2025 Spring Math

HTOP,PDE,COMPLEX,ALGO

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Chapter 1

Functions of Complex Variables

§1.1 Lecture 1 (02-03)Complex numbers

§1.1.1 Introduction

You know the sets

$$(R,+,\times)and(Q,+,\times)$$

$Q(\sqrt{2}) = \{a + b\sqrt{2}|a, b \in Q\}$ $Q(\sqrt{2})$, they are infinite sets.

Definition 1.1.1: Fields

We say that a set F together with two binary operations, called addition $+$,$F\times F \to F$, and multiplication $\times$, $F \times F \to F$ satisfying:

- associativity $(a + b) + c = a + (b + c)$
- commutativity $a + b = b + a, ab = ba$
- existence of identities
  there is a $O \in F$ such that $a + O = a$ for all $a \in F$
  there is a $1 \in F$ such that $a \times 1 = a$ for all $a \in F$
- existence of inverses
  for every $a \in F$ there is a $b \in F$ such that $a + b = O$
  for every $a \in F$ there is a $b \in F$ such that $a \times b = 1$
- distributivity
  $a \times (b + c) = a \times b + a \times c$

Example 1: Fields

- For every prime number p, and $n \geq 0$,$n \in N$, there is a field $(F_{p^n}, +, \times)$ with $p^n = q$ elements (cardinality q).
- (C,+,x) is a field

Question: $(R^d, +)$,$d \geq 2$, is there a multiplication, $\times$ such that $(R^d, +, x)$ is a field?

- R^2 , sclar product, $x \cdot y = x_1y_1 + x_2y_2 \in R$ is not a field because $R^2 \times R^2 \to R$.
- cross product, not satisfying Definition 1 and 5.
- d=2, there is a good defination of multiplication, $(R^2, +, \times) \cong (C, +, \times)$.

Lemma 1.1.1

Consider the vector space $(R^2, +)$
Define the multiplication: $(a, b) \times (c, d) = (ac - bd, ad + bc)$
Then $(R^2, +, \times)$ is a field.

Proof:

- You can check that $\times$ is commutative and satisfies associativity and distributivity.
- We can check $(1, 0)$ is a multiplication identity.
- also if $(a, b) \neq (0, 0)$, then

$$(a, b) \times (\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}) = (\frac{a^2 + b^2}{a^2 + b^2}, \frac{-ab + ab}{a^2 + b^2}) = (1, 0)$$

§1.2 Lecture 2 (02-05)Complex numbers

§1.2.1 The algebra of complex numbers

It is important to classify sets:
in Set Theory: bijection
in Topology: homeomorphism
For **Fields**: isomorphism

Definition 1.2.1: isomorphism

We say that fields F and F' are isomorphic if there exists a funtion

$$i : F \to F'$$

which is a bijection and preserve the binary operations

$$i(a + b) = i(a) + i(b)$$

and

$$i(ab) = i(a)i(b)$$

.
i is called an isomorphism and we write $F \cong F'$

Comments:
There is a unque field up to isomorphism of order (its cardinality)

$$p^n : F_{p^n}$$

Definition 1.2.2: subfield

Given a field (F,+,x), we say that (E,+,x) is a subfield of (F,+,x) if E $\subset$ F and the addition of multiplication of E is the same addition and multiplication of F.

Example 1: subfield

$$Q \subset R$$
$$Q \subset Q(\sqrt{2}) \subset R \subset R^2$$

Definition 1.2.3: complex number

Let F be a field such that:

- R is a subfield of F
- there is an element $i \in F$ which is the root of the equation

$$x^2 + 1 = 0(i^2 = -1)$$

Then we define the field of complex numbers C as

$$C = \{x \in F : x = \alpha + i\beta, \alpha, \beta \in R\}$$

Comments:

- We need to show that there is at least one field F satisfying these properties. **Actually,** $(F, +, \times) = (R, +, \times)$ **satisfies these properties.**
- $R' = \{x \in R : x = (\alpha, 0), \alpha \in R\} \cong R \rightarrow$ R is a subfield of R^2 .
- Define $i := (0, 1) \in R^2$

$$(a, b) \times (c, d) := (ac - bd, ad + bc)$$

  Then,

$$i^2 = (0, 1)^2 = (0, 1) \times (0, 1) = (-1, 0)$$

Example 2: exercise

Set of 2×2 real matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in R$$

This set together with the standard addition and multiplication of matrices is a field and it is isomorphic to $(R^2, +, \times)$.
$F = R^2, C = R^2$

Comments:
It turns out that any defination of C is unique (up to isomorphisms).

Lemma 1.2.1

Let F and F' be two fields satisfying the properties of the definition of complex numbers, then call $\mathbb{C} and \mathbb{C}'$ the corresponding complex number fields.
Then $\mathbb{C} \cong \mathbb{C}'$.

Proof:
Call $i \in \mathbb{C}$ and $i' \in \mathbb{C}'$the square roots -1.
Then:

$$f : \mathbb{C} \to \mathbb{C}' \text{ defined as } f(\alpha + i\beta) = \alpha + i'\beta \text{ is an isomorphism}$$

$$z = \alpha + i\beta, z' = \alpha' + i'\beta', w = \gamma + i\delta, w' = \gamma' + i'\delta'$$

Lemma 1.2.2

Consider the vetor space $(\mathbb{R}, +)$.Then any multiplication '$\times$' defined in this vector space which transforms it into a field $(\mathbb{R}, +, \times)$,is such that $(\mathbb{R}, +, \times) \cong \mathbb{C}$.

Proof:
Consider the elements

$$(1, 0) \text{ and } j = (0, 1) \in \mathbb{R}^2$$

Then any elemnt of $\mathbb{R}^2$ can be written as$\alpha + j\beta, \alpha, \beta \in \mathbb{R}$
Then

$$j^2 = a + jb, a, b \in \mathbb{R}$$

Then,

$$j^2 - jb = a, j^2 - 2\frac{jb}{2} = a \to j^2 - 2\frac{jb}{2} + \frac{b^2}{4} = a + \frac{b^2}{4}$$

$$(j + \frac{b}{2})^2 = a + \frac{b^2}{4} \tag{1.1}$$

Define:$i = \frac{j - \frac{b}{2}}{\sqrt{a + \frac{b^2}{4}}}$
Claim:

$$a + \frac{b^2}{4} < 0$$

If:

$$a + \frac{b^2}{4} \geq 0$$

Then (1.1) can be written as

$$\to ((j - \frac{b}{2}) + \sqrt{a + \frac{b^2}{4}})((j - \frac{b}{2}) - \sqrt{a + \frac{b^2}{4}}) = 0$$

either

$$j - \frac{b}{2} = \sqrt{a + \frac{b^2}{4}}, j - \frac{b}{2} = -\sqrt{a + \frac{b^2}{4}}$$

§1.3 Complex numbers

The field of rationals $\mathcal{Q}$ is the fractional field of the integral domian $\mathcal{Z}$.
With the metrice

$$d(x, y) = |x - y|$$
$$|x| = max\{x, -x\}, \forall x, y \in \mathcal{R}$$

The fielf $\mathcal{Q}$ is not a complete ,metrice space.
Its completion $\mathbb{R}$, nevertheless is not algebraically closed.
For instance, $f(x) = x^2 + 1$, doesn't split in $\mathbb{R}$.
So we define the spliting field for $f(x)$ over $\mathbb{R}$
i.e. the smallest field extension of $\mathbb{R}$ such that $f(x)$ has roots.

Consider the principle ideal $< f(x) >$ in $\mathbb{R}[x]$
since $f(x)$ is irreducible, then $< f(x) >$ si a maximal ideal.
Then the quotient ring $\mathbb{C} = \frac{\mathbb{R}[x]}{<f(x)>}$ is a field!
which is the spilting field of $f(x)$
The field $\mathbb{C}$ has no further spilting extensions! All polynomials in $\mathbb{C}[x]$ split in $\mathbb{C}$.
So as soon as we forced one polynomial. $f(x) = x^2 + 1$ to have roots, we in fact imposed that all polynomials have roots.

Topology:
The Topology of $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$.
Thus $\mathbb{C}$ is locally compact, but not compact space.
The Alexandroff compactification says that we can add a point at infinity to $\mathbb{C}$ so that $\mathbb{C} \cup \{\infty\}$ compact.

Definition 1.3.1: Alexandroff compactification

- A set $U \supseteq \mathbb{C} \cup \{\infty\}$ is an open neighborhood of $\{\infty\}$ if $U = \mathbb{C} \cup \{\infty\}\ K$ When $K \supseteq \mathbb{C}$.

$\mathbb{C} \cup \{\infty\}$ is called the Riemann sphere.

