac{e^{i\lambda\xi}}{i\lambda} q_{(1)}^{j\ell} + \tilde{Q}_{(1)}^{j\ell}$$

Take ∇,

$$\nabla_j Q^{j\ell} = e^{j\lambda\xi} \nabla_j \xi q_{(1)}^{j\ell} + \frac{e^{i\lambda\xi} \nabla_j q_{(1)}^{j\ell}}{i\lambda} + \nabla_j \tilde{Q}_{(1)}^{j\ell}.$$

Choose $q_{(1)}^{j\ell}$ s.t.

$$\nabla_j \xi q_{(1)}^{j\ell} = u^\ell$$

Choose correction $\tilde{Q}^{j\ell}_{(1)}$ s.t.

$$\nabla_j \tilde{Q}^{j\ell}_{(1)} = e^{i\lambda\xi}(\frac{-\nabla q^{j\ell}_{(1)}}{i\lambda}) = \nabla_j(\frac{e^{i\lambda\xi}q^{j\ell}_{(1)}}{i\lambda}) - e^{i\lambda\xi}u^\ell - u^\ell, \text{with integral } 0$$

How to solve $\nabla_j \xi q^{j\ell}_{(1)} = u^\ell$. If we did not need q be symmetric, i.e. $q^{j\ell}_{(1)} = q^{\ell j}_{(1)}$ could use $q^{j\ell}_{(1)} = \frac{\nabla^j \xi u^\ell}{|\nabla\xi|^2}$. Decomposing $u^\ell = u^\ell_\perp + u^\ell_\parallel = u^\ell_\perp + \frac{(u\cdot\nabla\xi)\cdot\nabla^\ell\xi}{|\nabla\xi|^2}$.
Choose $q^{j\ell}_{(1)} = \frac{1}{|\nabla\xi|^2}(\nabla^j\xi u^\ell_\perp + \nabla^\ell \xi u^j_\perp) + \frac{(u\cdot\nabla\xi}{|\nabla\xi|^2}\delta^{j\ell}$. Check that $\nabla_j\xi q_{(1)} = u^\ell_\perp + 0 + u^\ell_\parallel = u^\ell$. This equation is under-determinate.

$$q^{j\ell}_{(1)} = q^{(jl)}_a(\nabla\xi)u^a$$

$$q^{j\ell}_a(\alpha p) = \alpha^{-1}q^{j\ell}(p) \quad \text{Homogeneous}$$

As for solving $\nabla_j \tilde{Q}^{j\ell}_{(1)} = e^{i\lambda\xi}\frac{\nabla_j q^{i\ell}_{(1)}}{i\lambda}$, $\left\|\tilde{Q}\right\|_0 \lesssim \frac{1}{\lambda}$. $\nabla_j \tilde{Q}^{j\ell}_{(1)} = e^{i\lambda\xi}u^\ell_{(2)}$. By induction, $\tilde{Q}^{j\ell} = \frac{e^{i\lambda\xi}}{i\lambda}q^{j\ell}_{(2)} + \tilde{Q}^{j\ell}_{(2)}$, $q^{j\ell}_{(2)} = q^{j\ell}_{(a)}(\nabla\xi)u^a_{(2)}$

$$\nabla_j \tilde{Q}^{j\ell}_{(1)} = -e^{i\lambda\xi}\frac{\nabla_j q^{j\ell}_{(2)}}{(i\lambda)}$$

gives $\mathcal{O}(\lambda^{-2})$ at the cost of one more derivative.
If we didn't require that $\tilde{Q}^{j\ell} = \tilde{Q}^{\ell j}$, $\nabla_j \tilde{Q}^{j\ell}_{(2)} = u^\ell_{(2)}$, $\tilde{Q}^{j\ell}_{(2)} = \mathbb{P}_{I_0}u_{(2)}$[2]. Instead we decompose $u^\ell_{(2)} = \mathcal{H}u^\ell + \nabla^\ell(\text{divergence free} + \nabla^\ell\nabla^{-1}\nabla_a u^a \text{gradient part, so } \nabla_\ell H u^\ell = 0$.
Set

$$\tilde{Q}^{j\ell}_{(2)} = \Delta^{-1}(\nabla^j \mathcal{H}u^\ell + \nabla^\ell\mathcal{H}u^j) + \Delta^{-1}\nabla_a u^a \delta^{j\ell} := R^{j\ell}_a[u^a_{(2)}]$$

