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*2025 Spring Math*

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HTOP,PDE,COMPLEX,ALGO

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## Chapter 1

# Functions of Complex Variables

### §1.1 Lecture 1 (02-03) Complex numbers

#### §1.1.1 Introduction

You know the sets

$$(R, +, \times) \text{ and } (Q, +, \times)$$

$Q(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in Q\}$   $Q(\sqrt{2})$ , they are infinite sets.

#### Definition 1.1.1: Fields

We say that a set  $F$  together with two binary operations, called addition  $+, F \times F \rightarrow F$ , and multiplication  $\times, F \times F \rightarrow F$  satisfying:

- associativity  $(a + b) + c = a + (b + c)$
- commutativity  $a + b = b + a, ab = ba$
- existence of identities  
there is a  $O \in F$  such that  $a + O = a$  for all  $a \in F$   
there is a  $1 \in F$  such that  $a \times 1 = a$  for all  $a \in F$
- existence of inverses  
for every  $a \in F$  there is a  $b \in F$  such that  $a + b = O$   
for every  $a \in F$  there is a  $b \in F$  such that  $a \times b = 1$
- distributivity  
 $a \times (b + c) = a \times b + a \times c$

#### Example 1: Fields

- For every prime number  $p$ , and  $n \geq 0, n \in N$ , there is a field  $(F_{p^n}, +, \times)$  with  $p^n = q$  elements (cardinality  $q$ ).
- $(C, +, \times)$  is a field

Question:  $(R^d, +), d \geq 2$ , is there a multiplication,  $\times$  such that  $(R^d, +, \times)$  is a field?

- $R^2$ , scalar product,  $x \cdot y = x_1y_1 + x_2y_2 \in R$  is not a field because  $R^2 \times R^2 \rightarrow R$ .
- cross product, not satisfying Definition 1 and 5.
- $d=2$ , there is a good definition of multiplication,  $(R^2, +, \times) \cong (C, +, \times)$ .

**Lemma 1.1.1**

Consider the vector space  $(\mathbb{R}^2, +)$   
 Define the multiplication:  $(a, b) \times (c, d) = (ac - bd, ad + bc)$   
 Then  $(\mathbb{R}^2, +, \times)$  is a field.

Proof:

- You can check that  $\times$  is commutative and satisfies associativity and distributivity.
- We can check  $(1, 0)$  is a multiplication identity.
- also if  $(a, b) \neq (0, 0)$ , then

$$(a, b) \times \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = \left( \frac{a^2 + b^2}{a^2 + b^2}, \frac{-ab + ab}{a^2 + b^2} \right) = (1, 0)$$

## §1.2 Lecture 2 (02-05)Complex numbers

### §1.2.1 The algebra of complex numbers

It is important to classify sets:

in Set Theory: bijection

in Topology: homeomorphism

For **Fields**: isomorphism

**Definition 1.2.1: isomorphism**

We say that fields  $F$  and  $F'$  are isomorphic if there exists a function

$$i : F \rightarrow F'$$

which is a bijection and preserve the binary operations

$$i(a + b) = i(a) + i(b)$$

and

$$i(ab) = i(a)i(b)$$

.

$i$  is called an isomorphism and we write  $F \cong F'$

**Comments:**

There is a unique field up to isomorphism of order (its cardinality)

$$p^n : F_{p^n}$$

**Definition 1.2.2: subfield**

Given a field  $(F, +, \times)$ , we say that  $(E, +, \times)$  is a subfield of  $(F, +, \times)$  if  $E \subset F$  and the addition and multiplication of  $E$  is the same addition and multiplication of  $F$ .

**Example 1: subfield**

$$Q \subset R$$

$$Q \subset Q(\sqrt{2}) \subset R \subset R^2$$

**Definition 1.2.3: complex number**

Let  $F$  be a field such that:

- $R$  is a subfield of  $F$
- there is an element  $i \in F$  which is the root of the equation

$$x^2 + 1 = 0 (i^2 = -1)$$

Then we define the field of complex numbers  $C$  as

$$C = \{x \in F : x = \alpha + i\beta, \alpha, \beta \in R\}$$

**Comments:**

- We need to show that there is at least one field  $F$  satisfying these properties.  
**Actually,  $(F, +, \times) = (R, +, \times)$  satisfies these properties.**
- $R' = \{x \in R : x = (\alpha, 0), \alpha \in R\} \cong R \rightarrow R$  is a subfield of  $R^2$ .
- Define  $i := (0, 1) \in R^2$

$$(a, b) \times (c, d) := (ac - bd, ad + bc)$$

Then,

$$i^2 = (0, 1)^2 = (0, 1) \times (0, 1) = (-1, 0)$$

**Example 2: exercise**

Set of  $2 \times 2$  real matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in R$$

This set together with the standard addition and multiplication of matrices is a field and it is isomorphic to  $(R^2, +, \times)$ .

$$F = R^2, C = R^2$$

**Comments:**

It turns out that any definition of  $C$  is unique (up to isomorphisms).

**Lemma 1.2.1**

Let  $F$  and  $F'$  be two fields satisfying the properties of the definition of complex numbers, then call  $\mathbb{C}$  and  $\mathbb{C}'$  the corresponding complex number fields. Then  $\mathbb{C} \cong \mathbb{C}'$ .

**Proof:**

Call  $i \in \mathbb{C}$  and  $i' \in \mathbb{C}'$  the square roots  $-1$ .

Then:

$f : \mathbb{C} \rightarrow \mathbb{C}'$  defined as  $f(\alpha + i\beta) = \alpha + i'\beta$  is an isomorphism

$$z = \alpha + i\beta, z' = \alpha' + i'\beta', w = \gamma + i\delta, w' = \gamma' + i'\delta'$$

**Lemma 1.2.2**

Consider the vector space  $(\mathbb{R}, +)$ . Then any multiplication ' $\times$ ' defined in this vector space which transforms it into a field  $(\mathbb{R}, +, \times)$  is such that  $(\mathbb{R}, +, \times) \cong \mathbb{C}$ .

**Proof:**

Consider the elements

$$(1, 0) \text{ and } j = (0, 1) \in \mathbb{R}^2$$

Then any element of  $\mathbb{R}^2$  can be written as  $\alpha + j\beta, \alpha, \beta \in \mathbb{R}$

Then

$$j^2 = a + jb, a, b \in \mathbb{R}$$

Then,

$$\begin{aligned} j^2 - jb = a, j^2 - 2\frac{jb}{2} = a &\rightarrow j^2 - 2\frac{jb}{2} + \frac{b^2}{4} = a + \frac{b^2}{4} \\ \left(j - \frac{b}{2}\right)^2 &= a + \frac{b^2}{4} \end{aligned} \tag{1.1}$$

**Define:**  $i = \frac{j - \frac{b}{2}}{\sqrt{a + \frac{b^2}{4}}}$

**Claim:**

$$a + \frac{b^2}{4} < 0$$

If:

$$a + \frac{b^2}{4} \geq 0$$

Then (1.1) can be written as

$$\rightarrow \left(j - \frac{b}{2}\right) + \sqrt{a + \frac{b^2}{4}} \left(j - \frac{b}{2}\right) - \sqrt{a + \frac{b^2}{4}} = 0$$

either

$$j - \frac{b}{2} = \sqrt{a + \frac{b^2}{4}}, j - \frac{b}{2} = -\sqrt{a + \frac{b^2}{4}}$$

## §1.3 Complex numbers

The field of rationals  $\mathcal{Q}$  is the fractional field of the integral domain  $\mathcal{Z}$ .  
With the metric

$$d(x, y) = |x - y|$$

$$|x| = \max\{x, -x\}, \forall x, y \in \mathcal{R}$$

The field  $\mathcal{Q}$  is not a complete metric space.  
Its completion  $\mathbb{R}$ , nevertheless is not algebraically closed.  
For instance,  $f(x) = x^2 + 1$ , doesn't split in  $\mathbb{R}$ .  
So we define the splitting field for  $f(x)$  over  $\mathbb{R}$   
i.e. the smallest field extension of  $\mathbb{R}$  such that  $f(x)$  has roots.

Consider the principal ideal  $\langle f(x) \rangle$  in  $\mathbb{R}[x]$   
since  $f(x)$  is irreducible, then  $\langle f(x) \rangle$  is a maximal ideal.  
Then the quotient ring  $\mathbb{C} = \frac{\mathbb{R}[x]}{\langle f(x) \rangle}$  is a field!  
which is the splitting field of  $f(x)$   
The field  $\mathbb{C}$  has no further splitting extensions! All polynomials in  $\mathbb{C}[x]$  split in  $\mathbb{C}$ .  
So as soon as we forced one polynomial.  $f(x) = x^2 + 1$  to have roots, we in fact imposed that all polynomials have roots.

### Topology:

The Topology of  $\mathbb{C}$  is isomorphic to  $\mathbb{R}^2$ .  
Thus  $\mathbb{C}$  is locally compact, but not compact space.  
The Alexandroff compactification says that we can add a point at infinity to  $\mathbb{C}$  so that  $\mathbb{C} \cup \{\infty\}$  compact.

#### Definition 1.3.1: Alexandroff compactification

- A set  $U \supseteq \mathbb{C} \cup \{\infty\}$  is an open neighborhood of  $\{\infty\}$  if  $U = \mathbb{C} \cup \{\infty\} \setminus K$   
When  $K \supseteq \mathbb{C}$ .

$\mathbb{C} \cup \{\infty\}$  is called the Riemann sphere.

#### Definition 1.3.2: Holomorphic Functions

A function  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic where  $\Omega \subseteq \mathbb{C}$  is holomorphic iff:

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists, } \forall z_0 \in \Omega$$

$$f(z) = f(x, y) = u(x, y) + iv(x, y), \text{ where } z = x + iy \in \Omega$$

$$u(z) = \operatorname{Re}(f(z)), v(z) = \operatorname{Im}(f(z))$$

$f : \Omega \rightarrow \mathbb{C}$  being holomorphic of course implies that  $u, v : \Omega \rightarrow \mathbb{R}$  differentiable.  
If we take  $f(z_0) = z_0 = 0$  and we write

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$



## 1.4. LECTURE 3 (02-10) – ALGEBRAIC STRUCTURE OF COMPLEX NUMBERS 8

Then,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{f(z)}{z} = \left(\frac{\partial f}{\partial z}\right)_{(z_0=0)} + \frac{\bar{z}}{z} \left(\frac{\partial f}{\partial \bar{z}}\right)_{(z_0=0)} + o(z)$$

For real  $z \rightarrow 0$ , we have  $\frac{\bar{z}}{z} = 1$ , and for purely imaginary  $z \rightarrow 0$ , we have  $\frac{\bar{z}}{z} = -1$ .  
we have to restrict  $\frac{\partial f}{\partial \bar{z}} = 0$

**Cauchy-Riemann equations:**

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

$$f'(z) = \frac{\partial f}{\partial z} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)(x, y), \text{ where } z = x + iy \in \Omega$$

Nevertheless,  $f$  being holomorphic is not equivalent to say  $f$  satisfies the Cauchy-Riemann equations.

Indeed we take  $f(z) = \frac{z^5}{|z|^4}$  on  $\mathbb{C} \setminus \{0\}$  and  $f(0) = 0$ .

We can observe that  $f(z)$  satisfies the Cauchy-Riemann equations on  $\mathbb{C}$  but  $f$  is not even continuous at  $z = 0$ .

### Theorem 1.3.1: Looman Menchoff Theorem

A continuous function  $f : \Omega \cup \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic iff  $f$  satisfies the Cauchy-Riemann equations.

$$\begin{aligned} \forall f, g \in H(\Omega) \\ (fg)' &= f'g + fg' \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \\ (g \circ f)'(z) &= g'(f(z)) \cdot f'(z) \end{aligned}$$

## §1.4 Lecture 3 (02-10) – Algebraic Structure of Complex numbers

### §1.4.1 Algebraic Structure of Complex numbers

#### Lemma 1.4.1

Consider  $(\mathbb{R}^2, +)$  as a vector space. Then any definition of multiplication on  $\mathbb{R}^2$  compatible with the vector space structure, which transforms  $\mathbb{R}^2$  into a field, defines a field isomorphism with  $\mathbb{C}$ .

#### Comments:

The assumption that the multiplication is compatible with the vector space structure means that the scalar multiplication coincides with the field multiplication in the following space.

$$(\mathbb{R}^2, +, \times)$$

#### 1.4. LECTURE 3 (02-10) – ALGEBRAIC STRUCTURE OF COMPLEX NUMBERS 9

and  $1 \in \mathbb{R}$  is the multiplicative identity  
then

$$au = (a1) \times u, a \in \mathbb{R}$$

Let  $L = \{a1 : a \in \mathbb{R}\}$

Then  $L$  is isomorphic with  $\mathbb{R}$

The addition in  $\mathbb{R}^2$  as a vector space coincides with the addition in  $\mathbb{R}^2$  as a field.

**Proof of the Lemma:**

Essentially we have to find a  $i \in \mathbb{R}^2$  such that  $i^2 = -1$

Since  $\mathbb{R}^2$  is a two-dimensional vector space, if we pick any  $j \in \mathbb{R}^2 \setminus L$

We know that  $1, j, j^2$  must be linearly dependent.

Then there must exist  $a, b \in \mathbb{R}$  such that  $aj^2 + bj + c1 = 0$

Then,

$$\begin{aligned} a(j^2) + 2\frac{b}{2a}j + c &= 0 \\ a\left(j + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2 &= 0 \\ \left(j + \frac{b}{2a}\right)^2 &= \frac{1}{a}\left(-c + a\left(\frac{b}{2a}\right)^2\right) \end{aligned}$$

Necessarily,  $-c + a\left(\frac{b}{2a}\right)^2 < 0$

Define

$$i = \frac{\sqrt{a}\left(j + \frac{b}{2a}\right)}{\sqrt{-a\left(\frac{b}{2a}\right)^2 + c}}$$

#### §1.4.2 Square roots

Every complex number has a square root.

Also we can write the 2 square roots in a quite explicit way (related to fundamental theorem of algebra).

##### Lemma 1.4.2

$$\sqrt{\alpha + i\beta} = \pm \left( \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + \frac{\beta}{|\beta|} \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \right)$$

**Proof:**

(Notation for complex numbers:  $\alpha + i\beta, x + iy, \dots$ )

We need to find  $z = x + iy$  such that  $z^2 = (x + iy)^2 = \alpha + i\beta$

So  $x^2 - y^2 = \alpha, 2xy = \beta$

Now,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2$$

So,

$$\begin{aligned}
 x^2 + y^2 &= \sqrt{\alpha^2 + \beta^2} \\
 x^2 - y^2 &= \alpha \\
 \Rightarrow x^2 &= \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2} \\
 y^2 &= \frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2} \\
 x &= \pm \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \\
 y &= \pm \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \\
 \sqrt{\alpha + i\beta} &= \pm \left( \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + \frac{\beta}{|\beta|} \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \right)
 \end{aligned}$$

#### Definition 1.4.1: Algebraic Extension

Let  $F$  be a field and  $E$  be a subfield of  $F$ .

We say that  $x \in F$  is algebraic in the field  $E$  if  $x$  satisfies the following equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \text{ for } a_n, a_{n-1}, \dots, a_1 \in E, a_n \neq 0$$

if every  $x \in F$  is algebraic in  $E$ , then we say that  $F$  is an algebraic extension of  $E$ . We say that a field  $F$  is algebraically closed if it does not have any strictly larger algebraic extension.

A field containing  $F$  is called an algebraic closure if it is a closed algebraic extension of  $F$ .

#### Comments:

- $\mathbb{R}$  is not an algebraic extension of  $\mathbb{Q}$ .  
If this was true, then every real number would be a root of a polynomial with rational coefficients:

$$q_n x^n + q_{n-1} x^{n-1} + \cdots + q_1 x + q_0 = 0 \text{ (an algebraic number)}$$

So the cardinality of the set of algebraic numbers is countable (transcendental numbers are real numbers which are not algebraic,  $e$  and  $\pi$  for example).

- $\mathbb{Q}(\sqrt{2}) = \{q + \sqrt{2}q', q, q' \in \mathbb{Q}\}$  is algebraic over  $\mathbb{Q}$ .
- $\mathbb{R}$  is not algebraically closed.
- The fundamental theorem of algebra states that any nonconstant polynomial with complex coefficients has at least one complex root.  
Actually this is equivalent to saying that all the roots of such a polynomial are complex.
- The field of complex numbers is algebraically closed.

## §1.5 Lecture 4 (02-12) – Conjugation and absolute value

### Definition 1.5.1: Complex Conjugation

Given a complex number  $z = x + iy \in \mathbb{C}$ , we define its complex conjugation  $\bar{z} = x - iy$ . This transformation is called complex conjugation.

#### Comments:

- $\bar{\bar{z}} = z$
- We define the real part of  $z = x + iy$  as  $Re(z) = x = \frac{z + \bar{z}}{2}$  and the imaginary part of  $z$  as  $Im(z) = y = \frac{z - \bar{z}}{2i}$
- $\overline{z + w} = \bar{z} + \bar{w}$ ,  $\overline{zw} = \bar{z}\bar{w}$ : complex conjugation defines an isomorphism  $\mathbb{C} \rightarrow \mathbb{C}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$
- if  $R$  is any rational function, then  $\overline{R(z_1, z_2, \dots, z_n)} = R(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$   
In particular,

$$\begin{aligned} a_n z^n + \dots + a_1 z + a_0 &= 0 \\ \Rightarrow \bar{a}_n \bar{z}^n + \dots + \bar{a}_1 \bar{z} + \bar{a}_0 &= 0 \end{aligned}$$

### Definition 1.5.2: absolute value

Given a complex number  $z \in \mathbb{C}$ , we define its absolute value, denoted by

$$|z| = \sqrt{z\bar{z}}$$

#### Comments:

- If  $z$  is real, then  $\bar{z} = z$
- we say that  $z$  is purely imaginary if  $Re(z) = 0$ , iff  $\bar{z} = -z$
- Then if  $z$  is real,  $|z|$  coincides with the traditional definition of absolute value on  $\mathbb{R}$
- $z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 \geq 0$
- Vector space  $(V, +, \text{scalar multiplication})$  over a field  $\mathbb{F}(Q, R, C)$
- $|zw| = |z||w|$ ,  $|\bar{z}| = |z|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ ,  $w \neq 0$
- $|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$
- $|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + 2Re(z\bar{w})$
- $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$

**Lemma 1.5.1: Triangle inequality**

If  $z, w \in \mathbb{C}$  then,  $|z + w| \leq |z| + |w|$

**Proof:**

Note that if  $a = \alpha + i\beta$  then

$$\begin{aligned} -|a| &\leq \operatorname{Re}(a) = \alpha \leq |a| = \sqrt{\alpha^2 + \beta^2} \\ \Rightarrow |z + w|^2 &= |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

**Comments:**

Assume that  $z$  and  $w$  are such that  $|z + w| = |z| + |w|$

$$\Rightarrow |z||w| = \operatorname{Re}(z\bar{w}) \Leftrightarrow z\bar{w} \geq 0$$

We proved that absolute value is a norm (which is a good notion of distance):

- $|z| = 0 \Leftrightarrow z = 0$  and  $|z| \geq 0, \forall z \in \mathbb{C}$
- $|az| = |a||z|, \forall a, z \in \mathbb{C}$
- $|z + w| \leq |z| + |w|$
- $(\mathbb{R}^d, +, \underbrace{\|\cdot\|}_{\text{Euclidean norm}}), \|(x_1, \dots, x_d)\| = \sqrt{x_1^2 + \dots + x_d^2}$

**Lemma 1.5.2: Cauchy inequality**

If  $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$ ,

$$\left| \sum_{i=1}^n z_i w_i \right| \leq \sqrt{\sum_{i=1}^n |z_i|^2} \sqrt{\sum_{i=1}^n |w_i|^2}$$

**Proof:**

Let  $\lambda \in \mathbb{R}$  and consider

$$\sum_{i=1}^n |z_i - \lambda w_i|^2 = \sum_{i=1}^n |z_i|^2 - 2\operatorname{Re}(\bar{\lambda} \sum_{i=1}^n z_i \bar{w}_i) + |\lambda|^2 \sum_{i=1}^n |w_i|^2$$

Choose

$$\lambda = \frac{\sum_{i=1}^n z_i \bar{w}_i}{\sum_{i=1}^n |w_i|^2}$$

Then,

$$0 \leq \sum_{i=1}^n |z_i|^2 - \frac{|\sum_{i=1}^n z_i \bar{w}_i|^2}{\sum_{i=1}^n |w_i|^2}$$

## §1.6 Recitation 2 (02-14)

holomorphic functions  $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$  s.t.

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0), \forall z_0 \in U$$

### Definition 1.6.1: Analytic function

Let  $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ , Then  $f(\cdot)$  is said to be analytic at  $z_0 \in U$  if  $\exists \{a_n\} \in \mathbb{C}$  and radius  $r > 0$  such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall z \in U \text{ with } |z - z_0| < r$$

where  $r \leq R = \lim_{n \rightarrow +\infty} |a_n|^{-\frac{1}{n}}$  the radius of convergence

Notice that if  $f(\cdot)$  is analytic, then it is holomorphic. ( $\Leftrightarrow$ )

$$f'(z) = \frac{\partial f}{\partial z} = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \forall z \in U \text{ with } |z - z_0| < r \leq R = \lim_{n \rightarrow +\infty} (|a_n|^{-\frac{1}{n}})$$

Analytic function can be constructed:

Take a sequence  $\{a_n\} \in \mathbb{C}$  and we consider the power series.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } |z| < R = \lim_{n \rightarrow +\infty} |a_n|^{-\frac{1}{n}}$$

This function  $f(z)$  is actually analytic on the disk  $D(o, R)$ .

$\forall z_0 \in D(0, R)$  and  $0 < s < R - |z_0|$  Let  $|z - z_0| < s$  i.e.  $z \in D(z_0, s)$  Then,  $|z_0| + |z - z_0| < R$  and we want to express the  $f(z)$  into a desired form.

Note that

$$\sum_{n=1}^{\infty} \underbrace{(|a_n| + |z - z_0|)^n}_{< R} = \sum_{n=0}^{\infty} |a_n| \left( \sum_{k=0}^n \binom{n}{k} |z_0|^{n-k} |z - z_0|^k \right) < \infty$$

converges on  $z \in D(z_0, s)$

which yields

$$\sum_{n=0}^{\infty} a_n \left( \sum_{k=0}^n \binom{n}{k} z_0^{n-k} (z - z_0)^k \right) \text{ converges absolutely in } D(z_0, s)$$

Then we can interchange the order of summation and obtain

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^n a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k = f(z) = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k$$

showing  $f(z)$  is the sum of a power series on  $D(z_0, s)$  and is thus analytic at  $\forall z_0 \in D(0, R)$

**Definition 1.6.2: Inverse mapping theorem(local)**

Take  $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$  analytic, Assume  $z_0 \in U$  with  $f'(z_0) \neq 0$   
 Then  $f(\cdot)$  is a local isomorphism at  $z_0 \in U$  This means that  $\exists W \subseteq U$  s.t.  $f(\cdot)$  is injective on  $W$  and is thus a bijection from  $W$  to  $f(W)=V$ .  
 And such that  $V \subseteq \mathbb{C}$  and its inverse  $g = f^{-1} : V \rightarrow W$  is also analytic.

Indeed, we know  $f : U \rightarrow \mathbb{C}$  is holomorphic. And its Jacobian is nonzero at  $z_0$   
 Hence,  $\exists W \subseteq U$  such that  $f$  is diffeomorphism from  $W$  onto  $f(W) = V \subseteq \mathbb{C}$   
 Since the Jacobian of  $g = f^{-1}$  on  $V$  is the inverse of the Jacobian of  $f$  on  $W$ , by Cauchy-Riemann equations,  $g : V \rightarrow W$  is also holomorphic.  
 Assume w.l.o.g.  $f(z_0) = z_0, f'(z_0) = 1$ , then, we can write

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, z \in U \cap B_R(0)$$

Now that  $f(g(z)) = z, \forall z \in V$  on some  $B_\zeta(0)$  we can solve this equation by picking

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

with coefficients  $b_n = P_n(a_2, \dots, a_n, b_1, \dots, b_{n-1})$  Here,  $b_n$  is a polynomial with nonnegative coefficients.

Solving this recursively, we show the existence and uniqueness for the power series expansion of the inverse map  $g : V \rightarrow W$

It remains to show that the positive radius of convergence of  $g(\cdot)$

Select some  $A > 0$  sufficiently large such that

$$|a_n| \leq A^n, \forall n \geq 1$$

Then, we consider the sum

$$F(z) = z - \sum_{n=2}^{\infty} A^n z^n = z - \frac{A^2 z^2}{1 - Az}, \forall z \in \mathbb{C} \text{ with } |z| < \frac{1}{A}$$

Its inverse  $G(z)$  is computed with the form

$$G(z) = \frac{1 + Az - \sqrt{(1 + Az)^2 - 4zA(A + 1)}}{2A(A + 1)}$$

$G(\cdot)$  is analytic with the expression

$$G(z) = z + \sum_{n=2}^{\infty} B_n z^n$$

Here  $B_n = P_n(A^2, \dots, A^n, B_1, \dots, B_{n-1})$  where  $P_n$  is the same polynomial with non-negative coefficients.

Thus  $|a_n| \leq A^n$  implies  $|b_n| \leq B \forall n \in \mathbb{N}$

Thus  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$  has at least the radius of convergence as large as that of  $G(\cdot)$

$$\begin{aligned} g'(f(z_0)) &\neq 0 \\ &= \frac{1}{f'(x)} \end{aligned}$$

**Definition 1.6.3: open mapping theorem**

Take analytic  $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ , s.t.  $f$  is nonconstant in a neighborhood of  $z, \forall z \in U$ , then  $f$  is an open mapping, i.e.  $\forall V \subseteq U, f(V)$  is open subset in  $\mathbb{C}$

Take any  $z_0 \in U$  and we denote

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall |z - z_0| < R = \lim_{n \rightarrow +\infty} |a_n|^{-\frac{1}{n}}$$

And we let  $u(z) = f(z + z_0) - f(z_0)$ , then, clearly the power series

$$u(z) = \sum_{n=1}^{\infty} a_n z^n \text{ has the same radius of convergence } R > 0$$

Since  $f(z)$  is nonconstant in at  $z_0$ , we can find some  $m \geq 1$  s.t.

$$u(z) = \sum_{n=m}^{\infty} a_n z^n \text{ with } a_m \neq 0$$

If we can show exists analytic function  $v(z)$  with  $v'(0) \neq 0$  s.t.  $u(z) = v(z)^m$  then the assertion is verified.

