M393C and CSE396 **Mathematical treatment of Inviscid Fluid Mechanics**
Unique number: 54290 and 61815
Spring 2022
TTH: 9:30-11:00am, PMA 11.176.

Instructor: Alexis F. Vasseur, PMA 11.172
Office Hours: by appointment
Website: https://utexas.instructure.com

Course overview: The course will focus on the incompressible Navier-Stokes equation and its inviscid limit, the Euler equation. We will develop the theory of existence, regularity, partial regularity, and non-uniqueness for the solutions for these models. We will study challenges due to boundary effects in detail, beginning with the linear Stokes equation.

Requirement: This course expects students to have a strong background in Real analysis (equivalent to Graduate Real Analysis) and Functional analysis (Graduate Applied Math I and II) and to have some basic knowledge of Partial differential equations (Graduate PDE I).

Courses Procedure and Covid 19: The class is scheduled to be face to face. However, we will follow the safety recommendation of the central administration due to the latest wave of Covid 19. Accommodation in the course delivery will be made according to the changing situation. Choices will be made with two goals in mind: (1) keeping a safe environment, (2) providing the best teaching experience possible. Changes will be communicated via **announcements** in Canvas. The hope is to revert to a classical in-class-only, face-to-face lecture as soon as possible.

Accommodations for the first two weeks: Up to the end the month, the lectures will be delivered on Zoom only. Recordings will be provided on Canvas.

University recommendation about classroom safety and Covid 19: As soon as face to face teaching will resume, I will ask (legal disclaimer: but not require) students to adhere to the university recommendation. As of today, the university recommends the following.
-Adhere to university mask guidance. Masks are strongly recommended (still, by law, optional), inside university buildings for vaccinated and unvaccinated individuals, except when alone in a private office or single-occupant cubicle.
-Vaccinations are widely available, free and not billed to health insurance. The vaccine will help protect against the transmission of the virus to others and reduce serious symptoms in those who are vaccinated.
-If you develop COVID-19 symptoms or feel sick, stay home and contact the University Health Services' Nurse Advice Line at 512-475-6877. If you need to be absent from class, contact Student Emergency Services and they will notify your professors. In addition, to help understand what to do if you have been had close contact with someone who tested positive for COVID-19, see this University Health Services link.

Lecture Notes on Mathematical Treatment of Incompressible Fluid Mechanics

Alexis F. Vasseur

May 22, 2022

Contents

Chapter 1

Introductions

1.1 Notations

- Domain $\Omega \subset \mathbb{R}^3$.
- For any $p : \Omega \to \mathbb{R}$, we denote

$$\nabla p = \begin{pmatrix} \partial_1 p \\ \partial_2 p \\ \partial_3 p \end{pmatrix} : \Omega \to \mathbb{R}^3$$

$$\Delta p = \partial_{11} p + \partial_{22} p + \partial_{33} p : \Omega \to \mathbb{R}$$

- For any $u : \Omega \to \mathbb{R}^3$, we denote

$$\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 : \Omega \to \mathbb{R}$$

$$\operatorname{curl} u = \nabla \times u = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} : \Omega \to \mathbb{R}.$$

Recall that $\Delta p = \operatorname{div}(\nabla p)$. For $u, v \in \mathbb{R}^3$,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ v_1 u_3 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

The Jacobian is defined as

$$(\nabla u)_{ij} = (\partial_j u_i) = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix}$$

Note that $\operatorname{div} u = \operatorname{tr}(\nabla u)$.

- For $v, u : \Omega \to \mathbb{R}^3$,
$$(v \cdot \nabla)u : \Omega \to \mathbb{R}^3.$$
For $i \in 1, 2, 3$, $[(v \cdot \nabla)u]_i = v_1\partial_1 u_i + v_2\partial_2 u_i + v_3\partial_3 u_i = v \cdot \nabla u_i$. So $(v \cdot \nabla)u = (\nabla u)v$.

- Symmetric Jacobian $Du = \frac{\nabla u + \nabla u^\top}{2}$

- Antisymmetric part $Ru = \frac{\nabla u - \nabla u^\top}{2}$.
$$\begin{aligned} Ru &= \frac{1}{2}\begin{pmatrix} 0 & \partial_2 u_1 - \partial_1 u_2 & \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 & 0 & \partial_3 u_2 - \partial_2 u_3 \\ \partial_1 u_3 - \partial_3 u_1 & \partial_2 u_3 - \partial_3 u_2 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \end{aligned}$$

- For $v \in \mathbb{R}^3$, suppose $\omega = \operatorname{curl} u$ be the vorticity, then
$$Ru \cdot v = \frac{1}{2}\begin{pmatrix} -\omega_3 v_2 + \omega_2 v_3 \\ v_1\omega_3 - \omega_1 v_3 \\ -\omega_2 v_1 + \omega_1 v_2 \end{pmatrix} = \frac{1}{2}\omega \times v = -\frac{1}{2} v \times \omega$$

1.2 Navier–Stokes and Euler without boundary

Consider $\Omega = \mathbb{T}^3$ (or $\mathbb{R}^3$), $\nu > 0$ is the viscosity of the fluid. The Navier–Stokes equation is given by
$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p - \nu\Delta u = 0 & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 \text{ incompressibility} & \text{in } (0,T) \times \Omega \\ u\big|_{t=0} = u^0 \text{ given} & \text{in } \Omega \end{cases}$$

Unknown

- $u : (0,T) \times \Omega \to \mathbb{R}^3$ velocity of the fluid
- $p : (0,T) \times \Omega \to \mathbb{R}$: is also unkhown

The pressure is the Lagrange multiplier associated to the incompressibility constraint $\operatorname{div} u = 0$.

$$\partial_t u + \underbrace{(u\cdot\nabla)u}_{\text{transport}} + \underbrace{\nabla p}_{\text{projection}} - \underbrace{\nu\Delta u}_{\text{heat}} = 0.$$

Euler equation: $\nu = 0$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } (0,T)\times\Omega \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega \\ u\big|_{t=0} = u^0 & \text{in } \Omega \end{cases}$$

1.3 Transport equation

Take $v : \mathbb{R}\times\Omega \to \mathbb{R}^3$ given, smooth. We define the flow map $X(s,t,x)$ satisfying the ODE ($X_{s\to t}(x)$)

$$\begin{cases} X(s,s,x) = x \\ \partial_t X = v(t,X) \end{cases}$$

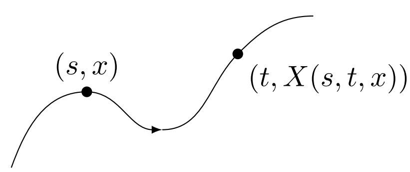


Figure 1.1: Flow map

This comes from the **Lagrange point of view**.

Theorem 1.1. *For any $\theta^0 \in C_c^\infty(\Omega)$, the unique solution to*

$$\begin{cases} \partial_t\theta + (v\cdot\nabla)\theta = 0 & t>0, x\in\Omega \\ \theta\big|_{t=0} = \theta^0 \end{cases} \tag{1.1}$$

is $\theta(t,x) = \theta^0(X(t,0,x))$.

Remark 1.2. (1.1) is called a transport equation.

Remark 1.3. θ is preserved along the flow:

$$\theta(t,x) = \theta^0(X(t,0,x))$$
$$\theta(0,x) = \theta^0(X(0,0,x)) = \theta^0(x).$$

Proof. Flow-map identity $X(t,0,X(0,t,x)) = x$. This shows that

$$[X(t,0,\cdot)]^{-1} = X(0,t,\cdot).$$

so $X(t,0,\cdot)$ is invertible.

$$\begin{aligned} \theta(t, X(0,t,x)) &= \theta^0(X(t,0,X(0,t,x)) \\ &= \theta^0(x). \end{aligned}$$

We differentiate in time.

$$\begin{aligned} (\partial_t\theta + \partial_t X(0,t,x)\cdot\nabla\theta)(t,X(0,t,x)) = 0. \\ (\partial_t\theta + v\cdot\nabla\theta)(t,X(0,t,x)) = 0 \qquad \forall (t,x). \end{aligned}$$

We get $\partial_t\theta + v\cdot\nabla\theta = 0$. □

Remark 1.4. In 1D, incompressible $v \equiv$ constant, so

$$\begin{aligned} X(s,t,x) &= x + (t-s)v, \\ X(t,s,x) &= x - (t-s)v, \\ \theta(t,x) &= \theta^0(x - tv). \end{aligned}$$

1.3.1 Density equation

Let v be a given particle velocity. ρ is the density of the particle, with $\rho(0,x) = \rho^0(x)$ at $t = 0$.

In area $A \subset \mathbb{R}^d$, the total mass in A is

$$\begin{aligned} M_A(t) &= \int_A \rho(t,x)\,\mathrm{d}x \\ \frac{\mathrm{d}}{\mathrm{d}t} M_A(t) &= -\int_{\partial A} n\cdot(\rho v)\,\mathrm{d}\hat{x} \\ &= -\int_A \mathrm{div}(\rho v)\,\mathrm{d}x. \end{aligned}$$

$$\int_A \{\partial_t\rho + \mathrm{div}(\rho v)\}\,\mathrm{d}x = 0 \qquad \forall A.$$

Then

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho v) = 0 & \text{a.e. } (t,x) \\ \rho(0,x) = \rho^0(x) \end{cases}$$

Theorem 1.5. *The solution to*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 & a.e.\ (t,x) \\ \rho(0,x) = 1 \end{cases}$$

is $\rho(t,x) = 1$ *if and only if* $\operatorname{div} v \equiv 0$.

Remark 1.6. $\rho(t,x) \equiv 1$ corresponds to incompressibility.

Proof.

$$\begin{aligned} \operatorname{div}(\rho v) &= \partial_1(\rho v_1) + \partial_2(\rho v_2) + \partial_3(\rho v_3) \\ &= \rho(\partial_1 v_1 + \partial_2 v_2 + \rho_3 v_3) + v_1 \partial_1 \rho + v_2 \partial_2 \rho + v_3 \partial_3 \rho \\ &= \rho \operatorname{div} v + v \cdot \nabla \rho. \end{aligned}$$

So

$$\partial_t \rho + v \cdot \nabla \rho + \rho \operatorname{div} v = 0.$$

1. If $\rho \equiv 1$ for all (t,x), then $\partial_t \rho = 0$, $\nabla \rho = 0$. So $\operatorname{div} u = 0$.
2. If $\operatorname{div} u \equiv 0$ then $\partial_t \rho + u \cdot \nabla \rho = 0$ so $\rho(t,x) = \rho^0(X(t,0,x)) = 1$.

□

Remark 1.7. For fluids (ρ, u),

$$\partial_t \rho + \operatorname{div}(\rho u) = 0.$$

For incompressible flows: $\operatorname{div} u = 0$, $\rho \equiv 1$.

Remark 1.8. Recall that

$$\partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u = 0. \tag{1.2}$$

Notation:

- $M = (M_{ij})_{ij}$, we denote
  $$(\operatorname{div} M)_i = (\partial_1 M_{i1} + \partial_2 M_{i2} + \partial_3 M_{i3})_i.$$
  where $\operatorname{div} M : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$.
- Given $u, v : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$,
  $$\begin{aligned} u \otimes v &: \mathbb{R}^+ \times \Omega \to \mathbb{R}^9 \\ (u \otimes v)_{ij} &= u_i v_j \end{aligned}$$

- $\operatorname{div}(u \otimes v) = (v \cdot \nabla)u + (\operatorname{div} v)u$.

For $i = 1, 2, 3$, $[\operatorname{div}(u \otimes v)]_i = \operatorname{div}(u_i v) = u_i \operatorname{div} v + (v \cdot \nabla)u_i$. So (1.2) can be rewritten as

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u = 0.$$

1.4 Pressure effect

Consider the case $\Omega = \mathbb{R}^3$.

Theorem 1.9 (Hodge decomposition)**.**

$$\begin{aligned} [L^2(\mathbb{R}^3)]^3 &= H \oplus G \\ H &= \{u \in [L^2(\mathbb{R}^3)]^3 : \operatorname{div} u = 0\} \\ G &= \left\{\nabla p : p \in \dot{H}(\mathbb{R}^3)\right\} \end{aligned}$$

where

$$\dot{H}(\mathbb{R}^3) = \{v : \nabla v \in L^2(\mathbb{R}^3)\}.$$

Define the projection onto H,

$$\mathbb{P}_{\text{div}} : [L^2(\mathbb{R}^3)]^3 \to H$$

Then for any $u \in [L^2(\mathbb{R}^3)]^3$, there exists $p \in \dot{H}(\mathbb{R}^3)$ such that

$$u = \mathbb{P}_{\text{div}} u + \nabla p.$$

This decomposition extends to $[H^s(\mathbb{R}^3)]^3$ for any $s \in \mathbb{R}$.

Remark 1.10. $u = \mathbb{P}_{\text{div}} u + \nabla p$ is uniquely defined.

Remark 1.11. $u \in [C_c^\infty(\mathbb{R}^3)]^3$, the decomposition is also valid in $[H^s(\mathbb{R}^3)]^3$

$$\mathbb{P}_{\text{div}} u = \mathbb{P}^s_{\text{div}} u$$

where $\mathbb{P}^s_{\text{div}}$ is the projection on

$$\{u \in [H^s(\mathbb{R}^3)]^3 : \operatorname{div} u = 0\}.$$

Proof. Use Fourier transform.

$$H = \left\{u \in [L^2(\mathbb{R}^3)]^3 : \xi \cdot \hat{u} = 0\right\}.$$

This means

$$\begin{aligned}
\widehat{\mathbb{P}_{\mathrm{div}} u}(\xi) &= \hat{u}(\xi) - \frac{\xi \cdot \hat{u}(\xi)}{|\xi|^2}\xi \\
\hat{u}(\xi) &= \hat{u}(\xi) - \frac{\xi \cdot \hat{u}(\xi)}{|\xi|^2}\xi + \frac{\xi \cdot \hat{u}(\xi)}{|\xi|^2}\xi \\
u(x) &= \mathbb{P}_{\mathrm{div}} u + \nabla p \text{ where } \hat{p} = i\frac{\xi \cdot \hat{u}(\xi)}{|\xi|^2}.
\end{aligned}$$

□

Now look at the Navier–Stokes equation,

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0.$$

Note that ∂_t and Δ commutes with div, so

$$\mathrm{div}(\partial_t u - \nu \Delta u) = \partial_t \,\mathrm{div}\, u - \nu \Delta \,\mathrm{div}\, u = 0.$$

Thus we can write

$$u \cdot \nabla u = \underbrace{-(\partial_t u - \nu \Delta u)}_{\mathrm{div}=0} - \nabla p$$

Take the $\mathbb{P}_{\mathrm{div}}$ of (1.2),

$$\begin{cases} \partial_t u + \mathbb{P}_{\mathrm{div}}((u \cdot \nabla)u) - \nu \Delta u = 0 \\ u = \mathbb{P}_{\mathrm{div}} u \end{cases}$$

Moreover, take divergence of (1.2),

$$\mathrm{div}\,\mathrm{div}(u \otimes u) + \Delta p = 0.$$

So

$$\begin{aligned}
-\Delta p &= \mathrm{div}\,\mathrm{div}(u \otimes u) \\
p &= (-\Delta)^{-1} \,\mathrm{div}\,\mathrm{div}(u \otimes u).
\end{aligned}$$

Note that $(-\Delta)^{-1}\,\mathrm{div}\,\mathrm{div}$ is a Riesz operator of order 0.

1.5 Vorticity

v: velocity and $\omega = \operatorname{curl} u$

Lemma 1.12. *If u is solution to N.S. equation(or Euler if $\nu = 0$). Then the equation can be rewritten:*

$$\begin{cases} \partial_t u + \omega \times u + \nabla q - \nu \Delta u = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{1.3}$$

we have $q = p + \frac{|u|^2}{2}$.

Remark 1.13. If $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$: Taking $\mathbb{P}_{\text{div}}$ on (1.3)

$$\partial_t u + \mathbb{P}_{\text{div}}(\omega \times u) - \nu \Delta u = 0 \qquad \operatorname{div} u = 0$$

Consider the ideal model: $\Omega \overrightarrow{e_3}$

$$\begin{cases} \dot{u} = \Omega \overrightarrow{e_3} \times u & t > 0 \\ u(t=0) = u^0 = (U \cos \Psi, U \sin \Psi, u_3^0) \end{cases}$$

$u = \bar{u} + u_3 e_3$

$$\Omega \overrightarrow{e_3} \times u = \Omega \overrightarrow{e_3} \times \bar{u} = (-u_2 \Omega, u_1 \Omega, 0)$$

$$\begin{cases} \dot{u_1} = -u_2 \Omega \\ \dot{u_2} = u_1 \Omega \\ \dot{u_3} = 0 \end{cases}$$

and

$$\ddot{\bar{u}} + \Omega^2 \bar{u} = 0 \qquad \dot{u}_3 = 0$$

$$u(t) = \big(U \cos(\Psi + \Omega r), U \sin(\Psi + \Omega r), u_3^0\big)$$

$$\begin{cases} \dot{x}(t) = u(t, x) \\ x(t=0) = x^0 \end{cases}$$

Helix:

$$X(t) = \left(\frac{U}{\Omega} \sin(\Psi + \Omega t); -\frac{U}{\Omega} \cos(\Psi + \Omega t), u_3^0 t\right) + \left(x_1^0 - \frac{U}{\Omega} \sin \Psi; x_2^0 + \frac{U}{\Omega} \cos \Psi; x_3^0\right).$$

Some computation then change a particle into a magnetic field $\overrightarrow{B} = \Omega \overrightarrow{e_3}$.

Proof.

$$\partial_t u + (u - \nabla)u + \nabla p - \gamma \Delta u = 0$$

$$\begin{aligned}(u, \nabla)u = (\nabla u) \cdot u = \nabla u \cdot u - \nabla u)^{\perp} u + \nabla u)^{\perp} u \\ = 2Ru \cdot u + \sum_i (\nabla u_i) u_i \\ = 2Ru \cdot u + \nabla \frac{|u|^2}{2} \qquad 2RU \cdot v = \omega \times v\end{aligned}$$

$$(u \cdot \nabla)u = \omega \times u + \nabla \frac{|u|^2}{2}$$

Therefore,

$$\partial_t u + \omega \times u + \nabla \frac{|u|^2}{2}$$

□

Lemma 1.14. *(Technical Lemma)*

1. $\forall a, b, c \in \mathbb{R}^3$: $a \cdot (b \times c) = \det(a, b, c)$
2. $\Omega \subset \mathbb{R}^3$: $u \in [C^\infty(\Omega)]^3$, $v \in [C^\infty(\Omega)]^3$

$$\int_\Omega \operatorname{curl} u \cdot v dx = \int_\Omega u \cdot \operatorname{curl} v dx$$

3. $\forall \Psi \in C^\infty(\Omega) : \operatorname{curl}(\nabla \Psi) = 0$

$$\forall u \in C^\infty(\Omega) : \operatorname{div}(\operatorname{curl} u) = 0$$

4. $a, b \in [C^\infty(\Omega)]^3$: $\operatorname{curl}(a \times b) = \operatorname{div}(a \otimes b - b \otimes a)$

Remark 1.15.

$$\operatorname{div}(a \otimes b) = (b \cdot \nabla)a + (\operatorname{div} b)a$$

$$\left([\operatorname{div}(a \times b)]_i = \sum_j \partial_j (a_i b_j) \right)$$

4. can be rewritten:

$$\operatorname{curl}(a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + (\operatorname{div} b)a - (\operatorname{div} a)b$$

Proof. 1. Let's define $F(a,b,c) = a \cdot (b \times c)$. F is a linear form and it is alternating:
$$F(a,a,c) = F(a,b,b) = F(a,c,c) = 0$$
So $F(a,b,c) = \lambda \det(a,b,c)$. But $F(e_1,e_2,e_3) = 1$. Thus $\lambda \equiv 1$.
$$|b \times c| = |b||c| \sin\psi$$
$$\det(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = Vol(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}) = \text{Surf}(\overrightarrow{b}, \overrightarrow{c})|a|\cos\psi = a \cdot (b \times c)$$

2.
$$\begin{aligned}\int_\Omega \operatorname{curl} u \cdot v &= \int_\Omega v_1(\partial_2 u_3 - \partial_3 u_2) + \int_\Omega v_2(\partial_3 u_1 - \partial_1 u_3) + \int_\Omega v_3(\partial_1 u_2 - \partial_2 u_1)\\ &= \int_\Omega (\partial_1 v_2 - \partial_2 v_1)u_3 + \int_\Omega (\partial_3 v_1 - \partial_1 v_3)u_2 + \int_\Omega (\partial_2 v_3 - \partial_3 v_2)u_1\\ &= \int_\Omega (\operatorname{curl} v) \cdot u\end{aligned}$$

3.
$$\operatorname{curl} \nabla P = 0 = (\partial_2\partial_3 P - \partial_3\partial_2 P, \partial_3\partial_1 P - \partial_1\partial_3 P, \partial_1\partial_2 P - \partial_2\partial_1 P)$$
$\forall v \in \mathcal{D}(\Omega)$:
$$\int_\Omega \operatorname{div}(\operatorname{curl} u) v dx = -\int_\Omega (\operatorname{curl} u) \cdot \nabla v dx = -\int_\Omega u \cdot \operatorname{curl}(\nabla v) dx = 0$$
So, $\operatorname{div}(\operatorname{curl} u) = 0$.

4. $\forall v \in [\mathcal{D}(\Omega)]^3$:
$$\begin{aligned}\int_\Omega \operatorname{curl}(a \times b) \cdot v &= \int_\Omega (a \times b) \cdot \operatorname{curl} v \qquad \text{by 2.}\\ &= \int_\Omega \det(\operatorname{curl} v, a, b) \qquad \text{by 1.}\\ &= \int_\Omega b \cdot (\operatorname{curl} v \times a)\\ &= \int_\Omega b^\perp (2Rv \cdot a) = \int_\Omega b^\perp(\nabla v - [\nabla v]^\perp) \cdot a\\ &= \sum_{ij} \int_\Omega b_i(\partial_j v_i - \partial_i v_j)a_j = \sum_{ij} \int_\Omega \partial_i(a_j b_i)v_j - \partial_j(b_i a_j)v_i\\ &= \int_\Omega \operatorname{div}(a \otimes b) \cdot v - div(a \otimes b) \cdot v = \int_\Omega v \cdot [\operatorname{div}(a \otimes b - b \otimes a)]dx\end{aligned}$$

True for any $v \in \mathcal{D}(\Omega)$, so $\operatorname{curl}(a \times b) = \operatorname{div}(a \otimes b - b \otimes a)$

$\square$

Remark 1.16. if $\operatorname{div} a = \operatorname{div} b \equiv 0$, then

$$\operatorname{curl}(a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b$$

Theorem 1.17. *If u is a solution to N.S. equation (Euler if $\nu = 0$) in Ω : the $\omega = \operatorname{curl} u$ satisfies:*

$$\partial_t \omega + (u \cdot \nabla)\omega - Du \cdot \omega - \nu \Delta w = 0$$

Remark 1.18.

- $Du = \nabla u + (\nabla u)^{\perp}$
- $Ru \cdot \omega = \omega \times \omega = 0$, $(\nabla u - \nabla u^{\perp}) \cdot \omega = 0$ and $Du \cdot \omega = \nabla u \cdot \omega = (\omega \cdot \nabla)u$.

Proof. Apply curl to incompressible N.S. equation (1.2) and $\operatorname{curl}(\nabla q) = 0$.

$$\partial_t \omega + \operatorname{curl}(\omega \times u) - \nu \Delta \omega = 0$$

$$\partial_t \omega + \operatorname{div}((\omega \otimes u) - (u \otimes w)) - \gamma \Delta w = 0$$

Since $\operatorname{div} u = 0$, $\operatorname{div}(\operatorname{curl} u) = 0$,

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0$$

$\square$

Question

- $\operatorname{curl} \nabla P = 0$ is the "dual to" $\operatorname{div} \operatorname{curl} \phi = 0$

$$\int \operatorname{curl} \nabla P \cdot \phi = \langle \operatorname{curl} \nabla P ; \phi \rangle = -\langle ; \operatorname{div} \operatorname{curl} \phi \rangle$$

- $\operatorname{curl}(a \times b) = \operatorname{div}(a \otimes b - b \otimes a)$, where $a.b.c$: $a \cdot (b \times c) = \det(a, b, c)$

$$\begin{aligned}
\int v \cdot \operatorname{curl}(a \times b) = \int \operatorname{curl} v \cdot (a \times b) \\
& \int b \cdot (\operatorname{curl} v \times a) = 2 \int b \cdot Rv \cdot a \\
& = \int b(\nabla v - (\nabla v)^{\perp})a = \sum_{ij} \int b_i(\partial_j v_i - \partial_i v_j)a_j \\
& = \sum_{ij} \left(\int (b_i a_j)\partial_j v_i - \int (b_i a_j)\partial_i v_j \right)
\end{aligned}$$

Since

$$\operatorname{curl} v \times a = 2Rva = \sum_{ij} \int v_j \partial_i (b_i a_j) - \int v_i \partial_j (b_i a_j) = \int v \cdot \operatorname{div}(a \otimes b) - \int v \cdot \operatorname{div}(b \otimes a)$$

Let u be solution to (1.2) in Ω (or Euler if $\nu = 0$) then $\omega = \operatorname{curl} u$ is solution to

$$\partial_t w + (u \cdot \nabla)\omega - Du \cdot \omega - \nu \Delta \omega = 0$$

2D case we have $\overrightarrow{\omega} = (\partial_1 u_2 - \partial_2 u_2)\overrightarrow{e_3} = \omega \overrightarrow{e_3}$ and $Du \cdot \overrightarrow{\omega} = \omega(Du \cdot \overrightarrow{e_3}) = 0$, then

$$\partial_t \omega + (u \cdot \nabla)\omega = \nu \Delta \omega$$

Proposition 1.19. $\mathbb{R}^2 : \omega^0(|x|) \in \mathcal{D}(\mathbb{R}^2)$, $\nu \equiv 0$, *then* $u(t,x) = U(|x|)\frac{x^\perp}{|x|}$, *here* $x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ *with* $U(r) = \frac{1}{r}\int \omega^0(\rho)\rho d\rho$. *Then* u *us a steady solution to Euler with* $\operatorname{curl} u = \omega^0$.

Proof. $-\Delta\phi = \omega^0$ in $\mathbb{R}^2$and $u = \nabla^\perp \phi$. $\operatorname{curl} u = \partial_2 u_2 - \partial_1 u_2 = -\partial_2^2 \phi - \partial_1^2 \phi = -\Delta\phi = \omega^0$. If ω^0 is radial then ϕ is also radial: $\phi(x) - \phi(|x|)$

- $u = \nabla^\perp \phi - \phi'(x)\dfrac{|x|^\perp}{|x|}$
- $\nabla \omega^0 = \omega^{0'}(|x|)\dfrac{x}{|x|}$
- $u \cdot \nabla \omega^0 \equiv 0$

So $u(t,x) = u(|x|)\dfrac{x^\perp}{|x|}$ is a steady solution to Euler. □

Stokes formula: γ a closed cone in $\mathbb{R}^3$; S a surface with boundary γ. $\overrightarrow{u} : \mathbb{R}^3 \to \mathbb{R}^3$, then $\oint_\gamma \overrightarrow{u} \cdot \overrightarrow{d\rho} = \iint_S \operatorname{curl} \overrightarrow{u}\, d\overrightarrow{S}$, where $d\rho = \dfrac{x^\perp}{|x|} d\rho$. When we consider γ_r, circle of radius r:

$$d\overrightarrow{\rho}(\theta) = \frac{x^\perp}{|x|} r d\theta$$

and

$$d\overrightarrow{\rho} = (d\theta r)(-\sin\theta, \cos\theta)$$

$$\oint_{\gamma_r} \overrightarrow{u} \cdot d\overrightarrow{\rho} = \oint U(r)\left(\frac{x^\perp}{|x|} \cdot \frac{x^\perp}{|x|}\right) d\rho = U(r)\oint_{\gamma_r} d\rho = 2\pi r U(r)$$

S_r disk inside γ_r.

$$\overrightarrow{dS} = dS\overrightarrow{e_3} \qquad \text{curl}\,\overrightarrow{u} = \omega^0\overrightarrow{e_3}$$

$$\iint_{S_r} \text{curl}\,\overrightarrow{u}\,\overrightarrow{dS} = \iint_{S_r} \omega^0(x)dS(x)$$

In polar coordinates

$$= 2\pi \int_0^r \omega^0(\rho)\rho d\rho = 2\pi r U(r)$$

So:

$$U(r) = \frac{1}{r}\int_0^r \omega^0(\rho)\rho d\rho$$

If ω^0 is compactly supported and $\iint \omega w = 0$. $\forall n \geqslant R\ U(0) = 0$. The model of Hurricane.

Theorem 1.20. *1. If u is a regular solution to Euler in a domain Ω with $\omega|_{t=0} = 0$ then $\omega = 0$ for all time.*

2. If u is a regular solution to N.S. the result is still true if there is no boundary.

Such solution is called irrotational.