$$\left\|\tilde{Q}_{(2)}\right\|_0 \lesssim \left\|u_{(2)}\right\|_0.$$

Since $\Delta^{-1}\nabla$ is an order -1 operator, it's bounded on C^0. Check $\nabla_j \tilde{Q}^{j\ell}_{(2)} = \mathbb{P}_{I_0}\mathcal{H}u^\ell_{(2)} + 0 + \nabla^\ell\Delta^{-1}\nabla_a u^a_{(2)} = \mathbb{P}_{I_0}u^\ell_{(2)} = u^\ell_{(2)}$.
To summarize $Q^{j\ell} = \sum_{k=1}^{2}\frac{e^{i\lambda\xi}}{i\lambda}q^{j\ell}_{(k)} + \tilde{Q}^{j\ell}_{(2)}$, $q^{j\ell}_{(k)} = q^{j\ell}_\alpha(\nabla\xi)u^a$, $q^{j\ell}_\alpha(\alpha p) = \alpha^{-1}q^{j\ell}_\alpha(p)$. $\nabla_j\xi q^{j\ell}_{(k)} = u^\ell_{(k)}$, $u^\ell_{(1)} = -u^\ell_{(1)}$, $u^\ell_{(k+1)} = \frac{-\nabla_j q^{j\ell}_{(k)}}{(i\lambda)}$

Proof. Error term

$$\begin{aligned}
&\nabla_j(V^jV^\ell + p\delta^{j\ell} + R^{j\ell})\\
&= \nabla_j(\sum_I V^j_I V^\ell_I + p\delta^{j\ell} + R^{j\ell})\\
&= \nabla_j(\sum_I \tilde{V}^j_I \tilde{V}^\ell_I + p\delta^{j\ell} + R^{j\ell})\\
&= \nabla_j(\sum_I V^j_I V^\ell_I + p\delta^{j\ell} + R^{j\ell} + \mathcal{O}(\delta V_I V_I) + ...)\\
&= \nabla_j(\underbrace{(\sum_I V^j_I V^\ell_I + p\delta^{j\ell} + R^{j\ell})}_{\text{small because low frequency concels.}} + (\underbrace{R^{j\ell} - R^{j\ell}_\varepsilon}_{\text{small with small } \varepsilon}) + \mathcal{O}(\delta V_I \bar{V}_I))
\end{aligned}$$

□

$$\sum W_I \bar{V}^\ell_I + p_0\delta^{j\ell} + R^{j\ell}_\varepsilon = 0$$

$$\partial_t V^\ell + \nabla_j(v^jv^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}$$

$$\partial_t \overset{*}{\mathrm{v}}{}^\ell + \nabla_j(\overset{*}{\mathrm{v}}{}^j\overset{*}{\mathrm{v}}{}^\ell) + \nabla^\ell \overset{*}{\mathrm{p}} = \nabla_j E^{j\ell} + \text{linear term(v)} + \nabla_j(vv^\ell) + \nabla^\ell p$$

[2] I_0 means integral 0.

Since $V_I^\ell = e^{i\lambda\xi_I}(V_I^\ell + \delta V_I^\ell)$,

$$\nabla_j(\sum W_I \bar{V}_I^\ell + p_0\delta^{j\ell} + R_\varepsilon^{j\ell}) = \sum_I V_I^j \bar{V}_I^\ell + p_0\delta^{j\ell} + R_\varepsilon^{j\ell} - (R^\ell - R_\varepsilon^{j\ell}) + \mathcal{O}(\delta V_I)$$

$$\sum_I V_I^j \bar{V}_{=}^\ell e(t)\frac{\delta^{j\ell}}{d} - \overset{o}{\mathrm{R}}{}_\varepsilon^{j\ell\,3} \quad \text{(Must be positive definite)} \tag{13}$$

Because $p_0 = \frac{e(t)}{d} - \frac{\operatorname{tr} R_\varepsilon}{d}$

$$\int \left|\overset{*}{\mathrm{v}}\right|^2 - |v|^2 dx = \int e(t)dx + \mathcal{O}(\lambda^{-1})$$

