Part A

1. $(x+y)^2 = O(x^2+y^2)$

Always True

prove : $(x+y)^2 \le c\,(x^2+y^2)$ for $n=\begin{bmatrix} x \\ y \end{bmatrix} \ge \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = n_0$

$$x^2+y^2+2xy \le cx^2+cy^2$$

$$\frac{x^2+y^2+2xy}{x^2+y^2} \le c \quad (x^2+y^2 \ne 0)$$

$$1+\frac{2xy}{x^2+y^2} \le c$$

$$1+\frac{2xy}{2xy} \le 1+\frac{2xy}{x^2+y^2} \le c \quad (x \ne 0, y \ne 0)$$

$\therefore$ so it when $c \ge 2$ and $\forall n_0\ (x_0 \ne 0, y_0 \ne 0)$,

$(x+y)^2 = O(x^2+y^2)$ Always True

2. $f(n) = O(f(n)^2)$ when $f(n) = \Omega(1)$

Always True

$\because f(n) = \Omega(1)$

$\therefore$ means $f(n) \ge c \ge 0$, when $\forall n \ge n_0$

$\because \lim_{n\to\infty} \frac{f(n)}{1} = \infty$ or c'

$\therefore \lim_{n\to\infty} \frac{f(n)}{f(n)^2} = \frac{1}{f(n)} = \frac{1}{\lim_{n\to\infty} f(n)}$

$= 0$ or $\frac{1}{c'}$

So it is always True

3. $\log(n!) = O(n\log n)$

Always True

find n_0, c to $\log(n!) \le c \cdot n\log n$ when $n \ge n_0$

$\because \log(n \cdot n-1 \cdot \cdots 2 \cdot 1) = \log n + \log n-1 + \cdots + \log 2 + \log 1$

$$= \sum_{i=1}^{n} \log i$$

$\therefore \sum_{i=1}^{n} \log i \le c \cdot n\log n$

and $\because \sum_{i=1}^{n} \log i \le \sum_{i=1}^{n} \log n = n\log n$

So $c=1$, it is always True

4. $f(n) + g(n) = \Theta(\max(f(n), g(n)))$

Always True

$$\lim_{n\to\infty} \frac{f(n)+g(n)}{\max(f(n),g(n))} \ ?$$

① When $f(n) > g(n)$

$$\lim_{n\to\infty} \frac{f(n)+g(n)}{f(n)} = 1+\frac{g(n)}{f(n)} = c$$

② when $f(n) = g(n)$

$$\lim_{n\to\infty} \frac{2f(n)}{f(n)} = 2$$

③ when $f(n) < g(n)$

$$\lim_{n\to\infty} \frac{f(n)+f(n)}{g(n)} = \frac{f(n)}{g(n)}+1 = c$$

$\therefore f(n)+g(n) = \theta(\max(f(n), g(n)))$

5. $\sqrt{n} = O(\log n)$

False

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log(n)} = \lim_{n\to\infty} \frac{\frac{1}{2}\frac{1}{\sqrt{n}}}{\frac{1}{n\ln 2}} = \lim_{n\to\infty} \frac{n\ln 2}{2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\ln 2}{2}\frac{n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\ln 2}{2}\sqrt{n} = \infty$$

So it is false

Part B

6.

- $f_1(n) = n^{\sqrt{n}}$ $=2^{\sqrt{n}\log n}$
- $f_2(n) = \sum_{i=0}^{n} i^3$ $=2^{4\log n}$
- $f_3(n) = 2^n$ $=2^n$
- $f_4(n) = \binom{n}{5}$ $=2^{5\log n}$
- $f_5(n) = \sqrt{2}^{\sqrt{n}}$ $=2^{\frac{\sqrt{n}}{2}}$
- $f_6(n) = (\log n)^5$
  $[\log n, n]$
- $f_7(n) = n^{100}2^{0.5n}$ $=2^{100\log n+0.5n}$
- $f_8(n) = n^{(5\log n)^2}$ $=2^{25(\log n)^3}$

$f_1(n)=n^{\sqrt{n}}=2^{\sqrt{n}\log n}$ $\quad f_3(n)=2^n$ $\quad f_5(n)=2^{\frac{\sqrt{n}}{2}}$ $\quad f_8(n)=2^{25(\log n)^3}$

$f_2(n)=\sum_{i=0}^{n} i^3 \approx \frac{1}{4}n^4 = 2^{4\log n-\log 4}$ $\quad f_4(n)=\frac{1}{120}n(n-1)(n-2)(n-3)(n-4) \approx \frac{1}{120}n^5$ $\quad f_7(n)=2^{100\log n+0.5n}$

for $f_7(n)$ and $f_3(n)$

$\because f_7(n)=n^{100}2^{0.5n}=2^{100\log n+0.5n}$

$\because f_3(n)=2^n$

so compare $100\log n+0.5n$ with n

$\because \lim_{n\to\infty}\frac{n}{100\log n+0.5n}=\lim_{n\to\infty}\frac{1}{\frac{100}{n\ln 2}+n}=0$