Definition 1.3.2: Holomorphic Functions

A function $f : \Omega \to \mathbb{C}$ is holomorphic where $\Omega \subseteq \mathbb{C}$ is holomorphic iff:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists}, \forall z_0 \in \Omega$$
$$f(z) = f(x, y) = u(x, y) + iv(x, y), wherez = x + iy \in \Omega$$
$$u(z) = Re(f(z)), v(z) = Im(f(z))$$

$f : \Omega \to \mathbb{C}$ being holomorphic of course implies that $u, v : \Omega \to \mathbb{R}^2$ differentiable.
If we take $f(z_0) = z_0 = 0$ and we write

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Then,
$$\frac{f(x)-f(z_0)}{z-z_0}=\frac{f(z)}{z}=(\frac{\partial f}{\partial z})_{(z_0=0)}+\frac{\bar{z}}{z}(\frac{\partial f}{\partial \bar{z}})_{(z_0=0)}+o(z)$$

For real $z \to 0$, we have $\frac{\bar{z}}{z}=1$, and for purely imaginary $z \to 0$, we have $\frac{\bar{z}}{z}=-1$.
we have to restrict $\frac{\partial f}{\partial z}=0$
Cauchy-Riemann equations:

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \end{cases}$$

$$f'(z)=\frac{\partial f}{\partial z}=(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})(x,y), \text{where } z=x+iy\in\Omega$$

Nevertheless, f being holomorphic is not equivalent to say f satisfies the Cauchy-Riemann equations.
Indeed we take $f(z)=\frac{z^5}{|z|^4}$ on $\mathbb{C}\backslash\{0\}$ and $f(0)=0$.
We can observe that $f(z)$ satisfies the Cauchy-Riemann equations on $\mathbb{C}$ but f is not even coutinuous at $z=0$.

Theorem 1.3.1: Looman Menchoff Theorrem

A continuous function $f:\Omega\cup\mathbb{C}\to\mathbb{C}$ is holomorphic iff f satisfies the Cauchy-Riemann equations.

$$\begin{aligned} &\forall f,g\in H(\Omega)\\ &(fg)=f'g+fg'\\ &(\frac{f}{g})=\frac{f'g-fg'}{g^2}\\ &(g\circ f)'(z)=g'(f(z))\cdot f'(z)\end{aligned}$$

§1.4 Lecture 3 (02-10) – Algebraic Structure of Complex numbers

§1.4.1 Algebraic Structure of Complex numbers

Lemma 1.4.1

Consider $(\mathbb{R}^2,+)$ as a vector space. Then any definition of multiplication on $\mathbb{R}^2$ compatible with the vector space structure, which transforms $\mathbb{R}^2$ into a field, defines a field isomorphism with $\mathbb{C}$.

Comments:
The assumption that the multiplication is compatible with the vector space structure means that the scalar multiplication coincides with the field multiplication in the following space.
$$(\mathbb{R}^2,+,\times)$$

and $1 \in \mathbb{R}$ is the multiplicative identity
then

$$au = (a1) \times u, a \in \mathbb{R}$$

Let $L = \{a1 : a \in \mathbb{R}\}$
Then L is isomorphic with $\mathbb{R}$
The addition in $\mathbb{R}^2$ as a vector space coincides with the addition in $\mathbb{R}^2$ as a field.
Proof of the Lemma:
Essentially we have to find a $i \in \mathbb{R}^2$ such that $i^2 = -1$
Since $\mathbb{R}^2$ is a two-dimentional vector space , if we pick any $j \in \mathbb{R}^2 \backslash L$
We know that $1, j, j^2$ must be linearly dependent.
Then thus must exist $a, b \in \mathbb{R}$ such that $aj^2 + bj + c1 = 0$
Then,

$$\begin{aligned} a(j^2) + 2\frac{b}{2a}j + c = 0 \\ a(j + \frac{b}{2a})^2 + c - a(\frac{b}{2a})^2 = 0 \\ (j + \frac{b}{2a})^2 = \frac{1}{a}(-c + a(\frac{b}{2a})^2) \end{aligned}$$

Necesaarily, $-c + a(\frac{b}{2a})^2 < 0$
Define

$$i = \frac{\sqrt{a}(j + \frac{b}{2a})}{\sqrt{-a(\frac{b}{2a})^2 + c}}$$

§1.4.2 Square roots

Every complex number has a square root.
Also we can write the 2 square roots in a quite explicit way(related to fundamental theorem of algebra).

Lemma 1.4.2

$$\sqrt{\alpha + i\beta} = \pm(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + \frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}})$$

Proof:
(Notation for complex numbers: $\alpha + i\beta, x + iy, \cdots$)
We need to find $z = x + iy$ such that $z^2 = (x + iy)^2\alpha + i\beta$
So $x^2 - y^2 = \alpha, 2xy = \beta$
Now,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2$$

So,

$$
\begin{aligned}
x^2 + y^2 &= \sqrt{\alpha^2 + \beta^2} \\
x^2 - y^2 &= \alpha \\
\Rightarrow x^2 &= \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2} \\
y^2 &= \frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2} \\
x &= \pm\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \\
y &= \pm\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \\
\sqrt{\alpha + i\beta} &= \pm(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + \frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}})
\end{aligned}
$$

Definition 1.4.1: Algebraic Extension

Let F be a field and E be a subfield of F.
We say that $x \in F$ is algebraic in the field E if x satisfies the following equation:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \text{ for } a_n, a_{n-1}, \cdots, a_1 \in E, an \neq 0$$

if every $x \in F$ is algebraic in E, then we say that F is an algebraic extension of E.
We say that a field F is algebraically closed if it does not have any strictly larger algebraic extension.
A field containing F is called an algebraic closure if it is a closed algebraic extension of F .

Comments:

- $\mathbb{R}$ is not an algebraic extension of $\mathbb{Q}$.
  If this was true, then every real number would be a root of a polynomial with rational coefficients:

  $$q_n x^n + q_{n-1}x^{n-1} + \cdots + q_1 x + q_0 = 0 \text{ (an algebraic number)}$$

  So the cardinality of the set of algebraic numbers is countable (trancsendental unmbers are real numbers which are not algebraic, e and π for example).

- $Q(\sqrt{2}) = \{q + \sqrt{2}q', q, q' \in Q\}$ is algebraic over Q.

- $\mathbb{R}$ is not algebraically closed.

- The fundamental theorem of algebra states that any nonconstant polynomial with complex coefficients has at least one complex root.
  Actually this is equivalent to saying that all the roots of such a polynomial are complex.

- The field of complex numbers is algebraically closed.

§1.5 Lecture 4 (02-12) – Conjugation and absolute value

Definition 1.5.1: Complex Conjugation

Given a complex number $z = x + iy \in \mathbb{C}$, we define its complex conjugation $\bar{z} = x - iy$ This transformation is called complex conjugation.

Comments:

- $\bar{\bar{z}} = z$
- We define the real part of $z = x + iy$ as $Re(z) = x = \frac{z+\bar{z}}{2}$ and the imaginary part of z as $Im(z) = y = \frac{z-\bar{z}}{2i}$
- $\overline{z+w} = \bar{z} + \bar{w}, \overline{zw} = \bar{z}\bar{w}$: complex conjugation defines an isomorphism $\mathbb{C} \to \mathbb{C}$
- $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$
- if R is any rational function, then $\overline{R(z_1, z_2, \cdots, z_n)} = R(\bar{z_z}, \bar{z_2}, \cdots, \bar{z_n})$
  In particular,

$$\begin{aligned} a_n z^n + \cdots + a_1 z + a_0 &= 0 \\ \Rightarrow \bar{a_n} \bar{z}^n + \cdots + \bar{a_1}\bar{z} + \bar{a_0} &= 0 \end{aligned}$$

Definition 1.5.2: absolute value

Given a complex number $z \in \mathbb{C}$, we define its absolute value, denoted by

$$|z| = \sqrt{z\bar{z}}$$

Comments:

- If z is real, then $\bar{z} = z$
- we say that z is purely imaginary if $Re(z) = 0$, iff $\bar{z} = -z$
- Then if z is real, |z| coincides with the traditional definition of absolute value on $\mathbb{R}$
- $z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 \geq 0$
- Vector space (V,+,scalar multiplication) over a field $\mathbb{F}(Q, R, C)$
- $|zw| = |z||w|, |\bar{z}| = |z|$
- $|\frac{z}{w}| = \frac{|z|}{|w|}, w \neq 0$
- $|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|$
- $|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + 2Re(z\bar{w})$
- $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$

Lemma 1.5.1: Triangle inequality

If $z, w \in \mathbb{C}$ then, $|z + w| \leq |z| + |w|$

Proof:
Note that if $a = \alpha + i\beta$ then

$$-|a| \leq Re(a) = \alpha \leq |a| = \sqrt{\alpha^2 + \beta^2}$$
$$\Rightarrow |z + w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$$

Comments:
Assume that z and w are such that $|z + w| = |z| + |w|$

$$\Rightarrow |z||w| = Re(z\bar{w}) \Leftrightarrow z\bar{w} \geq 0$$

We proved that absolute value is a norm (which is a good notion of distance):

- $|z| = 0 \Leftrightarrow z = 0$ and $|z| \geq 0, \forall z \in \mathbb{C}$
- $|az| = |a||z|, \forall a, z \in \mathbb{C}$
- $|z + w| \leq |z| + |w|$
- $(R^d, +, \underbrace{||\ ||}_{\text{Euclidean norm}}), ||(x_1, \cdots, x_d)|| = \sqrt{{x_1}^2 + \cdots + {x_d}^2}$

Lemma 1.5.2: Cauchy inequality

If $z_1, \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$,

$$|\sum_{i=1}^{n} z_i w_i| \leq \sqrt{\sum_{i=1}^{n} |z_i|^2} \sqrt{\sum_{i=1}^{n} |w_i|^2}$$

Proof:
Let $\lambda \in \mathbb{R}$ and consider

$$\sum_{i=1}^{n} |z_i - \lambda w_i|^2 = \sum_{i=1}^{n} |z_i|^2 - 2Re\bar{\lambda}(\sum_{i=1}^{n} z_i \bar{w_i}) + |\lambda|^2 \sum_{i=1}^{n} |w_i|^2$$

Choose

$$\lambda = \frac{\sum_{i=1}^{n} z_i \bar{w_i}}{\sum_{i=1}^{n} |w_i|^2}$$

Then,

$$0 \leq \sum_{i=1}^{n} |z_i|^2 - \frac{|\sum_{i=1}^{n} z_i \bar{w_i}|^2}{\sum_{i=1}^{n} |w_i|^2}$$

§1.6 Recitation 2 (02-14)

holomorphic functions $f : U \subseteq \mathbb{C} \to \mathbb{C}$ s.t.

