Property of Expectation and Variance

If $\mathbb{E}(X_i)$ finite, $\mathbb{E}\left(\sum a_i X_i\right) = \sum a_i \mathbb{E}(X_i)$

If $X_1 \cdots X_n$ independent, $\mathbb{E}\left(\prod X_i\right) = \prod \mathbb{E}(X_i)$

If $X_1 \cdots X_n$ independent and have finite means:

$$\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}(X_i)$$

Moment Generating Function

For RV X, $\quad \psi(t) = \mathbb{E}(e^{tX})$

$$= \mathbb{E}\left(1 + tX + \frac{(tX)^2}{2!} + \cdots\right)$$

And, $\psi^{(n)}(0) = \mathbb{E}(X^n)$

Note, the derivative is wrt t, not X.

Sketch proof: $\psi^{(n)}(0) = \left[\frac{d^n}{dt^n}\mathbb{E}(e^{tX})\right]_{t=0} = \mathbb{E}\left[\left(\frac{d^n}{dt^n} e^{tX}\right)_{t=0}\right]$

$$= \mathbb{E}\left[\left(X^n e^{tX}\right)_{t=0}\right] = \mathbb{E}(X^n)$$

MGF under linear transformation:

Let X be RV having MGF $\psi(\,)$. let $Y = aX + b$.

For every t s.t. $\psi(at) < \infty$,

MGF of Y, ψ_2, satisfy $\psi_2(t) = e^{bt}\psi_1(at)$

(can be derived from rule of sum of RV)

- $\mathcal{N}(0,1)$: $\psi(t) = e^{\frac{1}{2}t^2}$, $\mathcal{N}(\mu, \sigma^2)$: $\psi(t) = e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$

MGF of sum of RV:

For RV $X_1 \cdots X_n$, MGF $\psi_1 \cdots \psi_n$, let $Y = X_1 + \cdots + X_n$

for every t s.t. $\psi_i(t) < \infty$

MGF of y, ψ satisfy: $\psi(t) = \prod_1^n \psi_i(t)$

Probability Integral Theorem

Let X have cdf $F(\cdot)$, let $Y = F(X)$,

we say transformation $X \to Y$ is probability integral transformation.

pdf of Y: uniform on $[0,1]$

Inverse Sampling:

Given $X \sim$ cdf $F(\cdot)$, to sample X, get $Y \sim U(0,1)$,

let $Z = F^{-1}(Y)$, $Z \sim X$.

Distribution of a Monotonic Function of RV

(a, b can be ∞)

Let X be RV with pdf $f(\cdot)$, for which $\Pr(a < X < b) = 1$.

Let $Y = r(X)$, r is differentiable and 1-to-1 for (a,b).

Let (α, β) be image of (a,b) under r.

r increasing: add $+$

r decreasing: add $-$

Then, pdf of Y is:

$$g(y) = \begin{cases} f(r^{-1}(y)) \left| (r^{-1})'(y) \right| & y \in (\alpha, \beta) \\ 0 & o/w \end{cases}$$

Covariance

Let X, Y be RV with finite means, $\mathbb{E}(X) = \mu_X$, $\mathbb{E}(Y) = \mu_Y$.

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)].$$

If $\sigma_X^2, \sigma_Y^2 < \infty$,

$$\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

$$\text{cov}(X, Y+Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$$

And, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X, Y)$

Correlation: $$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

Variance-covariance Matrix (Covariance Matrix)

For RV $X_{1 \times n}$, $\text{Var}(X_i) < \infty$,

covariance matrix K: $K_{ij} = \text{cov}(X_i, X_j)$

Cross-covariance Matrix K_{XY}: $K_{XY}(i, j) = \text{cov}(X_i, Y_j)$.

- $\text{cov}(X, X) = \mathbb{E}(XX^T) - \mathbb{E}(X)\mathbb{E}(X)^T$