Problem 1

A fair coin is tossed repeatedly. Show that the following two statements are equivalent:
(a) the outcomes of different tosses are independent,
(b) for any given finite sequence of heads and tails, the chance of this sequence occurring in the first m tosses is 2^{-m}, where m is the length of the sequence.

Proof.

- (a) $\Rightarrow$ (b):
  Since the coin tosses are independent and fair, for any finite sequence $(x_1, x_2, \ldots, x_m)$ where each x_i is either H or T, we have

$$P(X_1 = x_1,\, X_2 = x_2,\, \ldots,\, X_m = x_m) = \prod_{i=1}^{m} P(X_i = x_i) = \left(\frac{1}{2}\right)^m = 2^{-m}.$$

- (b) $\Rightarrow$ (a):
  Consider any finite collection of coin tosses at positions $i_1, i_2, \ldots, i_m$, with corresponding outcomes $u_j \in \{H, T\}$ for $1 \le j \le m$.

  Let $M = \max\{i_j : 1 \le j \le m\}$ be the position of the last toss.

  For each j, define E_j to be the set of all possible sequences of length M where the i_j-th position shows the outcome u_j.

  For each E_j, we have fixed exactly one position (the i_j-th position) to have value u_j, while the other $M - 1$ positions can each be either H or T. Therefore:

$$|E_j| = 2^{M-1}$$

  The intersection $\bigcap_{j=1}^{M} E_j$ represents sequences where the i_j-th position shows u_j for every j. This means we've fixed exactly m positions, leaving $M - m$ positions that can be either H or T. Thus:

$$|\bigcap_{j=1}^{M} E_j| = 2^{M-m}$$

  Then, we can calculate:

$$P(E_j) = \frac{|E_j|}{2^M} = \frac{2^{M-1}}{2^M} = \frac{1}{2}$$

  Similarly:

$$P\left(\bigcap_{j=1}^{M} E_j\right) = \frac{|\bigcap_{j=1}^{M} E_j|}{2^M} = \frac{2^{M-m}}{2^M} = \frac{1}{2^m}$$

  We also know that:

$$\prod_{j=1}^{M} P(E_j) = \prod_{j=1}^{M} \frac{1}{2} = \left(\frac{1}{2}\right)^m = \frac{1}{2^m}$$

Therefore:

$$P\left(\bigcap_{j=1}^{M} E_j\right) = \prod_{j=1}^{M} P(E_j)$$

This equality demonstrates that the events E_j are mutually independent. Since each E_j represents "the i_j-th toss shows outcome u_j", we have proven that different tosses are independent, regardless of which positions or outcomes we select.

□

Problem 2

A symmetric random walk takes place on the integers 0, 1, 2, ..., N with absorbing barriers at 0 and N, starting at k. Show that the probability that the walk is never absorbed is zero.

Proof. Imagine we continue tracking the theoretical directions of our random walk even after absorption occurs. We consider an infinite sequence of potential steps with probability $\frac{1}{2}$ for each direction.

If a sequence of N consecutive right steps ever occurs, the walker will either reach position N at that point (if not yet absorbed) or would have been absorbed earlier.

We now prove such a sequence must eventually occur with probability 1.

Divide our infinite sequence into disjoint groups of length N:

$$(s_1, s_2, \ldots, s_N), (s_{N+1}, \ldots, s_{2N}), \ldots$$

Let s denote the specific sequence of N consecutive right steps. For any group, the probability that all steps are to the right is 2^{-N}.

The probability that s eventually occurs:

$$\begin{aligned}\mathbb{P}(s \text{ occurs eventually}) &\geq \lim_{n\to\infty} \mathbb{P}(s \text{ occurs as one of the first } n \text{ groups}) \\ &= 1 - \lim_{n\to\infty} (1 - 2^{-N})^n = 1\end{aligned}$$

Since $(1 - 2^{-N})^n \to 0$ as $n \to \infty$, the sequence s must eventually occur with probability 1.

Therefore:

$$\mathbb{P}(\text{walker is eventually absorbed}) = 1$$

Thus, the probability that the random walk is never absorbed is 0. □

Problem 3

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ satisfies $\mathbb{P}(B) > 0$. Let $\mathbb{Q} : \mathcal{F} \to [0, 1]$ be defined by $\mathbb{Q}(A) = \mathbb{P}(A \mid B)$. Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A \mid C) = \mathbb{P}(A \mid B \cap C)$;discuss.

Proof. To show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space:

- $\mathbb{Q}(\emptyset) = \mathbb{P}(\emptyset|B) = 0$
- $\mathbb{Q}(\Omega) = \mathbb{P}(\Omega|B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$
- For disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$, since $A_i \cap B \in \mathcal{F}$ then we have:

$$\begin{aligned}\mathbb{Q}\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) \\ &= \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty}(A_i \cap B)\right)}{\mathbb{P}(B)} \\ &= \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} \\ &= \sum_{i=1}^{\infty} \mathbb{Q}(A_i)\end{aligned}$$

Therefore, $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space.

