Question 1 **Properties of Eigenvalues** (5+5=10 credits)

Let $\mathbf{A}$ be an invertible matrix.

1. Prove that all the eigenvalues of $\mathbf{A}$ are non-zero.

we know that if a matrix A is invertible, then $\det(A) \neq 0$ (lecture slide)

$\because$ A is an invertible matrix, so $\det(A) \neq 0$

$\because$ the eigenvalue of A is: $\det(A - \lambda I) = 0$

$\therefore$ if $\lambda = 0$, $\det(A - \lambda I) = \det(A) \neq 0$

$\therefore$ λ can not be zero

2. Prove that for any eigenvalue λ of $\mathbf{A}$, λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.

from the eigenvalue equation $Ax = \lambda x$; which says if λ is an eigenvalue, and $x \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of A, then $Ax = \lambda x$ (from lecture slide.

$\because$ $AA^{-1} = I$ since A is invertible

$\therefore$ $A^{-1}Ax = A^{-1}\lambda x$

$x = A^{-1}\lambda x$

$\lambda^{-1} x = A^{-1} x$

$\therefore$ $A^{-1}x = \lambda^{-1} x$

from the eigenvalue equation, λ^{-1} is an eigenvalue of A^{-1}

Question 2 **Properties of Eigenvalues II** (10 credits)

Let $\mathbf{B}$ be a square matrix. Let $\mathbf{x}$ be an eigenvector of $\mathbf{B}$ with eigenvalue λ. Prove that for all integers $n \geq 1$, $\mathbf{x}$ is an eigenvector of $\mathbf{B}^n$ with eigenvalue λ^n.

$\because$ if x is an eigenvector of B with eigenvalue λ.

$\therefore$ $Bx = \lambda x$

$\because$ for any $c \in \mathbb{R} / \{0\}$, we have $B(c \cdot x) = cBx = c \cdot \lambda x = \lambda(c \cdot x)$

$\therefore$ if $c = \lambda$, then $B(\lambda \cdot x) = \lambda(\lambda \cdot x)$

$B(B \cdot x) = \lambda^2 x$

$B^2 x = \lambda^2 x$

$\therefore$ We can use the induction to prove: $B^n x = \lambda^n x$

for base case: $n = 1$

$Bx = \lambda x$ ✓

Inductive Hypothesis: Assume $n = k$, and $B^k x = \lambda^k x$ for $k > 1$

step cases: $B^{k+1} x = B(B^k x)$

$= B(\lambda^k x)$

$= \lambda^k (Bx)$

$= \lambda^{k+1} x$

proved!

Question 3 **Distinct eigenvalues and linear independence** (20+5 credits)

Let $\mathbf{A}$ be a $n \times n$ matrix. $1 \cdots\cdots n$

1. Suppose that $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, and corresponding non-zero eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Prove that $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is linearly independant.

   Hint: You may use without proof the following property: If $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ is linearly dependent then there exists some p such that $1 \le p < m$, $\mathbf{y}_{p+1} \in \operatorname{span}\{\mathbf{y}_1, \dots, \mathbf{y}_p\}$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_p\}$ is linearly independent.

Assume that $\{x_1 \cdots x_n\}$ is linearly dependent, then according to the property: there exists some p such that $1 \le p < n$, $\{x_1 \cdots x_p\}$ is linearly independent

$\because \{x_1 \cdots x_p\}$ are independent, $x_{p+1} \in \operatorname{span}\{x_1 \cdots x_p\}$

$\therefore x_{p+1} = \sum_{i=1}^{p} c_i x_i = C_1 x_1 + C_2 x_2 + \cdots C_n x_n$

$\therefore A x_{p+1} = \sum_{i=1}^{p} c_i A x_i = \sum_{i=1}^{p} c_i \lambda_i x_i$

$\because A x_{p+1} = \lambda_{p+1} x_{p+1} = \sum_{i=1}^{p} c_i \lambda_{p+1} x_i$

$\therefore$ In order to let $\sum_{i=1}^{p} c_i \lambda_i x_i = \sum_{i=1}^{p} c_i \lambda_{p+1} x_i$

We have to let $\forall i$, $\lambda_i = \lambda_{p+1}$ $\Rightarrow$ which says that $\lambda_1 = \lambda_2 = \lambda_3 \cdots \lambda_{p+1}$

However, We know $\lambda_1 \neq \lambda_2 \neq \lambda_3 \cdots \neq \lambda_{p+1}$, so there is a contradiction

So $\{x_1 \cdots x_n\}$ is linearly independent

2. Hence, or otherwise, prove that A can have at most n distinct eigenvalues.

The $p_A(\lambda) = \det(A - \lambda I) = C_0 + C_1\lambda + C_2\lambda^2 + \cdots + (-1)^n \lambda^n$

So $P_A(\lambda)$ is a polynomial of degree n, so from the Fundamental Theorem of Algebra we know that n degree polynomial can have at most n roots.

