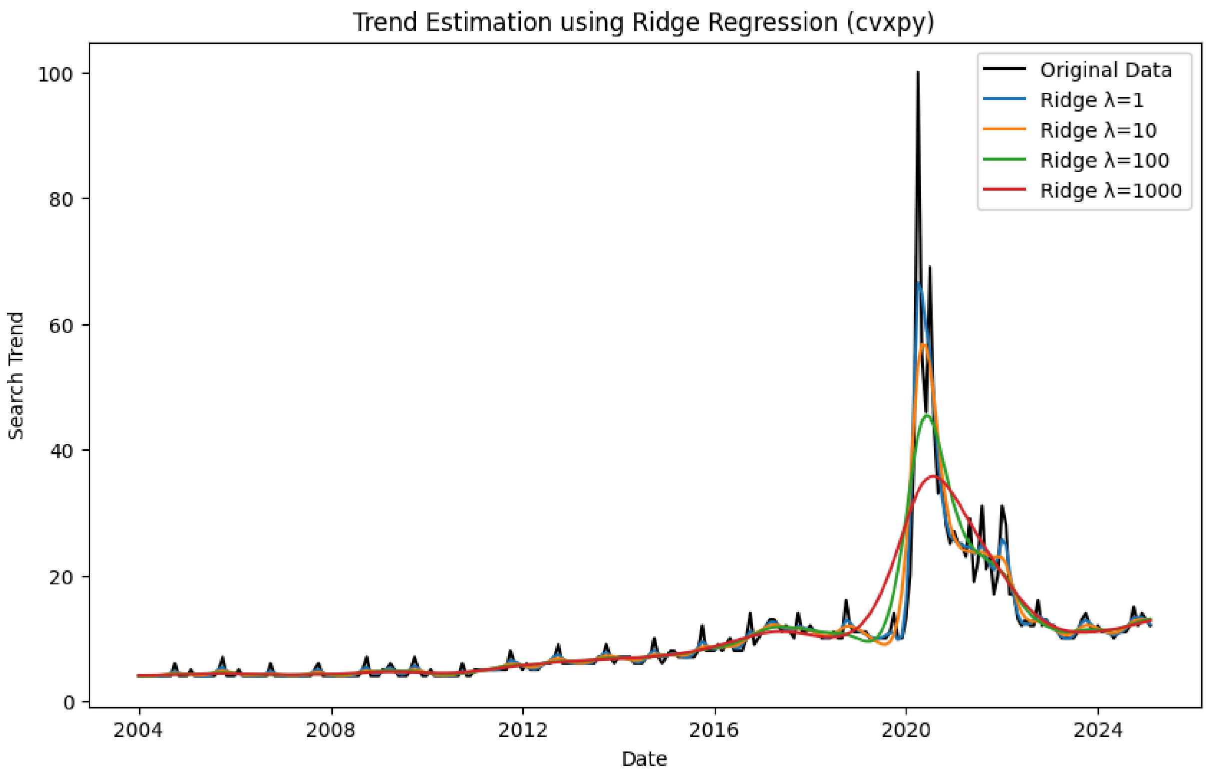
HW3

Problem 1

Part a

In [48]:
```
import numpy as np
import pandas as pd
import cvxpy as cp
import matplotlib.pyplot as plt

file_path = "multiTimeline_mask.csv"
df = pd.read_csv(file_path, skiprows=1)

df.columns = ['Month', 'Trend']
df['Month'] = pd.to_datetime(df['Month'])
df = df.set_index('Month')

y = df['Trend'].values
n = len(y)

x = np.arange(1, n + 1)
X = np.column_stack([np.ones(n), x - 1])
for i in range(n - 2):
    c = i + 2
    xc = ((x > c).astype(float)) * (x - c)
    X = np.column_stack([X, xc])

def solve_ridge(X, y, lambda_val, penalty_start=1):
    """ Solves Ridge Regression using cvxpy """
    n, p = X.shape
    beta = cp.Variable(p)
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.sum_squares(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)
    prob = cp.Problem(objective)
    prob.solve()
    return beta.value

lambda_values = [1, 10, 100, 1000]
ridge_estimates = {lmb: solve_ridge(X, y, lmb) for lmb in lambda_values}

plt.figure(figsize=(10, 6))
plt.plot(df.index, y, label='Original Data', color='black')

for lmb, beta in ridge_estimates.items():
    plt.plot(df.index, X @ beta, label=f'Ridge λ={lmb}')

plt.xlabel('Date')
plt.ylabel('Search Trend')
plt.title('Trend Estimation using Ridge Regression (cvxpy)')
```

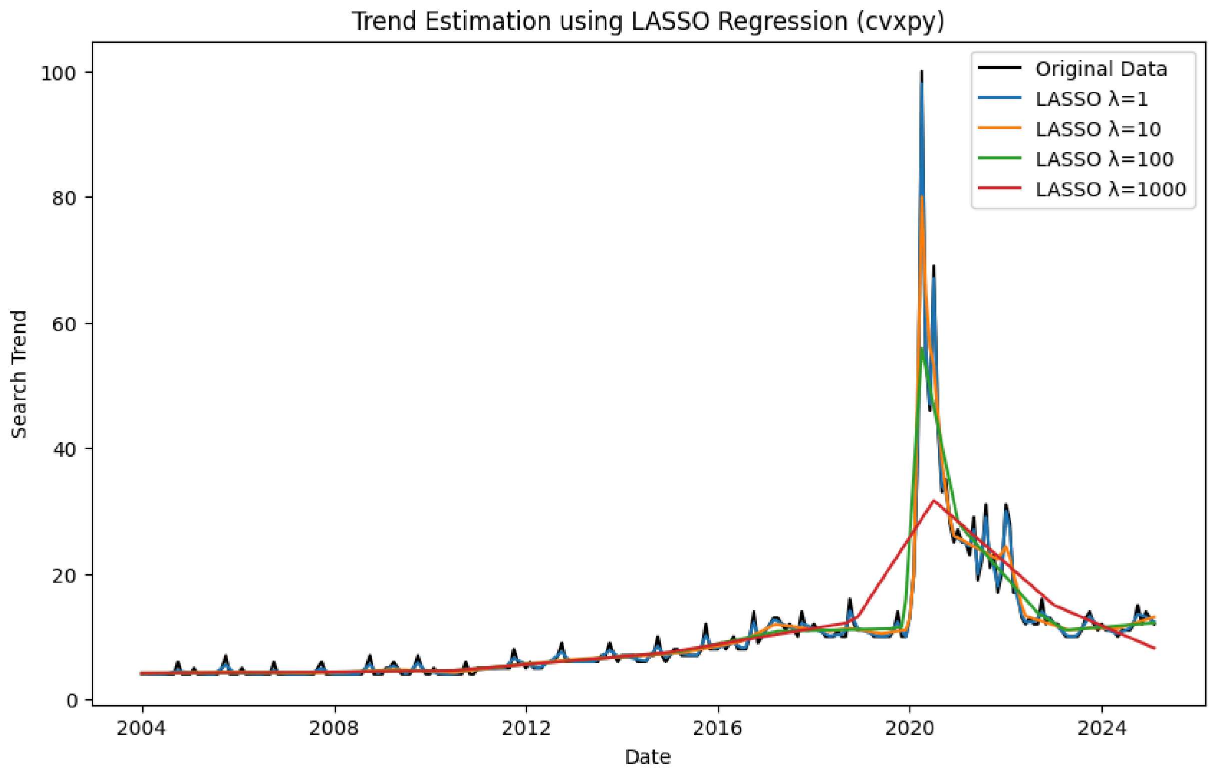
```
plt.legend()
plt.show()
```



From the graph we can see $\lambda = 1000$ is the best interpret the trend.

Part b

In [49]:
```
def solve_lasso(X, y, lambda_val, penalty_start=1):
    """ Solves LASSO Regression using cvxpy """
    n, p = X.shape
    beta = cp.Variable(p)
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.norm1(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)
    prob = cp.Problem(objective)
    prob.solve()
    return beta.value

lasso_estimates = {lmb: solve_lasso(X, y, lmb) for lmb in lambda_values}

plt.figure(figsize=(10, 6))
plt.plot(df.index, y, label='Original Data', color='black')

for lmb, beta in lasso_estimates.items():
    plt.plot(df.index, X @ beta, label=f'LASSO λ={lmb}')

plt.xlabel('Date')
plt.ylabel('Search Trend')
plt.title('Trend Estimation using LASSO Regression (cvxpy)')
plt.legend()
plt.show()
```



From the graph we can see $\lambda = 1000$ provides the best fit

Part c

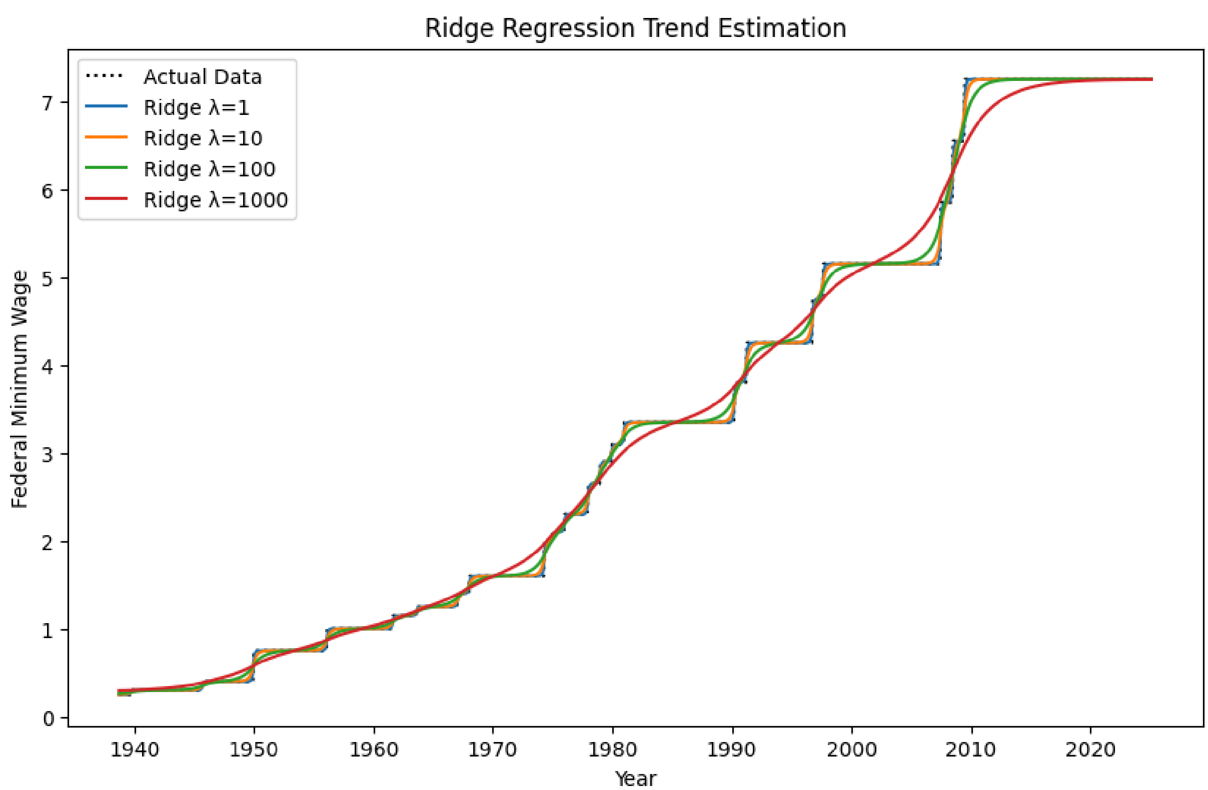
I would say Ridge($\lambda = 1000$) best estimate the trend.