$\therefore n=O(100\log n+0.5n)$

$\therefore f_7(n)>f_3(n)$

for $f_3(n)$ and $f_1(n)$

$\because \lim_{n\to\infty}\frac{\sqrt{n}\log n}{n}=\lim_{n\to\infty}\frac{\log n}{\sqrt{n}}=\lim_{n\to\infty}\frac{\frac{1}{n\ln 2}}{\frac{1}{2\sqrt{n}}}$

$\lim_{n\to\infty}\frac{2}{\ln 2}\frac{1}{\sqrt{n}}=0$

$\therefore \sqrt{n}\log(n)=O(n)$

$\therefore f_3(n)>f_1(n)$

for $f_1(n)$ and $f_5(n)$

$\because \lim_{n\to\infty}\frac{\frac{1}{2}\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\log n}=0$

$\therefore \frac{\sqrt{n}}{2}=O(\sqrt{n}\log n)$

$\therefore f_1(n)>f_5(n)$

for $f_5(n)$ and $f_8(n)$

$\because \lim_{n\to\infty}\frac{\frac{1}{2}\sqrt{n}}{25(\log n)^3}=\lim_{n\to\infty}\frac{\frac{1}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{n}}}{25\cdot\frac{1}{n\ln 2}(\log n)^2}=\frac{\frac{1}{4}\sqrt{n}}{\frac{25}{\ln 2}(\log n)^2}$

$=\lim_{n\to\infty}\frac{1}{4}\cdot\frac{\ln 2}{25}\cdot\frac{\frac{1}{2\sqrt{n}}}{2\cdot\frac{1}{n\ln 2}\log n}=\lim_{n\to\infty}\frac{\ln 2^2}{100\cdot 4}\frac{\sqrt{n}}{\log n}=\infty$

$\therefore f_5(n)>f_8(n)$

for $f_8(n)$ and $f_4(n)$ and $f_2(n)$

$\because f_4(n)=\frac{n^5}{120}\approx 2^{120\log n-\log 120}$

$\because 120\log n-\log 120=O(25(\log n)^3)$

$\therefore f_8(n)=2^{25(\log n)^3}>f_4(n)$

$\because f_4(n)=O(n^5)$, $f_2(n)=O(n^4)$

$\therefore f_4(n)>f_2(n)$

for $f_2(n)$ and $f_6(n)$

$\because \lim_{n\to\infty}\frac{25(\log n)^3}{\frac{1}{4}n^4}=100\cdot\frac{(\log n)^3}{n^4}=\frac{3\cdot\frac{1}{n\ln 2}(\log n)^2}{4n^3}\cdot 100$

$\lim_{n\to\infty}\frac{300}{4\ln 2}\frac{(\log n)^2}{n^4}=\lim_{n\to\infty}\frac{300}{4\ln 2}\frac{2\cdot\log n\cdot\frac{1}{n\ln 2}}{4n^3}$

$\lim_{n\to\infty}\frac{600}{16(\ln 2)^2}\frac{\log n}{n^4}=\lim_{n\to\infty}\frac{600}{16(\ln 2)^2}\frac{\frac{1}{n\ln 2}}{4n^3}=0$

$\therefore f_2(n)>f_6(n)$

$\therefore f_7(n)>f_3(n)>f_1(n)>f_5(n)>f_8(n)>f_4(n)$
$>f_2(n)>f_6(n)$

7.

(a)

	cost	Time
1: Let $n = A.length$	C_1	1
2: **for** $i = 1$ to $n - 1$ **do**	C_2	n
3: Let $mIdx = i$	C_3	$n-1$
4: **for** $j = i + 1$ to n **do**	C_4	$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n+1}$
5: **if** $A[j] < A[mIdx]$ **then**	C_5	$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}$
6: $mIdx = j$	C_6	$\sum_{i=1}^{\frac{n-1}{2}} 2i$ or $\sum_{i=1}^{\frac{n}{2}} 2i-1$
7: **if** $mIdx \neq i$ **then**	C_7	$n-1$
8: $tmp = A[i]$	C_8	$n-1$
9: $A[i] = A[mIdx]$	C_9	$n-1$
10: $A[mIdx] = tmp$	C_{10}	$n-1$

explain: assume A. len = n

Line4: $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n+1} = \sum_{i=1}^{n-1} n-i+1 = \sum_{i=1}^{n-1} n+1 - \sum_{i=1}^{n-1} i$

$= (n-1)(n+1) - \frac{(1+n-1)\,n-1}{2} = \frac{(n-1)(n+2)}{2}$

$\therefore T_4(n) = C_4\frac{(n-1)(n+2)}{2} = \Theta(n^2)$

Line 5: $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} = \sum_{i=1}^{n-1} n-i = \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = (n-1)n - \frac{(1+(n-1))\,n-1}{2}$