$e(t) \geqslant Ke_R$ on $\operatorname{supp} R_\varepsilon$. We make a choice of K later. Then we localize and renormalize it by choosing,

$$\eta_{K_0}(t) = \eta(\frac{t - K_0 t}{\tau}),$$

τ is the time cut off and $\sum_{K_0} \eta_{K_0}^2(t) = 1$. (Start with $\tilde{\eta}(t) \geq 1$ on $[-\frac{2}{3}, \frac{2}{3}]$ and $\tilde{\eta}(t) \geq 0$ and $\in C_c^\infty((-\frac{3}{4}, \frac{3}{4}))$. $\eta := \dfrac{\tilde{\eta}(t)}{(\sum_{K\in\mathbb{Z}} \tilde{\eta}(t-K_0)^2)^{1/2}}$. Thus $\eta^2 := \dfrac{\tilde{\eta}(t)^2}{(\sum_{K\in\mathbb{Z}} \tilde{\eta}(t-K_0)^2)} = 1$ gives us a partition of unity. Let's localize (13) to have

$$\text{RHS} = \sum_{K_0} \eta_{K_0}^2(t)(t(t)\frac{\delta^{j\ell}}{d} - R_\varepsilon^{j\ell})$$

Let's write $V_I^\ell = \eta_{K_0} \mathrm{V}^{o\,\ell}_{\;I}$ and $I \in \mathbb{I}(K_0)$ if V_I is positive at time $K_0\tau$. so

$$\text{LHS} = \sum_{K_0} \eta_{K_0}^2 \sum_{I\in\mathbb{I}(K_0)} \overset{o}{\mathrm{V}}{}^j \bar{\overset{o}{\mathrm{V}}}{}^\ell_I.$$

So it's sufficient to solve

$$\sum_{I\in\mathbb{I}(K_0)} \overset{o}{\mathrm{V}}{}^j \bar{\overset{o}{\mathrm{V}}}{}^\ell_I = e(t)\frac{\delta^{j\ell}}{d} - \overset{o}{\mathrm{R}}{}_\varepsilon^{j\ell}$$

on every time slice. η_{K_0} only lives for $t \sim \tau$.
Cut off the space.
I need $\left|\nabla\hat{\xi}_I\right| = 1$ at initial time.

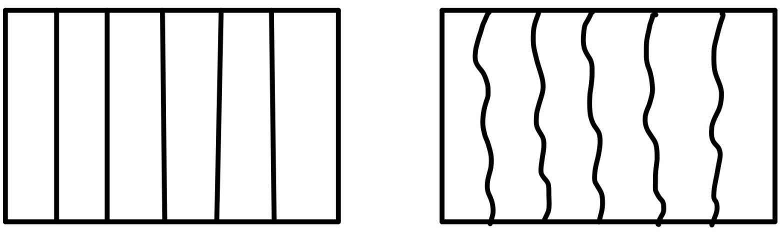


Figure 3: Initial and later phase function

At $t = 0$ Assume $\begin{cases} \nabla\hat{\xi}_I \in \{e_1, e_2, e_3\} \\ \hat{\xi}_I(t(I), x) = e_1 \cdot x \end{cases}$ is smooth enough. $e^{i\lambda\xi(t,x)}$ $\lambda \in 2\pi\mathbb{Z}$ and $e^{i\lambda(\xi_I+\xi_J)}$ with $\left\| |\nabla\xi_I + \nabla xi_j|^{-1} \right\|_{C^0} \leqslant A$. Let Ψ_I be spacial cut off. Write $v_I^\ell = \eta_{k_0}\mathring{v}_I^\ell = \eta_{k_0}\Psi_I\mathring{v}_I^\ell$, where $\eta_{k_0}\mathring{v}_I^\ell$ is the active part. Also $\sum_{I\in\mathcal{I}_(k_0)} = 1$.

[3] trace-free

The space cut off is depending on time $\sum\limits_{I\in\mathcal{I}(k_0)} \mathring{v}_I^j\bar{v}_I^\ell = e(\xi)\dfrac{\delta^{j\ell}}{d} - \mathring{R}_\varepsilon^{j\ell}$ on time slice.

$$\sum_{I\in\mathcal{I}(k_0)} \Psi_I^2 \mathring{v}_I^j \mathring{v}_I^\ell = \sum_{I\in\mathcal{I}(k_0)} \Psi_I^2 (e(t)\frac{\delta^{j\ell}}{d} - \mathring{R}_\varepsilon^{j\ell})$$

$$\Psi_I(t(I),x) = \Psi(k_0\tau, x)$$

$$\sum_{K\in(\mathbb{Z}/2\mathbb{Z})^3} \Psi(k_0 t, x) = 1$$

Partition of unit is also transported

$$(\partial_t + v\cdot\nabla)\Psi_I(t,x) = 0$$

$$\sum_{K\in(\mathbb{Z}/2\mathbb{Z})^3} \Psi^2 = 1$$

by uniqueness.

