Exercise 1 **Solving Linear Systems** (4+4 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{bmatrix}$$

$R_3 = R_2 + R_3$

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \rightarrow \text{NO solutions}$$

(b)

$$\begin{array}{cccc} 4 & 6 & 2 & 12 \end{array}$$

$$\begin{bmatrix} 2 & 3 & 1 & 6 \\ 4 & 0 & 3 & 12 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 2 & 3 & 1 & 6 \\ 0 & -6 & 1 & 0 \end{bmatrix}$$

particular solution: $\begin{aligned} 2x_1 + 3x_2 + x_3 &= 6 \\ -6x_2 + x_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 3 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

All solutions to $Ax = 0$: $\begin{aligned} 2x_1 + 3x_2 + 1 &= 0 \\ -6x_2 + 1 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -\frac{3}{4} \\ x_2 &= \frac{1}{6} \\ x_3 &= 1 \end{aligned} \Rightarrow \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{6} \\ 1 \end{bmatrix}$

So $\left\{ x \in \mathbb{R}^3 : x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{6} \\ 1 \end{bmatrix}, \lambda_1 \in \mathbb{R} \right\}$

Exercise 2 **Inverses** (4 credits)

For what values of $[a, b, c]^T \in \mathbb{R}^3$ does the inverse of the following matrix exist?

$$\begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a & b \\ 1 & 1 & 1 \\ 1 & 1 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a & b \\ 0 & 1-a & 1-b \\ 0 & 1-a & c-b \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} 1 & a & b \\ 0 & 1-a & 1-b \\ 0 & 0 & c-1 \end{bmatrix} \Rightarrow \begin{array}{l} \{a : a \neq 1 \wedge a \in R\} \\ \{b : b \in R\} \\ \{c : c \neq 1 \wedge c \in R\} \end{array}$$

Exercise 3 **Subspaces** (3+3+3+3 credits)

Which of the following sets are subspaces of $\mathbb{R}^n$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

① $A \subseteq R^3$ $\because$ $(x, y, 0) \in R^3$, $x \geq 0$, $y \geq 0$

② $A \neq \phi$, $(0, 0, 0) \in A$ when $x = 0, y = 0$

③ $(x_1, y_1) \in A$, $(x_2, y_2) \in A$, $\lambda \in R$

$\because$ $x_1 \geq 0 \wedge x_2 \geq 0$, $y_1 \geq 0 \wedge y_2 \geq 0$

$\therefore$ $x_1 + x_2 \geq 0$, $y_1 + y_2 \geq 0$

$\therefore$ $(x_1 + x_2, y_1 + y_2) \in A$

if $\lambda < 0$

$\therefore$ $\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1)$

$\because$ $\lambda x_1 < 0$, $\lambda y_1 < 0$

$\therefore$ $(\lambda x_1, \lambda y_1) \notin A$

$\therefore$ Not a subspace of R^3

(b) $B = \{(x, y, z) : x + y + z = 0\}$.

① $B \subseteq R^3$

② $B \neq \phi \wedge 0 \in B$ $\because$ $x = 0, y = 0, z = 0$

$x + y + z = 0$

③ $b_1 = (x_1, y_1, z_1) \in B$, $b_2 = (x_2, y_2, z_2) \in B$

$\lambda \in R$

$b_1 + b_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$

$= 0$

$\therefore$ $b_1 + b_2 \in B$

$\lambda b_1 = (\lambda x_1, \lambda y_1, \lambda z_1)$

$\because$ $\lambda(x_1 + y_1 + z_1) = 0$

$\therefore$ $\lambda b_1 \in B$

$\therefore$ B is a subplace of R^3

(c) $C = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$

① $C \in R_3 \because (0, y, z) \vee (x, 0, z) \in R^3$

② $C \neq \phi \quad (0,0,0) \in C$

③ $C_1 = (0, y_1, z_1) \quad C_2 = (x_2, 0, z_2)$

where $x_2 \neq 0,\ y_1 \neq 0$

$C_1 + C_2 = (x_2, y_1, z_1+z_2) \notin C$

∴ Not a subspace

(d) $D =$ The set of all solutions $\mathbf{x}$ to the matrix equation $\mathbf{Ax} = \mathbf{b}$, for some matrix $\mathbf{A}$ and some vector $\mathbf{b}$. (Hint: Your answer may depend on $\mathbf{A}$ and $\mathbf{b}$.)