For the second part, given $C \in \mathcal{F}$ with $\mathbb{Q}(C) > 0$:

- By definition of conditional probability:

$$\mathbb{Q}(A|C) = \frac{\mathbb{Q}(A \cap C)}{\mathbb{Q}(C)}$$

- Substituting the definition of $\mathbb{Q}$:

$$\begin{aligned}\mathbb{Q}(A|C) &= \frac{\mathbb{P}(A \cap C|B)}{\mathbb{P}(C|B)} \\ &= \frac{\mathbb{P}(A \cap C \cap B)/\mathbb{P}(B)}{\mathbb{P}(C \cap B)/\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} \\ &= \mathbb{P}(A|B \cap C)\end{aligned}$$

$\square$

Problem 4

A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses (zero is an even number). Show that $p_0 = 1$ and that $p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1}$ if $n \geq 1$. Solve this difference equation.

Proof.

- For p_0: After 0 tosses, we have 0 heads, which is even. Since this is the only possible outcome, $p_0 = 1$.
- For the recurrence relation when $n \geq 1$:
  After $n - 1$ tosses, we either have an even number of heads (probability p_{n-1}) or an odd number (probability $1 - p_{n-1}$)
  To get an even number after n tosses:
  - We need a tail on the n-th toss if we had an even number before: $p_{n-1} \cdot (1-p)$
  - We need a head on the n-th toss if we had an odd number before: $(1-p_{n-1}) \cdot p$

  Therefore: $p_n = p_{n-1}(1 - p) + (1 - p_{n-1})p = p(1 - p_{n-1}) + (1 - p)p_{n-1}$
- To solve this recurrence relation: Since $p_n = p + p_{n-1}(1 - 2p)$
  Let $q = 1 - 2p$, so $p_n = p + q \cdot p_{n-1}$
  Iterating this relation and using $p_0 = 1$:

$$\begin{aligned} p_n &= p + q(p + qp_{n-2}) \\ &= p + qp + q^2 p_{n-2} \\ &= \ldots \\ &= p(1 + q + q^2 + \ldots + q^{n-1}) + q^n \cdot p_0 \\ &= p\frac{1 - q^n}{1 - q} + q^n \end{aligned}$$

  Substituting $q = 1 - 2p$ and simplifying:

$$\begin{aligned} p_n &= p\frac{1 - (1 - 2p)^n}{2p} + (1 - 2p)^n \\ &= \frac{1 - (1 - 2p)^n}{2} + (1 - 2p)^n \\ &= \frac{1 + (1 - 2p)^n}{2} \end{aligned}$$

Therefore, $p_n = \frac{1+(1-2p)^n}{2}$ is the solution to the given recurrence relation. □

Problem 5

Loaded dice.

(a) Show that it is not possible to weight two dice in such a way that the sum of the two numbers shown by these loaded dice is equally likely to take any value between 2 and 12 (inclusive).

(b) Given a fair die and a loaded die, show that the sum of their scores, modulo 6, has the same distribution as a fair die, irrespective of the loading.

Proof. (a) We'll prove this by contradiction.

Suppose it's possible to weight two dice such that the sum is equally likely to be any value from 2 to 12. Let p_i be the probability that the first die shows i, and q_j be the probability that the second die shows j.

For the sum to be equally distributed, we need:

$$P(\text{sum} = k) = \frac{1}{11} \quad \forall k \in \{2, 3, \ldots, 12\}$$

Consider the sums 2 ,7 and 12:

$$\begin{aligned} P(\text{sum} = 2) &= p_1 q_1 = \frac{1}{11} \\ P(\text{sum} = 12) &= p_6 q_6 = \frac{1}{11} \\ P(\text{sum} = 7) &= p_1 q_6 + p_2 q_5 + p_3 q_4 + p_4 q_3 + p_5 q_2 + p_6 q_1 = \frac{1}{11} \end{aligned}$$

Since

$$p_1 q_6 + p_6 q_1 \geq 2\sqrt{p_1 q_6 p_6 q_1} = \frac{2}{11}$$

because $p_i, q_i > 0$
So we have $P(\text{sum} = 7) \geq \frac{2}{11}$, which is a contradiction.
Therefore, it is not possible to weight two dice in such a way that the sum of the two numbers shown by these loaded dice is equally likely to take any value between 2 and 12.
(b)Let X be the outcome of the fair die, with $P(X = i) = \frac{1}{6}$ for all $i \in \{1, 2, \ldots, 6\}$.

Let Y be the outcome of the loaded die, with arbitrary probabilities $P(Y = j) = q_j$ where $\sum_{j=1}^{6} q_j = 1$.

We want to show that $(X + Y) \bmod 6$ has a uniform distribution on $\{0, 1, 2, 3, 4, 5\}$.

For any $k \in \{0, 1, 2, 3, 4, 5\}$:

$$P((X + Y) \bmod 6 = k) = \sum_{j=1}^{6} P(Y = j) \cdot P((X + j) \bmod 6 = k)$$

The key insight is that for any fixed value of j, as X varies from 1 to 6, the value $(X + j) \bmod 6$ takes each value in $\{0, 1, 2, 3, 4, 5\}$ exactly once.

Therefore, $P((X + j) \bmod 6 = k) = \frac{1}{6}$ for all j and k, which gives us:

$$\begin{aligned} P((X + Y) \bmod 6 = k) &= \sum_{j=1}^{6} P(Y = j) \cdot \frac{1}{6} \\ &= \frac{1}{6} \sum_{j=1}^{6} P(Y = j) \\ &= \frac{1}{6} \end{aligned}$$

So we have $P((X + Y) \bmod 6 = k) = \frac{1}{6}$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.
This proves that the distribution of $(X + Y) \bmod 6$ is uniform, regardless of how the loaded die is weighted. $\square$