$= \frac{2n(n-1)}{2} - \frac{n(n-1)}{2} = \frac{(n-1)n}{2}$

$\therefore T_5(n) = C_5\frac{(n-1)n}{2} = \Theta(n^2)$

Line 6: Two situation $\begin{cases} n \text{ is even} \\ n \text{ is odd} \end{cases}$

① n is even so

$\sum_{i=1}^{\frac{n}{2}} 2i-1$ assignments $= \frac{n^2}{4}$

② n is odd so

$\sum_{i=1}^{\frac{n-1}{2}} 2i = \frac{n^2-1}{4}$

They are both belong to $\Theta(n^2)$

$\therefore$ assume n is even

$\therefore T_6(n) = \frac{n^2}{4}C_6$

line 7, 8, 9, 10: for the sorted Array A, if we make the number of index $\lfloor\frac{n}{2}\rfloor+1$ to the first num in A and the rest elements will be arrayed in reverse order then we can have "n−1" swap instructions which will bigger than $\frac{n}{2}$ (total reverse order)

So $T_{7,8,9,10}(n) = C_{7\text{-}10}(n-1) = \Theta(n)$

So total $T(n) = C_1 + C_2 n + \underline{C_3(n-1)} + \underline{C_4\frac{(n-1)(n+2)}{2} + C_5\frac{n(n-1)}{2} + \frac{n^2}{4}C_6} + \underline{C_{7\text{-}10}(n-1)}$

$T(n) = \frac{c'n^2 + c''n + c'''}{4}$

(b)

$\because \lim_{n\to\infty} \frac{\frac{c'n^2+c''n+c'''}{4}}{n^2} = \frac{c'n^2+c''n+c'''}{4n^2}$

$= \lim_{n\to\infty} \frac{c'}{4} + \frac{c''}{4n} + \frac{c'''}{4n^2}$ = will be a const

So we can say $T(n) = \Theta(n^2)$

8、

(a) [10 pts] Each infected person can be identified and isolated within a day after he/she is infected, such that the person only infects two other people.

(b) [10 pts] No isolation measure is imposed, which means a person who has infected two people today will infect yet another two people tomorrow, the day after tomorrow, etc..

my Interpretion and Assumption: Patient-0 infects two other people at day 1, and he will die Today (Not to live to see the day 2). if he doesn't get the medicine.
patient-1 and Patient-2 will not be contagious at day 1, because P_1, P_2 are in the incubation period, they will infect people at day 2 and they will die before the end of the day 2.
The reason why I interpretion like this is if P_1, P_2 are contagious, then they can infect people at day-1 Which means P_3, P_4 will be infected and so on, then everyone will be infected in one day.

(a)

day	Increased Infected person	All Infected people
1	1	1
2	2	3
3	4	7
4	8	15
⋮	⋮	⋮
n	2n	2^n-1

So at the day 10 All Infected people is $2^{10}-1 = 1023$ people

$\because 1023 < 10000$

$\therefore$ enough medicine, no one die

(b)

day	Increased Infected person	All Infected people
1	1	1
2	2	3
3	6	9
4	18	27
5	54	81
6	162	243
7	486	729
8	1458	2187
9	4374	6561
10	13122	19683
11		
⋮		
n		

if $T(n)$ is the Increased infected people at Day-n
and $S(n)$ is the All Infected people at day-n

$\because T(n) = 2\sum_{i=1}^{n-1} T(i)$ and $T(1) = 1$

$\therefore \begin{cases} T(n) = 3T(n-1) \\ T(1) = 1 \end{cases}$

$\therefore T(n) = 2\cdot 3^{n-2} \ (n \geq 2)$ and $T(1) = 1$

$\therefore S(n) = \dfrac{2(1-3^{n-1})}{-2} + 1 = 3^{n-1}$

When day 10, $S(n) = 3^9 = 19683$

$T(n) = 2\cdot 3^8 = 13122$

at Day-11, the Medicine will arrive but 13122 people Can't survive After day 10, SO YES, 9683 people will DIE At Day-10

9.

(a) [10 pts] $O(\log n)$ number of tries and unlimited number of plastic balls. Please provide an explanation that your strategy indeed takes $O(\log n)$ number of tries.

(b) [10 pts] $\Theta(\sqrt{n})$ number of tries and two plastic balls. Please provide an explanation that your strategy indeed takes $\Theta(\sqrt{n})$ number of tries and at most two plastic balls.

(a) strategy: let the player kick the ball at the median line, so it will have two situations. hit the bullseye or not, if so move the end line to m, otherwise move the start line to m. if e-s=1 then we can get the answer, else repeat the process again with the new s, e.

```
while(e-s≠1)
  if(k(s,e)=YES):
      s=m
  else:
      e=m
return s
```

analysis: for every steps, the size of e-s will halved, so when the problem size is 1, it will take log (e-s) steps. so it will have $O(\log n)$ tries

(b)