Apply the cut off $\Psi_I^2(\sum\limits_{I\in\mathbb{I}(K)} \mathring{v}^j\bar{\mathring{v}}_I^J) = \Psi_I^2(e(t)\frac{\delta^{j\ell}}{d} - \mathring{R}_\varepsilon^{j\ell})$

$$v_I^\ell = \eta_{t_0}(t)\Psi_I(t,x)\mathring{v}_I^\ell$$

where $I=(K,f)$ =(time,direction of oscillation), $K\in\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^3$ location index. Reason that we can use partition of unity: Homogeneous in amplitude.

$$v_I^\ell = \eta_{K_0}\Psi_I e^{1/2}(t)\mathring{v}_I^\ell$$

LHS $\sum\limits_I e(t)\mathring{v}^I\bar{\mathring{v}}_I^\ell = e(t)\frac{\delta^{j\ell}}{d} - \mathring{R}_\varepsilon^{j\ell}$. On $\operatorname{supp} R_\varepsilon$, $e(t) > Ke_R > 0$

$$\sum_I \mathring{v}_I^j\bar{\mathring{v}}_I^\ell = \frac{\delta^{j\ell}}{d} - \frac{\mathring{R}^{j\ell}}{e(t)} = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell}$$

where $\frac{\delta^{j\ell}}{d}$ is the dominated term.

$$\sum_I \mathring{v}_I^j\bar{\mathring{v}}_I^\ell = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell} \qquad \varepsilon = \mathcal{O}(\frac{1}{K})$$

$$\sum_I \mathring{v}_I^j\bar{\mathring{v}}_I^\ell = \frac{\delta^{j\ell}}{d} - \varepsilon^{j\ell} \tag{14}$$

What we know about v_I?

1. $v_I^\ell\cdot\xi_I = 0$
2. $(i\nabla\xi_I)\times V_I = |\nabla\xi_I|v_I$

$$\mathring{b}_I \in <\nabla\xi_I>^\perp$$

$$\mathring{a}_I = -\frac{\nabla\xi_I}{|\xi_I|}\times b_I \Leftarrow \text{rotation in plane } <\nabla\xi>^\perp$$

Thus (14)$=\sum\limits_I(\mathring{a}_I + i\mathring{b}_I^j)(\mathring{a}_I^\ell - i\mathring{b}_I^\ell) = 2\sum\limits_I(a_I^j\mathring{a}_I^\ell + \mathring{b}_I^j b_I^\ell) = \sum(\delta^{j\ell} - \dfrac{\nabla^j\xi_I\nabla^d\xi_I}{|\nabla\xi_I|^2}\cdot\left|\mathring{b}_I\right|^2$, is true because the imaginary part cancels. $(\frac{\mathring{a}_I}{|\mathring{b}_I|}, \frac{\mathring{b}_I}{\mathring{b}_I}, \frac{\nabla^l\xi_I}{|\nabla\xi_I|})$ are orthogonal frame. Thus the identity

$$\delta^{j\ell} = \frac{\mathring{a}_I^{j\ell}\mathring{a}_I^\ell}{\left|\mathring{b}_I\right|^2} + \frac{\mathring{b}_I^{j\ell}\mathring{b}_I^\ell}{\left|\mathring{b}_I\right|^2} + \frac{\nabla^j\xi_I\nabla^\ell\xi_I}{|\nabla\xi_I|^2} \tag{15}$$

by renormalization. Choose $\mathring{b}_I^\ell = \gamma_I \mathbb{P}_I^\perp(\nabla\xi_{\sigma_I}) = \gamma_I(\nabla^\ell \xi_I - \frac{(\nabla\xi_{\sigma_I}\nabla\xi_I)\nabla^\ell\xi_I}{|\nabla\xi_I|^2})$, since $\nabla\xi_{\sigma_I}$ is not parallel to $\nabla\xi_{\sigma_I}$.

$$\begin{aligned}\left|\mathring{b}_I\right|^2 &= \gamma_I^2\left(|\nabla\xi_{\sigma_I}|^2 - 2\frac{(\nabla\xi_{\sigma_I}\cdot\nabla\xi_I)^2}{|\nabla\xi_I|^2} + \frac{(\nabla\xi_{\sigma_I}\cdot\nabla\xi_I)^2}{|\nabla\xi_I|^2}\right)\\ &= \gamma_I^2\left(\frac{|\nabla\xi_{\sigma_I}|^2|\nabla\xi_I|^2 - (\nabla\xi_{\gamma_I}\nabla\xi_I)^2}{|\nabla\xi_I|^2}\right)\\ &= \gamma_I^2\left(\frac{|\nabla\xi_{\sigma_I}\wedge\nabla\xi_{\sigma_I}|^2}{|\nabla\xi_I|^2}\right)\end{aligned} \tag{16}$$

$$\sum_I \gamma_I^2 \frac{|\nabla\xi_{\sigma_I}\wedge\nabla\xi_I|^2}{|\nabla\xi_I|^2}\left(\delta^{j\ell} - \frac{\nabla^j\xi_I\nabla^\ell\xi_I}{|\nabla\xi_I|^2}\right) = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell}$$

where $(\nabla\xi_I^\perp)^{j\ell}$.

