Problem 1

Show that the assumption that $\mathbb{P}$ is countably additive is equivalent to the assumption that $\mathbb{P}$ is continuous. That is to say, show that if a function $\mathbb{P} : \mathcal{F} \to [0,1]$ satisfies $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$, then $\mathbb{P}$ is countably additive (in the sense of satisfying Definition (1.3.1b)) if and only if $\mathbb{P}$ is continuous (in the sense of Lemma (1.3.5)).

Proof.

- $\Rightarrow$ Suppose $\mathbb{P}$ is countably additive.
  Suppose w.l.o.g. that $\{A_n\}_{n=1}^{+\infty}$ be any increasing sequence of events and $A_1 \subset A_2 \cdots$
  Let $A = \bigcup_{n=1}^{\infty} A_n$
  Let $B_1 = A_1, B_2 = A_2 \backslash A_1, \cdots B_n = An \backslash A_{n-1}$ and they are all disjoint
  Then
  $$A = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$
  For every n, we have $A_n = \bigcup_{k=1}^{n} B_k$
  By countable additivity, we have $\mathbb{P}(A_n) = \sum_{k=1}^{n} \mathbb{P}(B_k)$
  Then, we have $\mathbb{P}(A) = \mathbb{P}(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \mathbb{P}(B_k)$
  So,
  $$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(B_k) = \sum_{k=1}^{\infty} \mathbb{P}(B_k) = \mathbb{P}(A)$$
  Thus, $\mathbb{P}$ is continuous from below. (similarly we can prove that $\mathbb{P}$ is continuous from above by assuming decreasing $\{A_n\}_{n=1}^{+\infty}$)

- $\Leftarrow$ Suppose $\mathbb{P}$ is continuous.
  Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{F}$. Define $B_n = \bigcup_{k=1}^{n} A_k$.
  Then $\{B_n\}_{n=1}^{\infty}$ is an increasing sequence with $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.
  By continuity, $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n)$. Since the A_n are pairwise disjoint,$\mathbb{P}(B_n) = \sum_{k=1}^{n} \mathbb{P}(A_k)$.
  Thus,
  $$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(A_k) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$
  This shows that $\mathbb{P}$ is countably additive.

□

Problem 2

The 'ménages' problem poses the following question. Some consider it to be desirable that men and women alternate when seated at a circular table. If n heterosexual couples are seated randomly according to this rule, show that the probability that nobody sits next to his or her partner is

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

You may find it useful to show first that the number of ways of selecting k non-overlapping pairs of adjacent seats is $\binom{2n-k}{k}2n(2n-k)^{-1}$.

Proof. Assume we have n couples labelled $1, 2, \ldots, n$. Define

$$A_k = \{\text{couple } k \text{ sits together}\}.$$

For any chosen k-tuple $(i_1, i_2, \ldots, i_k)$, we wish to count

$$\mathbb{N}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big).$$

To eliminate rotational symmetry, choose a couple not among $\{i_1, \ldots, i_k\}$ (say the one with the smallest index not chosen) and fix its man in seat 1. Since men and women alternate, the problem reduces to selecting k non-overlapping adjacent seat pairs from $2n$ seats. This is equivalent to choosing k "compressed" positions from $2n-k$ spots, which can be done in

$$\binom{2n-k-1}{k} \text{ ways.}$$

Next, assign the k couples to these pairs in $k!$ ways. The remaining $n-k-1$ men and $n-k$ women are arranged in $(n-k-1)!(n-k)!$ ways. Including the initial factor $2n$ from the possible choices before fixing, we have

$$\mathbb{N}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big) = 2n\binom{2n-k-1}{k} k!\,(n-k-1)!(n-k)!.$$

Using the inclusion–exclusion principle, the probability that no couple sits together is

$$\mathbb{P}\Big(\bigcap_{j=1}^{n} A_j^c\Big) = 1 - \mathbb{P}\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\,\mathbb{P}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big).$$

Substituting the count and dividing by the total number of arrangements, we obtain

$$\mathbb{P}\Big(\bigcap_{j=1}^{n} A_j^c\Big) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!.$$

□

Problem 3

The probabilistic method. 10 per cent of the surface of a sphere is coloured blue, the rest is red. Show that, irrespective of the manner in which the colours are distributed, it is possible to inscribe a cube in S with all its vertices red.

