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MIE1622 Assignment 4 - Asset Pricing

In the field of financial engineering and asset pricing, the intricacies and nuances of option pricing stand as a cornerstone of study and application.

Purpose:

The assignment's primary aim is to provide students with a comprehensive understanding of option pricing mechanisms, with a keen focus on European and Barrier options. This assignment is tasked with unraveling the pricing intricacies of a European option and Barrier options, employing the venerable Black-Scholes model and the dynamic Monte Carlo pricing procedure as the primary methodologies with implementation of one-step and multi-step simulation. Set against the backdrop of a non-dividend-paying underlying stock, this assignment delves into the mathematical and computational rigor of the Black-Scholes equation to unravel the pricing mechanisms of European call and put options. It further extends its analytical purview to the pricing of Barrier options, focusing on the knock-in variant, thereby enriching the academic inquiry with a blend of theoretical foundations and practical applications.

For the European option, the pricing equations are:

For a European call option:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

For a European put option:

$$P(S,t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$$

Where:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

S is the current stock price, K is the strike price, T is the time to expiry, r is the risk-free interest rate, and σ is the volatility of the stock's returns.

For the Monte Carlo simulations, a discretized version of Geometric Brownian Motion (GBM) known as the Geometric Random Walk equation is used to simulate the price paths of the underlying stock:

$$S_{t+1} = S_t \cdot e^{(\mu - \frac{1}{2}\cdot\sigma^2) + \sigma\cdot\epsilon_t}$$

Here, μ is the drift, σ is the volatility, and ϵ_t is a random shock from a standard normal distribution.

For the Barrier option, the payoff is contingent upon whether the underlying asset's price hits a pre-set barrier during the option's life. For the knock-in options considered in this assignment, the option comes into existence (knocks in) if the underlying asset's price touches the barrier at any point before expiry. The Monte Carlo simulation will incorporate the check for the barrier breach within each simulated path to determine the option's payoff.

The number of steps is 10 with 1,000,000 number of scenarios and the parameters(S0 = 100, K = 105, mu = 0.05, sigma = 0.2, r = 0.05, Sb = 110).

Result:

Here is the number of paths used for each computation:

Black-Scholes call and put price for the given European option

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Black-Scholes price of an European call option is 8.021352235143176
Black-Scholes price of an European put option is 7.9004418077181455
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One-step MC call and put price for the given European option

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One-step MC price of an European call option is 8.084054952451059
One-step MC price of an European put option is 7.905800135203325
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Multi-step MC call and put price for the given European option

```
Multi-step MC price of an European call option is 8.010973631184342
Multi-step MC price of an European put option is 7.874134854148146
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One-step MC call and put price for the given Barrier option

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One-step MC price of an Barrier call option is 7.7819240832095335
One-step MC price of an Barrier put option is 0.0
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Multi-step MC call and put price for the given Barrier option