$$\sum \gamma_I^2\frac{|\nabla\xi_{\sigma_I}\wedge\nabla\xi_{\sigma_I}|^2}{|\nabla\xi_I|^2}(\nabla\xi_I^\perp)^{j\ell}\nabla_J\xi_J\nabla_\ell\xi_J = \frac{|\nabla\xi_J|^2}{d} + \varepsilon^{j\ell}\nabla_J\xi_J\nabla_\ell\xi_J$$

Plug (16), (15) back in

$$\sum_I \gamma_I^2 A(\nabla\xi)_J^I = \frac{|\nabla\xi_J|^2}{d} + \varepsilon^{j\ell}\nabla_J\xi_J\nabla_\ell\xi_J$$

We need 12 waves in 6 direction and their conjugate. Letting $I = (k, f) \in \mathbb{Z}\times(\mathbb{Z}/\mathbb{Z}\tau)^3\times\mathbb{F}$, F are the faces of regular dodecahedron.

$$\sum_{I\in K\times\mathbb{F}} \gamma_I^2 A(\nabla\xi_J^I) = \frac{|\nabla\xi_J|^2}{2d} + \varepsilon^{j\ell}\frac{\nabla_j\xi_J\nabla\xi_\ell\xi_J}{2}$$

$\exists\ c > 0$, s.t. $\left\|A_J^T - A(\nabla\hat{xi}_J^I\right\| \leqslant C$, $\left\|\xi_J - \frac{|\nabla\xi_J|^2}{2d}\right\| \leqslant C$. The equation $\sum_{I\in\mathbb{F}} A_J^I\gamma_I^2 = y_I$ has a solution $\gamma_I = \gamma_f(A_J^I, y_I)$ depending smoothly with uniform bounds on $\partial_A\gamma_F$ and $\gamma_y\gamma_f$ and higher derivative.

$$\sum_I A(\nabla\xi)_J^I\gamma_I^2 = \frac{|\nabla\xi_I|^2}{2d} + \varepsilon^{j\ell}\nabla_J\xi_J\nabla_J\xi_J$$

where $\varepsilon^{j\ell} \sim \mathcal{O}(\frac{1}{k})$.

*Proof. (inverse function theorem)*Rewrite as $F(\gamma, A, y) = 0$, $F_J(\gamma, A, y) = \sum_I A_J^I\gamma_I^2 - y_J$. It is sufficient to check that $\frac{\partial F_J}{\partial\gamma_\gamma}(?)$ is invertible.

$$\gamma_I, A_J^T, y_J) = (\hat{\gamma_f}, A(\nabla\hat{\xi})_J^I, \frac{|\nabla\xi_J|^2}{2d})$$

If h_I is in the null space at this point

$$\frac{\partial f_J}{\partial\gamma}H_I = 2A(\nabla\hat{\xi})_J^I\hat{\gamma}_I h_I = 0 \quad\Rightarrow\quad \hat{\gamma}_I h_I = 0 \quad \forall I$$

since $A(\nabla\hat{\xi})^I$ is invertible. $\Rightarrow h_I = 0\ \forall\ I$ since $\hat{\gamma}_I \neq 0\ \forall I$. ☐

$$V_I^i = e^{i\lambda\xi_I}(V_I^\ell + \delta V_I^\ell) \qquad \sum_I V_I^j\overline{V}_I^\ell = e(t)\delta^{j\ell} - \mathring{R}_\varepsilon^{j\ell}$$

$$(\text{space cut off})v_I^\ell = \eta_{k_0}(t) \quad \Psi_I(t,x)e^{1/2}(t)\mathring{V}_I^\ell$$

$$\sum_I \mathring{V}_I^i\mathring{C}_I^\ell = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell},$$

where $\varepsilon^{j\ell} = \dfrac{-\mathring{R}_{\varepsilon}^{j\ell}}{e(t)} = \mathcal{O}(\frac{1}{k})$, $\quad e(t) \geqslant K e_R$ on supp R_ε.

$$(\partial_t + v^i \nabla_i)\xi_I = 0 \qquad < O_{[k]} f, x >$$

$$\hat{\xi}_I(t(i), x) = < f, x > \qquad \text{oscillation}$$

where $I = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{F} =$ time $\times$ space $\times$ direction.