Proof. Let the sphere S be partitioned into a blue region B and a red region, with

$$\frac{|B|}{|S|} = 0.1.$$

Fix a standard cube inscribed in S so that its 8 vertices lie on S. For every rotation R (with R uniformly distributed over SO(3)), denote its vertices by

$$v_1(R), v_2(R), \ldots, v_8(R).$$

Define the event

$$E_i = \{R \in \mathrm{SO}(3) \mid v_i(R) \in B\}, \quad i = 1, 2, \ldots, 8.$$

Since the distribution of each vertex is uniform on S, we have

$$P(E_i) = 0.1 \quad \text{for each } i.$$

Suppose for contradiction that every rotation R results in at least one vertex landing in B. Then

$$\mathrm{SO}(3) = \bigcup_{i=1}^{8} E_i.$$

By the Inclusion-Exclusion Principle,

$$P\Big(\bigcup_{i=1}^{8} E_i\Big) \leq \sum_{i=1}^{8} P(E_i) = 8 \times 0.1 = 0.8.$$

Thus, the probability that a rotation R yields no vertex in B is

$$P\Big(\bigcap_{i=1}^{8} E_i^c\Big) = 1 - P\Big(\bigcup_{i=1}^{8} E_i\Big) \geq 1 - 0.8 = 0.2 > 0.$$

This positive probability implies that there exists at least one rotation R such that none of the vertices fall in the blue region; that is, they all lie in the red region.

□

Problem 4

Poker. During a game of poker, you are dealt a five-card hand at random. With the convention that aces may count high or low, show that:

$$\begin{aligned}
&\mathbb{P}(1 \text{ pair }) \simeq 0.423,\\
&\mathbb{P}(\text{ straight }) \simeq 0.0039,\\
&\mathbb{P}(4 \text{ of a kind }) \simeq 0.00024,\\
&\mathbb{P}(2 \text{ pairs }) \simeq 0.0475,\\
&\mathbb{P}(\text{ flush }) \simeq 0.0020,\\
&\mathbb{P}(3 \text{ of a kind }) \simeq 0.021,\\
&\mathbb{P}(\text{ full house }) \simeq 0.0014,\\
&\mathbb{P}(\text{ straight flush }) \simeq 0.000015.
\end{aligned}$$

Proof.

$$\begin{aligned}
|\Omega| &= \binom{52}{5}\\
|1 \text{ pair}| &= 13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3\\
P[1 \text{ pair}] &= \frac{13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3}{\binom{52}{5}} \approxeq 0.423\\
|Straight| &= 10 \times (4^5 - 4)\\
P[Straight] &= \frac{10 \times (4^5 - 4)}{\binom{52}{5}} \approxeq 0.0039\\
|4 \text{ of a kind}| &= 13 \times (52 - 4)\\
P[4 \text{ of a kind}] &= \frac{13 \times 48}{\binom{52}{5}} \approxeq 0.00024\\
|2 \text{ pairs}| &= \binom{13}{2} \times \binom{4}{2}^2 \times 11 \times 4\\
P[2 \text{ pairs}] &= \frac{\binom{13}{2} \times \binom{4}{2}^2 \times 11 \times 4}{\binom{52}{5}} \approxeq 0.0475\\
|Flush| &= 4 \times \binom{13}{5} - 4 \times 10\\
P[Flush] &= \frac{4 \times \binom{13}{5} - 4 \times 10}{\binom{52}{5}} \approxeq 0.0020\\
|3 \text{ of a kind}| &= 13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2
\end{aligned}$$

$$P[\text{3 of a kind}] = \frac{13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2}{\binom{52}{5}} \approxeq 0.021$$

$$|FullHouse| = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$$

$$P[FullHouse] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approxeq 0.0014$$

$$|Straightflush| = 10 \times 4$$

$$P[Straightflush] = \frac{10 \times 4}{\binom{52}{5}} \approxeq 0.000015$$

□

Problem 5

Let m be Lebesgue measure on [0,1], and $0 \leq a \leq b \leq c \leq d \leq 1$ such that $a + d \geq b + c$. Give an example of a sequence of sets $A_1, A_2, \cdots$ in [0,1], such that $m(\liminf_n A_n) = a$, $\liminf_n m(A_n) = b$, $\limsup_n m(A_n) = c$ and $m(\limsup_n A_n) = d$.

