Department of Statistics, UC Berkeley
MA Exam (Statistics 201A)

21 January 2023

1 Instructions

1. This exam has three questions for 50 points (201B will count for the remaining 50 points).
2. This is an open-book exam. You are welcome to use any material for solving these questions. You can directly use facts from class and homework. If you are borrowing ideas from outside, you would need to provide justification for them.
3. You need to upload your solutions on Gradescope by 12:30 pm.

2 Questions

1. To regression data $(x_1, y_1), \ldots, (x_n, y_n)$, I want to fit the model

$$Y_i = \frac{Ae^{\omega x_i}}{1 + e^{\omega x_i}} + \epsilon_i \qquad \text{with } \epsilon_i \stackrel{\text{i.i.d}}{\sim} N(0, \sigma^2).$$

Here $x_1, \ldots, x_n$ are treated as deterministic constants (as is usually the case in regression). This model has three unknown parameters A, ω, σ and we are interested in their posterior densities under the prior:

$$A, \omega, \log \sigma \stackrel{\text{i.i.d}}{\sim} \text{Unif}(-C, C)$$

You may assume that C is very large for the calculations below. You may leave the normalization constant for each posterior density below in the form of an integral if it cannot be evaluated in closed form.

   a) Calculate the joint posterior density of A, ω, σ. (**4 points**)
   b) Calculate the joint posterior density of ω, σ. (**6 points**)
   c) Calculate the posterior density of ω. (**6 points**).

2. Given regression data $(x_1, y_1), \ldots, (x_n, y_n)$ with nonnegative integer valued $y_i \in \{0, 1, 2, \ldots\}$, I want to fit the model

$$Y_i \stackrel{\text{independent}}{\sim} \text{Poi}\left(e^{\theta x_i}\right)$$

for $i = 1, \ldots, n$ ($\text{Poi}(\lambda)$ refers to the Poisson distribution with mean λ which corresponds to the probabilities $\frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$). Here $x_1, \ldots, x_n$ are treated as deterministic constants (as is usually the case in regression). This model has one unknown parameter θ and we are interested in its posterior density under the prior:

$$\theta \sim \text{Unif}(-C, C)$$

for a very large C.

a) Calculate the likelihood function for the model. (**3 points**)

b) Calculate the posterior density of θ. (**3 points**)

c) Approximate the posterior by a suitable normal distribution. Express clearly the mean and variance of the approximating normal distribution in terms of the maximum likelihood estimator of θ and the observed dataset $(x_1, y_1), \dots, (x_n, y_n)$. (**4 points**).

3. Determine whether each of the following claims is true or false. Provide reasons in each case (**24 points; 3 for each question. No point will be awarded if no reason is provided**).

a) The number of accidents on Hearst Avenue in a given day can be modeled by the Poisson distribution.

b) Suppose X is a continuous random variable with density function f_X. Then $0 \leq f_X(x) \leq 1$ for every x.

c) The standard Cauchy density can be written as

$$\frac{1}{\pi}\frac{1}{1+x^2} = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) g(\sigma) d\sigma$$

for some density function g.

d) If X and Y are independent random variables, then $R := \sqrt{X^2 + Y^2}$ and Θ are independent random variables as well. Here Θ is the random variable defined as the angle made by the vector (X, Y) with the positive X-axis in the counterclockwise direction.

e) Suppose X and Y are two random variables such that $X + Y$ and $X - Y$ have the same variance. Then X and Y must be uncorrelated.

f) Suppose X is a random variable such that X and $1/X$ have exactly the same distribution. Then $\mathbb{P}\{X = 1\} = 1$.

g) Suppose X_1, X_2, Θ are three random variables such that X_1 and X_2 are conditionally independent given $\Theta = \theta$ for every value of θ. Then X_1 and X_2 are unconditionally independent as well.

h) Suppose $X_1, \dots, X_{500}$ are i.i.d with common discrete distribution giving probabilities $0.3, 0.2, 0.15, 0.3, 0.05$ to the values $8, 13, 20, 29, 35$. Then $X_1 + \dots + X_{500}$ will be approximately normally distributed.