Problem 2

Part a

In [50]:
```
import numpy as np
import pandas as pd
import cvxpy as cp
import matplotlib.pyplot as plt

# Load the dataset
file_path = "FEDMINNFRWG.csv"
df = pd.read_csv(file_path)

# Convert date column to datetime and sort data
df['observation_date'] = pd.to_datetime(df['observation_date'])
df = df.sort_values('observation_date')

# Extract the target variable (federal minimum wage)
y = df['FEDMINNFRWG'].values
n = len(y)

# Construct the design matrix X with indicator functions
X = np.tril(np.ones((n, n)))  # Lower triangular matrix
```

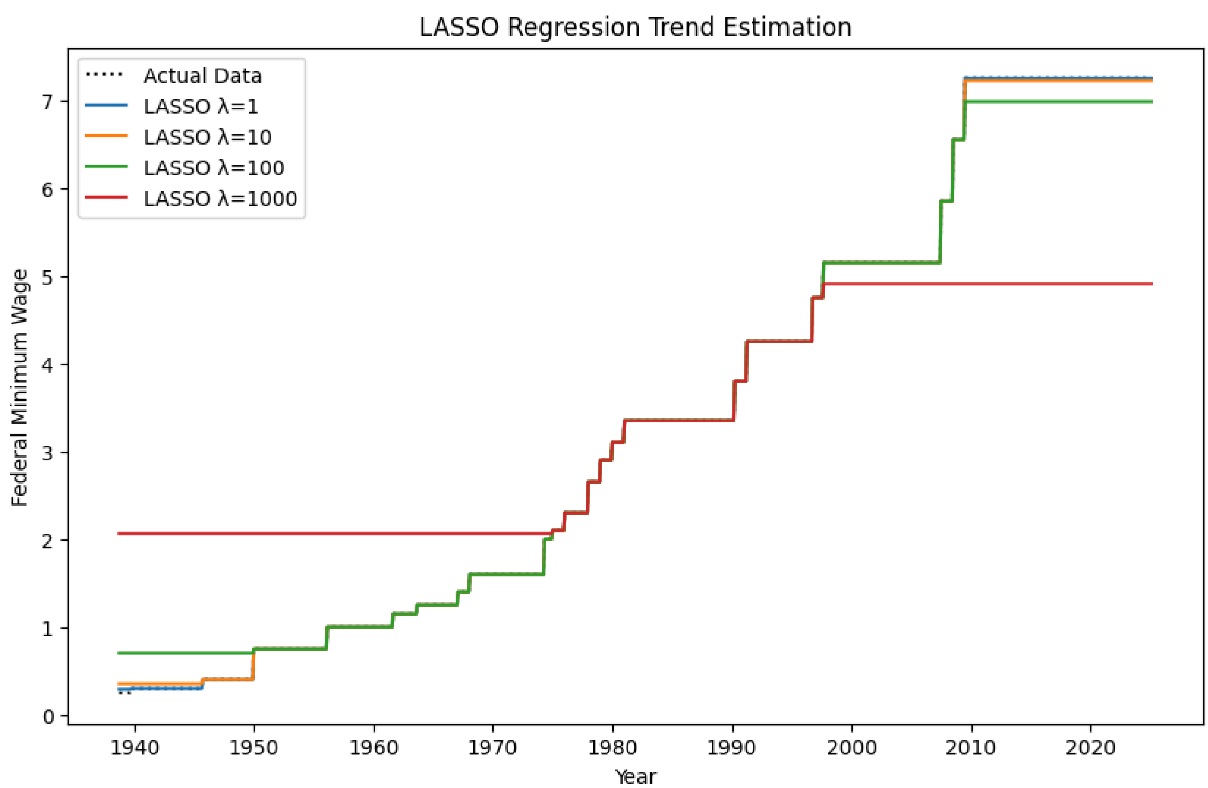
```python
# Define the Ridge Regression function using cvxpy
def solve_ridge(X, y, lambda_val, penalty_start=1):
    """Solves Ridge Regression with given lambda value."""
    n, p = X.shape
    beta = cp.Variable(p)
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.sum_squares(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)
    prob = cp.Problem(objective)
    prob.solve()
    return beta.value

# Ridge regression for lambda = 1, 10, 100, 1000
lambda_values = [1, 10, 100, 1000]
ridge_estimates = {lmb: solve_ridge(X, y, lmb) for lmb in lambda_values}

# Plot the results
plt.figure(figsize=(10, 6))
plt.plot(df['observation_date'], y, label='Actual Data', linestyle='dotted', color=

for lmb, beta in ridge_estimates.items():
    plt.plot(df['observation_date'], X @ beta, label=f'Ridge λ={lmb}')

plt.xlabel('Year')
plt.ylabel('Federal Minimum Wage')
plt.title('Ridge Regression Trend Estimation')
plt.legend()
plt.show()
```



$\lambda = 1000$ provides the best summary for the data.

Part b

In [51]:
```
# Define the LASSO Regression function using cvxpy
def solve_lasso(X, y, lambda_val, penalty_start=1):
    n, p = X.shape
    beta = cp.Variable(p)
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.norm1(beta[penalty_start:])  # L1 regularization
    objective = cp.Minimize(loss + reg)
    prob = cp.Problem(objective)
    prob.solve()
    return beta.value

# Perform LASSO Regression for different lambda values
lasso_estimates = {lmb: solve_lasso(X, y, lmb) for lmb in lambda_values}

# Plot the results for LASSO Regression
plt.figure(figsize=(10, 6))
plt.plot(df['observation_date'], y, label='Actual Data', linestyle='dotted', color=

for lmb, beta in lasso_estimates.items():
    plt.plot(df['observation_date'], X @ beta, label=f'LASSO λ={lmb}')

plt.xlabel('Year')
plt.ylabel('Federal Minimum Wage')
plt.title('LASSO Regression Trend Estimation')
plt.legend()
plt.show()
```



$\lambda = 1000$ provides the best summary.

Part c

Ridge($\lambda = 1000$) provides the best summay amongs these 8 function.

Problem 3

Part a

1. Ridge Regression Objective Function

The Ridge regression estimates (β_0) and (β_1) by minimizing the following objective function:

$$\min_{\beta_0,\beta_1} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 + \lambda\beta_1^2$$

where:

- ($\lambda > 0$) is the regularization parameter.

2. Solving for Ridge Coefficients

Step 1: Compute (β_0)

Taking the partial derivative with respect to (β_0):

$$\frac{\partial}{\partial\beta_0} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 + \lambda\beta_1^2 = 0$$

Expanding:

$$\sum_{i=1}^n 2(y_i - \beta_0 - \beta_1 x_i)(-1) = 0$$

$$\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^n y_i = n\beta_0 + \beta_1 \sum_{i=1}^n x_i$$

$$\beta_0 = \frac{\sum_{i=1}^n y_i}{n} - \beta_1 \frac{\sum_{i=1}^n x_i}{n} = \bar{y} - \beta_1\bar{x}$$

Step 2: Compute (β_1)

Taking the partial derivative with respect to (β_1):

$$\frac{\partial}{\partial\beta_1} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 + \lambda\beta_1^2 = 0$$

Expanding:

$$\sum_{i=1}^n 2(y_i - \beta_0 - \beta_1 x_i)(-x_i) + 2\lambda\beta_1 = 0$$

$$\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)x_i + \lambda\beta_1 = 0$$

Substituting $(\beta_0 = \bar{y} - \beta_1\bar{x})$:

$\sum_{i=1}^{n}(y_i - \bar{y} + \beta_1\bar{x} - \beta_1 x_i)x_i + \lambda\beta_1 = 0$

$\sum_{i=1}^{n}(y_i - \bar{y})x_i + \beta_1\bar{x}\sum_{i=1}^{n} x_i - \beta_1\sum_{i=1}^{n} x_i^2 + \lambda\beta_1 = 0$

Since $(\sum_{i=1}^{n}(y_i - \bar{y})\bar{x} = \bar{x}\sum_{i=1}^{n}(y_i - \bar{y}) = 0)$, we simplify:

$\sum_{i=1}^{n}(y_i - \bar{y})x_i - \beta_1\sum_{i=1}^{n}(x_i - \bar{x})^2 + \lambda\beta_1 = 0$

$\beta_1(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \lambda) = \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})$

$\beta_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\lambda + \sum_{i=1}^{n}(x_i - \bar{x})^2}$

Part b

1. LASSO Regression Objective Function

LASSO regression estimates (β_0) and (β_1) by minimizing the following objective function:

$\min_{\beta_0,\beta_1}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 + \lambda|\beta_1|$

where:

- $(\lambda > 0)$ is the regularization parameter.

