Problem 1

Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$

Proof. by Euler's formula, we have

$$i = e^{i(\frac{\pi}{2}+2k\pi)}, k \in \mathbb{Z}$$

Suppose $z^4 = i$,we have

$$\begin{aligned} z^4 &= e^{i(\frac{\pi}{2}+2k\pi)} \\ z &= e^{i(\frac{\pi}{8}+\frac{k\pi}{2})} \\ z &= \cos(\frac{\pi}{8}+\frac{k\pi}{2}) + i\sin(\frac{\pi}{8}+\frac{k\pi}{2}), k \in \{0,1,2,3,\cdots\} \end{aligned}$$

similarly, we have

$$-i = e^{i(\frac{3\pi}{2}+2k\pi)}, k \in \mathbb{Z}$$

Suppose $w^4 = -i$,we have

$$\begin{aligned} w^4 &= e^{i(\frac{3\pi}{2}+2k\pi)} \\ w &= e^{i(\frac{3\pi}{8}+\frac{k\pi}{2})} \\ w &= \cos(\frac{3\pi}{8}+\frac{k\pi}{2}) + i\sin(\frac{3\pi}{8}+\frac{k\pi}{2}), k \in \{0,1,2,3,\cdots\} \end{aligned}$$

□

Problem 2

Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0$$

solution:
by the quadratic formula, we have

$$z = \frac{-(\alpha+i\beta) \pm \sqrt{(\alpha+i\beta)^2 - 4(\gamma+i\delta)}}{2}.$$

and

$$\Delta = (\alpha+i\beta)^2 - 4(\gamma+i\delta).$$

Problem 3

Show that the system of all matrices

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with the standard matrix addition and scalar multiplication is isomorphic to $\mathbb{C}$

Proof. Define the function

$$\varphi\colon \mathbb{C} \to S, \quad \varphi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit. Denote the set of all matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ by S.

1. φ is Well-Defined and Linear

For any two complex numbers $z = a + bi$ and $w = c + di$, we have:

$$\varphi(z+w) = \varphi\big((a+c) + (b+d)i\big) = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}.$$

On the other hand,

$$\varphi(z) + \varphi(w) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix}.$$

Thus, $\varphi(z+w) = \varphi(z) + \varphi(w)$, showing that φ preserves addition.

Similarly, for any real scalar k,

$$\varphi(kz) = \varphi(ka + kbi) = \begin{pmatrix} ka & kb \\ -kb & ka \end{pmatrix} = k \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = k\,\varphi(z),$$

which shows φ is linear with respect to scalar multiplication.

2. φ Preserves Multiplication

For $z = a + bi$ and $w = c + di$, note that

$$z \cdot w = (a+bi)(c+di) = (ac - bd) + (ad + bc)i.$$

Then,

$$\varphi(z \cdot w) = \varphi\big((ac-bd) + (ad+bc)i\big) = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}.$$

Now, compute the product $\varphi(z)\,\varphi(w)$:

$$\varphi(z)\,\varphi(w) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}.$$

Since the two products are equal, $\varphi(z\cdot w) = \varphi(z)\,\varphi(w)$; hence, φ preserves multiplication.

3. φ is Bijective

Injectivity: Suppose $\varphi(a+bi) = \varphi(c+di)$. Then,

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}.$$

By comparing corresponding entries, we get $a = c$ and $b = d$. Therefore, $a+bi = c+di$, and φ is injective.

Surjectivity: Take any matrix in S of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$. Choosing $z = \alpha + \beta i \in \mathbb{C}$, we observe that

$$\varphi(z) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Thus, every element of S has a preimage in $\mathbb{C}$, proving that φ is surjective.
Hence, the set

$$S = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha,\, \beta \in \mathbb{R} \right\}$$

is isomorphic to the field of complex numbers, $\mathbb{C}$.

$$\mathbb{C} \cong S, \quad \varphi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

□

Problem 4

Prove that

$$|\frac{a-b}{1-\bar{a}b}| = 1$$

if either |a|=1 or |b|=1

Proof. Suppose that $|a| = 1$. Then, we can write $a = e^{i\theta}$.So, we have

$$|\frac{e^{i\theta}-b}{1-e^{-i\theta}b}| = |\frac{e^{i\theta}-b}{1-e^{-i\theta}b}||e^{-i\theta}| = |\frac{e^{-i\theta}e^{i\theta}(e^{i\theta}-b)}{e^{i\theta}-b}| = 1$$

Suppose that $|b| = 1$ write $b = e^{i\theta}$ Then, the expression becomes

$$\frac{a-b}{1-\bar{a}b} = \frac{a-e^{i\theta}}{1-\bar{a}e^{i\theta}}.$$

Now, compute the squared moduli of the numerator and denominator.