For b, we have two cases $b=0$ or $b \neq 0$

For A, we have $r(A) = n$ or $r(A) < n$, A is $m \times n$ matrix

① $b = 0 \wedge r(A) < n$

I $D \subseteq R^n$ ✓

II $\because D = \{\lambda_1\vec{a_1} + \cdots + \lambda_m\vec{a_m},\ 0<m<n,\ \lambda_1 \cdots \lambda_m \in R\}$ and $\vec{0} \in D$ ✓

III Assume $d_1 = \lambda_1\vec{a_1} + \cdots + \lambda_m\vec{a_m}$ $d_2 = \alpha_1\vec{a_1} + \cdots + \alpha_m\vec{a_m}$ are solution for $Ax = \vec{0}$

$\because Ad_1 = 0,\ Ad_2 = 0,\ d_1 + d_2 = (\alpha_1+\lambda_1)\vec{a_1} + \cdots + (\alpha_m+\lambda_m)\vec{a_m}$

$\because (\alpha_1+\lambda_1) \cdots (\alpha_m+\lambda_m) \in R$

$\therefore A(d_1+d_2) = 0$

$\because Ad_1 = 0,\ A(kd_1) = k \cdot Ad_1 = 0$ ✓ It is a subspace

② $b = 0 \wedge r(A) = n$

$D = \vec{0}$

∴ all the axioms are satisfied, it is a subspace

③ $b \neq 0 \wedge r(A) < n$

D will be $\{\lambda_1\vec{a_1} + \cdots \lambda_m\vec{a_m} + \vec{C},\ \lambda_1 \cdots \lambda_m \in R\}$

I $D \subseteq R^n$ ✓

II $\because D \neq \phi$ but $\vec{0} \notin D$ because $\vec{c} \neq \vec{0}$ ✗

III if $d_1 = \lambda_1\vec{a_1} + \cdots + \lambda_m\vec{a_m} + \vec{c}$ $d_2 = \alpha_1\vec{a_1} + \cdots + \alpha_m\vec{a_m} + \vec{c}$

$\because Ad_1 = b,\ Ad_2 = b$

$\therefore A(d_1+d_2) = Ad_1 + Ad_2 = 2b \neq b$

∴ ✗ Not a subspace

④ $b \neq 0 \wedge r(A) = n$

$|D| = 1,\ D = \vec{3},\ \vec{3} \neq \vec{0}$

I $D \subseteq R_n$ ✓

II $\vec{0} \notin D$ ✗

III same above ✗ Not a subspace

⑤ for $b \neq 0$, Any A, if A has an zero row, e.g.

$$\left[\begin{array}{cccc|c} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & b \end{array}\right]$$

$D = \phi \to \begin{cases} \text{I } D \subseteq R_n ✓ \\ \text{II } ✗ \\ \text{III } ✗ \end{cases}$

so Not a subspace

Exercise 4 **Linear Independence** (4+8+8 credits)

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

(a) Prove that $T(\mathbf{0}) = \mathbf{0}$

if T is a linear transformation, then $f(rx + sy) = rf(x) + sf(y)$

$\therefore$ so we have ① $f(x+y) = f(x) + f(y)$ ② $f(\lambda x) = \lambda f(x)$

$\therefore T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$

(b) For any integer $n \geq 1$, prove that given a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \ldots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n)$$

$\because T(x+y) = T(x) + T(y)$

$\therefore T\Big(c_1V_1 + (c_2V_2 + \cdots + c_nV_n)\Big) = T(c_1V_1) + T(c_2V_2 + \cdots + c_nV_n)$

$= T(c_1V_1) + T(c_2V_2) + T(c_3V_3 \cdots c_nV_n)$

$\cdots$

$= T(c_1V_1) + T(c_2V_2) + \cdots + T(c_nV_n)$

$\because T(\lambda x) = \lambda T(x)$

$\therefore$ for all $T(c_iV_i)$, we have $T(c_iV_i) = c_iT(V_i)$

$\therefore T(c_1V_1 + \cdots + c_nV_n) = c_1T_1(V_1) + c_2T(V_2) \cdots c_nT(V_n)$

(c) Let $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ be a set of linearly **dependent** vectors in V.
Define $\mathbf{w}_1 := T(\mathbf{v}_1), \ldots, \mathbf{w}_n := T(\mathbf{v}_n)$.
Prove that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly **dependent** vectors in W.