When you consider interaction with waves. $\nabla_j (v_I^j v_J^\ell)$

$$\begin{aligned} I_m \mathring{V}_I^\ell &= \mathring{b}_I^\ell = \gamma_I P_I^\perp (\nabla \xi_{\sigma I})^\ell \\ &= \gamma_I \left(\nabla^\ell \xi_{\sigma I} - \frac{\nabla \xi_{\sigma I} \cdot \nabla \xi_I)}{|\nabla \xi_I|^2} \nabla^\ell \xi_I \right) \end{aligned} \tag{17}$$

$$Re \mathring{V}_I^\ell = -\frac{(\nabla \xi_I) \times \mathring{b}_I}{|\nabla \xi_I|}$$

Plug above into (17)

$$\sum_{I \in k \times F} \gamma_I^2 \frac{|\nabla \xi_I \wedge \nabla \xi_{\sigma I}|^2}{|\nabla \xi_I|^2} \left(\delta^{j\ell} - \frac{\nabla^j \xi_I \nabla^\ell \xi_q}{|\nabla \xi_I|^2} \right) = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell}$$

$$V_I = e^{i\lambda \xi_I} (v_I^\ell + \delta v_I^\ell)$$

$$I_M(V_I) = \eta_{K_0}(t) \Psi_I e^{1/2}(t) (\gamma_I P_I^\perp (\delta \xi_{\sigma_I}))$$

$$R_\ell v_I = -\frac{\nabla \xi_I}{|\nabla \xi_I|} \times I_m V_I$$

$$\|V_I\|_0) \leqslant \|\eta\| \|\Psi\| \left\| e^{1/2} \right\| \|\gamma_I\| \|\nabla \xi_{\sigma_I}\|$$

$$\sum_{K_0} \eta_{K_0}^2(t) = 1 \qquad e(t) \geqslant K e_R$$

$$\sum_I \gamma_I^2 \frac{|\nabla \xi_I \wedge \nabla \xi_{\sigma_I}|^2}{|\nabla \xi_I|^2} \left(\delta^{j\ell} - \frac{\nabla^j \xi_I \nabla^\ell \xi_I}{|\nabla \xi_I|^2} \right) = \frac{\delta^{j\ell}}{d} + \varepsilon^{j\ell}$$

where $\varepsilon^{j\ell} = \dfrac{-\mathring{R}_{\varepsilon}^{j\ell}}{e(t)}$.

Introduce Frequency Energy levels

Let $\Theta \geqslant 2\ e_v > e_R > 0$, $L \geq 1$. Say $(v, p, R) \leqslant (\Theta, e_v, e_R)$ if

$$\|\nabla_{\overleftarrow{a}} v\|_o \leqslant \Theta^{|\overrightarrow{a}|} e_v^{1/2} \quad 1 \leqslant |\overrightarrow{a}| \leqslant L$$

$$\|\nabla_{\overleftarrow{a}} R\|_o \leqslant \Theta^{|\overrightarrow{a}|} e_R$$

we always begin with Renyolds flow from previous stage. $\|v\|_0$ close to/converge to real solutions.

$$\left\|\nabla \check{V}\right\|_0 = \|\nabla V\|_0 + \|\nabla V\|_0$$

The correction $\|V\|_0 \lesssim e_R^{1/2}$, $\|\nabla v\|_0 \lesssim \lambda e^{1/2}$ from the previous stage. $\|\nabla v\| \to \Theta e_v^{1/2}$. Taking derivative cost a factor of Θ.

Lemma 4.7. (Practice main lemma) $\exists\ \hat{c} \geqslant 1$ *such that the following holds. If* $(v, p, R) \leqslant (\Theta, e_v, e_R)$ *if* $N \geqslant (\frac{e_v}{e_R})$ $\exists (\check{v}, \check{p}, \check{R}) \leqslant (\check{\Theta}, \check{e_v}, \check{e_R})$. *Here we want* $\check{e}_R$ *as small as possible and* $\check{v} = v + V$ $\check{p} + P$

$$\|v\|_0 \leqslant \hat{c} e_R^{1/2} \qquad \|p\|_0 \leqslant c e_R$$

$$\|\nabla v\| \leqslant c(N\Theta) e_R^{1/2} \qquad \|\nabla P\|_0 \leqslant c(N\Theta) e_R$$

How big is $\check{e}_R$?