Numerator:

$$|a - e^{i\theta}|^2 = (a - e^{i\theta})(\overline{a} - e^{-i\theta}) = |a|^2 - ae^{-i\theta} - e^{i\theta}\overline{a} + 1.$$

Denominator:

$$|1 - \overline{a}e^{i\theta}|^2 = (1 - \overline{a}e^{i\theta})(1 - ae^{-i\theta}) = 1 - ae^{-i\theta} - e^{i\theta}\overline{a} + |a|^2.$$

Since the two squared moduli are equal, it follows that

$$\left|\frac{a - e^{i\theta}}{1 - \overline{a}e^{i\theta}}\right| = \frac{|a - e^{i\theta}|}{|1 - \overline{a}e^{i\theta}|} = 1.$$

Thus, we conclude that

$$\left|\frac{a - b}{1 - \overline{a}b}\right| = 1.$$

□

Problem 5

Prove that

$$|\frac{a - b}{1 - \bar{a}b}| < 1$$

if |a|<1 and |b|<1

Proof. Write

$$a = r_1 e^{i\alpha}, \quad b = r_2 e^{i\beta}, \quad \text{with } r_1, r_2 < 1.$$

Multiplying the numerator and denominator of

$$\frac{a - b}{1 - \overline{a}b}$$

by $e^{-i\alpha}$ (which does not change the modulus), we obtain

$$\frac{a - b}{1 - \overline{a}b} = \frac{r_1 - r_2 e^{i(\beta - \alpha)}}{1 - r_1 r_2 e^{i(\beta - \alpha)}}.$$

Set

$$\theta = \beta - \alpha,$$

so that

$$\left|\frac{a - b}{1 - \overline{a}b}\right| = \left|\frac{r_1 - r_2 e^{i\theta}}{1 - r_1 r_2 e^{i\theta}}\right|.$$

Taking the squared modulus gives

$$\left|\frac{r_1 - r_2 e^{i\theta}}{1 - r_1 r_2 e^{i\theta}}\right|^2 = \frac{|r_1 - r_2 e^{i\theta}|^2}{|1 - r_1 r_2 e^{i\theta}|^2} = \frac{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}{1 + r_1^2 r_2^2 - 2r_1 r_2 \cos\theta}.$$

Since

$$r_1^2 + r_2^2 < 1 + r_1^2 r_2^2 \quad \Longrightarrow \quad r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta < 1 + r_1^2 r_2^2 - 2r_1 r_2 \cos\theta,$$

we conclude that

$$\frac{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}{1 + r_1^2 r_2^2 - 2r_1 r_2 \cos\theta} < 1.$$

That is,

$$\left| \frac{a-b}{1-\overline{a}b} \right|^2 < 1,$$

and hence

$$\left| \frac{a-b}{1-\overline{a}b} \right| < 1.$$

$\square$

Problem 6

If $|a_i| < 1, \lambda_i \geq 0, i = 1, \cdots, n$ and $\lambda_1 + \cdots + \lambda_n = 1$
Show that

$$|\lambda_1 a_1 + \cdots + \lambda_n a_n| < 1$$

Proof. Since for each $i = 1, \ldots, n$ we have $|a_i| < 1$ and $\lambda_i \geq 0$ with

$$\lambda_1 + \cdots + \lambda_n = 1,$$

by the triangle inequality,

$$|\lambda_1 a_1 + \cdots + \lambda_n a_n| \leq \lambda_1 |a_1| + \cdots + \lambda_n |a_n|.$$

Because each $|a_i| < 1$, it follows that

$$\lambda_1 |a_1| + \cdots + \lambda_n |a_n| < \lambda_1 + \cdots + \lambda_n = 1.$$

Thus, we obtain

$$|\lambda_1 a_1 + \cdots + \lambda_n a_n| < 1.$$

$\square$