Proof. 1. Consider the flow X associated to

$$\begin{cases} X(s,s,x) = x \\ \partial_t X(\cdot,t,\cdot) = u(t, X(\cdot,t,\cdot)) \end{cases}$$

$$\partial_t \omega + (u\cdot\nabla)\omega - Du\cdot\omega = 0$$

Fix $s = 0$, $x \in \Omega$:

$$\tilde{\omega}(t) = \omega(t, X(0,t,x))$$

$$\partial_t\tilde{\omega}(t) = \left[\partial_t\omega\frac{\partial X}{\partial r}\cdot\nabla\omega\right](t, X(0,t,x)) = (\partial_t\omega + u\cdot\nabla\omega)(t, X(0,t,x)] = Du\cdot\omega(t, X)$$

$$\frac{d}{dt}\tilde{\omega}(t) = Du(t, X(0,t,x))\cdot\tilde{\omega}(t)$$

So $\left|\frac{d}{dt}\tilde{w}(t)\right| \leqslant \|Du\|_{L^\infty}|\tilde{w}(t)|$.
So $|\omega(t)| \leqslant |\omega(0)|\exp(\|Du\|_{L^\infty}t)$ by Gronwall inequality. If $\tilde{\omega}(0) = \dot{\omega}(x) \equiv 0$ $\tilde{\omega} \equiv 0$ then $\forall\ t,\ x : \omega(t,x) = 0$. Langrange point of view.

2. For N.S. in $\mathbb{R}^3$ or $\mathbb{T}^3$, ω is the solution to

$$\partial_t w + (u \cdot \nabla)\omega - Du \cdot \omega - \nu\Delta\omega = 0$$

$$\partial_t \frac{|w|^2}{2} + u \cdot \nabla \frac{|w|^2}{2} - \omega^{\perp} Du \cdot \omega - \omega\Delta\omega = 0$$

So

$$\partial_t \int \frac{|w|^2}{2} dx - \int \nabla \frac{|w|^2}{2} \operatorname{div} u + \int |\nabla\omega|^2 \leqslant \|DU\|_{L^\infty} \int |\omega|^2 dx$$

where $\int \frac{|w|^2}{2} dx$ is called enstrophy in physics. By Gronwall: $\int \frac{|w|^2}{2} \leqslant \int \frac{|\dot{\omega}|^2}{2} dx \exp(2\|\nabla u\|_{L^\infty} t)$. If then enstrophy at $t = 0$ is 0:

$$\int \frac{|w^0|^2}{2} dx \equiv 0$$

then $\int \frac{|w|^2}{2} dx \equiv 0$ so $\omega = 0$.

Noted that we use the fact that there is no boundary(integration by parts).

□

rot$(\nabla\phi) \equiv 0$. we proved it. Consider $u : \mathbb{R}^3 \to \mathbb{R}^3$, if $\operatorname{curl} u \equiv 0$ then $u = \nabla\phi$ for a $\phi : \mathbb{R}^3 \to \mathbb{R}$

1.5.1 Irrotational flow

Recall that a flow u is called

- irrotational, if $\operatorname{curl} u = 0$.
- incompressible, if $\operatorname{div} u = 0$.

If $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, we will show that irrotational incompressible flow is a potential:

$$u = \nabla\varphi \qquad \text{where } -\Delta\varphi = 0 \text{ in the domain.}$$

If assuming $\int_\Omega |u|^2 < +\infty$, then as a consequence, φ is constant, and $u \equiv 0$.

Definition 1.21. We say that Ω is a domain *without hole* if any closed curve in Ω is homotopic to a point.

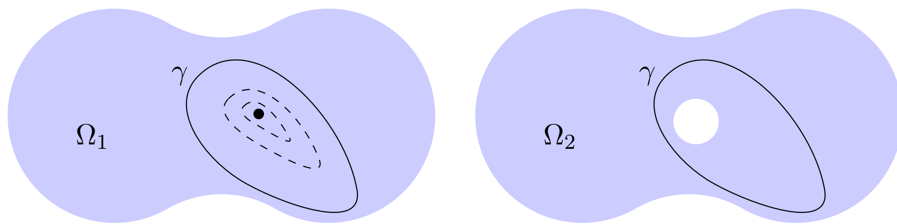


Figure 1.2: 2D domain without a hole (Ω_1) and with a hole (Ω_2)

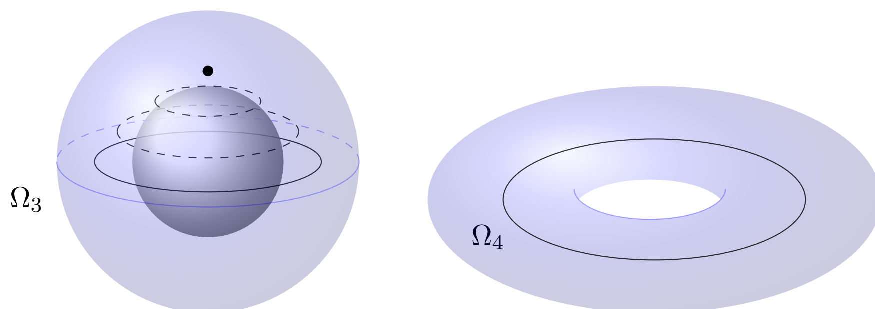


Figure 1.3: 3D domain without a hole (Ω_3) and with a hole (Ω_4)

Example 1.22. In two dimension, see Figure 1.2.

Example 1.23. In three dimension, see Figure 1.3.

Theorem 1.24. *Let Ω be a domain of $\mathbb{R}^3$ without holes and path-connected. Consider $\psi \in [C^1(\Omega)]^3$ that verifies $\operatorname{curl} \psi \equiv 0$ in Ω. Then there exists $\varphi \in C^2(\Omega)$ such that $\psi = \nabla \varphi$ in Ω.*

Proof. We first consider the case when Ω is star-shaped, then consider general domains.

Step 1. First, assume that Ω is star-shaped, i.e., there exists $x_0 \in \Omega$, such that for all $x \in \Omega$, $x_0 + t(x - x_0) \in \Omega$. Without loss of generality, take

$x_0 = 0$. then define

$$
\begin{aligned}
\varphi(x) &= \int_0^1 x \cdot \psi(tx)\,\mathrm{d}t \\
\nabla\varphi(x) &= \int_0^1 \psi(tx)\,\mathrm{d}t + \int_0^1 tx\nabla\psi(tx)\,\mathrm{d}t \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\{t\psi(tx)\}\,\mathrm{d}t + \int_0^1 tx\Big(\nabla\psi - \nabla\psi^\top\Big)\,\mathrm{d}t \\
&= \psi(x) + \int_0^1 tx\Big(\nabla\psi - \nabla\psi^\top\Big)\,\mathrm{d}t \\
&= \psi(x).
\end{aligned}
$$

In the last line we used for any v, $(\nabla\psi - \nabla\psi^\top)\cdot v = \operatorname{curl}\psi \times v = 0$.

To summary, we construct φ by first choosing γ as the line segment connecting x and x', and define

$$
\varphi(x) = \int_0^1 \psi(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t = \int_\gamma \psi(\ell)\,\mathrm{d}\ell.
$$

To verify this, note that $\gamma(t) = tx$, $\gamma'(t) = x$, so

$$
\varphi(x) = \int_0^1 \psi(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t = \int_0^1 \psi(tx)\cdot x\,\mathrm{d}t.
$$

Step 2. For general Ω, for any x_0, x, we define

$$
\varphi(x) = \int_\gamma \psi(\ell)\,\mathrm{d}\ell = \int_0^1 \psi(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t.
$$

with some curve γ connecting x_0 and x, i.e. $\gamma(0) = x_0, \gamma(1) = x$.

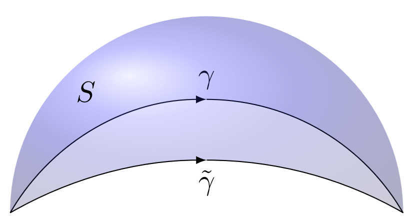


Figure 1.4: Find S with boundary $\partial S = \hat{\gamma}$

We claim that φ does not depend on the choice of the path. For two paths γ and $\tilde{\gamma}$, we can constract a close loop $\hat{\gamma}$. We need to show that for any closed curve $\hat{\gamma}$

$$\oint_{\hat{\gamma}} \psi(\ell)\,\mathrm{d}\ell \equiv 0.$$

To show this, first we find a surface S with $\hat{\gamma}$ as the boundary (S exists because Ω is without hole). By Stokes theorem,

$$\oint_{\hat{\gamma}} \psi(\ell)\,\mathrm{d}\ell = \iint_S \underbrace{\operatorname{curl}\psi}_{=0} \cdot \mathrm{d}S = 0.$$

So $\varphi(x) = \int_0^1 \psi(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t$ does not depend on γ. □

Soppose a solution of Euler

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

is an irrotational flow, i.e. $\operatorname{curl} u = 0$, then $u = \nabla\varphi$. Thanks to divergence free,

$$\Delta\varphi = 0.$$

The equation on φ is

$$\partial_t \nabla\varphi + \underbrace{\nabla\varphi \cdot \nabla\nabla\varphi}_{\nabla\left(\frac{|\nabla\varphi|^2}{2}\right)} + \nabla p = 0.$$

Remark 1.25. The potential is defined up to a constant.

$$\nabla\left\{ \partial_t\varphi + \frac{|\nabla\varphi|^2}{2} + p \right\} = 0.$$

This derives the Bernolli equation:

$$\begin{cases} \partial_t\varphi + \frac{|\nabla\varphi|^2}{2} + p = 0 \\ \Delta\varphi = 0. \end{cases}$$

The boundary condition is the Neumann condition due to impermiability:

$$\partial_n\varphi = \nabla\varphi \cdot n = u \cdot n = 0.$$

1.6 Domains with boundaries

Let Ω be a domain with boundary. Euler equation in Ω:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega \times [0,T] \\ \operatorname{div} u = 0 & \text{in } \Omega \times [0,T] \\ u(0,x) = u^0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \qquad \text{(impermiability condition)} \end{cases}$$

Navier–Stokes equation in Ω:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u & \text{in } \Omega \times [0,T] \\ \operatorname{div} u = 0 & \text{in } \Omega \times [0,T] \\ u(0,x) = u^0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \qquad \text{(no-slip condition)} \end{cases}$$

Example 1.26 (Prandtl Layer)**.** Consider $\Omega = \mathbb{R}^3_+$. $u(t,x) = Ae_1$ is a solution to the Euler (without pressure). With the viscosity effect, the Prandtl layer is

$$u_\nu(t,x) = \varphi_\nu(t,z) e_1, \qquad \varphi_\nu(t,z) := \varphi\left(t, \frac{z}{\sqrt{\nu}}\right)$$

where φ (φ_1) and φ_ν solves

$$\begin{cases} \partial_t \varphi - \partial_{zz}\varphi = 0 \\ \varphi\big|_{z=0} = 0 \\ \lim_{z\to\infty} \varphi(t,z) = A \\ \varphi\big|_{t=0} = A \end{cases}, \qquad \begin{cases} \partial_t \varphi_\nu - \nu\partial_{zz}\varphi = 0 \\ \varphi_\nu\big|_{z=0} = 0 \\ \lim_{z\to\infty} \varphi_\nu(t,z) = A \\ \varphi_\nu\big|_{t=0} = A \end{cases}.$$

To verify incompressibility,

$$\operatorname{div} u_\nu = (\partial_1 \varphi_\nu) = 0.$$

Moreover, the nonlinear term vanishes,

$$(u_\nu \cdot \nabla) u_\nu = \varphi_\nu \partial_1 u_\nu = 0.$$

So u solves the Navier–Stokes equation with $\nabla p = 0$.

Remark 1.27. For $\nu << 1$, $u_\nu \sim u_E$ (solution to Euler).

Remark 1.28. In 2D, $\omega^0 = \mathrm{curl}(Ae_1) = 0$, so solution of Euler is irrotational for all time. For Navier–Stokes,

$$\omega(t,x,z)\cdot e_2 = \partial_3 u_1 - \partial_1 u_3 = \partial_z \varphi_\nu(t,z) = \frac{1}{\sqrt{\nu}}\partial_z\varphi_1\left(t, \frac{z}{\sqrt{\nu}}\right).$$

Especially at $z=0$: $\omega(t,x,0)\cdot e_2 = \partial_z\varphi_\nu(t,0) \neq 0$. This shows a production of vorticity at the boundary, then propagated inside the domain. In the Euler case, we can think there is a vorticity which stays a dirac on the boundary and does not diffuse into the domain.

Poten

1.6.1 D'Alembert Paradox (1752)

An object in an irrotational, incompressible, inviscid steady flow does **not** experience any drag force. This means: *birds (or planes) do NOT fly.*

Let $v =$ constant, let Ω be a smooth bounded domain, which is a solid moving with velocity v in a fluid. The fluid surrounding the object satisfies the Euler equation:

$$\begin{cases} (u-v)\cdot\nabla u + \nabla p = 0 & \text{in } \Omega^c \\ \operatorname{div} u = 0 & \text{in } \Omega^c \\ \operatorname{curl} u = 0 & \text{in } \Omega^c \text{ irrotational} \\ (u-v)\cdot n = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

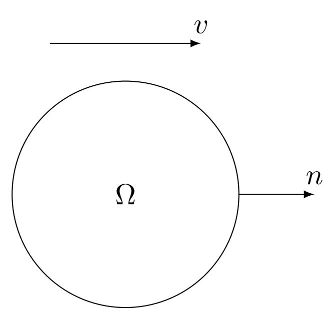


Figure 1.5: D'Alembert Paradox

Theorem 1.29. *There exists a solution to* (1.4) *and*

$$\int_{\partial\Omega} p\cdot n = 0$$

This means: no drag forces.

Idea of the proof:

$$
\begin{aligned}
\int_{\partial\Omega} p \cdot n_i \,\mathrm{d}\hat{x} &= -\int_{\Omega^c} \partial_i p \,\mathrm{d}x \\
&= \int_{\Omega^c} (u-v)\cdot \nabla u_i \,\mathrm{d}x \\
&= \int_{\Omega^c} \operatorname{div}((u-v)u_i)\,\mathrm{d}x \\
&= -\int_{\partial\Omega} u_i (u-v)\cdot n = 0.
\end{aligned}
$$

Let $\bar{u}$ be a solution to Euler equation to (1.4) satisfying boundary condition $(\bar{u}-v)\cdot n = 0$ on $\partial\Omega$. Let

$$
u(t,x) = \bar{u}(t, x - vt).
$$

That says Ω is stationary in the frame of v. Look for a stationary solution in this frame:

$$
\partial_t u = 0 \Rightarrow \partial_t \bar{u} - v \cdot \nabla_x \bar{u} = 0,
$$

So the potential of $\bar{u}$ solves

$$
\begin{cases} \Delta\varphi = 0 \\ \partial_n \varphi = v \cdot n \end{cases}
$$

satisfies the Neumann boundary condition. By Lax-Milgram, there exists a unique solution such that

$$
\int_{\Omega^c} |\nabla\varphi|^2 \,\mathrm{d}x \le C|v|^2.
$$

$$
\boxed{\text{Bernoulli equation: } \frac{|\nabla\varphi|^2}{2} - v\cdot\nabla\varphi + p = 0.}
$$

Lemma 1.30. *There exists a constant $C > 0$ such that*

$$
|\nabla\varphi(y)| \le \frac{C}{|y|^3}
$$

for $y \in \mathbb{R}^3 \setminus \Omega$.

Proof. **Step 1.** If $\Omega \subset B_R$, then

$$\int_{\partial B_R} \nabla\varphi \cdot n \,\mathrm{d}\hat{x} = 0.$$

To see this,

$$\begin{aligned} 0 = \iint_{B_R\setminus\Omega} \Delta\varphi \,\mathrm{d}x &= -\int_{\partial\Omega} \nabla\varphi \cdot n \,\mathrm{d}\hat{x} + \int_{\partial B_R} \nabla\varphi \cdot n \,\mathrm{d}\hat{x} \\ &= -\int_{\partial\Omega} v \cdot n \,\mathrm{d}\hat{x} + \int_{\partial B_R} \nabla\varphi \cdot n \,\mathrm{d}\hat{x}. \end{aligned}$$

Here we applied the boundary condition $\partial_n\varphi = v \cdot n$ where v is a constant,

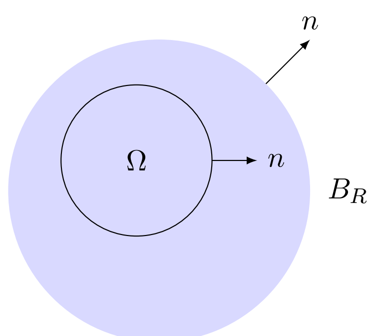


Figure 1.6: Find R large such that $\Omega \subset B_R$

and

$$\int_{\partial\Omega} v \cdot n \,\mathrm{d}\hat{x} = \int_{\Omega} \operatorname{div} v \,\mathrm{d}x = 0$$

by symmetry. So

$$\int_{\partial B_R} \nabla\varphi \cdot n \,\mathrm{d}\hat{x} = 0.$$

Step 2. Remember the fundamental solution:

$$-\Delta_x \frac{1}{|x-y|} = c_0\delta_{x=y}.$$

Consider $y \in \mathbb{R}^3 \setminus B_R$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3\setminus B_R} \Delta\varphi(x) \frac{1}{|x-y|} \,\mathrm{d}x \\ &= -c_0\varphi(y) - \int_{\partial B_R} \frac{\partial_n\varphi(\hat{x})}{|\hat{x}-y|} \,\mathrm{d}\hat{x} + \int_{\partial B_R} \partial_n\left(\frac{1}{|\hat{x}-y|}\right)\varphi(\hat{x}) \,\mathrm{d}\hat{x} \end{aligned}$$

for R fixed such that $\Omega \subset B_R$.

For the second integral, if $|y| \geq 2R$, then

$$\left|\partial_n \frac{1}{|\hat{x}-y|}\right| \leq \frac{1}{|\hat{x}-y|^2} \leq \frac{C}{||y|+R|^2}$$

and φ is smooth on ∂B_R.

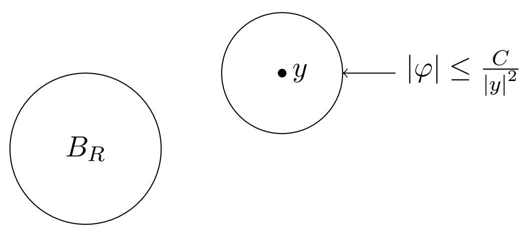
For the first integral, using Step 1,

$$\int_{\partial B_R} \frac{\partial_n \varphi(\hat{x})}{|y|} \, \mathrm{d}\hat{x} \equiv 0$$

So

$$\left|\int_{\partial B_R} \partial_n \varphi(\hat{x}) \left(\frac{1}{|\hat{x}-y|} - \frac{1}{|y|}\right) \mathrm{d}\hat{x}\right| \leq \frac{C_R}{|y|^2}.$$

So $|\varphi(y)| \leq \frac{C_R}{|y|^2}$.



Step 3. Since φ is harmonic, $\varphi(y) = \eta * \varphi(y)$ for any η radial with $\int \eta \equiv 1$ and $\operatorname{supp} \eta \subset B_R^c$. So

$$\nabla \varphi = \nabla \eta * \varphi(y).$$

Let η be

$$\eta(x) = \frac{1}{|y|^3} \eta_0\left(\frac{x}{|y|}\right).$$

with η_0 has integral 1 and $\operatorname{supp} \eta_0 \subset B_y$.

$$|\nabla \eta| \leq \frac{1}{|y|} \underbrace{\frac{1}{|y|^3} \nabla \eta\left(\frac{x}{|y|}\right)}_{\in L^1}$$

So

$$|\nabla\varphi(y)| \leq \|\nabla\eta\|_{L^1}\|\varphi\|_{L^\infty} \leq \frac{C}{|y|}\frac{1}{|y|^2} \leq \frac{C}{|y|^3}.$$

This proof can be found in Gilbarg-Trudinger, for higher derivative.
Step 4. Denote $\Omega_R = B_R \setminus \Omega$.

$$\begin{aligned}
\int_{\partial\Omega} p \cdot n \,\mathrm{d}\hat{x} &= -\int_{\partial\Omega} \left\{ \frac{|\nabla\varphi|^2}{2} - (v \cdot \nabla\varphi) \right\} n \,\mathrm{d}\hat{x} \\
&= -\int_{\Omega^c} \nabla\left(\nabla(v \cdot \nabla\varphi - \frac{|\nabla\varphi|^2}{2}) \right) \mathrm{d}x \\
&= -\int_{\Omega_R^c} v \cdot \nabla(\nabla\varphi) - \nabla\varphi \cdot \nabla^2\varphi \,\mathrm{d}x + \mathrm{Rest} \\
&= -\int_{\Omega_R^c} (v-u) \cdot \nabla u \,\mathrm{d}x + \mathrm{Rest} \\
&= -\int_{\Omega_R^c} \mathrm{div}(u \otimes (v-u)) \,\mathrm{d}x + \mathrm{Rest} \\
&= \int_{\partial\Omega} (v-u) \cdot nu \,\mathrm{d}\hat{x} + \mathrm{Rest} = 0 + \mathrm{Rest}.
\end{aligned}$$

Rest terms:

$$\int_{\partial B_R} \left\{ \frac{|\nabla\varphi|^2}{2} - v \cdot \varphi \right\} n \,\mathrm{d}x \approx R^2 \cdot (R^{-6} - R^{-3}) \to 0$$

as $R \to \infty$. □

Conclusion

- As $\nu << 1$, a Navier–Stokes flow is more and more like an Euler flow
- BUT the effect close to the boundary is of order 1
- Non trivial interaction between pressure, vorticity, viscosity, and the boundary

This is the end of Chapter 1: Introduction!

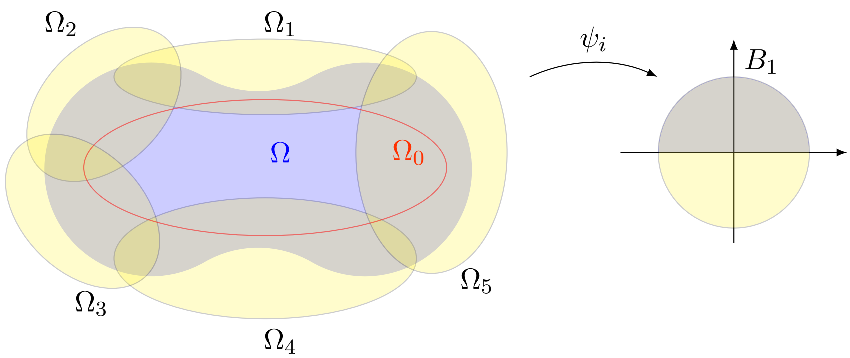
Chapter 2

Linear Stokes equation

2.1 Hodge decomposition in bounded domain

Let Ω be a bounded domain with $\partial\Omega$ Lipschitz, it means $\partial\Omega \subset \cup_{i=1}^{N} \Omega_i$:

- Ω_i open in $\mathbb{R}^3$.
- there exists $\Psi_i : \Omega_i \to B_1(0)$ Ψ_i is one to on and onto.
- Ψ and Ψ^{-1} are both Lipschitz.
- $\Psi_i(\partial\Omega \cap \Omega_i) = B_1(0) \cap \{z = 0\}$.
- $\psi_i(\Omega_i \cap \Omega) = B_1(0) \cap \{z > 0\}$



Especially the Dirichlet case

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = u^0 & \text{on } \partial\Omega \end{cases}$$

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \left(\|f\|_{H^{-1}(\Omega)} + \left\|u^0\right\|_{H^{1/2}(\partial\Omega)}\right)$$

$$\left\|u^0\right\|_{H^{1/2}(\partial\Omega)} = \inf_{v|_{\partial\Omega}=u^0} \|v\|_{H^1(\Omega)}$$

$$\begin{cases} -\Delta u = g & \text{in } \Omega \\ \partial_n u = h \end{cases} \tag{2.1}$$

The system need to satisfy the compatibility condition $\int_\Omega g dx = \int_{\partial\Omega} h d\hat{x}$ and $\|\nabla u\|_{L^2(\Omega)} \leqslant C\left(\|g\|_{(H^1(\Omega))^*} + \|h\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right)$ Define $\tilde{H} = \{u \in [L^2(\Omega)]^3; \operatorname{div} u \in L^2(\Omega)\}$ and $\|u\|^2_{\tilde{H}\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)}$

Proposition 2.1.
- *$\tilde{H}(\Omega)$ is a Banach space.*
- *$C^\infty(\bar{\Omega})$ dense in $\tilde{H}(\Omega)$.*
- *Trace operator $T_{\vec{n}} : u \to \vec{u} \cdot \vec{n}$ on $\partial\Omega$. And $T_{\vec{n}}$ is bounded from $\tilde{H}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$.*

Remark 2.2.
- $\forall \ \ \phi \in C^\infty(\overline{\Omega})$: $T_{an}\phi : \vec{\phi} \cdot \vec{n}$ on Ω. Then extension on $\tilde{H}(\Omega)$ is bounded, i.e. $\forall \phi \in C^\infty(\overline{\Omega})$:

$$\|T_{\vec{n}}\phi\|_{H^{\frac{1}{2}}(\Omega)} \leqslant C\,\|\phi\|_{\tilde{H}(\Omega)}$$

- We have a "full trace operator":

$$T: \quad H^1(\Omega) \to H^{\frac{1}{2}}(\Omega) \qquad Tu = u|_{\partial\Omega}$$

- The trace for $u \in L^2(\Omega)$ does NOT exist. $C^\infty(\overline{\Omega})$ are dense in $L^2(\Omega)$. Consider $f \equiv 1$ in Ω. If $f = 1$ on $\partial\Omega$, $f_m \in C^\infty$ $f_m \to f$ in L^2 while $Tf_m = 0$, $Tf = 1$.
- $(\nabla u)_{3\times3}$ is a matrix and we really have 1 component for $\operatorname{div} u = \partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3$ in L^2.

Proof. Assume $u \in [C^\infty(\overline{\Omega})]^3$, $v \in C^\infty(\overline{\Omega})$:

$$\iint_\Omega u \cdot \nabla v dx = -\iint_\Omega (\operatorname{div} u) v dx + \int_{\partial\Omega} v(u \cdot \vec{n}) dx$$

$$\begin{aligned}\left|\int v(u\cdot\overrightarrow{n})dx\right| &\leqslant \|u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)}\|\operatorname{div}u\|_{L^2(\Omega)}\\ &\leqslant \|u\|_{\tilde{H}(\Omega}\|v\|_{H^1(\Omega)}\\ &\leqslant \|u\|_{\tilde{H}(\Omega)}\|v|_{\partial\Omega}\|_{H^{\frac{1}{2}}(\Omega)}\end{aligned}$$

We cab extend this to $v \in H^1(\Omega)$ by density. $\forall\, \tilde{v} \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a extension an extension: $v \in H^1(\Omega)$ s.t. $v|_{\partial\Omega} = \tilde{v}\left|\langle(u,n);v\rangle_{H^{-1/2}(\partial\Omega)}, H^{3/2}(\partial\Omega)\right| \leqslant \|\overrightarrow{u}\cdot\overrightarrow{n}\|_{H^{-1/4}(\partial\Omega)}\cdot\|\tilde{v}\|_{H^{1/2}(\partial\Omega)}$ So $\|\overrightarrow{u}\cdot\overrightarrow{n}\|_{H^{-1/2}(\partial\Omega)} \leqslant \|u\|_{\tilde{H}(\partial\Omega)}$ □

Define $H = \{u \in [L^2(\Omega)]^3; \operatorname{div} u = 0 \quad \overrightarrow{u}\cdot\overrightarrow{n} = 0 \text{in } \partial\Omega\}$ and $\|u\|_H = \|u\|_{L^2(\Omega)}$, a natural space for solution of Euler.

Proposition 2.3. *H is closed in $\tilde{H}$ and so is a Banach space:*

Proof. Define $F:\ \tilde{H} \to L^2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$, $F(u) = (\operatorname{div} u; u\cdot\overrightarrow{n})$. Then F is continuous from $\tilde{H} \to L^2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ and so $F^{-1}(0,0) = H$ is closed. □

Let's define $G = \{\nabla P \in L^2(\Omega)\}, \|u\|_G = \|\nabla p\|_{L^2}$, then G is a Banach space.

Theorem 2.4. *(Hodge Decomposition) $H \oplus G = [L^2(\Omega)]^3$. So $\forall\ f \in [L^2(\Omega)]^3$ there exists a unique $u \in H(\Omega)$ $\nabla P \in G(\Omega)$ s.t. $f = u + \nabla P$ and $\forall\ u \in H(\Omega)$ and $\nabla P \in G$: $\int u\cdot\nabla P dx = 0$*

Proof.