And we define  $b_k = \frac{a_{m+k}}{a_m}$  for  $k \geq 1$  and consider the power

$$g(z) = \sum_{k=1}^{\infty} b_k z^k$$

and  $u(z) = a_m z^m (1 + g(z))$ , Now we write  $k(z) = (1 + g(z))^{\frac{1}{m}} - 1$  then  $(1 + k(z))^m = 1 + g(z)$ , let

$$v(z) = a_m^{\frac{1}{m}} z (1 + k(z)) = a_m^{\frac{1}{m}} z + a_m^{\frac{1}{m}} z k(z)$$

Then we have  $v(z)^m = u(z)$  and  $v(z)$  is analytic with  $v'(0) \neq 0$

Hence,  $\exists$  open disk  $D(0, r)$  s.t.  $V(D) \supseteq D' := D(o, r')$

And then, the image of  $D'$  under  $z \rightarrow z^m$  is again an open disk, verifying the assertion.

## §1.7 Lecture 5 (02-17) – The geometric representation of complex numbers

**Definition 1.7.1: Euler's formula**

$$e^{i\theta} = \cos\theta + i\sin\theta, \theta \in \mathbb{R}$$

Stereographic projection:



### §1.7.1 Geometric multiplication

$$\begin{aligned}
 a &= (\alpha, \beta) = \alpha + i\beta \\
 &= r\cos\theta + ir\sin\theta \\
 &= r(\cos\theta + i\sin\theta) \\
 r &= |a| = \sqrt{\alpha^2 + \beta^2} \\
 \cos\theta &= \frac{\alpha}{r}, \sin\theta = \frac{\beta}{r} \\
 \sin\theta &= \frac{\beta}{r}
 \end{aligned}$$

#### Definition 1.7.2: argument of a complex number

Given  $a \in \mathbb{C}$  and its polar representation

$$a = r(\cos\theta + i\sin\theta)$$

we call  $\theta$  the argument of  $a$  and denote it by

$$\theta = \arg(a)$$

#### Lemma 1.7.1

Let  $a_1 = r_1(\cos\theta_1 + i\sin\theta_1)$  and  $a_2 = r_2(\cos\theta_2 + i\sin\theta_2)$   
 $\arg(a_1 a_2) = \arg(a_1) + \arg(a_2)$

**Proof:**

$$\begin{aligned}
 a_1 a_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
 &= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2) \\
 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))
 \end{aligned}$$

**Comments:**

- we will later define the function  $e^z, z \in \mathbb{C}$   
 We will show that  $e^{z_1} e^{z_2} = e^{z_1 + z_2}$   
 And also  $e^{i\theta} = \cos\theta + i\sin\theta$   
 Then  $a_1 a_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

### §1.7.2 The binomial equation

$$\begin{aligned}
 (a + b)^n &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\
 &= \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} a^{n-k} b^k
 \end{aligned}$$

Note that if  $a = r(\cos\theta + i\sin\theta)$

Then for  $n \geq 0$  and we want to check it is true for  $n = -1$  and  $n \in \mathbb{Z}$

$$\begin{aligned} a^n &= r^n(\cos n\theta + i\sin n\theta) \\ a^{-1} &= \frac{1}{r(\cos\theta + i\sin\theta)} \\ &= r^{-1} \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \\ &= r^{-1}(\cos\theta - i\sin\theta) \\ &= r^{-1}(\cos(-\theta) + i\sin(-\theta)) \end{aligned}$$

**Lemma 1.7.2: de Moivre**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

### §1.7.3 Roots of complex numbers

$$\begin{aligned} -1 &= 1(\cos\pi + i\sin\pi) \\ 1 &= (-1)^2 = (\cos\pi + i\sin\pi)^2 = \cos 2\pi + i\sin 2\pi \end{aligned}$$

**Cube roots of 1:**

$$\begin{aligned} z^3 - 1 &= 0 \\ z^3 - 1 &= (z - 1)(z^2 + z + 1) = 0 \\ \Delta &= 1 - 4 = -3 < 0 \\ z_1 &= \cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3} \\ z_2 &= \cos \frac{4\pi}{3} + i\sin \frac{4\pi}{3} \end{aligned}$$

**Lemma 1.7.3**

Let  $a \in \mathbb{C}$  and  $n \geq 1$ , Then the equation

$$z^n = a, a = r(\cos\theta + i\sin\theta)$$

has only  $n$  distinct solutions. (called the  $n$ -th roots of  $a$ ) given by

$$a = r^{\frac{1}{n}} \left( \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right), k = 0, 1, \dots, n-1$$

**Proof.**

$$z = r^{\frac{1}{n}} \left( \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right), \text{ where } e^n = r$$

■

## §1.8 Lecture 6 (02-19) – Riemann sphere

### Definition 1.8.1: stereographic projection

We will define a mapping bwtween the Riemann sphere and the extended complex plane  $\mathbb{C} \cup \{\infty\}$  called the stereographic projection.

Riemann sphere has radius 1

$$\begin{aligned} z &= a(x_1 + ix_2), a \in \mathbb{R} \\ &= \frac{x_1 + ix_2}{1 - x_3} \end{aligned}$$

### Definition 1.8.2: stereographic projection

we define the stereographic projection as the mapping

$$S : S - \{(0, 0, 1)\} \rightarrow \mathbb{C}$$

and

$$S(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

**Comments:** We can extend the domain of the stereographic projection to all of  $S$  if we add a point " $\infty$ " to  $\mathbb{C}$  and define  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  extended complex plane,

$$S(0, 0, 1) = \infty$$

- $S : S \rightarrow \bar{\mathbb{C}}$  is a bijection
- It is possible to define a topology ( this is the collection of open sets) on  $\bar{\mathbb{C}}$  s.t. its restriction to  $\mathbb{C}$  coincides with the open sets in  $\mathbb{C}$

$$\begin{aligned} z &= \frac{x_1 + ix_2}{1 - x_3} \\ |z|^2 &= \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3} \\ &\rightarrow (1 - x_3)|z|^2 = 1 + x_3 \\ &\rightarrow x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \\ z + \bar{z} &= 2\frac{x_1}{1 - x_3} \\ x_1 &= \frac{(z + \bar{z})(1 - x_3)}{2} \\ x_1 &= \frac{z + \bar{z}}{|z|^2 + 1} \\ x_2 &= \frac{z - \bar{z}}{i(|z|^2 + 1)} \end{aligned}$$

Consider a circle on S:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$$

show that this is the general equation of a circle on the Riemann sphere.

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

We want to find the locus of this circle in  $\mathbb{C}$  under S.

$$\Leftrightarrow \alpha_1(z + \bar{z}) - i\alpha_2(z - \bar{z}) + \alpha_3(|z|^2 - 1) = 2\alpha_0(|z|^2 + 1), z = x + iy$$

$$2\alpha_1 x + 2\alpha_2 y + \alpha_3(x^2 + y^2 - 1) = \alpha_0(x^2 + y^2 + 1)$$

$$(\alpha_3 - \alpha_0)(x^2 + y^2) + 2\alpha_1 x + 2\alpha_2 y = \alpha_0 + \alpha_3$$

For  $\alpha_3 \neq \alpha_0$ , this is the equation of a circle in  $\mathbb{C}$

For  $\alpha_3 = \alpha_0$ , this is the equation of a line in  $\mathbb{C}$

**Comments:**

Formula for the distance between two points  $z, z' \in \mathbb{C}$  in S:

$$d(z, z') = \|S^{-1}(z) - S^{-1}(z')\|_2 = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

This notion of distance is a metric which is equivalent to the metric in  $\mathbb{C}$  included by the Euclidean norm.

## §1.9 Lecture 7 (02-24) – Complex function

Analysis of complex functions: differentiable, etc  $R \rightarrow R, R \rightarrow C, C \rightarrow R, C \rightarrow C$

$C \rightarrow C$ : continuity, differentiability  $\Leftrightarrow$  analyticity

twice differentiability,  $\infty$  differentiability ( $C^\infty(\mathbb{R})$ )

### Definition 1.9.1: analyticity

$$f(x + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

### Definition 1.9.2: continuity

Consider a function  $f: C \rightarrow C$ , then we say  $f$  has a limit  $L$  when  $x$  tends to  $a$ , written as

$$\lim_{z \rightarrow a} f(z) = L$$

if  $\forall \epsilon > 0, \exists \delta$  s.t.  $|f(z) - L| < \epsilon$  whenever  $|z - a| < \delta$

we say that  $f$  is continuous at  $a$  if  $\lim_{z \rightarrow a} f(z) = f(a)$

we say that  $f$  is continuous if it is continuous at every point where it is defined.

**Comments:** This definition of continuity is the standard definition of continuity in a topology space. In this case, the topological space is

$$(C, T) \rightarrow (C, T) \Leftrightarrow f \text{ is continuous iff } f^{-1}(0) \text{ is open in } 0 \in T \text{ open}$$

$T$  is the subset of  $C$  which are open sets: a union of open balls  $B(z, r) = \{z' \in C, |z - z'| < r\}$

## §1.9.1 analiticity

**Definition 1.9.3: differentiability and analiticity**

Given  $f : C \rightarrow C$  we say that  $f$  is differentiable at  $a \in C$  if the limit exists:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = f'(a)$$

We say that  $f$  is analytic if it is differentiable at every point where it is defined.

**Comments:** In general, we will assume that the set of point where  $f : C \rightarrow C$  is defined is an open set ( $O = \bigcup B(x, \epsilon_x)$ )

In particular, this is being assumed in the definition given above.

If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

$$\begin{aligned} f(z+h) - f(z) &= \frac{f(z+h) - f(z)}{h} h \\ \rightarrow |f(z+h) - f(z)| &\leq \underbrace{\left| \frac{f(z+h) - f(z)}{h} \right|}_{|f'(z)|} \underbrace{||h||}_{\rightarrow 0} \end{aligned}$$

Of course  $f'(z)$  when it exists is called the derivative of  $f$  at  $z$ .

$f : C \rightarrow C$   $f$  is differentiable at  $z = x + iy$

$$f(z) = u(x, y) + iv(x, y)$$

Then

$$\begin{aligned} f'(z) &= \lim_{h \in \mathbb{R} \rightarrow 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \rightarrow 0} \frac{v(x+h, y) - v(x, y)}{h} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

On the other hand,

$$\begin{aligned} f'(z) &= \lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} \\ &= \lim_{k \rightarrow 0} \frac{u(x, y+k) + iv(x, y+k) - u(x, y) - iv(x, y)}{ik} \\ &= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

**Lemma 1.9.1: Cauchy-Riemann equations**

Assume that  $f : C \rightarrow C$  is differentiable at  $z=x+iy$  with  $f=u+iv$ , Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Comments:**

- Assume that the existence of  $f'(z)$  lets me differentiate again  $u_x, u_y, v_x, v_y$ . Then,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$u$  and  $v$  should be harmonic.

- The existence of  $f'(z)$  implies the existence of  $u_x, u_y, v_x, v_y$ 
  - $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i \frac{\partial u}{\partial y}$
  - $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

## §1.10 Lecture 8 (02-26)–polynomial

### Definition 1.10.1: harmonic function

We say that a function  $u : R^d \rightarrow R, d \geq 1$  is harmonic if it satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

**Comments:** Weyl's Lemma(PDE):

If  $u$  is a function that is weakly harmonic,

$$\int u \Delta \phi dx = 0, \forall \phi \text{ smooth}$$

Then  $u$  is pointwise harmonic and infinitely differentiable.

### Lemma 1.10.1

Assume that  $u$  and  $v$  are functions which have continuous first order partial derivatives, and that satisfy the Cauchy-Riemann equations  
Then  $f = u + iv$  is analytic, with a continuous derivative  $f'(z)$

**Proof:** The fact that  $u$  and  $v$  have continuous partial derivative implies that  $f = (u, v)$  is differentiable

$$\frac{f(x+h, y+k) - f(x, y)}{h+ik} = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} ik + \mathcal{E}(h, k) \text{ (goes to 0 faster than } h+ik)$$

### §1.10.1 polynomials

The polynomial functions will give us some framework which will enable us to develop some efficient integration techniques.

Two concepts will be important: singularities and zeros of functions.

$a$  is a zero of  $f$  if  $f(a) = 0$

$a$  is a singularity of  $f$  if  $f$  is not defined at  $a$ . **Comments:**

- The constant function  $f : C \rightarrow C$   $f(z) = \text{constant}$  is analytic
- The function  $f(z) = z$  is analytic

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = 1$$

- $f(z) = z^2$  and  $f'(z) = \lim_{h \rightarrow 0} \frac{(z+h)^2 - z^2}{h} = 2z$
- Similarly, using the binomial theorem, you can show that  $(z^n)' = nz^{n-1}$
- You can check that the product or the sum of analytic functions is analytic

#### Lemma 1.10.2

Every polynomial  $P : C \rightarrow C$ ,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, a_n \neq 0$$

is analytic.

Futhermore,

$$P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$$

**Comments:** We will prove later the fundamental theorem of algebra which states that every complex polynomial has at least one root in  $C$ .

Then given the polynomial  $P(z)$  of degree  $n$  (which means that  $P(z) = a_0 + a_1 z + \cdots + a_n z^n, a_n \neq 0$ ) can be written as

$$P(z) = (z - \alpha_1)Q(z), \text{ where } Q \text{ is a polynomial of degree } n-1$$

Repeating this argument for  $Q$ , and so on,

$$P(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

where  $\alpha_1, \dots, \alpha_n$  are the roots of  $P$ , which are not necessarily distinct.

$$P(z) = (z - \alpha_1)^{h_1} (z - \alpha_2)^{h_2} \cdots (z - \alpha_k)^{h_k}, \sum_{i=1}^k h_i = n, 1 \leq k \leq n$$

#### Definition 1.10.2

Given a polynomial  $P$  written in that form with distinct roots  $\alpha_1, \dots, \alpha_k, 1 \leq k \leq n$

we call  $h_i, 1 \leq i \leq k$  the order of the zero  $\alpha_i$ , when  $h_i = 1$  we will say that  $\alpha_i$  is simple.

**Theorem 1.10.1: Luca's theorem**

If all the zeros of a polynomial  $P$  lies in a half-plane  $H$ , then all the zeros of  $P'(z)$  lies in the same half plane.

**§1.11 Lecture 9 (03-02)**

**Proof.**

$$\begin{aligned}
 P(z) &= (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_k) \\
 &= \prod_{i=1}^k (z - \alpha_i) \\
 P'(z) &= \sum_{i=1}^k \prod_{j \neq i} (z - \alpha_j) \\
 \frac{P'(z)}{P(z)} &= \sum_{i=1}^k \frac{1}{z - \alpha_i}
 \end{aligned}$$

■

**Fact:** any half Plane  $H \in \mathbb{C}$  is determined by two complex numbers  $a$  and  $b$  so that

$$z \in H \text{ iff } \operatorname{Im}\left(\frac{z-a}{b}\right) < 0$$

Hint: use the fact that  $\forall a, b \in \mathbb{C}, bt + a, t \in \mathbb{R}$  is a line in  $\mathbb{C}$ .

Assume that  $a_k \in H$  and  $z \notin H$  then

$$\begin{aligned}
 \operatorname{Im}\left(\frac{a_k - a}{b}\right) &< 0 \text{ and } \operatorname{Im}\left(\frac{z - a}{b}\right) \geq 0 \\
 \Rightarrow \operatorname{Im}\left(\frac{b}{z - a_k}\right) &< 0 \\
 \Rightarrow \operatorname{Im}\left(\frac{z - a_k}{b}\right) &= \operatorname{Im}\left(\frac{z - a}{b} + \frac{a - a_k}{b}\right) > 0 \\
 \operatorname{Im}\left(b \frac{P'(z)}{P(z)}\right) &= \sum_{k=1}^n \operatorname{Im}\left(\frac{b}{z - a_k}\right) < 0 \text{ whenever } z \notin H \\
 &\Rightarrow P'(z) \neq 0
 \end{aligned}$$

2.4 Rational functions

2.5 Power series

2.6 Exponential, trigonometric function and logarithm



### §1.11.1 2.4 Rational functions

We will begin the discussion about how to classify singularities:

$f$  analytic

singularities: poles(isolated), essential

$$P = (z - a)^h Q$$

$$f(z) \approx \frac{g(z)}{(z - a)^h}$$

#### Definition 1.11.1: rational function

A rational function  $R$  is a function of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where  $P$  and  $Q$  are polynomials which do not have common factors.

Any zeros of  $Q$  are called poles of  $R$ .

We say that a pole of  $R$  has order  $h$  if the corresponding zero of  $Q$  has order  $h$ .

#### Comments:

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}$$

$$= \frac{P'(z)}{Q(z)} - P(z) \frac{Q'(z)}{Q(z)^2}$$

Suppose that  $\alpha$  is a pole of  $R$  of order  $h$ : then

$$Q(z) = (z - \alpha)^h Q_1(z)$$

$$\Rightarrow Q'(z) = (z - \alpha)^{h-1} Q_2(z), Q_1(\alpha), Q_2(\alpha) \neq 0$$

$$R'(z) = \frac{P'(z)}{Q(z)} - P(z) \frac{Q'(z)}{Q(z)^2}$$

$$\frac{P'(z)}{Q(z)} \text{ has a pole } \alpha \text{ of order } h \text{ (} P'(z) \neq 0 \text{)}$$

#### Lemma 1.11.1

If  $R$  has a pole  $\alpha$  of order  $h$ ,  $R'$  has a pole  $\alpha$  of order  $h+1$

#### Definition 1.11.2: pole or zero at infinity

Let  $R$  be a rational function, Consider the rational function

$$R_1(z) := R\left(\frac{1}{z}\right)$$

We say  $R$  has a zero at infinity of order  $h$  if  $R_1$  has a zero at  $0$  of order  $h$ .

Similarly, we say that  $R$  has a pole at infinity of order  $h$  if  $R_1$  has a pole at  $0$  of order  $h$ .

**Comments:**  $R$  is not defined at infinity. Nevertheless if

$$\lim_{z \rightarrow \infty} R(z) = R(\infty)$$

exists, call it  $R(\infty)$ , I can extend the definition of  $R$  to  $\bar{\mathbb{C}}$

In this case, we could say that  $\infty$  is a removable singularity

If  $R(\infty) = 0$ , the  $\infty$  is a removable singularity which is a zero

## Chapter 2

# Honors Theory of Probability

## §2.1 Lecture 1 (02-03) Introduction to Probability

Office Hour: Wed 12:30-1:30pm, Fri 3:30-4:30pm W910

- Homework: weekly
- Grades: 5% participation, 15% homework, 40% midterm, final group project (Max 4 people a group, presentation 20%, Final report 20% 10 pages)

### §2.1.1 Intro

Why do we need modern theory of probability?

#### Example 1

- Coin flip 7 times, what is  $P[\text{first outcome and fifth outcome are Head}]$ ?  
sol:

$$\Omega = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

$$A = (x_1, x_2, x_3, x_4, x_5, x_6, x_7), x_1 = x_5 = H$$

$$P(A) = \frac{|\Omega|}{|A|} = \frac{1}{4}$$

- Stock Price (mathematically, geometric Brownian motion, Stochastic process, such that  $t \rightarrow S_t$  is continuous but nowhere differentiable)  
What is  $P[S_T \geq 100]$ ?

$$\Omega = C[0, T] = \{\text{continuous function}, f[0, T] \rightarrow R\}$$

$$A = \{f \in C[0, T] | f(t) \geq 100\}$$

requires measure theory that defines  $P[S_T \geq 100]$

In modern Prob, Probability Space:  $(\Omega, F, P)$ ,  $\Omega$  is sample space,  $F$  is  $\sigma$ -algebra (meaningful subset of  $\sigma$ ),  $P$  is probability measure ( $P : F \rightarrow [0, 1]$ ).

Topics covered:

- Probability space,  $\sigma$ -algebra, measure, Conditional Probability and Independence
- Random variables (measurable functions), distribution
- Expectation (Lebesgue integral), Conditional distribution and expectation, functions of random variables, Radon-Nikodym Derivative
- Random walks

- generating functions, characteristic functions
- Branching processes
- Convergence of random variables, Law of large numbers, Monte-Carlo Method
- Central Limit Theorem
- Time permitting: Large deviations, Markov Chains

### §2.1.2 Probability Space

$(\Omega, F, P)$

#### Example 2: Coinflip and Stock Price

- Coin flip infinite times,

$$\Omega = \{(x_1, x_2, \dots), x_i = H, T\}$$

Let  $A$  be the event of  $10^6$  consecutive Tails,

$$A = \bigcup_{i=1}^{+\infty} \{x_i = x_{i+1} = \dots = x_{i+10^6-1} = 0, (x_i \in \Omega)\}$$

$$P[A] = 1$$

- Discrete Stock Price model,  $t=0,1,2,\dots,T$   
 $T$  is maturing time, time step  $\Delta t \ll T$

$$N = \frac{T}{\Delta t}$$

$$\text{price go } \begin{cases} \nearrow \text{ by factor : } e^{\sigma\sqrt{\Delta t}} \\ \searrow \text{ by factor : } e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$\Omega = \{(x_1, x_2, \dots, x_N), x_i = 0 \text{ or } 1\}$$

Stock price at time  $t$ :  $\forall \omega \in \Omega$

$$S_N(\omega) = S_0 e^{\sum_{i=1}^N x_i \sigma \sqrt{\Delta t} e^{(N - \sum_{i=1}^N x_i)(-\sigma\sqrt{\Delta t})}}$$

$$S : \Omega \rightarrow R(\text{Random Variable})$$

Event: return at  $T$  is positive but not more than 10%:

$$\{\omega \in \Omega : S_N(\omega) \in (S_0, 1.1S_0]\}$$

#### Example 3: Gambling

- Gambling:

- start with  $\{0,1,2,\dots\}$  each time bet an integer amount
  - if amount of money =0, stays at 0
- wealth process:  $\Omega = \{0,1,2,\dots\} \times \{0,1,2,\dots\} \times \dots$   
wealth after time n: (random variable)  $X_n : \Omega \rightarrow N, (x_1, x_2, \dots, x_n) \rightarrow x_n$   
 $(X_n)$  is a Markov Chain (future only depends on present state, but not the past)

$$\begin{aligned} \text{Event: } \{ \text{State } j \text{ is reached from state } i \} \\ &= \{ \omega \in \Omega : \exists n \in N, X_0(\omega) = i, X_n(\omega) = j \} \\ &= \bigcup_{m=1}^{+\infty} \{ \omega \in \Omega : X_0(\omega) = i, X_m(\omega) = j \} \end{aligned}$$

## §2.2 Lecture 2 (02-05) – Algebra

### §2.2.1 algebra and $\sigma$ -algebra

in practice, want  $F$  to be closed under  $\bigcap, \bigcup, ^c$

#### Definition 2.2.1

Let  $\mathcal{A}$  be a collection of subsets of  $\Omega$ ,  $\mathcal{A}$  is an algebra iff:

- $\Omega \in \mathcal{A}$
- if  $A, B \in \mathcal{A}$  then  $A \cup B \in \mathcal{A}$
- if  $A \in \mathcal{A}$  then  $A^c \in \mathcal{A}$

#### Remark 1

if  $A, B \in \mathcal{A} \rightarrow A \cap B \in \mathcal{A}$ , because  $A \cap B = (A^c \cup B^c)^c$

**Fact:**

- ①  $P(\Omega)$ (powerset) =  $\{A : A \subset \Omega\}$  is an algebra
- ② smallest algebra/trivial algebra:  $\{\emptyset, \Omega\}$
- ③ Let  $\mathcal{A}_1, \mathcal{A}_2$  be two algebras of  $\Omega$

$\mathcal{A}_1 \cap \mathcal{A}_2 = \{B \in \Omega : B \in \mathcal{A}_1 \text{ and } B \in \mathcal{A}_2\}$  is an algebra

if  $(\mathcal{A}_j)_{j \in J}$  is a family of algebras, then  $\bigcap_{j \in J} \mathcal{A}_j$  is an algebra

- ④ Let  $\mathcal{E}$  be any collection of subsets of  $\Omega$

$$a(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{E} \\ \mathcal{A} \text{ is an algebra}}} \mathcal{A} \text{ is an algebra}$$

$a(\mathcal{E})$  is an algebra by ③

in fact,  $a(\mathcal{E})$  is the smallest algebra containing  $\mathcal{E}$

It is called the algebra generated by  $\mathcal{E}$

- ⑤ Let  $A \subseteq \Omega, \mathcal{E} = \{A\}$   
Then

$$a(\mathcal{E}) = \underbrace{\{A, A^c, \Omega, \emptyset\}}_f$$

**Proof:**

- $a(\mathcal{E}) \subseteq f$   
notice that  $f$  is an algebra since  $f \supseteq \mathcal{E}$ , therefore  $f \supseteq a(\mathcal{E})$  because  $a(\mathcal{E})$  is the smallest algebra containing  $\mathcal{E}$ .
- $f \subseteq a(\mathcal{E})$ :

$$A \in a(\mathcal{E}), A^c \in a(\mathcal{E})$$

because  $a(\mathcal{E})$  is an algebra,  $\emptyset, \Omega \in a(\mathcal{E})$

- ⑥  $\pi = \{A_1, A_2, \dots, A_n\}, \Omega = \bigcup_{i=1}^n A_i, A_i \cap A_j = \emptyset$  Then:

$$a(\pi) = \left\{ \bigcup_{i \in I} A_i, \text{ for } I \subset 1, 2, \dots, n \right\} = \text{finite disjoint union of } (A_i)_{i=1}^n$$

- ⑦ Let  $\mathcal{A}$  be an algebra of  $\mathbb{R}$   
Let  $X : \Omega \rightarrow \mathbb{R}$  be a function  
Then

$$\underbrace{\{X^{-1}(A), A \in \mathcal{A}\}}_{\{\omega \in \Omega : X(\omega) \in A, \text{ for some } A \in \mathcal{A}\}} \text{ is an algebra of } \Omega$$