1. High-Low interaction term

$$\nabla_j R_L^{j\ell} = \nabla_j(v^j v^\ell) = v^j \nabla_j v^\ell = \Theta_I e^{i\lambda\xi_I} \tilde{v}_I^j \nabla_j v^\ell$$

   Non-stationary phase

$$\|R_L\|_0 \lesssim \frac{1}{\lambda}\|v\|_0\|\nabla v\|_0 \leqslant (N\Theta)^{-1} e_R^{1/2}(\Theta e_v^{1/2}) = \frac{e_v^{1/2} e_R^{1/2}}{N}$$

   We hope that $\check{e}_R = \dfrac{e_v^{1/2} e_R^{1/2}}{N}$

2. transport $\nabla_j R_T^{j\ell} = \partial_t v^\ell + \nabla_j(v^j v^\ell) = (\partial_t + v^j \nabla_j(v^j v^\ell) = (\partial_t + v^j \nabla_j) v^\ell$

$$\|R_T\|_0 \leqslant \lambda^{-1}\|v\|_0\|\nabla v_1\|_0 \leqslant (N\Theta)^{-}1 \cdot 1 \cdot \Theta e_R^{1/2} = N^{-1} e^{1/2}$$

   where $R_T^{j\ell} = \sum_I \dfrac{e^{i\lambda\xi_I}}{i\lambda} q_a^{j\ell}(\nabla\xi_I) D_t V_1$. In the result we missing $e_v^{1/2}$ which make this even worse to $\frac{1}{3+\sqrt{8}}$.

To optimizing, we keep the transport term.

$$\|R_T\|_0 \lesssim \lambda^{-1}\big\|(\partial_t + v^j\nabla_j)V_I\big\|_0 = (N\Theta)^{-1}(\|D_t\|) e_R^{1/2}$$

$$(\partial_t + v^i\nabla_i)\xi_I = 0$$

$$(\partial_t + v^i\nabla_i)\nabla_a\xi_I = -\nabla_a v^i \nabla_i \xi \sim \Theta e_v^{1/2} \cdot 1$$

The result $\|R_T\|_0 = \frac{e_v^{1/2} e_R^{1/2}}{N}$ is ideal

$$\sum_I v_I^j v_I^\ell = e(t)\frac{\delta^{j\ell}}{d} - \mathring{R}_\varepsilon^{j\ell}$$

And more for frequency level:

$$\|\nabla_{\vec{a}}(\partial_t + v\nabla)R\|_0 \leqslant \Theta^{|\vec{a}|+1} e_v^{1/2} e_R \quad 0 \leqslant |\vec{a}| \leqslant L-1$$

better for space derivative.

Actual Frequency Levels

Note that e_R, $e_v \sim R$, $p \sim \frac{m^2}{s^2}$, and $\Theta \sim \nabla \sim m^{-1}$ $\partial_t \sim s^{-1}$ $v \sim \frac{m}{s}$ and $(\Theta e_v^{1/2}) \sim (\partial_t + v\cdot\nabla) \sim s^{-1}$. No assumed bound for $\|v\|_0$. The bounds are invariant under

$$\begin{cases} v^\ell & & c^\ell + v^\ell(t, x - tc) \\ p(t,x) & \Rightarrow & p(t, x - tc) \\ R^{j\ell}(t,x) & & R^{j\ell}(t, x - tc) \end{cases}$$

Thus well-posed.

$$\nabla_j R_L^{j\ell} = v^j \nabla_j v^\ell = \sum_I e^{i\lambda\xi_I} \tilde{v}_I^j \nabla_j v^\ell = \sum_I e^{i\lambda\xi_I} u_I^\ell$$

$$R^{j\ell} = \sum_I \frac{e^{i\lambda\xi_I}}{i\lambda} q_a^{j\ell}(\nabla\xi_I) u_I^a + L.O.T$$

$$\overline{D}_t R_T^{j\ell} = \frac{e^{i\lambda\xi_I}}{i\lambda} D_t(q_a^{j\ell}(\nabla\xi_I) u_I^a)$$

seems not give a λ.