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0), \forall z_0 \in U$$

Definition 1.6.1: Analytic function

Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$,Then $f(\cdot)$ is said to be analytic at $z_0 \in U$ if $\exists\{a_n\} \in \mathbb{C}$ and radius r>0 such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \forall z \in U \text{ with } |z-z_0| < r$$

where $r \le R = \lim_{n\to+\infty} |a_n|^{-\frac{1}{n}}$ the radius of convergenece

Notice that if $f(\cdot)$ is analytic, then it is holomorphic.($\Leftrightarrow$)

$$f'(z) = \frac{\partial f}{\partial z} = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}, \forall z \in U \text{ with } |z-z_0| < r \le R = \lim_{n\to+\infty}(|a_n|^{-\frac{1}{n}})$$

Analytic function can be constructed:
Take a sequence $\{a_n\} \in \mathbb{C}$ and we consider the power series.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } |z| < R = \lim_{n\to+\infty} |a_n|^{1\frac{1}{n}}$$

This function f(z) is actually analytic on the disk $D(o, R)$.
$\forall z_0 \in D(0, R)$ and $0 < s < R - |z-0|$ Let $|z-z_0| < s$ i.e. $z \in D(z_0, s)$ Then,$|z_0| + |z - z_0| < R$ and we want ot express the f(z) into a desired form.
Note that

$$\sum_{n=1}^{\infty} \underbrace{(|a_n| + |z-z_0|)^n}_{<R} = \sum_{n=0}^{\infty} |a_n|(\sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|z-z_0|^k) < \infty$$

converges on $z \in D(z_0, s)$
which yields

$$\sum_{n=0}^{\infty} a_n(\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z-z_0)^k) \text{ converges absolutely in } D(z_0, s)$$

Then we can interchange the order of summation and obtain

$$f(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k = f(z) = \sum_{k=0}^{\infty}(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k})(z-z_0)^k$$

showing f(z) is the sum of a power series on $D(z_0, s)$ and is thus analytic at $\forall z_0 \in D(0, R)$

Definition 1.6.2: Inverse mapping theorem(local)

Take $f : U \subseteq \mathbb{C} \to \mathbb{C}$ analytic, Assume $z_0 \in U$ with $f'(z_0) \neq 0$
Then $f(\cdot)$ is a local isomorphism at $z_0 \in U$ This means that $\exists W \subseteq U$ s.t. $f(\cdot)$ is injective on W and is thus a bijection from W to f(W)=V.
And such that $V \subseteq \mathbb{C}$ and its inverse $g = f^{-1} : V \to W$ is also analytic.

Indeed, we know $f : U \to \mathbb{C}$ is holomorphic. And its Jacobian is nonzero at z_0
Hence, $\exists W \subseteq U$ such that f is diffieomorphism from W onto $f(W) = V \subseteq \mathbb{C}$
Since the Jacobian of $g = f^{-1}$ on V is the inverse of the Jacobian of f on W,by Cauchy-Riemann equations, $g : V \to W$ is also holomorphic.
Assume w.l.o.g. $f(z_0) = z_0, f'(z_0) = 1$, then, we cna write

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, z \in U \cap B_R(0)$$

Now that f(g(z)) =z,$\forall z \in V$ on some $B_\zeta(0)$ we can solve this equation by picking

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

with coefficients $b_n = P_n(a_2, \cdots, a_n, b_1, \cdots, b_{n-1})$ Here,b_n is a polynomial with nonnegative coefficients.
Soving this recursively, we show the existence and uniqueness for the power series expansion of the inverse map $g : V \to W$
It remains to show that the positive radius of convergence of $g(\cdot)$
Select some A>0 sufficiently large such that

$$|a_n| \leq A^n, \forall n \geq 1$$

Then, we consider the sum

$$F(z) = z - \sum_{n=2}^{\infty} A^n z^n = z - \frac{A^2 z^2}{1 - Az}, \forall z \in \mathbb{C} \text{ with } |z| < \frac{1}{A}$$

Its inverse G(z) is computed with the form

$$G(z) = \frac{1 + Az - \sqrt{(1 + Az)^2 - 4zA(A + 1)}}{2A(A + 1)}$$

$G(\cdot)$ is analytic with the expression

$$G(z) = z + \sum_{n=2}^{\infty} B_n z^n$$

Here $B_n = P_n(A^2, \cdots, A^n, B_1, \cdots, B_{n-1})$ where P_n is the same polynomial with nonnegative coefficients.
Thus $|a_n| \leq A^n$ implies $|b_n| \leq B \forall n \in N$
Thus $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ has at least th eradius of convergence as large as that of $G(\cdot)$

$$\begin{aligned} g'(f(z_0)) &\neq 0 \\ &= \frac{1}{f'(x)} \end{aligned}$$

Definition 1.6.3: open mapping theorem

Take analytic $f : U \subseteq \mathbb{C} \to \mathbb{C}$, s.t. f is nonconstant in a neighborhood of z,$\forall z \in U$, then f is an open mapping, i.e. $\forall V \subseteq U f(V)$ is open subset in $\mathbb{C}$

Take any $z_0 \in V V \subseteq U$ and we denote

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \forall |z - z - 0| < R = \lim_{n\to+\infty} |a_n|^{-\frac{1}{n}}$$

And we let $u(z) = f(z + z_0) - f(z_0)$, then , clearly the power series

$$u(z) = \sum_{n=1}^{\infty} a_n z^n \text{ has the same radius of convergence } R > 0$$

Since f(z) is nonconstant in at z_0 , we can find some $m \geq 1$ s.t.

$$u(z) = \sum_{n=m}^{\infty} a_n z^n \text{ with } a_m \neq 0$$

If we can show exists analytic function v(z) with $v'(0) \neq 0$ s.t.$u(z) = v(z)^m$ then the assertion is verified.
And we define $b_k = \frac{a_{m+k}}{a_m}$ for $k \geq 1$ and consider the power

$$g(z) = \sum_{k=1}^{\infty} b_k z^k$$

and $u(z) = a^m z^m (1 + g(n))$, Now we write $k(z) = (1 + g(z))^{\frac{1}{m}} - 1$
then $(1 + k(z))^m = 1 + g(z)$, let

$$v(z) = a_m^{\frac{1}{m}} z(1 + k(z)) = a_m^{\frac{1}{m}} z + a_m^{\frac{1}{m}} zk(z)$$

Then we have $v(z)^m = u(z)$ and v(z) is analytic with $v'(0) \neq 0$
Hence,$\exists$ open disk $D(0, r)$ s.t. $V(D) \supseteq D' := D(o, r')$
And then, the image of D' under $z \to z^m$ is again an open disk, verifying the assertion.

§1.7 Lecture 5 (02-17) – The geometric representation of complex numbers

Definition 1.7.1: Euler's formula

$$e^{i\theta} = cos\theta + isin\theta, \theta \in \mathbb{R}$$

Stereographic projection:

§1.7.1 Geometric multiplication

$$
\begin{aligned}
a &= (\alpha, \beta) = \alpha + i\beta \\
&= rcos\theta + irsin\theta \\
&= r(cos\theta + isin\theta) \\
r = |a| &= \sqrt{\alpha^2 + \beta^2} \\
cos\theta &= \frac{\alpha}{r}, sin\theta = \frac{\beta}{r} \\
sin\theta &= \frac{\beta}{r}
\end{aligned}
$$

Definition 1.7.2: argument of a complex number

Given $a \in C$ and its polar representation

$$a = r(cos\theta + isin\theta)$$

we call θ the argument of a and denote it by

$$\theta = arg(a)$$

Lemma 1.7.1

Let $a_1 = r_1(cos\theta_1 + isin\theta_1)$ and $a_2 = r_2(cos\theta_2 + isin\theta_2)$
arg(a_1a_2)=arg(a_1)+arg(a_2)

Proof:

$$
\begin{aligned}
a_1a_2 &= r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2) \\
&= r_1r_2(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2 + isin\theta_1cos\theta_2 + icos\theta_1sin\theta_2) \\
&= r_1r_2(cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))
\end{aligned}
$$

Comments:

- we will later define the function $e^z, z \in \mathbb{C}$
  We will show that $e^{z_1}e^{z_2} = e^{z_1+z_2}$
  And also $e^{i\theta} = cos\theta + isin\theta$
  Then $a_1a_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