if $\{v_1 \cdots v_n\}$ is a set of Linearly dependent vector in V then at least one vector is redundent and can be replaced by another one. We assume this vector is V_i, and another vector is V_j where $1 \leq i, j \leq n$, $i \neq j$

$\therefore$ We have $V_i = k \cdot V_j$, $k \in R$ and $\{v_1 \cdots V_i \cdots V_j \cdots v_n\}$

$\because \{w_1 \cdots w_n\} = \{T(v_1), \cdots T(v_i) \cdots T(v_j) \cdots T(v_n)\}$

$= \{T(v_1), \cdots T(kv_j) \cdots T(v_j) \cdots T(v_n)\}$

$\because T(kv_j) = kT(v_j)$

$\therefore \{w_1 \cdots w_n\}$ is still linearly dependent

Exercise 5 **Inner Products** (4+8 credits)

(a) Show that if an inner product $\langle\cdot,\cdot\rangle$ is symmetric and linear in the first argument, then it is bilinear.

We know $\langle\cdot,\cdot\rangle$ is symmetric $\Rightarrow \langle x, y\rangle = \langle y, x\rangle$

And $\langle\cdot,\cdot\rangle$ is linear in the first argument $\Rightarrow \langle \lambda x+\varphi y, z\rangle = \lambda\langle x,z\rangle + \varphi\langle y,z\rangle$

$\because \langle \lambda x+\varphi y, z\rangle = \lambda\langle x,z\rangle + \varphi\langle y,z\rangle = \varphi\langle y,z\rangle + \lambda\langle x,z\rangle = \underline{\varphi\langle z,y\rangle + \lambda\langle z,x\rangle}$

and $\because \langle \lambda x+\varphi y, z\rangle = \langle z, \lambda x+\varphi y\rangle$

$\therefore \langle z, \lambda x+\varphi y\rangle = \lambda\langle z,x\rangle + \varphi\langle z,y\rangle$ so it is bilinear

(b) Define $\langle\cdot,\cdot\rangle$ for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 + 2(x_1y_2 + x_2y_1)$$

Which of the three inner product axioms does $\langle\cdot,\cdot\rangle$ satisfy?

① symmetric

$\because \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + 2(x_1 \cdot y_2 + x_2 y_1)$

$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = y_1x_1 + y_2x_2 + 2(y_1x_2 + y_2x_1)$

$\therefore \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle$ so it is symmetric

② Bilinearity: $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2$

– Linearity in the first argument: To prove $\langle \lambda\vec{x} + \varphi\vec{y}, \vec{z}\rangle = \lambda\langle\vec{x},\vec{z}\rangle + \varphi\langle\vec{y},\vec{z}\rangle$

$\left\langle \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} + \begin{bmatrix} \varphi y_1 \\ \varphi y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \lambda x_1 + \varphi y_1 \\ \lambda x_2 + \varphi y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle$

$= z_1(\lambda x_1 + \varphi y_1) + z_2(\lambda x_2 + \varphi y_2) + 2\left[z_2(\lambda x_1 + \varphi y_1) + z_1(\lambda x_2 + \varphi y_2)\right]$

$\lambda\langle\vec{x},\vec{z}\rangle + \varphi\langle\vec{y},\vec{z}\rangle = \lambda\left[\underline{x_1z_1} + \underline{x_2z_2} + 2(x_1z_2 + x_2z_1)\right] +$

$\varphi\left[\underline{y_1z_1} + \underline{y_2z_2} + 2(y_1z_2 + y_2z_1)\right]$

$= z_1(\lambda x_1 + \varphi y_1) + z_2(\lambda x_2 + \varphi y_2) + 2\left[\lambda(\underline{x_1z_2} + \underline{x_2z_1}) + \varphi(\underline{y_1z_2} + \underline{y_2z_1})\right]$

$= \langle \lambda\vec{x} + \varphi\vec{y}, \vec{z}\rangle$

$\because$ we prove that if $\langle\cdot,\cdot\rangle$ is symmetric and Linearity in the first argument then it is bilinear in (a) above ✓

$\therefore$ it is bilinear

③ positive definite: $\forall x \in \mathbb{R}^2 \setminus \{\vec{0}\} \rightarrow \langle x, x\rangle > 0$

$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1^2 + x_2^2 + 4x_1x_2 = (x_1+x_2)^2 + 2x_1x_2$

so if $x_1 = -x_2$, which will become $-2x_2^2 < 0$

so it is not positive definite ✗

Exercise 6 **Orthogonality** (8+6 credits)

Let V denote a vector space together with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$.