---

2. Solving for LASSO Coefficients

Step 1: Compute (β_0)

Taking the partial derivative with respect to (β_0):

$\frac{\partial}{\partial\beta_0}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 + \lambda|\beta_1| = 0$

Expanding:

$\sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-1) = 0$

$\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0$

$\sum_{i=1}^{n} y_i = n\beta_0 + \beta_1\sum_{i=1}^{n} x_i$

$\beta_0 = \frac{\sum_{i=1}^{n} y_i}{n} - \beta_1\frac{\sum_{i=1}^{n} x_i}{n} = \bar{y} - \beta_1\bar{x}$

Thus, the LASSO estimate for (β_0) is:

$\hat{\beta}_0^{\text{lasso}}(\lambda) = \bar{y} - \bar{x}\hat{\beta}_1^{\text{lasso}}(\lambda)$

Step 2: Compute (β_1)

Taking the partial derivative with respect to (β_1):

$\frac{\partial}{\partial \beta_1} \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 + \lambda|\beta_1| = 0$

Expanding:

$\sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-x_i) + \lambda \cdot \text{sign}(\beta_1) = 0$

Substituting $(\beta_0 = \bar{y} - \beta_1 \bar{x})$:

$\sum_{i=1}^{n}(y_i - \bar{y} + \beta_1 \bar{x} - \beta_1 x_i)x_i + \frac{\lambda}{2}\text{sign}(\beta_1) = 0$

$\sum_{i=1}^{n}(y_i - \bar{y})x_i - \beta_1 \sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{\lambda}{2}\text{sign}(\beta_1) = 0$

Rearranging for (β_1):

$\beta_1 \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) - \frac{\lambda}{2}\text{sign}(\beta_1)$

$\beta_1 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) - \frac{\lambda}{2}\text{sign}(\beta_1)}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

---

3. Piecewise Solution for (β_1)

Since the absolute value function $(|\beta_1|)$ causes non-differentiability at $(\beta_1 = 0)$, we analyze three cases:

Case 1: $(\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) > \frac{\lambda}{2})$

$\hat{\beta}_1^{\text{lasso}}(\lambda) = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) - \frac{\lambda}{2}}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Case 2: $(\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) < -\frac{\lambda}{2})$

$\hat{\beta}_1^{\text{lasso}}(\lambda) = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) + \frac{\lambda}{2}}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Case 3: $(-\frac{\lambda}{2} \leq \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) \leq \frac{\lambda}{2})$

$\hat{\beta}_1^{\text{lasso}}(\lambda) = 0$

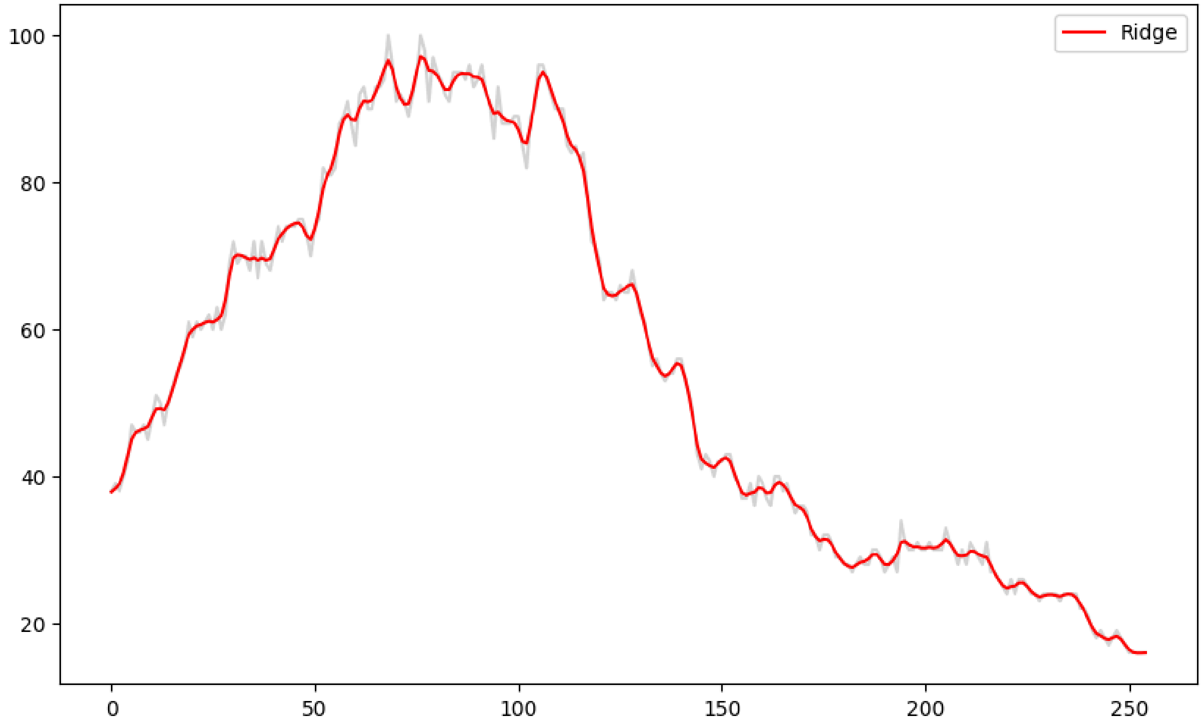
Problem 4

Part a

In [52]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import statsmodels.api as sm
import cvxpy as cp

#below penalty_start = 2 means that b0 and b1 are not included in the penalty
def solve_ridge(X, y, lambda_val, penalty_start=2):
    n, p = X.shape

    # Define variable
    beta = cp.Variable(p)

    # Define objective
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.sum_squares(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)

    # Solve problem
    prob = cp.Problem(objective)
    prob.solve()

    return beta.value

def ridge_cv(X, y, lambda_candidates):
    n = len(y)
    folds = []
    for i in range(5):
        test_indices = np.arange(i, n, 5)
        train_indices = np.array([j for j in range(n) if j % 5 != i])
        folds.append((train_indices, test_indices))
    cv_errors = {lamb: 0 for lamb in lambda_candidates}

    for train_index, test_index in folds:
        X_train = X[train_index]
        X_test = X[test_index]
        y_train = y[train_index]
        y_test = y[test_index]

        for lamb in lambda_candidates:
            beta = solve_ridge(X_train, y_train, lambda_val = lamb)
            y_pred = np.dot(X_test, beta)
            squared_errors = (y_test - y_pred) ** 2
            cv_errors[lamb] += np.sum(squared_errors)
    for lamb in lambda_candidates:
        cv_errors[lamb] /= n

    best_lambda = min(cv_errors, key = cv_errors.get)

    return best_lambda, cv_errors


file_path = 'multiTimeline_yahoo.csv'
df = pd.read_csv(file_path, skiprows=1)  # Skip description row
df.columns = ['date', 'value']
```

```
df['date'] = pd.to_datetime(df['date'])
df['time_index'] = np.arange(1, len(df) + 1)

# Extract data
y = df['value'].values
x = df['time_index'].values
n = len(y)
x = np.arange(1, n+1)
X = np.column_stack([np.ones(n), x-1])
for i in range(n-2):
    c = i+2
    xc = ((x > c).astype(float))*(x-c)
    X = np.column_stack([X, xc])

lambda_candidates = np.array([0.1, 1, 10, 100, 1000, 10000, 100000])
print(lambda_candidates)

best_lambda, cv_errors = ridge_cv(X, y, lambda_candidates)
print(best_lambda)
print("CV errors for each lambda:")
for lamb, error in sorted(cv_errors.items()):
    print(f"Lambda = {lamb:.2f}, CV Error = {error:.6f}")

b_ridge = solve_ridge(X, y, lambda_val = best_lambda)
ridge_fitted = np.dot(X, b_ridge)
plt.figure(figsize = (10, 6))
plt.plot(y, color = 'lightgray')
plt.plot(ridge_fitted, color = 'red', label = 'Ridge')
plt.legend()
plt.show()
```

```
[1.e-01 1.e+00 1.e+01 1.e+02 1.e+03 1.e+04 1.e+05]
1.0
CV errors for each lambda:
Lambda = 0.10, CV Error = 3.641771
Lambda = 1.00, CV Error = 3.529195
Lambda = 10.00, CV Error = 4.035778
Lambda = 100.00, CV Error = 5.041541
Lambda = 1000.00, CV Error = 6.701243
Lambda = 10000.00, CV Error = 8.873522
Lambda = 100000.00, CV Error = 15.392089
```



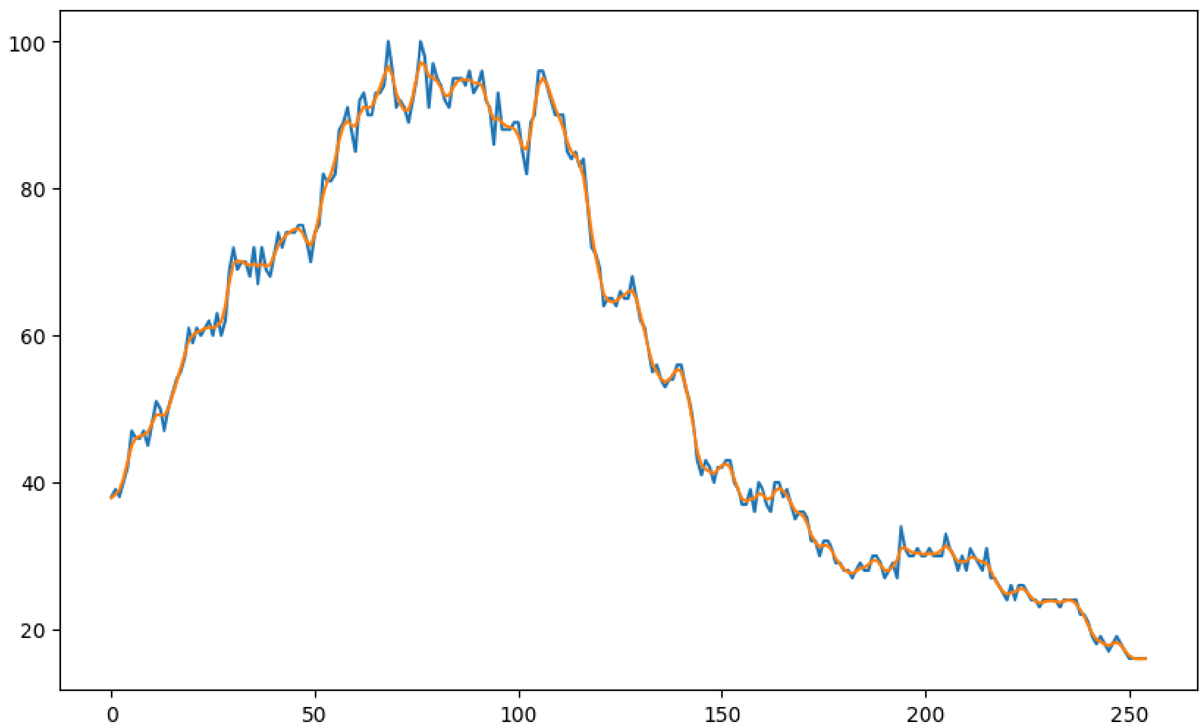
Part b

i

In [53]:
```
C = 10**4

