Problem 1

Let $\mathcal{F}$ and $\mathcal{G}$ be σ-fields of subsets of Ω.

(*a*) Use elementary set operations to show that $\mathcal{F}$ is closed under countable intersections; that is, if $A_1, A_2, \dots$ are in $\mathcal{F}$, then so is $\bigcap_i A_i$.

(*b*) Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{H}$ is a σ-field.

(*c*) Show that $\mathcal{F} \cup \mathcal{G}$, the collection of subsets of Ω lying in either $\mathcal{F}$ or $\mathcal{G}$, is not necessarily a σ-field.

Proof.

(*a*) :Let $A_1, A_2, \dots \in \mathcal{F}$. Then $A_1^c, A_2^c, \dots \in \mathcal{F}$.

Since $\mathcal{F}$ is a σ-field, we have $\bigcup_i A_i \in \mathcal{F}$.

By De Morgan's Law, $\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c \in \mathcal{F}$.

Hence, $\bigcap_i A_i \in \mathcal{F}$.

(*b*) :1. Since $\Omega \in \mathcal{F}$ and $\Omega \in \mathcal{G}$, we have $\Omega \in \mathcal{H}$.

2. $\forall A \in \mathcal{H}$, we have $A \in \mathcal{F}$ and $A \in \mathcal{G}$, so $A^c \in \mathcal{F}$ and $A^c \in \mathcal{G}$, hence $A^c \in \mathcal{H}$.

3. $\forall A_1, A_2, \dots \in \mathcal{H}$, we have $A_1, A_2, \dots \in \mathcal{F}$ and $A_1, A_2, \dots \in \mathcal{G}$, so $\bigcup_i A_i \in \mathcal{F}$ and $\bigcup_i A_i \in \mathcal{G}$, hence $\bigcup_i A_i \in \mathcal{H}$.

(*c*) :Counterexample: Let $\Omega = \{1,2,3\}, \mathcal{F} = \{\emptyset, \{1,2\}, \{3\}, \Omega\}, \mathcal{G} = \{\emptyset, \{1\}, \{2,3\}, \Omega\}$.

Then $\{1,2\} \in \mathcal{F}$ and $\{1\} \in \mathcal{G}$, but $\{1,2\} \cup \{1\} = \{1,2\} \notin \mathcal{F} \cup \mathcal{G}$.

Hence, $\mathcal{F} \cup \mathcal{G}$ is not necessarily a σ-field.

□

Problem 2

Let $A_1, A_2, \dots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m, \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Clearly $C_n \subseteq A_n \subseteq B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \bigcap_n B_n = \bigcap_n \bigcup_{m \geq n} A_m, \quad \lim C_n = C = \bigcup_n C_n = \bigcup_n \bigcap_{m \geq n} A_m$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that

(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,

(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$,

We say that the sequence $\{A_n\}$ converges to $A = \lim A_n$ if B and C are the same set A.

Suppose that $A_n \to A$ and show that

(c) A is an event, i.e. $A \in \mathcal{F}$,

(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Proof.

(a) :Since by defination: $B = \bigcap_n \bigcup_{m \geq n} A_m$

$\Rightarrow$Suppose $\omega \in B$, then $\omega \in \bigcup_{m \geq n} A_m$ for all $n \in \mathbb{N}$

This means for each n, we can find an $m \geq n$ such that $\omega \in A_m$

Hence, $\omega \in A_n$ for infinitely many values of n

$\Leftarrow$Suppose $\omega \in A_n$ for infinitely many values of n,

then for each n, we can find an $m \geq n$ such that $\omega \in A_m$

This means $\omega \in \bigcup_{m \geq n} A_m$ for all $n \in \mathbb{N}$

Hence, $\omega \in \bigcap_n \bigcup_{m \geq n} A_m = B$

(b) :$\Rightarrow$Suppose $\omega \in C$, then $\exists n \in \mathbb{N}$ such that $\omega \in \bigcap_{m \geq n} A_m$

This means $\exists n \in \mathbb{N}$ such that $\omega \in A_m$ for all $m \geq n$

Hence, $\omega \in A_n$ for all but finitely many values of n

$\Leftarrow$Suppose $\omega \in A_n$ for all but finitely many values of n,

then $\exists n \in \mathbb{N}$ such that $\omega \in A_m$ for all $m \geq n$

This means $\omega \in \bigcap_{m \geq n} A_m$

Hence, $\omega \in \bigcup_n \bigcap_{m \geq n} A_m = C$

(c) :Since $A_n \to A$, so $\lim_{n\to+\infty} A_n = \limsup_{n\to+\infty} A_n = \liminf_{n\to+\infty} A_n$

So we have $A = B = C$

Since B_n are countable unions of A_m, so $B_n \in \mathcal{F}$ for all $n \in \mathbb{N}$