§1.7.2 The binomial equation

$$
\begin{aligned}
(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \\
&= \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} a^{n-k}b^k
\end{aligned}
$$

Note that if $a = a = r(cos\theta + isin\theta)$
Then for $n \geq 0$ and we want to check it is true for $n = -1$ and $n \in Z$

$$\begin{aligned} a^n &= r^n(cosn\theta + isinn\theta) \\ a^{-1} &= \frac{1}{r(cos\theta - isin\theta)} \\ &= r^{-1}\frac{cos\theta - isin\theta}{(cos\theta + isin\theta)(cos\theta - isin\theta)} \\ &= r^{-1}(cos\theta - isin\theta) \\ &= r^{-1}(cos(-\theta) + isin(-\theta)) \end{aligned}$$

Lemma 1.7.2: de Maivre

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

§1.7.3 Roots of complex numbers

$$\begin{aligned} -1 &= 1(\cos\pi + i\sin\pi) \\ 1 = (-1)^2 = (\cos\pi + i\sin\pi)^2 &= \cos 2\pi + i\sin 2\pi \end{aligned}$$

Cube roots of 1:

$$\begin{aligned} z^3 - 1 &= 0 \\ z^3 - 1 = (z-1)(z^2+z+1) &= 0 \\ \Delta = 1 - 4 = -3 &< 0 \\ z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \\ z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \end{aligned}$$

Lemma 1.7.3

Let $a \in \mathbb{C}$ and $n \geq 1$, Then the equation

$$z^n = a, a = r(\cos\theta + i\sin\theta)$$

has only n distinct solutions.(called the n-th roots of a) given by

$$a = r^{\frac{1}{n}}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}), k = 0, 1, \cdots, n-1$$

Proof.

$$z = e(\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}), \text{ where } e^n = r$$

∎

§1.8 Lecture 6 (02-19) – Riemann sphere

Definition 1.8.1: stereographic projection

We will define a mapping bwtween the Riemann sphere and the extended complex plane $\mathbb{C} \cup \{\infty\}$ called the stereographic projection.

Riemann sphere has radius 1

$$\begin{aligned} z &= a(x_1 + ix_2), a \in \mathbb{R} \\ &= \frac{x_1 + ix_2}{1 - x_3} \end{aligned}$$

Definition 1.8.2: stereographic projection

we define the stereographic projection as the mapping

$$S : S - \{(0,0,1)\} \to \mathbb{C}$$

and

$$S(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

Comments: We can extend the domain of the stereographic projection to all of S if we add a point "∞ " to $\mathbb{C}$ and define $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ extended complex plane,

$$S(0,0,1) = \infty$$

- $S : S \to \bar{\mathbb{C}}$ is a bijection
- It is possible to define a topology (this is the collection of open sets) on $\bar{\mathbb{C}}$ s.t. its restriction to $\mathbb{C}$ coincides with the open sets in $\mathbb{C}$

$$\begin{aligned} z &= \frac{x_1 + ix_2}{1 - x_3} \\ |z|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1 - x_3^2}{(1-x_3)^2} &= \frac{1 + x_3}{1 - x_3} \\ \to (1 - x_3)|z|^2 &= 1 + x_3 \\ \to x_3 &= \frac{|z|^2 - 1}{|z|^2 + 1} \\ z + \bar{z} &= 2\frac{x_1}{1 - x_3} \\ x_1 = \frac{(z + \bar{z})(1 - x_3)}{2} \\ x_1 &= \frac{z + \bar{z}}{|z|^2 + 1} \\ x_2 &= \frac{z - \bar{z}}{i(|z|^2 + 1)} \end{aligned}$$

Consider a circle on S:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$$

show that this is the general equation of a circle on the Riemann sphere.

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

We want to find the locus of this circle in $\mathbb{C}$ under S.

$$\begin{aligned} \Leftarrow \alpha_1(z+\bar{z}) - i\alpha_2(z-\bar{z}) + \alpha_3(|z|^2-1) = 2\alpha_0(|z|^2+1), z = x+iy \\ 2\alpha_1 x + 2\alpha_2 y + \alpha_3(x^2+y^2-1) = \alpha_0(x^2+y^2+1) \\ (\alpha_3-\alpha_0)(x^2+y^2) + 2\alpha_1 x + 2\alpha_2 y = \alpha_0+\alpha_3 \end{aligned}$$

For $\alpha_3 \neq \alpha_0$, this is the equation of a circle in $\mathbb{C}$
For $\alpha_3 = \alpha_0$, this is the equation of a line in $\mathbb{C}$
Comments:
Formula for the distance between two points $z, z' \in \mathbb{C}$ in S:

$$d(z,z') = ||S^{-1}(z) - S^{-1}(z')||_2 = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$

This notion of distance is a metric which is equivalent to the metric in $\mathbb{C}$ included by the Euclidean norm.

§1.9 Lecture 7 (02-24) – Complex function

Analysis of complex functions:differentiable,etc $R \to R, R \to C, C \to R, C \to C$
$C \to C$:continuity, differentiability$\Leftrightarrow$analiticity
twice differentiability,∞ differentiability($C^\infty(\mathbb{R})$)

Definition 1.9.1: analiticity

$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$

Definition 1.9.2: continuity

Consider a funtion f:$C \to C$, then we say f has a limit L when x tends to a ,written as

$$\lim_{z \to a} f(z) = L$$

if $\forall \epsilon > 0, \exists \delta$ s.t. $|f(z) - L| < \epsilon$ whenever $|z-a| < \delta$
we say that f is continuous at a if $\lim_{z \to a} f(z) = f(a)$
we say that f is continuous if it is continuous at every point where it is defined.

Comments: This defination of continuity is the standard defination of continuity in a topology space. In this case, the topological space is

$$(C,T) \to (C,T) \Leftrightarrow \text{ f is continuous iff } f^{-1}(0) \text{ is open in } 0 \in T \text{ open}$$

T is the subset of C which are open sets: a union of open balls $B(z,r) = \{z' \in C, |z-z'| < v\}$

§1.9.1 analiticity

Definition 1.9.3: differentiability and analiticity

Given $f : C \to C$ we say that f is differentiable at $a \in C$ if the limit exists:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$$

We say that f is analytic if it is differentiable at every point where it is defined.

Comments: In general, we will assume that the set of point where $f : C \to C$ is defined is an open set $(O = \bigcup B(x, \epsilon_x)\ )$
In particular, this is being assumed in the defination grown above.
If f is differentiable at a, then f is continuous at a.

$$f(z+h) - f(z) = \frac{f(z+h)-f(z)}{h} h$$
$$\to |f(z+h) - f(z)| \le \underbrace{|\frac{f(z+h)-f(z)}{h}}_{|f'(z)|} \underbrace{||h|}_{\to 0}$$

Of course f'(z) when it exists is called the derivative of f at z.
$f : C \to C$ f is differentiable at $z = x + iy$

$$f(z) = u(x,y) + iv(x,y)$$

Then

$$\begin{aligned} f'(z) &= \lim_{h\in\mathbb{R}\to 0} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

On the other hand,

$$\begin{aligned} f'(z) &= \lim_{k\to 0} \frac{f(z+ik)-f(z)}{ik} \\ &= \lim_{k\to 0} \frac{u(x,y+k)+iv(x,y+k)-u(x,y)+iv(x,y)}{ik} \\ &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}$$

Lemma 1.9.1: Cauchy-Riemann equations

Assume that $f : C \to C$ is differentiable at z=x+iy with f=u+iv,Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Comments:

- Assume that the existence of f'(z) lets me differentiate again u_x, u_y, v_x, v_y. Then,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

  and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

  u and v should be harmonic.

- The existence of f'(z) implies the existence of u_x, u_y, v_x, v_y
  - $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y}$
  - $|f'(z)|^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 = (\frac{\partial v}{\partial y})^2 + (\frac{\partial v}{\partial x})^2$

§1.10 Lecture 8 (02-26)–polynomial

Definition 1.10.1: harmonic function

We say that a function $u : R^d \to R, d \geq 1$ is harmonic if it satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Comments: Weyl's Lemma(PDE):
If u is a function that is weakly harmonic,

$$\int u \Delta \phi dx = 0, \forall \phi \text{ smooth}$$

Then u is pointwise harmonic and infinitely differentiable.