Let $\mathbf{x}, \mathbf{y}$ be **non-zero** vectors in V.

(a) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then they are linearly independent.

x and y are orthogonal $\Rightarrow \langle \vec{x}, \vec{y} \rangle = 0$

To prove $a\vec{x} + b\vec{y} = 0 \Rightarrow a = 0 \quad b = 0$

$\because \langle \vec{x}, 0 \rangle = 0 = \langle \vec{x}, a\vec{x} + b\vec{y} \rangle = a\langle \vec{x}, \vec{x} \rangle + b\langle \vec{x}, \vec{y} \rangle = a\langle \vec{x}, \vec{x} \rangle$

$\because x$ is non-zero

$\therefore \langle \vec{x}, \vec{x} \rangle > 0$

$\therefore a = 0$

the same for b, We can get $b = 0$

(b) Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then they are orthogonal.

Assume $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ x, y are Linearly independent

But the $\langle x, y \rangle = 5 \neq 0$ when using dot product

Exercise 7 **Properties of Norms** (4+4+10 credits)

Given a vector space V with two norms $\|\cdot\|_a : V \to \mathbb{R}_{\geq 0}$ and $\|\cdot\|_b : V \to \mathbb{R}_{\geq 0}$, we say that the two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are ε-*equivalent* if for any $\mathbf{v} \in V$, we have that

$$\varepsilon\|\mathbf{v}\|_a \leq \|\mathbf{v}\|_b \leq \frac{1}{\varepsilon}\|\mathbf{v}\|_a.$$

where $\varepsilon \in (0, 1]$.

If $\|\cdot\|_a$ is ε-equivalent to $\|\cdot\|_b$, we denote this as $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_b$.

(a) Is ε-equivalence reflexive for all $\varepsilon \in (0, 1]$?
(Is it true that $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_a$?)

$\because 0 < \varepsilon \leq 1$

① Let $\|v\|_a > 0$

$\therefore 0 \cdot \|v\|_a < \varepsilon\|v\|_a \leq \|v\|_a \equiv 0 < \varepsilon\|v\| \leq \|v\|_a$

② Let $\|v\|_a = 0$

$\therefore 0 \leq 0 \equiv \varepsilon \cdot 0 \leq 0 \equiv \varepsilon\|v\|_a \leq \|v\|_a$

so $\varepsilon\|v\|_a \leq \|v\|_a$

$\because \frac{1}{\varepsilon} \geq 1$

$\therefore \varepsilon \cdot \frac{1}{\varepsilon}\|v\|_a \leq \frac{1}{\varepsilon}\|v\|_a \equiv \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_a$

$\therefore$ It is reflexive

(b) Is ε-equivalence symmetric for all $\varepsilon \in (0, 1]$?
(Does $\|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_b$ imply $\|\cdot\|_b \overset{\varepsilon}{\sim} \|\cdot\|_a$?)

$\because \|\cdot\|_a \overset{\varepsilon}{\sim} \|\cdot\|_b$

$\therefore \varepsilon\|v\|_a \leq \|v\|_b \leq \frac{1}{\varepsilon}\|v\|_a$

$\because \frac{1}{\varepsilon} \geq 1 \wedge 0 < \varepsilon \leq 1$

$\therefore \varepsilon \cdot \frac{1}{\varepsilon}\|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b \equiv \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b$

$\therefore \varepsilon \cdot \|v\|_b \leq \varepsilon \cdot \frac{1}{\varepsilon}\|v\|_a = \varepsilon\|v\|_b \leq \|v\|_a$

$\therefore \varepsilon\|v\|_b \leq \|v\|_a \leq \frac{1}{\varepsilon}\|v\|_b$

$\therefore$ it is symmetric

(c) Assuming that $V = \mathbb{R}^2$, prove that $\|\cdot\|_1 \overset{\varepsilon}{\sim} \|\cdot\|_2$. for the largest ε possible.