tau_gr = np.logspace(np.log10(0.0001), np.log10(1), 20)
sig_gr = np.logspace(np.log10(0.1), np.log10(1), 20)
#sig_gr = np.array([0.16])

t, s = np.meshgrid(tau_gr, sig_gr)

g = pd.DataFrame({'tau': t.flatten(), 'sig': s.flatten()})

for i in range(len(g)):
    tau = g.loc[i, 'tau']
    sig = g.loc[i, 'sig']
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, n-2)]))
    Mat = np.linalg.inv(Q) + (X.T @ X)/(sig ** 2)
    Matinv = np.linalg.inv(Mat)
    sgn, logcovdet = np.linalg.slogdet(Matinv)
    sgnQ, logcovdetQ = np.linalg.slogdet(Q)
    g.loc[i, 'logpost'] = (-n-1)*np.log(sig) - np.log(tau) - 0.5 * logcovdetQ + 0.5

#Posterior maximizers:
max_row = g['logpost'].idxmax()
print(max_row)
tau_opt = g.loc[max_row, 'tau']
sig_opt = g.loc[max_row, 'sig']
print(tau_opt, sig_opt)
ratio = sig_opt**2 / tau_opt**2
print(ratio)
```

```
# Posterior mean of beta with tau_opt and sig_opt
Q = np.diag(np.concatenate([[C, C], np.repeat(tau_opt**2, n-2)]))

XTX = np.dot(X.T, X)
TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig_opt ** 2)))
XTy = np.dot(X.T, y)

betahat = np.dot(TempMat, XTy/(sig_opt ** 2))
muhat = np.dot(X, betahat)

plt.figure(figsize = (10, 6))
plt.plot(y)
plt.plot(muhat)
plt.show()
```

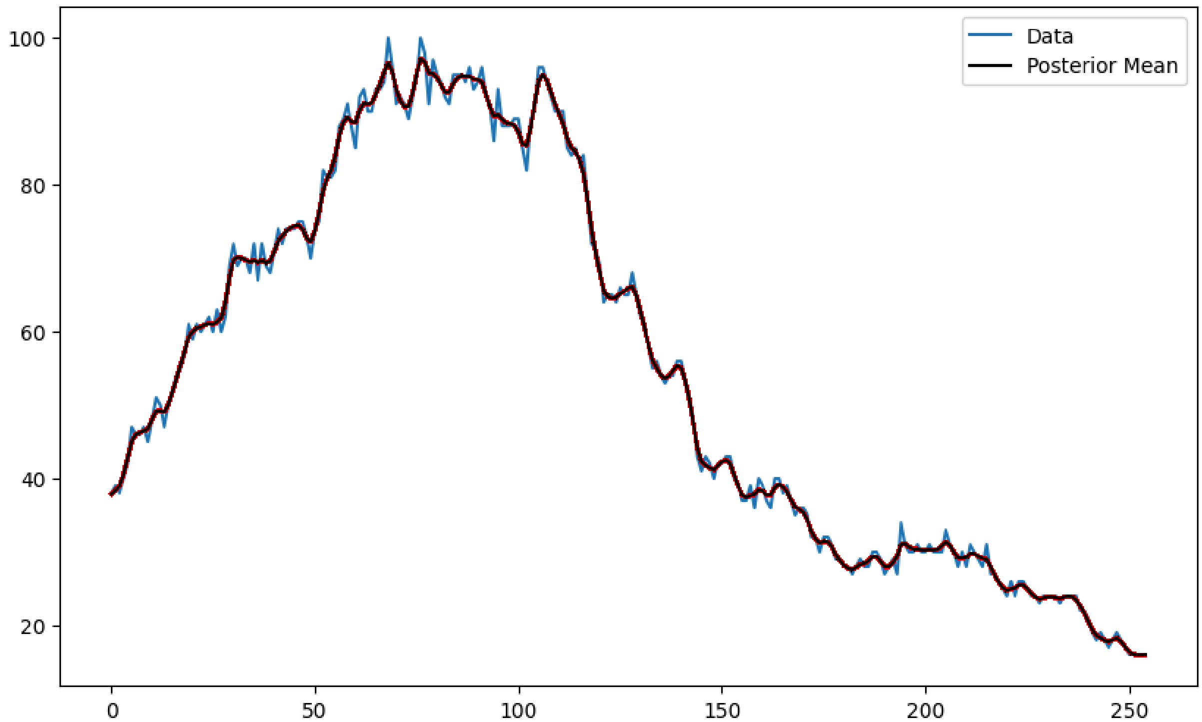
```
399
1.0 1.0
1.0
```



ii

In [54]:
```
g['post'] = np.exp(g['logpost'] - np.max(g['logpost']))
g['post'] = g['post']/np.sum(g['post'])
N = 1000
samples = g.sample(N, weights = g['post'], replace = True)
tau_samples = np.array(samples.iloc[:,0])
sig_samples = np.array(samples.iloc[:,1])
betahats = np.zeros((n, N))
muhats = np.zeros((n, N))
for i in range(N):
    tau = tau_samples[i]
    sig = sig_samples[i]
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, n-2)]))
```

```
    XTX = np.dot(X.T, X)
    TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig ** 2)))
    XTy = np.dot(X.T, y)
    betahat = np.dot(TempMat, XTy/(sig ** 2))
    muhat = np.dot(X, betahat)
    betahats[:,i] = betahat
    muhats[:,i] = muhat

beta_est = np.mean(betahats, axis = 1)
mu_est = np.mean(muhats, axis = 1) #these are the fitted values
plt.figure(figsize = (10, 6))
plt.plot(y, label = 'Data')
for i in range(N):
    plt.plot(muhats[:,i], color = 'red')
plt.plot(mu_est, color = 'black', label = 'Posterior Mean')
plt.legend()
plt.show()
```



iii

Ridge regression from Part a provides a smoother trend estimate, so I prefer the ridge regression.

Problem 5

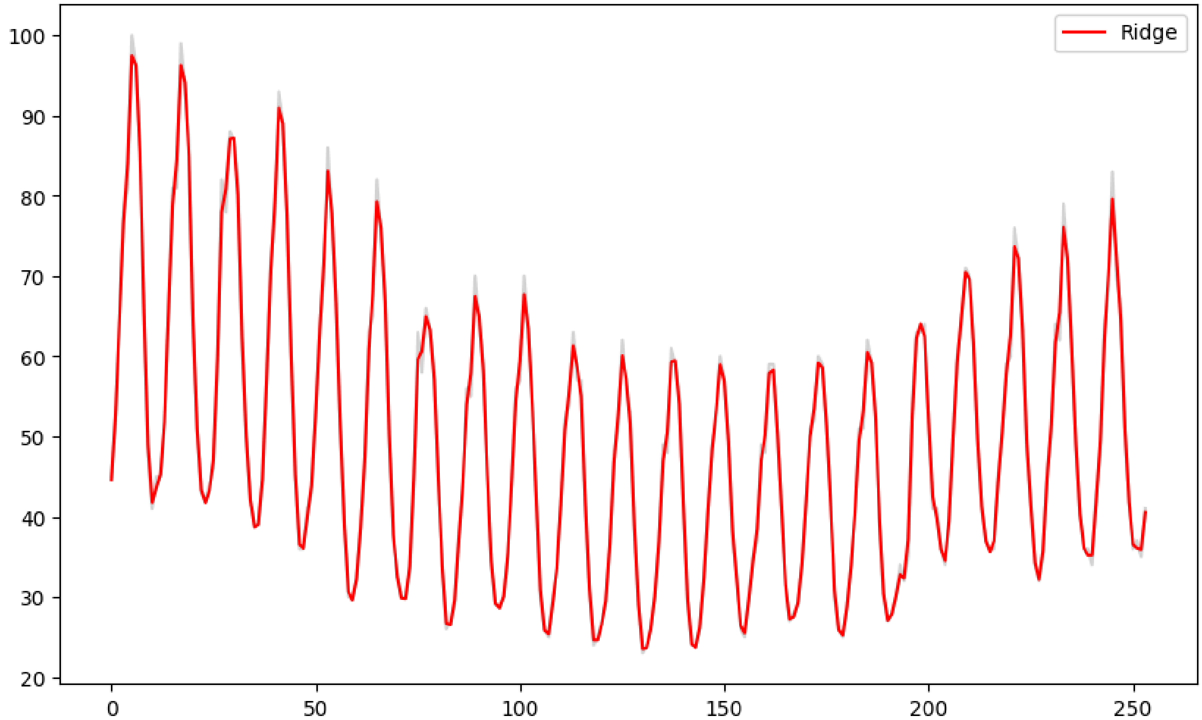
Part a

In [55]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import statsmodels.api as sm
```

```python
import cvxpy as cp

#below penalty_start = 2 means that b0 and b1 are not included in the penalty
def solve_ridge(X, y, lambda_val, penalty_start=2):
    n, p = X.shape

    # Define variable
    beta = cp.Variable(p)

    # Define objective
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.sum_squares(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)

    # Solve problem
    prob = cp.Problem(objective)
    prob.solve()

    return beta.value

def ridge_cv(X, y, lambda_candidates):
    n = len(y)
    folds = []
    for i in range(5):
        test_indices = np.arange(i, n, 5)
        train_indices = np.array([j for j in range(n) if j % 5 != i])
        folds.append((train_indices, test_indices))
    cv_errors = {lamb: 0 for lamb in lambda_candidates}

    for train_index, test_index in folds:
        X_train = X[train_index]
        X_test = X[test_index]
        y_train = y[train_index]
        y_test = y[test_index]

        for lamb in lambda_candidates:
            beta = solve_ridge(X_train, y_train, lambda_val = lamb)
            y_pred = np.dot(X_test, beta)
            squared_errors = (y_test - y_pred) ** 2
            cv_errors[lamb] += np.sum(squared_errors)
    for lamb in lambda_candidates:
        cv_errors[lamb] /= n

    best_lambda = min(cv_errors, key = cv_errors.get)

    return best_lambda, cv_errors


file_path = 'multiTimeline_golf.csv'
df = pd.read_csv(file_path, skiprows=1)
df.columns = ['date', 'value']
df['date'] = pd.to_datetime(df['date'])
df['time_index'] = np.arange(1, len(df) + 1)

y = df['value'].values
x = df['time_index'].values
```

```python
n = len(y)
x = np.arange(1, n+1)
X = np.column_stack([np.ones(n), x-1])
for i in range(n-2):
    c = i+2
    xc = ((x > c).astype(float))*(x-c)
    X = np.column_stack([X, xc])

lambda_candidates = np.array([0.1, 1, 10, 100, 1000, 10000, 100000])
print(lambda_candidates)

best_lambda, cv_errors = ridge_cv(X, y, lambda_candidates)
print(best_lambda)
print("CV errors for each lambda:")
for lamb, error in sorted(cv_errors.items()):
    print(f"Lambda = {lamb:.2f}, CV Error = {error:.6f}")

b_ridge = solve_ridge(X, y, lambda_val = best_lambda)
ridge_fitted = np.dot(X, b_ridge)
plt.figure(figsize = (10, 6))
plt.plot(y, color = 'lightgray')
plt.plot(ridge_fitted, color = 'red', label = 'Ridge')
plt.legend()
plt.show()
```

```
[1.e-01 1.e+00 1.e+01 1.e+02 1.e+03 1.e+04 1.e+05]
0.1
CV errors for each lambda:
Lambda = 0.10, CV Error = 18.819188
Lambda = 1.00, CV Error = 18.866577
Lambda = 10.00, CV Error = 71.209054
Lambda = 100.00, CV Error = 189.635108
Lambda = 1000.00, CV Error = 223.805138
Lambda = 10000.00, CV Error = 228.491390
Lambda = 100000.00, CV Error = 230.453150
```