Lemma 1.10.1

Assume that u and v are functions which have continuous first order partial derivatives, and that satisfy the Cauchy-Riemann equations
Then $f = u + iv$ is analytic, with a continuous derivative $f'(z)$

Proof: The fact that u and v have continuous partial derivative implies that $f = (u, v)$ is differentiable

$$\frac{f(x+h, y+k) - f(x,y)}{h+ik} = \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}ik + \mathcal{E}(h,k) \text{ (goes to 0 faster than h+ik)}$$

§1.10.1 polynomials

The polynomial functions will give us some framework which will enable us to develop some efficient integration techniques.
Two concepts will be important: singularities and zeros of functions.
a is a zero of f if $f(a) = 0$
a is a singularity of f if f is not defined at a. **Comments:**

- The constant function $f : C \to C f(z) = constant$ is analytic
- The function $f(z) = z$ is analytic

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = 1$$

- $f(z) = z^2$ and $f'(z) = \lim_{h \to \infty} \frac{(z+h)^2 - h^2}{h} = 2z$
- Similarly, using the binomial theorem, you can show that $(z^n)' = nz^{n-1}$
- You can check that the product or the sum of analytic functions is analytic

Lemma 1.10.2

Every polynomial $P : C \to C$,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, a_n \neq 0$$

is analytic.
Futhermore,

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

Comments: We will prove later the fundamental theorem of algebra which states that every complex polynomial has at least one root in C.
Then given the polynomial P(z) of degree n (which means that $P(z) = a_0 + a_1 z + \cdots + a_n z^n, a_n \neq 0$) can be written as

$$P(z) = (z - \alpha_1) Q(z), \text{ where } Q \text{ is a polynomial of degree n-1}$$

Repeating this argument for Q, and so on,

$$P(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

where $\alpha_1, \cdots, \alpha_n$ are the roots of P, which are not necessarily distinct.

$$P(z) = (z - \alpha_1)^{h_1} (z - \alpha_2)^{h_2} \cdots (z - \alpha_k)^{h_k}, \sum_{i=1}^{k} h_i = n, 1 \leq k \leq n$$

Definition 1.10.2

Given a polynomial P written in that form with distinct roots $\alpha_1, \cdots, \alpha_k$, $1 \leq k \leq n$
we call $h_i, 1 \leq i \leq k$ the order of the zero α_i, when $h_i = 1$ we will say that α_i is simple.

Theorem 1.10.1: Luca's theorem

If all the zeros of a polynomial P lies in a half-plane H, then all the zeros of $P'(z)$ lies in the same half plane.

§1.11 Lecture 9 (03-02)

Proof.

$$\begin{aligned} P(z) &= (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_k) \\ &= \prod_{i=1}^{k}(z-\alpha_i) \\ P'(z) &= \sum_{i=1}^{k}\prod_{j\neq i}(z-\alpha_j) \\ \frac{P'(z)}{P(z)} &= \sum_{i=1}^{k}\frac{1}{z-\alpha_i} \end{aligned}$$

■

Fact: any half Plane H $\in \mathbb{C}$ is determined by two complex numbers a and b so that

$$z \in H \text{ iff } Im(\frac{z-a}{b}) < 0$$

Hint: use the fact that $\forall a, b \in \mathbb{C}, bt + a, t \in \mathbb{R}$ is a line in C.
Assume that $a_k \in H$ and $z \notin H$ then

$$Im(\frac{a_k - a}{b}) < 0 \text{ and } Im(\frac{z-a}{b}) \geq 0$$

$$\Rightarrow Im(\frac{b}{z-a_k}) < 0$$

$$\begin{aligned} &\Rightarrow Im(\frac{z-a_k}{b}) = Im(\frac{z-a}{b} + \frac{a-a_k}{b}) > 0 \\ Im(b\frac{P'(z)}{P(z)}) &= \sum_{k=1}^{n} Im(\frac{b}{z-a_k}) < 0 \text{ whenever } z \notin H \\ &\Rightarrow P'(z) \neq 0 \end{aligned}$$

2.4 Rational functions
2.5 Power series
2.6 Exponential,trigonometric function and logarithm

§1.11.1 2.4 Rational functions

We will begin the discussion about how to classify singularities:
f analytic
singularities: poles(isolated), essential

$$P = (z-a)^h Q$$
$$f(z) \approx \frac{g(z)}{(z-a)^h}$$

Definition 1.11.1: rational function

A rational function R is a function of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are polynomials which do not have common factors.
Any zeros of Q are called poles of R.
We say that a pole of R has order h if the corresoponding zero of Q has order h.

Comments:

$$\begin{aligned} R'(z) &= \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2} \\ &= \frac{P'(z)}{Q(z)} - P(z)\frac{Q'(z)}{Q(z)^2} \end{aligned}$$

Suppose that α is a pole of R of order h: then

$$\begin{aligned} Q(z) &= (z-\alpha)^h Q_1(z) \\ \Rightarrow Q'(z) &= (z-\alpha)^{h-1} Q_2(z), Q_1(\alpha), Q_2(\alpha) \neq 0 \end{aligned}$$

$$R'(z) = \frac{P'(z)}{Q(z)} - P(z)\frac{Q'(z)}{Q(z)^2}$$

$\frac{P'(z)}{Q(z)}$ has a pole α of order h ($P'(z) \neq 0$)

Lemma 1.11.1

If R has a pole α of order h, R' has a pole α of order h+1

Definition 1.11.2: pole or zero at infinity

Let R be a rational function, Consider the rational function

$$R_1(z) := R(\frac{1}{z})$$

We say R has a zero at infinity of order h if R_1 has a zero at 0 of order h.

Similarly, we say that R has a pole at infinity of order h if R_1 has a pole at 0 of order h.

Comments: R is not defined at infinity. Nevertheless if

$$\lim_{z\to\infty} R(z) = R(\infty)$$

exists, call it $R(\infty)$, I can extend the defination of R to $\bar{\mathbb{C}}$
In this case, we could say that ∞ is a removable singularity
If $R(\infty) = 0$, the ∞ is a removable singularity which is a zero

§1.12 Lecture 10 (03-05)

Lemma 1.12.1

consider a rational function

Definition 1.12.1: order of a rational function

The order of a rational function is equal to the common number of zeros or poles(with repeatition) of R

Comments: The order is max{m,n}

§1.12.1 Power series

We will recall some properties of power series.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is the context of complex numbers
To study complex power series, we need to develop some topology and some notions of convergence in C.
$D(\mathbb{C}, ||)$ this is a normed vector space over C.

- $|z| \geq 0$ and $|z| = 0 \Leftrightarrow z = 0$
- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$

We can then define the concept of limits of sequence

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \lim_{n\to\infty} |a_n - a| = 0$$

We will need to construct limits of sequences under the assumption that a sequence is a Cauchy sequence
In Q, it is not true that

$$\lim_{n\to\infty} q_n \text{ exists as a rational number even though it might exists as a real number}$$

Cauchy sequence in C:
We say that $\{a_n\}$ is a Cauchy sequence if

$$\lim_{n\to\infty} \sup_{m\geq n} |a_n - a_m| = 0$$

In R any Cauchy sequence is convergent.

Definition 1.12.2

A normed vector space is called complete if every Cauchy sequences are convergent.

- Q is not complete
- R is complete
- C is complete
- Then to study whether or not a series is convergent, I don't have to necessarily identify its limit.

Definition 1.12.3: power series

A power series is a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n \in C$ and z is a complex number or of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n \in C$ and $z, z_0 \in C$
We say that a power series is convergent at z if

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k z^k \text{ exists}$$

otherwise we say the power series is divergent at z.

Comments:
A power series is a complex function (not only a complex number)

- Given a sequence of complex functions (f_n) , we say that it converges pointwise to f if
$$\lim_{n\to\infty} f_n(z) = f(z)$$
- we say that if f_n converges uniformly to f if

$$\lim_{n\to\infty} \sup_{z\in D} |f_n(z) - f(z)| = 0$$

§1.13 Lecture 11 (03-12)–Exponential, trigonometric functions and logarithm

$$\begin{aligned} \bar{e^z} &= e^{\bar{z}} \\ \bar{e^{ix}} &= e^{-ix} \\ |e^{ix}|^2 &= e^{ix}e^{-ix} = 1 \\ e^{ix} &:= R \to S = \{z : |z| = 1\} \end{aligned}$$

Definition 1.13.1

Define the trigonometric functions as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Comments:

- These definitions, extend the defination of cossine and sine functions, to complex arguments.
- Euler's formula:
$$e^{iz} = \cos z + i \sin z$$
- From Euler's formula we can have
$$\cos(z)^2 + \sin(z)^2 = 1$$

Definition 1.13.2: period of a function

We say that $c \in C$ is a period of a complex function f if

$$f(z + c) = f(z), \forall z \in C$$

Definition 1.13.3

The smallest positive period of

$$e^{iz}$$

will be called 2π

Using some analysis including the intermediate value theorem, we can prove that there exists a number $y_0 > 0$ s.t.

$$e^{iy_0} = 1$$

Comments:

Definition 1.13.4: logarithm

A logarithm of $w \in C$ is defined as a root of th eequation

$$e^z = w$$

denoted by $z = \log w$

Notice that if z is a logarithm of w, then

$$e^{z+2\pi i} = e^z$$

$z + 2\pi i$ is also a logarithm of w.
Comments:

- 0 has no logarithm
- for $w \neq 0$ we want to find a $z = x + iy$ such that

  $$e^z = w \Rightarrow e^x = |w|$$

  so

  $$x = \log |w|$$

§1.14 Lecture 12 (03-19)–Complex integration

§1.14.1 line integral

The integral of a complex function $f : [a, b] \to c$ as

$$\int_a^b f(z)dz = \int_a^b u(x)dx + i\int_a^b v(x)dx$$

We also proved that

$$|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$$

Comments:
We want to define integrals alone a complex curve

$$\gamma = \{z(x) : \alpha \leq x \leq \beta\}$$

Definition 1.14.1: complex curve

We say that γ is an arc or curve (also we might use the contour) in the complex plane, if it can be represented as

$$\gamma = \{z(t) : a \leq x \leq b\}$$

where z(t)=x(t)+iy(t) and $x : [a, b] \to R, y : [a, b] \to R$ are continuous

Comments:
Peano curve: a continuous curve that fills the square
We say that z is differentiable if z in the representation of the arc is differentiable, so that

$$z'(t) = x'(t) + iy'(t)$$

We will say that an arc is regular if $z'(t) \neq 0$ for all $a \leq t \leq b$ (this is convenient because then at the point z(t) the arc will have a tangent line with slope argz'(t))
We will also say that an arc is piecewise differentiable (or regular) if the arc is differentiable (regular) at all points except for a finite number of points where itis still continuous and it has right and left derivatives which coincide with the left and right limits of z'(t).