**Hint:**

$$X^{-1}(A \cup B) = (X^{-1}(A)) \cup (X^{-1}(B))$$

- ⑧  $\Omega = \mathbb{R}, \mathcal{E} = \{\text{left open right closed intervals}\} = \begin{cases} (a, b], & -\infty \leq a < b < +\infty \\ (a, +\infty) \end{cases}$

Then:

$$\begin{aligned} a(\mathcal{E}) &= \text{"finite disjoint union of elements in } \mathcal{E}\text{"} \\ &= \underbrace{\{I_1 \cup \dots \cup I_k; I_j \in \mathcal{E}, I_i \cap I_j = \emptyset\}}_f \end{aligned}$$

**Hint:**

- $f \subseteq a(\mathcal{E})$  straightforward
- $a(\mathcal{E}) \subseteq f$ 
  - check  $f$  is an algebra
  - since  $\mathcal{E} \subset f$ ,  $a(\mathcal{E})$  is the smallest algebra containing  $\mathcal{E}$ ,  $a(\mathcal{E}) \subseteq f$
  - $(a, b]^c = \underbrace{(-\infty, a] \cup (b, +\infty)}_{\in f}$

### Lemma 2.2.1

In Probability, if  $(A_k)_{k=1}^n$  are disjoint events, we have  $P(\bigcup_{k=1}^{+\infty} A_k) = \sum_{k=1}^{+\infty} P(A_k)$

**Definition 2.2.2:**  $\sigma$  – algebra

A  $\sigma$  – algebra ( $\sigma$  – field)  $\mathcal{A}$  is an collection of subsets of  $\Omega$  such that

- $\Omega \in \mathcal{A}$
- if  $A_1, A_2, \dots \in \mathcal{A}$  then  $\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$
- if  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$

**Remark 2**

if  $A_1, A_2, \dots \in \mathcal{A}$  then  $\bigcap_{i=1}^{+\infty} A_i \in \mathcal{A}$

$\sigma$  – algebra represents the collection of information determined by partial derivatives

**Example 1**

- coinflips infinity many times

$$\Omega = \{(x_1, x_2, \dots) | x_i = 0, 1\} = (0, 1)^\infty$$

After observing first outcome,  $f_1 = \{\emptyset, \Omega, A_0, A_1\}$

Where  $A_0 = \{(0, x_2, x_3, \dots), x_i = 0, 1\}$ ,  $A_1 = \{(1, x_2, x_3, \dots), x_i = 0, 1\}$

After observing second outcome,  $f_2 = \{\emptyset, \Omega, A_0, A_1, A_{00}, A_{01}, A_{10}, A_{11}\}$

Where  $A_{00} = \{(0, 0, x_3, x_4, \dots), x_i = 0, 1\}$ ,  $A_{01} = \{(0, 1, x_3, x_4, \dots), x_i = 0, 1\} \dots$

After observing first n outcomes,

$$f_n = \{\emptyset, \Omega, (A_j)_{j \in (\sigma 1)^n} \text{ and finite disjoint unions}\}$$

and

$$f_1 \subseteq f_2 \subseteq \dots \subseteq f_n \subseteq \dots$$

**Proposition 2.2.0**

- ① intersection of  $\sigma$  – algebras is a  $\sigma$  – algebra
- ② Let  $\mathcal{E}$  be a collection of subsets of  $\Omega$

$$\sigma(\mathcal{E}) = \bigcap_{\substack{f \supseteq \mathcal{E} \\ f \text{ is an algebra}}} f \text{ is a smallest } \sigma \text{ – algebra that contains } \mathcal{E}$$

- ③ For any collection  $\mathcal{E}$ , we have  $a(\mathcal{E}) \subseteq \sigma(\mathcal{E})$

- ④ For any collection  $\mathcal{E}$ , we have  $\sigma(a(\mathcal{E})) = \sigma(\mathcal{E})$

**Hint:**

$$a(\mathcal{E}) \subseteq \sigma(\mathcal{E}) \rightarrow \sigma(a(\mathcal{E})) \subseteq \sigma(\mathcal{E})$$

$$\mathcal{E} \subseteq \sigma(a(\mathcal{E})) \rightarrow \sigma(\mathcal{E}) \subseteq \sigma(a(\mathcal{E}))$$

## §2.3 Recitation 1 (02-07) – Problem Solving

### §2.3.1 Basic Set Theory

**Definition 2.3.1:**  $\cup, \cap, ^c$

**De Morgan's Law:**

$$\left(\bigcup_{j \in J} A_j\right)^c = \bigcap_{j \in J} A_j^c$$

$$\left(\bigcap_{j \in J} A_j\right)^c = \bigcup_{j \in J} A_j^c$$

**Definition 2.3.2:**  $\setminus$

$$A \setminus B = A \cap B^c$$

$$\text{Then } A \setminus B = A \cap B^c = A \setminus (A \cap B) = B^c \setminus A^c$$

**Remark 1**

$$\bigcap_{n \geq 1} A_n = A_1 \setminus \left(\bigcup_{j \geq 2} (A_1 - A_j)\right) \text{ (ex)}$$

### §2.3.2 Limits of Sets

**Definition 2.3.3: Limit Sets**

Let  $(A_n)_{n \geq 1}$  be a sequence of sets, then

$$B_k = \bigcup_{n \geq k} A_n, C_k = \bigcap_{n \geq k} A_n$$

Then  $B_k$  is increasing,  $C_k$  is decreasing

Define:

$$\limsup_{n \rightarrow +\infty} A_n = \lim_{k \rightarrow +\infty} B_k = \bigcap_{k \geq 1} \bigcup_{n \geq k} A_n$$

$$\liminf_{n \rightarrow +\infty} A_n = \lim_{k \rightarrow +\infty} C_k = \bigcup_{k \geq 1} \bigcap_{n \geq k} A_n$$



**Definition 2.3.4: liminf, limsup of sequence**

$$\limsup a_n = \inf_{k \geq 1} \sup_{n \geq k} a_n, \liminf a_n = \sup_{k \geq 1} \inf_{n \geq k} a_n$$

When  $\limsup A_n = \liminf A_n$ , Then we say  $\lim_{n \rightarrow +\infty} A_n$  exist and  $\lim_{n \rightarrow +\infty} A_n = \limsup A_n = \liminf A_n$

In probability,

$$\begin{aligned} \limsup A_n &= \{A_n \text{ occurs infinitely often}\} \\ &= \{A_n.i.o\} \\ x \in \limsup A_n &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k \text{ such that } x \in A_n \\ &\Leftrightarrow A_n.i.o \\ \liminf A_n &= \{A_n \text{ occurs eventually}\} \\ &\Leftrightarrow \exists k \in \mathbb{N}, \forall n \geq k, x \in A_n \end{aligned}$$

**Remark 2**

1. if  $A_n$  increases, then  $\limsup A_n = \lim_{n \rightarrow \infty} A_n = \bigcup_{n \geq 1} A_n$   
if  $A_n$  decreases, then  $\liminf A_n = \lim_{n \rightarrow \infty} A_n = \bigcap_{n \geq 1} A_n$
2.  $(\limsup A_n)^c = \liminf A_n, (\liminf A_n)^c = \limsup A_n$

**§2.3.3 Exercise**

1.  $A_k = \begin{cases} E, & \text{if } k \text{ is odd} \\ F, & \text{if } k \text{ is even} \end{cases}$   
Then  $\limsup A_n = E \cup F, \liminf A_n = E \cap F$
2. Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ , Let  $A = \{x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) = f(x)\}$  Then,

$$\begin{aligned} A^c &= \bigcup_{k=1}^{+\infty} \bigcap_{m=1}^{+\infty} \bigcup_{n=m}^{+\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\} \\ &= \bigcup_{k=1}^{+\infty} [\limsup_n \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}] \end{aligned}$$

3. Suppose that  $\lim_{n \rightarrow \infty} f_n(x) = f(x), \forall x \in \mathbb{R}$   
Then

$$\begin{aligned} \{x : f(x) \leq t\} &= \bigcap_{k=1}^{+\infty} \bigcup_{m=1}^{+\infty} \bigcup_{n=m}^{+\infty} \{x \in \mathbb{R} : f_n(x) < t + \frac{1}{k}\} \\ &= \bigcap_{k=1}^{+\infty} [\liminf_n \{x : f_n(x) < t + \frac{1}{k}\}] \end{aligned}$$

**Proof:**

1. ex2: Want

$$A = \bigcup_{k \geq 1} \bigcap_{n \geq 1} \bigcup_{m \geq n} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}$$

$$f_n(x) \rightarrow f(x) \text{ iff } \forall \epsilon > 0, \exists N \geq N, |f_n(x) - f(x)| < \epsilon$$

$$\{x : f_n(x) \rightarrow f(x)\} = \bigcap_{\epsilon > 0} \bigcup_N \bigcap_{n \geq N} \{x : |f_n(x) - f(x)| < \epsilon\},$$

USE that  $\{|f_n - f| < \epsilon\}$  is monotone increasing in  $n$

**Review on mapping:**

$$f : X \rightarrow Y, f^{-1} : Y \rightarrow X$$

**Basic properties:**

- $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$
- $f(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} f(A_i)$

For  $(B_i)$  subset of  $Y$ :

- if  $B_1 \subset B_2, f^{-1}(B_1) \subset f^{-1}(B_2)$
- $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$
- $f^{-1}(B^c) = (f^{-1}(B))^c$

#### Definition 2.3.5: Indicator Mapping

$$1_A : X \rightarrow \{0, 1\}$$

$$1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

**Exercise:**

1.  $1_{\limsup A_n} = \limsup 1_{A_n}$
2.  $1_{\liminf A_n} = \liminf 1_{A_n}$

**Proof:**

if

$$\begin{aligned} 1_{\limsup A_n}(x) = 1 &\Leftrightarrow x \in \limsup A_n \\ &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k, x \in A_n \\ &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k, 1_{A_n}(x) = 1 \\ &\Leftrightarrow \limsup 1_{A_n}(x) \geq 1 \text{ (Definition of limsup)} \\ &\Leftrightarrow 1_{\limsup A_n}(x) = 1 \end{aligned}$$

**Exercise:** Let  $\mathcal{A}_1, \mathcal{A}_2$  be algebras of  $\Omega$

1. show that  $\underbrace{\mathcal{A}_1 \cap \mathcal{A}_2}_{B \subset \Omega: B \in \mathcal{A}_1, B \in \mathcal{A}_2}$  is an algebra

2. show that  $\underbrace{\mathcal{A}_1 \cup \mathcal{A}_2}_{B \subset \Omega: B \in \mathcal{A}_1 \text{ or } B \in \mathcal{A}_2}$  is an algebra iff  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  or  $\mathcal{A}_2 \subseteq \mathcal{A}_1$

**Proof:** Suppose by contradiction that

$$\exists A_1 \in \mathcal{A}_1 \text{ but } A_1 \notin \mathcal{A}_2 \text{ and } A_2 \in \mathcal{A}_2 \text{ but } A_2 \notin \mathcal{A}_1 \text{ and } \mathcal{A}_1 \cup \mathcal{A}_2 \text{ is an algebra}$$

Therefore:

$$A_1 \cup A_2 \in \mathcal{A}_1 \cup \mathcal{A}_2, A_1 \setminus A_2 = A_1 \cap \underbrace{A_2^c}_{\mathcal{A}_1 \cup \mathcal{A}_2} \in \mathcal{A}_1 \cup \mathcal{A}_2$$

$$A_2 \setminus A_1 = A_2 \cap A_1^c \in \mathcal{A}_1 \cup \mathcal{A}_2$$

$\Rightarrow$  at least two of

$$(A_1 \setminus A_2) \cup (A_2 \setminus A_1), A_1 \setminus A_2, A_2 \setminus A_1 \text{ are in } \mathcal{A}_1 \text{ or } \mathcal{A}_2$$

**Assume in  $\mathcal{A}_1$**

$\Rightarrow$  (by  $\mathcal{A}_1$  is an algebra, all three sets are in  $\mathcal{A}_1$ )

$$\Rightarrow A_2 = \underbrace{(A_1 \cup A_2)}_{\in \mathcal{A}_1} \setminus \underbrace{(A_1 \setminus A_2)}_{\in \mathcal{A}_1} \in \mathcal{A}_1$$

**Contradiction!**

## §2.4 Lecture 3 (02-10)–(Content and Measure)

Recall  $\sigma$ -algebra:

### Example 1

- ① We know  $\Omega = \mathbb{R}$ ,

$$\mathcal{E} = \{\text{left open right closed intervals}\} = \begin{cases} (a, b], & -\infty \leq a < b < +\infty \\ (a, +\infty), & a \in \mathbb{R} \end{cases}$$

Then we know:

$$a(\mathcal{E}) = \{\text{"finite disjoint union of elements in } \mathcal{E}\}$$

What is  $\sigma(a(\mathcal{E}))$ ?

$\sigma(\epsilon) = \text{Borel Sets } \mathcal{B}(\mathbb{R})$

Any "reasonable" subset of  $\mathbb{R}$  is in  $\sigma(\epsilon)$

- $(a, b) \in \sigma(\epsilon) : (a, b) = \bigcup_{n \geq 1} \underbrace{(a, b - \frac{1}{n}]}_{\in \sigma(\epsilon)} \in \sigma(\epsilon)$
- any singleton  $\{a\} \in \sigma(\epsilon)$ , because  $\{a\} = \bigcap_{n \geq 1} \underbrace{(a - \frac{1}{n}, a + \frac{1}{n})}_{\in \sigma(\epsilon)} \in \sigma(\epsilon)$
- any countable set is in  $\sigma(\epsilon)$  (e.g.  $\mathbb{Q}$ )
- The set of transcendental numbers is in  $\sigma(\epsilon)$ , because the set of algebraic numbers is countable

**Definition 2.4.1: measurable set**

A pair  $(\Omega, F)$ , where  $F$  is a  $\sigma$ -algebra of  $\Omega$ , is called a measurable space  
Any set  $A \in F$  is called a measurable set

**§2.4.1 Content and Measure****Definition 2.4.2: Content**

Let  $\mathcal{A}$  be an algebra of  $\Omega$ , A set function  $\mu : \mathcal{A} \rightarrow [0, +\infty)$  is called a content iff:

- $\mu(\emptyset) = 0$
- if  $A, B \in \mathcal{A}$  and  $A \cap B = \emptyset$ , then  $\mu(A \cup B) = \mu(A) + \mu(B)$  (finite additivity)

**Lemma 2.4.1**

Let  $\mu : \mathcal{A} \rightarrow [0, +\infty)$  be a content,  $\forall A, B \in \mathcal{A}$  then:

- ①  $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$
- ② if  $A \subset B$ , and  $\mu(A) < +\infty$ , then  $\mu(B \setminus A) = \mu(B) - \mu(A)$
- ③ if  $A \subset B$ , then  $\mu(A) \leq \mu(B)$
- ④ if  $A_1, A_2, \dots \in \mathcal{A}$ , then  $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$
- ⑤ if  $A_1, A_2, \dots \in \mathcal{A}$ ,  $A_i \cap A_j = \emptyset$  and  $\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$  then  $\mu(\bigcup_{i=1}^{+\infty} A_i) \geq \sum_{i=1}^{+\infty} \mu(A_i)$

**Proof:**

finite add:

1:

$$\begin{aligned}\mu(B) &= \mu(A \cup B) + \mu(B \setminus A) \\ \mu(A) + \mu(B \setminus A) &= \mu(A \cup B) \\ \Rightarrow \mu(A) + \mu(B) &= \mu(A \cup B) + \mu(A \cap B)\end{aligned}$$

4:

Let  $B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus (A_1 \cup A_2), \dots$

Then  $B_j$  are disjoint and  $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$

By finite add for  $B_j$ :

$$\mu\left(\bigcup_{i=1}^n A_i\right) = \mu\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n (\mu(A_i) \setminus \bigcup_{j=1}^{i-1} A_j) \leq \sum_{i=1}^n \mu(A_i)$$

5:

For  $\forall n$ ,

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) \geq \mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i) \text{ send } n \nearrow +\infty \text{ to conclude}$$

**Remark 1**

In general.  $\mu(\bigcup_{i=1}^{+\infty} A_i) \neq \sum_{i=1}^{+\infty} \mu(A_i)$  although  $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$  require continuity of  $\mu$

**Counterexample:**

- Let  $\Omega = \mathbb{R}, \mathcal{A} = \mathcal{a}(\epsilon), \forall A \in \mathcal{A}$ ,

$$\mu(A) = \lim_{L \rightarrow +\infty} \frac{\overbrace{[A \cup [0, L]]}^{\text{Length of interval}}}{L} \text{ "Density of A in } [0, +\infty)\text{"}$$

Then  $\mu$  is a content.

Take  $A_i = (i, i+1]$ , then  $\mu((i, i+1]) = 0$ , But  $\mu(\bigcup_{i=0}^{+\infty} A_i) = \mu((0, +\infty)) = 1$

- For  $\mathbb{R}, \mathcal{a}(\epsilon)$ , Given  $A \in \mathcal{a}(\epsilon)$

$$\begin{aligned} \mathring{A} &= \{x \in A : \exists r_x > 0, s.t. (x - r_x, x + r_x) \subset A\} \\ \partial A &= \bar{A} \setminus \mathring{A} \end{aligned}$$

$$\mu(A) = \begin{cases} 2, & \text{if } 0 \in \mathring{A} \\ 1, & \text{if } 0 \in \partial A \\ 0, & \text{else} \end{cases}$$

Then  $\mu$  is a content

However,  $A_i = (\frac{1}{i+1}, \frac{1}{i}]$ ,  $\mu(A_i) = 0$ , But  $\mu(\bigcup_{i=1}^{+\infty} A_i) = \mu((0, 1]) = 1$

**Example:**

- (Discrete Probability)

$$\Omega = \{\omega_1, \dots, \omega_n\} \text{ finite set, } P = P(\Omega)$$

Set  $A_i = \{\omega_i\}, P(A_i) = P(\omega_i) = P_i$  such that  $\sum_{i=1}^n P_i = 1$

Then  $P : P(\Omega) \rightarrow [0, 1]$  defines a content on  $P(\Omega)$  by extending the P using finite additivity:

$$\forall A \in P(\Omega), P(A) = \sum_{\omega \in A} P(\omega)$$

- $\Omega = \mathbb{R}$ , algebra  $\mathcal{A} = \mathcal{a}(\epsilon) = \{ \text{finite disjoint union of elements in } \epsilon \}$   
Define  $m : \mathcal{a}(\epsilon) \rightarrow [0, +\infty)$ , set  $m([a, b]) = b-a$ ,  
and extend by additivity:

$$m(I) = \sum_{i=1}^n m(I_i), \text{ if } I = I_1 \cup \dots \cup I_n, I_j \cap I_i = \emptyset$$

$\Rightarrow m$  is a content on  $(\mathbb{R}, a(\epsilon))$

$m$  can be further extended to  $(\mathbb{R}, \sigma(\epsilon))$ , called Lebesgue Measure

**Definition 2.4.3: countably additive**

A content  $\mu : \mathcal{A} \rightarrow [0, +\infty)$  is countably additive if:

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i) \text{ for every disjoint } A_1, A_2, \dots \in \mathcal{A}$$

**Definition 2.4.4: measure**

Let  $(\Omega, F)$  be an measurable space, then a content  $\mu : F \rightarrow [0, +\infty)$  that is countably additive is called a measure

**Lemma 2.4.2**

$m : a(\epsilon) \rightarrow [0, +\infty)$  is countably additive

**Proof:**

Let  $A_1, A_2, \dots \in a(\epsilon)$ ,  $(A_k)$  disjoint,  $A := \bigcup_{i=1}^{+\infty} A_i \in a(\epsilon)$

Want to show :

$$m(A) = \sum_{i=1}^{+\infty} m(A_i)$$

we can write, using  $A, (A_k) \in a(\epsilon)$ ,  $A = \bigcup_{j=1}^n I_j$ , where  $I_j \in \epsilon$  and  $(I_j)$  disjoint

$A_j = \bigcup_{k=1}^{n_i} J_{ik}$ , where  $J_{ik} \in \epsilon, (J_{ik})$  disjoint

Then:

$$\begin{aligned} m(A) &= \sum_{j=1}^n m(I_j) \\ &= \sum_{j=1}^n \sum_{k=1}^{+\infty} \sum_{i=1}^{n_i} m(I_j \cap J_{ik}) \\ &= \sum_{i=1}^{+\infty} m\left(\underbrace{\bigcup_{j=1}^n I_j}_A \cap \underbrace{\left(\bigcup_{k=1}^{n_i} J_{ik}\right)}_{A_i}\right) \\ &= \sum_{i=1}^{+\infty} m(A_i) \end{aligned}$$

## §2.5 Lecture 4 (02-12)–Measure and Extension

$$\Omega = \mathbb{R}, m : \epsilon \rightarrow [0, +\infty), \text{ such that } \begin{cases} m([a, b]) = b - a, \\ m((a, +\infty)) = +\infty \end{cases}$$

extend  $m$  to  $a(\epsilon) : \forall A \in a(\epsilon),$

$$\text{if } A = \bigcup_{i=1}^n I_j, I_j \text{ disjoint, } m(A) = \sum_{j=1}^n m(I_j)$$

**Fact:**

if  $I \in \epsilon$  s.t.  $I = \bigcup_{i=1}^{+\infty} I_i, (I_i) \text{ disjoint and } I_j \in \epsilon$   
Then

$$\begin{aligned} m(I) &= |I| \\ &= \sum_{i=1}^{+\infty} |I_i| = \bigcup_{i=1}^{+\infty} m(I_i) \end{aligned}$$

### Lemma 2.5.1

we want to prove  $m$  is a countably additive content on  $(\mathbb{R}, a(\epsilon))$

$$\begin{aligned} \text{Let } A_j \in a(\epsilon), A = \bigcup_{j=1}^{\infty} A_j \in a(\epsilon), (A_j) \text{ disjoint} \\ \exists (I_i) \text{ such that } I_i \in \epsilon, \text{ disjoint } A = \bigcup_{i=1}^n I_i \\ \exists (J_{ij}) \text{ such that } J_{ij} \in \epsilon, \text{ disjoint } A_j = \bigcup_{k=1}^j J_{ik} \\ m(A) \underset{\text{def}}{=} \sum_{i=1}^n m(I_i) = \sum_{i=1}^n m\left(\bigcup_{j,k} \underbrace{(I_i) \cap (J_{ik})}_{\in \epsilon}\right) \\ \underset{\text{fact}}{=} \sum_{i=1}^n \sum_{j=1}^{+\infty} \sum_{k=1}^{n_j} m(I_i \cap J_{ik}) \\ \underset{\text{finite add}}{=} \sum_{j=1}^{+\infty} m(A_j) \end{aligned}$$

### Theorem 2.5.1

$m$  extends to a measure on  $(\mathbb{R}, \sigma(\epsilon) = \mathcal{B}(\mathbb{R}))$  It is the unique measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that

$$m([a, b]) = b - a$$

**Definition 2.5.1**

If  $\mu$  is a measure on  $(\Omega, F)$ , then  $(\Omega, F, \mu)$  is said to be a measure space  
 If  $\mu(\Omega) = 1$ , then  $(\Omega, F, \mu)$  is said to be a probability space

**Lemma 2.5.2**

Let  $(\Omega, F, \mu)$  be a measure space, then:

- ① stability: Let  $A_1, A_2, \dots \in F$ , then  $\mu(\bigcup_{i=1}^{+\infty} A_i) \leq \sum_{i=1}^{+\infty} \mu(A_i)$   
 ② continuity from below: Let  $A_1, A_2, \dots \in F$ ,  $A_1 \subseteq A_2 \subseteq \dots$ , then

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \lim_{i \rightarrow +\infty} \mu(A_i) = \mu\left(\lim_{i \rightarrow +\infty} A_i\right)$$

- ③ continuity from above: Let  $A_1, A_2, \dots \in F$ ,  $A_1 \supseteq A_2 \supseteq \dots$ ,  
 ( $\mu(A_i) < +\infty$  we need this for the Counterexample  $A_i = [i, +\infty)$ ) then

$$\mu\left(\bigcap_{i=1}^{+\infty} A_i\right) = \lim_{i \rightarrow +\infty} \mu(A_i) = \mu\left(\lim_{i \rightarrow +\infty} A_i\right)$$

**Proof:**

- ① Let  $B_1 = A_1, B_2 = A_2 \setminus A_1, \dots, B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ ,  
 Then  $(B_i)$  disjoint,  $\bigcup_{i=1}^{+\infty} A_i = \bigcup_{i=1}^{+\infty} B_i$

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{+\infty} B_i\right) = \sum_{i=1}^{+\infty} \mu(B_i) = \sum_{i=1}^{+\infty} \mu\left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right) \leq \sum_{i=1}^{+\infty} \mu(A_i)$$

- ② Let  $B_1 = A_1, B_2 = A_2 \setminus A_1, \dots, B_n = A_n \setminus A_{n-1} \dots$ ,

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{+\infty} A_i\right) &= \mu\left(\bigcup_{i=1}^{+\infty} B_i\right) = \sum_{i=1}^{+\infty} \mu(B_i) \\ &= \mu(A_1) + \sum_{i=2}^{+\infty} (\mu(A_i) - \mu(A_{i-1})) \\ &= \lim_{n \rightarrow +\infty} \left( \mu(A_1) + \sum_{i=2}^n (\mu(A_i) - \mu(A_{i-1})) \right) \\ &= \lim_{n \rightarrow +\infty} \mu(A_n) \end{aligned}$$

- ③

$$\mu(A_1) - \mu\left(\bigcap_{i=1}^{+\infty} A_i\right) = \mu\left(A_1 \setminus \bigcap_{i=1}^{+\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{+\infty} A_1 \setminus A_i\right)$$

$$\text{Since } A_1 \nearrow, \text{ by 2 : } \lim_{n \rightarrow +\infty} (\mu(A_1) - \mu(A_i)) = \mu(A_1) - \lim_{n \rightarrow +\infty} \mu(A_i)$$