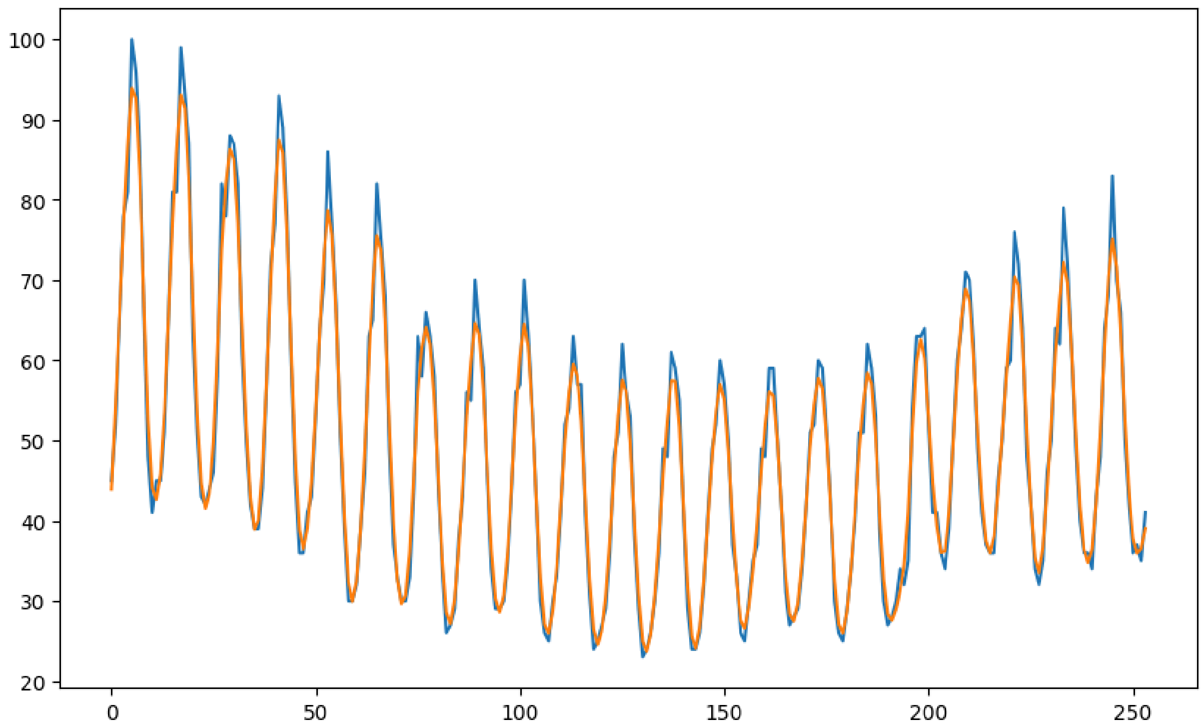
Part b

i

In [56]:
```
C = 10**4

tau_gr = np.logspace(np.log10(0.0001), np.log10(1), 20)
sig_gr = np.logspace(np.log10(0.1), np.log10(1), 20)
#sig_gr = np.array([0.16])

t, s = np.meshgrid(tau_gr, sig_gr)

g = pd.DataFrame({'tau': t.flatten(), 'sig': s.flatten()})

for i in range(len(g)):
    tau = g.loc[i, 'tau']
    sig = g.loc[i, 'sig']
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, n-2)]))
    Mat = np.linalg.inv(Q) + (X.T @ X)/(sig ** 2)
    Matinv = np.linalg.inv(Mat)
    sgn, logcovdet = np.linalg.slogdet(Matinv)
    sgnQ, logcovdetQ = np.linalg.slogdet(Q)
    g.loc[i, 'logpost'] = (-n-1)*np.log(sig) - np.log(tau) - 0.5 * logcovdetQ + 0.5

#Posterior maximizers:
max_row = g['logpost'].idxmax()
print(max_row)
tau_opt = g.loc[max_row, 'tau']
sig_opt = g.loc[max_row, 'sig']
print(tau_opt, sig_opt)
ratio = sig_opt**2 / tau_opt**2
print(ratio)
```

```
# Posterior mean of beta with tau_opt and sig_opt
tau = tau_opt
sig = sig_opt
Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, n-2)]))

XTX = np.dot(X.T, X)
TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig ** 2)))
XTy = np.dot(X.T, y)

betahat = np.dot(TempMat, XTy/(sig ** 2))
muhat = np.dot(X, betahat)

plt.figure(figsize = (10, 6))
plt.plot(y)
plt.plot(muhat)
plt.show()
```

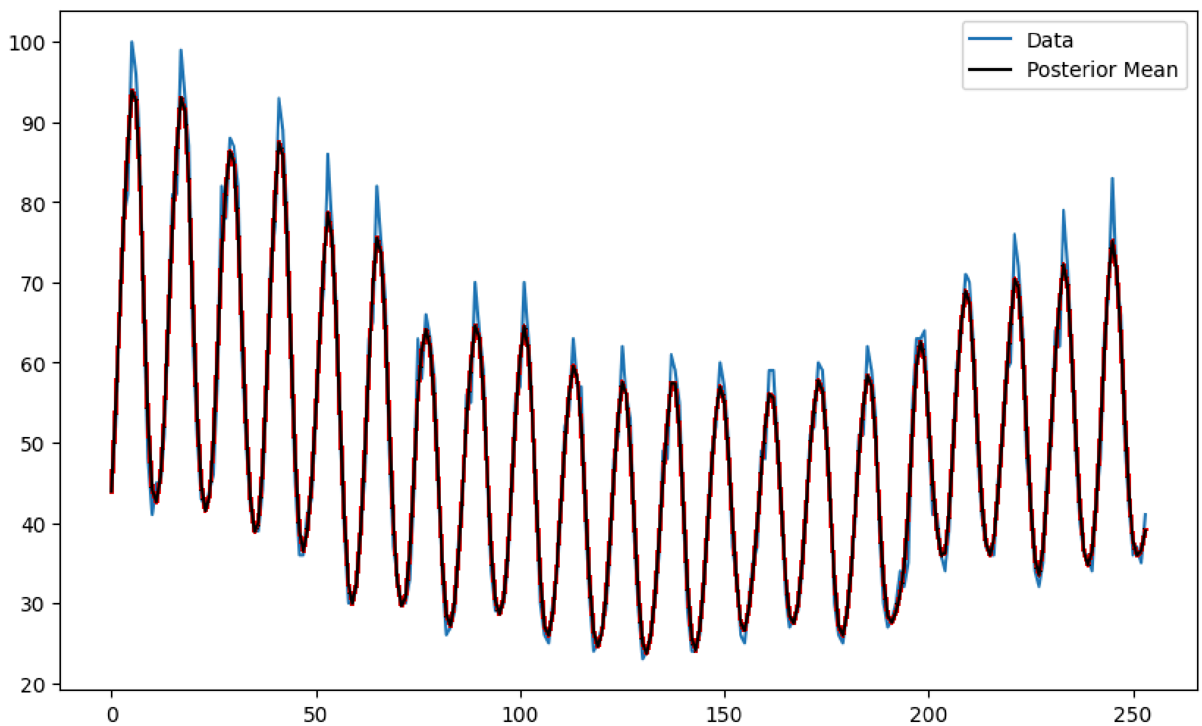
```
399
1.0 1.0
1.0
```



ii

In [57]:
```
g['post'] = np.exp(g['logpost'] - np.max(g['logpost']))
g['post'] = g['post']/np.sum(g['post'])
N = 1000
samples = g.sample(N, weights = g['post'], replace = True)
tau_samples = np.array(samples.iloc[:,0])
sig_samples = np.array(samples.iloc[:,1])
betahats = np.zeros((n, N))
muhats = np.zeros((n, N))
for i in range(N):
    tau = tau_samples[i]
```

```
    sig = sig_samples[i]
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, n-2)]))
    XTX = np.dot(X.T, X)
    TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig ** 2)))
    XTy = np.dot(X.T, y)
    betahat = np.dot(TempMat, XTy/(sig ** 2))
    muhat = np.dot(X, betahat)
    betahats[:,i] = betahat
    muhats[:,i] = muhat

beta_est = np.mean(betahats, axis = 1)
mu_est = np.mean(muhats, axis = 1) #these are the fitted values
plt.figure(figsize = (10, 6))
plt.plot(y, label = 'Data')
for i in range(N):
    plt.plot(muhats[:,i], color = 'red')
plt.plot(mu_est, color = 'black', label = 'Posterior Mean')
plt.legend()
plt.show()
```



iii

From two graphes, Bayesian trend estimate is smoother, so I prefer the Bayesian trend estimate.