Definition 1.14.2: closed and simple arc

We say that an arc is simple or a Jordan arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$.
We say that an arc is closed or a closed curve if its initial point coincides with its final points, so that $z(\alpha) = z(\beta)$
We define a simple closed arc or Jordan curve as an arc which is simple (except $z(\alpha) = z(\beta)$) and closed.

Comments:
A Jordan curve is closed, continuous, 1-1 mapping from $[\alpha, \beta]$ with α indentified with β to C
Jordan curve cannot be space-filling
A Jordan curve separate C into the interior and the exterior of the curve.(Jordan curve theorem)
There exists Jordan curve which have positive area.

Definition 1.14.3: Line integral

Let γ be a piecewise differentiable arc in C, and let f be a complex function defined on γ. We define the line integral of f over γ as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Comments:
1. This line integral is invariant under a change in the parameterization of the arc.
So if $z(t) : \alpha \leq t \leq \beta$ and $t = t(\tau), a \leq \tau \leq b$,differentiable, then

$$\int_\alpha^\beta f(z(t))z'(t)dt = \int_a^b f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_a^b f(w(\tau))w'(\tau)d\tau$$

where $w(\tau) = z(t(\tau)), t(a) = \alpha, t(b) = \beta$
2.In this discussion arc have an orientation

§1.15 Lecture 13 (04-14)–Cauchy integral theorem

Definition 1.15.1: Linear fraction or transformation

A linear fraction or linear transformation is a function from C to C of the form

$$f(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$$

where a,b,c,d are complex numbers, $ac - bd \neq 0$

Comments:

- A linear fraction is a rational function of order 1: common number of zeros and poles.
- Each linear fraction is a bijection from C to C. As a matter of fact, the inverse of

$$\frac{az+b}{cz+d} \text{ is } \frac{dw-b}{-cw+a}$$

- group representation
- linear fraction map circles into circles

Lemma 1.15.1

Let $a \neq b$,Then for every z in the line segment between a and b ,

$$\frac{z-a}{z-b} \in [-\infty, 0]$$

Proof:

$$\begin{aligned} z &= a + (b-a)t \text{ with } t \in [0,1] \\ \text{Then } \frac{z-a}{z-b} &= \frac{(b-a)t}{(b-a)(1-t)} = \frac{t}{1-t} \\ \text{Then } \frac{z-a}{z-b} &\in [0, \infty) \end{aligned}$$

Recall the defination of the winding number

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$$

Lemma 1.15.2

As a function of a, the winding number $n(\gamma, a)$ is constant in each region determined by γ and O in the bounded region.

Proof: There is a

Chapter 2

Honors Theory of Probability

§2.1 Lecture 1 (02-03)Introduction to Probability

Office Hour: Wed 12:30-1:30pm,Fri 3:30-4:30pm W910

- Homework: weekly
- Grades: 5% participation, 15% homework, 40% midterm, final group project(Max 4 people a group,presetation 20%, Final report 20% 10 pages)

§2.1.1 Intro

Why do we need modern theory of probability?

Example 1

- Coin flip 7 times, what is P[first outcome and fifth outcome are Head]?
  sol:
  $$\Omega = (x1, x2, x3, x4, x5, x6, x7)$$
  $$A = (x1, x2, x3, x4, x5, x6, x7), x1 = x5 = H$$
  $$P(A) = \frac{|\Omega|}{|A|} = \frac{1}{4}$$
- Stock Price (mathematically,geometric Browmion motion, STochastic process, such that $t \to S_t$ is continuous but nowehere differentiable)
  What is P[$S_T \geq 100$]?
  $$\Omega = C[0, T] = \{\text{continuous function}, f[0, T] \to R\}$$
  $$A = \{f \in C[0, T] | f(t) \geq 100\}$$
  requires measure theory that defines P[$S_T \geq 100$]
  In modern Prob, Probability Space:(Ω, F, P), Ω is sample space, F is σ-algebra(meaningful subset of σ), P is probability measure($P : F \to [0, 1]$).

Topics covered:

- Probability space, σ-algebra, measure ,Conditional Probability and Independence
- Random variables (measurable functions),distribution
- Expectation (Lebesgue integral),Conditional distribution and expectation, functionos of random variables, Radom-Nikodym Derivative
- Random walks

- generating functions, characteristic functions
- Branching processes
- Convergence of random variables, Law of large numbers, Morte-Carlo Method
- Central Limit Theorem
- Time permitting: Large deviations, Markov Chains

§2.1.2 Probability Space

(Ω, F, P)

Example 2: Coinflip and Stock Price

- Coin flip infinite times,

$$\Omega = \{(x1, x2, \cdots), x_i = H, T\}$$

  Let A be the event of 10^6 consecutive Tails,

$$A = \bigcup_{i=1}^{+\infty} \{x_i = x_{i+1} = \cdots = x_{i+10^6-1} = 0, (x_i \in \Omega)\}$$

  $P[A] = 1$

- Discrete Stock Price model, t=0,1,2,$\cdots$,T
  T is maturing time, time step $\Delta t \ll T$

$$N = \frac{T}{\Delta t}$$

$$\text{price go} \begin{cases} \nearrow \text{by factor} : e^{\sigma\sqrt{\Delta t}} \\ \searrow \text{by factor} : e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$\Omega = \{(x_1, x_2, \cdots, x_N), x_i = 0 \text{ or } 1\}$$

  Stock price at time t: $\forall \omega \in \Omega$

$$S_N(\omega) = S_0 e^{\sum_{i=1}^{N} x_i \sigma\sqrt{\Delta t} e^{(N-\sum_{i=1}^{N} x_i)(-\sigma\sqrt{\Delta t})}}$$

$$S : \Omega \to R(\text{Random Variable})$$

  Event: return at T is positive but not more than 10%:

$$\{\omega \in \Omega : S_N(\omega) \in (S_0, 1.1S_0]\}$$

Example 3: Gambling

- Gambling:

- start with {0,1,2,$\cdots$} each time bet an interger amount
- if amount of money =0, stays at 0
  wealth process: $\Omega - \{0,1,2,\cdots\} \times \{0,1,2,\cdots\} \times \cdots$
  wealth after time n: (random variable) $X_n : \Omega \to N, (x_1, x_2, ...x_n) \to x_n$
  $(X_n)$is a Markov Chain (future only depends on present state, but not the past)

$$\begin{aligned}&\text{Event: }\{\text{State j is reached from state i}\}\\ &= \{\omega \in \Omega : \exists n \in N, X_0(\omega) = i, X_n(\omega) = j\}\\ &= \bigcup_{m=1}^{+\infty} \{\omega \in \Omega : X_o(\omega) = i, X_n(\omega) = j\}\end{aligned}$$

§2.2 Lecture 2 (02-05) – Algebra

§2.2.1 algebra and σ-algebra

in practice, want F to be closed under $\bigcap, \bigcup, ^c$

Definition 2.2.1

Let $\mathcal{A}$ be a collection of subsets of Ω , $\mathcal{A}$ is an algebra iff:

- $\Omega \in \mathcal{A}$
- if A,B$\in \mathcal{A}$ then A$\cup$B $\in \mathcal{A}$
- if A$\in \mathcal{A}$ then A$^c \in \mathcal{A}$

Remark 1

if A,B$\in \mathcal{A} \to$ A $\cap$ B$\in \mathcal{A}$, because A$\cap$B = (A$^c\cup$B^c)^c

Fact:

① $P(\Omega)$(powerset)={A:A $\subset \Omega$} is an algebra

② smallest algebra/trivial algebra: $\{\emptyset, \Omega\}$

③ Let $\mathcal{A}_1, \mathcal{A}_2$ be two algebras of Ω

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \{B \in \Omega : B \in \mathcal{A}_1 \text{ and } B \in \mathcal{A}_2\} \text{ is an algebra}$$

$$\text{if}(\mathcal{A}_j)_{j \in J}\text{is a family of algebras, then } \bigcap_{j \in J} \mathcal{A}_j \text{ is an algebra}$$

④ Let $\mathcal{E}$ be any collection of subsets of Ω

$$a(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{E} \\ \mathcal{A} \text{ is an algebra}}} \mathcal{A} \text{ is an algebra}$$

$a(\mathcal{E})$ is an algebra by ③
in fact, $a(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$
It is called the algebra generated by $\mathcal{E}$

⑤ Let $A \subseteq \Omega$,$\mathcal{E} = \{A\}$
Then

$$a(\mathcal{E}) = \{\underbrace{A, A^c, \Omega, \emptyset}_{f}\}$$

Proof:

- $a(\mathcal{E}) \subseteq f$
notice that f is an algebra since $f \supseteq \mathcal{E}$, therefore $f \supseteq a(\mathcal{E})$ because $a(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$.
- $f \subseteq a(\mathcal{E})$:

$$A \in a(\mathcal{E}), A^c \in a(\mathcal{E})$$

beacuse $a(\mathcal{E})$ is an algebra, $\emptyset, \Omega \in a(\mathcal{E})$

⑥ $\pi = \{A_1, A_2, \cdots, A_n\}$, $\Omega = \bigcup_{i=1}^{n} A_i, A_i \cap A_j = \emptyset$ Then:

$$a(\pi) = \{\bigcup_{i \in I} A_i, for I \subset 1, 2, \cdots, n\} = \text{ finite disjoint union of } (A_i)_{i=1}^{n}$$

⑦ Let $\mathcal{A}$ be an algebra of $\mathbb{R}$
Let $X : \Omega \to$ be a function
Then

$$\underbrace{\{X^{-1}(A), A \in \mathcal{A}\}}_{\{\omega \in \Omega : X(\omega) \in A, \text{ for some } A \in \mathcal{A}\}} \qquad \text{is an algebra of } \Omega$$

Hint:

$$X^{-1}(A \cup B) = (X^{-1}(A)) \cup (X^{-1}(B))$$

⑧ $\Omega = \mathbb{R}$, $\mathcal{E} = \{\text{left open right closed intervals}\} = \begin{cases} (a, b], & -\infty \leq a < b < +\infty \\ (a, +\infty) & \end{cases}$

Then:

$$\begin{aligned} a(\mathcal{E}) &= \text{"finite disjoint union of elements in } \mathcal{E}\text{"} \\ &= \underbrace{\{I_1 \cup \cdots \cup I_k; I_j \in \mathcal{E}, I_i \cap I_j = \emptyset\}}_{f} \end{aligned}$$

Hint:

- $f \subseteq a(\mathcal{E})$ strightforward
- $a(\mathcal{E}) \subseteq f$
  - check f is an algebra
  - since $\mathcal{E} \subset f$, $a(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$, $a(\mathcal{E}) \subseteq f$
  - $(a, b]^c = \underbrace{(-\infty, a] \cup (b, +\infty)}_{\in f}$

Lemma 2.2.1

In Probability, if $(A_k)_{k=1}^{n}$ are disjoint events, we have $P(\bigcup_{k=1}^{+\infty} A_k) = \sum_{k=1}^{+\infty} P(A_k)$

Definition 2.2.2: $\sigma - algebra$

A $\sigma - algebra$ ($\sigma - field$) $\mathcal{A}$ is an collection of subsets of Ω such that

- $\Omega \in \mathcal{A}$
- if $A_1, A_2, \cdots \in \mathcal{A}$ then $\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$
- if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

Remark 2

if $A_1, A_2, \cdots \in \mathcal{A}$ then $\bigcap_{i=1}^{+\infty} A_i \in \mathcal{A}$
σ -algebra represents the collection of information determined by partial derivatives

Example 1

- coinflips infinity many times

$$\Omega = \{(x_1, x_2, \cdots)x_i = 0, 1\} = (0, 1)^{\infty}$$

After observing first outcome, $f_1 = \{\emptyset, \Omega, A_0, A_1\}$
Where $A_0 = \{(0, x_2, x_3, \cdots), x_i = 0, 1\}, A_1 = \{(1, x_2, x_3, \cdots), x_i = 0, 1\}$

After observing second outcome, $f_2 = \{\emptyset, \Omega, A_0, A_1, A_{00}, A_{01}, A_{10}, A_{11}\}$
Where $A_{00} = \{(0, 0, x_3, x_4, \cdots), x_i = 0, 1\}, A_{01} = \{(0, 1, x_3, x_4, \cdots), x_i = 0, 1\} \cdots$

After observing first n outcomes,

$$f_n = \{\emptyset, \Omega, (A_j)_{j \in (\sigma 1)^n} \text{and finite disjioint unions}\}$$

and

$$f_1 \subseteq f_2 \subseteq \cdots \subseteq f_n \subseteq \cdots$$

Proposition 2.2.0

① intersection of σ-algebras is a σ-algebra

② Let $\mathcal{E}$ be a collecion of subsets of Ω

$$\sigma(\mathcal{E}) = \bigcap_{\substack{f \supseteq \mathcal{E} \\ f \text{ is an algebra}}} f \text{ is a smallest } \sigma\text{-algebra that contains } \mathcal{E}$$

③ For any collection $\mathcal{E}$, we have $a(\mathcal{E}) \subseteq \sigma(\mathcal{E})$

④ For any collection $\mathcal{E}$, we have $\sigma(a(\mathcal{E})) = \sigma(\mathcal{E})$
Hint:

$$a(\mathcal{E}) \subseteq \sigma(\mathcal{E}) \to \sigma(a(\mathcal{E})) \subseteq \sigma(\mathcal{E})$$
$$\mathcal{E} \subseteq \sigma(a(\mathcal{E})) \to \sigma(\mathcal{E}) \subseteq \sigma(a(\mathcal{E}))$$

§2.3 Recitation 1 (02-07) – Problem Solving

§2.3.1 Basic Set Theory

Definition 2.3.1: $\cup, \cap, ^c$

De Morgan's Law:

$$(\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$$
$$(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c$$

Definition 2.3.2: $\backslash$

$A\backslash B = A \cap B^c$
Then $A\backslash B = A\cap B^c = A\backslash(A\cap B) = B^c\backslash A^c$

Remark 1

$$\bigcap_{n\geq 1} A_n = A_1\backslash(\bigcup_{j\geq 2}(A_1 - A_j)) \text{ (ex)}$$

§2.3.2 Limits of Sets

Definition 2.3.3: Limit Sets

Let $(A_n)_{n\geq 1}$ be a sequence of sets, then

$$B_k = \bigcup_{n\geq k} A_n, C_k = \bigcap_{n\geq k} A_n$$

Then B_k is increasing, C_k is decreasing
Define:

$$\limsup_{n\to+\infty} A_n = \lim_{k\to+\infty} B_k = \bigcap_{k\geq 1}\bigcup_{n\geq k} A_n$$
$$\liminf_{n\to+\infty} A_n = \lim_{k\to+\infty} C_k = \bigcup_{k\geq 1}\bigcap_{n\geq k} A_n$$

Definition 2.3.4: liminf, limsup of sequence

$$\limsup an = \inf_{k\geq 1}\sup_{n\geq k} a_n, \liminf an = \sup_{k\geq 1}\inf_{n\geq k} a_n$$

When $\limsup A_n = \liminf A_n$, Then we say $\lim_{n\to+\infty} A_n$ exist and $\lim_{n\to+\infty} A_n = \limsup A_n = \liminf A_n$

In probability,

$$\begin{aligned}
\limsup A_n &= \{A_n \text{ occurs infinitely often}\}\\
&= \{An.i.o\}\\
x \in \limsup A_n &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k \text{ such that } x \in A_n\\
&\Leftrightarrow A_n.i.o\\
\liminf A_n &= \{A_n \text{ occurs eventually}\}\\
&\Leftrightarrow \exists k \in \mathbb{N}, \forall n \geq k, x \in A_n
\end{aligned}$$

Remark 2

1. if A_n increases, then $\limsup A_n = \lim_{n\to\infty} A_n = \bigcup_{n\geq 1} A_n$
   if A_n decreases, then $\liminf A_n = \lim_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n$
2. $$(\limsup A_n)^c = \liminf A_n, (\liminf A_n)^c = \limsup A_n$$

§2.3.3 Exercise

1. $A_k = \begin{cases} E, \text{if k is odd} \\ F, \text{if k is even} \end{cases}$
   Then $\limsup A_n = E \cup F, \liminf A_n = E \cap F$
2. Let $f_n : \mathbb{R} \to \mathbb{R}$, Let A=$\{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) = f(x)\}$ Then,
$$\begin{aligned}
A^c &= \bigcup_{k=1}^{+\infty}\bigcap_{m=1}^{+\infty}\bigcup_{n=m}^{+\infty}\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}\\
&= \bigcup_{k=1}^{+\infty}[\limsup_n\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}]
\end{aligned}$$
3. Suppose that $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in \mathbb{R}$
   Then
$$\begin{aligned}
\{x : f(x) \leq t\} &= \bigcap_{k=1}^{+\infty}\bigcup_{m=1}^{+\infty}\bigcup_{n=m}^{+\infty}\{x \in \mathbb{R} : f_n(x) < t + \frac{1}{k}\}\\
&= \bigcap_{k=1}^{+\infty}[\liminf_n\{x : f_n(x) < t + \frac{1}{k}\}]
\end{aligned}$$

Proof:

1. ex2: Want

$$A = \bigcup_{k\geq 1}^{+\infty} \bigcap_{\geq =1}^{+\infty} \bigcup_{n\geq m}^{+\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}$$

$$f_n(x) \to f(x) \text{ iff } \forall \epsilon > 0, \exists N \geq N, |f_n(x) - f(x)| < \epsilon$$

$$\{x : f_n(x) \to f(x)\} = \bigcap_{\epsilon>0} \bigcup_{N} \bigcap_{n\geq N} \{x : |f_n(x) - f(x)| < \epsilon\},$$

$$\text{USE that } \{|f_n - f| < \epsilon\} \text{ is monotone increasing in } \epsilon$$