**Definition 2.5.2:  $\sigma$ -finite measure**

Given a measure space  $(\Omega, \mathcal{F}, \mu)$ ,  $\mu$  is said to be finite if  $\mu(\Omega) < +\infty$   
 $\mu$  is  $\sigma$ -finite if there exists  $(E_i)_{i=1}^{+\infty}$  such that  $\bigcup_{i=1}^{+\infty} E_i \in \Omega$  and  $\mu(E_i) < +\infty$

**Example 1**

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$  is  $\sigma$ -finite

**Definition 2.5.3**

if  $F \in \mathcal{F}$  is such that  $\mu(F) = 0$ , then  $F$  is called a  $\mu$ -null set

**Example 2**

$m(\{a\}) = 0$  because

$$m(\{a\}) = m\left(\bigcap_{n \geq 1} \left(a - \frac{1}{n}, a\right]\right) = (\text{continuous from above}) \lim_{n \rightarrow +\infty} m\left(\left(a - \frac{1}{n}, a\right]\right) = 0$$

$$m(Q) = \sum_{q \in Q} m(\{q\}) = 0$$

**Recall:**

Start with  $(R, a(\epsilon), m)$   $m$  is content + countably additive

extention:  $(R, a(\epsilon), m) \rightarrow (R, \sigma(\epsilon), m)$   $m$  is measure

**Theorem 2.5.2: Caratheodory Extention Theorem**

Let  $\mathcal{F}$  be an algebra on  $\Omega$ ,  $\mu$  be a countably additive content on  $(\Omega, \mathcal{F})$ ,  
 If  $\mu$  is  $\sigma$ -finite, then  $\mu$  extends to a measure on  $(\Omega, \sigma(\mathcal{F}))$

**Example 3**

Let  $\epsilon = \begin{cases} (a, b], & -\infty \leq a < b < +\infty \\ (a, +\infty), & a \in \mathbb{R} \end{cases}$  Let  $m_F : a(\epsilon) \rightarrow [0, +\infty)$  be a content,  
 such that

$$m_F([a, b]) = F(b) - F(a), m_F((a, +\infty)) = F(+\infty) - F(a)$$

Where  $F$  is a right continuous increasing function on  $\mathbb{R}$ ,

$$F(\pm\infty) = \lim_{x \rightarrow \pm\infty} F(x)$$

Then  $m_F$  is a countably additive content on  $(\mathbb{R}, a(\epsilon))$

By the extension theorem,  $m_F$  extends to a measure on  $(\mathbb{R}, \sigma(a(\epsilon)) = \sigma(\epsilon) = \mathcal{B}(\mathbb{R}))$

( $F(x)=x$ ) gives Lebesgue)

#### Definition 2.5.4: Lebesgue-Stieltjes mesures

think of  $m_F(A) = \int_A \underbrace{dF(x)}_{\text{R-S integral}}$ ,  $F$  is the distribution function of the measure

### §2.5.1 $\pi$ and $\lambda$ system

#### Definition 2.5.5: $\pi$ and $\lambda$ system

Let  $C$  be a collection of sets of  $\Omega$   
 $C$  is a  $\pi$ -system if:

- $\emptyset \in C$
- $\forall A, B \in C, A \cap B \in C$

$C$  is a  $\lambda$ -system if:

- $\Omega \in C$
- if  $A, B \in C$ , and  $A \subseteq B$ , then  $B \setminus A \in C$
- if  $A_1, A_2, \dots \in C$ , and  $A_1 \subseteq A_2 \subseteq \dots$ , then  $\bigcup_{i=1}^{+\infty} A_i \in C$

#### Example 4

$\epsilon = \left\{ \begin{array}{l} (a, b] \\ (a, +\infty) \end{array} \right\}$  is a  $\pi$ -system

**Exercise:** if  $C$  is both a  $\pi$ -system and a  $\lambda$ -system, then it is a  $\sigma$ -algebra

#### Lemma 2.5.3: Dynkins Lemma

Let  $C$  be a  $\pi$ -system, then any  $\lambda$ -system containing  $C$  also contains the  $\sigma(C)$   
**Hint:** show that any such  $\lambda$ -system is also a  $\pi$ -system

#### Theorem 2.5.3: Uniqueness Theorem

Let  $C$  be a  $\pi$ -system, Let  $\mu_1, \mu_2$  be two finite measures on  $(\Omega, \sigma(C))$   
 Suppose that  $\mu_1(A) = \mu_2(A)$  and  $\mu_1(\Omega) = \mu_2(\Omega)$  on  $C$ , then  $\mu_1(A) = \mu_2(A)$  on  $\sigma(C)$

**Proof.**

$$D = \{A \in \sigma(C) : \mu_1(A) = \mu_2(A)\}$$

We know  $C \in D$ . what to show  $D$  is a  $\lambda$ -system

If so, by Dynkins Lemma,  $\sigma(C) \subseteq D$ , so that  $D = \sigma(C)$

Check  $D$  is a  $\lambda$ -system:

- $\Omega \in D$  follows from  $\mu_1(\Omega) = \mu_2(\Omega)$
- if  $A, B \in D, A \subseteq B \rightarrow \mu_1(A) = \mu_2(A), \mu_1(B) = \mu_2(B)$   
 $\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$   
 So,  $\Rightarrow B \setminus A \in D$
- if  $A_n \in D$  and  $A_n$  increasing,  $\lim_{n \rightarrow +\infty} A_n = A$

$$\mu_2(A) = \mu_2 \lim_{n \rightarrow +\infty} (A_n) \underbrace{=}_{\text{cont.}} \lim_{n \rightarrow +\infty} \mu_2(A_n) = \lim_{n \rightarrow +\infty} \mu_1(A_n) = \mu_1(A)$$

■

#### Remark 1

Also hold for  $\mu_1, \mu_2, \sigma$ -finite

## §2.6 Recitation 2 (02-14)-Exercise

### EX1

Let  $\Omega$  be a countable set,

$$\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite, or } A^c \text{ is finite}\}$$

- ① show that  $\mathcal{A}$  is an algebra
- ② Let  $P : \mathcal{A} \rightarrow [0, +\infty)$  that  $P(A) = \begin{cases} 0, & \text{if } A \text{ is finite} \\ 1, & \text{if } A^c \text{ is finite} \end{cases}$

Is  $P$  a content/measure?

Solution: (1):

- $\emptyset \in \mathcal{A}$  because  $\emptyset$  is finite
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$  by definition
- $A, B \in \mathcal{A}$ 
  - if  $A, B$  are finite, then  $A \cup B$  is finite
  - if one of  $A, B$  is countably infinite, say  $B$

$$(A \cup B)^c = A^c \cap B^c \text{ is finite} \Rightarrow A \cup B \in \mathcal{A}$$

(2):  $\Omega = \{\omega_1, \omega_2, \dots\}$

Let  $A_i = \{\omega_i\}$  so that  $P(\bigcup_i \{\omega_i\}) = P(\Omega) = +\infty$  But  $\sum P(\omega_i) = 0$

**EX2**

Let  $\Omega$  be an uncountable set,  $A = \{\{\omega\}, \omega \in \Omega\}$ , compute  $\sigma(A)$  and justify:

Solution:

$$\sigma(A) = \underbrace{\{A \subseteq \Omega : A \text{ is countable or } A^c \text{ is countable}\}}_f$$

**Proof.** •  $f$  is a  $\sigma$ -algebra

- $\emptyset \in f$  because  $\emptyset$  is countable
- $A \in f \Rightarrow A^c \in f$  by definition
- if  $A_1, A_2, \dots \in f$ 
  - \* if  $A_i$  are countable, then  $\bigcup_{i=1}^{+\infty} A_i$  is countable
  - \* if one of  $A_i$  is uncountable, say  $A_1$  then  $A_1^c$  countable, then  $(\bigcup_{i=1}^{+\infty} A_i)^c$  is uncountable
- $\sigma(A) \subseteq f$  by  $\sigma(A)$  is minimal
- $\sigma(A) \supseteq f$ 
  - \* if  $A \in f$  countable,  $A = \{\omega_1, \omega_2, \dots\} =$
  - \* if  $A^c \in f$  countable, then

■

**EX3**

Let  $\Omega = \mathbb{R}$

$$C_1 = \{(-\infty, b], b \in \mathbb{R}\}$$

$$C_2 = \{(a, b], -\infty \leq a < b < +\infty\}$$

$$C_3 = \{(a_1, b_1] \cup (a_2, b_2] \cup (a_3, b_3] \cdots (a_n, b_n], -\infty \leq a_1 < b_1 \leq a_2 < \dots < b_n < +\infty\}$$

show that  $\sigma(C_1) = \sigma(C_2) = \sigma(C_3)$

Solution:

**Proof.**  $\sigma(C_1) = \sigma(C_2)$

- $C_1 \subseteq \sigma(C_2)$  because  $(-\infty, b] \in \sigma(C_2) \Rightarrow \sigma(C_1) \subseteq \sigma(C_2)$  because  $\sigma(C_1)$  is minimal
- $C_2 \subseteq \sigma(C_1)$  because  $(a, b] = (-\infty, b] \setminus (-\infty, a] \in \sigma(C_1) \Rightarrow \sigma(C_2) \subseteq \sigma(C_1)$  because  $\sigma(C_2)$  is minimal

■

**§2.6.1 Special Case:**

$$\Omega = \{1, 2, \dots, N\}, F = P(\Omega), P(\{1\}) = \dots = P\{N\} = \frac{1}{N}$$

$$\text{For event: } E \in \Omega, P(E) = \frac{|E|}{|\Omega|}$$

**Definition 2.6.1: Inclusion-Exclusion Principle**

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i) - \sum_{i < j} P(E_i \cap E_j) + \sum_{i < j < k} P(E_i \cap E_j \cap E_k) + \dots + (-1)^{r-1} \sum_{i_1 < \dots < i_r} P(E_{i_1} \cap \dots \cap E_{i_r}) + \dots + (-1)^{n-1} P(E_1 \cap E_2 \cap \dots \cap E_n)$$

**Remark 1**

- ① Prove by induction
- ② 
$$\begin{aligned} P(E_1 \cup E_2 \cup \dots \cup E_n) &\leq \sum P(E_i) \\ P(E_1 \cup E_2 \cup \dots \cup E_n) &\geq \sum P(E_i) - \sum P(E_i \cup E_j) \\ P(E_1 \cup E_2 \cup \dots \cup E_n) &\leq \sum P(E_i) - \sum P(E_i \cap E_j) + \sum P(E_i \cap E_j \cap E_k) \end{aligned}$$

**Example 1: Brithday Problem**

N people, what is the P[at least two people have the same birthday]?  
Solution:

$$\begin{aligned} \Omega &= \{(x_1, \dots, x_n, x_i \in \{1, \dots, 365\})\} |\Omega| = 365^N \\ A^c &= \{(x_1, \dots, x_N) \in \Omega : x_i \neq x_j\} |A^c| = 365 \cdot 364 \cdots (365 - N + 1) \\ P(A) &= \frac{|A^c|}{|\Omega|} = 1 \cdots (1 - \frac{1}{365}) \cdots (1 - \frac{N-1}{365}) (use 1 - x \leq e^{-x}) \\ &\leq e^{-\sum_{i=0}^{N-1} \frac{i}{365}} \\ &= e^{-\frac{N(N-1)}{730}} \end{aligned}$$

in fact if  $N > 23$ , then  $P(A^c) < \frac{1}{2}$

**§2.7 Lecture 5 (02-17)**

Useful  $\pi$ -system that generates  $\mathbf{B}(\mathbf{R})$ :

- ①  $\mathcal{E} = \begin{cases} \{(a, b], -\infty \leq a < b < +\infty\} \\ \{(a, +\infty), a \in \mathbb{R}\} \end{cases}$
- ②  $\mathcal{E}_1 = \{(a, b], -\infty \leq a < b < +\infty\}$
- ③  $\mathcal{E}_2 = \{(a, b), -\infty \leq a < b \leq +\infty\}$
- ④  $\mathcal{E}_{\text{open}} = \{A \in \mathcal{E}, A \text{ open}\}$
- ⑤  $\mathcal{E}_{\text{closed}} = \{A \in \mathcal{E}, A \text{ closed}\}$
- ⑥  $\mathcal{E}_{\text{half}} = \{(-\infty, b], b \in \mathbb{R}\}$

easy to check  $\pi$ -system

Note:

- A open iff  $\forall x \in A, \exists \varepsilon_x > 0$ , s.t.  $(x - \varepsilon_x, x + \varepsilon_x) \subseteq A$
- A closed iff  $A^c$  open

**Proof.** (2)

$$\begin{aligned}\mathcal{E}_1 &\subseteq \sigma(\mathcal{E}) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}) \\ \mathcal{E} &\subseteq \sigma(\mathcal{E}_1), (a, +\infty) = \bigcup_{n \geq 1} \underbrace{(a, a+n]}_{\mathcal{E}_1} \in \sigma(\mathcal{E}_1) \\ \Rightarrow \sigma(\mathcal{E}) &\subseteq \sigma(\mathcal{E}_1)\end{aligned}$$

(3)

$$\begin{aligned}\mathcal{E}_1 &\subseteq \sigma(\mathcal{E}_2) : (a, b] = \bigcap_{n \geq 1} \underbrace{(a, b + \frac{1}{n})}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2) \\ \mathcal{E}_2 &\subseteq \sigma(\mathcal{E}_1) : (a, b) = \bigcup_{n \geq 1} \underbrace{(a, b - \frac{1}{n})}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2)\end{aligned}$$

(4)

$$\mathcal{E}_2 \subseteq \mathcal{E}_{\text{open}} \Rightarrow \sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{E}_{\text{open}})$$

**Fact:**

every open set  $A \subseteq \mathbb{R}, A = \bigcup_{i=1}^{+\infty} \underbrace{(x_i - \mathcal{E}_i, x_i + \mathcal{E}_i)}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2)$

$$\Rightarrow \mathcal{E}_{\text{open}} \subseteq \sigma(\mathcal{E}_2) \Rightarrow \sigma(\mathcal{E}_{\text{open}}) \subseteq \sigma(\mathcal{E}_2)$$

(e)

$A \text{ open} \Leftrightarrow A^c \text{ closed implies}$

$$\mathcal{E}_{\text{closed}} \subseteq \sigma(\mathcal{E}_{\text{open}}) \Rightarrow \sigma(\mathcal{E}_{\text{closed}}) \subseteq \sigma(\mathcal{E}_{\text{open}})$$

$$\mathcal{E}_{\text{open}} = \sigma(\mathcal{E}_{\text{closed}})$$

■

### Example 1: mismatch

N men and N hats

P[no one finds his own hats]=?

Solution:

$$E_i = \{i^{\text{th}} \text{ letter in } i^{\text{th}} \text{ envelope}\}$$

Want to compute  $P(\bigcap_{i=1}^n E_i^c) = 1 - P(\bigcup_{i=1}^n E_i)$  By inclusion-exclusion:

$$P(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i < j} P(E_i \cap E_j) + \cdots + (-1)^{n-1} P(\sum_{i=1}^n E_{i_1} \cap \cdots \cap E_{i_r})$$

$$P(\sum_{i=1}^n E_{i_1} \cap \cdots \cap E_{i_r}) = \frac{|E_{i_1} \cap \cdots \cap E_{i_r}|}{|\Omega|} = \frac{(n-r)!}{n!}$$

$$\sum_{i_1 < \cdots < i_r} P(E_{i_1} \cap \cdots \cap E_{i_r}) = \binom{n}{r} \frac{(n-r)!}{n!} = \frac{1}{r!}$$

$$P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N-1} \frac{1}{N!}$$

$$P(\bigcap_{i=1}^n E_i^c) \rightarrow 1 - \frac{1}{e}$$

### Exercise:

- Circle: 10 couples,  
P[no couple sit next to each other]=?
- Texas Holder:  
P[Straight]=? P[Full House]=? sol:

$$|\Omega| = \binom{52}{5}$$

$$|S_{\text{straight}}| = 10 \times (4^5 - 4)$$

$$P[\text{Straight}] = \frac{10 \times (4^5 - 4)}{\binom{52}{5}} \approx 0.0039$$

$$|F_{\text{full house}}| = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$$

$$P[\text{Full House}] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approx 0.0014$$

Conditional Probability:

"If the event B has occurred, what is the probability of event A?"  $P[A|B]$

N experiment: natural

$$P[A|B] = \frac{\text{number of occurrence of both A and B}}{\text{number of occurrence of B}} = \frac{P[A \cap B]}{P[B]}$$

### Definition 2.7.1

If  $P[B] > 0$ , then the conditional probability of A given B is  $P[A|B] = \frac{P[A \cap B]}{P[B]}$ .

### Example 2

① 2 kids

- $P[\text{two boys} | \text{at least one boy}] = ?$

- $P[\text{two boys} \setminus \text{younger kid is boy}] = ? \ 1/2$
- $P[\text{two boys} \setminus \text{at least one boy born on Tuesday}] = ?$

(1)

$$A = \{BB\}, B = \{BG, GB, BB\}$$

$$P[A \setminus B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]}{P[B]} = \frac{1/4}{3/4} = \frac{1}{3}$$

(3)

$$\Omega = \{B_i B_j, B_i G_j, G_i B_j, G_i G_j, i, j = 1, \dots, 7\}$$

$$A = \{B_i B_j, i, j = 1, \dots, 7\}$$

$$B = \{B_2 B_j, B_i B_2, B_2 G_j, i, j = 1, \dots, 7\} \text{ 13+14 elements}$$

$$A \cap B = \{B_2 B_j, B_i B_2, i, j = 1, \dots, 7\} \text{ 13 elements}$$

$$P[A \setminus B] = \frac{13}{27}$$

## §2.8 Lecture 6 (02-19)-Conditional Probability

### Definition 2.8.1: Law of total probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

$$P[A] = P[A|B] \cdot P[B] + P[A|B^c]P[B^c]$$

More generally, Let  $(B_i)_{i=1}^n$  be a partition of  $\Omega$  (i.e.  $B_i \cap B_j = \emptyset$  and  $\bigcup_{i=1}^n B_i = \Omega$ ), then:

$$P[A] = \sum_{i=1}^n P[A|B_i]P[B_i]$$

**Proof:**

$$P[A] = P[A \cap B] + P[A \cap B^c] = P[A|B]P[B] + P[A|B^c]P[B^c]$$

### Example 1

- ① draw balls randomly from box A(3B2W) to B(4B3W) ,then draw from B randomly,  $P[\text{The second draw is Black}]$   
 By Law of total probability:  $P = P[2\text{nd Black} | 1\text{st Black}]P[1\text{st Black}] + P[2\text{nd Black} | 1\text{st White}]P[1\text{st White}]$   
 $= \frac{5}{8} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{23}{40}$

**Reverse Question:**



If A happens, which  $B_i$  is the most likely? **Bayes' Formula:**

$$P[B_i|A] = \frac{P[B_i \cap A]}{P[A]} = \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^n P[A|B_i]P[B_i]}$$

### Example 2

① Covid Test:

- false negative:  $P[\text{negative}|\text{infected}] = 0.05$
- false positive:  $P[\text{positive}|\text{not infected}] = 0.01$

Suppose 5% of the population are infected, then  $P[\text{infected}|\text{positive}] = ?$

*Solution.*

$$P[V|P] = \frac{P[P|V]P[V]}{P[P|V]P[V] + P[P|V^c]P[V^c]} = \frac{5}{6}$$

② Prisoner parados: A, B, C. 2 executed, 1 pardoned.

A asked: "Please tell me the name of someone else who will be executed"

Guard: "B will be executed"

$P[A \text{ survive} | B \text{ will be executed}] = ?$

*Solution.*

$$\begin{aligned} P[A | \text{Guard says B}] &= \frac{P[\text{Guard says B} | A]P[A]}{P[\text{Guard says B} | A]P[A] + P[\text{Guard says B} | B]P[B] + P[\text{Guard says B} | C]P[C]} \\ &= \frac{\frac{1}{6}}{\frac{1}{6} + 0 + \frac{1}{3}} = \frac{1}{3} \end{aligned}$$

*Solution.*

$$\begin{aligned} \Omega &= \{(\text{survive person}, \text{name mentioned by Guard})\} \\ &= \left\{ \begin{array}{l} (A, B), \frac{1}{6} \\ (A, C), \frac{1}{6} \\ (B, C), \frac{1}{3} \\ (C, B), \frac{1}{3} \end{array} \right. P[A|B] = \frac{P(A, B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3} \end{aligned}$$

③ Two envelope problem: X, 2X, switch or not?

•

## §2.9 Lecture 7 (02-21)-Independence

### Definition 2.9.1: Independence

$$P[A|B] = P[A]$$

then we say A and B are independent  
Two events A and B are independent iff

$$P[A \cap B] = P[A]P[B]$$

(A,B independent  $\Rightarrow A^c, B$  independent)

### Definition 2.9.2: Multiple events Independence

The events  $A_1, A_2, \dots, A_n$  are independent iff

$$P[A_1 \cap A_2 \cap \dots \cap A_n] = P[A_1]P[A_2] \dots P[A_n]$$

Notice: Stronger than the condition  $P[A \cap B] = P[A]P[B]$  (pairwise independence)

### Example 1: pairwise independence < Independence

$x_1, x_2, x_3$  are coin flips:

$$P(x_i = 1) = P(x_i = 0) = \frac{1}{2}$$

$$A_1 = \{x_2 = x_3\}, A_2 = \{x_1 = x_3\}, A_3 = \{x_1 = x_2\}$$

$$P[A_i] = \frac{1}{2}, P[A_i \cap A_j] = P[x_1 = x_2 = x_3] = \frac{1}{4} = P[A_i]P[A_j] \text{ pairwise independent}$$

$$P[A_1 \cap A_2 \cap A_3] = P[x_1 = x_2 = x_3] = \frac{1}{4} \text{ not independent}$$

### Example 2: Independence

Independent trials, each with success probability p, fail probability 1-p

$P_{n,m}[\underbrace{\text{first } n \text{ success}}_{A_{n,m}} \text{ occurred before first } m \text{ failures}] = ?$

*Solution.* (Pascal)

By Law of total probability:

$$\begin{aligned} P[A_{n,m}] &= P[A_{n,m} | \text{1st success}] P[\text{1st success}] + P[A_{n,m} | \text{1st failure}] P[\text{1st failure}] \\ &= P[A_{n-1,m}] \cdot P + P[A_{n,m-1}] \cdot (1 - P) \\ \Rightarrow P_{n,m} &= P_{n-1,m} \cdot P + P_{n,m-1} \cdot (1 - P) \end{aligned}$$

Boundary condition:  $P_{0,m} = (1 - P)^m, P_{n,0} = 0$

Or building generating function

(Fermat):

{First n success before first m failure}

$\Leftrightarrow$  {at least n success in the first m+n-1 trials} (ex)

$P[\text{exact } k \text{ success in } n+m-1 \text{ trials}] = \binom{n+m-1}{k} p^k (1-p)^{m+n-1-k}$  Binomial distribution

$\Rightarrow P[\text{at least } n \text{ success in the first } m+n-1 \text{ trials}] = \sum_{k=n}^{m+n-1} \binom{n+m-1}{k} p^k (1-p)^{m+n-1-k}$

**Example 3**

Multiple choice test, m options, p-knows the answer, 1-p random guess

$$P[\text{knows the answer} | \text{correct}] = \frac{p}{p + (1-p)(\frac{1}{m})} = \frac{mp}{mp + 1 - p} \text{ Bayes}$$

**Example 4**

Gambler's ruin:

bet 1 dollar each time, p-win, 1-p-lose, initial amount of money= $i \in [0, N]$

$P_i[\text{Reach } N \text{ before reaching } 0] = \text{win times} = N - i + \text{lose times}$ , Method2:  $p_i = pp_{i+1} + (1-p)p_{i-1}$

characteristic polynomials: take  $p_i = cr^i$

$$pr^2 - r + (1-p) = 0$$

$$r = 1, \frac{1-p}{p}$$

$$\text{if } p \neq 1-p, \text{ then } p_i = c_1 + c_2 \left(\frac{1-p}{p}\right)^i$$

$$\text{if } p = 1-p, \text{ then } p_i = c_1 + c_2 i, c_1, c_2 \text{ determined by } p_0, p_N$$

One dimensional random walk:

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n$$

$$X_i \text{ are i.i.d. } P(X_i = 1) = P, P(X_i = -1) = 1 - P$$

**Example 5**

① secretary problem:

N candidates

After each interview, immediately make offer or rejection

what is the best strategy

maximize  $P[\text{best candidate is offered}]$

*Solution.* Not making offer to first r candidate, make an offer to the next candidate that is better than  $\{1, 2, \dots, r\}$

$P[\text{Best candidate is offered}]$

$$= \sum_{i=0}^N \underbrace{P[\text{best candidate is } i]}_{\frac{1}{N}} P[\text{Best candidate is offered} | \text{Best candidate} = i]$$

**§2.10 Lecture 7 (02-24)-Random walk****Definition 2.10.1: 1-dimensional random walk**

$$S_n = S_0 + x_1 + x_2 + \cdots + x_n \quad (x_i) \text{ i.d.d (independent + identically distributed)}$$

$$P(x_i = 1) = p, P(x_i = -1) = 1 - p$$

Generalization: 2-D(simple) random walk  $S_n = S_0 + x_1 + x_2 + \cdots + x_n$  ( $x_i$ ) i.d.d

$$P(x_i = \pm e_1) = P(x_i = \pm e_2) = \frac{1}{4}$$

P[random walk(starting at (i,j)) exit the boundary through A]=?