Then, $B = \bigcap_n B_n \in \mathcal{F}$

Hence, we have $A \in \mathcal{F}$

(d) :Since $C_n \subseteq A_n \subseteq B_n$, we have $\mathbb{P}(C_n) \le \mathbb{P}(A_n) \le \mathbb{P}(B_n)$

Since C_n increases to C, B_n decreases to B,

we have $\mathbb{P}(C_n) \to \mathbb{P}(C) = \mathbb{P}(A), \mathbb{P}(B_n) \to \mathbb{P}(B) = \mathbb{P}(A)$

Hence by squeeze theorem, $\mathbb{P}(A_n) \to \mathbb{P}(A)$

□

Problem 3

Let $\mathcal{E}$ be the left open right closed intervals of $\Omega := \mathbb{R}$ defined in class. Write down the algebra generated by $\mathcal{E}$, and prove your result.

We have $\mathcal{E} = \{\text{left open right closed intervals}\} = \begin{cases} (a,b], & -\infty \le a < b < +\infty \\ (a,+\infty) & \end{cases}$

$$a(\mathcal{E}) = \text{"finite disjioint union of elements in } \mathcal{E}\text{"}$$
$$= \underbrace{\{I_1 \cup \cdots \cup I_k; I_j \in \mathcal{E},\ \text{and } I_i \cap I_j = \emptyset\}}_{f}$$

Proof. Denote the collection of finite disjioint union of elements in $\mathcal{E}$ by f

- $f \subseteq a(\mathcal{E})$
  - Since $a(\mathcal{E})$ is an algebra containing $\mathcal{E}$, it is closed under finite unions.
  - Any element of f is a finite disjoint union of intervals from $\mathcal{E}$, hence belongs to $a(\mathcal{E})$.
  - Thus, $f \subseteq a(\mathcal{E})$
- $a(\mathcal{E}) \subseteq f$
  - we need to prove that f is an algebra
    - since $\Omega = \mathbb{R} = (-\infty,+\infty)$, choose $I_1 = (-\infty, a], I_2 = (a,+\infty) \in \mathcal{E}$, then $I_1 \cup I_2 = \Omega \in f$
    - for $\forall I_i, I_j \in f, I_i \cup I_j \in f$ by defination

  * for $\forall (a,b] \in f$, we have $(a,b]^c = (-\infty, a] \cup (b, +\infty) \in f$
    for $\forall (a,+\infty) \in f$, we have $(a,+\infty)^c = (-\infty, a] \in f$

- since $\mathcal{E} \subseteq f$, f is an algebra and $a(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$, Hence, $a(\mathcal{E}) \subseteq f$

$\square$

Problem 4

Let Ω be a sample space. Show that any finite algebra on Ω is a $\sigma-$algebra.

Proof. Let $\mathcal{A}$ be a finite algebra on Ω, and we can write $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$.

- Since $\mathcal{A}$ is an algebra, $\Omega \in \mathcal{A}$.
- Since $\mathcal{A}$ is an algebra, if $A_i \in \mathcal{A}$, then $A_i^c \in \mathcal{A}$.
- Since $\mathcal{A}$ is an algebra, if $A_i, A_j \in \mathcal{A}$, then $A_i \cup A_j \in \mathcal{A}$.
  Let $\{B_i\}_{i\in I}$ be a countable family of subsets of Ω, where I is a countable index set, and each $B_i \in \mathcal{A}$. Since $\mathcal{A}$ is finite, the image of the function $f : I \to \mathcal{A}$ defined by $f(i) = B_i$ is finite. That is, there are only finitely many distinct sets in $\{B_i\}_{i\in I}$. Let these distinct sets be $C_1, C_2, \ldots, C_m$, where $m \leq n$ and $C_k \in \mathcal{A}$ for all $k = 1, 2, \ldots, m$.

  The countable union $\bigcup_{i\in I} B_i$ can therefore be written as:

  $$\bigcup_{i\in I} B_i = C_1 \cup C_2 \cup \cdots \cup C_m.$$

  Since $\mathcal{A}$ is an algebra, it is closed under finite unions. By induction:

  - For $m = 1$, $C_1 \in \mathcal{A}$.
  - Assume $C_1 \cup C_2 \cup \cdots \cup C_k \in \mathcal{A}$ for some $m = k \geq 1$.
  - Then, $(C_1 \cup C_2 \cup \cdots \cup C_k) \cup C_{k+1} \in \mathcal{A}$ because $\mathcal{A}$ is closed under binary unions.

  Thus, $C_1 \cup C_2 \cup \cdots \cup C_m \in \mathcal{A}$, and we conclude:

  $$\bigcup_{i\in I} B_i \in \mathcal{A}.$$

Therefore, $\mathcal{A}$ is a σ-algebra. $\square$