So $P_A(\lambda) = (\lambda - C_1)(\lambda - C_2)\cdots(\lambda - C_n)$ Where $C_1 \neq C_2 \neq \cdots \neq C_n$

So we have at most n distinct eigenvalues

Question 4 **Properties of Determinants** (10+15=25 credits)

1. Prove $\det(A^T) = \det(A)$.

Base case: $n = 1$

$A = [A_0] = A^T \Rightarrow \det(A) = \det(A^T)$

Inductive hypothesis: for $n = t-1$, for $\forall A \in R^{t-1 \times t-1}$, it is true that $\det(A_{t-1}^T) = \det(A_{t-1})$

Next step: for $n = t$, let A_t and A_t^T be:

$$A = \begin{array}{r} \\ 21 \\ \\ \\ \end{array} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{bmatrix}, \quad A^T = \begin{array}{c} \qquad 12 \\ \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{t1} \\ A_{12} & A_{22} & \cdots & A_{t2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1t} & A_{2t} & \cdots & A_{tt} \end{bmatrix} \end{array}$$

using the Laplace Expansion:

$\det(A) = A_{11}\det(A_{r11}) - A_{12}\det(A_{r12}) + \cdots + (-1)^{t+1} A_{t1}\det(A_{rt1})$ — first row

$\det(A^T) = A_{11}\det(A^T_{r11}) - A_{12}\det(A^T_{r21}) + \cdots + (-1)^{t+1} A_{t1}\det(A^T_{r1t})$ — first column

Where A_{rij} means the matrix of A with i^{th} row removed j^{th} column removed

We find that $A_{rij}^T = A^T_{rji}$ since $A_{ij} = A^T_{ji}$, so removing i^{th} row, j^{th} col of A = removing j^{th} row, i^{th} col of A^T

$\because A_{rij} \in R^{t-1 \times t-1}$

$\therefore \det(A_{rij}) = \det(A_{rij}^T) = \det(A^T_{rji})$ (IH)

$\therefore \det(A^T) = A_{11}\det(A_{r11}) - A_{12}\det(A_{r12}) + \cdots + (-1)^{t+1} A_{t1}\det(A_{rt1}) = \det(A)$

$\therefore \det(A^T) = \det(A)$

2. Prove $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

We know $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0_n \\ 0 & 1 & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & 1_n \end{bmatrix}$

Base step: $n=1$ $\quad I_1 = 1 \quad \det(I_1) = 1$

Inductive hypothesis: Assume $\det(I_{n-1}) = 1$

next step: using the Laplace Expansion:

$\det(I_n) = 1 \cdot \det(I_{r11})$ — first row

$I_{r11} \in R^{n-1 \times n-1} = I_{n-1}$

$\therefore \det(I_n) = 1 \cdot \det(I_{n-1}) = 1 \cdot 1 = 1$

So $\det(I_n) = 1$

Question 5 **Eigenvalues of symmetric matrices** (15 credits)

$A^T = A$ $\qquad Av_1 = \lambda_1 v_1 \quad Av_2 = \lambda_2 v_2$

1. Let $\mathbf{A}$ be a symmetric matrix. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_1, and let $\mathbf{v}_2$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_2. Assume that $\lambda_1 \neq \lambda_2$. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. (Hint: Try proving $\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2$. Recall the identity $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$.) $\quad v_1^T \cdot v_2 = 0$

$\because A$ is symmetric $\Rightarrow A^T = A$

$\because AV_1 = \lambda_1 V_1,\ AV_2 = \lambda_2 V_2 \Rightarrow V_1 = \frac{AV_1}{\lambda_1},\ V_2 = \frac{AV_2}{\lambda_2}$

$\because V_1^T V_2 = V_2^T V_1$

$\because \frac{V_1^T A}{\lambda_1} V_2 = V_2^T \frac{AV_1}{\lambda_1} \Rightarrow \lambda_2 V_1^T V_2 = \lambda_1 V_2^T V_1$

$\therefore \lambda_1 V_1^T V_2 = \lambda_2 V_1^T V_2$

$\lambda_1 \langle V_1, V_2 \rangle = \lambda_2 \langle V_1, V_2 \rangle$

$\because \langle V_1, V_2 \rangle = \langle V_1, V_2 \rangle$ and $\lambda_1 \neq \lambda_2$

$\therefore \langle V_1, V_2 \rangle = 0 \Rightarrow V_1$ and V_2 are orthogonal

Question 6 **Computations with Eigenvalues** (3+3+3+3+3=15 credits)

Let $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$.

1. Compute the eigenvalues of $\mathbf{A}$.

$\det(A - \lambda I) = 0$, the roots is eigenvalues

$\begin{vmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$ $\quad 4 \cdot -1 \ (\lambda+1)(\lambda-4)$

$= (-1-\lambda)(4-\lambda) - 6 = \lambda^2 - 3\lambda - 4 - 6 = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$

$\therefore \lambda = -2 \quad \lambda = 5$

2. Find the eigenspace E_λ for each eigenvalue λ. Write your answer as the span of a collection of vectors.

$A x = \lambda x \Rightarrow (A - \lambda I) x = 0$

for $\lambda = -2$:

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} x = 0 \Rightarrow E_2 = \text{Span}\left[\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$

for $\lambda = 5$:

$$\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} x = 0 \quad E_5 = \text{span}\left[\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right]$$

3. Verify the set of all eigenvectors of $\mathbf{A}$ spans $\mathbb{R}^2$.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

So $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$

4. Hence, find an invertable matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

$$P = [P_1\ P_2] = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & -7 & 3 & -1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} \end{array}\right] = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

So $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$

5. Hence, find a formula for efficiently [1] calculating $\mathbf{A}^n$ for any integer $n \geq 0$. Make your formula as simple as possible.

We know $A = PDP^{-1}$ so $A^n = PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1} = PD^nP^{-1}$

$$\therefore A^n = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & -2^n \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 5^n & (-2)^{n+1} \\ 3 \cdot 5^n & (-2)^n \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

$$A^n = \begin{bmatrix} \frac{5^n - 3(-2)^{n+1}}{7} & \frac{2 \cdot 5^n + (-2)^{n+1}}{7} \\ \frac{3 \cdot 5^n - 3(-2)^n}{7} & \frac{6 \cdot 5^n + (-2)^n}{7} \end{bmatrix}$$