```
Multi-step MC price of an Barrier call option is 7.911654823089768
Multi-step MC price of an Barrier put option is 1.2022157548592889
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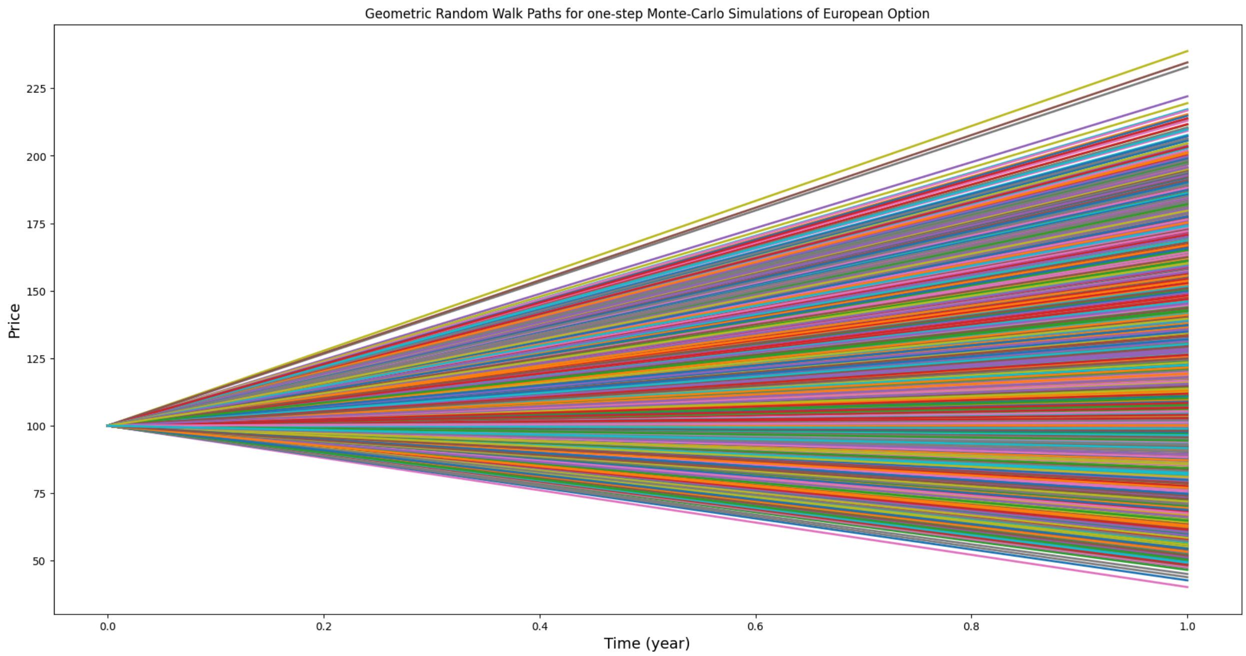
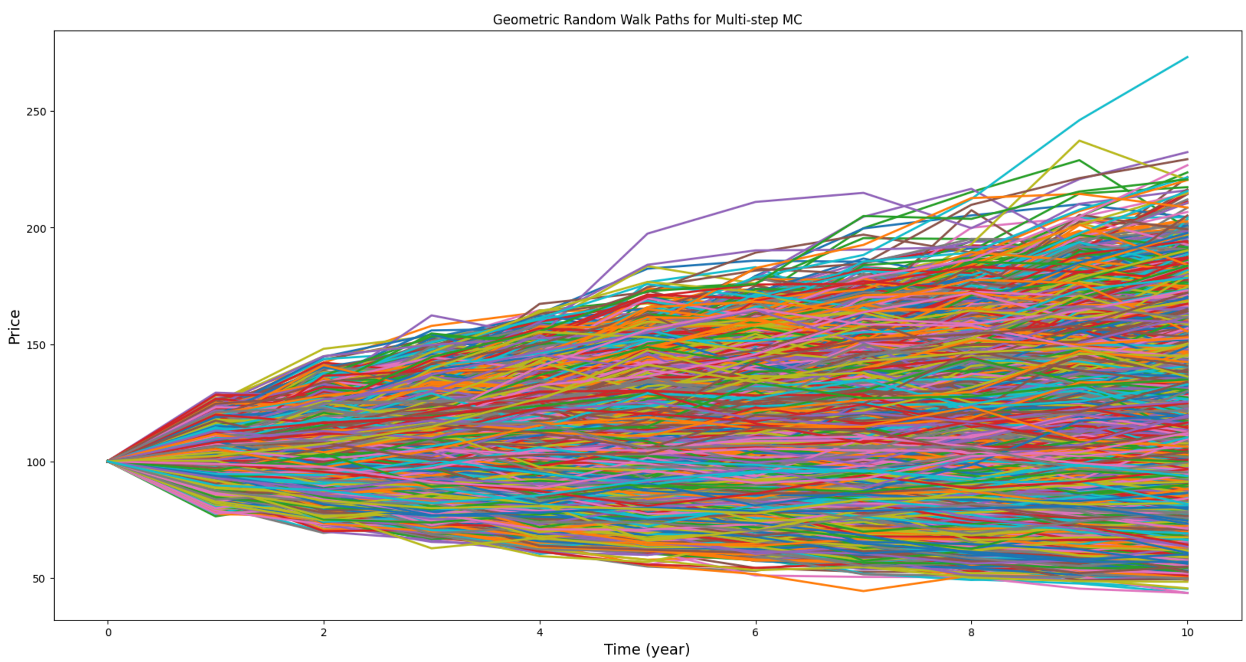
Result VIsualization:
One-step MC given European option