- $\iint u\cdot\nabla P dx = -\iint P \operatorname{div} u + \int_{\partial\Omega} P(u\cdot\overrightarrow{n})d\hat{x}$, if $u \in H(\Omega)$.
- Consider $f \in [L^2(\Omega)]^3$ we construct ∇P first; and formally:

$$\begin{cases}\Delta = \operatorname{div} f\\ \partial_n P = \overrightarrow{f}\cdot\overrightarrow{n} \qquad \text{not defined}\end{cases}$$

first

$$\begin{cases}\Delta P = \operatorname{div} f & \text{in } \Omega\\ P = 0 & \text{on } \partial\Omega\end{cases} \tag{2.2}$$

where $\operatorname{div} f \in H^{-1}(\Omega)$ and $\langle v, \operatorname{div} f\rangle = \left|-\int_\Omega \nabla v f dx\right| \leqslant \|f\|_{L^2}\|\nabla v\|_{L^2} \leqslant \|f\|_{L^2}\|v\|_{H^1(\Omega)}$ for all $v \in H^1_0(\Omega)$ and $\|\operatorname{div} f\|_{H^{-1}(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$ and

$\exists P_1 \in H^1(\Omega)$: $\|\nabla P_1\|_{L^2} \leqslant C\|f\|_{L^2(\Omega)}$ and P_1 is a solution to (2.2). Formally, $P - P_1$ satisfies

$$\begin{cases} \Delta P = 0 & \text{in } \Omega \\ -\partial_n P_1 = (f - \gamma P_1) \cdot \overrightarrow{n} \end{cases}$$

where $\operatorname{div}(f - \nabla P_1) = 0$. Therefore $f - \nabla P_1 \in \tilde{H}(\Omega)$ and $\|(f - \nabla P_1)\overrightarrow{n}\|_{H^{1/2}(\Omega)} \leqslant \|f - \nabla P_1\|_{L^2} \leqslant C\|f\|_{L^2(\Omega)}$. We can construct such P_2: $\|\nabla P_2\|_{L^2(\Omega)} \leqslant C \left\| f - \nabla P_1 \right\|_{H^{-\frac{1}{2}}(\partial\Omega} \leqslant \tilde{C}\|f\|_{L^2(\Omega)}$.
Define $u = f - \nabla(P_1 + P_2) \in [L^2(\Omega)]^3$ $\operatorname{div} u = (\operatorname{div} f - \Delta P_1) - \Delta P_2 = 0$ on $\partial\Omega$: $u \cdot \overrightarrow{n} = (f - \nabla P_1) \cdot \overrightarrow{n} - \partial_n P_2 = 0$, so $u \in H(\Omega)$

□

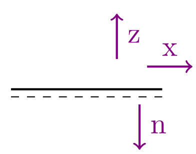
Remark 2.5. Euler: $u \in H^1(\Omega) \cap L^\infty(\Omega)$

$$\begin{cases} \partial_t u + \nabla P = f = -u \cdot \nabla u \\ \operatorname{div} u = 0 \\ u \cdot \overrightarrow{n} = 0 \text{ on } \partial\Omega \end{cases}$$

Then $\mathbb{P}_H$ of the equation satisfies

1. $\mathbb{P}_H u = u$
2. ∂_t and $\mathbb{P}_H$ commute
3. $\partial_t u = \mathbb{P}_H f$ and $\partial_t u + \mathbb{P}_H(u - \nabla u) = 0$

Let $\Omega = \mathbb{R}3$ $\Delta P = -\sum_{ij} \partial_i u_j \partial_j u_i = -\operatorname{div}\operatorname{div}(u \otimes u)$ and $\partial_n P = -u_1\partial_1 u_3 - u_3\partial_3 u_3 = 0$ due to $u_1\partial_1 u_3 = u_3 = 0$, while $\partial_3 u_3 \neq 0$



If we consider Navier-Stokes now:

$$\partial_t u + \nabla P - \Delta u = f = -(u \cdot \nabla u)$$

$\mathbb{P}_H$: $\partial_t u - \mathbb{P}_H(\Delta u) = \mathbb{P}_H f = -\mathbb{P}_H(u \cdot \nabla u)$ where $\mathbb{P}_H$ doesn't commute with Δ.

$$\begin{aligned} (\Delta u) \cdot \overrightarrow{n} &= \partial_{11} u_3 + \partial_{22} u_3 + \partial_3 3 u_3 = \partial_3(\partial_3 u_3) = -\partial_3(\partial_1 u_1) \operatorname{div} u = 0 \\ &= \partial_1(-\partial_3 u_1) = \partial_1(\partial_1 u_3 - \partial_3 u) = \partial_1 w \end{aligned}$$

where $\partial_1 u_3 = 0$ and w is the curl. $\mathbb{P}$ is defined through the linear Stokes equation:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u \cdot \overrightarrow{n} = 0 \end{cases}$$

2.2 More of vectorial differential identities

Lemma 2.6. *1. For any $a, b, c \in \mathbb{R}^3$,*

$$a \times (b \times c) = (b \otimes c - c \otimes b)a = (a \cdot c)b - (a \cdot b)c.$$

2. For $v \in C^\infty(\mathbb{R}^3)$, $\phi \in \left[C^\infty(\mathbb{R}^3)\right]^3$,

$$\operatorname{curl}(v\phi) = v \operatorname{curl} \phi + \nabla v \times \phi.$$

3. For $\phi, \psi \in \left[C^\infty(\mathbb{R}^3)\right]^3$,

$$\operatorname{div}(\phi \times \psi) = \psi \operatorname{curl} \phi - \phi \operatorname{curl} \psi.$$

Proof. 1. Since $a \times (b \times c) \perp (b \times c)$, while $b \times c \perp b$ and c, so

$$a \times (b \times c) = \lambda(a \cdot c)b + \mu(a \cdot b)c$$

for some constants $\lambda, \mu \in \mathbb{R}$, unless in the degenerate case b and c are colinear in which case the identity is trivially zero.

Take $a = e_1$, $b = e_1$, $c = e_2$:

$$\begin{cases} a \times (b \times c) = e_1 \times e_3 = -e_2, \\ (a \cdot c)b = 0, (a \cdot b)c = e_2 \end{cases} \quad \Rightarrow \mu = -1.$$

Take $a = e_2$, $b = e_1$, $c = e_2$:

$$\begin{cases} a \times (b \times c) = e_2 \times e_3 = e_1, \\ (a \cdot c)b = e_1, (a \cdot b)c = 0 \end{cases} \quad \Rightarrow \lambda = 1.$$

2. Take any fixed $w \in \mathbb{R}^3$,

$$\begin{aligned} \operatorname{curl}(v\phi) \times w &= \left[\nabla(v\phi) - \nabla(v\phi)^\top\right] w \\ &= v(\nabla\phi - \nabla\phi^\top)w + (\phi \otimes \nabla v - \nabla v \otimes \phi)w \\ &= v \operatorname{curl} \phi \times w + \phi(\nabla v \cdot w) - \nabla v(\phi \cdot w) \\ &= v \operatorname{curl} \phi \times w + w \times (\phi \times \nabla v) \\ &= (v \operatorname{curl} \phi + \nabla v \times \phi) \times w. \end{aligned}$$

3. Using duality again, take $v \in C_c^\infty(\mathbb{R}^3)$, integration by part twice,

$$\begin{aligned}
\int_{\mathbb{R}^3} \operatorname{div}(\phi \times \psi) v \, \mathrm{d}x &= -\int \nabla v \cdot (\phi \times \psi) \, \mathrm{d}x \\
&= \int \phi \cdot (\nabla v \times \psi) \, \mathrm{d}x \\
&= \int \phi \cdot (\operatorname{curl}(v\psi) - v \operatorname{curl} \psi) \, \mathrm{d}x \\
&= \int v(\psi \cdot \operatorname{curl} \phi - \phi \operatorname{curl} \psi) \, \mathrm{d}x.
\end{aligned}$$

This is true for any v, so

$$\operatorname{div}(\phi \times \psi) = \psi \cdot \operatorname{curl} \phi - \phi \cdot \operatorname{curl} \psi.$$

□

2.3 Solving divergence equation

Theorem 2.7. *Let Ω be bounded and $\partial\Omega$ is a $C^{1,1}$ boundary. For any $1 < p < \infty$, there exists a constant, depending on Ω and p, such that for any $b \in L^p(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ with*

$$\operatorname{div} u = b \textit{ in } \Omega.$$

and

$$\|\nabla u\|_{L^p(\Omega)} \leq C \|b\|_{L^p(\Omega)}.$$

Remark 2.8. The u is not unique: you can add $\tilde{u}$ such that $\operatorname{div} \tilde{u} = 0$ and $\tilde{u} \in C_c^\infty(\Omega)$. For instance, if Ω is axisymmetric, denote $r^2 = x^2 + y^2$, you can take the rotation flow

$$\tilde{u}(x, y, z) = \phi(r, z)(-y, x, 0).$$

i.e.

$$\tilde{u}(\boldsymbol{x}, z) = (\phi(|\boldsymbol{x}|, z)\boldsymbol{x}^\perp, 0).$$

We have

$$\operatorname{div}(\tilde{u}) = \phi \operatorname{div}(\boldsymbol{x}^\perp) + \nabla\phi \cdot \boldsymbol{x}^\perp = 0 + \phi_r \frac{\boldsymbol{x}}{r} \cdot \boldsymbol{x}^\perp = 0.$$

Proof of Theorem 2.7. Consider

$$\begin{cases} -\Delta P = b & \text{in } \Omega \\ \partial_n P = 0 & \text{on } \partial\Omega \end{cases}$$

It exists a solution by Lax–Milgram. And the regularity theory shows that

$$\left\|\nabla^2 P\right\|_{L^p(\Omega)} \le C_{\Omega,p}\|b\|_{L^p(\Omega)}.$$

If we write $u_1 = \nabla P$, then $\operatorname{div} u_1 = b$. However, on the boundary, we only have $u_1 \cdot n = 0$, but we want $u = 0$ on $\partial\Omega$.

We need to correct the boundary by a correction u_2 satisfying:

$$\begin{cases} \operatorname{div} u_2 = 0 & \text{in } \Omega \\ u_2 = (\nabla P - (\nabla_n P)n) = n \times (\nabla P \times n) & \text{on } \Omega \end{cases}$$

The correction of the boundary has to go through the charts, but perturbs the divergence structure.

We are looking for $u_2 = \operatorname{curl} A$.

Lemma 2.9. *Fix Ω bounded with $C^{1,1}$ boundary. There exists $\varphi \in C^{1,1}(\Omega)$ such that*

- *$\nabla\varphi = n$ on $\partial\Omega$.*
- *$|\varphi(x)| \le Cd(x, \partial\Omega)$ in Ω.*

Proof. Let $\{\Omega_i\}_i$ be a covering of the boundary. For any i, ψ_i, ψ_i^{-1} are C^1 and $\nabla\psi_i, \nabla\psi_i^{-1}$ are Lipschitz. Moreover $-\partial_3\psi_i = n$.

Take $\Omega_0 \subset\subset \Omega$ such that

$$\Omega \subset \bigcup_{i=0}^{N} \Omega_i$$

and partition of unity: $\varphi_i \in C_c^\infty(\Omega_i)$, and $\sum_{i=0}^N \varphi_i(x) \equiv 1$.

For $x \in \Omega$, define

$$\varphi(x) = \sum_{i=1}^{N} \varphi_i(x)(\psi_i)_3(x).$$

Then

$$\nabla\varphi(x) = \sum_{i=1}^{N} \nabla\varphi_i(x)(\psi_i)_3(x) + \sum_{i=1}^{N} \varphi_i(x)\nabla(\psi_i)_3(x).$$

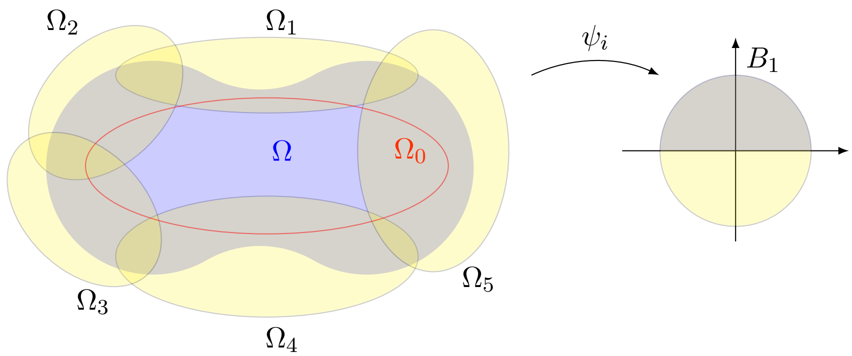


Figure 2.1: Partition of unity

On the boundary, $(\varphi_i)_3 = 0$, and $\nabla(\psi_i)_3 = n$, so

$$\nabla\varphi(x) = \left(\sum_{i=1}^{N} \varphi_i(x)\right) n(x) = n(x).$$

□

Remark 2.10. If we take $\varphi(x) = d(x, \partial\Omega)$, then $\nabla\varphi(x) = n$ on $\partial\Omega$. But even if $\partial\Omega \in C^\infty$, $d(x, \partial\Omega)$ is not more than Lipschitz. For example, $\Omega = \mathbb{R}^d \setminus B_1$, then d is merely Lipschitz at the origin.

Introduce $V : \Omega \to \mathbb{R}^d$ which solves the elliptic system

$$\begin{cases} -\Delta V = 0 & \text{in } \Omega \\ V = \nabla P \times n & \text{on } \partial\Omega \end{cases}$$

Note that $\nabla P \times n$ is the trace of $W^{2,p}(\Omega)$.

Set $A = \varphi V$ and $u_2 = \operatorname{curl} A = \operatorname{curl}(\varphi V)$. Then

- $\operatorname{div} u_2 = 0$.
- $u_2 = \varphi \operatorname{curl} V + \nabla\varphi \times V = 0 + n$, because $|\varphi| \leq d(x, \partial\Omega)$, so $\varphi \equiv 0$ on $\partial\Omega$.

This finishes the construction of u_2 and the solution of $\operatorname{div} u = b$. □

♣ ♣ ♣

Using theorem we have proved previously. Let's consider the following systems: Neumann

$$\begin{cases} \Delta P = b & \text{in } \Omega \\ \partial_n P = 0 & \text{on } \Omega \end{cases}$$

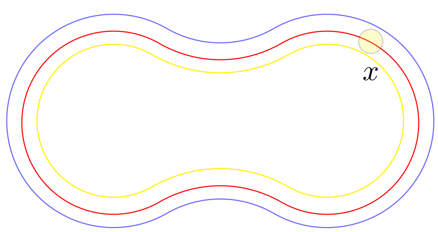
where $\nabla P|_{\partial\Omega} = \overrightarrow{n} \times (\nabla P|_{\partial\Omega} \times \overrightarrow{n})(= \nabla P - \partial_n P \overrightarrow{n})$.
Dirichlet:

$$\begin{cases} \Delta F = 0 & \text{in } \Omega \\ F|_{\partial\Omega} = (-\nabla P \times \overrightarrow{n}) & \text{on } \partial\Omega \end{cases}$$

- $\operatorname{div} u = b$
- $u = \nabla P + \phi \operatorname{curl} F + \nabla\phi \times F$ on $\partial\Omega$: $u|_{\partial\Omega} = \nabla P + \overrightarrow{n} \times F = \nabla P - \overrightarrow{n} \times (\nabla P \times \overrightarrow{n}) = 0$ since $\phi = 0$ on $\partial\Omega$, $\nabla\phi = \overrightarrow{n}$ in $\partial\Omega$.
- By elliptic regularity $\nabla P \in W^{1,p}(\Omega)$
- $\nabla P \in W^{1-1/p,p}(\partial\Omega)$ on the boundary: by elliptic regularity: $F \in W^{1,p}$. but $\partial_i u = \partial\nabla P + \partial\phi \operatorname{curl} F + \phi\partial_i \operatorname{curl} F + \partial_i \nabla\phi \times F + \nabla \times \partial_i F$. but $\partial_i u = \partial_i \nabla P$
- $\partial_i u$ is bounded in $L^p(\Omega)$ if $\phi\partial_i \operatorname{curl} F$ is.

$$\Delta F = 0 \text{ in } \Omega :$$

Consider $\eta \in C_c^\infty(B_{1/2})$ such that $\int \eta dx = 1$ and radial. $\eta_r = \frac{1}{r^3}\eta(\frac{x}{r})$ and $\operatorname{supp} \eta_r \in B_{r/2}$ for all x: $d(x;\partial) > 2r$ and $\phi(x) = \operatorname{curl} F * \eta_r(x)$.



$$\varphi|\partial_i \operatorname{curl} F(x)| = \leqslant \frac{\varphi}{r}[\operatorname{curl} F * \frac{1}{r^3}\partial_i\eta(\frac{\cdot}{r})](x) \leqslant C((\operatorname{curl} F * (\frac{1}{r^3}\partial_i\eta(\frac{\cdot}{r}))(x)$$

$$I_n = \int_{2^{-n} \leqslant d(x,\partial\Omega) \leqslant 2^{-n+1}} |\varphi(\partial_i \operatorname{curl} F)|^p dx$$

and

$$J_n = \int_{2^{-n} \leqslant d(x,\partial\Omega) \leqslant 2^{-n+1}} |\operatorname{curl} F|^p dx$$

$$I_n \leqslant C \int_{2^{-n-1} \leqslant d(x,\partial\Omega) \leqslant 2^{-n+2}} |\operatorname{curl} F|^p dx \leqslant J_{n-1} + J_n + J_{n+1}$$

Therefore

$$\begin{aligned}\int_\Omega |\phi \partial_r \operatorname{curl} F|^p dx &= C \sum_n I_n \\ &\leqslant \sum_n (J_{n-1} + J_n + J_{n+1}) \\ &\leqslant 3 \sum_n J_n \\ &\leqslant \tilde{C} \int_\Omega |\operatorname{curl} F|^p dx\end{aligned}$$

Therefore, $\phi \partial_i \operatorname{curl} F$ is bounded in $L^p(\Omega)$.
So $\exists C > 0 : \|\nabla u\|_{L^p} \leqslant C \|b\|_{L^p}$

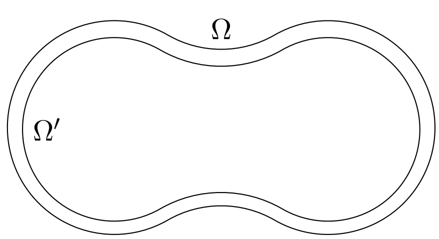
2.4 Gradient distribution

Theorem 2.11. *Ω bounded $C^{1,1}$ $\partial\Omega$. Consider $p \in \mathcal{D}'(\Omega)$ such that: $\nabla p \in W^{-1,p}(\Omega)$ for $p \in (1, \infty)$. Then there exist a $P \in L^p(\Omega)$ such that $\forall u \in W_0^{1,p}(\Omega) : \langle p, u \rangle = -\int_\Omega (\operatorname{div} u) P dx$.*

Remark 2.12. means $\nabla p := \nabla P$, i.e. $p = P$ up to a constant.

Proof. $q \in L^{p^*}(\Omega')$ $\Omega' \subset\subset \Omega$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Solve in Ω': $\operatorname{div} u = q$ $\exists u \in W^{1,p^*}(\Omega')$: $\|\nabla u\|_{L^\infty} \leqslant \|q\|_{L^{p^*}}$ where $d(x, \partial\Omega) \geqslant r > 0$. for $x \in \Omega'$: $q_r = q * \eta_r$ and $u_r = u * \eta_r$, $q_r \in C_c^\infty(\Omega)$ $u_r \in C_c^\infty(\Omega)$ and $\operatorname{div} u_r = q_r$.

$$C \|\nabla P\|_{W^{-1,p}(\Omega)} \|q\|_{L^{p^*}(\Omega)} |\langle \nabla p; u_r \rangle| = |-\langle p; \operatorname{div} u_r \rangle| = |-\langle p; q_r \rangle|$$



So $|\langle p; q_r \rangle| \leqslant C \|q_r\|_{L^{p^*}(\Omega)}$. So for a time set of $\{a \in L^{p^*}(\Omega); \int q dx = 0\}$ we have $|\langle p; q \rangle| \leqslant C \|q\|_{L^{p^*}(\Omega)}$ $p \in$ dual for $L^{p^*}(\Omega) := L^p(\Omega)$. (Riesz). then

$\exists P \in L^p(\Omega)$. such that $\{p, q\} = \int Pqdx\ \forall q : \int qdx = 0$ Especially if $q = \operatorname{div} u$; $u \in W_0^{1,p}(\Omega)$ then: $-\langle \nabla p; u\rangle = \langle p; \operatorname{div} u\rangle = \int P \operatorname{div} udx \in L^p(\Omega)$

□

Define $J^{1,p}(\Omega) = \{u \in [W_0^{1,p}(\Omega)]^3; \operatorname{div} u = 0\}$.

Theorem 2.13. $J^{1,p}(\Omega) := \overline{\{u \in [C_c^\infty(\Omega)]^3; \operatorname{div} u = 0\}}$ *in* $J^{1,p}(\Omega)$.

Remark 2.14. In applied math 2: $C_c^\infty(\Omega)$ we dense? for $W_0^{1,p}(\Omega)$ by approximation method. $\operatorname{div} u = 0$ is a strong contrast. We show theorem via abstract settings.

Theorem 2.15. $l \in W^{-1,p}(\Omega)$ *satisfying:*

- $\forall v \in W^{1,p^*}(\Omega)$
$$l(v) \leqslant \|\nabla v\|_{L^{p^*}}$$
- $\forall v \in J^{1,p^*}(\Omega)$:
$$l(v) = 0$$

Then there exists $P \in L^p(\Omega)$*:* $\langle; r\rangle = \int P \operatorname{div} v$ *for* $\forall\ v \in W_0^{1,p^*}(\Omega)$ *and* $\|P\|_{L^p(\Omega)} \leqslant C\,\|p\|_{W^{-1,p}(\Omega)}$

Proof. For all $p \in L^{p^*}(\Omega)$: we define $G(p)$ in the following way: Consider $u \in W^{1,p^*}(\Omega)$ such that $\operatorname{div} u = P$ and define $G(P) = (l, u)$.

- $u^1, u^2 \in W_0^{1,p^*}(\Omega)$ satisfying $\operatorname{div} u_1 = \operatorname{div} u_2 = P$. $u_1 - u_2 \in J^{1,p^*}$ $l(u_1 - u_2) = 0$ or $l(a_1) = l(a_2) = P$ So G does not depend on the representation of u.
- $|G(P)| \leqslant |(l, u)| \leqslant C\,\|\nabla u\|_{L^{p^*}} \leqslant C\,\|P\|_{L^{p^*}}$ by Riesz theorem $\exists q \in L^p(\Omega)$ such that $\forall\ P \in L^{p^*} : \int Pdx \equiv 0$.
$$G(p) = \int pq$$

Consider $u \in W^{1,p^*}(\Omega)$:

$$G(\operatorname{div} u) = (\rho, u) = \int \operatorname{div} uqdx$$

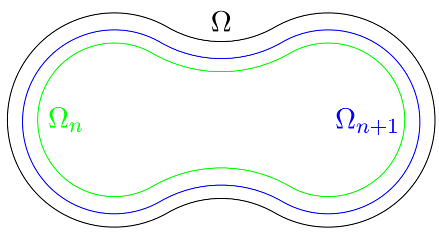
□

Next step consider

$$\mathring{J}^{1,p}(\Omega) = \overline{\{u \in [C_c^\infty(\Omega)]^3; \operatorname{div} u = 0\}} \text{ in } J^{1,p}(\Omega)$$

The theorem is still true replacing $J^{1,p}(\Omega)$ by $\mathring{J}^{1,p}$ We work on $\Omega' \subset\subset \Omega$, l still verifies 1 and 2 on $W^{1,p^*}(\Omega')$.
But $\forall\ v \in J^{1,p}(\Omega')$, $v^*\eta_r \in \mathring{J}^{1,p}(\Omega)$ for r small enough.

So $0 \in \mathring{J}^{1,p}$. Then $\forall v \in J^{1,p}(\Omega') : \{l, v\} = 0$. From the previous theorem, $\exists P_n$, such that $l(v) = \int P_{\Omega'} \operatorname{div} v$ for all $v \in W_0^{1,p}(\Omega')$.
Take Ω_n: $\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega \quad \cup_n \Omega_n = \Omega$. $P_{\Omega_{n+1}} = P_{\Omega_n} + c_n$



$$\nabla P_{\Omega_n} \equiv \text{const, for all } n.$$

So $l(v) = \int P \operatorname{div} v dx$ for the same function P.

♣ ♣ ♣

Recall that Ω is a $C^{1,1}$ bounded domain, define

$$J^{1,s}(\Omega) = \left\{u \in \left[W_0^{1,s}(\Omega)\right]^3 : \operatorname{div} u = 0\right\},$$
$$\mathring{J}^{1,s}(\Omega) = \overline{\{u \in [C_c^\infty(\Omega)]^3 : \operatorname{div} u = 0\}}^{J^{1,s}(\Omega)}.$$

Theorem 2.16. *For all $\ell \in W^{-1,p^*}(\Omega)$ with $\frac{1}{p} + \frac{1}{p^*} = 1$ such that*

$$\langle \ell, \varphi \rangle = 0, \qquad \forall \varphi \in \mathring{J}^{1,s}(\Omega).$$

we can find $p \in L^{p^*}(\Omega)$ such that $\ell = \nabla p$. That is,

$$\langle \ell, \varphi \rangle = -\int_\Omega p \operatorname{div} \varphi \, dx, \qquad \forall \varphi \in \left[W_0^{1,p}(\Omega)\right]^3.$$

Theorem 2.17. $J^{1,s}(\Omega) = \mathring{J}^{1,s}(\Omega)$.

Proof. Assume that it is not the case. Then there exists $v^* \in J^{1,s}(\Omega)$ but not in $\mathring{J}^{1,s}(\Omega)$. By Hahn–Banach, there exists $\ell \in W^{-1,p^*}(\Omega)$ such that $\langle \ell, v^* \rangle = 1$ and $\langle \ell, v \rangle = 0$ for all $v \in \mathring{J}^{1,s}(\Omega)$. By Theorem 2.17, ℓ is defined by

$$\langle \ell, v \rangle = -\int_\Omega p \operatorname{div} v \, \mathrm{d}x, \qquad \forall v \in W_0^{1,s}(\Omega)$$

for a $p \in L^{s*}(\Omega)$. So:

$$\langle \ell, v^* \rangle = -\int_\Omega p \operatorname{div} v^* \, \mathrm{d}x = 0$$

because $v^* \in J^{1,s}(\Omega)$ so $\operatorname{div} v^* = 0$. Hence contradiction. $\square$

Definition 2.18. Define $V = J^{1,2}(\Omega)$, i.e.

$$V = \{u \in H_0^1(\Omega) : \operatorname{div} u = 0\}.$$

Define

$$H = \{u \in L^2(\Omega) : \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Proposition 2.19. *$H = \mathring{H}$. This means that $[C_c^\infty(\Omega)]^3 \cap H$ is dense in H.*

Proof. Recall $G(\Omega) = \{\nabla p \in L^2(\Omega)\}$.

Step 1: $\mathring{H} \perp G(\Omega)$ in $L^2(\Omega)$. Take $v \in C_c^\infty(\Omega)$ with $\operatorname{div} v = 0$,

$$\int \nabla p \cdot v \, \mathrm{d}x = -\int p \operatorname{div} v \, \mathrm{d}x = 0.$$

By density: can be extended to any $v \in \mathring{H}(\Omega)$.

Step 2: $G(\Omega) = \mathring{H}^\perp$ in $L^2(\Omega)$. Take $u \in [L^2(\Omega)]^3$, $u \in \mathring{H}^\perp$. We define $\ell \in H^{-1}(\Omega)$ by

$$\langle \ell, v \rangle = \int_\Omega u(x) v(x) \, \mathrm{d}x, \qquad \forall v \in H_0^1(\Omega).$$

Since $u \in \mathring{H}^\perp$, we have

$$\langle \ell, v \rangle = \int u(x) v(x) \, \mathrm{d}x = \langle u, v \rangle_{L^2(\Omega)} = 0, \qquad \forall v \in \mathring{H}.$$

By Theorem 2.16, there exists $p \in L^2(\Omega)$, such that $u = \nabla p \in L^2(\Omega)$. So $u \in G(\Omega)$.

Step 3: We proved $[L^2(\Omega)]^3 = G(\Omega) \oplus \mathring{H}$. Becuase $[L^2(\Omega)]^3 = G(\Omega) \oplus H$ and $H \supset \mathring{H}$, it must hold that $H = \mathring{H}$.

$\square$

2.5 Existence for the steady Stokes equation

Theorem 2.20. *Take Ω bounded with $C^{1,1}$ boundary $\partial\Omega$. For any $f \in H^{-1}(\Omega)$, there exists $u \in V(\Omega)$ and $p \in L^2(\Omega)$ such that*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{2.3}$$

Moreover,

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}$$

with C depending only in Ω.

Remark 2.21. This is the steady stokes equation. Variational form of (2.3): for all $v \in V$:

$$\int_\Omega \nabla v : \nabla u \,\mathrm{d}x = \sum_{ij} \int_\Omega \partial_i v_j \partial_i u_j \,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}. \tag{2.4}$$

Proof. Let us prove that (2.3) is equivalent to (2.4). If (u,p) is a solution to (2.3) in sense of distribution: $\forall \phi \in [C_c^\infty(\Omega)]^3$ with $\operatorname{div}\phi = 0$,

$$\int_\Omega \sum_{ij} \partial_i \phi_j \partial_i u_j \,\mathrm{d}x + \underbrace{\int_\Omega \nabla p \cdot \phi \,\mathrm{d}x}_{-\int p \operatorname{div}\phi = 0} = \langle f, \phi\rangle.$$

By density, for all $v \in V$,

$$\int_\Omega \nabla v : \nabla u \,\mathrm{d}x = \langle f, v\rangle.$$

To be more precise, there exists $\phi_n \to v$ in V with $\phi_n \in C_c^\infty(\Omega)$, and both $\phi \mapsto \int \nabla\phi : \nabla u$ and $\phi \mapsto \langle f, \phi\rangle$ are continuous in V (with $H_0^1(\Omega)$ norm).