Proof. Define

$$I = [0, a] \quad \text{and} \quad E = [a, d].$$

Let α be any irrational number(say $\sqrt{2}$) and let $\{n\alpha\}$ denote the fractional part of $n\alpha$. For each $n \in \mathbb{N}$, set

$$X_n = \begin{cases} [\, a + (d-a)\{n\alpha\},\ a + (d-a)\{n\alpha\} + (b-a)\,], & \text{if } n \text{ is even}, \\ [\, a + (d-a)\{n\alpha\},\ a + (d-a)\{n\alpha\} + (c-a)\,], & \text{if } n \text{ is odd}, \end{cases}$$

and define

$$A_n = I \cup X_n.$$

Then we can have

$$m\Big(\liminf_{n\to\infty} A_n\Big) = a, \quad \liminf_{n\to\infty} m(A_n) = b, \quad \limsup_{n\to\infty} m(A_n) = c, \quad m\Big(\limsup_{n\to\infty} A_n\Big) = d.$$

□

Problem 6

Let $\Omega = \mathbb{R}$ and consider the following subsets of $\mathcal{P}(\mathbb{R})$:

$$\mathcal{C}_1 := \{(-\infty, b] : b \in \mathbb{R}\}$$
$$\mathcal{C}_2 := \{(a, b] : a, b \in \mathbb{R}\}$$
$$\mathcal{C}_3 := \{A \subset \mathbb{R}, A \text{ is closed }\}.$$

Show that $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3)$.

Proof. $\sigma(C_1) = \sigma(C_2)$

- $C_1 \subseteq \sigma(C_2)$
  Since $(-\infty, b] \in \sigma(C_2)$
  So, $\sigma(C_1) \subseteq \sigma(C_2)$ because $\sigma(C_1)$ is minimal

- $C_2 \subseteq \sigma(C_1)$
  Writing $(a, b] = (-\infty, b] \backslash (-\infty, a]$
  Since$(-\infty, b]$ and $(-\infty, a]$ are in C_1, we have (a,b]$\in \sigma(C_1)$
  So, $\sigma(C_2) \subseteq \sigma(C_1)$ because $\sigma(C_2)$ is minimal

$\sigma(C_2) = \sigma(C_3)$

- $\sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{C}_3)$*:* Let $(a, b] \in \mathcal{C}_2$. Notice that for any $x \in \mathbb{R}$, the set $(-\infty, x]$ is closed.
  Hence, both $(-\infty, b]$ and $(-\infty, a]$ belong to $\mathcal{C}_3$.
  Writing $(a, b] = (-\infty, b] \setminus (-\infty, a]$, we conclude that $(a, b]$ is an element of $\sigma(\mathcal{C}_3)$.
  Therefore, $\mathcal{C}_2 \subseteq \sigma(\mathcal{C}_3)$.
  So $\sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{C}_3)$ because $\sigma(C_2)$ is minimal

- $\sigma(\mathcal{C}_3) \subseteq \sigma(\mathcal{C}_2)$*:* Every open set $U \subset \mathbb{R}$ can be written as $U = \bigcup_{n=1}^{\infty} I_n$,
  where each I_n is an open interval. For any open interval $I_n = (a, b)$, we have

$$(a, b) = \bigcup_{m=1}^{\infty} \left(a + \frac{1}{m}, b \right].$$

  Thus,

$$U = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(a_n + \frac{1}{m}, b_n \right],$$

  showing that U is a countable union of half–open intervals of the form $(a, b]$.
  Hence, $U \in \sigma(\mathcal{C}_2)$, where $\mathcal{C}_2 = \{(a, b] : a, b \in \mathbb{R}\}$.
  So we have $U^c \in \sigma(\mathcal{C}_2)$,where $U^c = \mathcal{C}_3$
  Therefore, $\sigma(\mathcal{C}_3) \subseteq \sigma(\mathcal{C}_2)$ because $\sigma(C_3)$ is minimal

□