By conditioning,  $P_{i,j} = \frac{1}{4}P_{i+1,j} + \frac{1}{4}P_{i-1,j} + \frac{1}{4}P_{i,j-1} + \frac{1}{4}P_{i,j+1}$

Boundary condition:  $P_{i,j} = 1$  if  $(i,j) \in A$ , and  $P_{i,j} = 0$  otherwise

We have

$$\begin{aligned} P_{i,j} &= \frac{1}{4}P_{i+1,j} + \frac{1}{4}P_{i-1,j} + \frac{1}{4}P_{i,j-1} + \frac{1}{4}P_{i,j+1} \\ &\Leftrightarrow \frac{1}{4}(P_{i+1,j} + P_{i-1,j} - 2P_{i,j}) + \frac{1}{4}(P_{i,j-1} + P_{i,j+1} - 2P_{i,j}) = 0 \\ &\Leftrightarrow \Delta P = 0 \end{aligned}$$

Discrete Laplacian on  $Z^2$ ,

$$\Delta P(x) = \sum_{y \approx x} \frac{1}{4}(P(y) - P(x))$$

### §2.10.1 Random Variable and Measurable functions

#### Example 1

- ① 2 coinflips,  $\Omega = \{HH, TT, HT, TH\}$ ,  $f = P(\Omega)$   $X = \# \text{heads}$ , then we may write  $X : \Omega \rightarrow \mathbb{N}$ ,  $X = 2 \cdot 1_{HH} + 1_{HT} + 1_{TH}$

$$P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x\})$$

It is called the distribution function of  $X$  (right continuous increasing function)

#### Definition 2.10.2

Let  $(\Omega_1, F_1), (\Omega_2, F_2)$  be two measurable space

- A map  $X : \Omega_1 \rightarrow \Omega_2$  is called measurable iff  $\forall A \in F_2, X^{-1} \in F_1$ ,  $X^{-1} = \{\omega \in \Omega_1 : X(\omega) \in A\}$
- if  $(\Omega_1, F_1, P)$  is a probability space, then a measurable function  $X : \Omega_1 \rightarrow \Omega_2$  is called a random variable
- if  $(\Omega_2, F_2) = (R, B(R))$  then a measurable function is called a Borel function
- if  $X : (\Omega_1, F_1, P) \rightarrow (R, B(R))$  then  $X$  is a  $R$ -valued random variable and

$$F_X(x) = P(X^{-1}(-\infty, x]) = P(X \leq x)$$

is called the distribution function of  $X$

**Remark:** By (a), we know that for any  $B \in B(R)$  we can define  $P(X^{-1}(B))$

**Example 2**

① Let  $(\Omega, F, P)$  be a probability space,  $A \in F$

Then  $1_A : \Omega \rightarrow \{0, 1\}$  is a random variable,  $1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in A^c \end{cases}$  We

need to show  $\forall B \in B(R), 1_A^{-1}(B) \in F$

$$\text{In fact, } 1_A^{-1}(B) = \begin{cases} \emptyset, & \text{if } 0, 1 \notin B \\ A, & \text{if } 1 \in B, 0 \notin B \\ A^c, & \text{if } 0 \in B, 1 \notin B \\ \Omega, & \text{if } 0, 1 \in B \end{cases}$$

In practice, only need to check the pre-image on a smaller set:

**Proposition 2.10.0**

Let  $(\Omega_1, F_1), (\Omega_2, F_2)$  be two measurable space

$\mathcal{E} \subseteq F_2$  such that  $\sigma(\mathcal{E}) = F_2$

Then  $X : \Omega_1 \rightarrow \Omega_2$  is measurable if  $\forall A \in \mathcal{E}, X^{-1}(A) \in F_1$

**Proof.** Let  $G = \{B \subseteq \Omega_2, X^{-1}(B) \in F_1\}$

Then  $G$  is a  $\sigma$ -algebra (ex)

Therefore if  $G \supseteq \mathcal{E}$  then  $G \supseteq \sigma(\mathcal{E}) = F_2$  ■

**Corollary 2.10.1**

Let  $(\Omega, F)$  be a measurable space, then  $X : \Omega \rightarrow R$  is a Borel function iff the following are true.

- $\{x < a\} \in F$  for  $\forall a \in R$
- $\{x \leq a\} \in F$  for  $\forall a \in R$
- $\{x > a\} \in F$  for  $\forall a \in R$
- $\{x \geq a\} \in F$  for  $\forall a \in R$

**Proof.**  $\{x < a, a \in \mathbb{R}\} = \{X^{-1}(-\infty, a), a \in \mathbb{R}\}$

suffices to show:  $\mathcal{E} = \{(-\infty, a), a \in \mathbb{R}\}$

satisfies  $\sigma(\mathcal{E}) = B(\mathbb{R})$

indeed:

$$[a, +\infty) = (-\infty, a)^c \in \sigma(\mathcal{E}) \quad (2.1)$$

$$[a, b) = [a, +\infty) \setminus (b, +\infty) = (-\infty, a)^c \setminus (-\infty, b)^c \in \sigma(\mathcal{E}) \quad (2.2)$$

$$\Rightarrow \sigma(\mathcal{E}) = B(R) \quad (2.3)$$

■

**Remark 1**

a  $\mathbb{R}$ -valued random variable is a function  $X : \Omega \rightarrow \mathbb{R}$  such that  $\forall a \in \mathbb{R}, \{X < a\} \in \mathcal{F}$

**Lemma 2.10.1**

A distribution function  $F$  satisfies:

- ①  $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow +\infty} F(x) = 1$
- ②  $F(x) \leq F(y)$  if  $x \leq y$
- ③  $F$  is right continuous,  $F(x+h) \rightarrow F(x)$  as  $h$  decreases to 0

**Proof.**  $F(x) = P(X \leq x)$  (2) is immediate

(1): Let  $A_n = \{x \leq -n\}$  then by continuous of measure:  $\lim_{n \rightarrow +\infty} P(A_n) = P(\bigcap_{n \geq 1} A_n) = 0$

Then by monotonicity of  $F$ ,  $\lim_{x \rightarrow -\infty} F(x) = 0$

(3): Let  $B_n = \{X \leq x + \frac{1}{n}\}$  (decreasing),  
 then  $F(x + \frac{1}{n}) = P(B_n) \xrightarrow[n \rightarrow +\infty, \text{cont.}]{} P(\bigcap_{n \geq 1} B_n) = P(X \leq x) = F(x)$

again use  $F$  increasing to conclude ■

**Lemma 2.10.2**

- ①  $F(y) - F(x) = P(x < X \leq y), \forall x \leq y$
- ②  $F(x) - \lim_{h \rightarrow +\infty} F(x-h) = P(X = x)$
- ③  $B_n = \{x - \frac{1}{n} < X \leq x\}$  (decreasing), Then  $\bigcap_{n \geq 1} B_n = \{X = x\}$   
 $\Rightarrow F(x) - F(x - \frac{1}{n}) = P(B_n) \xrightarrow[n \rightarrow +\infty]{} P(X = x)$   
 again use  $F$  increasing to conclude

Naturally:

- (absolute) continuous random variable
- discrete random variable

**§2.11 Lecture 8 (02-26)**

properties of measurable functions (R.V.s):

- Let  $\{X < Y\} = \{\omega \in \Omega_1 : X(\omega) < Y(\omega)\}$   
 $\{X > Y\} = \{\omega \in \Omega_1 : X(\omega) > Y(\omega)\}$

**Lemma 2.11.1**

Let  $(\Omega, F)$  be a measurable space,  $X, Y$  are Borel Functions

- $\{X < Y\}, \{X \leq Y\}, \{X = Y\}, \{X \neq Y\} \in F$
- $X + Y, X \cdot Y, X/Y$  are Borel functions

**Proof.** (1): Use that  $\mathbb{Q}$  is dense in  $\mathbb{R}$

$$\begin{aligned}\{X < Y\} &= \bigcup_{q \in \mathbb{Q}} \{X < q < Y\} = \bigcup_{q \in \mathbb{Q}} \underbrace{\{X < q\}}_{\in F} \cap \underbrace{\{q < Y\}}_{\in F} \in F \\ \{X = Y\} &= \bigcap_{n \geq 1} \{X < Y + \frac{1}{n}\} \cap \bigcap_{n \geq 1} \{X > Y - \frac{1}{n}\} \in F\end{aligned}$$

(2): Fact: if  $Y$  is Borel, then  $aY + b, a, b \in \mathbb{R}$  is Borel (ex)

Then

$$\forall a \in \mathbb{R}, \{X + Y < a\} = \{X < a - Y\} \in F \text{ by (1)}$$

so  $X + Y$  is Borel

$$\{X^2 < a\} = \begin{cases} \emptyset, \\ \{X < \sqrt{a}\} \cap \{X > -\sqrt{a}\} \in F \end{cases}$$

so  $X^2$  is Borel

$$X \cdot Y = \frac{1}{4}[(X + Y)^2 - (X - Y)^2] \text{ is Borel} \quad \blacksquare$$

**Lemma 2.11.2**

Let  $(X_n)_{n \geq 1}$  be a sequence of Borel functions on  $(\Omega, F)$ ,

Then the following are Borel functions:

- $\sup_{n \geq 1} X_n, \inf_{n \geq 1} X_n, \limsup_{n \geq 1} X_n, \liminf_{n \geq 1} X_n$

In particular, if  $\lim_{n \rightarrow +\infty} X_n$  exists, then  $\lim_{n \rightarrow +\infty} X_n$  is Borel

**Proof.**

$$\{\sup_{n \geq 1} X_n < a\} = \bigcap_{n \geq 1} \{X_n < a\} \in F \forall a \in \mathbb{R} \Rightarrow \sup_{n \geq 1} X_n \text{ is a Borel function}$$

$$\{\inf_{n \geq 1} X_n > a\} = \bigcap_{n \geq 1} \{X_n > a\} \in F \forall a \in \mathbb{R} \Rightarrow \inf_{n \geq 1} X_n \text{ is a Borel function}$$

$$\limsup_{n \geq 1} X_n = \inf_{m \geq 1} \sup_{n \geq m} X_n \text{ is a Borel function}$$

$$\liminf_{n \geq 1} X_n = \sup_{m \geq 1} \inf_{n \geq m} X_n \text{ is a Borel function} \quad \blacksquare$$

**Lemma 2.11.3**

Let  $(\Omega_1, F_1), (\Omega_2, F_2), (\Omega_3, F_3)$  be two measurable space  
 $X : \Omega_1 \rightarrow \Omega_2, Y : \Omega_2 \rightarrow \Omega_3$  are measurable, then  $Y \circ X : \Omega_1 \rightarrow \Omega_3$  is measurable

**Proof.**

$$\forall A \in F_3, (Y \circ X)^{-1}(A) = X^{-1}(\underbrace{Y^{-1}(A)}_{\in F_2}) \in F_1$$

■

**Definition 2.11.1:  $\sigma$ -algebra generated by r.v.**

If  $X : \Omega \rightarrow \mathbb{R}$  is a random variable, then

$$\sigma(x) = \{X^{-1}(A), A \in B(\mathbb{R})\}$$

is called the  $\sigma$ -algebra generated by  $X$

Let  $(X_i)_{i \in I}$  be a family of r.v.s

$$\sigma(X_i, i \in I) = \sigma\left(\bigcup_{i \in I} \sigma(X_i)\right)$$

is the  $\sigma$ -algebra generated by  $(X_i)_{i \in I}$

**Remark 1**

$\sigma(X)$  is the smallest  $\sigma$ -algebra such that  $X$  is measurable

**Example 1**

- ①  $(\Omega, F, P)$ , Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable,  $X = b_1 1_{A_1} + b_2 1_{A_2} + \dots + b_n 1_{A_n}$ ,  $b_i \in \mathbb{R}, A_i \in F, A_i \cap A_j = \emptyset$   
 if  $b_j \neq b_j$   
 Then  $\sigma(X) = \sigma\{A_1, A_2, \dots, A_n\}$  = "finite disjoint union of  $A_1 \dots A_n$ "

**Proof.** •  $\supseteq$ : Note that  $X^{-1}(\{b_1\}) = A_1, \dots, X^{-1}(\{b_n\}) = A_n$   
 $\Rightarrow A_1, \dots, A_n \in \sigma(x), \sigma(A_1, \dots, A_n) \in \sigma(x)$   
 •  $\subseteq$

**Lemma 2.11.4**

$$\sigma(X) = \sigma(\{X \leq a\}, a \in \mathbb{R})$$

it suffices to show  $\forall a \in \mathbb{R}, \{X \leq a\} \in \sigma(\{A_1, \dots, A_n\})$   
 $\{X \leq a\}$  = "finite disjoint union of  $A_i$  and  $(\bigcup A_i)^c$ "





Two specific cases: Discrete and (Absolutely) continuous random variables

### Definition 2.11.2

A r.v. is discrete if it takes values in a countable set  $\{X_1, X_2, \dots\}$   
 prob mass function:  $f(x) = P(X = x)$

### Remark 2

We say  $x_1, x_2, \dots$ , are atoms of  $F_x$

### Definition 2.11.3

A r.v. is (absolutely) continuous if

$$F_x(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

for some integrable function  $f: \mathbb{R} \rightarrow [0, +\infty)$ ,  $f(x) = F'_x(x)$  is called the probability density function of  $X$

### Remark 3

$F_x$  is absolutely continuous  
 $F$  is absolutely continuous iff

$$\forall \epsilon > 0, \exists \delta > 0$$

s.t. for any finite collection of intervals  $a_i, b_i$  s.t.

$$\sum_{i=1}^n |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$$

$$C^1 \Rightarrow \text{absolutely continuous} \Rightarrow \text{uniformly continuous}$$

### Remark 4

$X$  is a singular Continuous R.V. if  $F_x$  is continuous but  $F_x$  is not absolutely continuous (ex: Cantor function)

Discrete R.V.:

$$F_x(x) = P(X \leq x) = \sum_{x_i \leq x} \underbrace{f(x_i)}_{\text{prob mass function}}$$

$$f(x) = F_x(x) - \lim_{y \nearrow x} F_x(y)$$

**Definition 2.11.4**

The expectation/mean of a discrete r.v. with prob mass function  $f$  is

$$E[X] = \sum_{x: f(x) > 0} x_i f(x_i)$$

whenever the sum is absolutely convergent

**Remark 5**

absolutely convergent  $\Rightarrow$  order of the sum does not matter

**Example 2**

① 2 coin flips,  $X = \# \text{heads}$ ,  $f(0) = f(2) = \frac{1}{4}$ ,  $f(1) = \frac{1}{2}$   
 $E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

**Lemma 2.11.5**

Let  $X$  be a r.v. taking values in  $\mathbb{N}$ .  
 Then

$$E[X] = \sum_{n=1}^{+\infty} P(X \geq n)$$

**Proof.**

$$\begin{aligned} E[x] &= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + \dots \\ &= P(X = 1) + P(X = 2) + P(X = 3) + \dots \\ &\quad P(X = 2) + P(X = 3) + \dots \\ &\quad P(X = 3) + \dots \end{aligned}$$

$$P(X \geq n)$$

■

## §2.12 Lecture 9 (02-28)-Expectation

### EX1

we have offers  $X_1, X_2, \dots$  continuous R.V. i.i.d

$T = \inf\{n > 1, X_n > X_1\}$  Compute  $E[T]$

*Solution.*

$$E[T] = \sum_{n=2}^{+\infty} nP(T = n)$$

By lemma,

$$\begin{aligned} E[T] &= \sum_{n \geq 1} P(T \geq n) = \sum_{n \geq 1} P(X_1 \geq X_2 \geq \dots \geq X_n) = \frac{1}{n-1}, n \geq 2 \\ &= +\infty \end{aligned}$$

**EX2**

Independent trials, each with success probability  $p$ , fail probability  $1-p$   
 Compute  $P[\text{There is } n \text{ consecutive successes before } m \text{ consecutive failures}]$  Condition on the first trial:

$$P[A] = p \cdot P[A|H] + (1-p) \cdot P[A|T]$$

Multiple conditioning:  $P(E_1 \cap E_2 \cdots E_n) = P(E_1)P(E_2|E_1) \cdots P(E_n|E_1 \cap \cdots E_{n-1})$   
 Let  $B = \{HH \cdots H\}$  (2nd- $n$ -th)

$$P(A|H) = P(B)P(A|H \cap B) + P(B^c)P(A|H \cap B^c) = p^{n-1} \cdot 1 + (1-p^{n-1}) \cdot P(A|T)$$

Let  $C = \{TT \cdots T\}$  (2nd- $m$ -th)

$$P(A|T) = P(C)P(A|T \cap C) + P(C^c)P(A|T \cap C^c)$$

Let  $q=1-p$ ,

$$\begin{aligned} P(A|H) &= \frac{p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}} \\ P(A|T) &= \frac{(1-q^{m-1})p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}} \\ P(A) &= \frac{p^{n-1}(1-q^{m-1})}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}} \end{aligned}$$

**§2.13 Lecture 10 (03-02)-Var****Lemma 2.13.1: change of variable**

Let  $g: R \rightarrow R$   $X$  be a r.v. with probability mass function  $f$ , then

$$E[g(x)] = \sum_{x: f(x) > 0} g(x)f(x)$$

**Proof.**

$$\begin{aligned} E[g(x)] &= \sum_y y \cdot P(g(x) = y) = \sum_y \sum_{x: g(x)=y} y \cdot P(X = x) \\ &= \sum_x g(x)P(X = x) = \sum_x g(x)f(x) \end{aligned}$$

■

**Definition 2.13.1**

Let  $k \in N$ , the  $k$ -th moment of  $X$  is  $m_k := E[X^k]$  as long as the expectation exists  
 Let  $k \in N$ , the  $k$ -th central moment of  $X$  is  $\sigma_k := E[X - E[X]]^k$  as long as the expectation exists

- $\sigma_2 = \text{Var}(X) = E(X - E[X])^2$  variance, "deviation fluctuation" from the mean

- $\sigma = \sqrt{\text{Var}(X)}$  standard deviation

**Fact:**

- $E[aX + bY] = aE[X] + bE[Y], a, b \in \mathbb{R}$   
 $\rightarrow \text{Var}(X) = E(X - E[X])^2 = E[X^2] - (E[X])^2$

### Example 1

- ① Bernoulli( $p$ )  $P(X = 1) = p, P(X = 0) = 1 - p$   
 $E[X] = 1 \cdot P(X = 1) = p, \text{Var}(X) = p - p^2$
- ② binomial( $n, p$ )  $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

$$E[X] = \sum_{k=0}^n k P(X = k) = \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1 - p)^{n-k}$$

Recall

$$(1 + x)^n = \sum_{k=0}^n x^k \cdot \binom{n}{k}$$

,differentiate both sides w.r.t  $x$ , then

$$n(1 + x)^{n-1} = \sum_{k=1}^n k \cdot x^{k-1} \cdot \binom{n}{k}$$

Let  $q = 1 - p, x = \frac{p}{q}$ , then

$$\begin{aligned} E[X] &= \sum_{k=1}^n k \cdot \binom{n}{k} \left(\frac{p}{q}\right)^{k-1} \cdot q^n \cdot \frac{p}{q} \\ &= n \cdot \frac{p}{1 - p} \cdot \left(1 + \frac{p}{1 - p}\right)^{n-1} = np \\ \text{Var}(X) &= np(1 - p) \end{aligned}$$

sol n': we have  $X = Y_1 + Y_2 + \dots + Y_n$  such that  $Y_i \sim \text{Bernoulli}(p)$  and  $Y_i$  independent

$$\begin{aligned} E[X] &= \sum_{i=1}^n E[Y_i] = np \\ \text{Var}(X) &= E[X^2] - (E[X])^2 = E\left(\sum_{i=1}^n Y_i\right)^2 - (np)^2 \\ &= \sum_{i=1}^n E[Y_i]^2 - 2 \sum_{i < j} E[Y_i] E[Y_j] - (np)^2 \\ &= np + 2p^2 \frac{n(n-1)}{2} - (np)^2 \\ &= np(1 - p) \end{aligned}$$

If  $x, y$  are independent, then  $E[XY] = E[X]E[Y]$

Proof:

$$\begin{aligned} E[XY] &= \sum_{x,y} xyP(X=x, Y=y) = \sum_{x,y} xyP(X=x)P(Y=y) \\ &= \sum_x xP(X=x) \sum_y yP(Y=y) = E[X]E[Y] \end{aligned}$$

### Definition 2.13.2

If  $E[XY] = E[X]E[Y]$ , then we say  $X, Y$  are uncorrelated

Covariance

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E[Y])]$$

Correlation

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

### Lemma 2.13.2

$|\rho(X, Y)| \leq 1$ ,  $\rho(X, Y) = \pm 1$  iff  $Y = aX + b$  for some  $a, b \in \mathbb{R}$

**Proof.**

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{E[(X - E[X])(Y - E[Y])]}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

by Cauchy-Schwarz inequality:  $|\rho(X, Y)| \leq 1$

■

Cauchy-Schwarz inequality:

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

**Proof.**

$$\begin{aligned} \text{Since } 0 &\leq E[(aX - bY)^2] \forall a, b \in \mathbb{R} \Rightarrow a^2 E[X^2] + 2abE[XY] + b^2 E[Y^2] \geq 0 \\ &\Rightarrow \frac{1}{4} \Delta = E[XY]^2 - E[X^2]E[Y^2] \leq 0 \end{aligned}$$

■

### Example 2

$$\textcircled{1} \quad X \sim \text{Geometric}(p), P(X=k) = (1-p)^{k-1} \cdot p$$

$$E[X] = \frac{1}{p}, \text{Var}(x) = \frac{1-p}{p^2}$$



## Chapter 3

# Partial Differential Equations

### §3.1 Lecture 1 (02-03) Initial and Boundary Conditions

#### §3.1.1 What are the PDEs?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u \quad (3.1)$$

$$u = u(x, y, t) \quad (3.2)$$

(3.1) is a second order PDE

#### §3.1.2 Classical solutions

To simplify, we consider here equations of one unknown function  $u$  of one or more variable

By a solution we mean a sufficiently smooth function  $u$  that satisfies the PDE at every point of the domain of its definition( $D$ ).

Sometimes, we will have to impose some boundary conditions on the domain  $D$ .

$C^n = \{u : D \rightarrow R : n\text{-times continuous differentiable w.r.t all variables} \}$   $n$  is the order of the equation

#### Example 1

- Heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u = u(x, t)$$

$$(t, x) \in D \subset R^2$$

$$u(t, x), \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}$$

$$n = 2$$

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} \in C^2$$

$$u \in C^2(D)$$

- Some Solution:

$$u(t, x) = t + \frac{x^2}{2}, D = R^2$$

$$u(t, x) = \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}, D = \{(t, x) : t > 0\}$$

$$u(t, x) = e^{-t+ix} = e^{-t}\cos(x) + ie^{-t}\sin(x), D = \mathbb{R}^2$$

### §3.1.3 Initial and Boundary Conditions

What is Initial data?

$$u(0, x) = u_0(x)$$

What are Boundary Conditions? rigorous definition of Boundary

- Dirichlet Boundary Condition:  $u(t, x)|_{\partial D} = g(t, x)$
- Neumann Boundary Condition: When the normal derivative of  $u$  is specified  $\frac{\partial u}{\partial n}$  where  $n$  is the normal to the boundary  $\partial(D)$

$$\frac{\partial u}{\partial n}(t, x) = g(t, x)$$

- Mixed Boundary Condition: Combination of Dirichlet and Neumann

### §3.1.4 Linear and Nonlinear waves

**Stationary waves**

$$\frac{\partial u}{\partial t} = 0 \tag{3.3}$$

$$u = u(t, x) \tag{3.4}$$

This is a first order linear homogeneous PDE

Solution of the form  $u = u(t) = c, \forall c \in \mathbb{R}$

Let's integral (3.3) :

$$0 = \int_0^t \frac{\partial u}{\partial t}(s, x) ds = u(t, x) - u(0, x)$$

Thus if an initial conditional

$$u(0, x) = f(x)$$

$$u(t, x) = f(x), \forall t, x$$

Thus for any given  $f \in C^1(\mathbb{R})$ , we have a solution of (3.3) and this is a stationary wave

The domain of this solution is  $\mathbb{R}$  if  $f$  is defined everywhere on  $\mathbb{R}$

Power of domain to change the solution:

Consider the ODE:

$$\frac{du}{dt} = 0, D = (-\infty, 0) \cup (0, \infty)$$

The solution is:

$$u(t) = \begin{cases} c_1, & t < 0 \\ c_2, & t > 0 \end{cases} \quad \forall c_1, c_2 \in \mathbb{R}$$

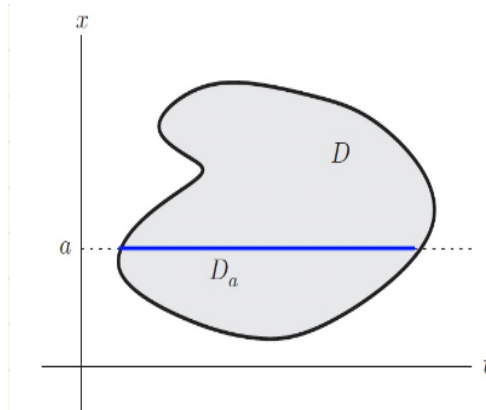
similarly, for the PDE in example (3.3):

$$u(t, x) = \begin{cases} 0, & x > 0 \\ x^2, & x \leq 0, t > 0 \\ -x^2, & x \leq 0, t < 0 \end{cases}$$



### 3.2. LECTURE 2 (02-05) – UNIFORM TRANSPORT AND TRAVELING WAVES 64

it is a  $\mathbb{C}^1$  solution of (3.3) on  $\mathbb{R}^2 \setminus \{(0, x) : x \leq 0\}$ , so it is not a function of  $x$  alone. Actually, it is not hard to check that if  $u$  is a classical solution to (3.3), defined on domain  $D \subset \mathbb{R}^2$  whose intersection with any horizontal line  $D_a = D \cap \{(t, a) : t \in \mathbb{R}\}, \forall a \in \mathbb{R}$  is either empty or a connected interval, then  $u(t, x) = f(x)$  is only a function of  $x$ .



## §3.2 Lecture 1 (02-05) Transport and Traveling waves

### §3.2.1 uniform transport

Let us consider the linear, homogeneous first order PDE:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (3.5)$$

$$u = u(0, x) = f(x) \quad (3.6)$$

$$x \in \mathbb{R}, t_0 = 0, t - t_0 = t, c \in \mathbb{R}$$

**use change of variable:**

$$\xi = x - ct$$

$\xi$  is called characteristic variable

let's look for a solution of (3.5) under the form

$$u(t, x) = v(t, \xi) = v(t, x - ct)$$

**use chain rule:**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial v}{\partial t} - c \frac{\partial v}{\partial \xi} \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial \xi} \end{aligned}$$

Thus:

$$0 = \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} - c \frac{\partial v}{\partial \xi} + c \frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial t}$$

**Conclusion:**

$$\frac{\partial v}{\partial t} = 0$$

Solution is constant  $v(\xi)$

$$u(t, x) = v(\xi) = v(x - ct)$$

**Proposition 3.2.0**

if  $u$  solves (3.5), then  $u$  has the form  $u(t, x) = v(x - ct)$  where  $v$  is a function of one variable that is  $C^1$ -smooth.

