Markov Inequality

For RV X s.t. $\Pr(X \geq 0) = 1$,

$$\Pr(X \geq t) \leq \mathbb{E}(X)/t$$

Chebyshev Inequality

If $Var(X) < \infty$,

$$\Pr(|X - \mathbb{E}(X)| \geq t) \leq \frac{Var(X)}{t^2}$$

Proof: $\Pr((X - \mathbb{E}(X))^2 \geq t^2) \leq \mathbb{E}((X - \mathbb{E}(X))^2)/t^2 = \frac{Var(X)}{t^2}$

Sample Mean

For n RV $X_1 \cdots X_n$, sample mean $\bar{X}_n = \frac{1}{n} \Sigma X_i$

If $X_1 \cdots X_n$ random sample from distr w/ mean μ and variance σ^2, $\bar{X}_n$ is sample mean,

$$\mathbb{E}(\bar{X}_n) = \mu \quad , \quad Var(\bar{X}_n) = \sigma^2/n$$

(By Chebyshev, $\Pr(|\bar{X}_n - \mu| \geq t) \leq \sigma^2/nt^2$

Convergence of RV

- Convergence in distribution

  Sequence of RV $Z_1, Z_2 \cdots$ w/ CDF $F_1, F_2 \cdots$
  converge to RV Z w/ CDF F if:
  $$\lim_{n \to \infty} F_n(x) = F(x), \text{ for every } x \text{ s.t. } F \text{ continuous}$$

- Convergence in Probability

  Sequence of RV $Z_1, Z_2 \cdots$ converges to b in prob if:
  for every $\varepsilon > 0$, $\lim_{n \to \infty} \Pr(|Z_n - b| < \varepsilon) = 1$
  $(Z_n \xrightarrow{P} b)$ only need Z_n be close to b

- Convergence with Probability 1 (almost surely)

  ... converges to b if: $\Pr(\lim_{n \to \infty} Z_n = b) = 1$ need any sequence of $\{Z_n\}$ converge to b
  $(Z_n \xrightarrow{a.s.} b)$

Almost sure convergence $\Rightarrow$ Convergence in prob $\Rightarrow$ Convergence in distr

Example of converge in distr but not w/ prob 1:

Let $Z_n = \begin{cases} 1 & \text{with } \Pr \frac{1}{n} \\ 0 & 1 - \frac{1}{n} \end{cases}$

Then $Z_n \to 0$ in prob.

~~However, since $\sum_{n \geq 1} \Pr(Z_i = 1) = +\infty$, events $\{Z_i = 1\}$ independent,~~

However, since $\sum_{n \geq 1} \Pr(Z_i = 1) = +\infty$, and events $\{Z_i = 1\}$
by Second Borel-Cantelli lemma, $\Pr(\lim_{n \to \infty} Z_n = b) \neq 1$.

Weak Law of Large Numbers

Suppose $X_1 \cdots X_n$ form a random sample from a distr w/ mean μ, and finite variance. Let $\overline{X}_n$ be sample mean,

then, $\overline{X}_n \xrightarrow{P} \mu$.

- If $Z_n \xrightarrow{P} b$, $g(z)$ is a function continuous at b,

$$g(Z_n) \xrightarrow{P} g(b).$$

Strong Law of Large Numbers

$$\overline{X}_n \xrightarrow{a.s} \mu$$

Chernoff Bounds

Idea: Want to generate a family of bounds for $Pr(X)$.

Consider in Chebyshev, we applied Markov on $[X-\mathbb{E}(X)]^2$

We can repeat that with a functional family of X with parameter λ. This will give a family of bounds parameterized by λ. Then optimize over λ to get the best bound.

Derivation: for $t \in \mathbb{R}$,

MGF of X

$$Pr(X \geq t) = Pr(e^{\lambda X} \geq e^{\lambda t}) \leq \frac{\mathbb{E}(e^{\lambda X})}{e^{\lambda t}} = e^{-\lambda t} \psi(\lambda), \ \forall \lambda > 0.$$

Take union bound : $Pr(X \geq t) \leq \min_{\lambda > 0} e^{-\lambda t} \psi(\lambda)$.

Central Limit Theorem (Lindberg & Lévy)

If RV $X_1 \cdots X_n$ form a random sample from a given distr w/ mean μ, var $\sigma^2 < \infty$, then for each fixed number x.

$$\lim_{n \to \infty} \Pr\left(\frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \leq x \right) = \Phi(x)$$

CDF of $N(0,1)$

(Standardized sample mean $\sim N(0,1)$)

Interpretation: The distribution of standardized sample mean converges in distribution to normal.

Usage: Estimate $\Pr(|\bar{X} - E(X)| \leq e) \simeq \Phi\left(\frac{e}{\sigma/\sqrt{n}}\right) - \Phi\left(-\frac{e}{\sigma/\sqrt{n}}\right)$

The Delta Method

(Computing approx distr for function of RV, esp. function of sample mean.)

Let $Y_1, Y_2 \cdots$ be sequence of RV, F^* be a cdf.

Let $\theta \in \mathbb{R}$, $\{a_i\}$ s.t. $a_i > 0$, $a_n \to \infty$.

Suppose $a_n(Y_n - \theta)$ converges in distr to F^*,

let $\alpha(\cdot)$ be a function w/ continuous derivative s.t. $\alpha'(\theta) \neq 0$,

then, $a_n[\alpha(Y_n) - \alpha(\theta)] / \alpha'(\theta)$

converges in distr to F^*.

Proof outline:

$a_n \to \infty$, $a_n(Y_n - \theta) \xrightarrow{d} F^*$, then Y_n close to θ as $n \to \infty$.

(Because for any CDF, $F^*(N) \to 0$ as $N \to \infty$. Thus if Y_n not close to θ, F^* will have non-negligible mass at ∞.)

Use Taylor Exp of $\alpha(Y_n)$ around θ:

$$\alpha(Y_n) \simeq \alpha(\theta) + \alpha'(\theta)(Y_n - \theta)$$

$$\Rightarrow \alpha(Y_n) - \alpha(\theta) \simeq \alpha'(\theta)(Y_n - \theta)$$

$$\Rightarrow \frac{a_n}{\alpha'(\theta)}(\alpha(Y_n) - \alpha(\theta)) \simeq a_n(Y_n - \theta)$$

Thus, CDF of LHS $\simeq$ CDF of RHS $= F^*$

Distribution of Function of Sample Mean

Consider $\alpha(\overline{X_n})$.

By CLT, $\frac{\sqrt{n}}{\sigma}(\overline{X_n} - \mu) \xrightarrow{d} \Phi$.

Thus, $\frac{\sqrt{n}}{\sigma}(\alpha(\overline{X_n}) - \alpha(\mu)) / \alpha'(\mu) \to \Phi$.

(By Delta Method, $Y_n = \overline{X_n}$, $\theta = \mu$, $a_n = \frac{\sqrt{n}}{\sigma}$, $F^* = \Phi$)