Now consider $u \in V$ solution to (2.4). Then

$$f + \Delta u \in H^{-1}(\Omega).$$

Moreover, for any $v \in H_0^1(\Omega)$,

$$\langle f + \Delta u, v\rangle = \langle f, v\rangle - \int_\Omega \nabla u : \nabla v \,\mathrm{d}x = 0$$

because u is a solution to (2.4). So by Theorem 2.15, $f + \Delta u = \nabla p$ with $p \in L^2(\Omega)$ and

$$\|p\|_{L^2(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}.$$

Moreover, $u \in V$ means $u \in H_0^1(\Omega)$ and $\operatorname{div} u = 0$.

Next we show existence and uniqueness of u in (2.4). This is by Lax–Milgram theorem: define

$$\begin{aligned} B(u,v) &= \int \nabla u : \nabla v \,\mathrm{d}x \qquad \forall u, v \in V(\Omega), \\ F(v) &= \langle f, v\rangle. \end{aligned}$$

F is a linear continuous functional in V. B is a bilinear form that is continuous, and

$$B(u,u) = \int \nabla u : \nabla u \,\mathrm{d}x \geq \|u\|^2_{H_0^1(\Omega)}$$

by Poincaré theorem. So

$$\begin{aligned} \|u\|_{V(\Omega)} &\leq C\|f\|_{H^{-1}(\Omega)} \\ \|p\|_{L^2(\Omega)} &\leq C\|f\|_{H^{-1}(\Omega)}. \end{aligned}$$

□

2.6 Higher regularity

2.6.1 Interior result

Theorem 2.22. *Consider Ω bounded, $\Omega' \subset\subset \Omega$, $f \in L^2(\Omega)$. If $p \in L^2(\Omega)$ and $u \in H^1(\Omega)$ with* $\operatorname{div} u = 0$, *$g \in H^1(\Omega)$ are solution to*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \end{cases}.$$

then: $u \in H^2(\Omega')$, $p \in H^1(\Omega')$, and

$$\|u\|_{H^2(\Omega')} + \|p\|_{H^1(\Omega')} \leq C\Big(\|f\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\Big).$$

with C depending on Ω and Ω'.

$$\operatorname{div}(u \circ \psi) = \sum_i \partial_i (u_i \circ \psi) = \sum_i \nabla u_i \circ \psi \cdot \partial_i \psi.$$

Proof. For all $v \in V$, $h = (h_1, h_2, h_3)$, define

$$D_h v(x) = \frac{v(x+h) - v(h)}{|h|}$$

This will be well-defined for $|h| \leq d(\Omega', \partial\Omega)$. □

Lemma 2.23. *Consider $\Omega' \subset\subset \Omega$.*

1. *For any $v \in H^1(\Omega)$,*

$$\|D_h v\|_{L^2(\Omega')} \leq \|\nabla v\|_{L^2(\Omega)}$$

2. *For any $v \in L^2(\Omega)$,*

$$\|D_h v\|_{H^{-1}(\Omega')} \leq C\|v\|_{L^2(\Omega)}$$

3. *If for any $|h| \leq d(\Omega', \partial\Omega)$,*

$$\|D_h v\|_{L^2(\Omega')} \leq C^*$$

then $\nabla v \in L^2(\Omega')$ and

$$\|Dv\|_{L^2(\Omega')} \leq C^*.$$

Proof. Note that for $v \in C_c^\infty(\Omega)$,

$$D_h v(x) = \int_0^1 \nabla v(x+th) \cdot \frac{h}{|h|} \,\mathrm{d}t \leq \left(\int_0^1 |\nabla v(x+th)|^2 \right)^{\frac{1}{2}}.$$

This can be verified by define $f(t) = v(x+th)$, $f'(t) = \nabla v(x+th) \cdot h$, then

$$\begin{aligned} \int_0^1 \nabla v(x+th) \cdot \frac{h}{|h|} \,\mathrm{d}t &= \frac{1}{|h|} \int_0^1 f'(t) \,\mathrm{d}t \\ &= \frac{1}{|h|}(f(1) - f(0)) \\ &= \frac{v(x+h) - v(x)}{h}. \end{aligned}$$

We can also take $\nabla v \in L^2(\Omega)$, and the inequality is still true a.e. by density. Now we integrate,

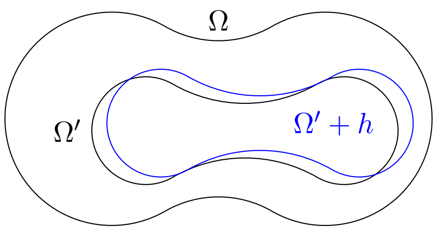
$$\begin{aligned}\int_{\Omega'} |D_h v(v)|^2 \,\mathrm{d}x &\leq \int_{\Omega'} \int_0^1 |\nabla v(x+th)|^2 \,\mathrm{d}t \,\mathrm{d}x \\ &\leq_{\text{ by Fubini }} \int_0^1 \int_{\Omega'} |\nabla v(x+th)|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &\leq \int_0^1 \int_{\Omega} |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}t = \int_{\Omega} |\nabla v|^2 \,\mathrm{d}x.\end{aligned}$$

□

♣ ♣ ♣

Let $e = (e_1, e_2, e_3)$, $|e| = 1$ and $D_h u(x) = \dfrac{u(x+he) - u(x)}{h}$, $\Omega' \subset\subset \Omega$.

Remark 2.24.
- If $u \in H^s(\Omega)$ then $D_h u \in H^s(\Omega)$ if $|h| < d(\Omega'; \partial\Omega)$



- $D_h(uv)(x) = vD_h u + u_h D_h v$ where $v_h(x) = v(x+he)$

$$\begin{aligned}D_h(uv)(x) &= \frac{u(x+he)v(x+he) - u(x)v(x)}{h} \\ &= v(x)[\frac{u(x+he) - u(x)}{h}] + u(x+h)[\frac{v(x+he) - v(x)}{h}]\end{aligned}$$

- $u, v \in C_c^\infty(\Omega')$.

$$\begin{aligned}
\int_\Omega D_h uv dx &= -\int_\Omega u D_{-h} v dx \\
&= \int \frac{u(x+he)-u(x)}{h} v(x) dx \\
&= \frac{1}{h}\int_\Omega u(x+he)v(x) - \frac{1}{h}\int u(x)v(x)dx \qquad x \sim x+he \\
&= \frac{-1}{-h}\int_\Omega u(x)v(x-he)dx - \frac{1}{h}\int u(x)v(x)dx \\
&= -\int_\Omega u(x)\frac{v(x-he)-v(x)}{-h} \\
&= -\int_\Omega u D_{-h} v dx
\end{aligned}$$

Lemma 2.25. *$\Omega' \subset\subset \Omega$: $|h| < d(\Omega'; \partial\Omega)$. $\forall u \in H^1(\Omega)$:*

1. *$\|D_h u\|_{L^2(\Omega')} \leqslant \|\nabla u\|_{L^2(\Omega)}$.*
2. *$\forall\ u \in L^2(\Omega) : \|D_h u\|_{H^{-1}(\Omega')} \leqslant \|u\|_{L^2(\Omega)}$.*
3. *If $\|D_h u\|_{L^2(\Omega')} \leqslant C$ for $|h| < d(\Omega'; \partial\Omega)$ then $u \in H^1(\Omega')$ and $\|\nabla u\|_{L^2} \leqslant C$.*

Proof. (2) $\forall\ \varphi \in C_c^\infty(\Omega')$,

$$\begin{aligned}
|\langle D_h u; \varphi\rangle| &= \left|\int_\Omega D_h u(x)\varphi(x)dx\right| \\
&= \left|-\int_\Omega u(x) D_h\varphi(x)dx\right| \\
&\leqslant \|u\|_{L^2(\Omega)} \|D_h\varphi\|_{L^2(\Omega)} \\
&\leqslant \|u\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\leqslant \|u\|_{L^2(\Omega)} \|\varphi\|_{H_0^1(\Omega')}
\end{aligned}$$

by density it is true for any $\varphi \in H_0^1(\Omega')$ $\|D_h u\|_{H^{-1}(\Omega')} \leqslant \|u\|_{L^2(\Omega)}$.

(3)Assume $\|D_h u\|_{L^2(\Omega)} \leqslant C$ for $h << 1$. Up to a subsequence $D_h u \to v$ weakly in L^2, so $D_h u \to v$ weakly in $\mathcal{D}'$.

$\forall\, \varphi \in C_c^\infty(\Omega') : D_{-h}\varphi(x) \to \nabla\phi(x)$ point-wise

$$\begin{aligned}\langle D_h u; \varphi\rangle &= -\langle u; D_h\varphi\rangle \\ &\to_{h\to 0} -\langle u; \nabla\varphi\rangle.\end{aligned}$$

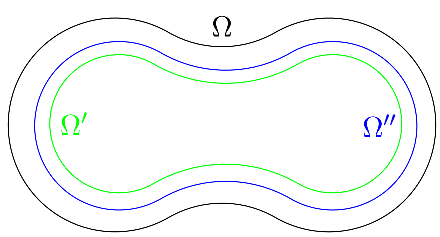
So $D_h u \to \nabla u$ in $\mathcal{D}'$ and $v = \nabla u$.
So $D_h u \to \nabla u$ weakly in L^2.
Then $\|\nabla\|_{L^2(\Omega)} \leqslant \limsup_{|h|<<1} \|D_h u\|_{L^2(\Omega)} \leqslant C$ □

Theorem 2.26. *(Local regularity, interior regularity)* $\Omega' \subset\subset \Omega$ *bounded set.* $f \in L^2(\Omega)$, $\nabla u \in L^2(\Omega)$, $g \in H^1(\Omega)$

$$\begin{cases} -\Delta u + \nabla p = f & \textit{in } \Omega \\ \operatorname{div} u = g & \textit{in } \Omega. \end{cases}$$

Then: $\left\|\nabla^2 u\right\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega')} \leqslant C(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|\nabla u\|_{L^2(\Omega)})$, *where* C *depends only on* $\Omega'' \subset\subset \Omega$.

Proof. Consider $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and $\varphi \in C^\infty(\Omega)$ $\operatorname{supp}\varphi \subset \Omega''$ and $\varphi \equiv 1$ on Ω'. $|h| \leqslant d(\Omega''; \partial\Omega)$:



Consider $v = D_{-h}(\varphi D_h u) \in H_0^1(\Omega)$

$$\int_\Omega \nabla v : \nabla u dx - \int_\Omega (p \operatorname{div} v) dx = \int f v dx$$

□

Remark 2.27. $\nabla p = f + \Delta u = f + \operatorname{div}(\nabla w)$ when $f \in L^2(\Omega) \in H^{-1}(\Omega)$ and $\operatorname{div}(\nabla w) \in H^{-1}(\Omega)$
So $p - ⨍_\Omega p dx \in L^2(\Omega)$:

$$\left\|p - ⨍_\Omega p dx\right\|_{L^2(\Omega)} \leqslant C \|\nabla p\|_{H^{-1}(\Omega)}.$$

Taking $p - \fint p dx$ as the new pressure term:

$$
\begin{aligned}
\int_\Omega \nabla v : \nabla u dx &= \int_\Omega -\nabla D_{-h}(\varphi^2 D_h u) : \nabla u dx \\
&= \int_\Omega \nabla(\varphi^2 D_h u) : D_h(\nabla u) dx \\
&= \int_\Omega \varphi \nabla(\varphi D_h u) : \nabla D_h u dx + \int_\Omega (\nabla \phi \otimes \varphi D_h u) : \nabla D_h u dx \\
w = \nabla(\varphi D_h u) \qquad &= \int_\Omega |\nabla(\varphi D_h u)|^2 dx - \int_\Omega \nabla(\varphi D_h u) : (\nabla \varphi \otimes D_h u) dx \\
&+ \int_\Omega \nabla(\varphi D_h u) : (\nabla \varphi \otimes D_h u) dx - \int_\Omega |\nabla \varphi|^2 |D_h u|^2 dx
\end{aligned}
$$

where

$$
\left| \int_\Omega |\nabla \varphi|^2 |D_h u|^2 dx \right| \leqslant C \|\nabla u\|_{L^2}
$$

(2)

$$
\begin{aligned}
\left| \int p \operatorname{div} v \right| &= \left| \int p \operatorname{div}(D_{-h} \varphi^2 D_h u) \right| \\
&\leqslant \left| \int p D_{-h}(\varphi \nabla \varphi \cdot D_h u) \right| + \left| \int p D_{-h}(\varphi^2 D_h g) \right| \\
&= I_1 + I_2.
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \left| \int p (2\nabla \varphi)_h D_{-h}(\varphi D_h u) dx \right| + I_3 \\
&\leqslant C \|p\|_{L^2} \|D_{-h}(\varphi D_h u)\|_{L^2} + I_3 + I_3 \\
&\leqslant C \|p\|_{L^2} \|\nabla(\varphi D_h u)\|_{L^2} + I_3 \\
&\leqslant C(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|w\|_{L^2(\Omega)} + I_3
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leqslant \|p\|_{L^2(\Omega)} \|g\|_{H^2(\Omega)} \\
&\leqslant C(\|f\|_{L^2} + \|\nabla u\|_{L^2} + \|g\|_{H^2(\Omega)})
\end{aligned}
$$

$$
I_3 \leqslant C \|p\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leqslant C(\|f\|_{L^2} + \|\nabla u\|_{L^2} + \|g\|_{H^2(\Omega)})
$$

(3)

$$\begin{aligned}
\int_\Omega fv dx &= \int_\Omega f D_{-h}(\varphi(\varphi D_h \varphi)) \\
&= \left| \int_\Omega f \varphi_h D_{-h}(\varphi D_h u) + \int_\Omega f (D_{-h}\varphi)\varphi D_h u \right| \\
&\leqslant \|f\|_{L^2(\Omega)} \|D_{-h}(\varphi D_h u)\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^\infty} \|f\|_{L^2} \|D_h u\|_{L^2(\Omega)} \\
&\leqslant C \|f\|_L^2 \left(\|w\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \right)
\end{aligned}$$

Let's put everything together, we have

$$\|w\|^2_{L^2(\Omega)} \leqslant C(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|w\|_{L^2(\Omega)} + C(\|f\|_{L^2} + \|\nabla u\|^2_{L^2(\Omega)})$$

So

$$\|w\|^2_{L^2} \leqslant C^*(\|f\|^2_{L^2} + \|\nabla u\|^2_{L^2})$$

$$\|w\|^2_{L^2} = \int_\Omega \left|\nabla(\varphi^2 D_h u\right|^2 dx \geqslant \int_{\Omega'} |\nabla D_h u|^2 dx$$

where $\int_{\Omega'} |\nabla D_h u|^2 dx = \int_{\Omega'} |D_h(\nabla u)|^2 dx \leqslant C$, using 3 of the lemma then $\nabla^2 \in L^2(\Omega')$ and $\left\|\nabla^2 u\right\|_{L^2(\Omega')} \lesssim (\|f\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})$

$\nabla p = \Delta u + f$ where $\Delta u \in L^2(\Omega')$ and $f \in L^2(\Omega')$.

$$\|\nabla p\|^2_{L^2} \leqslant C^{**}(\|f\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})$$

♣ ♣ ♣

hint of homework

Recore that u is divergence-free, and a and b satisfies the transport equation

$$\partial_t v + u \cdot \nabla v = 0,$$

then $v = (\nabla a)^\top b$ satisfies

$$\partial_t v + u \cdot \nabla v + (\nabla u)^\top v = 0.$$

Taking the curl $\omega = \operatorname{curl} v$,

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0. \tag{2.5}$$

If u is a solution to Euler, then $\operatorname{curl}(u - (\nabla a^\top b))$ verifies (2.5). If the initial value matches the one with Euler equation, then $\operatorname{curl}(\nabla a^\top b) = 0$.

To match with the initial values, take $Y(t,0,x)$ to be the solution to

$$\begin{cases} \partial_t Y + u \cdot \nabla Y = 0 \\ Y\big|_{t=0}(x) = x \end{cases}$$

Then ∇Y is a matrix, and $\nabla Y\big|_{t=0}(x) = \nabla x = \mathrm{Id}$.

Take $v(t, Y(t,0,x)) = \tilde{v}(t,x)$, then

$$\partial_t \tilde{v} + (\nabla u)^\top (t, Y(t,0,x))\tilde{v} = 0.$$

♣ ♣ ♣

Last time:

Theorem 2.28. *Let (u,p) be a solution to*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \end{cases}$$

Then for $\Omega' \subset\subset \Omega$, there exists $C = C_{\Omega,\Omega'}$ such that

$$\left\|\nabla^2 u\right\|_{L^2(\Omega')} + \|\nabla p\|_{L^2(\Omega')} \le C\Big(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\Big).$$

2.6.2 Boundary result

Theorem 2.29. *Let*

$$B_r^+ = \{|x| \le r; x_3 \ge 0\}.$$

If (u,p) be a solution to

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } B_1^+ \\ \operatorname{div} u = g & \text{in } B_1^+ \\ u = 0 & \text{on } \{x_3 = 0\} \cap B_1^+ \end{cases} \tag{2.6}$$

Then for $r < 1$,

$$\begin{aligned} &\left\|\nabla^2 u\right\|_{L^2(B_r^+)} + \|\nabla p\|_{L^2(B_r^+)} \\ &\qquad \le C_r\Big(\|f\|_{L^2(B_1^+)} + \|g\|_{H^1(B_1^+)} + \|\nabla u\|_{L^2(B_1^+)}\Big). \end{aligned}$$

Proof. We prove in several steps.

Step 1. $\nabla\bar{\nabla}u \in L^2$. Consider a test function $\Phi(x) = 1$ in B_r and on B_1^+.

$$D_h^e(v) = \frac{v(x+he) - v(x)}{h}.$$

Consider flat direction $e = (e_1, e_2, 0)$, u solution to (2.6).

$$v = -D_{-h}^e(\Phi^2 D_h^e u) \in H_0^1\big(B_1^+\big).$$

Using v as a test function, we can integrate by part and

$$\int_{B_1^+} \nabla v : \nabla u \, \mathrm{d}x + \int_{B_1^+} v \cdot \nabla p \, \mathrm{d}x = \int_{B_1^+} f v \, \mathrm{d}x.$$

From the equation, $\nabla P \in H^{-1}\big(B_1^+\big)$, so we can find $P \in L^2\big(B_1^+\big)$ with

$$\|P\|_{L^2(B_1^+)} \lesssim \|f + \Delta u\|_{H^{-1}(B_1^+)} \leq \text{RHS}.$$

Next, we estimate the bilinear term,

$$\begin{aligned} &\left| \int_{B_1^+} \nabla v : \nabla u \, \mathrm{d}x - \int_{B_1^+} |\nabla(\Phi D_h^e u)|^2 \, \mathrm{d}x \right| \\ &\qquad \leq C \int_{B_1^+} |\nabla \Phi|^2 |\nabla u|^2 \, \mathrm{d}x + \frac{1}{3} \int_{B_1^+} |\nabla(\Phi D_h^e u)|^2 \, \mathrm{d}x \end{aligned}$$

Other terms: commutators are bounded by

$$\leq C\Big(\|p\|_{L^2}^2 + \|f\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\Big) + \frac{1}{3} \int_{B_1^+} |\nabla(\Phi D_h^e u)|^2 \, \mathrm{d}x.$$

Again we use the Caccioppoli trick

$$\begin{aligned} &\int \nabla(\Phi\Phi D_h^e u)(\nabla D_h^e u) \, \mathrm{d}x \\ &\quad = \int (\nabla \Phi)(\Phi D_h^e u)(\nabla D_h^e u) \, \mathrm{d}x + \int \nabla(\Phi D_h^e u)\Phi(\nabla D_h^e u) \, \mathrm{d}x \\ &\quad = \int |\nabla(\Phi D_h^e u)|^2 \, \mathrm{d}x - \int \nabla(\Phi D_h^e u) \nabla \Phi D_h^e u + \int \nabla(\Phi D_h^e u) \nabla \Phi D_h^e u. \end{aligned}$$

We get

$$\int_{B_1^+} |\nabla(\Phi D_h^e u)|^2 \, \mathrm{d}x \leq C\Big(\|f\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|g\|_{H^1}^2\Big).$$

So $\nabla D_h^e u$ is uniformly bounded in $L^2(B_r^+)$ with respect to h. By our lemma,

$$\int_{B_r^+} \big|\nabla\bar{\nabla}u\big|^2 \, \mathrm{d}x \leq C\Big(\|f\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|g\|_{H^1}^2\Big).$$

Here $\bar{\nabla} = (\partial_1, \partial_2, 0)$.

Step 2. $\partial_3 p \in L^2$. We need to control $\partial_3^2 u$. Note that the third component is controlled using the divergence equation,

$$\partial_3^2 u_3 = \partial_3(\partial_3 u_3) = -\partial_3(g - \partial_1 u_1 - \partial_2 u_2) \in L^2\big(B_1^+\big).$$

From the equation,

$$-\Delta u_3 + \partial_3 p = f_3.$$

Now $\Delta u_3, f_3 \in L^2$, so $\partial_3 p \in L^2$.

Step 3. $\bar{\nabla} p \in L^2$. Denote $\bar{u} = (u_1, u_2, 0)$. Then $\bar{u}$ satisfies

$$-\Delta \bar{u} + \bar{\nabla} p = \bar{f}.$$

By differentiating in the flat direction,

$$-\Delta \bar{\nabla} \bar{u} + \bar{\nabla}^2 p = \bar{\nabla} \bar{f}.$$

Note that

$$\Delta \bar{\nabla} \bar{u} = \operatorname{div}(\nabla \bar{\nabla} \bar{u}) \in H^{-1}, \qquad \bar{\nabla} \bar{f} \in H^{-1}$$

so $\bar{\nabla}^2 p \in H^{-1}$. Combined with last step, we have $\bar{\nabla} \nabla p \in H^{-1}$, so $\bar{\nabla} p \in L^2$.

Step 4. $u \in H^2$. By previous two steps, $\nabla p \in L^2$, so the theorem follows using elliptic theory.

□

2.6.3 Global result

The next step to show higher regularity is to flatten the boundary. Let $\Omega \subset \bigcup_{i=0}^N \Omega_i$. In a chart we define

$$\bar{u}(y) = u(\psi_i^{-1}(y)), \qquad u(y) = \bar{u}(\psi_i(y)).$$

If u satisfies

$$\begin{cases} -\Delta u + \nabla p = f \\ \operatorname{div} u = g \end{cases}$$

then $\bar{u}$ satisfies

$$\begin{cases} -\operatorname{div}(A(x)\nabla \bar{u}) + B(x) \cdot \nabla \bar{u} + (\nabla \bar{p})^\top \nabla \psi = \bar{f} \\ \operatorname{tr}(\nabla \bar{u} \nabla \psi_i) = \bar{g}. \end{cases}$$

Here $A(x) = (\nabla\psi_i \otimes \nabla\psi_i)$.

♣ ♣ ♣

Theorem 2.30. *Ω a bounded domain, with $\partial\Omega$ $C^{2,1}$ take $f \in L^2(\Omega)$, $g \in H^1(\Omega)$. Then $\exists!$ solution $u \in V(\Omega)$ to*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $u \in V$ and $\left\|\nabla^2 u\right\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\omega)} \leqslant C(\|f\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}$ and C depends only on Ω.

Remark 2.31. Local results depend on $\|\nabla u\|_{L^2}$ too. The control on $\|\nabla u\|_{L^2}$ comes from the uniqueness from Lax-milgram.

Theorem 2.32. *Local high regularity: fix $k \in \mathbb{N}$: Ω bounded $\Omega \subset\subset \Omega$ $\partial\Omega$ $C^{k+2,1}$, $f \in H^k(\Omega)$, $g \in H^{k+1}(\Omega)$, $\nabla u \in L^2(\Omega)$: such that*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \end{cases}$$

and $\left\|\nabla^2 u\right\|_{H^{k+2}(\Omega')} + \|\nabla P\|_{H^{k+1}(\omega')} \leqslant C(\|f\|_{H^k(\Omega)} + \|\nabla g\|_{H^k(\Omega)} + \|\nabla u\|_{L^2(\Omega)})$ and C depends only on k, Ω, Ω'.

Proof. Consider a sequence of set $\Omega' = \Omega_k \subset\subset \ldots \subset\subset \Omega_{-1} = \Omega$. We want to show that $\forall n \in \{0, 1, \ldots, k\}$, we have $\|u\|_{H^{n+2}(\Omega_n)} + \|P\|_{H^{n+1}(\Omega_n)} \leqslant C_n(\|f\|_{H^k(\Omega)} + \|\nabla g\|_{H^k(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. For $n = 0$: see H^2 result. Assume true for n,

$$\tilde{v} = \partial^\alpha v \qquad \text{for } |\alpha| = n + 1$$

$$\begin{cases} -\Delta\tilde{u} + \nabla\tilde{p} = \tilde{f} \\ \operatorname{div}\tilde{u} = \tilde{g} \end{cases}$$

Then $\left\|\nabla^2\tilde{u}\right\|_{L^2(\Omega_{n+1})} + \|\nabla\tilde{p}\|_{L^2(\Omega_{n+1})} \leqslant C(\left\|\tilde{f}\right\|_{L^2(\Omega_n)} + \|\nabla\tilde{g}\|_{L^2(\Omega_n)} + \|\nabla\tilde{u}\|_{L^2(\Omega)})$ $\leqslant C(\|f\|_{H^{n+1}(\Omega)} + \|g\|_{H^{n+1}(\Omega)} + \|u\|_{H^{n+2}(\Omega_n)}$. Therefore

$$\|u\|_{H^{n+3}(\Omega_{n+1})} + \|\nabla p\|_{H^{n+2}(\Omega_{n+1})} \leqslant \left\|\nabla^2\tilde{u}\right\|_{L^2(\Omega_{n+1})} + \|\nabla\tilde{p}\|_{L^2(\Omega_{n+1})} + \|\nabla u\|_{L^2(\Omega_{n+1})} + \|\nabla p\|_{L^2(\Omega_{n+1})}$$

, where $+\|\nabla u\|_{L^2(\Omega_{n+1})} + \|\nabla p\|_{L^2(\Omega_{n+1})}$ has already been controlled. $\square$

Theorem 2.33. *Global Higher regularity: $k \in \mathbb{N}$, Ω bounded $\partial\Omega$ $C^{k+2}, 1$ $f \in H^k(\Omega)$, $g \in H^{k+1}(\Omega)$ $\int_\Omega g dx = 0$. Then there exist a unique solution: $u \in V(\Omega) \cap H^{k+2}(\Omega)$ of*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$u \in V(\Omega)$ and $\|u\|_{H^{k+2}(\Omega)} + \|\nabla p\|_{H^{k+1}(\Omega)} \leqslant C \|f\|_{H^k(\Omega)}$ where C depends only on Ω and k.

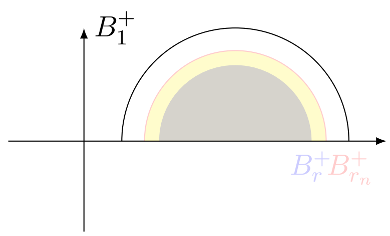
Proposition 2.34. *Fix $k \in \mathbb{N}$: $f \in H^k(B_1^+)$; $g \in H^{k+1}(B_1^+)$. Consider $u \in H^1(B_1^+)$ solution to*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } B_1^+ \\ \operatorname{div} u = 0 & \text{in } B_1^+ \end{cases}$$

Then for any $0 < r < 1$, there exists a constant C such that

$$\|u\|_{H^{k+2}(B_r^+)} + \|\nabla P\|_{H^{k+1}(B_r^+)} \leqslant C(\|f\|_{H^k(B_1^+)} + \|\nabla u\|_{L^2(B_1^+)}$$

C depends only on k, r.