Actually,  $v$  is satisfying

$$u(0, x) = v(x) = f(x)$$

**Conclusion:**

$$\begin{cases} \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \\ u = u(0, x) = f(x) \end{cases}$$

The solution is

$$u(t, x) = f(x - ct)$$

$f(x - ct)$  is a traveling wave solution moving at velocity  $c$ :

if  $c > 0$ , the wave moves to the right

if  $c < 0$ , the wave moves to the left

if  $c = 0$ , the wave is stationary

**Example 1**

$$\begin{cases} \frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial x} = 0 \\ u = u(0, x) = \frac{1}{1+x^2} \end{cases}$$

The solution is

$$u(t, x) = \frac{1}{1 + (x - 2t)^2}$$

The lines  $x = ct + k$  are called the characteristic lines for the equation (3.5)

on the characteristic lines, the solution is always constant

on geometric interpretation,  $u(t, x) = f(x - ct) = f(k)$

Consider the equation:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + au = 0 \quad (3.7)$$

where  $a$  and  $c$  are given numbers.

**Use change of variable:**

$$u(t, x) = v(t, \xi) = v(t, x - ct)$$

we will get by the same computation

$$\frac{\partial v}{\partial t} + av = 0$$

The solution will be

$$\begin{aligned} v(\xi) &= e^{-at} f(\xi) \\ u(t, x) &= v(t, \xi) = v(x - ct) = e^{-at} f(x - ct) \end{aligned}$$

So,

$$\begin{cases} u(t, x) = e^{-at} f(x - ct) \\ u(0, x) = f(x) \end{cases}$$

**§3.2.2 Nonuniform transport**

$$\frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = 0 \quad (3.8)$$

In the uniform case ( $c$  constant), characteristic line is  $x=ct+k$  (the solution of ODE below)

$$\frac{dx}{dt} = c$$

In the general non-uniform case, we can still define characteristic curves

$$\frac{d}{dt}x(t) = c(x(t))$$

Let's show that the solutions of (3.8) are constants along the characteristic curves  $(t, x(t))$ . We need to check that  $u(t, x(t)) = h(t)$  is constant of  $t$ . compute the derivative of  $h(t)$

$$\begin{aligned} \frac{d}{dt}h(t) &= \frac{d}{dt}u(t, x(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} \\ &= \frac{\partial u}{\partial t} + c(x(t)) \cdot \frac{\partial u}{\partial x} = 0 \end{aligned}$$

$h(t)$  and thus  $u(t, x(t))$  is constant along the characteristic curves

**Definition 3.2.1**

The graph of a solution of the ODE

$$\frac{d}{dt}x(t) = c(x(t))$$

is called a characteristic curve for the transport equation (3.8) with speed  $c(x)$

Thus at each point  $(t, x)$  the slope of the characteristic curve is  $c(x)$  (the wave speed). Thus, we proved:

**Proposition 3.2.0**

The solutions to the linear transport equation (3.8) are constant along the characteristic curves

In the particular case  $c(x)=c$ , the characteristic curves are straight lines of slope  $c$

Moreover,

(3.8) is an autonomous first order ODE which can be "solved" using the method of separation of variables

Assume that  $c(x) \neq 0$

$$\begin{aligned} \frac{dx(t)}{dt} &= c(x(t)) \\ \frac{dx}{c(X)} &= dt \rightarrow \\ \beta(x) &= \int \frac{dx}{c(x)} = t + k \end{aligned}$$

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Thus characteristic curves satisfies the implicit equation

$$\beta(x) = t + k$$

We can find  $x$  from the equality using  $\beta^{-1}$  (the inverse of  $\beta$ )

$$\begin{aligned}\beta(x) &= t + k \\ \beta^{-1}\beta(y) &= y \\ x(t) &= \beta^{-1}(t + k)\end{aligned}$$

#### Remark 1

How to determine where  $c(x_0) = 0$ ?

Solve this ODE

$$\begin{cases} \frac{dx}{dt} = c(x) \\ x(t_0) = x_0 \end{cases}$$

and  $x(t) = x_0$  will be the characteristic curve going through  $(t_0, x_0)$

From the proposition, we know that  $u(t, x(t)) = \text{constant}$

$$\begin{aligned}u(t, x(t)) &= v(k) \\ &= v(\beta(x(t)) - t)\end{aligned}$$

where  $k$  is the constant determining the characteristic curve  $(t, x(t))$

$$\begin{aligned}u(t, x) &= v(\xi) \\ \text{where } \xi &= \beta(x) - t\end{aligned}$$

which we will call characteristic variable

$$u(t, x) = v(\beta(x) - t) \tag{3.9}$$

This formula may fail at points where  $c$  vanishes

$$\begin{aligned}t &= 0 \\ u(0, x) &= f(x) = v(\beta(x) - 0) \\ f(x) &= v(\beta(x)) \\ v(\xi) &= f(\beta^{-1}(\xi)) \\ &= f \cdot \beta^{-1}(\xi) \\ u(t, x) &= f \cdot \beta^{-1}(\beta(x) - t)\end{aligned}$$

This is the formula for the solution of (3.8) in the general case which is:

$$\begin{cases} \frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = 0 \\ u(0, x) = f(x) \end{cases}$$

**Example 2**

Let us apply the method of characteristics to solve the nonuniform transport equation:

$$\frac{\partial u}{\partial t} + \frac{1}{x^2 + 1} \frac{\partial u}{\partial x} = 0$$

The characteristic curves are the graphs of the solutions of the ODE:

$$\frac{dx}{dt} = \frac{1}{x^2 + 1}$$

$$\begin{aligned} \Rightarrow \beta(x) &= \int (x^2 + 1) dx \\ &= \frac{x^3}{3} + x \\ &= t + k \end{aligned}$$

It is not particularly helpful to find  $x(t)$  from this equation.

characteristic variable  $\xi$  is given by  $\xi = \frac{x^3}{3} + x - t$  and the general solution is of the form

$$u = v\left(\frac{x^3}{3} + x - t\right)$$

where  $v$  is any  $\mathbb{C}^1$  function.

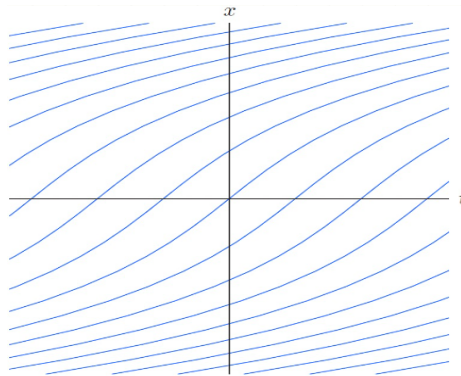
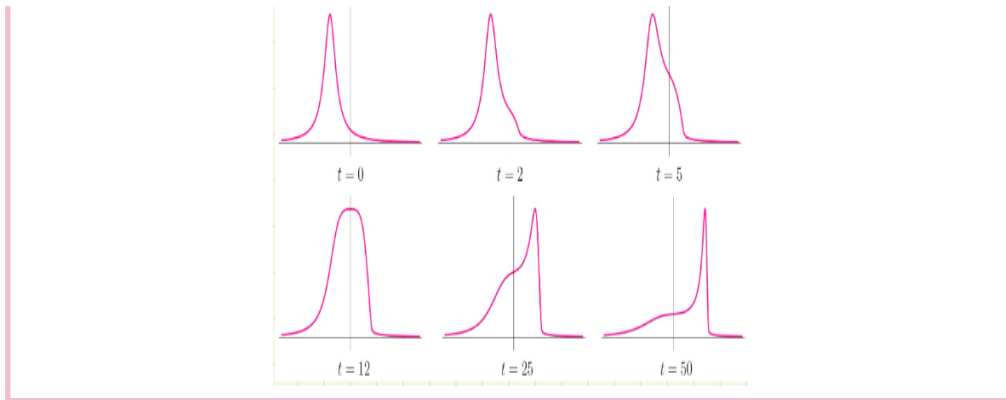


Figure 3.1: Characteristic curves for  $u_t + (x^2 + 1)^{-1} u_x = 0$

In the case of the initial condition:

$$u(0, x) = \frac{1}{1 + (x + 3)^2}$$

The solution looks as follows:



### §3.3 Lecture 3 (02-10)-Nonlinear transport and shocks

Nonlinear transport equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad (3.10)$$

characteristic equation:

$$\frac{dx(t)}{dt} = u(t, x(t)), \quad x(t) \text{ is the characteristic curve} \quad (3.11)$$

Let's show that  $u$  the solution of (3.10) is constant along the characteristic curves  $x(t)$  we need to check that  $h(t) = u(t, x(t))$  is constant of  $t$

$$\begin{aligned} \frac{d}{dt} h(t) &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} \\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot u(t, x(t)) \\ &= 0 \quad (\text{by (3.10)}) \end{aligned}$$

As  $u$  is constant along the characteristic curves, it must be of the form function of characteristic variable:

$$\begin{aligned} \xi &= x - ut \\ u(t, x) &= f(x - tu(t, x)) \end{aligned} \quad (3.12)$$

#### Example 1

Suppose the initial data is:

$$f(\xi) = \alpha\xi + \beta$$

Then (3.12) becomes:

$$u = \alpha(x - tu) + \beta$$

$$u(t, x) = \frac{\alpha x + \beta}{1 + \alpha t} \quad (3.13)$$

As far as  $1 + \alpha t \neq 0$ , the graph of the solution is line

If  $\alpha > 0$ ,

$$\begin{aligned} u(t, x) &\rightarrow 0 \\ t &\rightarrow +\infty \end{aligned}$$

If  $\alpha < 0$ , then the solution blows up as  $t \rightarrow -\frac{1}{\alpha}$  since  $\dot{x} = u(t, x(t))$   
As  $u(t, x(t))$  is a constant,

$$\begin{aligned} u(t, x(t)) &= u(0, x(0)) \\ &= u(0, y) \\ &= f(y) \end{aligned}$$

Our characteristic ODE is:

$$\begin{aligned} \dot{x} &= f(x) \\ \Rightarrow x(t) &= tf(y) + y \\ \Rightarrow u(t, x(t)) &= u(t, tf(y) + y) \\ &= u(0, y) \\ &= f(y), \forall t \\ u(t, tf(y) + y) &= f(y) \forall t, y \\ u(t, x) &= f(y) \\ x &= tf(y) + y \end{aligned}$$

I find  $y \in \mathbb{R}$  Suppose let  $f(y) = y$

$$\begin{aligned} x &= ty + y, y = \frac{x}{t+1} \\ \Rightarrow u(t, x) &= f(y) = y = \frac{x}{t+1} \\ \Rightarrow u(t, x) &= \frac{x}{t+1} \end{aligned}$$

which agrees with (3.13) for  $\alpha = 1, \beta = 0$

There is a problem with this construction that becomes clear when considering the following example:

$$u(0, x) = f(x) = \frac{1}{2}\pi - \tan^{-1}(x)$$

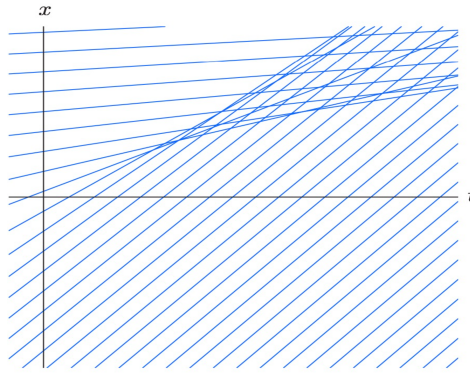


Figure 3.2: Characteristic lines for  $u(0, x) = \frac{1}{2}\pi - \tan^{-1}(x)$

Two characteristic lines that are not parallel must cross and the value of the solution is supposed to be equal the slope of the characteristic line passing through the point.

At a crossing point, the solution will have to be equal to two different values, one corresponding to each characteristic line.

While mathematically, such a multiply valued solutions are possible, they are physically untenable.

One needs to decide which (if any) of the possible values is the physically appropriate.

### §3.3.1 The wave equation:d'Alembert's formula

Let us consider the 1D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (3.14)$$

where  $c > 0$  is a given constant

the wave speed, the initial data are of the form:

$$u(0, x) = f(x), \quad \frac{\partial u}{\partial t} = g(x) \quad (3.15)$$

The initial value problem is to find a  $C^2$  function that solves (3.14) and satisfies (3.15)

In this section, we will consider only the 1D case posed on the whole line  $\mathbb{R}$

Let us define the wave operator:

$$\square = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$$

Thus the wave equation becomes:

$$\square u = 0$$

In analogy with the elementary factorization

$$t^2 - c^2 x^2 = (t - cx)(t + cx)$$

We can write

$$\square = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right)$$



Now, if

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \quad (3.16)$$

then  $u$  is automatically a solution of the wave equation, because

$$\square u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

Clearly, (3.16) is a transport equation with constant wave speed  $c$ , and its solutions are traveling waves with speed  $c$ :

$$u(t, x) = p(\xi) = p(x - ct)$$

where  $p$  is an arbitrary function.

If  $p \in \mathbb{C}^2$ ,  $u(t, x)$  is a classical solution of the wave equation (3.14).

The factorization of  $\square$  can be written in the reverse order:

$$\square = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)$$

Solving the transport equation:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0$$

with  $-c$  instead of  $c$ , we get:

$$u(t, x) = q(x + ct)$$

where  $q$  is an arbitrary function.

the solution represent traveling waves moving to the left with constant speed  $c > 0$

If  $q \in \mathbb{C}^2$ , we get a second family of solutions to the wave equation.

Thus there are both left and right traveling-wave solution,

Thus, we have two families of solutions of the wave equation:

$$\begin{aligned} u(t, x) &= p(x - ct) \\ u(t, x) &= q(x + ct), \forall p, q \in \mathbb{C}^2 \end{aligned}$$

Any linear combination of these solutions is still a solution of (3.14):

$$u(t, x) = p(x - ct) + q(x + ct), c > 0$$

### Theorem 3.3.1

Every solution to (3.14) can be written in the form

$$u(t, x) = p(x - ct) + q(x + ct), \forall p, q \in \mathbb{C}^2 \quad (3.17)$$

**Proof:**

$$\begin{aligned}\xi &= x - ct \\ \eta &= x + ct \\ x &= \frac{\xi + \eta}{2}, t = \frac{\eta - \xi}{2c} \\ u(t, x) &= u\left(\frac{\eta - \xi}{2c}, \frac{\xi + \eta}{2}\right) \\ \text{so, } u(t, x) &= v(\xi, \eta)\end{aligned}$$

Lets find a PED for v:

$$\begin{aligned}\frac{\partial u}{\partial t} &= c\left(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) \\ \frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) \\ \frac{\partial^2 u}{\partial x^2} &= c^2\left(\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) \\ 0 &= \square u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \\ &= -4c^2 \frac{\partial^2 v}{\partial \xi \partial \eta} = 0 \\ \Leftrightarrow \frac{\partial^2 v}{\partial \xi \partial \eta} &= 0 \\ \frac{\partial}{\partial \xi} \frac{\partial v}{\partial \eta} &= 0 \\ \Leftrightarrow \frac{\partial w}{\partial \xi} &= 0 \\ \Leftrightarrow w(\xi, \eta) &= w(\eta) \\ \Leftrightarrow w &= r(\zeta)\end{aligned}$$

where  $\zeta$  is an arbitrary function of  $\eta$

$$w = \frac{\partial v}{\partial \eta} = r(\zeta)$$

Integrating we get:

Now we want to solve the initial value problem (3.17):

$$\begin{aligned}
 u(t, x) &= p(x - ct) + q(x + ct) \\
 u(0, x) &= p(x) + q(x) = f(x) \\
 \frac{\partial u}{\partial t} &= -cp'(x) + cq'(x) = g(x) \\
 \begin{cases} p'(x) + q'(x) &= f(x) \\ -cp'(x) + cq'(x) &= g(x) \end{cases} \\
 2p'(x) &= f(x) + \frac{g(x)}{c} \\
 p(x) &= \frac{1}{2}f(x) - \frac{1}{2c} \int_0^x g(z)dz + a \\
 &\text{where } a \text{ is an integration constant} \\
 q(x) &= f(x) - p(x) \\
 &= \frac{1}{2} + \frac{1}{2c} \int_0^x g(z)dz - a \\
 u(t, x) &= p(\xi) + q(\eta) \\
 &= \frac{f(\xi) + f(\eta)}{2} - \frac{1}{2c} \int_0^\xi g(z)dz + \frac{1}{2c} \int_0^\eta g(z)dz \\
 &= \frac{f(\xi) + f(\eta)}{2} + \frac{1}{2c} \int_\eta^\xi g(z)dz
 \end{aligned}$$

This is called d'Alembert's solution to the IVP (3.15)

### Theorem 3.3.2: d'Alembert's solution

The solution of the problem:

$$\begin{aligned}
 \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\
 u(0, x) &= f(x) \\
 \frac{\partial u}{\partial t} &= g(x), x \in \mathbb{R}
 \end{aligned}$$

is given by:

$$u(t, x) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)dz, \forall f \in \mathbb{C}^2, g \in \mathbb{C}^1$$

Note that this formula provides a classical solution if  $f \in \mathbb{C}^2$  and  $g \in \mathbb{C}^1$

### External forcing:

If a homogeneous medium is subject to an external force  $F$ , then the system is described as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \quad (3.18)$$

We apply a generalized version of d'Alembert's technique. To simplify, we first impose homogeneous initial data:

$$\begin{aligned} u(0, x) &= 0 \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}$$

We search for a solution in the form  $u(t, x) = v(\xi, \eta)$  and from the chain rule, we get

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{1}{-4c^2} F\left(\frac{\eta - \xi}{2c}, \frac{\eta + \xi}{2}\right)$$

Integrating with respect to  $\eta$ , we get

$$\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \xi} = \frac{1}{-4c^2} \int_{\xi}^{\eta} F\left(\frac{\zeta - \xi}{2c}, \frac{\zeta + \xi}{2}\right) d\zeta$$

From the identities:

$$\frac{\partial u}{\partial t} = c\left(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right), \quad \frac{\partial u}{\partial x} = c\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right)$$

We get:

$$\frac{\partial v}{\partial \xi} = -\frac{1}{2c} \frac{\partial u}{\partial t} + \frac{1}{2c} \frac{\partial u}{\partial x}$$

Taking  $\xi = \eta$  :

$$\frac{\partial v}{\partial \xi} = -\frac{1}{2c} \frac{\partial u}{\partial t} + \frac{1}{2c} \frac{\partial u}{\partial x} = 0$$

because of homogeneous initial conditions. Thus (3.18) becomes:

$$\frac{\partial v}{\partial \xi} = \frac{1}{-4c^2} \int_{\xi}^{\eta} F\left(\frac{\zeta - \xi}{2c}, \frac{\zeta + \xi}{2}\right) d\zeta$$

Now, we integrate this with respect to  $\xi$ :

$$-v(\xi, \eta) = v(\eta, \eta) - v(\xi, \eta) = \frac{1}{-4c^2} \int_{\xi}^{\eta} \int_{\xi}^{\zeta} F\left(\frac{\zeta - \xi}{2c}, \frac{\zeta + \xi}{2}\right) d\zeta d\xi$$

since  $v(\xi, \xi) = 0$ , Thus:

$$\begin{aligned} v(\xi, \eta) &= \frac{1}{4c^2} \int_{\xi}^{\eta} \int_{\phi}^{\zeta} F\left(\frac{\zeta - \phi}{2c}, \frac{\zeta + \phi}{2}\right) d\zeta d\phi \\ &= \frac{1}{4c^2} \iint_{T(\xi, \eta)} F\left(\frac{\zeta - \phi}{2c}, \frac{\zeta + \phi}{2}\right) d\zeta d\phi \end{aligned}$$

That is the double integral takes place over the triangle

$$T(\xi, \eta) = \{(\phi, \zeta) : \xi \leq \phi \leq \zeta \leq \eta\}$$

Recalling that  $\xi = x - ct$  and  $\eta = x + ct$  and setting  $\phi = y - cs$  and  $\zeta = y + cs$ , the inequality  $\xi \leq \phi \leq \zeta \leq \eta$  becomes  $x - ct \leq y - cs \leq y + cs \leq x + ct$ , So  $T(\xi, \eta)$  can be rewritten as  $D(t, x) = \{(s, y) | x - c(t-s) \leq y \leq x + c(t-s), 0 \leq s \leq t\}$

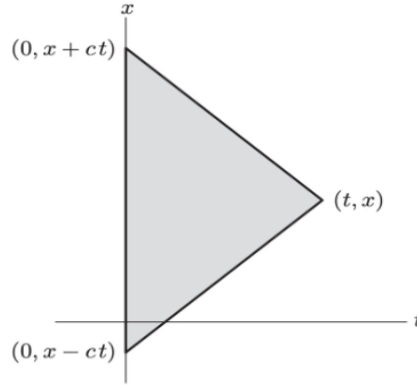


Figure 3.3: The Domain of integration  $D(t, x)$

Using the change of variables  $\frac{\zeta - \phi}{2c} = s$  and  $\frac{\zeta + \phi}{2} = y$  for the double integrals and computing the Jacobian, we get:  $\det \begin{pmatrix} \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial s} \\ \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial s} \end{pmatrix} = \det \begin{pmatrix} 1 & -c \\ 1 & c \end{pmatrix} = 2c$

Thus,

$$\begin{aligned} u(t, x) &= \frac{1}{2c} \iint_D (t, x) F(s, y) ds dy \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s, y) dy ds \end{aligned}$$

is the formula for the solution to the forced wave equation with homogeneous initial conditioni.

To solve the case of general initial condition, one needs to take the sum of the solution to the forced equation with homogeneous initial conditions plus the d'Alembert solution to the unforced equation subject to inhomogeneous boundary conditions.

## §3.4 Lecture 4 (02-12)

### Theorem 3.4.1

The solution of the problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \\ u(0, x) = f(x) \\ \frac{\partial u}{\partial t} = g(x) \end{cases}$$

is given by

$$u(t, x) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s, y) dy ds$$

The triangle  $D(t, x)$  is called the domain of dependence of the point  $(t, x)$

### Example 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \sin \omega t \sin x, u(0, x) = 0, \frac{\partial u}{\partial t} = 0$$

So:

$$\begin{aligned} F(t, x) &= \sin \omega t \sin x \\ u(t, x) &= \frac{1}{2c} \int_0^t \int_{x-t+s}^{x+t-s} \sin \omega s \sin y dy ds \\ &= \frac{1}{2} \int_0^t \sin \omega s [\cos(x - t + s) - \cos(x + t - s)] ds \\ &= \begin{cases} \frac{\sin \omega t - \omega \sin t}{1 - \omega^2} \sin x, & \omega \neq 1 \\ \frac{\sin t - t \cos t}{2} \sin x, & \omega = 1 \end{cases} \end{aligned}$$

When  $\omega \neq 1$ , the solution is bounded.

If  $\omega \in \mathbb{Q}$ , then the solution  $u$  is periodic in time.

If  $\omega \in \mathbb{I}$ , then  $u$  is quasi-periodic (it never exactly repeats) in time.

When  $\omega = 1$  the solution is unbounded.

### §3.4.1 Exercises

- ① Supposed  $u(t, x)$  is defined for all  $(t, x) \in \mathbb{R}^2$  and solves  $\frac{\partial u}{\partial t} + 2u = 0$ .  
Prove that  $\lim_{t \rightarrow +\infty} u(t, x) = 0$  for all  $x$   
SOL:

$$\begin{aligned} u(0, x) &= f(x) \\ u(t, x) &= e^{-2t} f(x) \\ \lim_{t \rightarrow +\infty} u(t, x) &= 0 \end{aligned}$$

- ② Let  $u(t, x)$  solve the initial value problem  $\frac{\partial u}{\partial t} + u^c = 0, u(0, x) = f(x)$ , where  $f(x)$  is a bounded  $\mathbb{C}^1$  function of  $x \in \mathbb{R}$
- (a): show that if  $f(x) \geq 0$  for all  $x$ , then  $u(t, x)$  is defined for all  $t > 0$ , and  $\lim_{t \rightarrow +\infty} u(t, x) = 0$
- (b): On the other hand, if  $f(x) < 0$ , then the solution  $u(t, x)$  is not defined for all  $t > 0$ , but in fact,  $\lim_{t \rightarrow r^-} u(t, x) = -\infty$ , for some  $0 < r < \infty$ . Given  $x$ , what is the corresponding value of  $r$ ?
- (c): Given  $f(x)$  as in part (b), what is the longest time interval  $0 < t < t_*$  on which  $u(t, x)$  is defined for all  $x \in \mathbb{R}$ ?