Problem 6

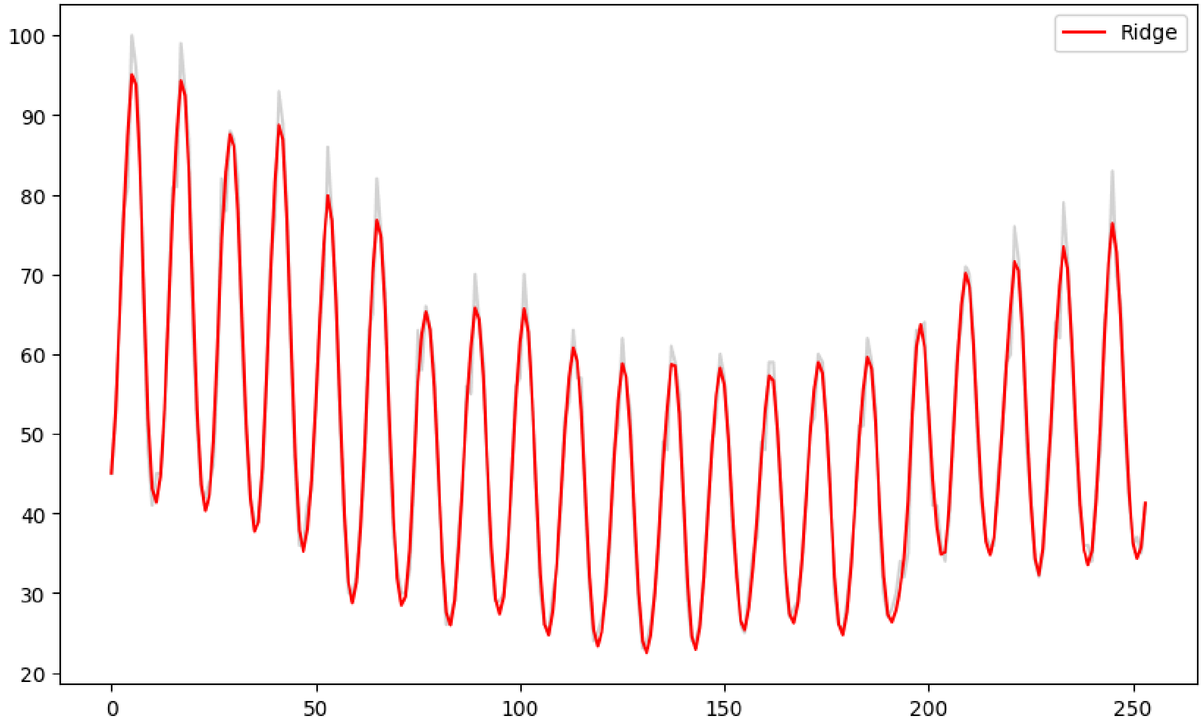
Part a

In [58]:
```
import numpy as np
import pandas as pd
```

```python
import matplotlib.pyplot as plt
import cvxpy as cp

file_path = 'multiTimeline_golf.csv'
df = pd.read_csv(file_path, skiprows=1)
df.columns = ['date', 'value']
df['date'] = pd.to_datetime(df['date'])
df['time_index'] = np.arange(1, len(df) + 1)

# Extract data
y = df['value'].values
x = df['time_index'].values
n = len(y)


x = np.arange(1, n+1)
X = np.column_stack([np.ones(n), x-1])
for i in range(n-2):
    c = i+2
    xc = ((x > c).astype(float)) * (x - c)
    X = np.column_stack([X, xc])
X = np.column_stack([
    X,
    np.cos(2 * np.pi * x / 12),  # Cosine term for seasonality
    np.sin(2 * np.pi * x / 12)   # Sine term for seasonality
])

#note that penalty_start is now set to 1 (instead of 2 as in the model used in clas
def solve_ridge(X, y, lambda_val, penalty_start=1):
    n, p = X.shape

    # Define variable
    beta = cp.Variable(p)

    # Define objective
    loss = cp.sum_squares(X @ beta - y)
    reg = lambda_val * cp.sum_squares(beta[penalty_start:])
    objective = cp.Minimize(loss + reg)

    # Solve problem
    prob = cp.Problem(objective)
    prob.solve()

    return beta.value

def ridge_cv(X, y, lambda_candidates):
    n = len(y)
    folds = []
    for i in range(5):
        test_indices = np.arange(i, n, 5)
        train_indices = np.array([j for j in range(n) if j % 5 != i])
        folds.append((train_indices, test_indices))
    cv_errors = {lamb: 0 for lamb in lambda_candidates}

    for train_index, test_index in folds:
        X_train = X[train_index]
```

```
        X_test = X[test_index]
        y_train = y[train_index]
        y_test = y[test_index]

        for lamb in lambda_candidates:
            beta = solve_ridge(X_train, y_train, lambda_val = lamb)
            y_pred = np.dot(X_test, beta)
            squared_errors = (y_test - y_pred) ** 2
            cv_errors[lamb] += np.sum(squared_errors)
    for lamb in lambda_candidates:
        cv_errors[lamb] /= n

    best_lambda = min(cv_errors, key = cv_errors.get)

    return best_lambda, cv_errors

lambda_candidates = np.array([0.1, 1, 10, 100, 1000, 10000, 100000])
print(lambda_candidates)

best_lambda, cv_errors = ridge_cv(X, y, lambda_candidates)
print(best_lambda)
print("CV errors for each lambda:")
for lamb, error in sorted(cv_errors.items()):
    print(f"Lambda = {lamb:.2f}, CV Error = {error:.6f}")

b_ridge = solve_ridge(X, y, lambda_val = best_lambda)
ridge_fitted = np.dot(X, b_ridge)
plt.figure(figsize = (10, 6))
plt.plot(y, color = 'lightgray')
plt.plot(ridge_fitted, color = 'red', label = 'Ridge')
plt.legend()
plt.show()
```

```
[1.e-01 1.e+00 1.e+01 1.e+02 1.e+03 1.e+04 1.e+05]
1.0
CV errors for each lambda:
Lambda = 0.10, CV Error = 18.339430
Lambda = 1.00, CV Error = 15.846146
Lambda = 10.00, CV Error = 19.399858
Lambda = 100.00, CV Error = 67.834483
Lambda = 1000.00, CV Error = 188.599596
Lambda = 10000.00, CV Error = 224.199922
Lambda = 100000.00, CV Error = 229.833292
```



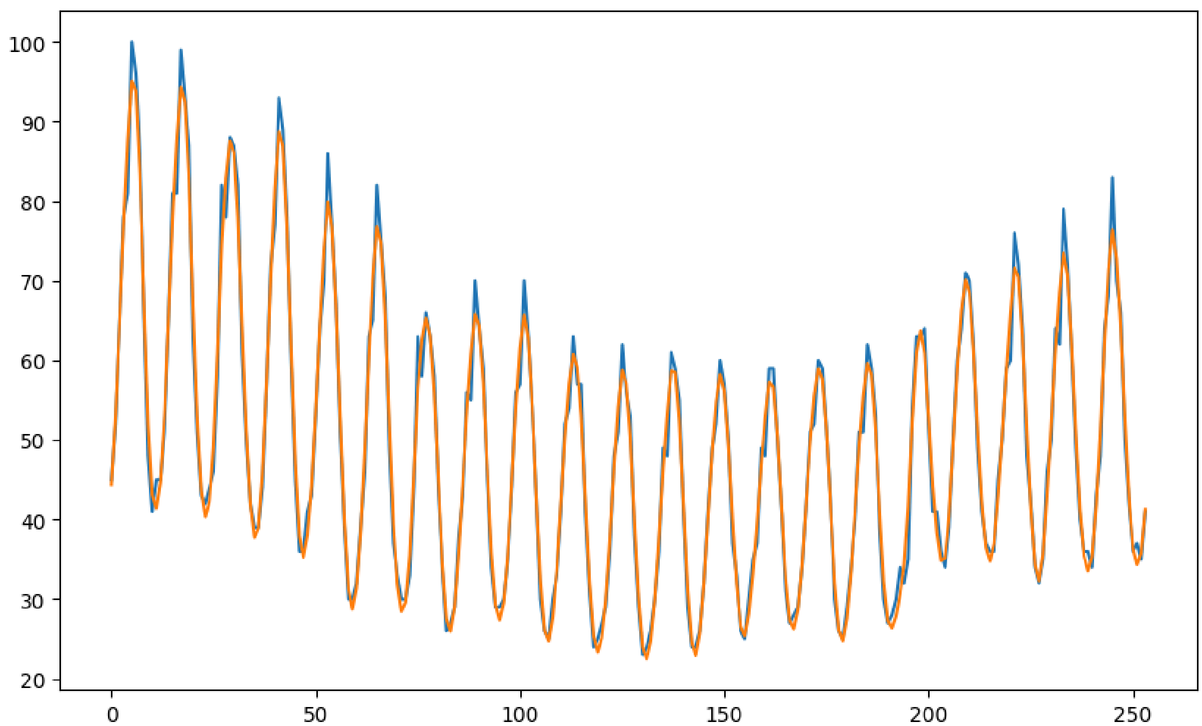
Part b

i

In [59]:
```
C = 10**4

tau_gr = np.logspace(np.log10(0.0001), np.log10(1), 20)
sig_gr = np.logspace(np.log10(0.1), np.log10(1), 20)
#sig_gr = np.array([0.16])

t, s = np.meshgrid(tau_gr, sig_gr)

g = pd.DataFrame({'tau': t.flatten(), 'sig': s.flatten()})

for i in range(len(g)):
    tau = g.loc[i, 'tau']
    sig = g.loc[i, 'sig']
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, X.shape[1]-2)]))
    Mat = np.linalg.inv(Q) + (X.T @ X)/(sig ** 2)
    Matinv = np.linalg.inv(Mat)
    sgn, logcovdet = np.linalg.slogdet(Matinv)
    sgnQ, logcovdetQ = np.linalg.slogdet(Q)
    g.loc[i, 'logpost'] = (-n-1)*np.log(sig) - np.log(tau) - 0.5 * logcovdetQ + 0.5

#Posterior maximizers:
max_row = g['logpost'].idxmax()
print(max_row)
tau_opt = g.loc[max_row, 'tau']
sig_opt = g.loc[max_row, 'sig']
print(tau_opt, sig_opt)
ratio = sig_opt**2 / tau_opt**2
print(ratio)
```

```
# Posterior mean of beta with tau_opt and sig_opt
tau = tau_opt
sig = sig_opt
Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, X.shape[1]-2)]))

XTX = np.dot(X.T, X)
TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig ** 2)))
XTy = np.dot(X.T, y)

betahat = np.dot(TempMat, XTy/(sig ** 2))
muhat = np.dot(X, betahat)

plt.figure(figsize = (10, 6))
plt.plot(y)
plt.plot(muhat)
plt.show()
```