Proof. Consider $r_k = r < r_k - 1 < ... < r_{-1} = 1$ We prove by induction that $\forall n : 0 \leqslant n \leqslant k$, $\|\partial_1^{\alpha_1}\partial_2^{\alpha_2} u\|_{H^2(B_{r_n}^+)} \leqslant C(\|f\|_{H^k(B_1^+)} + \|\nabla g\|_{H^k(B_1^+)} + \|\nabla u\|_{L^2(B_1^+)})$, $\alpha_1 + \alpha_2 = n$. Same proof as before, let $\tilde{v} = \partial_1^{\alpha_1}\partial_2^{\alpha_2} v$

$$\begin{cases} -\Delta \tilde{u} + \nabla \tilde{p} = \tilde{f} & \text{in } \Omega \\ \operatorname{div} \tilde{u} = \tilde{g} & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } x_3 = 0 \end{cases}$$

We obtain the claim.
We want to prove by induction that for any $0 \leqslant n \leqslant k$: $\forall \alpha_1, \alpha_2$ s.t. $\alpha_1 + \alpha_2 = k - n$

$$\left\|\partial_{x_3}^{n+2}\partial_1^{\alpha_1}\partial_2^{\alpha_2} u\right\|_{L^2(B_r)} + \left\|\partial_{x_3}^{n+1}\partial_1^{\alpha_1}\partial_2^{\alpha_2} p\right\|_{L^2(B_r)} \leqslant RHS$$

True for $n = 0$ from flat derivative process.
Assume that it is true for n. Consider $\forall \alpha_1, \alpha_2$ s.t. $\alpha_1 + \alpha_2 = k - (n+1)$

1. $\partial_{x_3}^{n+3}\partial_1^{\alpha_1}\partial_2^{\alpha_2} u_3 = \partial_{x_3}^{n+2}\partial_1^{\alpha_1}\partial_2^{\alpha_2}(\partial_{x_3} u_3)$.
   Since $\operatorname{div} u = 0$,

   $$\partial_{x_3}^{n+3}\partial_1^{\alpha_1}\partial_2^{\alpha_2} u_3 = -\partial_{x_3}^{n+2}\partial_1^{\alpha_1+1}\partial_2^{\alpha_2} u_1 - \partial_{x_3}^{n+2}\partial_1^{\alpha_1}\partial_2^{\alpha_2+1} u_2$$

   where $\alpha_1 + \alpha_2 + 1 = k - n$. So by induction property: $\left\|\partial_{x_3}^{n+3}\partial_1^{\alpha_1}\partial_2^{\alpha_2} u_3\right\|_{L^2}$ bounded

2. third component of the Stokes equation

   $$[\partial_3^{n}\partial_1^{\alpha_1}\partial_2^{\alpha_2}] \quad -\Delta u_3 + \partial_3 p = f_3$$

   So $\partial_3^{n+2}\partial_1^{\alpha_1}\partial_2^{\alpha_2} p \leqslant RHS$

3. Need $\partial_3^{n+3}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ 2 first component of the stokes equation:

   $$[\partial_3^{n+1}\partial_1^{\alpha_1}\partial_2^{\alpha_2}] \begin{cases} -\Delta u_1 + \partial_1 p = f_1 \\ -\Delta u + \partial_2 p = f_2 \end{cases}$$

   Thus $\partial_3^{n+3}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_3^{n+1}\partial_1^{\alpha_1+1}\partial_2^{\alpha_2} p \\ \partial_3^{n+1}\partial_1^{\alpha_1}\partial_2^{\alpha_2+1} p \end{pmatrix}$ By induction it is controlled with RHS.

□

Additional result:

Theorem 2.35. $1 < p < +\infty$ $k \in \mathbb{N}$: *Let Ω be bounded with smooth enough boundary. Assume that $f \in W^{p,k}(\Omega)$ there exist a unique solution $u \in W^{p,k+2}(\omega)$ solution to*

$$\begin{cases} -\Delta u + \nabla P = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and $\|u\|_{W^{p,k+2}(\Omega)} + \|\nabla p\|_{W^{p,k+1}(\Omega)} \leqslant \|f\|_{W^{p,k}(\Omega)}$ C depends only on Ω, k, p.

2.7 Stokes operator

Ω fixed regular

Remark 2.36.
$$\begin{cases} -\Delta u + \nabla P = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

u is the same solution with f repalced by $\mathbb{P}f$. $\mathbb{P}f$ being the projection from L^2 to H,

$$H = \{f \in L^2(\Omega); \operatorname{div} f = 0, \overrightarrow{u} \cdot \overrightarrow{n} = 0 \text{ on } \partial\Omega\}$$

$$f = \mathbb{P}f + \nabla g, \qquad -\Delta u + \nabla(pg) = \mathbb{P}f.$$

Consider some terms in H the equation can be rewritten as

$$\mathbb{P}(\Delta u) = f \in H$$

$\mathbb{P}\Delta$ is called the Stokes operator, $\mathbb{P}\Delta : H^2(\Omega) \cap V(\Omega) \to H$ bijection.

♣ ♣ ♣

2.7.1 Spectral properties of the Stokes operator

Denote $\tilde{\Delta} = \mathbb{P}\Delta$. It has the following properties.

1. $\tilde{\Delta}$ has a discrete spectrum:

$$-\tilde{\Delta} u = \lambda u \qquad u \neq 0, u \in H(\Omega)$$

   has solutions for $0 < \lambda_1 < \lambda_2 \leq \cdots \to \infty$.

2. For all k,

$$\dim(\ker(-\tilde{\Delta} - \lambda_k)) < \infty.$$

3. There exists $\{\varphi_k\}_{k=1}^{\infty}$ with $\varphi \in H(\Omega) \cap V(\Omega)$ with
   - $-\tilde{\Delta}\varphi_k = \lambda_k \varphi_k$
   - $\{\varphi_k\}$ is an orthonormal basis of $H(\Omega)$. For any $j \neq k$, it holds that $\int_\Omega \left|\varphi_k^2\right| \mathrm{d}x = 1$ and $\int_\Omega \varphi_k(x)\varphi_j(x) \,\mathrm{d}x = 0$.

4. $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal basis for $V(\Omega)$:

$$\int_\Omega |\nabla\varphi_k|^2 \,\mathrm{d}x = \lambda_k, \qquad \int_\Omega \nabla\varphi_k : \nabla\varphi_j \,\mathrm{d}x = 0 \iff j \neq k.$$

Remark 2.37. The orthogonality is easy to show.

- With
$$-\tilde{\Delta}\varphi_j = \lambda_j\varphi_j$$
we use $\varphi_k \in V(\Omega)$ as a test function:
$$\int_\Omega \nabla\varphi_k : \nabla\varphi_j \,\mathrm{d}x = \lambda_j \int \varphi_j\varphi_k \,\mathrm{d}x.$$
By symmetry, it is also equal to $\lambda_k \int \varphi_j\varphi_k \,\mathrm{d}x$. So $(\lambda_j - \lambda_k)\int \varphi_j\varphi_k \,\mathrm{d}x = 0$. If $\lambda_j \neq \lambda_k$, then $\int \varphi_j\varphi_k = 0$, and $\int \nabla\varphi_j : \nabla\varphi_k = 0$.
- Since $\dim(\ker(-\tilde{\Delta} - \lambda\mathrm{Id}))$ is finite, we can choose a orthogonal family in $\ker(-\tilde{\Delta} - \lambda\mathrm{Id})$.
- If $\int_\Omega \varphi_j\varphi_k = 0$ then $\int_\Omega \nabla\varphi_j : \nabla\varphi_k \,\mathrm{d}x = 0$.
- $\int_\Omega |\nabla\varphi_k|^2 \,\mathrm{d}x = \lambda_k \int_\Omega |\varphi_k|^2 \,\mathrm{d}x = \lambda_k$.

Remark 2.38. Other properties

- for all $f \in H(\Omega)$, there existss c_k such that
$$f = \sum_{k=1}^{\infty} c_k\varphi_k \qquad c_k = (f;\varphi_k) = \int_\Omega f(x)\varphi_k(x)\,\mathrm{d}x$$
and
$$\|f\|_H^2 = \sum_{k=1}^{\infty} |c_k|^2.$$
- for all $f \in V(\Omega)$, $c_k = (f, \varphi_k)$,
$$\|\nabla f\|_{L^2} = \|f\|_{V(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k |c_k|^2.$$
Moreover,
$$\|\nabla f\|_{L^2}^2 = \lim_{K\to\infty} \sum_{k=1}^{K}\sum_{j=1}^{K} \int c_k c_j (\underbrace{\nabla\varphi_k : \nabla\varphi_j}_{\lambda_k\delta_{kj}})^2 \,\mathrm{d}x = \lim_{k\to\infty} \sum_{k=1}^{K} |c_k|^2\lambda_k.$$

Remark 2.39. In $[0,1]$ one can have a Fourier decomposition of a function with $f(0)=f(1)=0$ by

$$f(x)=\sum_{k=1}^{\infty} c_k \sin(k\pi x) \qquad\qquad c_k=\int_0^1 f(x)\sin(k\pi x).$$

Laplacian in 1D: $-\Delta f=-f''$. So

$$\varphi_k=\sin(k\pi x) \qquad\qquad -\varphi_k''=\underbrace{(k\pi)^2}_{=\lambda_k}\varphi_k$$

Proposition 2.40. *$\tilde{\Delta}$ is an isometry between $V(\Omega)$ and its dual.*

1. *$f\in[V(\Omega)]'$ if and only if $\sum_{k=1}^{\infty}\frac{|f_k|^2}{\lambda_k}$ is finite, where $f_k=\langle f,\varphi_k\rangle$.*
2. *If $f\in[V(\Omega)]'$ then*
$$\|f\|^2_{[V(\Omega)]'}=\sum_{k=1}^{\infty}\frac{|f_k|^2}{\lambda_k}.$$
3. *$\tilde{\Delta}:V(\Omega)\to[V(\Omega)]'$ is a bijection.*

Proof. For all $a\in V(\Omega)$, define $a_J=\sum_{k=1}^{J}(a,\varphi_k)\varphi_k$. Then $a_J\to a$ in $V(\Omega)$ as $J\to\infty$.

- For all $a\in V(\Omega)$, $f\in[V(\Omega)]'$,

$$\begin{aligned}
\langle f;a_J\rangle &= \left\langle f;\sum_{k=1}^{J}(a,\varphi_k)\varphi_k\right\rangle\\
&=\sum_{k=1}^{J}\frac{\langle f,\varphi_k\rangle}{\sqrt{\lambda_k}}(a,\varphi_k)\sqrt{\lambda_k}\\
&\le\sqrt{\sum_{k=1}^{J}\frac{|f_k|^2}{\lambda_k}}\sqrt{\sum_{k=1}^{J}(a,\varphi_k)^2\lambda_k}\\
&\le\sqrt{\sum_{k=1}^{\infty}\frac{|f_k|^2}{\lambda_k}}\|a_J\|_{V(\Omega)}
\end{aligned}$$

Take $J \to \infty$, we have for all $a \in V(\Omega)$,

$$|\langle f; a\rangle| \leq \|a\|_{V(\Omega)} \sqrt{\sum_{k=1}^{\infty} \frac{|f_k|^2}{\lambda_k}}$$

so $\|f\|^2_{[V(\Omega)]'} \leq \sum_{k=1}^{\infty} \frac{|f_k|^2}{\lambda_k}$.

- Now take $a_J = \sum_{k=1}^{J} \frac{f_k}{\lambda_k} \varphi_k$, so

$$\|a_J\|^2_{V(\Omega)} = \sum_{k=1}^{J} \frac{|f_k|^2}{\lambda_k^2} \lambda_k = \sum_{k=1}^{J} \frac{|f_k|^2}{\lambda_k} = \langle f; a_J\rangle.$$

So

$$\langle f; a_J\rangle = \|a_J\|_{V(\Omega)} \sqrt{\sum_{k=1}^{J} \frac{|f_k|^2}{\lambda_k}}.$$

Taking sup in J yields

$$\langle f; a\rangle = \|a\|_{V(\Omega)} \sqrt{\sum_{k=1}^{\infty} \frac{|f_k|^2}{\lambda_k}}.$$

So $\|f\|_{[V(\Omega)]'} \geq \sqrt{\sum_{k=1}^{J} \frac{|f_k|^2}{\lambda_k}}$. By the first part of the proof, equality holds.

- Combined we have $f \in V'(\Omega)$ iff $\sum_{k=1}^{\infty} \frac{|f_k|^2}{\lambda_k} < \infty$, and

$$\|f\|^2_{[V(\Omega)]'} = \sum_{k=1}^{J} \frac{|f_k|^2}{\lambda_k}.$$

- Take $a \in V(\Omega)$, $\sum_{k=1}^{\infty} |(a, \varphi_k)|^2 \lambda_k < +\infty$,

$$-\tilde{\Delta} a = \sum_{k=1}^{\infty} (a, \varphi_k) - \tilde{\Delta} \varphi_k = \sum_{k=1}^{\infty} \lambda_k (a, \varphi_k) \varphi_k = \sum_{k=1}^{\infty} d_k \varphi_k$$

and

$$\sum_{k=1}^{\infty} \frac{d_k^2}{\lambda_k} = \sum_{k=1}^{\infty} (a, \varphi_k)^2 \lambda_k = \|a\|^2_{V(\Omega)}.$$

This means

$$\begin{aligned}
-\tilde{\Delta} : V(\Omega) &\to (V(\Omega))' \\
\sum_k a_k \varphi_k &\mapsto \sum_k \lambda_k a_k \varphi_k \\
\sum_k \frac{1}{\lambda_k} d_k \varphi_k &\leftarrow\!\shortmid \sum_k d_k \varphi_k
\end{aligned}$$

is an isometry.

$\square$

Chapter 3

Linear non-stationary problems

Take V to be a Banach space, and V' to be its dual space. We define the space $L^1_{\text{loc}}(a,b;V')$ by the following.

Definition 3.1. We say that $v^* \in L^1_{\text{loc}}(a,b;V')$ if $v^*(t,\cdot) \in V'$ for almost all $t \in [a,b]$, and $t \mapsto \|v^*(t,\cdot)\|_{V'}$ is in $L^1_{\text{loc}}(a,b)$, i.e. for any $\mathcal{I} \subset\subset [a,b]$,

$$\int_{\mathcal{I}} \|v(t,\cdot)\|_{V'} \,\mathrm{d}t < +\infty.$$

We call $u^* \in \mathcal{D}'(a,b;V')$ time derivative of v^* if and only if for every $X \in \mathcal{D}(a,b)$, and any $v \in V$:

$$\langle \langle u^*, v\rangle_{V',V}; X\rangle_{\text{time}} = -\int_a^b \langle u^*(t,\cdot); v\rangle \partial_t X \,\mathrm{d}t.$$

Formally it means

$$\int_a^b \langle u^*(t,\cdot), v\rangle X(t) \,\mathrm{d}t.$$

Remark 3.2. We test v^* against $\langle \partial_t v^*, v(x)X(t)\rangle$ with $v \in V$ function in x only and $X \in C_c^\infty(a,b)$ function in t only.

3.1 Construction of solutions

Let Ω be bounded with $\partial\Omega$ smooth, $T > 0$. Consider the problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } (0,T)\times\Omega \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = a & \text{in } \Omega \end{cases}.$$

Given data: $a \in H(\Omega)$, with $\|a\|_{L^2}^2$ the kinetic energy of initial value. $f \in L^2(0,T;[V(\Omega)]')$. We want to find a solution in the form of

$$u(t,x) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x).$$

Equation can be rewritten as

$$\partial_t u - \tilde{\Delta} u = f.$$

Formally:

$$\sum_{k=1}^{\infty} \big(c_k'(t) + \lambda_k c_k(t) - f_k(t)\big)\varphi_k(x)$$

where $f_k(t) = \langle f(t), \varphi_k\rangle$. We want to construct the coefficients c_k:

$$\begin{cases} c_k'(t) + \lambda_k c_k(t) = f_k \\ c_k(0) = (a,\varphi_k) = a_k \end{cases} \tag{3.1}$$

(3.1) has a unique solution given by

$$c_k(t) = e^{-\lambda_k t}\left(a_k + \int_0^t e^{\lambda_k s} f_k(s)\,\mathrm{d}s\right).$$

♣ ♣ ♣

Last time we wanted to construct solution when Ω bounded and smooth

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } (0,T)\times\Omega \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega \\ u = 0 & \text{in } (0,T)\times\partial\Omega \\ u|_{t=0} = a^0 & \text{on } \Omega \end{cases}$$

$$\begin{cases} f \in L^2(0,T;(V(\Omega)) \\ a^0 \in H(\Omega) \end{cases}$$

We want a solution in the sense: $u \in L^2(0,T;V(\Omega))$ ($u = 0$ on ,$\partial_t u \in L^2(0,T;[V(\Omega)]')$ a.e. t: $\forall w \in V(\Omega)$:

$$\langle \partial_t u; w\rangle_{V',V} + (\nabla u : \nabla w)_{L^2,L^2} = \langle f, w\rangle_{V',V}$$

We will show: $u \in C(0,T;L^2(\Omega))$. Actually we rewrite the equation as follows:

$$\partial_t u - \tilde{\Delta} u = f$$

(the Stokes operator)
We define $c_k(t)$ as solution to:

$$\begin{cases} c_k' + \lambda_k c_k = f_k(t) \\ c_k(0) = a_k^0 \end{cases} \qquad \begin{cases} -\tilde{\Delta}\varphi_k = \lambda_k \varphi_k \\ \|\varphi_k\|_{L^2} = 1 \\ \varphi \in V(\Omega). \end{cases}$$

$a_k^0 = \int_\Omega a^0 \varphi_k dx$ and $f_k(t)$" = " $\int_\Omega f(t,x)\varphi_k(x)dx = \langle f(t,\cdot);\varphi_k\rangle_{V',V}$ $c_k = e^{-\lambda_k t}\left(a_k^0 + \int_o^t e^{\lambda_k s} f_k(s)ds\right)$. $\forall N \in \mathbb{N}$:

$$U_N(t,x) = \sum_{k=1}^N c_k \varphi_k(x) \qquad f_N(t,x) = \sum_{k=1}^N f_k \varphi_k(x) \qquad a_N(t,x) = \sum_{k=1}^N a_k^0 \varphi_k(x)$$

$$a_N \in C^\infty(\Omega),\ f_N \in L^2(0,T;C^\infty(\Omega))\ u_N \in C(0,T;C^\infty(\Omega))$$

$$\partial_t u_N = \sum_{k=1}^N c_k'(t)\varphi_k(x)$$

$$-\tilde{\Delta} u_N = \sum c_k(t)(-\tilde{\Delta})\varphi_k(x) = \sum c_k(t)\lambda_k \varphi_k(x)$$

Then

$$\begin{cases} \partial_t u_N - \tilde{\Delta} u_N = f_N \\ u_N|_{t=0} = a_N \end{cases}$$

Then

$$\frac{d}{dt}\int_\Omega \frac{|u_N|^2}{2} dx + \int_\Omega |\nabla u_N|^2 = \int_\Omega f_N u_N$$

$$\leqslant \|f_N(t)\|_{V'} \|u_N(t)\|_V \leqslant \frac{1}{2}\int_\Omega |\nabla u_N|^2 dx + 2\|f_N\|_{V'}$$

gives $\forall t \in [0,T)$:

$$\|u_N(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \int_\Omega |\nabla u_N(s)| dx ds \leqslant 2\,\|f_N\|^2_{L^2(0,t,V(\Omega)'} + \|a_N\|^2_{L^2(\Omega)}$$

$$\leqslant 2\,\|f\|^2_{L^2(0,t,V(\Omega)'} + \left\|a^0\right\|^2_{L^2(\Omega)}$$

$$\|f_N(t)\|^2_{V(\Omega)} = \sum_{k=1}^{N} \frac{|f_k(t)|^2}{\lambda_k} \leqslant \sum_{k=1}^{\infty} \frac{|f_k(t)|^2}{\lambda_k} = \|f(t,\cdot)\|^2_{V'}$$

$$\left\|a_k^0\right\| \leqslant \left\|a^0\right\|_L^2$$

u_N is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^2(0,\infty;V(\Omega))$.

$$u_N \to u = \sum_{k=0}^{\infty} c_k(t)\varphi_k(x) = \sum_{k=0}^{\infty}\left[\int_0^T |c_k(t)|^2 \lambda_k dt\right]$$

weakly in $L^2(0,\infty;V(\Omega))$, so $\|u\|_{L^2(0,T;V(\Omega))}$. The series is absolutely convergent. Thus, $\|u_N - u\|_{L^2(0,T;V(\Omega))} = \sum_{k=N+1}^{\infty}\left[\int_0^T |c_k(t)|^2 \lambda_k dt\right] \to 0$ as $N \to +\infty$.

Therefore $u_N \to u$ strongly in $L^2(0,T;V(\Omega))$

$$\partial_t u_N = \tilde{\Delta} u_N + f_N$$

$$\|\partial_t u_N\|^2_{L^2(0,T;(V(\Omega))} \leqslant 2\left\|\tilde{\Delta} u_N\right\|_{L^2(0,V(\Omega)'} + \|f_N\|_{L^2(V')} \leqslant \|u_N\|^2_{L^2(0,T;V(\Omega)}$$

$$(-\tilde{\Delta} : V(\Omega) \to V'(\Omega) \quad \text{isometry })$$

So: $\|\partial_t u_N\|^2_{L^2(0,T;V(\Omega)')} \leqslant C$. $\forall\ w \in V(\Omega): \quad \chi \in C_0^1((0,T))$

$$\int_0^T \langle \partial_t u_N; w\rangle \chi(t) dt = -\int_0^T \langle u_N(t,\cdot), w\rangle \partial_t \chi dt$$

$$\int_0^T \langle v; w\rangle \chi(t) dt = -\int_0^T \langle u(t,\cdot), w\rangle \partial_t \chi dt$$

then $v : \partial_t u$ and $\partial_t v_N \to \partial_t u$ strongly in $L^2(V')$. Convergence in $L^2(0,T)$ implies convergence a.e. (up to a subsequence). a.e. t:

$$u_N(t,\cdot) \to u(t,\cdot) \text{strongly in } V(\Omega)$$
$$\partial_t u_N(t,\cdot) \to \partial_t u(t,\cdot) \text{strongly in } V(\Omega)'$$

a.e. t: $\forall w \in V(\Omega)$: for N fixed:

$$\langle \partial_t u_N, w \rangle_{V',V} + (\nabla u_N, \nabla w)_{L^2,L^2} = \langle f_N, w \rangle_{V',V}$$

passing into the limit gives: $\langle \partial_t; w \rangle + (\nabla u : \nabla w) = \langle f, w \rangle$. For now we have only $u \in L^\infty(0, ; L^2(\Omega))$

Lemma 3.3. *Assume that* $u \in L^2(0,T; V(\Omega))$ $\partial_t u \in L^2(0,T; (V(\Omega)'))$*, then up to a change of values on a set of measure* 0*:*

1. $u \in C(0,T; L^2(\Omega) - w$
2. $u \in C(0,T; L^2(\Omega))$
3. $\int_0^t \langle \partial_t u; u \rangle_{V',V} ds = \dfrac{\|u(t,\cdot)\|_{L^2}^2}{2} - \dfrac{\|u^0\|_{L^2}^2}{2}$

Proof. 1. (Lebesgue point on $\partial_t u$ and u) For almost every time t:

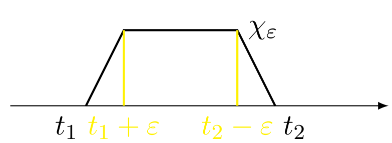
$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \|\partial_t u(s) - \partial_t u(t)\|_{V(\Omega)'} ds \to 0 \text{ as } \varepsilon \to 0$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \|u(s) - u(t)\|_{V(\Omega)'} ds \to 0 \text{ as } \varepsilon \to 0$$

Lebesgue's points in time for both u , $\partial_t u$. $\forall w \in V(\Omega), \chi_\varepsilon \in \text{Lipsh}(\Omega)$

$$\int_0^T \langle u, w \rangle \chi_c ds = -\int_0^T \chi_\varepsilon'(t) \langle u, w \rangle.$$

for t_1, t_2 Lebesgue's points:



Take $\varepsilon \to 0$

$$\frac{1}{2\varepsilon} \int_{t_2-\varepsilon}^{t_2+\varepsilon} \langle u, w \rangle ds - \frac{1}{2\varepsilon} \int_{t_1-\varepsilon}^{t_1+\varepsilon} \langle u, w \rangle ds \to \int_{t_1}^{t_2} \langle \partial_t u, w \rangle dt$$

Therefore: $\forall w \in V(\Omega)$: a.e. t_1, t_2 s.t.

$$\langle u(t_2, \cdot), w \rangle - \langle u(t_1, \cdot), w \rangle \to 0$$

$$\langle u(t_2,\cdot),w\rangle - \langle u(t_1,\cdot),w\rangle = \int_{t_1}^{t_2} \langle \partial_t u, w\rangle dt \to 0$$

uniformly as $t_2 \to t_1$

$$\int_{t_1}^{t_2} \langle \partial_t u, w\rangle \leqslant \sqrt{t_2 - t_1}\, \|\partial_t u\|_{L^2(V')} \|w\|_V$$

$\forall V \in L^2(\Omega)$: $v = \tilde{v} + \nabla P$ where $\tilde{v} in H$, by density, $\forall \varepsilon > 0$, $\exists\, v_\varepsilon \in V(\Omega)$: such that $\left\|\tilde{V} - V_\varepsilon\right\|_{L^2} \leqslant \varepsilon$.

$$|\langle u(t_2,\cdot);\tilde{v}\rangle - \langle u(t_2,\cdot);v\rangle| \leqslant \|u\|_{L^2}\, \varepsilon$$

hence $\lim_{t_2\to t_1}(u(t_2,\cdot);t) = (u(t_1,\cdot);t)$So: $u \in C(0,T;L^2 - weak)$

2.

3. Consider $d_k(t) = \langle u;\varepsilon_k\rangle$ $u_N = \sum_{k=1}^N d_k(t)\varphi_k(x)$ $u'_N = \sum_{k=1}^N d'_k(t)\varphi_k(x) \in L^2_t(C^\infty_x)$.

$$\mathrm{dt}\frac{|u_N|^2}{2} = (u'_N, u_N) = \sum_{k=1}^N d'_k d_k$$

$$\int_0^t \langle u'_N, u_N\rangle ds = \frac{\|u_N(t)\|^2_{L^2}}{2} - \frac{\left\|u^0\right\|_{L^2}}{2}$$

$u'_N \to u'$ strongly in $L^2(V')$ and $u_N \to u$ strongly in $L^2(V)$.

$$\int_0^t \langle u', u\rangle ds = \frac{\|u(t)\|^2_{L^2}}{2} - \frac{\left\|u^0\right\|_{L^2}}{2}$$

then: $\|u(t,:)\|_{L^2}$ is $C([0,T])$. Therefore with 1, we know that $u \in C(0,T;L^2(\Omega))$.

□

We have shown

Theorem 3.4. *Consider a smooth bounded Ω, $T > 0$, $f \in L^2(0,T;(V(\Omega))')$, $a^0 \in H(\Omega)$ then there exist a unique u:*

$$u \in L^2(0,T;V(\Omega)) \qquad \partial_t u \in L^2((0,T;(V(\Omega))')$$

a.e. t, $\forall w \in V(\Omega)$:

$$\langle \partial_t u, w\rangle + (\nabla u : \nabla w) = \langle f, w\rangle$$

we have also: $u \in C(0,T;L^2(\Omega))$ *and* $u(t=0)=u^0$.

$$\|u\|^2_{L^2(0,T,t)} + \|\partial_t u\|^2_{L^2(0,T;t')} \leqslant C\left(\|f\|^2_{L^2(t)} + \left\|a^0\right\|^2_{L^2}\right)$$

C *depends only on* Ω.

♣ ♣ ♣

Theorem 3.5. *For any* $a \in H(\Omega)$, $f \in L^2(0,T;V')$, *there exists a unique solution* $u \in L^2(0,T;V)$ *with* $\partial_t u \in L^2(0,T;V')$ *such that*

$$\langle \partial_t u, v\rangle + \langle \nabla u : \nabla v\rangle = \langle f, v\rangle \text{ a.e. } t \in [0,T], v \in V(\Omega). \tag{3.2}$$

And $u \in C(0,T;L^2(\Omega))$ *with* $u\big|_{t=0} = a$.

Proof of uniqueness. Consider u_1, u_2 two solutions to (3.2). Set $u = u_1 - u_2$ and for a fixed t consider $v = u(t)$.

$$\underbrace{\langle \partial_t u, u\rangle}_{\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|u|^2}{2}\,\mathrm{d}x} + \int |\nabla u|^2 \,\mathrm{d}x = 0.$$

(because $u \in L^2(0,T;V)$ and $\partial_t u \in L^2(0,T;V')$.) So

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 \,\mathrm{d}x \leq 0.$$

Since $u = 0$ at $t = 0$, we have $u_1 = u_2$ for all time. □

Theorem 3.6. *For* $a \in V(\Omega)$, $f \in L^2(0,T;H(\Omega))$. *The solution* u *of* (3.2) *verifies*

$$\|\partial_t u\|_{L^2(0,T;L^2(\Omega))} + \left\|\nabla^2 u\right\|_{L^2(0,T;L^2(\Omega))} \leq C\Big(\|f\|_{L^2(0,T;L^2(\Omega))} + \|a\|_{H^1(\Omega)}\Big).$$

where C *depends only on* Ω. *Moreover, there exists* $\nabla p \in L^2((0,T)\times\Omega)$ *such that*

$$\partial_t u + \nabla p - \Delta u = f \text{ in } \Omega$$

and

$$\|\nabla p\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \leq C\Big(\|f\|_{L^2(0,T;L^2(\Omega))} + \|a\|_{H^1(\Omega)}\Big).$$

Idea of the proof.