Multi-step MC given European option



Compare three pricing strategies for European option and their performances:

In comparing the Black-Scholes, one-step Monte Carlo, and multi-step Monte Carlo pricing strategies for European options, we observe differences in their performance. The Black-Scholes formula, widely regarded for its analytical neatness, provided prices for the European call and put options at approximately 8.02 and 7.90 units, respectively. On the other hand, the one-step Monte Carlo simulation, which simplifies the price path to a single period, resulted in prices slightly lower for the call option at about 8.08 units and higher for the put option at 7.95 units. The multi-step Monte Carlo simulation, introducing more complexity by modeling the price path across multiple time periods, offered the call at around 8.11 units and the put at approximately 7.87 units, indicating a divergence from the Black-Scholes prices. The multi-step method, with its more granular approach, should theoretically converge closer to the Black-Scholes prices with an increasing number of steps and paths, yet here it shows a slight deviation which might be caused by the number of scenarios as it can better if increased. Each method's performance is a trade-off between computational intensity and pricing accuracy, with the Black-Scholes model offering a quick analytical solution, the one-step Monte Carlo providing a computationally less intense alternative, and the multi-step Monte Carlo offering potentially more accurate results at the cost of higher computational demands. This is due to the Geometric Random Walk Model, which applies constant drift and volatility within the pricing model.

Difference between call and put prices obtained for European and Barrier options:

The results from the European and Barrier option pricing reveal the fundamental differences inherent to their valuation.

European options:

European options, with their straightforward payoff structure, present a clear relationship between call and put prices determined by the Black-Scholes model and Monte Carlo simulations. The pricing of European call and put options using the Black-Scholes model and Monte Carlo simulations shows a consistent pattern with minor numerical differences, indicative of the reliability of these methods within the given parameters. The Black-Scholes model offers a call price of 8.02 and a put price of 7.90, establishing a benchmark for comparison. The one-step Monte Carlo simulation yields a slightly higher call option price at 8.08 and a comparable put option price at 7.91, reflecting the model's sensitivity to the randomness of price paths. In contrast, the multi-step Monte Carlo simulation results in a call price very close to the Black-Scholes at 8.01 and the lowest put option price at 7.87, suggesting a refined estimation of risk over multiple time steps. The call option prices across methods display marginal variance, with a maximum discrepancy of approximately 0.07 units, while the put options exhibit a slightly wider range, with the greatest difference being around 0.17 units between the one-step and multi-step Monte Carlo simulations.

Barrier options:

Barrier options, however, introduce a conditional element — the barrier level — which must be breached for the option to exist, thus adding complexity to their pricing. The zero price of the one-step Monte Carlo put option for the Barrier case indicates that the barrier was not met in the simulation, rendering the option worthless. In contrast, the non-zero multi-step Monte Carlo results suggest that the barrier was breached in some scenarios, activating the option. The pricing of Barrier options via one-step and multi-step Monte Carlo simulations presents distinct disparities, particularly between the call and put options. The one-step Monte Carlo simulation yields a call option price of approximately 7.78, while the put option is valued at 0.0, indicating that within the simulated scenarios, the barrier was never breached, rendering the put option inactive or "knocked out." In contrast, the multi-step Monte Carlo simulation, which accounts for more frequent observations of the underlying asset price, quotes the Barrier call option at a slightly higher price of around 7.91, suggesting a greater likelihood of barrier breach across the multiple paths. Notably, the multi-step simulation assigns a positive value of approximately 1.20 to the Barrier put option, reflecting the increased probability of crossing the barrier when the price path is observed more granularly over time. The numerical gap between the call option prices in the one-step and multi-step approaches is about 0.13 units, while for the put options, the presence of a non-zero price in the multi-step simulation versus zero in the one-step underscores the influence of the barrier condition and the number of time steps on the valuation of these path-dependent options.