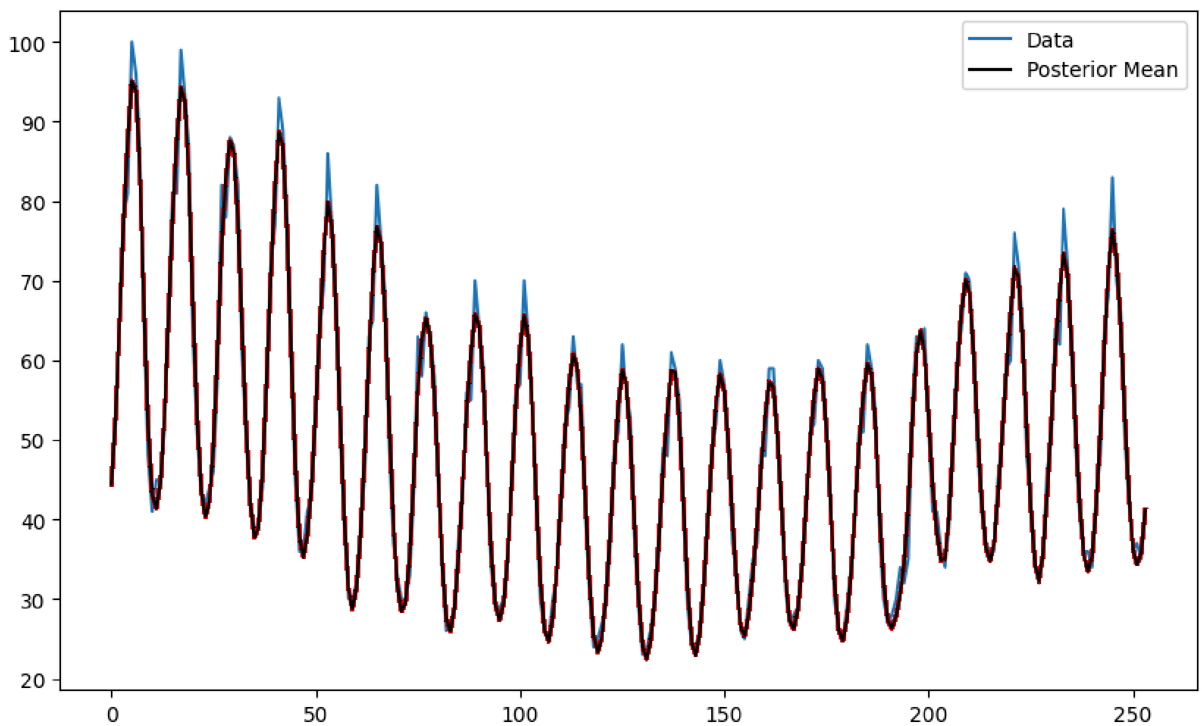
```
399
1.0 1.0
1.0
```



ii

In [60]:
```
g['post'] = np.exp(g['logpost'] - np.max(g['logpost']))
g['post'] = g['post']/np.sum(g['post'])
N = 1000
samples = g.sample(N, weights = g['post'], replace = True)
tau_samples = np.array(samples.iloc[:,0])
sig_samples = np.array(samples.iloc[:,1])
betahats = np.zeros((X.shape[1], N))
muhats = np.zeros((n, N))
for i in range(N):
    tau = tau_samples[i]
```

```
    sig = sig_samples[i]
    Q = np.diag(np.concatenate([[C, C], np.repeat(tau**2, X.shape[1] - 2)]))
    XTX = np.dot(X.T, X)
    TempMat = np.linalg.inv(np.linalg.inv(Q) + (XTX/(sig ** 2)))
    XTy = np.dot(X.T, y)
    betahat = np.dot(TempMat, XTy/(sig ** 2))
    muhat = np.dot(X, betahat)
    betahats[:,i] = betahat
    muhats[:,i] = muhat

beta_est = np.mean(betahats, axis = 1)
mu_est = np.mean(muhats, axis = 1) #these are the fitted values
plt.figure(figsize = (10, 6))
plt.plot(y, label = 'Data')
for i in range(N):
    plt.plot(muhats[:,i], color = 'red')
plt.plot(mu_est, color = 'black', label = 'Posterior Mean')
plt.legend()
plt.show()
```



iii

From two graphs, bayesian trend estimate is smoother, so I prefer bayesian trend estimate

Problem 7

Part a

1. Model Specification

We consider the following Bayesian regression model:

$$y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I)$$

where:

- (y) is the $(n \times 1)$ response vector,
- (X) is the $(n \times p)$ design matrix,
- (β) is the $(p \times 1)$ vector of regression coefficients,
- (ϵ) is an independent Gaussian noise term.

2. Prior Distributions

The Bayesian model assumes the following prior distributions:

$$\beta|\tau, \sigma \sim \mathcal{N}(0, Q)$$

where (Q) is a diagonal matrix given by:

$$Q = \mathrm{diag}(C, C, \tau^2, \tau^2, \ldots, \tau^2)$$

Additionally, the hyperparameters (τ) and (σ) follow uniform priors:

$$\log \tau \sim \mathrm{Uniform}(-C, C), \quad \log \sigma \sim \mathrm{Uniform}(-C, C)$$

3. Likelihood Function

The likelihood function for (y) given (X), (β), and (σ^2) is:

$$p(y|X, \beta, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^\top (y - X\beta)\right)$$

4. Prior Density of (β)

The prior density of (β) given (τ) and (σ) is:

$$p(\beta|\tau, \sigma) = (2\pi)^{-\frac{n}{2}} |Q|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\beta^\top Q^{-1}\beta\right)$$

Since (Q) is diagonal:

$$Q^{-1} = \mathrm{diag}(1/C, 1/C, 1/\tau^2, \ldots, 1/\tau^2)$$

Thus, the quadratic form simplifies to:

$$\beta^\top Q^{-1}\beta = \frac{\beta_0^2}{C} + \frac{\beta_1^2}{C} + \sum_{j=2}^{p} \frac{\beta_j^2}{\tau^2}$$

5. Posterior Distribution of (β)

By Bayes' theorem, the posterior distribution is proportional to the product of the likelihood and prior:

$$p(\beta|y, X, \tau, \sigma) \propto p(y|X, \beta, \sigma)p(\beta|\tau, \sigma)$$

Substituting the expressions:

$$p(\beta|y, X, \tau, \sigma) \propto \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^\top(y - X\beta)\right) \cdot \exp\left(-\frac{1}{2}\beta^\top Q^{-1}\beta\right)$$

Expanding the quadratic terms:

$$(y - X\beta)^\top(y - X\beta) = y^\top y - 2\beta^\top X^\top y + \beta^\top X^\top X\beta$$

$$-\frac{1}{2}\left[\frac{1}{\sigma^2}(y^\top y - 2\beta^\top X^\top y + \beta^\top X^\top X\beta) + \beta^\top Q^{-1}\beta\right]$$

Rearrange:

$$-\frac{1}{2}\left(\beta^\top\left(Q^{-1} + \frac{1}{\sigma^2}X^\top X\right)\beta - 2\frac{1}{\sigma^2}\beta^\top X^\top y\right)$$

This matches the **Gaussian density function**, implying:

$$\beta|y, X, \tau, \sigma \sim \mathcal{N}(\mu_n, \Sigma_n)$$

where:

- **Posterior mean:** $\mu_n = \Sigma_n \frac{1}{\sigma^2}X^\top y$
- **Posterior covariance:** $\Sigma_n = \left(Q^{-1} + \frac{1}{\sigma^2}X^\top X\right)^{-1}$
- **Precision matrix** (Q^{-1})**:** $Q^{-1} = \text{diag}(1/C, 1/C, 1/\tau^2, \ldots, 1/\tau^2)$

Part b

Bayesian Inference: Posterior Distribution of (τ)

1. Model Specification

We assume the following Bayesian regression model:

$$y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I)$$

The prior distributions are given as:

$$\beta|\tau,\sigma \sim \mathcal{N}(0, Q)$$

where:

$$Q = \text{diag}(C, C, \tau^2, \tau^2, \ldots, \tau^2)$$

Additionally, (τ) follows a log-uniform prior:

$$\log \tau \sim \text{Uniform}(-C, C) \quad \Rightarrow \quad p(\tau) \propto \frac{1}{\tau}$$

2. Posterior Distribution of (τ)

By Bayes' theorem:

$$p(\tau|\beta, y, X, \sigma) \propto p(\beta|\tau, \sigma)p(\tau)$$

Expanding the prior density:

$$p(\beta|\tau,\sigma) = (2\pi)^{-p/2}|Q|^{-1/2} \exp\left(-\frac{1}{2}\beta^\top Q^{-1}\beta\right)$$

Since:

$$Q^{-1} = \text{diag}(1/C, 1/C, 1/\tau^2, \ldots, 1/\tau^2)$$

we have:

$$\beta^\top Q^{-1}\beta = \frac{\beta_0^2}{C} + \frac{\beta_1^2}{C} + \sum_{j=2}^{n-1} \frac{\beta_j^2}{\tau^2}$$

