Problem 1

Solve the partial differential equation $\frac{\partial^2 u}{\partial x \partial y} = 0$ for $u(x, y)$.

Problem 2

Solve the following initial value problems and graph the solutions at times t=1,2,and 3.

$$u_t + u_x + \frac{1}{2}u = 0, u(0, x) = \tan^{-1} x$$

Problem 3

Graph some of the characteristic lines for the following equations, and write down a formula for the general solution:

$$u_t + u_x + 3u = 0$$

Problem 4

Let $c \neq 0$. Prove that if the initial data satisfies $u(0, x) = v(x) \to 0$ as $x \to \pm\infty$, then, for each fixed x, the solution to the transport equation (2.4) satisfies $u(t, x) \to 0$ as $t \to \infty$.

Problem 5

Write down a solution to the transport equation $u_t + 2u_x = 0$ that is defined on a connected domain $D \subset R^2$ and that is not a function of the characteristic variable alone.

Problem 6

Consider the linear transport equation $u_t + (1 + x^2)u_x = 0$.(a) Find and sketch the characteristic curves. (b) Write down a formula for the general solution. (c) Find the solution to the initial value problem $u(0, x) = f(x)$ and discuss its behavior as t increases.

Problem 7

Consider the transport equation $\frac{\partial u}{\partial t} + c(t,x)\frac{\partial u}{\partial x} = 0$ with time-varying wave speed.

Define the corresponding characteristic ordinary differential equation to be $\frac{dx}{dt} = c(t,x)$, the graphs of whose solutions $x(t)$ are the characteristic curves. (a) Prove that any solution $u(t,x)$ to the partial differential equation is constant on each characteristic curve.

(b) Suppose that the general solution to the characteristic equation is written in the form $\xi(t,x) = k$, where k is an arbitrary constant. Prove that $\xi(t,x)$ defines a characteristic variable, meaning that $u(t,x) = f(\xi(t,x))$ is a solution to the time-varying transport equation for any continuously differentiable scalar function $f \in C^1$.

Problem 8

Write the following solutions to the wave equation $u_{tt} = u_{xx}$in d'Alembert form (2.82). Hint: What is the appropriate initial data?

$$(a)\cos x\cos t$$

Problem 9

Suppose $u(t,x)$ solves the initial value problem $u_{tt} = 4u_{xx} + \sin\omega t\cos x,\ u(0,x) = 0$, $u_t(0,x) = 0$. Is $h(t) = u(t,0)$ a periodic function?

Problem 10

(a) Use Exercise 2.4.13 to prove that the only classical solution to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = 0$, $u_t(0,x) = 0$, satisfying the indicated decay assumptions is the trivial solution $u(t,x) \equiv 0$. (b) Establish the following Uniqueness Theorem for the wave equation: there is at most one such solution to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$.