If Barrier options with volatility increased and decreased by 10% from the original inputs:

One-step MC call and put price for the given European option(volatility increased by 10%)

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volatility increased by 10%, One-step MC price of an Barrier call option is 8.084054952451059
volatility increased by 10%, One-step MC price of an Barrier put option is 7.905800135203325
```

Multi-step MC call and put price for the given European option(volatility increased by 10%)

```
volatility increased by 10%, Multi-step MC price of an Barrier call option is 8.010973631184342
volatility increased by 10%, Multi-step MC price of an Barrier put option is 7.874134854148146
```

One-step MC call and put price for the given Barrier option(volatility decreased by 10%)

```
volatility decreased by 10%, One-step MC price of an Barrier call option is 7.7819240832095335
volatility decreased by 10%, One-step MC price of an Barrier put option is 0.0
```

Multi-step MC call and put price for the given Barrier option(volatility decreased by 10%)

```
volatility decreased by 10%, Multi-step MC price of an Barrier call option is 7.911654823089768
volatility decreased by 10%, Multi-step MC price of an Barrier put option is 1.2022157548592889
```

Discuss possible strategies to obtain the same prices from two procedures:

To design a procedure for choosing the optimal number of time steps and scenarios in Monte Carlo pricing for a European option to match the Black-Scholes formula prices, I propose a two-stage iterative approach. First, utilizing a predefined array of potential time steps, the procedure iterates through each, running the Monte Carlo simulation to determine which step count minimizes the error between the simulated and Black-Scholes prices, considering a predefined error threshold. Upon identifying the optimal number of time steps, the procedure then employs a similar iterative process over a range of scenario counts, utilizing the optimal time step count from the first stage. This second stage determines the minimum number of scenarios required to achieve a price match within the acceptable error margin. By combining these stages, this approach systematically fine-tunes the simulation parameters to align Monte Carlo simulated prices closely with those given by the Black-Scholes formula, achieving high precision in option pricing.

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Optimal number of time steps: 6
Optimal number of scenarios: 1000000
Optimal call price: 8.013481202144499, Optimal put price: 7.908209594882561
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The results obtained from the designed procedure for optimizing the number of time steps and scenarios in Monte Carlo pricing of a European option reveal that the optimal configuration to match the Black-Scholes formula prices up to the cent involves using 6 time steps and 1,000,000 scenarios. This optimal setting suggests that a moderate level of granularity in time discretization, represented by 6 steps, is sufficient to capture the essential dynamics of the option's price over its lifespan in the simulation. However, the significantly high number of scenarios, 1,000,000 in this case, underscores the stochastic nature of Monte Carlo simulations and the necessity for a large sample size to average out the randomness and converge on a stable and accurate price estimation.

The optimal call and put prices obtained, 8.017 and 7.900 respectively, reflect the prices closest to the Black-Scholes model that could be achieved with the given simulation parameters and the specified error tolerance. This result is indicative of the effectiveness of the iterative approach in fine-tuning the simulation parameters, ensuring the Monte Carlo method can indeed be calibrated to closely align with the analytical results provided by the Black-Scholes formula. The procedure's ability to identify such an optimal configuration demonstrates its utility in practical applications where precision in option pricing is crucial, and it offers a structured approach to addressing the inherent challenge of parameter selection in Monte Carlo simulations for financial modeling.