MA Comprehensive Exam
Stat 201B, January 2024
Elizabeth Purdom

Directions

- You will find 4 questions in this exam and each question counts equally
- This is an open-book exam. You are free to consult any material for answering the questions. But you are not allowed to consult any person during the course of the exam.
- You must provide calculations and/or justification in every question.
- You may use without proof any fact proved or used in class, lecture notes, or the textbook. If you are using facts that have not been proved or used in class, in lecture notes, or in the textbook then you need to provide justification for them.

Assume $Y_1, \ldots, Y_n$ are i.i.d. from a distribution with pdf given by

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k}, x \geq 0.$$

where $k > 0$, $\lambda > 0$. The mean of this distribution is $\lambda\Gamma(1+\frac{1}{k})$, where $\Gamma(m)$ is the gamma function, and for m an integer we have $\Gamma(m) = (m-1)!$.

We will assume for all the following problems that k is known.

1. Show that the maximum likelihood estimate of λ is

$$\hat{\lambda} = \left(\frac{1}{n}\sum_i Y_i^k\right)^{\frac{1}{k}}$$

2. Give an (asymptotic) $100(1-\alpha)\%$ confidence interval for λ when $k = 1$.

3. Provide an asymptotic α-level Likelihood Ratio Test (LRT) for testing $H_0 : \lambda = \lambda_0$, for $\lambda_0 > 0$.

4. Consider a prior distribution for λ with pdf

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{x}\right)^{\alpha+1} e^{-\frac{\beta}{x}}$$

with mean $\frac{\beta}{\alpha-1}$ and variance $\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$. Derive the posterior distribution when $k = 1$ and determine whether this a conjagate prior for λ. Give the Bayes estimator of λ under squared error loss with this prior.

STAT 201A - Introduction to Probability at an advanced level
Exam

UC Berkeley

December 15th, 2023

1. Let $\{b_n^{(i)}\}_{i\in\{1,\dots,n\}}$ be a sequence of independent and identically distributed Bernoulli random variables with parameter $p_n \in [0,1]$. Calculate the following limits in distribution:

   a) (**10 points**) Assume that $\lim_{n\to\infty} p_n = 0.1$. Calculate the limit in distribution when $n \to \infty$ of
   $$\frac{\sum_{i=1}^n b_n^{(i)}}{n}$$

   b) (**10 points**) Assume that $p_n = 1/n$. Calculate the limit in distribution when $n \to \infty$ of
   $$\sum_{i=1}^n b_n^{(i)}$$

2. (**10 points**) Consider $U_1, U_2, U_3, U_4, U_5, U_6$ independent random variables with a uniform distribution in $[0,1]$. Find the distribution and the expected value of:

   a) (**5 points**) U_1 given that the minimum of $U_1, U_2, U_3, U_4, U_5, U_6$ is greater than $1/2$.

   b) (**5 points**) The third-order statistic of $U_1, U_2, U_3, U_4, U_5, U_6$.

3. (**10 points**) A random variable X is said to have a log-normal distribution with parameters μ and σ^2, represented as $X \sim \text{LogNormal}(\mu, \sigma^2)$, if $\log X \sim N(\mu, \sigma^2)$ or, equivalently, it has density
   $$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma\, x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \mathbb{1}_{\{x>0\}}. \tag{1}$$

   Show that $X^\alpha \sim \text{LogNormal}(\alpha\mu, \alpha^2\sigma^2)$ for every power $\alpha \neq 0$.

4. (**10 points**) Consider the Markov process with four states described by the graph in Figure 1. Note that state 1 is absorbing. Compute the expected absorption time, given that the starting position is state 2.

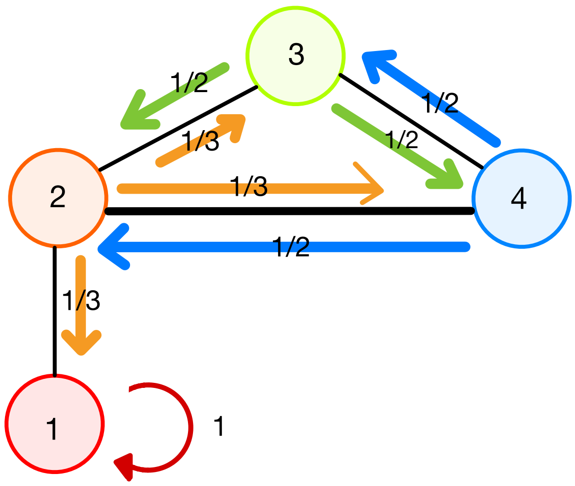

Figure 1: Graph representing the transitions of a Markov chain