Step 1. Multiply the equation against $\partial_t u$, then

$$\int |\partial_t u|^2 \,\mathrm{d}x + \int (\partial_t \nabla u : \nabla u) = \int f \partial_t u.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{|\nabla u|^2}{2} \,\mathrm{d}x + \int |\partial_t u|^2 = \int f \partial_t u \leq \int \frac{|\partial_t u|^2}{2} + \frac{1}{2} \|f\|_{L^2(\Omega)}.$$

Then for any time t, integration in $(0, T)$ gives

$$\left(\int \frac{|\nabla u|^2}{2} \,\mathrm{d}x \right)(t) + \frac{1}{2} \int_0^t \int_\Omega |\partial_t u|^2 \,\mathrm{d}x \leq \frac{1}{2} \|\nabla a\|_{L^2}^2 + \frac{1}{2} \|f\|_{L^2((0,T)\times\Omega)}^2.$$

Step 2 For a.e. t,

$$\nabla p - \Delta u = f - \partial_t u \in L^2\big(0, T; L^2(\Omega)\big).$$

So

$$\big\|\nabla^2 u\big\|_{L^2(\Omega)}^2(t) \leq C\Big(\|f(t)\|_{L^2(\Omega)}^2 + \|\partial_t u(t)\|_{L^2(\Omega)}^2 \Big).$$

We now integrate in time. □

Proof. u is a solution to (3.2), so

$$u(t, x) = \sum_{k=1}^{\infty} c_k(t) \varphi_k(x)$$

where φ_k are eigenfunctions of the Stokes operator,

$$-\tilde{\Delta} \varphi_k = \lambda_k \varphi_k; \qquad \operatorname{div} \varphi_k = 0.$$

Then

$$c_k'(t) + \lambda_k c_k = f_k(t)$$

with

$$f_k(t) = \int_\Omega f(t, x) \varphi_k(x) \,\mathrm{d}x.$$

Multiply by $c_k'(t)$:

$$\lambda_k \frac{\mathrm{d}}{\mathrm{d}t}\frac{|c_k|^2}{2} + |c_k'(t)|^2 = f_k c_k' \leq \frac{|f_k|^2}{2} + \frac{|c_k'|^2}{2}.$$

So

$$\lambda_k |c_k|^2(t) + \int_0^t |c_k'(t)|^2 \,\mathrm{d}t \leq \lambda_k a_k^2 + \int_0^t |f_k(t)|^2 \,\mathrm{d}t.$$

Summation in k from 1 to ∞, we have

$$\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega |\partial_t u(t)|^2 \,\mathrm{d}x \,\mathrm{d}t \leq \|\nabla a\|^2_{L^2(\Omega)} + \|f\|^2_{L^2((0,T)\times\Omega)}.$$

Step 2 For all $v \in H_0^1(\Omega)$, time fixed, define

$$\ell_t(v) = \int_\Omega v(x)(f(t,x) - \partial_t u(t,x)) \,\mathrm{d}x - \int_\Omega \nabla v : \nabla u \,\mathrm{d}x$$

So $\|\ell_t(v)\| \leq C_t \|v\|_{H_0^1(\Omega)}$ is in H^{-1}. And $\ell_t(v) \equiv 0$ for any $v \in V(\Omega)$ because u is a weak solution to (3.2). Then we know that $\ell_t(v) = -\int p \operatorname{div} v$, for any $v \in H_0^1(\Omega)$, a.e. t,

$$\langle \nabla v : \nabla u \rangle - \langle p, \operatorname{div} v \rangle = \langle f - \partial_t u; v \rangle$$

It is the weak form of the steady Stokes with sourse term $f - \partial_t u \in L^2(\Omega)$. By steady Stokes regularity, for a.e. t,

$$\left\|\nabla^2 u\right\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)} \leq 2C\Big(\|\partial_t u\|^2_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\Big).$$

Integrating in time:

$$\left\|\nabla^2 u\right\|^2_{L^2((0,T)\times\Omega)} + \|\nabla p\|^2_{L^2((0,T)\times\Omega)} \leq 2C\Big(\|\partial_t u\|^2_{L^2((0,T)\times\Omega)} + \|f\|^2_{L^2((0,T)\times\Omega)}\Big)$$

□

We state without proof for the general case other than the L^2 setting.

Theorem 3.7. *Let Ω be bounded and regular. Let $a \in W^{1,q}(\Omega)$ with $\operatorname{div} a = 0$. For $f \in L^p(0,T;L^q(\Omega))$. Then u is a solution to (3.2) verifies*

$$\begin{aligned}&\|\partial_t u\|_{L^p(0,T;L^q(\Omega))} + \left\|\nabla^2 u\right\|_{L^p(0,T;L^q(\Omega))} + \|\nabla p\|_{L^p(0,T;L^q(\Omega))} \\ &\qquad \leq C\Big(\|a\|_{W^{1,q}(\Omega)} + \|f\|_{L^p(0,T;L^q(\Omega))}\Big).\end{aligned}$$

We can decompose $u = u_1 + u_2$ with

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla p_1 = 0 \\ u_1\big|_{t=0} = a \end{cases} \qquad \begin{cases} \partial_t u_2 - \Delta u_2 + \nabla p_2 = f \\ u_2\big|_{t=0} = 0 \end{cases}$$

For u_1, one has a smoothing effect due to Stokes: look at Fourier

$$u_1(t,x) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x),$$
$$c_k(t) = e^{-\lambda_k t} a_k.$$

For u_2,

$$\|\partial_t u_2\|_{L^p(0,T;L^q(\Omega))} + \left\|\nabla^2 u_2\right\|_{L^p(0,T;L^q(\Omega))} + \|\nabla p_2\|_{L^p(0,T;L^q(\Omega))} \\ \leq C\|f\|_{L^p(0,T;L^q(\Omega))}$$

Up to now, evolutionary Stokes equation behave similar as the heat equation.

3.2 Local regularity

Define

$$Q_1 = (-1,0) \times B_1.$$

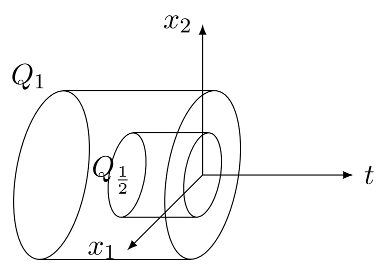


Figure 3.1: Time-space cylinder

Remark 3.8. For heat equation, $\partial_t u - \Delta u = 0$ in Q_1. If $u \in L^2(Q_1)$ then $u \in C^\infty(Q_{\frac{1}{2}})$. However this is not true for Stokes.

There exists u a solution to

$$\begin{cases} \partial_t u + \nabla p - \Delta u = f \text{ in } Q_1 \\ \operatorname{div} u = 0 \end{cases}$$

but u is not smooth. Consider $q \neq 0$ with $\Delta q = 0$ in B_1 and consider any function $f(t) \in L^2(0,T)$. Then

$$u(t,x) = f(t)\nabla q(x)$$

verifies Stokes equation (3.2):

$$\begin{aligned} \operatorname{div} u &= 0 \qquad (\Delta q = 0) \\ \partial_t u - \Delta u &= f'(t)\nabla q(x) = \nabla \underbrace{(f'(t)q)}_{\text{pressure}}. \end{aligned}$$

♣ ♣ ♣

Theorem 3.9. *(Local regularity)*

$$\begin{aligned} \partial_t u - \Delta u + \nabla P &= f \qquad \text{in } Q_2 \\ \operatorname{div} u &= 0 \end{aligned}$$

If $u \in L^p(-2,0;W^{1,q}(B_2))$ *and* $p \in L^p(-2,0;L^p(B_2))$ *and* $f \in L^m(-2,0;L^q(B_2))$ *for* $m \geqslant p$. *Then* $\left\|\nabla^2 u\right\|_{L^p(-1,0;L^q(B_1))} + \|\partial_t u\|_{L^p(-1,0;L^q(B_1))} + \|\nabla p\|_{L^p(-1,0;L^q(B_1))} \leqslant C\,\|f\|_{L^p(-2,0;L^q(B_2))} + \|u\|_{L^p(0,2;W^{1,q}(B_2))} + \|p\|_{(-2,0;L^1(B_2))}$

Remark 3.10. 1. Regularity in time cannot be gained more than that.

2. you need a control on the pressure inside Q_2

Example 3.11. Take $\Delta h(x) = 0$; $f(t) \in L^p(-2,0)$. $u(t,x) = f(t)\nabla h(x)$ $\partial_t u - \Delta h = \nabla[f'(t)h(x)]$

Proof. $\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ and $\varphi \equiv 0$ on Q_2^c and $\varphi \equiv 1$ on Q_1. Consider $v = \varphi u$ and $\operatorname{div} v = u \cdot \nabla\varphi$.
For $w(t)$:

$$\begin{aligned} -\Delta w + \nabla r = 0 &\text{ in } B_2 \\ w = 0 &\text{ on } \partial B_2 \\ \operatorname{div} w = u \cdot \nabla\varphi&. \end{aligned}$$

Steady Stokes estimate:

$$\left\|\nabla^2 w\right\|_{L^q(B_2)} + \|\nabla r\|_{L^q(B_2)} \leqslant \|u \cdot \varphi\|_{W^{1,q}(B_2)} \leqslant C_\varphi \|u\|_{W^{1,q}(B_2)}$$

$\int \left(\left\| \nabla^2 w \right\|^p_{L^q(B_2)} + \| \nabla r \|^p_{L^q(B_2)} \right) dt \leqslant \int C \| u \cdot \varphi \|^p_{W^{1,q}(B_2)} dt \leqslant \int C_\varphi \| u \|^p_{W^{1,q}(B_2)} dt$

Therefore:

$\| \nabla w \|_{L^p(L^q)(Q_2)} + \| \nabla r \|_{L^p(L^q)(Q_2)} \leqslant C \| f \|_{L^p(-2,0;L^q(B_2))} + \| u \|_{L^p(0,2;W^{1,q}(B_2))} + \| p \|_{(-2,0;L^1(B_2))}$

Let $V = v - w$,

$\partial_t V - \Delta V + \nabla(\varphi P - r) = \varphi\{\partial_t u - \Delta u + \nabla p\} - \partial_t w + \Delta w - \nabla r - u \Delta \varphi - 2 \nabla u \cdot \nabla \varphi + P \nabla \varphi$

where $P = \varphi P - r$. Therefore,

$$\begin{aligned} \partial_t V - \Delta V + \nabla P = F. \\ v \equiv 0 \text{ on } \partial B_2 \\ v \equiv 0 \text{ at } t = -2 \end{aligned}$$

where $F = \partial_t w - u \Delta \varphi - 2(\nabla u \cdot \nabla) u P \nabla \varphi$. From the global regularity result

$\| \partial_t v \|_{L^p(L^q)(Q_2)} + \left\| \nabla^2 V \right\|_{L^p(L^q)(Q_2)} + \| \nabla P \|_{L^p(L^q)(Q_2)} \leqslant \left\| \nabla^2 u \right\|_{L^p(L^q)(Q_1)} + \| \nabla p \|_{L^p(L^q)(Q_1)}$

$\leqslant C \left(\| f \|_{L^p(-2,0;L^q(B_2))} + \| u \|_{L^p(0,2;W^{1,q}(B_2))} + \| p \|_{(-2,0;L^1(B_2))} \right) + \| \partial_t w \|_{L^p(L^q)(Q_2)}$

Now we are trying to control $\partial_t w$. Let $\overline{w} = \partial_t w$, $\bar{r} = \partial_t r$

$$\begin{cases} -\Delta \bar{w} + \nabla \bar{r} = 0 \\ \bar{w} = 0 on \partial B_2 \\ \operatorname{div} \bar{w} = \partial_t u \cdot \nabla \varphi + u \cdot \partial_t \nabla \varphi ??? \end{cases}$$

Duel equation $q*$ is the conjugate of q with fixed t:

$$\begin{cases} -\Delta \tilde{w} + \nabla \tilde{r} = 0 \\ \tilde{w} = 0 \text{ on } \partial B_2 \\ \operatorname{div} \tilde{w} = 0 \end{cases}$$

Thus

$$\left\| \nabla^2 \tilde{w} \right\|_{L^{q*}(B_2)} + \| \nabla \tilde{r} \|_{L^{q*}(B_2)} \leqslant C \| g \|_{L^{q*}(B_2)}$$

Then

$$\begin{aligned} \int_{B_2} \bar{w} g(x) &= \int \bar{w}(-\Delta \tilde{w} + \nabla \tilde{r}) \\ &= \int \nabla \bar{w} \nabla \tilde{w} - \int \tilde{r} \cdot (\partial_t u \nabla \varphi + u \partial_t \varphi \nabla \phi) \\ &= - \int \tilde{w} \nabla \bar{r} - ... \end{aligned}$$

Since

$$\left|\int_{B_2} \bar{w}g(x)dx\right| \leqslant C_\varphi \|g\|_{L^{q*}} \|\tilde{r}\|_{L^{q*}} \|u\|_{L^q} + \left|-\int \tilde{r}\cdot\partial_t u \nabla\varphi\right|$$

$$= \left|-\int \tilde{r}(\Delta u - \nabla p + f)\nabla\varphi\right| \leqslant \|\nabla\tilde{r}\|_{L^{q*}} \left(\|w\|_{W^{1,q}} + \|p\|_{L^q} + \|f\|_{L^q}\right)$$

With t fixed,

$$\|\partial_t w(t)\|^p_{L^q} \leqslant C_p \left(\|u(t)\|^{W^{1,q}(B_2)} + \|f(t)\|^p_{L^q(B_2)} + \|p(t)\|^p_{L^q(B_2)}\right)$$

Hence

$$\|\partial_t w\|^p_{L^p(L^q)(Q_2)} \leqslant C\left(\|f\|_{L^p(-2,0;L^q(B_2))} + \|u\|_{L^p(0,2;W^{1,q}(B_2))} + \|p\|_{(-2,0;L^1(B_2))}\right)^p$$

□

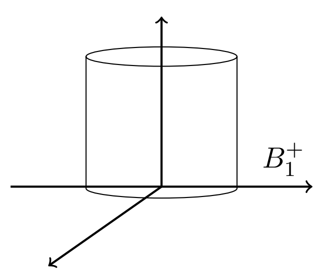


Figure 3.2: Space cylinder

Theorem 3.12. *The time space cylinder* $Q_r^+ = (-r,0)\times B_r^+$

$$\begin{cases} \partial_t - \Delta u + \nabla p = f \ \ in\ Q_2^+ \\ \operatorname{div} u = 0 \ \ in\ Q_2^+ \\ u \equiv 0 \ \ at\ z = 0 \end{cases}$$

If for $m \geqslant p$, $A := \|u\|_{L^p(-2,0;W^{1,q}(B^+))} + \|P\|_{L^p(L^q(Q_2^+)} + \|f\|_{L^m(L^q(Q_2^+)} < +\infty$
Then $\|\partial_t u\|_{L^m(L^q)(Q_1^+)} + \left\|\nabla^2 u\right\|_{L^m(L^q)(Q_1^+)} + \|\nabla p\|_{L^m(L^q(Q_1^+)} \leqslant CA$

Remark 3.13. 1. We regularize up to L^m in time of the force.

2. We cannot get higher regularization in x even if f is more regular in x.

Chapter 4

Nonlinear Navier-Stokes equation

4.1 Nonlinear problem

1. Compactness
   Typical example: $\partial_t u,\ \nabla u \in L^p([0,t] \times \Omega)$ By Sobolev: $\forall q < q*$:
   $$\frac{1}{q*} = \frac{1}{q} - \frac{1}{d}$$
   Compactness in $L^q((0,t) \times \Omega)$.

2. For evolution problem visually you have more information on ∇u than $\partial_t u$. Linear Stokes: typical spaces: $u \in L^2(0,T;V)(\|\cdot\|_V = \|\nabla u\|_{L^2})$ and $\partial_t \in L^2(0,T;V')$ weak.
   Still true that: there is compactness in $L^2(0,T;L^q(\Omega))$ $\forall q < q^*$ $\frac{1}{q*} = \frac{1}{2} - \frac{1}{3}$.

Framework:

- V, V_0, V_1 are Banach spaces. V_1 reflexive.
- $V_0 \subset\subset V \subset V_1$. The embedding $V_0 \subset V$ is compact and embedding $V \in V_1$ is continuous.

Lemma 4.1. $\forall \eta > 0\ \exists C_\eta > 0$ *such that* $\forall v \in V_0$*:* $\|v\|_V \leqslant \eta \|v\|_{V_0} + C_\eta \|v\|_{V_1}$

Proof. (Similar to Poincare, by contradiction).$\forall n \in \mathbb{N}\ \exists v_N \in V_0$ such that :

$$\|v_N\|_V > \eta \|v_N\|_{V_0} + N \|v_N\|_{V_1}$$

Set $w_N = \dfrac{v_N}{\|v_N\|_V}$:

$$1 > \eta \|w_N\|_{V_0} + N \|w_N\|_{V_1}$$

Thus:

$$\|w_N\|_{V_0} \leqslant \frac{1}{\eta} \qquad \forall\ N$$

$$\|w_N\|_{V_1} \leqslant \frac{1}{N} \to 0 \text{ as } N \to 0$$

While $\|w_n\|_V = 1$. Therefore, $w_N \to 0$ in V_1 as $N \to \infty$ w_N uniformly bounded in V_0 compact in V. So up to a subsequence: w_N converges strongly in V_0. By identification of the limit $\|w_N\|_V \to 0$. Contradiction because: $\|w_N\|_V = 1$. □

Theorem 4.2. *(Aubin-Lions Lemma) Take V, V_0, V_1 as in the Framework of the lemma and V_0 is reflexive. For any $1 < p_1, p_0 < +\infty$, for any $t < \infty$. $W = \left\{u : \|u\|_W = \|u\|_{L^{p_1}(0,T;V_1)} + \|u_0\|_{L^{p_0}(0,T;V_0)} < \infty\right\}$. W is compact in $L^{p_0}(0,T;V_0)$.*

Proof. Consider $\{v_j\} : \|v_j\|_W \leqslant C$.

1. Up to a subsequence:
   $v_j \to v$ weakly, in $L^{p_0}(0,T;V_0)$
   $\partial_t v_j \to \partial_t v$ weakly, in $L^{p_1}(0,T;V_1)$
   $v \in W$. $w_j = v_j - v \in W$. We want to show that $w_j \to 0$ in $L^p(0,T;V)$.

2. Claim: it is enough to show that $w_j \to 0$ in $L^{p_1}(0,T;V_1)$.
   Assume convergence in $L^{p_1}(0,T;V_1)$, $\forall \eta$ and a.e. $t \in (0,T]$:

$$\|w_j(t,\cdot)\|_V \leqslant \eta \|w_j(t,\cdot)\|_{V_0} + C_\eta \|w_j(t,\cdot)\|_{V_1}$$

$$\|w_j\|_{L^p(0,T;V)} \leqslant \eta \|w_j(t,\cdot)\|_{L^{p_0}(0,T;V_0)} + C_\eta \|w_j(t,\cdot)\|_{L^{p_1}(0,T;V_1)}$$

   So $\limsup_{j\to+\infty} \|w_j\|_{L^{p_0}(0,T;V)} \leqslant \eta C$ True for $\forall \eta > 0$, so $\|w_j\|_{L^{p_0}(0,T;V)} \to 0$

3. Show that actually $w_j \to 0$ in $L^{p_0}(0,T;V_1)$. $\forall t < s$:

$$w_j(t) = w_j(s) - \int_t^s \partial_t w_j(\tau) d\tau$$

   New value in time: $\frac{1}{s_1 - t} \int_t^{s_1} ds$.

$$w_j(t) = \frac{1}{s_1 - t} \int_t^{s_1} w_j(s) ds - \int_t^{s_1} \int_t^s \partial_t w_j(\tau) d\tau$$

$$\sup_{t\in[0,T]} \|w_j(t)\|_{V_1} \leqslant \|w_j\|_{L^{p_1}(0,T;V_1)} (s_1 - t)^{\frac{1}{p*}-1}$$

while

$$\left|\int_t^{s_1} \int_t^s \partial_t w_j(\tau) d\tau\right| \leqslant \int_t^{s_1} (s-t)^{1/p*} \|\partial_t w_j\|_{L^{p_1}(V_1)} \leqslant C(s-t)^{\frac{1}{p_1*}} \|\partial_t w_j\|_{L^{p_1}(V_1)}.$$

So:

$$\sup_{t\in(0,T]} \|w_j(t)\|_{V_1} \leqslant C$$

uniformly. While $\varepsilon = s_1 - t$

$$\|w_j\|_{V_1} \leqslant C \|\partial_t w_j\|_{L^{p_1}(0,T;V_1)} (s_1 - t)^{1/p*} + \left\|\frac{1}{s_1 - t}\int_t^{s_1} w_0(s) ds\right\|_{V_1}$$
$$C\varepsilon^{1/p*} + \left\|W_j^\varepsilon\right\|_{V_1}$$

For ε fixed, t fixed: $\left\|W_j^\varepsilon\right\|_{V_0} \leqslant C_\varepsilon$. For ε fixed: $\left\|W_j^\varepsilon\right\|_{V_1} \to 0$ as $j \to \infty$ and compact in V_1. For ε fixed : $\left\|W_j^\varepsilon\right\|_{V_1} \to 0$ as $j \to +\infty$. $\text{esssup}_j \|w_j(t)\|_{V_1} \leqslant C\varepsilon^{1/p_1*}$ true $\forall\ \varepsilon$. So: $\|w_j(t)\|_{V_1} \to 0$ as $j \to +\infty$. By the Lebesgue's Dominated convergence theorem, we have $w_j \to 0$ $L^{p_0}(0,T;V)$ compactly.

□

♣ ♣ ♣

Lemma 4.3. *If $u \in L^\infty\big(0,T;L^2(\Omega)\big)$ and $\nabla u \in L^2((0,T)\times\Omega)$, then $u \in L^p(0,T;L^q(\Omega))$ for*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \qquad \textit{and} \qquad 2 \le p \le \infty.$$

If Ω is bounded then

$$u \in L^p(0,T;L^q(\Omega)) \textit{ for } \frac{2}{p} + \frac{3}{q} \ge \frac{3}{2}.$$

Proof. By Sobolev, $\nabla u \in L^2_t L^2_x$ then

$$\|u\|_{L^2(0,T;L^6(\Omega))} \le C\|\nabla u\|_{L^2}.$$

Here we use Sobolev, $\frac{1}{p} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ so $H^1 \hookrightarrow L^6$. By interpolation, for all $0 \le \alpha \le 1$, $u \in L^{p_\alpha}_t L^{q_\alpha}_x$ with

$$\frac{1}{p_\alpha} = \frac{\alpha}{2} + \frac{1-\alpha}{\infty}, \qquad \frac{1}{q_\alpha} = \frac{\alpha}{6} + \frac{1-\alpha}{2}.$$

So

$$\frac{2}{p_\alpha} + \frac{3}{q_\alpha} = \frac{3}{2}, \qquad 2 \le p_\alpha \le \infty.$$

□

Remark 4.4. Note that

$$\begin{aligned} \|u\|_{L^{p_\alpha}(0,T;L^{q_\alpha}(\Omega))} &\le C\|\nabla u\|^{\alpha}_{L^2((0,T)\times\Omega)}\|u\|^{1-\alpha}_{L^\infty(0,T;L^2(\Omega))}, \\ \|u\|_{L^{\frac{10}{3}}_{t,x}} &\le C\|\nabla u\|^{\bar\alpha}_{L^2((0,T)\times\Omega)}\|u\|^{1-\bar\alpha}_{L^\infty(0,T;L^2(\Omega))} \end{aligned}$$

and $\frac{10}{3} > 3$.

Now consider Navier–Stokes equation: Ω is a bounded smooth domain, $T > 0$,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p - \Delta u = f & \text{in } (0,T)\times\Omega \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u\big|_{t=0} = u^0. \end{cases} \tag{NS}$$

where

$$\begin{aligned} &u^0 \in H = \big\{u \in L^2(\Omega) : \operatorname{div} u = 0, u\cdot n = 0 \text{ on } \partial\Omega\big\}, \\ &f \in L^1\big(0,T;L^2(\Omega)\big) + L^2\big(0,T;V(\Omega)'\big). \end{aligned}$$

Definition 4.5. We say that u is a Leray–Hopf weak solution to (NS) if

- $u \in L^\infty\big(0,T;L^2(\Omega)\big)$ and $\nabla u \in L^2(0,T;V(\Omega))$.
- $\forall \varphi \in C^\infty_{0,0}((0,T)\times\Omega) = \{\varphi \in C^\infty_c : \operatorname{div}\varphi = 0\}$,

$$\int_0^T\int_\Omega -u\partial_t\varphi - u^\top\cdot\nabla\varphi\cdot u + \nabla\varphi : \nabla u - f\cdot\varphi = 0.$$

- $u \in C\big(0, T; L^2_{\mathrm{w}}(\Omega)\big)$, that means for all $t \in (0, T)$ and for all $\psi \in L^2(\Omega)$,

$$\operatorname*{ess\,lim}_{s \to t} \int_\Omega u(s, x)\psi(x)\,\mathrm{d}x = \int_\Omega u(t, x)\psi(x)\,\mathrm{d}x.$$

- $\lim\limits_{t \to 0} \int_\Omega \big|u(t, x) - u^0(x)\big|^2\,\mathrm{d}x = 0.$

- Energy inequality: for all $0 \le t \le T$,

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \frac{1}{2}\big\|u^0\big\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f \cdot u\,\mathrm{d}x\,\mathrm{d}s.$$

Remark 4.6. Formally, multiply (NS) by u and then integrate,

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \Delta u = f.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|u|^2}{2}\,\mathrm{d}x + \int_\Omega (u \cdot \nabla)\frac{|u|^2}{2} + \int u \cdot \nabla p - \int \Delta u \cdot u = \int f \cdot u.$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|u|^2}{2}\,\mathrm{d}x \\
&\quad - \int_\Omega (\operatorname{div} u)\frac{|u|^2}{2} + \int_{\partial\Omega} \frac{|u|^2}{2} u \cdot n \\
&\quad - \int_\Omega (\operatorname{div} u)p + \int_{\partial\Omega} p(u \cdot n) \\
&\quad + \int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega n^\top \nabla u \cdot u \\
&\quad = \int_\Omega f_1 \cdot u + \int_\Omega f_2 \cdot u.
\end{aligned}$$

4.1.1 Galerkin method

For any $j \in \mathbb{N}$, ,we want to construct

$$u_j = \sum_{k=1}^{j} u_j^k(t)\psi_k(x),$$

where $\{\psi_k\}_{k=1}^{\infty}$ are the eigenfunctions of Stokes in Ω:

$$\begin{cases} -\tilde{\Delta}\psi_k = \lambda_k \psi_k, & \\ \operatorname{div}\psi_k = 0 & \text{in } \Omega \\ \psi_k = 0 & \text{on } \partial\Omega \end{cases}$$

We want: for all $\varphi \in \big\{C_{0,0}^{\infty}((0,T)\times\Omega) : \forall t, \varphi(t) \in \operatorname{Span}\{\psi_1,\dots,\psi_j\}\big\}$.

$$\int_0^T \int_\Omega \Big(-\partial_t\varphi \cdot u_j - u_j^\top \cdot \nabla\varphi \cdot u_j + \nabla u_j : \nabla\varphi - f\cdot\varphi\Big)\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{4.1}$$

Claims that: for all j, there exists a unique such u_j.

(4.1) is equivalent to $\forall\tilde{\varphi} \in C_0^\infty(0,T)$: for all $k \in \{1,\dots,j\}$,

$$\begin{aligned} &\int_0^T -\tilde{\varphi}'(t)\sum_{i=1}^j \int \psi_k(x)\psi_i(x)\,\mathrm{d}x\; u_j^i(t) \\ &\quad - \int_0^T \tilde{\varphi} \sum_{i,l=1}^j \int_\Omega [\psi_i(x)\cdot\nabla\psi_k(x)\cdot\psi_l(x)]\,\mathrm{d}x\; u_j^i(t)u_j^l(t) \\ &\quad + \sum_{i=1}^j \int_0^T \Big(\int_\Omega \nabla\psi_k : \nabla\psi_l\,\mathrm{d}x\Big)\tilde{\varphi}(t)u_j^l(t) \\ &\quad - \sum_{i=1}^j \int_0^T \int_\Omega f\cdot u\,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned}$$

Suppose $f = \sum_{k=1}^j f_k(t)\psi_k(x)$. By orthogonality,

$$\begin{aligned} &-\int_0^T \tilde{\varphi}'(t)u_j^k(t)\,\mathrm{d}t \\ &\qquad - \int_0^T \tilde{\varphi}(t)\sum_{i,l=1}^j a_{ikl}u_j^i(t)u_j^l(t)\,\mathrm{d}t \\ &\qquad + \int_0^T \tilde{\varphi}(t)\lambda_k u_j^k(t) \\ &\qquad - \int_0^t \tilde{\varphi}(t)f_k(t) = 0. \end{aligned}$$

So we want to solve for $1 \le k \le j$,

$$(u_j^k)'(t) + \lambda_k u_j^k(t) = \sum_{i,l=1}^j a_{ikl}u_j^i(t)u_j^l(t) + f_k(t).$$

♣ ♣ ♣

Reminder: we want to construct a solution to

$$\begin{aligned}\partial_t u + (u\cdot\nabla)u + \nabla P - \Delta u &= f \in L^2(0,T,[V(\Omega)]')\\ \operatorname{div} u &= 0\\ u|_{t=0} &= u^0 \in L^2(H)\\ u &\in L^2(0,T;V(\Omega)))\end{aligned}$$

In the sense of distribution: $\forall\ \phi \in C_c^\infty((0,T)\times\Omega)$:

$$-\iint (\partial_t\phi u + u^T\cdot\phi u - \nabla : \nabla\phi - f\cdot u) = 0 \tag{4.2}$$

$$W_j = \text{Span}\{\Psi_1, ..., \Psi_j\}$$

and

$$-\tilde\Delta\Psi_k = \lambda_k\Psi_k$$

Goal: (Galerkin Method) $\exists!\ u_j \in C(0,T;W_j)$, $\forall\ \phi\in C^1(0,T;W_j)$: Projection on Ψ_k

$$u_j^k = \sum_{k=1}^j u_j^k\Psi_k \qquad a_{i,k,l} = \int \Psi_j^T\nabla\Psi_k\Psi_l(x)dx \qquad f_k(t) = \int_\Omega f(t,x)\Psi_k(x)dx$$

$$\partial_t u_j + \sum_{k,l} u_j^k u_j^l a_{i,k,l} - \lambda_k h_j^k = f_k \qquad u_j^k(0) = \int_\Omega u^0(x)\Psi_k(x)dx$$

By Cauchy-Lipschitz (for ODE's): there exists a small time t_0 such that u_j^k has a unique solution on $[0,t_0)$. If $t_0 < T$, then $\|u_j\|_{l^\infty}\to+\infty$ as $t\to t_0$.