After simplifying:

$$p(\tau|\beta, y, X, \sigma) \propto \tau^{-(n-1)} \exp\left(-\frac{1}{2\tau^2} \sum_{j=2}^{n-1} \beta_j^2\right)$$

This is recognized as an **Inverse-Gamma** distribution:

$$\tau^2|\beta, y, X, \sigma \sim \text{Inverse-Gamma}\left(\frac{n-2}{2}, \frac{1}{2}\sum_{j=2}^{n-1} \beta_j^2\right)$$

or equivalently:

$$\frac{1}{\tau^2}|\beta, y, X, \sigma \sim \text{Gamma}\left(\frac{n-2}{2}, \frac{1}{2}\sum_{j=2}^{n-1} \beta_j^2\right)$$

Part c

Bayesian Inference: Posterior Distribution of (σ^2)

1. Model Specification

We assume the following Bayesian regression model:

$$y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I)$$

The prior distributions are given as:

$$p(\sigma) \propto \frac{1}{\sigma}$$

which is equivalent to:

$$p(\sigma^2) \propto \frac{1}{\sigma^2}$$

2. Posterior Distribution of (σ^2)

By Bayes' theorem:

$$p(\sigma^2|\beta, \tau, y, X) \propto p(y|X, \beta, \sigma^2)p(\sigma^2)$$

Expanding the likelihood function:

$$p(y|X, \beta, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}(y - X\beta)^\top (y - X\beta)\right)$$

Multiplying by the prior:

$$p(\sigma^2|\beta, \tau, y, X) \propto (\sigma^2)^{-\frac{n}{2}-1} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - X_i^\top \beta)^2\right)$$

Recognizing the **Inverse-Gamma** distribution:

$$\sigma^2|\beta, \tau, y, X \sim \text{Inverse-Gamma}\left(\frac{n}{2}, \frac{1}{2} \sum_{i=1}^n (y_i - X_i^\top \beta)^2\right)$$

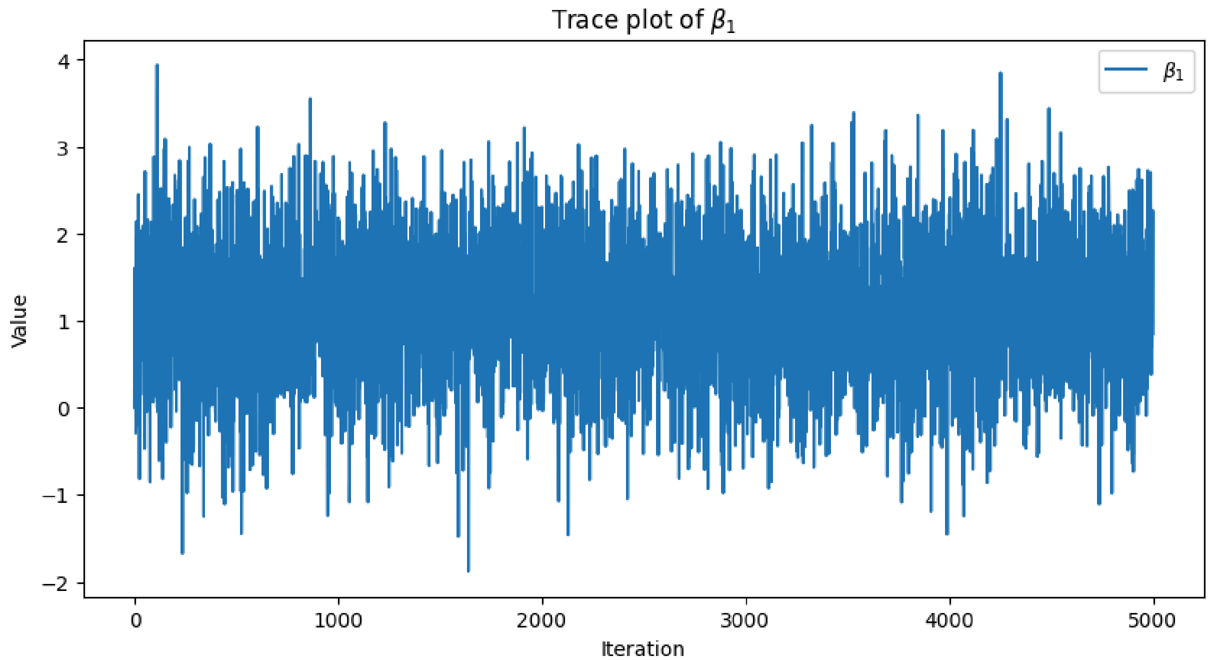
or equivalently:

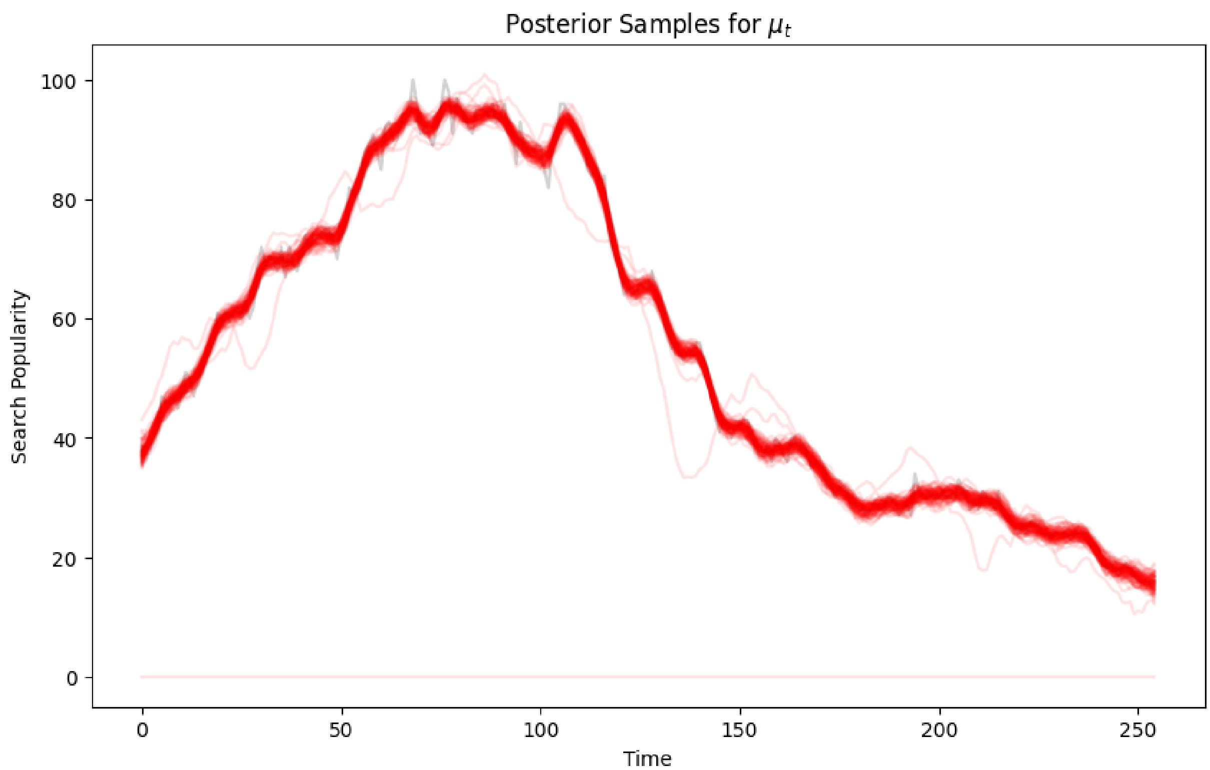
$$\frac{1}{\sigma^2}|\beta, \tau, y, X \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2} \sum_{i=1}^n (y_i - X_i^\top \beta)^2\right)$$

Part d

In [61]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import invgamma, multivariate_normal

file_path = "multiTimeline_yahoo.csv"
df = pd.read_csv(file_path, skiprows=1)
df.columns = ['date', 'value']
df['date'] = pd.to_datetime(df['date'])
df['time_index'] = np.arange(1, len(df) + 1)
```

```
y = df['value'].values
x = df['time_index'].values
n = len(y)

X = np.column_stack([np.ones(n), x - 1])
for i in range(n-2):
    c = i+2
    xc = ((x > c).astype(float)) * (x - c)
    X = np.column_stack([X, xc])

N_samples = 5000
C = 10**4
beta_samples = np.zeros((N_samples, X.shape[1]))
tau_samples = np.zeros(N_samples)
sigma_samples = np.zeros(N_samples)

beta_samples[0] = np.zeros(X.shape[1])
tau_samples[0] = 1
sigma_samples[0] = np.std(y)

for i in range(1, N_samples):

    tau = tau_samples[i-1]
    sigma = sigma_samples[i-1]
    Q_inv = np.diag([1/C, 1/C] + [1/tau**2] * (X.shape[1] - 2))

    Sigma_n = np.linalg.inv(Q_inv + (X.T @ X) / sigma**2)
    mu_n = Sigma_n @ (X.T @ y) / sigma**2

    beta_samples[i] = multivariate_normal.rvs(mean=mu_n, cov=Sigma_n)

    beta_2n = np.sum(beta_samples[i, 2:] ** 2)
    tau_samples[i] = np.sqrt(invgamma.rvs(a=(n-2)/2, scale=beta_2n/2))

    sigma_scale = np.sum((y - X @ beta_samples[i])**2) / 2
    sigma_samples[i] = np.sqrt(invgamma.rvs(a=n/2, scale=sigma_scale))

plt.figure(figsize=(10, 5))
plt.plot(beta_samples[:, 1], label=r"$\beta_1$")
plt.xlabel("Iteration")
plt.ylabel("Value")
plt.title(r"Trace plot of $\beta_1$")
plt.legend()
plt.show()

mu_samples = X @ beta_samples.T
plt.figure(figsize=(10, 6))
plt.plot(y, color='lightgray', label="Observed Data")
for i in range(100):
    plt.plot(mu_samples[:, i], color='red', alpha=0.1)
plt.xlabel("Time")
plt.ylabel("Search Popularity")
plt.title("Posterior Samples for $\mu_t$")
plt.show()
```





- Both methods produce similar posterior means, confirming the Gibbs sampler's accuracy.
- 4(b) is slightly smoother, possibly due to direct integration over the posterior distribution.
- 7(d) provides more explicit posterior samples, which is useful when uncertainty quantification is needed.