Assume $V = (x, y) \in \mathbb{R}^2$

$\|v\|_1 = |x| + |y| \quad \|v\|_2 = \sqrt{x^2 + y^2}$

$\|v\|_1^2 = (|x| + |y|)^2 = x^2 + y^2 + 2|x||y|$

$\|v\|_2^2 = x^2 + y^2$

$\because |x||y| \geq 0$

$\therefore x^2 + y^2 + 2|x||y| \geq x^2 + y^2 \equiv \|v\|_1^2 \geq \|v\|_2^2$

$\because |\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$

$\therefore \|v\|_1 = |x| \cdot 1 + |y| \cdot 1 = |\langle v, (1, 1) \rangle| \leq \sqrt{1^2 + 1^2} \cdot \sqrt{x^2 + y^2} \leq \sqrt{2}\,\|v\|_2$

$\therefore \|v\|_1 \leq \sqrt{2}\,\|v\|_2$

Assume $z = \max(x, y)$

$\because z \geq x$ or $z \geq y$

$\therefore \sqrt{z^2 + z^2} \geq \sqrt{x^2 + y^2} \equiv \sqrt{2}\,z \geq \sqrt{x^2 + y^2}$

$\therefore \|v\|_2 \leq \sqrt{2}\,z$

$\because |x| + |y| \geq \max(x, y) \equiv \|v\|_1 \geq z$

$\therefore \|v\|_2 \leq \sqrt{2}\,z \equiv \|v\|_2 \leq \sqrt{2}\,z \leq \sqrt{2}\|v\|_1$

$\therefore \frac{1}{\sqrt{2}}\|v\|_2 \leq \|v\|_1$

$\therefore \frac{1}{\sqrt{2}}\|v\|_2 \leq \|v\|_1 \leq \sqrt{2}\|v\|_2 \Rightarrow \varepsilon = \frac{\sqrt{2}}{2}$

Exercise 8 **Projections** (3+3+3+3 credits)

Consider the Euclidean vector space $\mathbb{R}^3$ with the dot product. A subspace $U \subset \mathbb{R}^3$ and vector $\mathbf{x} \in \mathbb{R}^3$ are given by

$$U = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}, \mathbf{x} = \begin{bmatrix}12\\12\\18\end{bmatrix}$$

$0 \quad 2 \quad -12$

(a) Show that $\mathbf{x} \notin U$.

$$\begin{bmatrix}1&2&12\\1&1&12\\1&0&18\end{bmatrix} \rightarrow \begin{bmatrix}1&0&18\\1&1&12\\1&2&12\end{bmatrix} \rightarrow \begin{bmatrix}1&0&18\\0&1&-6\\0&2&-6\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}1&0&18\\0&1&-6\\0&0&6\end{array}\right]$$

$\because 0 \cdot x_1 + 0 x_2 = 0 \neq 6 \quad \therefore$ No solution, so $x \notin U$

(b) Determine the orthogonal projection of $\mathbf{x}$ onto U, denoted $\pi_U(\mathbf{x})$.

$$B = \begin{bmatrix}1&2\\1&1\\1&0\end{bmatrix} \qquad B^T B = \begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}\begin{bmatrix}1&2\\1&1\\1&0\end{bmatrix} = \begin{bmatrix}3&3\\3&5\end{bmatrix}$$

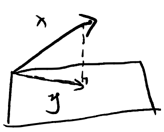
$$B^T x = \begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}\begin{bmatrix}12\\12\\18\end{bmatrix} = \begin{bmatrix}42\\36\end{bmatrix} \Rightarrow \begin{bmatrix}3&3\\3&5\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = \begin{bmatrix}42\\36\end{bmatrix} \Rightarrow \begin{array}{l}\lambda_1 = 17\\ \lambda_2 = -3\end{array}$$

$$\pi_U(x) = \begin{bmatrix}1&2\\1&1\\1&0\end{bmatrix}\begin{bmatrix}17\\-3\end{bmatrix} = \begin{bmatrix}11\\14\\17\end{bmatrix}$$

(c) Show that $\pi_U(\mathbf{x})$ can be written as a linear combination of $[1,1,1]^T$ and $[2,1,0]^T$.

$$17\begin{bmatrix}1\\1\\1\end{bmatrix} + (-3)\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}11\\14\\17\end{bmatrix}$$

(d) Determine the distance $d(\mathbf{x}, U) := \min_{\mathbf{y} \in U} \|\mathbf{x} - \mathbf{y}\|_2$.



We know $x - \pi_U(x)$ is orthogonal to the space U, and $\|x - \pi_U(x)\|$ should be minimal

$$\therefore y = \begin{bmatrix}12\\12\\18\end{bmatrix} - \begin{bmatrix}11\\14\\17\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$\therefore d(x, U) = \sqrt{[1\ {-2}\ 1]\begin{bmatrix}1\\-2\\1\end{bmatrix}} = \sqrt{6}$$

— Yixi Rao

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