Taking $\phi = \varphi u_j = \varphi\sum_{k=1}^j u_j^k\Psi_k$

$$-\iint \varphi\left(\partial_t u_j u_j + u_j^T\nabla u_j\cdot u_j + |\nabla u_j|^2 - f\cdot u_j\right) = 0$$

$$-\iint\varphi(t)\left(\partial_t\int\frac{|u_j|^2}{2}dx\right)dt - \int\varphi\int(u_j\cdot\nabla u_j)\cdot u_j dxdt + \int\varphi(t)(|\nabla u_j|^2dx)dt - \int\varphi(t)(f\cdot u_j dx)dt = 0$$

The second term $= -\int \varphi(t) \int u_j \nabla \frac{|u_j|^2}{2} dxdt = 0$. Take $\varphi = \mathbb{I}_{[0,\bar{t}]}$. Then

$$\begin{aligned}\int_\Omega \frac{|u_j(t,x)|^2}{2} dx + \iint |\nabla u_j|^2 dxdt &= \int_\Omega \left|u_j^0(x)\right|^2 dx + \int_0^{\bar{t}} \int_\Omega (f u_j) dx \\ &\leqslant \int_\Omega \left|u^0\right|^2 dx + \|f\|_{L^2(0,T;[V(\Omega)]')} \|u_j\|_{L^2(0,T;(V(\Omega))} \\ &\leqslant \int_\Omega \left|u^0\right|^2 dx \, \|f\|_{L^2(0,T;[V(\Omega)]')} \|u_j\|_{L^2(0,T;(V(\Omega))},\end{aligned}$$

using

$$\|f\|_{L^2(0,T;[V(\Omega)]')} \|u_j\|_{L^2(0,T;(V(\Omega))} \leqslant \frac{1}{2} \iint |\nabla u_j|^2 dxdt + 2 \|f\|^2_{L^2(0,T;(V(\Omega))}$$

So

$$\sup_{\bar{t} \leqslant t_0} \int \frac{|u_j(\bar{t},x)|}{2} dx + \frac{1}{2} \int_0^t \int_\Omega |\nabla u_j|^2 dxdt \leqslant \int_\Omega \left|u^0\right|^2 xd + 2 \|f\|^2_{L^2(0,T;V'(\Omega))} := \tilde{C}$$

So $\{u_j^k(t)\}$ are uniformly bounded on $(0, t_0)$. So the solution exists up to T.
We showed that a priori estimate:

$$\|u_j\|^2_{L^\infty(0,T;H(\Omega))} + \|u_j\|^2_{L^2(0,T;V(\Omega))} \leqslant \tilde{C}$$

Define

$$\tilde{H}^k(\Omega) = \{\sum_{l=1}^{+\infty} \alpha_l \Psi_l(x); \sum_{l=1}^{\infty} \alpha_l^2 \lambda_l^{2k} < \infty\}$$

$\tilde{H}^0(\Omega) = H(\Omega)$, $\tilde{H}^1 = V(\Omega)$ and $\tilde{H}^2(\Omega) \subset H^2(\Omega)$.
Claim $\partial_t u_j$ is uniformly bounded in $L^2(0,T;[\tilde{H}^2(\Omega)]')$. $\forall\, \phi \in L^2(0,T;\tilde{H}^2(\Omega))$:

$$\iint \partial_t u_j \phi = -\int u_j \partial_t \phi = -\iint \sum_{k=1}^{\infty} u_j^k(t) \partial_t \phi(t) = -\iint u_j \partial_t \phi^j$$

where $\phi^j = \sum_{k=1}^j \phi_k \Psi_k(x) \in W_j$.
Therefore,

$$-\iint u_j \partial_t \phi^j = \iint u_j^T \nabla \phi^j u_j - \iint \nabla u_j \nabla \phi^j + \iint f \phi^j$$

$u \in L^\infty(L^2) \cap L^2(H^1)$ then $u \in L^p(L^q)$, where $\frac{2}{p} + \frac{3}{q} \geqslant 3/2$, and $u \in L^4(0,T;L^3(\Omega))$ with $|u|^2 \in L^2(0,T;L^{3/2}(\Omega))$. Thus

$$\begin{aligned}\left|\iint u_j^T \nabla\phi^j u_j\right| &\leqslant \left\|u^2\right\|_{L^2(0,T;L^{3/2}(\Omega))} \left\|\nabla\phi^i\right\|_{L^2(0,T;L^3(\Omega))}\\ &\leqslant \tilde{C}\left\|\nabla\phi^j\right\|_{L^2(0,T;L^6(\Omega)))}\\ &\leqslant \tilde{C}\left\|\phi^j\right\|_{L^2(0,T;\tilde{H}^2(\Omega))}\\ &\leqslant \tilde{C}\left\|\phi\right\|_{L^2(0,T;\tilde{H}^2(\Omega))}\end{aligned}$$

$$\begin{aligned}\left|\iint \nabla u_j \cdot \nabla\phi^j\right| &\leqslant \left\|\nabla u_j\right\|_{L^2(0,T;L^2(\Omega))} \left\|\nabla\phi^j\right\|_{L^2(0,T;L^2(\Omega))}\\ &\leqslant \left\|\nabla^2\phi^j\right\|_{L^2(0,T;L^2(\Omega))} \qquad (Poincare)\\ &\leqslant \tilde{C}\left\|\phi\right\|_{L^2(\tilde{H}^2)}\end{aligned}$$

$$\begin{aligned}\left|\iint f\phi^j\right| &\leqslant \left\|f\right\|_{L^2(V(\Omega)')} \left\|\phi^j\right\|_{L^2 0,T;V(\Omega))}\\ &\leqslant \tilde{C}\left\|\phi\right\|_{L^2(0,T;\tilde{H}^2(\Omega))}\end{aligned}$$

$$|\langle\partial_t u;\phi\rangle| \leqslant \tilde{C}\left\|\phi\right\|_{L^2(0,T;\tilde{H}^2(\Omega))}$$

So $\partial_t u$ is uniformly bounded in $L^2(0,T;(\tilde{H}^2(\Omega))')$. We use the Aubin-Lions Lemma with $V_0 \subset V \subset V_1$, where $V_0 = V(\Omega)$; $V_1 = (\tilde{H}^2(\Omega))'$; $V = H$.

We have that u^j uniformly bounded in $L^2(0,T;V_0)$ and $\partial_t u^j$ uniformly bounded $L^2(0,T;V_1)$. So $\{u^j\}$ is compact in $L^2(0,T;H)$. (in $L^2((0,T)\times\Omega)$).

Up to a subsequence, $u_j \to u$ strongly in $L^2(L^2)$, and $\nabla u_j \to \nabla u$ weakly in $L^2(L^2)$ and $u \in L^\infty(L^2)$. And pushing into the limit (weakly) in the energy equality, for a.e. t:

$$\frac{1}{2}\left\|u(t,\cdot)\right\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\nabla u|^2 dxds \leqslant \frac{1}{2}\left\|u^0\right\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega f\cdot udsdx$$

If we fix $\phi \in C_c^\infty(0,T)\times\Omega$; $\forall j \geqslant j_0$: u^j verifies (4.2) for ϕ.

$$-\iint \partial_t phi u_j + u_j^T\nabla\phi\cdot u_j - \nabla\phi : \nabla u_j + f\cdot\phi = 0$$

converges to the following equation as $j\to\infty$

$$-\iint \partial_t\phi + u^T\nabla\phi u - \nabla\phi : \nabla u + f\cdot\phi = 0$$

Then true for any $W_{j,0}$, the time for any $\phi \in C_c^\infty((0,T) \times \Omega)$.

♣ ♣ ♣

Recall that for the Navier–Stokes equation,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p - \Delta u = f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u\big|_{t=0} = u^0 & \text{in } \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \end{cases} \tag{4.3}$$

We have already constructed

- $u \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2(0,T; V(\Omega))$. Recall that $V(\Omega)$ implies $\operatorname{div} u = 0$ in the sense of distribution and $u = 0$ on $\partial\Omega$ in the sense of trace.
- $\partial_t u \in L^2\Big(0,T; [\tilde{H}^2(\Omega)]'\Big)$.
- u is a solution to (4.3) in the sense of distribution
- For a.e. t, $\|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \le \left\|u^0\right\|_{L^2(\Omega)} + \int_0^t \int_\Omega f \cdot u \,\mathrm{d}s\,\mathrm{d}x$.
- $\forall p \ge 2$ and q such that $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, $u \in L^p(0,T; L^q(\Omega))$.

4.1.2 Initial value

Proposition 4.7. *Denote $L^2_{\mathrm{w}}(\Omega)$ to be the $L^2(\Omega)$ with weak topology. Then $u \in C(0,T; L^2_{\mathrm{w}}(\Omega))$.*

Remark 4.8. For now, $u \in L^p(0,T; L^q(\Omega))$, $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, $p \ge 2$. So for a.e. t, $u(t) \in L^q(\Omega)$. We claim that there is ONE representation of u which is $C(0,T; L^2_{\mathrm{w}}(\Omega))$.

Proof. **Step 1.** Consider $\varphi \in \tilde{H}^2(\Omega)$. Claim that

$$h_\varphi(t) = \int_\Omega u(t,x)\varphi(x)\,\mathrm{d}x \in C^{\frac{1}{2}}(0,T).$$

This is because, for a.e. s, t,

$$\begin{aligned} \frac{|h_\varphi(t) - h_\varphi(s)|}{|t-s|^{\frac{1}{2}}} &\le \frac{\left|\int_s^t \int_\Omega \partial_t u \varphi \,\mathrm{d}x\,\mathrm{d}\tau\right|}{|t-s|^{\frac{1}{2}}} \\ &\le \|\partial_t u\|_{L^2\left(s,t;[\tilde{H}^2(\Omega)]'\right)} \|\varphi\|_{\tilde{H}^2(\Omega)} \frac{\|1\|_{L^2(s,t)}}{|t-s|^{\frac{1}{2}}}. \end{aligned}$$

Here $\|1\|_{L^2(s,t)} = |t-s|^{\frac{1}{2}}$. Therefore, for all $\varphi \in \tilde{H}^2(\Omega)$, $h_\varphi(t) \in C(0,T)$.

Step 2. Fix $0 \le \tilde{t} \le T$, consider $t_n \to \tilde{t}$ with $n \to \infty$, $u(t_n) \in L^2(\Omega)$. Then $u(t_n)$ is uniformly bounded in $L^2(\Omega)$, so up to a subsequence it converges in $L^2_{\mathrm{w}}(\Omega)$, denote $u(t_n) \xrightharpoonup{L^2(\Omega)} g \in L^2(\Omega)$. Especially, for every $\varphi \in \tilde{H}^2(\Omega)$,

$$\int_\Omega \varphi(x) u(t_n, x)\,\mathrm{d}x \to \int_\Omega \varphi(x) g(x)\,\mathrm{d}x$$

It also converges to $h_\varphi(\tilde{t})$. Combined we have

$$h_\varphi(\tilde{t}) = \int_\Omega \varphi(x) g(x)\,\mathrm{d}x$$

for every $\varphi \in \tilde{H}^2(\Omega)$. We have showed that

$$\operatorname*{ess\,lim}_{t\to\tilde{t}} u(t_n) = g \text{ in } [\tilde{H}^2(\Omega)]'.$$

Step 3. $\forall \varphi \in L^2(\Omega)$, $\forall \varepsilon > 0$, $\exists \varphi_\varepsilon \in \tilde{H}^2(\Omega)$ with $\|\varphi_\varepsilon - \varphi\|_{L^2(\Omega)} \le \varepsilon$ by density. For a.e. $t \in [0,T]$,

$$\begin{aligned}
&\overline{\operatorname*{ess\,lim}_{t\to\tilde{t}}}\left|\int u(t,x)\varphi(x)\,\mathrm{d}x - \int u(\tilde{t},x)\varphi(x)\,\mathrm{d}x\right| \\
&\quad\le \overline{\operatorname*{ess\,lim}_{t\to\tilde{t}}}\left|\int u(t,x)\varphi_\varepsilon(x)\,\mathrm{d}x - \int u(\tilde{t},x)\varphi_\varepsilon(x)\,\mathrm{d}x\right| \\
&\qquad + 2\|\varphi - \varphi_\varepsilon\|_{L^2(\Omega)}\|u\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad\le 0 + 2\varepsilon\|u\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}$$

(esslim means take limits over t where $u(t,x)$ exists) This is true for all $\varepsilon > 0$, so

$$\int u(t,x)\varphi(x)\,\mathrm{d}x \to \int u(\tilde{t},x)\varphi(x)\,\mathrm{d}x \qquad \text{as } t \to \tilde{t}$$

for all $\varphi \in L^2(\Omega)$. This means that $u \in C(0,T; L^2_{\mathrm{w}}(\Omega))$. □

Proposition 4.9. $\lim_{t\to 0}\left\|u(t) - u^0\right\|_{L^2(\Omega)} = 0$.

Remark 4.10. u is continuous at $t = 0$ in $L^2(\Omega)$ strongly. It is known only at $t = 0$.

Proof. **Step 1.** By weak continuity,

$$\lim_{t\to 0} \int u(t,x)u^0(x)\,\mathrm{d}x = \int_\Omega |u^0(x)|^2\,\mathrm{d}x.$$

Step 2. For a.e. $t>0$,

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{2}\left\|u^0\right\|^2_{L^2(\Omega)} + \underbrace{\int_0^t \int_\Omega f\cdot u\,\mathrm{d}x\,\mathrm{d}s}_{\le \int_0^t \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^t \|f\|_{[V(\Omega)]'}\,\mathrm{d}s}.$$

So

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\left\|u^0\right\|^2_{L^2(\Omega)} + \underbrace{\int_0^t \|f\|_{[V(\Omega)]'}\,\mathrm{d}s}_{\to 0,\text{ dominated convergence theorem}}.$$

$$\left\|u(t)-u^0\right\|^2_{L^2(\Omega)} = \|u(t)\|^2_{L^2(\Omega)} + \left\|u^0\right\|^2_{L^2(\Omega)} - 2\int_\Omega u(t,x)u^0(t,x)\,\mathrm{d}x.$$

As $t\to 0$,

$$\limsup_{t\to 0}\left\|u(t)-u^0\right\|^2_{L^2(\Omega)} \le 2\left\|u^0\right\|^2_{L^2(\Omega)} - 2\left\|u^0\right\|^2_{L^2(\Omega)} = 0.$$

□

Theorem 4.11 (Ladyžhenskaya–Prodi–Serrin criterion)**.** *Consider $u^0 \in V(\Omega)$, $f \in L^2((0,T)\times\Omega)$, and any weak solution u to* (4.3)*. If*

$$u \in L^p(0,T;L^q(\Omega)), \qquad \frac{2}{p}+\frac{3}{q}=1$$

then

$$\nabla u \in L^\infty\big(0,T;L^2(\Omega)\big), \qquad \nabla^2 u, \partial_t u \in L^2((0,T)\times\Omega).$$

Remark 4.12. If $u^0 \in C^\infty(\Omega)$, and $f\in C^\infty((0,T)\times\Omega)$, then by bootstrapping we have $u \in C^\infty((0,T)\times\Omega)$.

Remark 4.13. Recall that any weak solution to (4.3) verifies $u\in L^p(0,T;L^q(\Omega))$ for $p\ge 2$ and $\frac{2}{p}+\frac{3}{q}=\frac{3}{2}$. The gap is called super–critical.

Proof. Take the equation

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \Delta u = f.$$

We multiply by $-\Delta u$ and integrate by part,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla u|^2}{2}\,\mathrm{d}x + \int |\Delta u|^2\,\mathrm{d}x \\
&\quad = -\int f\cdot\Delta u\,\mathrm{d}x - \int u\cdot\nabla u\cdot\Delta u\,\mathrm{d}x \\
&\quad \le \frac{\|f(t)\|^2_{L^2(\Omega)}}{2} + \frac{\|\Delta u\|^2_{L^2(\Omega)}}{2} + \frac{\|\Delta u\|^2_{L^2(\Omega)}}{4} + 2\||u||\nabla u|\|^2_{L^2(\Omega)} \\
&\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla u|^2}{2}\,\mathrm{d}x + \int |\Delta u|^2\,\mathrm{d}x \\
&\quad \le \frac{\|f(t)\|^2_{L^2(\Omega)}}{2} + C\|u(t)\|^2_{L^q(\Omega)}\|\nabla u\|^2_{L^{q*}(\Omega)} \\
&\quad \le \frac{\|f(t)\|^2_{L^2(\Omega)}}{2} + C\|u(t)\|^2_{L^q(\Omega)}\|\nabla u\|^{2\alpha}_{L^2(\Omega)}\left\|\nabla^2 u\right\|^{2(1-\alpha)}_{L^2(\Omega)}
\end{aligned}$$

with $\frac{1}{q}+\frac{1}{q*}=\frac{1}{2}$, $\frac{1}{q^*}=\frac{\alpha}{2}+\frac{1-\alpha}{6}$.

Now we use $a^{2(1-\alpha)}b^{2\alpha} \le \varepsilon a^2 + C_\varepsilon b^2$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla u|^2}{2}\,\mathrm{d}x + \frac{1}{4}\int |\Delta u|^2\,\mathrm{d}x \\
&\quad \le \frac{\|f(t)\|^2_{L^2(\Omega)}}{2} + \varepsilon\left\|\nabla^2 u\right\|^2_{L^2(\Omega)} + C_\varepsilon\left(\|u(t)\|^{\frac{2}{\alpha}}_{L^q(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\right)
\end{aligned}$$

Need: $u(t) \in L^p_t L^q_x$ with $p = \frac{2}{\alpha}$. □

♣ ♣ ♣

Proposition 4.14. *Under the same assumption, u is unique.*

Remark 4.15. If $u \in L^\infty_t L^2_x$, $\nabla u \in L^2_t L^2_x$, and $f \in L^2_t(H^{-1}_x)$, then we may have non-uniqueness. This is a very recent work of Albritton, Brie, Colombo, based on the work of Misha Vishik.

Proof. Consider two solutions u_1, u_2, with $u_2 \in L^p_t L^q_x$, and $\frac{2}{p}+\frac{3}{q}=1$.

$$\partial_t(u_1-u_2) + (u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2 + \nabla(p_1-p_2) = \Delta(u_1-u_2).$$

Multiply by $u_1 - u_2$ and then integrate,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_1-u_2|^2\,\mathrm{d}x + \int_\Omega (u_1\cdot\nabla)\frac{|u_1-u_2|^2}{2}\,\mathrm{d}x + \int_\Omega (u_1-u_2)\cdot\nabla u_2\cdot(u_1-u_2)\,\mathrm{d}x + \int_\Omega |\nabla(u_1-u_2)|^2\,\mathrm{d}x \le 0.$$

So

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int |u_1-u_2|^2\,\mathrm{d}x &+ \int |\nabla(u_1-u_2)|^2\,\mathrm{d}x \\ &\le \frac{1}{2}\|\nabla(u_1-u_2)\|_{L^2}^2 + 2\int |u_2|^2|u_1-u_2|^2\,\mathrm{d}x \\ &\le \frac{1}{2}\|\nabla(u_1-u_2)\|_{L^2}^2 + \|u_2\|_{L^q}^2\|u_1-u_2\|_{L^{q^*}}^2\end{aligned}$$

with $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$.

We claim that $\frac{1}{q^*}=\frac{\alpha}{2}+\frac{1-\alpha}{6}$. With this,

$$\begin{aligned}\|u_2\|_{L^q}^2\|u_1-u_2\|_{L^{q^*}}^2 &\le \|u_2\|_{L^q}^2\|u_1-u_2\|_{L^2}^{2\alpha}\|u_1-u_2\|_{L^6}^{2(1-\alpha)} \\ &\le \frac{1}{4}\|\nabla(u_1-u_2)\|_{L^2}^2 + C\|u_2\|_{L^q}^{\frac{2}{\alpha}}\|u_1-u_2\|_{L^2}^2.\end{aligned}$$

We want to show that $\|u_2\|_{L^q}^{\frac{2}{\alpha}}$ is integrable in time. Then

$$\int |u_1-u_2|^2\,\mathrm{d}x \le \int |u_1^0-u_2^0|^2\,\mathrm{d}x\exp\left(\tilde{C}\int_0^T \|u_2\|_{L^q(\Omega)}^{\frac{2}{\alpha}}\,\mathrm{d}t\right)$$

provided that $\int\|u_2\|_{L^q}^{\frac{2}{\alpha}}\,\mathrm{d}t<\infty$. By computation, $\frac{2}{\alpha}=p$. □

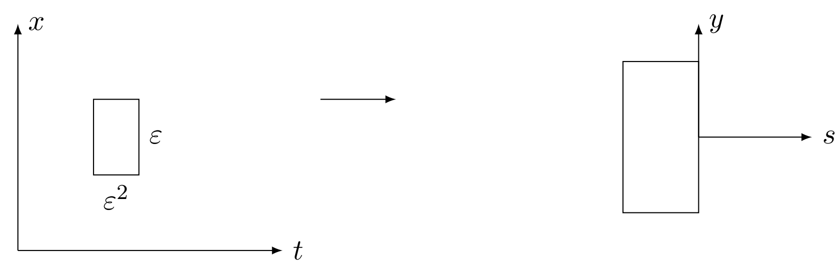
4.2 Universal scaling and applications

4.2.1 Universal scaling of Navier–Stokes equation

u is a solution to Navier–Stokes in $(t-\varepsilon^2,t)\times B_\varepsilon(x)$, if and only if $\tilde{u}_\varepsilon(s,y)=\varepsilon u(t+\varepsilon^2 s, x+\varepsilon y)$ is a solution to Navier–Stokes in $Q_1(0,0)$.

Proof.

$$\begin{aligned}\partial_s\tilde{u}_\varepsilon &= \varepsilon^3\partial_t u(t+\varepsilon^2 s, x+\varepsilon y) \\ (\tilde{u}\cdot\nabla)\tilde{u} &= \varepsilon^3(u\cdot\nabla u)(t+\varepsilon^2 s, x+\varepsilon y) \\ \nabla\tilde{p}_\varepsilon &= \varepsilon^3 p(t+\varepsilon^2 s, x+\varepsilon y) \\ \Delta\tilde{u}_\varepsilon &= \varepsilon^{1+2}\Delta u(t+\varepsilon^2 s, x+\varepsilon y)\end{aligned}$$



So

$$\partial_s \tilde{u}_\varepsilon + (\tilde{u}_\varepsilon \cdot \nabla)\tilde{u}_\varepsilon + \nabla \tilde{p}_\varepsilon = \Delta \tilde{u}_\varepsilon.$$

□

4.2.2 Invariant spaces through the universal scaling

If $\frac{2}{p} + \frac{3}{q} = 1$, then

$$\|u\|_{L^p(L^q)(Q_\varepsilon(t,x))} = \|\tilde{u}_\varepsilon\|_{L^p(L^q)(Q_1(0,0))}.$$

Proof.

$$\begin{aligned}
\|\tilde{u}_\varepsilon\|_{L^p(L^q)(Q_1(0,0))} &= \left(\int_{-1}^{0}\left(\int_{B_1} |\tilde{u}_\varepsilon(s,y)|^q \,\mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}s\right)^{\frac{1}{p}} \\
&= \left(\int_{-1}^{0} \varepsilon^{-2}\left(\int_{B_1} \varepsilon^{q-3}|\tilde{u}_\varepsilon(t+\varepsilon^2 s, x+\varepsilon y)|^q \varepsilon^3 \,\mathrm{d}y\right)^{\frac{p}{q}} \varepsilon^2 \,\mathrm{d}s\right)^{\frac{1}{p}} \\
&= \varepsilon^{-\frac{2}{p}+1-\frac{3}{q}}\left(\int_{t-\varepsilon^2}^{t}\left(\int_{B_\varepsilon(x)} u^q(t,x)\,\mathrm{d}x\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}} \\
&= \varepsilon^{1-\frac{2}{p}-\frac{3}{q}}\|u\|_{L^p(L^q)(Q_\varepsilon(t,x))}.
\end{aligned}$$

□

Remark 4.16. In dimension N, the invariant space is $\frac{2}{p}+\frac{N}{q} = 1$. In dimension 3, $p = q = 5$ is the homogeneous space. In dimension 2, $p = q = 4$ is the homogeneous space.

Remark 4.17. In dimension 2, we control the universal scalingvia the a priori estimate. $u \in L^\infty(L^2)$, $\nabla u \in L^2(L^2)$. Then

$$\nabla(|u|^2) = u \cdot \nabla u \in L^2(L^1).$$

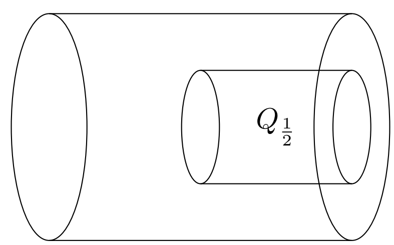
By Sobolev embedding, $u^2 \in L^2(L^2)$, so $u \in L^4(L^4)$, which is on the critical scaling.

4.2.3 ε-regularity theory

Lemma 4.18. *There exists $\eta > 0$ such that if u is a "suitable" weak solution to the Navier–Stokes equation in Q_1, such that*

$$\int_{Q_1} |u|^3 + |p|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t \le \eta,$$

then for all $n \in \mathbb{N}$, $|\nabla^n u(t,x)| \le C_n$ in $Q_{\frac{1}{2}}$.



Q_1

The idea is to control the flux of the energy. Note that

$$\partial_t u - \Delta u + \nabla p = -\operatorname{div}(u \otimes u)$$

where the right hand side is quadratic. So if we have enough smallness, the force is small, so we treat it perturbatively as a heat equation. There are several ways to prove it, and I prefer the De Giorgi method.

Definition 4.19. We say that u is a suitable solution to the Navier–Stokes equation, if it is a Leray–Hopf solution, and it verifies in the sense of distribution that

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \le 0$$

in the sense of distribution.

Construction: construct first the solution to

$$\begin{cases} \partial_t u_\varepsilon + \tilde{u}_\varepsilon \cdot \nabla u_\varepsilon + \nabla p_\varepsilon - \Delta u_\varepsilon = 0 \\ u_\varepsilon\big|_{t=0} = u^0 \\ \operatorname{div} u_\varepsilon = 0 \end{cases} \tag{P$_\varepsilon$}$$

where $\tilde{u}_\varepsilon = \phi_\varepsilon * u_\varepsilon$. Then a solution to (P$_\varepsilon$) can be constructed using the Galerkin method. Passing into the limit yields a solution to the Navier–Stokes equation and the local energy inequality.

Other proof of the Ladyžhenskaya–Prodi–Serrin for $p > 3$ in $\mathbb{R}^3$.

Hypothesis 4.20. *If u is a suitable solution to Navier–Stokes equation with $u \in L^p_t L^q_x$, $\frac{2}{p} + \frac{3}{q} = 1$, $p > 3$, then*

$$\begin{aligned} -\Delta p &= \operatorname{div}\operatorname{div}(u \otimes u) \\ p &= \operatorname{div}\operatorname{div}(-\Delta)^{-1}(u \otimes u) \end{aligned}$$

By Reisz,

$$\|P\|_{L^{\frac{p}{2}}\left(L^{\frac{q}{2}}\right)} \leq C_p \|u\|^2_{L^p(L^q)}.$$

So if $p, q \geq 3$, fix a $(t,x) \in (0,\infty) \times \mathbb{R}^3$, through the universal scaling, $\tilde{u}$ solves the Navier–Stokes equation in $\mathbb{R}^3$. Then

$$\begin{aligned} \int_{Q_\varepsilon} |\tilde{u}_\varepsilon|^3 + |\tilde{p}_\varepsilon|^{\frac{3}{2}} \,\mathrm{d}t\,\mathrm{d}x &\leq \|\tilde{u}_\varepsilon\|^3_{L^p(L^q)(Q_1)} + \|\tilde{p}_\varepsilon\|^{\frac{3}{2}}_{L^{\frac{p}{2}}\left(L^{\frac{q}{2}})(Q_1\right)} \\ &= \|\tilde{u}_\varepsilon\|^3_{L^p(L^q)(Q_\varepsilon(t,x))} + \|\tilde{p}_\varepsilon\|^{\frac{3}{2}}_{L^{\frac{p}{2}}\left(L^{\frac{q}{2}})(Q_\varepsilon(t,x)\right)}. \end{aligned}$$

By Lebesgue's dominated convergence theorem, for ε small enough,

$$\|u\|^3_{L^p(L^q)(Q_\varepsilon(t,x))} + \|\tilde{p}_\varepsilon\|^{\frac{3}{2}}_{L^{\frac{p}{2}}\left(L^{\frac{q}{2}})(Q_\varepsilon(t,x)\right)} \leq \eta.$$

so

$$\|\tilde{u}_\varepsilon\|^3_{L^p(L^q)(Q_1)} + \|\tilde{p}_\varepsilon\|^{\frac{3}{2}}_{L^{\frac{p}{2}}\left(L^{\frac{q}{2}})(Q_1\right)} \leq \eta.$$

Thanks to the ε-regularity lemma, $\tilde{u}_\varepsilon$ is smooth in x at $(0,0)$, so u is smooth at t, x.