Remark.

$$\overline{D}_t R_L^{j\ell} = \frac{e^{i\lambda\xi_I}}{i\lambda} D_t(q_a^{j\ell}(\nabla\xi_I)\tilde{v}_I^b \nabla_b v^a)$$

$$D_t \nabla_b v^a = (\partial_t + v^j\nabla_j)\nabla_b v^a$$

If we use Euler-Reynolds equation (12)

$$(\partial_t + v^j\nabla_j)\nabla_b v^\ell = -\nabla_b v^j \nabla_j v^\ell - \nabla_b \nabla^\ell P - \nabla_b \nabla_j R^{j\ell}$$

where $|D_t| \leqslant \Theta e_v^{1/2}$. The bound that I want is $(\Theta e^{1/2})(\Theta e^{1/2})$.

$$\check{D}_t = \partial_t + v \cdot \nabla + V \cdot \nabla$$

where V is small and there hidden a λ in the last ∇ from $e^{i\lambda\xi_I}$.

Frequency energy levels

We say (v, p, R) have energy levels below (Θ, e_0, e_R) to order L. $(v, p, R) \leqslant (\Theta, e_v, e_R)$ is $\|\nabla_{\overrightarrow{a}} v\|_0 \leqslant \Theta^{|\overrightarrow{a}|} e^{1/2}$, $1 \leqslant |\overrightarrow{a}| \leqslant L$. $\|\nabla_{\overrightarrow{a}} p\|_0 \leqslant \Theta^{|\overrightarrow{a}|} e_v$ and $\|\nabla_{\overrightarrow{a}} R\|_0 \leqslant \Theta^{|\overrightarrow{a}|} e_R$. $\|\nabla_{\overrightarrow{a}}(\partial_t + v\nabla)R\|_0 \leqslant \Theta^{|\overrightarrow{a}|}(\Theta e^{1/2})e_R$. $0 \leqslant |\overrightarrow{a}| \leqslant L$ and on the RHS we need $0 \leqslant |\overrightarrow{a}| \leqslant L - 1$

Lemma 4.8. *If $L \geq 2$ $\eta > 0$ and $M > 0$ $K > 0$ is as determined previously $\exists \hat{c} = \hat{c}(\eta, L, M)$ s,t, the following holds: Let $(v, p, R) \leqslant (\Theta, e_v, e_R)$ to order L. supp $R \subset I \cdot \mathbb{T}^3$.*
Let $e(t) : \mathbb{R} \to \mathbb{R}_+$ satisfying $e(t) \geqslant K e_R$ $\forall r \in N(I; (\Theta, e^{1/2})^{-1})/$ and for lifting function

$$\left\| \frac{d^r}{dt^r} e^{1/2} \right\|_0 \leqslant M(\Theta e_0^{1/2})^r e_R^{1/2}$$

Let $N \geqslant 0$ $N \geqslant (\frac{e_v}{e_R})$ be given $N > \Theta^\eta$ where η is small.
Then $\exists$ $(\check{v}, \check{p}, \check{R}) \leqslant (\check{\Theta}, \check{e}_v, \check{e}_R) = (\check{c}N\Theta, e_R, \dfrac{e_v^{1/2} e_R^{1/2}}{N})$

$\check{v} = v + V$ $\check{p} + P$

$$\|v\|_0 \leqslant \hat{c} e_R^{1/2} \qquad \|p\|_0 \leqslant \check{c} e_R$$

$$\|\nabla v\|_0 \leqslant \check{c}(N\Theta) e_R^{1/2} \qquad \|\nabla P\|_0 \leqslant \check{c}(N\Theta) e_R$$

$\mathrm{supp}(V, p, R) \subset N(\mathrm{supp}\, e, (\Xi e^{1/2})^{-1})$

$$\left\| \int |v|^2 dx - \int e(t) dx \right\| \leqslant \frac{e_R}{N}$$

small error. Letting $V^\ell = \nabla_j w^{j\ell}$ $\|w\|_0 \leqslant \dfrac{e_R^{1/2}}{N\Xi}$

Proof. (of Onsager's conjecture) Set $(v, p, R)_{(0)} = (0, 0, 0)$ initial (12) flow. Let $\Xi_{(0)} > 1$(very big) $e_v \geqslant e_R(0)$ be given. So we want new error $e_{R(K+1)} = \dfrac{e_{R(k)}^{1+\delta}}{z}$ z to be chosen. Force $\dfrac{e_{R(k)}^{1/2} e_{R(k)}^{1/2}}{N_{(k)}} : \dfrac{e_R^{1-\delta}}{z} \Rightarrow N_(k) = z \cdot (\frac{e_v}{e_R})_K^{1/2} e^{-\delta}$ shrinking exponentially. □

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