SOL:

$$\begin{aligned}\frac{\partial u}{\partial t} &= -u^2, u(0, x) = f(x) \\ \frac{du}{u^2} &= -dt \Rightarrow -\frac{1}{u} = -t - c(x) \\ \frac{1}{u} &= t + c(x) \Rightarrow u(t, x) = \frac{1}{t + c(x)} \\ u(0, x) &= f(x) = \frac{1}{c(x)} \\ u(t, x) &= \frac{1}{t + \frac{1}{f(x)}} = \frac{f(x)}{tf(x) + 1} \\ (a) : f(x) &\geq 0 \Rightarrow u(t, x) \text{ is defined for any } t \geq 0 \text{ and } \lim_{t \rightarrow +\infty} u(t, x) = 0 \\ (b) : f(x) &< 0 \Rightarrow \tau(x) = \frac{1}{f(x)} \Rightarrow u(t, x) = \frac{1}{t + \tau(x)} \\ (c) : If f(x) &< 0, [0, \tau(x)) \\ t_\star &= \inf\{\tau(x) : x \in \mathbb{R}\} = \inf_{x \in \mathbb{R}} \frac{1}{f(x)} = -\frac{1}{\inf_{x \in \mathbb{R}} f(x)}\end{aligned}$$

So we can have  $f(x) = \tan^{-1}(x)$

- ③ Solve the following initial value problem and graph the solution at times  $t=1, 2$ , and 3:

(a)  $u_t - 3u_x = 0, u(0, x) = e^{-x^2}$

Recall:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

We know that  $u(t, x) = v(\xi) = (x - ct)$

$$\forall v \in \mathbb{C}^1$$

$$u(0, x) = f(x) = v(x)$$

$$\text{Thus } u(t, x) = f(x - ct)$$

**In our case:**

$$u(t, x) = f(x + 3t) = e^{-(x+3t)^2}$$

- ④ Graph some of the characteristic lines for the following equations and write down a formula for the general solution:

(a)  $u_t - 3u_x = 0$ ,

SOL: The characteristic lines of the equation  $\frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = 0$  are the solution of the ODE:

$$\frac{dx}{dt} = c(x)$$

In our case  $\frac{dx}{dt} = -3 \rightarrow x(t) = -3t + k$

- ⑤ (a) Prove that if the initial data is bounded,  $|f(x)| \leq M$  for all  $x \in \mathbb{R}$ , then the solution to the damped transport equation  $\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + au = 0$  with  $a > 0$  satisfies

$u(t, x) \rightarrow 0$  as  $t \rightarrow \infty$

(b) Find a solution to it that is defined for all  $(t, x)$  but does not satisfy  $u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$

SOL: Recall the solution to it with initial condition  $u(0, x) = f(x)$  is given by

$$u(t, x) = f(x - ct)e^{-at}$$

$$|u(t, x)| = |f(x - ct)|e^{-at} \leq Me^{-at}$$

$$(b) f(x) = e^{Ax}$$

$$u(t, x) = e^{A(x-ct)-at} = e^{-(Ac+a)t+Ax}$$

$$\text{Choose } A \leq -\frac{a}{c}$$

⑥ (a) Write down a formula for the general solution to the nonlinear partial differential equation  $u_t + u_x + u^2 = 0$ .

(b) Show that if the initial data is positive and bounded,  $0 \leq u(0, x) = f(x) \leq M$ , then the solution exists for all  $t > 0$ , and  $u(t, x) \rightarrow 0$  as  $t \rightarrow \infty$ .

(c) On the other hand, if the initial data is negative somewhere, so  $f(x) < 0$  at some  $x \in \mathbb{R}$ , then the solution blows up in finite time:  $\lim_{t \rightarrow \tau^-} u(t, y) = -\infty$  for some  $\tau > 0$  and some  $y \in \mathbb{R}$ .

(d) Find a formula for the earliest blow-up time  $\tau_* > 0$ .

SOL: Writing  $u(t, x)$  in terms of the characteristic variable  $\xi = x - t$  we get:

$$u(t, x) = v(t, \xi) = u(t, \xi + t)$$

Let us find the equation for  $v$ :

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u^2 = -v^2 \\ \Rightarrow \frac{dv}{v^2} &= -dt \Rightarrow -\frac{1}{v} = -t - c(\xi) \\ \Rightarrow v(t, \xi) &= \frac{1}{t + c(\xi)} \\ \Rightarrow u(t, x) &= \frac{1}{t + c(x - t)} \end{aligned}$$

Where  $c$  is an arbitrary  $\mathbb{C}^1$  function.

(b): if there is an initial condition  $u(0, x) = f(x)$ , then we can determine  $c(x)$ :

$$\begin{aligned} u(0, x) &= f(x) = \frac{1}{c(x)} \\ \Rightarrow u(t, x) &= \frac{1}{t + \frac{1}{f(x-t)}} = \frac{f(x-t)}{tf(x-t) + 1} \end{aligned}$$

If  $0 \leq f(x)$ , then  $tf(x-t) + 1 \neq 0$ , so  $u(t, x)$  is defined for any  $t > 0$ .

The function  $\frac{s}{ts+1}$  is increasing in  $s$

Therefore:

$$0 \leq u(t, x) \leq \frac{M}{Mt + 1} \xrightarrow[\text{(as } t \rightarrow \infty)]{} 0 \text{ as } f(x) \leq M$$



(c): The solution formula implies that, if  $f(x) < 0$  then

$$\begin{aligned} u(t, x_*) &\rightarrow -\infty \\ \text{as } t \rightarrow \tau &= -\frac{1}{f(x)} \\ \text{where } x_* &= x - \frac{1}{f(x)} \end{aligned}$$

(d):

$$\tau_* = -\frac{1}{\inf_{x \in \mathbb{R}} f(x)}$$

### §3.5 Lecture 5 (02-17)-Exercise

#### Problem 1

2.2.17. (a) Solve the initial value problem  $u_t - xu_x = 0$ ,  $u(0, x) = (x^2 + 1)^{-1}$ . (b) Graph the solution at times  $t = 0, 1, 2, 3$ . (c) What is  $\lim_{t \rightarrow \infty} u(t, x)$ ?

*Solution.* (a)

(1) find characteristic curves by solving

$$\frac{dx}{dt} = -x$$

$$x(t) = x_0 e^{-t}$$

(2) find  $u$  from the information that  $u(t, x)$  is constant along the characteristics:

$$u(t, x(t)) = u(0, x(0)) = f(x(0))$$

$$u(t, x_0 e^{-t}) = f(x_0) = \frac{1}{x_0^2 + 1}$$

$$\text{Denote } x = x_0 e^{-t} \Rightarrow x_0 = e^t x$$

$$\begin{aligned} u(t, x) &= \frac{1}{e^{2t} x^2 + 1} \\ &= \frac{e^{-2t}}{x^2 + e^{-2t}} \end{aligned}$$

(b)  $t=0, 1, 2, 3$

$$(c) \lim_{t \rightarrow +\infty} u(t, x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

#### Problem 2

2.2.25. Suppose that  $c(x) \in C^1$  is continuously differentiable for all  $x \in \mathbb{R}$ . (a) Prove that the characteristic curves of the transport equation (2.16) cannot cross each other. (b) A point where  $c(x_*) = 0$  is known as a fixed point for the characteristic equation  $\frac{dx}{dt} = c(x)$ . Explain why the characteristic curve passing through a fixed point  $(t, x_*)$  is a horizontal straight line. (c) Prove that if  $x = g(t)$  is a characteristic curve, then so are all the horizontally translated curves  $x = g(t + \delta)$  for any  $\delta$ . (d) True or false:

Every characteristic curve has the form  $x = g(t + \delta)$ , for some fixed function  $g(t)$ . (e) Prove that each non-horizontal characteristic curve is the graph  $x = g(t)$  of a strictly monotone function. (f) Explain why a wave cannot reverse its direction. (g) Show that a non-horizontal characteristic curve starts, in the distant past,  $t \rightarrow -\infty$ , at either a fixed point or at  $-\infty$  and ends, as  $t \rightarrow +\infty$ , at either the next-larger fixed point or at  $+\infty$ .

*Solution.* (a)

$$\frac{dx}{dt} = c(x), x(t_0) = x_0$$

If  $x_1(t), x_2(t)$  are characteristic curves that cross each other at  $(t_0, x_0)$  then they both solve the same Cauchy problem.

By uniqueness, we have  $x_1(t) = x_2(t)$  for all  $t$

(b)

We know that  $x_*$  is a zero of  $C: C(x_*) = 0$  Let's consider the following problem:

$$\begin{cases} \frac{dx}{dt} = c(x) \\ x(0) = x_* \end{cases}$$

So,  $x(t) = x_*$

(c)

If  $g(t)$  is a characteristic curve

$$\frac{d}{dt}g(t) = c(g(t))$$

Then

$$g(t + \delta) = g_\delta(t) \text{ is also characteristic}$$

since it satisfies

$$\frac{d}{dt}g_\delta(t) = c(g_\delta(t))$$

since  $c$  does not depend on  $t$

(d)

False: If  $c$  admit a zero and  $c \neq 0$ , then  $\exists$  single  $g$  such that any characteristics of the form  $g(t + \delta)$ .

In deed, the fixed point solution and non-fixed point solutions are not given with the same  $g$ .

(e)

To determine the monotonicity of  $g$ , we need to check the sign of  $g'(t)$ .

it is always  $>0$  or  $<0$  in between the horizontal lines.

Then  $g'(t) = c(g(t))$ .

If  $\exists t_1$  and  $t_2$  so that  $g'(t_1) > 0$  and  $g'(t_2) < 0$  then  $\exists t_*$  between  $t_1, t_2$  s.t,  $c(g(t_*)) = 0$ , therefore  $x(t)$  is a fixed point solution and the graph should be a horizontal line

(f)

$c(x(t))$  is the speed of the wave

Reformulation: Can the function  $c(x(t))$  change its sign along a fixed characteristic curve?

If  $c(x(t_1)) > 0, c(x(t_2)) < 0$ , then exists  $c(x(t_*)) = 0$ , therefore  $x(t)$  is a fixed point solution and the graph should be a horizontal line

(g)

**Problem 3**

2.4.11.(a) Write down an explicit formula for the solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} = 0, \quad u(0, x) = \sin x, \quad \frac{\partial u}{\partial t}(0, x) = \cos x, \quad -\infty < x < \infty, \quad t \geq 0.$$

(b) True or false: The solution is a periodic function of  $t$ .

(c) Now solve the forced initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} = \cos 2t, \quad u(0, x) = \sin x, \quad \frac{\partial u}{\partial t}(0, x) = \cos x, \quad -\infty < x < \infty, \quad t \geq 0.$$

(d) True or false: The forced equation exhibits resonance. Explain.

(e) Does the answer to part (d) change if the forcing function is  $\sin 2t$ ?

*Solution.* (a)

$$\begin{aligned} u(t, x) &= \frac{\sin(x-2t) + \sin(x+2t)}{2} + \frac{1}{4}(\sin(x+2t) - \sin(x-2t)) \\ &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) \end{aligned}$$

(b)

T s.t.  $u(t+T, x) = u(t, x)$

$T = \pi$

(c)

$$\begin{aligned} u(t, x) &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4} \sum_0^t \sum_{x-2(t-s)}^{x+2(t-s)} \cos 2s \, dy \, ds \\ &= \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}(1 - \cos 2t) \end{aligned}$$

(d)

resonance:unbounded

False, no resonance since the solution is bounded

**Problem 4**

2.4.13. Let  $u(t, x)$  be a classical solution to the wave equation  $u_{tt} = c^2 u_{xx}$ . The total energy

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2} \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] dx$$

represents the sum of kinetic and potential energies of the displacement  $u(t, x)$  at time  $t$ . Suppose that  $\nabla u \rightarrow 0$  sufficiently rapidly as  $x \rightarrow \pm\infty$ ; more precisely, one can find  $\alpha > \frac{1}{2}$  and  $C(t) > 0$  such that  $|u_t(t, x)|, |u_x(t, x)| \leq C(t)/|x|^\alpha$  for each fixed  $t$  and all sufficiently large  $|x| \gg 0$ . For such solutions, establish the Law of Conservation of Energy by showing that  $E(t)$  is finite and constant. Hint: You do not need the formula for the solution.

*Solution.* We need to show that  $\frac{dE(t)}{dt} = 0$

$$\begin{aligned}
 \frac{d}{dt}E(t) &= \frac{d}{dt} \int_{-\infty}^{+\infty} \frac{1}{2} \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] dx \\
 &= \int_{-\infty}^{+\infty} \left[ \left( \frac{\partial u}{\partial t} \right) \cdot \frac{\partial^2 u}{\partial t^2} + c^2 \left( \frac{\partial u}{\partial x} \right) \frac{\partial^2 u}{\partial x \partial t} \right] dx \\
 &= c^2 \int_{-\infty}^{+\infty} \left[ \left( \frac{\partial u}{\partial t} \right) \cdot \frac{\partial^2 u}{\partial x^2} + \left( \frac{\partial u}{\partial x} \right) \frac{\partial^2 u}{\partial x \partial t} \right] dx \\
 &= c^2 \int_{-\infty}^{+\infty} 1 \cdot \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right) dx \text{ integration by parts} \\
 &= -c^2 \int_{-\infty}^{+\infty} \underbrace{\frac{\partial}{\partial x} 1}_{=0} \cdot \left( \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right) dx + 0(\text{boundary terms}) \\
 &= 0
 \end{aligned}$$

### §3.6 Lecture 6 (02-19)–Fourier Series

#### Problem 5

2.4.15

*Solution.*

$$\begin{aligned}
 E(t) &= \int_{-\infty}^{+\infty} \left[ \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] dx \\
 \frac{dE(t)}{dt} &= \int_{-\infty}^{+\infty} \left[ \frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x \partial t} \right] dx \\
 &= \int_{-\infty}^{+\infty} \left[ c^2 \left( \frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x \partial t} \right) - a \left( \frac{\partial u}{\partial t} \right)^2 \right] dx \\
 &= c^2 \int_{-\infty}^{+\infty} 1 \cdot \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right) dx - a \sum_{-\infty}^{+\infty} \left( \frac{\partial u}{\partial t} \right)^2 dx \\
 &\leq 0 \Leftrightarrow \frac{dE(t)}{dt} \leq 0
 \end{aligned}$$

So,  $E(t)$  is decreasing

We need to prove that there is a unique solution to the problem 
$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \\ u(0, x) = f(x) \\ \frac{\partial u}{\partial t} = g(x) \end{cases}$$

Suppose  $\exists u_1(t, x), u_2(t, x)$

$$u(t, x) = u_1(t, x) - u_2(t, x)$$

$u$  satisfies the equation with initial data

$$u(0, x) = \frac{\partial u}{\partial t} = 0$$

We need to show that  $u(t, x) = 0 \forall t, x$

$$E(0) = \int_{-\infty}^{+\infty} \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \left( \frac{\partial u}{\partial x} \right)^2 dx = 0$$

Since  $E(0)=0$ ,  $\frac{dE(t)}{dt} \leq 0$ ,  $E(t \geq 0) \Rightarrow E(t) = 0 \Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} = E$ ,  $\forall x, t \Rightarrow u$  is constant  $\Rightarrow u(t, x) = 0$

### Problem 6

2.4.17

*Solution.*

$$\square = \frac{\partial^2}{\partial t^2} - c^2(x) \frac{\partial^2}{\partial x^2}$$

$$\left(\frac{\partial}{\partial t} + c(x) \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - c(x) \frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - c^2(x) \frac{\partial^2}{\partial x^2} - c(x)c'(x) \frac{\partial}{\partial x}$$

since  $c'(x) \neq 0$ , the d'Alembert formula does not apply

### §3.6.1 Fourier Series

The linear heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial t} = L[u] \end{cases} \quad (3.19)$$

$$L[u] = \frac{\partial^2 u}{\partial x^2}$$

$$L[u + v] = L[u] + L[v]$$

$$L[\alpha u] = \alpha L[u]$$

These are true for any functions  $u$  and  $v$  in  $\mathbb{C}^2$

Let us find solutions to (3.19) in the form.

$$u(t, x) = e^{\lambda t} v(x)$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} [e^{\lambda t} v(x)] \\ &= \lambda e^{\lambda t} \\ L[u] &= L[e^{\lambda t} v] \\ &= \frac{\partial^2}{\partial x^2} [e^{\lambda t} v] \\ &= e^{\lambda t} \frac{\partial^2}{\partial x^2} v \\ &= e^{\lambda t} L[v] \\ \frac{\partial u}{\partial t} &= L[u] \end{aligned}$$

Thus  $u = e^{\lambda t} v$  will be a solution to (3.19) iff

$$L[v] = v'' = \lambda v$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x \in \mathbb{R}$$

Any nonzero linear combination of these particular solutions is still a solution of (3.19)  
For example

$$u(t, x) = c_1 e^{-t} \cos x + c_2 e^{-4t} \sin 2x + c_3 x + c_4$$

The heated ring

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(t, -\pi) &= u(t, \pi) \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x} \\ u(0, x) &= f(x) \\ u(t, x) &= e^{\lambda t} v(x) \\ v'' &= \lambda v \\ v(-\pi) &= v(\pi), v'(-\pi) = v'(\pi)\end{aligned}$$

find  $\lambda$  and  $v(x)$   
case 1:  $\lambda = \omega^2 > 0$

$$\begin{aligned}v(x) &= ae^{\omega x} + be^{-\omega x}, \forall a, b \in \mathbb{R} \\ ae^{\omega\pi} + be^{-\omega\pi} &= ae^{-\omega\pi} + be^{\omega\pi} \\ a\omega e^{-\omega\pi} - b\omega e^{\omega\pi} &= a\omega e^{\omega\pi} - b\omega e^{-\omega\pi} \\ \omega &\neq 0 \\ e^{\omega\pi}(b-a) + (a-b)e^{-\omega\pi} &= 0 \\ (e^{\omega\pi} - e^{-\omega\pi})(b-a) &= 0 \\ \Leftrightarrow \underbrace{(e^{2\omega\pi} - 1)}_{\neq 0}(b-a) &= 0 \\ &\Rightarrow a = b \text{ and } a = -b \\ &\Rightarrow a = b = 0 \Rightarrow v(x) = 0\end{aligned}$$

case 2:  $\lambda = 0$

$$\begin{aligned}v'' &= 0 \\ v(x) &= a + bx, \forall a, b \in \mathbb{R} \\ a - b\pi &= a + b\pi \\ &\Rightarrow b = 0, \forall a \in \mathbb{R} \\ \text{Thus } v(x) &= a \\ u(t, x) &= e^{\lambda t} v(x) = a\end{aligned}$$

case 3:  $\lambda = -\omega^2 < 0$

$$\begin{aligned}
 v'' &= \lambda v \\
 v(x) &= a \cos \omega x + b \sin \omega x, \forall a, b \in \mathbb{R} \\
 a \cos \omega \pi - b \sin \omega \pi &= a \cos \omega \pi + b \sin \omega \pi \\
 a \sin \omega \pi + b \cos \omega \pi &= -a \sin \omega \pi + b \cos \omega \pi \\
 2b \sin \omega \pi &= 0, 2a \sin \omega \pi = 0 \\
 \text{If } \sin \omega \pi &\neq 0 \\
 \text{then } a &= b = 0 \text{ If } \sin \omega \pi = 0 \\
 \Rightarrow \forall a, b \in \mathbb{R}, \omega &= 1, 2, \dots \\
 v(x) &= a \cos kx + b \sin kx \forall a, b \in \mathbb{R} \\
 \text{With } \lambda &= -k^2 \\
 u_k(t, x) &= e^{-k^2 t} (\cos kx) \\
 u'_k(t, x) &= e^{-k^2 t} (\sin kx)
 \end{aligned}$$

By Fourier:

$$u(t, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k e^{-k^2 t} \cos kx + b_k e^{-k^2 t} \sin kx]$$

With coefficients  $a_k, b_k$ , the parameters  $a_k, b_k$  are fixed such that

$$u(0, x) = f(x)$$

### §3.6.2 Fourier Series 2

We introduce  $L^2$ -inner product for continuous functions on the interval  $[-\pi, \pi]$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

Basic properties of the inner product:

- symmetry:  $\langle f, g \rangle = \langle g, f \rangle$
- bilinearity:  $\langle \alpha f_1 + \beta f_2, g \rangle = \alpha \langle f_1, g \rangle + \beta \langle f_2, g \rangle$
- positivity:  $\langle f, f \rangle > 0$  if  $f \neq 0$ , and  $\langle 0, 0 \rangle = 0$

The associated norm ( $L^2$ -norm) is defined by

$$\|f\| = \sqrt{\langle f, f \rangle}$$

#### Lemma 3.6.1

The trigonometric functions  $1, \cos x, \sin x, \cos(2x), \sin(2x), \dots$  satisfies the rela-

tions

$$\langle \cos(kx), \cos(lx) \rangle = \langle \sin(kx), \sin(lx) \rangle = 0, \forall k \neq l \text{ integers}$$

$$\langle \cos(ux), \sin(lx) \rangle = 0, \forall u, l$$

$$\|1\| = \sqrt{2}$$

$$\|\cos(kx)\| = \|\sin(kx)\| = 1, \forall k \neq 0$$

$$f \perp g \Leftrightarrow \langle f, g \rangle = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

$$\begin{aligned} \langle f, \cos(lx) \rangle &= \frac{a_0}{2} \langle 1, \cos(lx) \rangle + \sum_{k=1}^{\infty} [a_k \langle \cos kx, \cos lx \rangle + b_k \langle \sin kx, \cos lx \rangle] \\ &= a_l \langle \cos lx, \cos lx \rangle \\ &= a_l \\ b_l &= \langle f, \sin(lx) \rangle \end{aligned}$$

### §3.7 Lecture 7 (02-24)

#### Definition 3.7.1

The Fourier series of a function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  is

$$f \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

where  $a_k = \langle f, \cos(kx) \rangle$ ,  $b_k = \langle f, \sin(kx) \rangle$ ,  $k = 1, 2, \dots$

#### Example 1

$$\textcircled{1} \quad f(x) = x$$

$$a_k = 0, b_k = \frac{2}{k}(-1)^{k+1}$$

$$\text{Therefore, } x \approx 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right)$$

#### Definition 3.7.2: Piecewise continuous function

A function  $f : [a, b] \rightarrow \mathcal{R}$  is piecewise continuous if it is defined and continuous except possibly at finite number of points

#### Proposition 3.7.0: even and odd function

If  $f$  is even, then  $b_k = 0$



If  $f$  is odd, then  $a_k = 0$

### Example 2

①

$$f(x) = |x|, \text{ even}$$

$$a_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ -\frac{4}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}$$

by theorem, this series converges to the  $2\pi$  periodic extension of  $|x|$

Let  $x = 0$ , we have

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

② computing  $\sum_{k=1}^{\infty} \frac{1}{k^2} = S$

$$\frac{S}{4} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$

$$\frac{3}{4}S = S - \frac{S}{4} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{6}$$

### §3.7.1 Complex Fourier series

Euler formulas:

$$e^{-ikx} = \cos(kx) - i \sin(kx)$$

$$\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx})$$

$$\sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx})$$

For any  $f, g : [-\pi, \pi] \rightarrow \mathbb{C}$

$L^2$ -Hermitian inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

where  $\overline{g(x)}$  is the complex conjugate of  $g(x)$

$L^2$ -Hermitian norm is defined by

$$\|f\| = \sqrt{\langle f, f \rangle}$$

Direct computation shows that

$$\langle e^{ikx}, e^{ielx} \rangle = \begin{cases} 1, & \text{if } k = e \\ 0, & \text{if } k \neq e \end{cases}$$

and

$$||e^{ikx}|| = 1$$

If  $f$  is a given continuous (or piecewise continuous) complex function then

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

with  $c_k = \langle f, e^{ikx} \rangle$   
It is easy to see that

$$\begin{aligned} a_k &= c_k + c_{-k} \\ b_k &= i(c_k - c_{-k}) \\ c_k &= \frac{1}{2}(a_k - ib_k) \\ c_{-k} &= \frac{1}{2}(a_k + ib_k) \\ k &= 0, 1, 2, \dots \end{aligned}$$

### Example 3

① Let  $\sigma$  be the step function

$$\sigma(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

then

$$c_k = \begin{cases} \frac{1}{\pi i k}, & \text{if } k \text{ odd} \\ \frac{1}{2}, & \text{if } k = 0 \\ 0, & \text{if } k \text{ even} \end{cases}$$

$$\sigma(x) \approx \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{\pi i k} e^{ikx}$$

## §3.8 Lecture 8 (02-26)–Fourier Series

### Problem 1

2.2.13

*Solution.* Hint:

$$D = \{(x, t) \in \mathbb{R}^2 : x, t \geq 0\}$$

find  $u = u(t, \xi)$

### §3.8.1 Differentiation and integration of Fourier series

In general the integral of periodic functions is not periodic, eg:  $f(x)=1$

**Lemma 3.8.1**

if  $f$  is  $2\pi$ -periodic and piecewise continuous function, then

$$g(x) = \int_0^x f(y)dy$$

is also  $2\pi$ -periodic iff

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

**Theorem 3.8.1**

if  $f$  is piecewise continuous and has zero mean on  $[-\pi, \pi]$ , then its fourier series:

$$f(x) \approx \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

can be integrated term by term

$$g(x) = \int_0^x f(y)dy \approx m + \sum_{k=1}^{\infty} \frac{1}{k} [a_k \sin kx - b_k \cos kx]$$

$$\text{where } m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)dx \text{ (integration by parts)}$$

**Example 1**

$$\textcircled{1} \quad f(x) = x$$

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

$$x \approx 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

$$\frac{x^2}{2} \approx \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos kx$$

$\bar{f}$  theorem at  $x = 0$

$$0 = \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12}$$

**Theorem 3.8.2**

if  $f$  is piecewise  $C^2$  and  $2\pi$ -periodic extension and

$$f(x) \approx \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

then

$$f'(x) \approx \sum_{k=1}^{\infty} kb_k \cos kx - ka_k \sin kx = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$$

(computing  $\langle f'(x), \cos(kx) \rangle$  using integration by parts)

### Example 2

①  $f(x) = |x|$

$$\begin{aligned} \frac{d}{dx}|x| &= \text{sign}(x) \\ &= \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases} \end{aligned}$$

We know the fourier series of  $|x|$ ,

$$\text{sign}(x) \approx \frac{4}{\pi}(\sin(x) + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots)$$

### §3.8.2 change of scale

$$2\pi \rightarrow 2l$$

$$y \in [-\pi, \pi] \rightarrow x \in [-l, l]$$

$$x = \frac{l}{\pi}y$$

$$f : [-l, l] \rightarrow \mathbb{R} \text{ piecewise continuous}$$

### Definition 3.8.1

$$f \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x]$$

Fourier series of  $f$  where

$$a_k = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{k\pi}{l}x dx$$

$$b_k = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{k\pi}{l} x dx$$

All results seen before remain valid in this more general setting

$$f : [a, b] \rightarrow \mathbb{R}$$

$$l = \frac{b-a}{2}$$

$$\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$$

$b-a$  periodic

### §3.8.3 Convergence of Fourier series

A sequence of functions  $v_n$  converges pointwise to a function  $v_*$  on a set  $I \subset \mathbb{R}$  if

$$\lim_{n \rightarrow \infty} v_n(x) = v_*(x), \forall x \in I$$

$\Leftrightarrow$  For any  $\epsilon > 0, \forall x \in I, \exists N = N(\epsilon, x)$  such that  $|v_n(x) - v_*(x)| < \epsilon \forall n \geq N$