$\square$

Remark 4.21. The limit case $L_t^\infty L_x^3$ is proven much later (2003) than the other cases (1960's). This is because there is no dominated convergence theorem in L^∞.

♣ ♣ ♣

4.2.4 Partial regularity

Fix $0 \leqslant d \leqslant N$, in $\mathbb{R}^N$

$$H_d^\varepsilon(S) = \inf \sum_i r_i^d$$

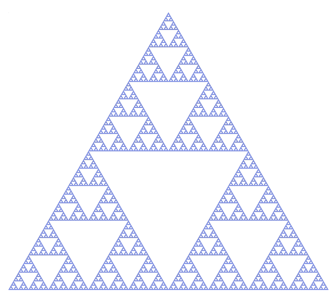
The limit is taken over $\{S_i\}$ is a covering of S, $r_i = \dim S_I$ and $\sup r_i \leqslant \varepsilon$. We define the d-Hausdorff dimension of S as $\mathcal{H}(S) = \lim_{\varepsilon\to 0} H^\varepsilon(S)$.

Remark 4.22.
- $\mathcal{H}_d(S) \in [0, +\infty]$. $\mathcal{H}_d(S) = \sup_{\varepsilon>0} H_d^\varepsilon(S)$, because $\mathcal{H}_d(S)$ decreases in ε.
- $S = B_1$:

$$\mathcal{H}_N(S) \approx \mathrm{Vol}(B_1)$$

While $N = 3$ and $d = 2$, $\mathcal{H}_2(S) \approx \mathrm{Area}_2(S)$.
While $N = 3$, $d = 1$: $\mathcal{H}_1(S) =$ length of S.

Remark 4.23. $\forall d \in [0, N]$, any $M > 0$, there exists $S \in \mathbb{R}^N$ such that $\mathcal{H}_d(S) = M$. The proof is using Cantor Sets.



Take S bounded,

- Then $\exists d \in [0, N]$ such that $\mathcal{H}_d(S) < +\infty$ take $\mathcal{H}_N(S) = \mathcal{H}(S) < +\infty$
- If $\mathcal{H}_d(S) < \infty$ then:

$$\forall\ d > d^* : \mathcal{H}_d(S) = 0. \qquad \sum_i r_i^d \leqslant \varepsilon^{d-d^*} \sum_i r_i^{d^*}$$

For $\{S_i\}$ covering of S, r_i is the diameter of S_i and $r_i < \varepsilon$

$$\mathcal{H}_d^{\varepsilon}(S) \leqslant \varepsilon^{d-d^*} \mathcal{H}_{d^*}^{\varepsilon}(S)$$

at the limit: $\mathcal{H}_d(S) \equiv 0$, for $d > d^*$.

- If $\mathcal{H}_{d^*} > 0$, then $\mathcal{H}_d = +\infty$ for all $d < d^*$.

Proposition 4.24. *$\forall S \in \mathbb{R}^N$, there exists $d^* \in [0, N]$ such that:*

$$\forall d > d^* : \mathcal{H}_d(S) = 0 \qquad \forall d < d^* : \mathcal{H}_d(S) = +\infty$$

we call $d^* = d_{\mathcal{H}}(S)$ the Hausdorff dimension of S. $d^* = \inf\{\phi : \mathcal{H}_d(S) \leqslant +\infty\}$ we call $d^* = d_{\mathcal{H}}(S)$ the Hausdorff dimension of S.

Here should be a pic
$d^* = \inf\{\phi : \mathcal{H}_d(S) < +\infty\}$.

Definition 4.25. Consider u a suitable weak solution to $N.S.$ equation in $\mathbb{R}^+ \times \mathbb{R}^3$ with $u^0 \in L^2(\mathbb{R}^3)$, $f \equiv 0$. Then $\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, we say that $(t, x) \in \Omega$, set of "regular points". If there exists a neighborhood of (t, x) on which u is bounded. We call $\mathbb{R}^+ \times \mathbb{R}^3 \Omega$ the set of "Singular points".

Theorem 4.26. *Partial regularity theorem (Caffarelli–Kohn–Nirenberg 1982) If u is a suitable weak soution to N.S. on $\mathbb{R}^+ \times \mathbb{R}^3$, $u_0 \in L^2(\mathbb{R}^3)$, $f \equiv 0$. Then Ω^c the set of "singular points" verifies: $d_{\mathcal{H}}(\Omega^c) \leqslant 1$ and $\mathcal{H}_1(\Omega^c) = 0$(thinner than a curve).*

Proposition 4.27. *A first step (Scheffer) with some Hypothesis: $d_{\mathcal{H}}(\Omega^c) \leqslant$* 5/3

Proof. Fix $\varepsilon > 0$: $Q_2 : \tilde{u}_\varepsilon(s, y) = \varepsilon u(\varepsilon^2 s + t, \varepsilon y + x)$.

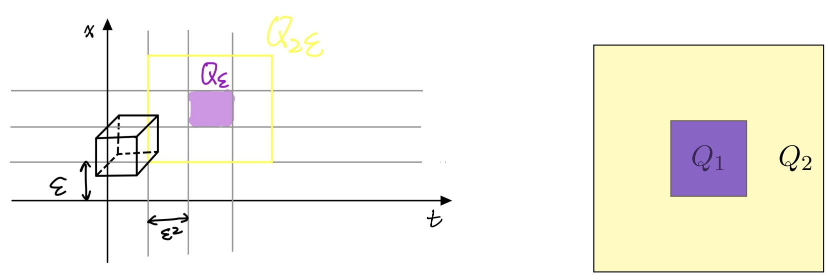


Figure 4.1: Rescaled Time-Space Cubes

$$\|u\|_{L^{10/3}(L^{10/3})} \leqslant +\infty \qquad 2/p + 3/q = 3/2$$

interpolate between $L^\infty(L^2)$ and $L^2(L^6)$ $\nabla u \in L^2(L^2)$. If $p = q$: $\varepsilon-$regularity lemma becomes that $\delta^* > 0$ such that if $\inf_{Q_2} |u|^{10/3} dxdt + \int_{Q_2} |P|^{5/3} dxdt \leqslant \delta^*$ then $|u| < 1$ in Q_1, since $10/3 > 3$ and $5/3 > 3/2$.

$$\int_{Q_2} (|u|^3 + |p|^{3/2}) dxdt \leqslant C \left(\int (|u|^{10/3} + |p|^{5/3}) dxdt \right)^{9/10} \leqslant \delta$$

where δ... Consider the rescaling $\tilde{u}_\varepsilon$... Claim if there exists $(t,x) \in Q_\varepsilon$ with (t,x) a singular point. Then $\iint_{Q_\varepsilon} \left(|u|^{10/3} + |p|^{5/3} \right) dxdt \geqslant \varepsilon^{5/3}$
This comes from a rescaling of the $\varepsilon-$regularity lemma: If the (t¿0)? singular point in Q_ε then the singular point in Q_1 is not locally bounded. Then

$$\int_{Q_2} |\tilde{u}_\varepsilon(s,y)|^{10/3} dxdy + \left|\tilde{P}_\varepsilon\right|^{5/3} dxdt \geqslant \delta^*,$$

$$10/3 - 5 = -5/3, \quad \int_{Q_{2\varepsilon}} \varepsilon^{`10/3-5} \left|u(\tilde{t},\tilde{x})\right|^{10/3} dxdy + \left|P(\tilde{t},\tilde{x})\right|^{5/3} dxdt \geqslant \delta^*.$$

$$\text{So,} \qquad \int_{Q_{2\varepsilon}} \varepsilon^{10/3-5} \left|u(\tilde{t},\tilde{x})\right|^{10/3} dxdy + \left|P(\tilde{t},\tilde{x})\right|^{5/3} dxdt \geqslant \delta^* \varepsilon^{`5/3}.$$

We consider a covering with cubes Q_ε (size ε^2 in time and ε in x) from the grid: of singular points of u

$$\begin{aligned}
\mathcal{H}_d^\varepsilon(\Omega^c) &\leqslant \sum_i r_i^{5/3} \qquad \text{for this covering} \\
&\leqslant \varepsilon^{5/3} \sum_i 1 \\
&\leqslant \frac{1}{\delta^*} \int_{Q_{2\varepsilon}^i} (\left|u(\tilde{t},\tilde{x})\right|^{10/3} + \left|P(\tilde{t},\tilde{x})\right|^{5/3}) d\tilde{t} d\tilde{x} \\
&\leqslant \frac{1}{\delta^\varepsilon} \int_{Q_{2\varepsilon}^i} (|u|^{10/3} + |P|^{5/3}) dtdx \qquad \text{Fubini} \\
&\leqslant \frac{C}{\delta^\varepsilon} \int_{\mathbb{R}^+ \times \mathbb{R}^3} \left(|u|^{10/3} + |P|^{5/3} \right) dtdx \leqslant C^*,
\end{aligned}$$

where C is the number of Q_1 in Q_2. So $\lim_{\varepsilon \to 0} \mathcal{H}_{5/3}^\varepsilon(\Omega^c) = \mathcal{H}_{5/3}(\Omega^c) \leqslant C^*$. So

$$d_{5/3}(\Omega^c) \leqslant 5/3$$

□

For Caffarelli–Kohn–Nirenberg result, use as a point :

$$\int_{Q_2} |\tilde{u}_\varepsilon(s,y)|^2 ds dy = \varepsilon^{4-5} \int_{Q_{2\varepsilon}} |\nabla u|^2 d\tilde{t} d\tilde{x}$$
$$= \frac{1}{\varepsilon} \int_{Q_{2\varepsilon}} |\nabla u|^2 d\tilde{t} d\tilde{x}$$

We need a $\varepsilon-$regularity result with

$$\int_{Q_2} |\nabla u|^2 dx dt$$

To have $\mathcal{H}_1(\Omega^c) \equiv 0$. Use covering lemmas.

♣ ♣ ♣

4.3 Boundary effect

Consider the Euler equation, and a shear flow at the boundary. Set $\Omega = \mathbb{T}^2 \times (0,1)$, where $\mathbb{T} = [0,1]$ is periodic. Initial value: $u^0 = Ae_1$. Then $u(t,x) \equiv Ae_1$ for all t, x is a solution to the Euler equation:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & p(t,x) = 0. \\ \operatorname{div} u = 0 & \\ u \cdot n = 0 & \text{at } z = 0 \text{ and } z = 1. \end{cases}$$

But the solution is not unique!

Theorem 4.28. *For any $C \in (0,1)$, there exists a solution to the Euler equation with*

1. $u^0 = Ae_1$
2. $\int_\Omega |u(t,x)|^2 \,\mathrm{d}x$ *is decreasing.*
3. $\|u(t,x) - Ae_1\|^2_{L^2(\Omega)} = CA^3 t$ *for* $0 < t < 1$.

This corresponds to the phenomenon of layer separation. The result is based on the convex integration method developed by De Lellis and Székelyhidi [Szé11] (L^∞ theory).

Proposition 4.29. *If u is a weak solution to Euler in $[0,T) \times \Omega$ such that*

1. $u^0(x) = Ae_1$
2. $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \,\mathrm{d}x \le 0$
3. *For a.e. t,*

$$\lim_{x_3\to 0}\int_{\mathbb{T}^2} u_1(t,x)u_3(t,x)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \lim_{x_3\to 1}\int_{\mathbb{T}^2} u_1(t,x)u_3(t,x)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0.$$

Then: $u(t,x) = Ae_1$ for all $t \in [0,T]$, $x \in \Omega$.

Remark 4.30. Boundary condition: $u_3(t,x_1,x_2,0) = u_3(t,x_1,x_2,1) = 1$.

$$\lim_{x_3\to 0} u_3(\cdot,\cdot,\cdot,x_3) = 0 \text{ in } L^\infty\Big(0,T;H^{-\frac{1}{2}}(\mathbb{T}^2)\Big).$$

It means that we don't have "strong traces" for non-unique solutions.

Proof. Energy proof:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(t,x) - Ae_1|^2\,\mathrm{d}x \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_\Omega |u(t,x)|^2\,\mathrm{d}x - 2A\int_\Omega u_1(t,x)\,\mathrm{d}x + A^2\Big\} \\
&\quad\le 0 - 2A\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_1(t,x)\,\mathrm{d}x \\
&\quad\le -2A\int_\Omega (\partial_t u_1)\,\mathrm{d}x \\
&\quad\le -2A\int_\Omega -\operatorname{div}(uu_1)\,\mathrm{d}x - \underbrace{\partial_1 p\,\mathrm{d}x}_{=0\text{ by periodicity}} \\
&\quad= 2A_1\Big(-\int_{\mathbb{T}^2} u_1(t,x_1,x_2,1)u_3(t,x_1,x_2,1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\qquad\qquad + \int_{\mathbb{T}^2} u_1(t,x_1,x_2,0)u_3(t,x_1,x_2,0)\,\mathrm{d}x_1\,\mathrm{d}x_2\Big)
\end{aligned}$$

The last expression makes sense because to be more precise we cut off the boundary and take the limits, but the property 3 ensures the limit exists. □

4.3.1 Convex integration

Consdier

- $v \in [C^{(0,T)\times\Omega}]^3$, $\operatorname{div} v = 0$

- $u \in C((0,T) \times \Omega; \mathscr{S}_0^{2\times 2})$ where $\mathscr{S}_0^{2\times 2}$ is the set of 2×2 symmetric traceless matrices:

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

  Note that the only nonnegative definite matrix in $\mathscr{S}_0^{2\times 2}$ is zero.

- $e, q \in C((0,T) \times \Omega)$.

We will say that v, u, q is a *subsolution* to Euler equation in $(0,T) \times \Omega$ with respect to e if

$$\begin{cases} \partial_t v + \operatorname{div} u + \nabla q = 0 & \text{in } (0,T) \times \Omega \\ v \otimes v - u \leq \frac{2e}{n}\mathrm{Id} & \text{in } (0,T) \times \Omega \ (n=2) \text{ is the dimension} \end{cases}$$

Remark 4.31. If A, B are symmetric matrices, then $A \leq B$ if for all $\xi \in \mathbb{R}^n$:

$$\xi^\top A \xi \leq \xi^\top B \xi.$$

It means that the smallest eigenvalue of $B - A$ is nonnegative.

Remark 4.32. If $v \otimes v - u = e\mathrm{Id}$, then

$$\partial_t v + \operatorname{div}(v \otimes v) - \operatorname{div}(e\mathrm{Id}) + \nabla q = 0$$

Since $\operatorname{div}(e\mathrm{Id}) = \operatorname{div} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} \partial_1 e \\ \partial_2 e \end{pmatrix} = \nabla e$, then

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla(q - e) = 0, \\ \operatorname{div} v = 0. \end{cases}$$

So v is a solution to the Euler equation.

Remark 4.33. Taking the trace of $v \otimes v - u \leq \frac{2e}{n}\mathrm{Id}$,

$$\operatorname{tr}(v \otimes v) = \operatorname{tr} \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_2 v_1 & v_2^2 \end{pmatrix} = |v|^2.$$

$\operatorname{tr} u = 0$ since $u \in \mathscr{S}_0^{2\times 2}$, so

$$\frac{|v|^2}{2} \leq e.$$

After convex integration, e is the target energy of the solution.

Proposition 4.34 (De Lellis–Székelyhidi, 2010 [DLS10])**.** *Consider (v, u, q) a subsolution to Euler in $(0,T) \times \Omega$ with respect to the energy e. Consider $\mathcal{U}$ an open subset of $(0,T) \times \Omega$, such that*

$$v \otimes v - u < e\mathrm{Id} \text{ on } \mathcal{U}.$$

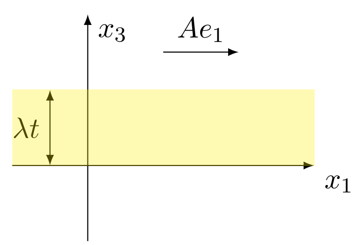
Then there exists infinitely many $\tilde{v} \in L^\infty((0,T) \times \Omega)$ such that

1. *$\tilde{v}$ is supported in $\mathcal{U}$.*
2. *$v + \tilde{v}$ is a solution to Euler in $\mathcal{U}$.*
3. *$\mathrm{div}(v + \tilde{v}) = 0$ in $[0,T) \times \Omega$.*

Remark 4.35. Consider $\mathcal{U}$ is a wedge (in time direction), subsolution: if $v \otimes v - u = e\mathrm{Id}$ in $\mathcal{U}^c$ and $v \otimes v - u < e\mathrm{Id}$ in $\mathcal{U}$, then $v + \tilde{v}$ is a solution to Euler everywhere, but $v + \tilde{v}\big|_{t=0} = v\big|_{t=0}$.

Construction of the subsolution

Define $\mathcal{U} = \{(t,x) : 0 < x_3 < \lambda t\}$, where λ is a coefficient. For $x_3 > \lambda t$, $v(t,x) = Ae_1$. (For simplicity, let us first fix $A \equiv 1$.)



We need to construct (v, u, q) in $\mathcal{U}$ as solution of (t, x_3).

$$v = \alpha(t, x_3) e_1$$
$$u = \begin{pmatrix} \beta(t,x_3) & \gamma(t,x_3) \\ \gamma(t,x_3) & \beta(t,x_3) \end{pmatrix}$$
$$\partial_t v + \mathrm{div}\, u + \nabla q = \begin{pmatrix} \partial_t \alpha + \partial_2 \gamma \\ 0 - \partial_3 \beta + \partial_3 q \end{pmatrix} = 0.$$

We choose $q = \beta$, so we need

$$\partial_t \alpha + \partial_x \gamma = 0.$$

Constraint: $0 < e\mathrm{Id} - (v \otimes v - u)$.

$$e\mathrm{Id} - (v \otimes v - u) = \begin{pmatrix} -\alpha^2 + \beta + e & \gamma \\ \gamma & -\beta + e \end{pmatrix}$$

We choose $\beta = \frac{|\alpha|^2}{2}$,

$$e\mathrm{Id} - (v \otimes v - u) = \begin{pmatrix} -\frac{|\alpha|^2}{2} + e & \gamma \\ \gamma & -\frac{|\alpha|^2}{2} + e \end{pmatrix}$$

We need trace to be positive and determinant to be positive, i.e. $\frac{|\alpha|^2}{2} < e$, $e - \frac{|\alpha|^2}{2} > |\gamma|$.

Take $\gamma = \lambda(\frac{|\alpha|^2}{2} - 1)$, $0 < \lambda < 1$.

$$\partial_t \alpha + \partial_x (\lambda \frac{|\alpha|^2}{2})$$

$\alpha = \frac{x_3}{\lambda t}$ for $0 < x_3 < \lambda t$ and $\alpha = 1$ for $x_3 > \lambda t$.

♣ ♣ ♣

Construction of e

$$\begin{aligned} \frac{|\alpha|^2}{2} + |\gamma| &= \frac{\lambda}{2} - \lambda\frac{|\alpha|^2}{2} + \frac{|\alpha|^2}{2} \\ &= \frac{1}{2} + \frac{\lambda - 1}{2} + (1-\lambda)\frac{|\alpha|^2}{2} \\ &= \frac{1}{2} + \frac{1-\lambda}{2}(\alpha^2 - 1) < 2 \qquad \text{where } 0 < \alpha < 1 \end{aligned}$$

$\forall\ 0 < \varepsilon < 1$: $e = \frac{1}{2} - \varepsilon(\frac{(1-\lambda)(1-\alpha^2)}{2})$, so we have $|\gamma| + \frac{|\alpha|^2}{2} < e$ in the wedge.

Let's take

$$\begin{aligned} \frac{|\alpha|^2}{2} - e &= \frac{|\alpha|^2}{2} - (\frac{1}{2} - \varepsilon\frac{(1-\alpha)(1-\alpha^2)}{2}) \\ &= \frac{\alpha^2 - 1}{2} + \varepsilon(1-\lambda)\frac{1-\alpha^2}{2} \\ &= -\frac{1-\alpha^2}{2}(1 - \varepsilon(1-\lambda)) \\ &\leqslant 0 \end{aligned}$$

And the equality doesn't hold while $0 < \alpha < 1$. We have $v \otimes v - u \leqslant eId$ with strict inequality for $0 < x_3 < \lambda t$.

Thanks to the convex integration properties: for $\varepsilon,\ \lambda$ between 0 and 1, there exist $\bar{v} = v + \tilde{v}$ solution to Euler in $(t, x_1, x_3) \in (0,1) \times \mathbb{T} \times [0,1]$.

We keep boundary condition

$$\begin{aligned}
\int_0^1 \int_{\mathbb{T}} \frac{|\bar{v}|^2}{2} dx_1 dx_3 &= \int_0^1 \int_0^1 (\frac{|\bar{v}|^2}{2} + e) dx_1 dx_3 \\
&= \int_0^1 \left(\frac{|v(t,x_3)|^2}{2} + e(t,x_3) \right) dx_3 \\
&= \int_0^{\lambda t} \left[\frac{1}{2} (\frac{x_3}{\lambda t})^2 + \frac{1}{2} - \frac{\varepsilon(1-\lambda)}{2} \left(1 - \frac{x_3}{\lambda t}^2 \right) \right] dx_3 + \int_{\lambda t}^1 \frac{1}{2}^2 dx_3 \\
&= \frac{1-\lambda t}{2} - \frac{\varepsilon(1-\lambda)}{2} \lambda t + \frac{1}{2} + \frac{\varepsilon(1-\lambda)}{2} [(\frac{x_3}{\lambda t})^2]_0^{\lambda t}
\end{aligned}$$

where $[(\frac{x_3}{\lambda t})^2]_0^{\lambda t} = \frac{\lambda t}{3}(\frac{1}{2} + \frac{\varepsilon(1-\lambda)}{2})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\bar{v}|^2 dx = -\frac{\varepsilon\lambda(1-\lambda)}{3}$$

Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{|\bar{v} - e_1|^2}{2} dx \\
&\quad = \mathrm{d}t \left(\int \frac{|v|^2}{2} dx - \int \bar{v}_1 dx \right) \\
&\quad = -\frac{\varepsilon\lambda(1-\lambda)}{3} - \int_0^1 \int_{\mathbb{T}} (\mathrm{div}(\bar{v} \otimes \bar{v}_1) + \partial_1 p) dx \\
&\quad = -\frac{\varepsilon\lambda(1-\lambda)}{3} - \int_0^1 \int_{\mathbb{T}} (\mathrm{div}(\bar{v} \otimes \bar{v}_1)) dx - \int_0^1 \int_{\mathbb{T}} \partial_3 (\bar{v_3} \bar{v_1}) dx
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \int_{\mathbb{T}} \frac{|\bar{v} - e_1|^2}{2} dx_1 dx_3 = -\frac{\varepsilon(1-\lambda)}{3} + \int_{\mathbb{T}} \lim_{x_3 \to 0} (\bar{v}_3 \bar{v}_1) dx,$$
$$\bar{v} \otimes \bar{v} = u + \tilde{v} \otimes \tilde{v} - |\tilde{v}|^2 Id.$$

at $x_3 = 0$: $\bar{v} \otimes \bar{v} = u$, so $v_3 v_1|_{x_2=0} = \gamma = \frac{\lambda}{2}(\alpha^2 - 1) = -\frac{\lambda}{2}$. So

$$\mathrm{d}t \int_0^1 \int_{\mathbb{T}} \frac{|\bar{v} - e|^2}{2} dx_1 dx_3 \leqslant -\frac{\varepsilon\lambda(1-\lambda)}{3} - \frac{\lambda}{2}.$$

Take $\lambda \to 1$, $\varepsilon \to 0$

$$\mathrm{d}t \int_0^1 \int_{\mathbb{T}} \frac{|\bar{v} - e|^2}{2} dx_1 dx_3 \leqslant \frac{1}{2}$$

$\int |v - e_1|^2 dx = Ct$, where $0 < C < 1$. $\forall A > 0$:
$u_A(t,x) = \frac{1}{A} u(\frac{t}{A}; x)$. If u is solution to Euler in $[0,1] \times \mathbb{T} \times [0,1]$ Then u_A is also solution in $[0, A] \times \mathbb{T} \times [0,1]$.

$$\partial_t u_A = \frac{1}{A^2} \partial_t u(\frac{t}{A}; x)$$

$$u_A \cdot \nabla_x u_A = \frac{1}{A^2}(u - \nabla u)(\frac{t}{A}; x)$$

$$q_A = \frac{1}{A^2} q(\frac{t}{A}; x)$$

then

$$\begin{cases} \partial_t u_A + u_A \cdot u_A + \nabla_x q_A = 0 \\ \operatorname{div} u_A = 0 \end{cases}$$

$u_A|_{t=0} = e_1$; $\forall\, C$: there exists: $u_{A,C}$ solution to Euler with: $\int |u_{A,C} - e_1|^2 = ct$ so $\int |u_C - Ae_1|^2 dx = CA^3 t$.

Inviscid asymptotic of N.S.

by denote limit:
Pattern: Shear Flow for the inviscid model.
To control the instability, we want to consider the fluctuation of the Pattern obtained as inviscid limit of N.S.:

$$\|u_0^\varepsilon - Ae_1\|_{L^2} \approx \varepsilon.$$

$$\begin{cases} \partial_t u_\nu^\varepsilon + u_\nu^\varepsilon \cdot \nabla u_\nu^\varepsilon + \nabla q_\nu^\varepsilon - \nu \Delta u_\nu^\varepsilon = 0, \\ \operatorname{div} u_\nu^\varepsilon = 0 \\ u_\nu^\varepsilon = 0 \text{ on } x_3 = 0\ , x_3 = 1. \\ u_\nu^\varepsilon|_{t=0} u_0^\varepsilon. \end{cases}$$

$\forall\ \nu$ fixed, there exist a Leray-Hopf solution For any $t \to 0$:

$$\int |u_\nu^\varepsilon|^2 dx \leqslant \int u_0^\varepsilon$$

up to a subsequence $\nu \to 0$: $\exists \bar{u} \in L^\infty(0, w; L^2(\Omega))$ such that $u_\nu^\varepsilon \to \bar{u}$ weakly in $L^2_{\nu,\alpha}$.

Remark 4.36.

- $\bar{u}$ may not be solution to Euler;
- but it is physical.

Theorem 4.37. $\exists$ *a universal constant $C > 0$ for any such weak limit through the double limit inviscid limit*

$$\|u(t) - Ae_1\|^2_{L^2(\Omega)} \leqslant CA^3 t$$

Remark 4.38. We do not know if $C \equiv 0$

Appendix A

Outroduction

Theorem A.1. *Let*

[BME]

$$\int_2^3 x \,\mathrm{d}x = \left.\frac{|x|^2}{2}\right|_{x=2}^{x=3} \tag{A.1}$$
$$= \frac{|3|^2}{2} - \frac{|2|^2}{2} \tag{A.2}$$
$$= \frac{5}{2} \tag{A.3}$$

[BSME]

$$\int_2^3 x \,\mathrm{d}x = \left.\frac{|x|^2}{2}\right|_{x=2}^{x=3}$$
$$= \frac{|3|^2}{2} - \frac{|2|^2}{2}$$
$$= \frac{5}{2}$$

[BMI]

$$\int_2^3 x \,\mathrm{d}x \tag{A.4}$$
$$= \frac{|x|^2}{2}\bigg|_{x=2}^{x=3} \tag{A.5}$$
$$= \frac{|3|^2}{2} - \frac{|2|^2}{2} \tag{A.6}$$
$$= \frac{5}{2} \tag{A.7}$$

[BSMI]

$$\int_2^3 x \,\mathrm{d}x$$
$$= \frac{|x|^2}{2}\bigg|_{x=2}^{x=3}$$
$$= \frac{|3|^2}{2} - \frac{|2|^2}{2}$$
$$= \frac{5}{2}$$

[BSV14]

Bibliography

[BSV14] K. Bardos, L. Sekelikhidi, Jr., and È. Videmann. On the absence of uniqueness for the Euler equations: the effect of the boundary. *Uspekhi Mat. Nauk*, 69(2(416)):3–22, 2014.

[DLS10] Camillo De Lellis and László Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[Szé11] László Székelyhidi. Weak solutions to the incompressible Euler equations with vortex sheet initial data. *C. R. Math. Acad. Sci. Paris*, 349(19-20):1063–1066, 2011.