A sequence of functions  $v_n$  converges uniformly to a function  $v_*$  on a set  $I \subset \mathbb{R}$  if for any  $\epsilon > 0, \exists N = N(\epsilon)$  such that

$$|v_n(x) - v_*(x)| < \epsilon, \forall x \in I$$

uniform convergence eg:

$$v_n(x) = \frac{1}{n} \sin(x)$$

$$v_n \rightarrow 0 \text{ uniformly}$$

$$|v_n(x) - 0| = \frac{1}{n} |\sin(x)|$$

$$n > N(\epsilon) = \left\lceil \frac{1}{\epsilon} \right\rceil + 1$$

uniform convergence  $\Leftrightarrow \sup_{x \in I}$

Recall that if a sequence of continuous functions  $v_n$  on  $I$  converges to  $v_*$  uniformly, then  $v_* : I \rightarrow \mathbb{R}$  is continuous on  $I$

If  $U_N(x) \rightarrow U_*(x)$  pointwise on some set  $I$ , we say that the series converges pointwise if

$$U_N(x) \rightarrow U_*(x) \text{ uniformly on } I$$

We say that the series converges uniformly

In particular if  $U_n(x)$  are continuous on a set  $I$  and the series converges uniformly, then the sum is continuous function

#### Theorem 3.8.3: Weierstrass m-test

Let  $I \subset \mathbb{R}$

$$|u_k(x)| \leq M_k, \forall x \in I, k \geq 1$$

where  $M_k \geq 0$  are numbers such that  $\sum_{k=1}^{\infty} M_k < \infty$

Then the series  $\sum_{k=1}^{\infty} u_k(x)$  is uniformly convergent to a function on  $I$ .  
In particular, if  $u_k(x)$  are continuous on  $I$ , then  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$

### Proposition 3.8.0

If  $\sum_{k=1}^{\infty} u_k$  converges uniformly on an interval  $I \subset \mathbb{R}$  then,

$$\int_0^x \left( \sum_{k=1}^{\infty} u_k(y) \right) dy = \sum_{k=1}^{\infty} \int_0^x u_k(y) dy$$

### Proposition 3.8.0

Suppose  $\sum_{k=1}^{\infty} u_k(x)$  converges pointwise on an interval  $I \subset \mathbb{R}$  and  $u_k$  are all  $C^1$  functions on  $I$  with  $\sum_{k=1}^{\infty} u'_k$  convergent uniformly.  
Then  $(\sum_{k=1}^{\infty} u_k)'$  exists and we have the equality

$$\left( \sum_{k=1}^{\infty} u_k \right)' = \sum_{k=1}^{\infty} u'_k$$

For

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

x-real variable

$$|e^{ikx}| = 1$$

$$|c_k e^{ikx}| = |c_k| = M_k, \forall x \in \mathbb{R}$$

### Theorem 3.8.4

Let the Fourier coefficients of  $f$  satisfies the condition

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty$$

then the Fourier series of  $f$  converges uniformly to a continuous function  $\bar{f}$  with same coefficients  $c_k = \langle f, e^{ikx} \rangle = \langle \bar{f}, e^{ikx} \rangle$

**Proof.** The Fourier series of

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

as  $\sum_{k=-\infty}^{\infty} |c_k| < \infty$

This series satisfies the conditions of the Weierstrass m-test with  $M_k = |c_k|, u_k(x) = c_k e^{ikx}$

$$\Rightarrow \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \bar{f}(x)$$

converges uniformly to a function  $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$  periodic continuous  
Warning:  $f$  is not necessarily equal to  $\bar{f}$  ■

**Corollary 3.8.1**

$f$  is  $2\pi$  periodic and piecewise  $C^1$  and  $f$  is continuous on the interval

$$[a, b] \subset [-\pi, \pi]$$

, then the Fourier series of  $f$  converge uniformly to  $f$  on any closed subinterval  $(a + \delta, b - \delta)$  for  $0 < \delta < \frac{b-a}{2}$

**§3.9 Lecture 9 (03-02)–Exercises****Problem 1**

3.1.4

eigensoltuion:  $u(t, x) = \phi(t)v(x)$

$$\begin{aligned}\frac{\partial u}{\partial t} &= \phi'(t)v(x) \\ \frac{\partial u}{\partial x} &= \phi(t)v'(x) \\ \Rightarrow \phi'(t)v(x) &= \phi(t)v'(x) \\ \Rightarrow \frac{\phi'(t)}{\phi(t)} &= \frac{v'(x)}{v(x)} = \lambda \\ \frac{\phi'(t)}{\phi(t)} = \lambda &\Rightarrow \phi(t) = c_1 e^{\lambda t} \\ \frac{v'(x)}{v(x)} = \lambda &\Rightarrow v(x) = c_2 e^{\lambda x} \\ u(t, x) &= c_1 c_2 e^{\lambda(t+x)} = c e^{\lambda(t+x)}, \forall c \in \mathbb{R}\end{aligned}$$

**Problem 2**

3.1.7

(a)

$$\begin{aligned}\frac{\partial u}{\partial t} &= \phi'(t)v(x) \\ \phi'(t)v(x) &= \phi(t)(x^2 v'(x))' \\ \frac{\phi'(t)}{\phi(t)} &= \frac{(x^2 v'(x))'}{v(x)} = \lambda \\ \frac{\phi'(t)}{\phi(t)} = \lambda &\Rightarrow \phi(t) = c_1 e^{\lambda t} \\ \frac{(x^2 v'(x))'}{v(x)} = \lambda &\Rightarrow x^2 v''(x) + 2x v'(x) = \lambda v(x)\end{aligned}$$

Cauchy-Euler equation

$$a(a-1) + 2a = \lambda$$

$$a^2 + a = \lambda$$

$$\Delta = 1 + 4\lambda$$

$$(1) : \lambda > -\frac{1}{4} \Rightarrow a_{1,2} = \frac{-1 \pm \sqrt{1+4\lambda}}{2}$$

$$v(x) = c_1 x^{a_1} + c_2 x^{a_2}$$

$$u(t, x) = e^{\lambda t} (c_1 x^{a_1} + c_2 x^{a_2}), \forall c_1, c_2$$

$$(2) : \lambda = -\frac{1}{4} \Rightarrow a = -\frac{1}{2}$$

$$v(x) = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$$

$$u(t, x) = e^{-\frac{1}{4}t} (c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x), \forall c_1, c_2$$

$$(3), \lambda < -\frac{1}{4} \Rightarrow a_{1,2} = \frac{-1 \pm i\sqrt{4\lambda+1}}{2}$$

$$v(x) = c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x$$

$$u(t, x) = e^{\lambda t} (c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x), \forall c_1, c_2$$

$$(b) : \text{case 1: } u(t, x) = e^{\lambda t} (c_1 x^{a_1} + c_2 x^{a_2})$$

$$u(t, 1) = 0 \Rightarrow c_1 + c_2 = 0$$

$$u(t, 2) = 0 \Rightarrow c_1 \cdot 2^{a_1} + c_2 \cdot 2^{a_2} = 0$$

$$\det = 2^{a_2} - 2^{a_1} \neq 0$$

$$\Rightarrow c_1 = c_2 = 0$$

$$\text{case 2: } u(t, x) = e^{\lambda t} (c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x)$$

$$u(t, 1) = 0 \Rightarrow c_1 + c_2 \ln 1 = 0 \rightarrow c_1 = 0$$

$$u(t, 2) = 0 \Rightarrow c_2 2^{-\frac{1}{2}} \ln 2 = 0 \rightarrow c_2 = 0$$

$$\text{case 3: } u(t, x) = e^{\lambda t} (c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x)$$

$$u(t, 1) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \rightarrow c_1 = 0$$

$$u(t, 2) = 0 \Rightarrow c_2 \sin \frac{\sqrt{4\lambda+1}}{2} \ln 2 = 0$$

$$c_2 \neq 0, \frac{1+4\lambda}{2} \ln 2 = k\pi$$

$$\lambda_k = -\frac{1}{4} - \frac{k^2 \pi^2}{(\ln 2)^2}$$

$$u_k(x, t) = x^{-\frac{1}{2}} \sin k\pi \frac{\ln x}{\ln 2} e^{\lambda_k t}, \forall k \geq 1$$



**Problem 3**

3.2.4

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

$$x \sim e^{ikx} = \cos(kx) + i \sin(kx)$$

$$a_k = \langle g, \cos(kx) \rangle = \frac{1}{2} p_0 \langle 1, \cos(kx) \rangle + \sum_{j=1}^{\infty} p_j \langle \cos(jx), \cos(kx) \rangle$$

$$= \begin{cases} 0, & \text{if } k > n \\ p_k, & \text{if } k \in [0, n] \end{cases}$$

$$b_k = \langle g, \sin(kx) \rangle = \frac{1}{2} p_0 \langle 1, \sin(kx) \rangle + \sum_{j=1}^{\infty} p_j \langle \cos(jx), \sin(kx) \rangle + q_j \langle \sin(jx), \sin(kx) \rangle$$

$$= \begin{cases} 0, & \text{if } k > n \\ q_k, & \text{if } k \in (0, n] \end{cases}$$

**Problem 4**

3.2.9

$$(a) \int_a^{a+l} = \int_l^{a+l} + \int_0^l - \int_0^a$$

$$\text{we need to prove } \int_l^{a+l} = \int_0^a$$

$$\int_l^{a+l} f(x) dx = \int_0^a f(s+l) ds$$

change of variable  $x = s + l$ 

$$\int_l^{a+l} f(x) dx = \int_0^a f(s) ds$$

$$(b) \int_0^l f(x+a) dx = \int_a^{a+l} f(s) ds \text{ (change } x+a=s) \\ = \int_0^l f(s) ds$$

## §3.9.1 Hilbert space

$L^2 = L^2[-\pi, \pi]$  is the space of square integrable functions

$$f : [-\pi, \pi] \rightarrow \mathbb{R}$$

$$\|f\| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x) dx} \text{ is called } L^2 \text{ norm}$$

$$\|1\| = 1$$

The triangle inequality  $\|f + g\| \leq \|f\| + \|g\| \forall f, g \in L^2$

$$\|cf\| = |c|\|f\|, \forall c \in \mathbb{R}$$

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(\bar{x})dx \text{ is the } L^2\text{-Hermitian inner product of } f, g \in L^2$$

Cauchy-Schwarz inequality  $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$

$$c_k = \langle f, e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-iux} dx$$

Every piecewise continuous function belongs to  $L^2$

$$\frac{1}{|x|^\alpha} \in L^2 \text{ if } \alpha < \frac{1}{2}$$

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\pi} + \int_{-\pi}^{-\epsilon} \frac{1}{|x|^{2\alpha}} \text{ limit exists iff } \alpha < \frac{1}{2}$$

**Definition 3.9.1**

Let  $v_n$  be a sequence in  $L^2$  we say that  $v_n \rightarrow f$  in  $L^2$  if

$$\lim_{n \rightarrow \infty} \|v_n - f\| = 0$$

**Definition 3.9.2**

Let  $\{\phi_n\}$  be a sequence in  $L^2$  we say that  $\{\phi_n\}$  is a an orthonormal system if

$$\langle \phi_j, \phi_k \rangle = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$$

$$f \sim \sum_{k=1}^{\infty} c_k \phi_k$$

**Theorem 3.9.1**

Let  $\{\phi_n\}$  be an orthonormal system in  $L^2$  and

$$\begin{aligned} V_n &= \text{span}\{\phi_1, \phi_2, \dots, \phi_n\} \\ &= \{\alpha_1 \phi_1, \dots, \alpha_n \phi_n\} \end{aligned}$$

is the  $n$ -dimensional space spanned by the independent elements  $\phi_1, \phi_2, \dots, \phi_n$ .  
Then for any  $f$  we have

$$\|f - s_n\| = \inf_{p \in V_n} \|f - p\|$$

where

$$s_n = \sum_{k=1}^n c_k \phi_k$$

$$c_n = \langle f, \phi_k \rangle$$

## Chapter 4

# Algorithm

### §4.1 Lecture 1 (02-03) Asymptotic bounds and big O

#### Definition 4.1.1: big-Theta notation(=)

Let  $f, g : N \rightarrow R^+$ , we say that  $f(n) = \Theta(g(n))$  if there  $\exists a, b > 0$  and  $n_0 \in \mathbb{N}$  such that  $a \cdot g(n) \leq f(n) \leq b \cdot g(n)$  for all  $n > n_0$ .

Intuitively,  $f(n) = \Theta(g(n))$  means that  $f(n)$  grows as fast as  $g(n)$  up to a constant when  $n$  tends to infinity.

Examples:  $3n = \Theta(n)$ ,  $2n^2 + 7n = \Theta(n^2)$ ,  $\ln n^2 = \Theta(\log n)$

#### Definition 4.1.2: big-O notation( $\leq$ )

Let  $f, g : N \rightarrow R^+$ , we say that  $f(n) = O(g(n))$  if  $\exists n_0 \in \mathbb{N}$  and  $b > 0$  such that  $f(n) \leq b \cdot g(n)$  for all  $n > n_0$ .  
 $\lim_{n \rightarrow \infty} \sup \frac{f(n)}{g(n)} < +\infty$

#### Definition 4.1.3: big-Omega notation( $\geq$ )

Let  $f, g : N \rightarrow R^+$ , we say that  $f(n) = \Omega(g(n))$  if  $\exists n_0 \in \mathbb{N}$  and  $a > 0$  such that  $a \cdot g(n) \leq f(n)$  for all  $n > n_0$ .  
 $\lim_{n \rightarrow \infty} \inf \frac{f(n)}{g(n)} > 0$

#### Remark 1

$$f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$
$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

#### Definition 4.1.4: little-o notation( $<$ )

We write  $f(n) = o(g(n))$  if for any  $b > 0$  there  $\exists n_0 \in \mathbb{N}$  such that  $f(n) < b \cdot g(n)$  for all  $n > n_0$ .

**Definition 4.1.5: little-omega notation( $>$ )**

We write  $f(n) = \omega(g(n))$  if for any  $a > 0$  there  $\exists n_0 \in \mathbb{N}$  such that  $a \cdot g(n) < f(n)$  for all  $n > n_0$ .

**Remark 2**

$f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$  and  $f(n) \neq \Theta(g(n))$   
 $f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n))$  and  $f(n) \neq \Theta(g(n))$   
 $f(n) = \omega(g(n)) \Leftrightarrow g(n) = \omega(f(n))$

$$n^{1+e}, n \log(n), e \rightarrow 0$$

**Theorem 4.1.1**

Any Algorithm has to check at least  $\log(n+1)$  entries of the sorted array for searching an element in the worst case

**§4.2 Recitation 1 (02-07)–Problem Solving**

Notation	Bounding Conditions	Relation	Type
$T(n) = O(f(n))$	$\exists c > 0, \exists n_0, \forall n \geq n_0, T(n) \leq cf(n)$	$\leq$	$O$
$T(n) = \Omega(f(n))$	$\exists c > 0, \exists n_0, \forall n \geq n_0, T(n) \geq cf(n)$	$\geq$	$\Omega$
$T(n) = \Theta(f(n))$	$\exists c_1, c_2 > 0, \exists n_0, \forall n \geq n_0, c_1 f(n) \leq T(n) \leq c_2 f(n)$	$=$	$\Theta$
$T(n) = o(f(n))$	$\forall c > 0, \exists n_0, \forall n \geq n_0, T(n) < cf(n)$	$<$	$o$
$T(n) = \omega(f(n))$	$\forall c > 0, \exists n_0, \forall n \geq n_0, T(n) > cf(n)$	$>$	$\omega$

**Possible useful formula:**

$$\int_1^n x^k dx = \frac{n^{k+1}}{k+1} - \frac{1}{k+1}$$

1. Multiplication Rule. Prove that for  $h(n) = \Omega(1)$ , if  $f(n) = \Theta(g(n))$  then  $f(n) \cdot h(n) = \Theta(g(n) \cdot h(n))$  (The same also holds for  $O, \Omega, \omega, \omega$ )

**Solution:**

Since  $f(n) = \Theta(g(n))$ ,  $\exists c_1, c_2 > 0, \exists n_0$  such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0$$

Since  $h(n) = \Omega(1)$ , there exist constants  $c_3 > 0$  and  $n_1$  such that

$$h(n) \geq c_3, \forall n \geq n_1$$

Combining the two, we have:

$$c_1 g(n) h(n) \leq f(n) h(n) \leq c_2 g(n) h(n)$$

$$\text{Therefore } f(n) h(n) = \Theta(g(n) h(n))$$

2. Addition Rule. Let  $f_1, \dots, f_k : N \rightarrow R_{\geq 0}$  for a fixed number  $k$  and  $f(n) = \sum_{i=1}^k f_i(n)$ . Prove that if  $f_i(n) = O(f_1(n))$  for all  $i \in \{1, \dots, k\}$ , then  $f(n) = \Theta(f_1(n))$ . Here

$f_1$  can be replaced with any one of  $f_1, \dots, f_k$

**Solution:**

Since  $f_i(n) = O(f_1(n))$ , there exist constants  $c_i$  and  $n_0$  such that

$$f_i(n) \leq c_i f_1(n), \forall n \geq n_0$$

Therefore,

$$\sum_{i=1}^k f_i(n) \leq \sum_{i=1}^k c_i f_1(n) = C f_1(n) \text{ where } C = \sum_{i=1}^k c_i$$

Additionally, there exists at least one  $f_j(n) = \Omega(f_1(n))$ , so  $\sum_{i=1}^k f_i(n) = \Theta(f_1(n))$

3. Let  $f$  and  $g$  be non-negative functions. Define  $T(n) = \max(f(n), g(n))$ . Prove that  $T(n) = \Theta(f(n) + g(n))$

**Solution:**

Since  $\max(f(n), g(n)) \leq f(n) + g(n)$ , we have  $T(n) = O(f(n) + g(n))$

Also,  $f(n) + g(n) \leq 2\max(f(n), g(n))$ , so  $T(n) = \Omega(f(n) + g(n))$

Therefore,  $T(n) = \Theta(f(n) + g(n))$

4. Let  $f$  and  $g$  be non-decreasing real-valued functions defined on positive integers and  $f(1) \geq 1$  and  $g(1) \geq 1$ . Prove or disprove that  $f(n) = O(g(n)) \Rightarrow 2^{f(n)} = O(2^{g(n)})$

**Solution:**

Since  $f(n) = O(g(n))$ , there exists a constant  $c > 0$  and  $n_0$  such that

$$f(n) \leq c \cdot g(n), \forall n \geq n_0$$

Therefore,

$$2^{f(n)} \leq 2^{c \cdot g(n)}$$

Since  $2^c$  is constant, we conclude that  $2^{f(n)} = O(2^{g(n)})$

5. Prove that  $\log(n!) = \Theta(n \log n)$ . Hint: To show an upper bound, compare  $n!$  with  $n^n$ . To show a lower bound, compare it with  $\left(\frac{n}{2}\right)^{\frac{n}{2}}$

**Solution:**

We know that:

$$\log(n!) = \log(1) + \log(2) + \dots + \log(n)$$

Since  $\log(n)$  is increasing, the sum can be bounded by:

$$\log(n!) \leq \sum_{k=1}^n \log(n) = n \log(n)$$

From the bound, we can use the fact that  $\log(k) \geq \log\left(\frac{n}{2}\right)$  for all  $k \geq \frac{n}{2}$

Hence, we can sum the lower half of the terms to get:

$$\log(n!) \geq \sum_{k=\frac{n}{2}}^n \log(k) \geq \sum_{k=\frac{n}{2}}^n \log\left(\frac{n}{2}\right) = \frac{n}{2} \log\left(\frac{n}{2}\right)$$

6. Show that  $c^n = o(n!)$  for any constant  $C$

**Solution:**

We need to prove that

$$\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$$

Using the recursive relation:

$$\frac{c^n}{n!} = \frac{c^{n-1}}{(n-1)!} \times \frac{c}{n}$$

As  $n \rightarrow \infty$ ,  $\frac{c}{n} \rightarrow 0$ , Thus,

$$\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$$

7. Show that  $c^{n^{1.001}} = \omega(n!)$  for any constant  $C$ . Hint: For any fixed numbers  $p, q > 0$ , we have  $n^p = \omega(\log^q n)$

**Solution:**

We need to prove that

$$\lim_{n \rightarrow \infty} \frac{c^{n^{1.001}}}{n!} = \infty$$

Taking the algorithm, we have:

$$\log\left(\frac{c^{n^{1.001}}}{n!}\right) = n^{1.001} \log(c) - \log(n!)$$

Since  $\log(n!) \approx n \log n$ , and  $n^{1.001}$  grows faster than  $n \log n$ , we conclude that

$$\lim_{n \rightarrow \infty} \log\left(\frac{c^{n^{1.001}}}{n!}\right) = \infty$$

$$\text{So, } \lim_{n \rightarrow \infty} \frac{c^{n^{1.001}}}{n!} = \infty$$

Therefore,  $c^{n^{1.001}} = \omega(n!)$

8. Show  $\sum_{i=1}^n i^k = \Theta(n^{k+1})$ . Hint: Show that  $n^{k+1} - (n-1)^{k+1} = \Theta(n^k)$

**Solution:**

Upperbound: Since  $i^k \leq n^k$ , we have

$$\sum_{i=1}^n i^k \leq \sum_{i=1}^n n^k = n^{k+1}$$

Lowerbound: Using the integral approximation

$$\sum_{i=1}^n i^k \geq \int_1^n x^k dx = \frac{n^{k+1}}{k+1} - \frac{1}{k+1}$$

Therefore,

$$\sum_{i=1}^n i^k = \Theta(n^{k+1})$$

9. Search for the peak. Suppose we have an array  $A[1..n]$  and we know that there exists some  $k$  (value unknown) such that  $A[1] < A[2] < \dots < A[k]$  and  $A[k] > A[k+1] > \dots > A[n]$ . We call  $A[k]$  the peak of  $A$ . Design an algorithm to find the peak and analyze it in  $O(\log n)$  time.

**Solution:**

Use Binary Search Algorithm:

1. Set  $l = 1, r = n$

2. While  $l < r$  do

3. Set  $m = \left\lfloor \frac{l+r}{2} \right\rfloor$

4. If  $A[m] < A[m+1]$  then  $l = m + 1$

5. Else  $r = m$

6. Return  $l$

Analysis:

Let  $T(n)$  be the time complexity of the algorithm

Let  $n_0 = 1$

Let  $c = 1$

Let  $T(n) = T\left(\frac{n}{2}\right) + c$

Let  $T(n) = O(\log n)$

10. 3SUM in a sorted array. Let  $A[1 \dots n]$  be a sorted array. We want to see  $A[i] + A[j] + A[k] = x$  for some different  $i, j, k$ , where  $x$  is a given number. Design an algorithm  $3SUM(n, a, x)$  which can return  $(i, j, k)$  or 0 if no such three elements exist. How fast can this problem be solved?



**Solution:**

fix one element  $A[i]$ , then use two pointers on the remaining array to find  $A[j] + A[k] = x - A[i]$   
 The time complexity is  $O(n^2)$

**Algorithm 3SUM(n,a,x):**

Input: Array  $A$  of  $n$  integers, integer  $x$

Output: indices  $i, j, k$  such that  $A[i] + A[j] + A[k] = x$  or return "No solution" if no such triplet exists  
 for  $i=0$  to  $n-1$  do

//Avoid duplicates by skipping repeated elements

if  $i > 0$  and  $A[i] == A[i-1]$

continue

left= $i+1$ , right= $n-1$

while left < right do

sum= $A[i] + A[left] + A[right]$

if sum ==  $x$

return  $i, left, right$  //Found the triplet

if sum <  $x$

left=left+1 //increase the sum by moving left pointer to the right

else

right=right-1 //decrease the sum by moving right pointer to the left

return "No solution" //no such triplet exists

**§4.3 Recitation 02**

Find tight big-O bounds for the following recurrences (and show your bounds are correct).

Example 1. Substitution.  $T(n) = \sqrt{2n} \cdot T(\sqrt{2n}) + \sqrt{n}$  with base case  $T(n) = 1$  for  $n \in [0, 1]$ .

Example 2. Recursion Tree.  $T(n) = T(n/3) + T(n/4) + T(n/5) + n$

Example 3. Recall The Master Theorem. Let  $a \geq 1$ ,  $b > 1$ , and  $f(n)$  be a function. If  $T(n) = aT(n/b) + f(n)$ , then we can solve  $T(n)$  as:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) = O(n^{\log_b(a-\epsilon)}) \text{ for some } \epsilon > 0 \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) = \Theta(n^{\log_b a}) \\ \Theta(f(n)) & \text{if } f(n) = \Omega(n^{\log_b(a+\epsilon)}) \text{ for some } \epsilon > 0 \end{cases}$$

Then what is the asymptotic bound for:  $T(n) = 4T(\frac{n}{2}) + n^2\sqrt{2}$ .

Find tight big-O bounds for the following recurrences (and show your bounds are correct).

Problem 1.  $T(n) = T(bn) + T(n - bn) + n$ , where  $b \in (0, 1)$  is a constant.

*Solution.* by recursion tree

$$H = \log(n) \\ T(n) = O(n) \cdot H = O(n \log(n))$$

Problem 2.  $T(n) = n^{\frac{3}{4}}T(n^{\frac{1}{4}}) + n$ .

*Solution.* Suppose  $T(n) = C \cdot n^k$

$$\begin{aligned} T(m^4) &= m^3 T(1) + m^4 \\ T(n) &= O(n) \end{aligned}$$

Problem 3.  $T(n) = 2T(2n/3) + T(n/3) + n^2$

*Solution.* by recursion tree

$$T(n) = O(n^2 \log n)$$

Problem 4. Let  $T(n) = 4T(n/3) + n$ . Use induction to prove  $T(n) = O(n^{\log_3 4})$ .

Problem 5.  $T(n) = T(n-2) + n^2$

*Solution.*

$$T(n) - T(n-1) = T(n-2) - T(n-1) + n^2$$

$$\text{Let } S(n) = T(n) - T(n-1)$$

$$S(n) = S(n-1) + n^2$$

$$S(n) = O(n^3)$$

$$T(n) = T(n-1) + O(n^3)$$

$$T(n) = O(n^3)$$

## §4.4 Lecture 2 (02-19) – Divide and Conquer