Review on mapping:

$$f : X \to Y, f^{-1} : Y \to X$$

Basic properties:

- $f(\bigcup_{i\in I}) = \bigcup_{i\in I} f(A_i)$
- $f(\bigcap_{i\in I}) = \bigcap_{i\in I} f(A_i)$

For (B_i) subset of Y:

- if $B_1 \subset B_2, f^{-1}(B_1) \subset f_-1(B_2)$
- $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$
- $f^{-1}(B^c) = (f^{-1}(B))$

Definition 2.3.5: Indicator Mapping

$$1_A : X \to \{0, 1\}$$

$$1_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

Exercise:

1. $1_{limsupA_n} = limsup 1_{A_n}$
2. $1_{liminfA_n} = liminf 1_{A_n}$

Proof:
if

$$\begin{aligned} 1_{limsupA_n}(x) = 1 &\Leftrightarrow x \in limsupA_n \\ &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k, x \in A_n \\ &\Leftrightarrow \forall k \in \mathbb{N}, \exists n \geq k, 1_{A_n}(x) = 1 \\ &\Leftrightarrow limsup 1_{A_n}(x) \geq 1 (\text{Definition of limsup}) \\ &\Leftrightarrow 1_{limsupA_n}(x) = 1 \end{aligned}$$

Exercise: Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras of Ω

1. show that $\underbrace{\mathcal{A}_1 \cap \mathcal{A}_2}_{B\subset\Omega : B\in\mathcal{A}_1, B\in\mathcal{A}_2}$ is an algebra

2. show that $\underbrace{\mathcal{A}_1 \cup \mathcal{A}_2}_{B\subset\Omega: B\in\mathcal{A}_1 \text{ or } B\in\mathcal{A}_2}$ is an algebra iff $\mathcal{A}_1 \subseteq \mathcal{A}_2$ or $\mathcal{A}_2 \subseteq \mathcal{A}_1$

Proof: Suppose by contradiction that

$$\exists A_1 \in \mathcal{A}_1 \text{ but } A_1 \notin \mathcal{A}_2 \text{ and } A_2 \in \mathcal{A}_2 \text{ but } A_2 \notin \mathcal{A}_1 \text{ and } \mathcal{A}_1 \cup \mathcal{A}_2 \text{ is an algebra}$$

Therefore:

$$A_1 \cup A_2 \in \mathcal{A}_1 \cup \mathcal{A}_2, A_1 \backslash A_2 = A_1 \cap \underbrace{A_2^c}_{\mathcal{A}_1\cup\mathcal{A}_2} \in \mathcal{A}_1 \cup \mathcal{A}_2$$

$$A_2 \backslash A_1 = A_2 \cap A_1^c \in \mathcal{A}_1 \cup \mathcal{A}_2$$

$\Rightarrow$ at least two of

$(A_1\backslash A_2) \cup (A_2 \backslash A_1), A_1\backslash A_2, A_2\backslash A_1$ are in $\mathcal{A}_1$ or $\mathcal{A}_2$

Assume in $\mathcal{A}_1$

$\Rightarrow$ (by $\mathcal{A}_1$ is an algebra, all three sets are in $\mathcal{A}_1$)

$$\Rightarrow A_2 = (\underbrace{A_1 \cup A_2}_{\in\mathcal{A}_1})\backslash(\underbrace{A_1\backslash A_2}_{\mathcal{A}_1}) \in \mathcal{A}_1$$

Contradiction!

§2.4 Lecture 3 (02-10)–(Content and Measure)

Recall σ-algebra:

Example 1

① We know $\Omega = \mathbb{R}$,

$$\mathcal{E} = \{\text{left open right closed intervals}\} = \begin{cases} (a,b], & -\infty \le a < b < +\infty \\ (a,+\infty), & a \in \mathbb{R} \end{cases}$$

Then we know:

$$a(\mathcal{E}) = \{ \text{"finite disjioint union of elements in } \mathcal{E}\text{"}\}$$

What is $\sigma(a(\mathcal{E}))$?
$\sigma(\epsilon) =$ Borel Sets $\mathcal{B}(R)$
Any "reasonable" subset of $\mathbb{R}$ is in $\sigma(\epsilon)$

- $(a,b) \in \sigma(\epsilon) : (a,b) = \bigcup_{n\ge1} \underbrace{(a, b - \frac{1}{n}]}_{\in\sigma(\epsilon)} \in \sigma(\epsilon)$
- any singleton $\{a\} \in \sigma(\epsilon)$, because $\{a\} = \bigcap_{n\ge1} \underbrace{(a - \frac{1}{n}, a + \frac{1}{n})}_{\in\sigma(\epsilon)} \in \sigma(\epsilon)$
- any countable set is in $\sigma(\epsilon)$ (e.g. $\mathbb{Q}$)
- The set of trancsendental numbers is in $\sigma(\epsilon)$, because the set of algebraic numbers is countable

Definition 2.4.1: measurable set

A pair (Ω, F) , where F is a σ-algebra of Ω, is called a measurable space
Any set $A \in F$ is called a measurable set

§2.4.1 Content and Measure

Definition 2.4.2: Content

Let $\mathcal{A}$ be an algebra of Ω, A set function $\mu : \mathcal{A} \to [0, +\infty)$ is called a content iff:

- $\mu(\emptyset) = 0$
- if A,B$\in \mathcal{A}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$ (finite additivity)

Lemma 2.4.1

Let $\mu : \mathcal{A} \to [0, +\infty)$ be a content, $\forall A, B \in \mathcal{A}$ then:

① $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$

② if $A \subset B$, and $\mu(A) < +\infty$,then $\mu(B \backslash A) = \mu(B) - \mu(A)$

③ if $A \subset B$, then $\mu(A) \leq \mu(B)$

④ if $A_1, A_2, \cdots \in \mathcal{A}$, then $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

⑤ if $A_1, A_2, \cdots \in \mathcal{A}$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{+\infty} \in A$ then $\mu(\bigcup_{i=1}^{+\infty} A_i) \geq \sum_{i=1}^{+\infty} \mu(A_i)$

Proof:
finite add:
1:

$$\begin{aligned} \mu(B) &= \mu(A \cup B) + \mu(B \backslash A) \\ \mu(A) + \mu(B \backslash A) &= \mu(A \cup B) \\ \Rightarrow \mu(A) + \mu(B) &= \mu(A \cup B) + \mu(A \cap B) \end{aligned}$$

4:

$$\begin{aligned} &\text{Let } B_1 = A_1, B_2 = A_2 \backslash A_1, B_3 = A_3 \backslash (A_1 \cup A_2), \cdots \\ &\text{Then } B_j \text{ are disjoint and } \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \\ &\text{By finite add for } B_j : \\ \mu(\bigcup_{i=1}^{n} A_i) = \mu(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} (\mu(A_i) \backslash \bigcup_{i=1}^{j=1} A_j) &\leq \sum_{i=1}^{n} \mu(A_i) \end{aligned}$$

5:

For $\forall n$,

$$\mu(\bigcup_{i=1}^{+\infty} A_i) \geq \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i) \text{ send n} \nearrow +\infty \text{ to conclude}$$

Remark 1

In general. $\mu(\bigcup_{i=1}^{+\infty} A_i) \neq \sum_{i=1}^{+\infty} \mu(A_i)$ although $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ require continuity of μ

Counterexample:

- Let $\Omega = \mathbb{R}, \mathcal{A} = a(\epsilon)$, $\forall A \in \mathcal{A}$,

$$\mu(A) = \lim_{L \to +\infty} \frac{\overbrace{|A \cup [0, L]|}^{\text{Length of interval}}}{L} \text{ "Density of A in } [0, +\infty)\text{"}$$

Then μ is a content.
Take $A_i = (i, i+1]$, then $\mu((i, i+1]) = 0$, But $\mu(\bigcup_{i=0}^{+\infty}) = \mu((0, +\infty)) = 1$

- For $\mathbb{R}, a(\epsilon)$, Given $A \in a(\epsilon)$

$$\mathring{A} = \{x \in A : \exists r_x > 0, s.t.(x - rx, x + rx) \subset A\}$$
$$\partial A = \bar{A} \backslash \mathring{A}$$

$$\mu(A) = \begin{cases} 2, & \text{if } 0 \in \mathring{A} \\ 1, & \text{if } 0 \in \partial A \\ 0, & \text{else} \end{cases}$$

Then μ is a content
However, $A_i = (\frac{1}{i+1}, \frac{1}{i}], \mu(A_i) = 0,$ But $\mu(\bigcup_{i=1}^{+\infty}) = \mu((0, 1])$

Example:

- (Discrete Probability)

$$\Omega = \{\omega_1, \cdots, \omega_n\} \text{ finite set, } , F = P(\Omega)$$

Set $A_i = \{\omega_i\}, P(A_i) = P(\omega_i) = P_i$ such that $\sum_{i=1}^{n} P_i = 1$
Then $P : P(\Omega) \to [0, 1]$ defines a content on $P(\Omega)$ by extending the P using finite additivity:

$$\forall A \in P(\Omega), P(A) = \sum_{\omega \in A} P(\omega)$$

- $\Omega = \mathbb{R},$ algebra $\mathcal{A} = a(\epsilon) = \{$ finite disjoint umiom of elements in $\epsilon\}$
Define $m : a(\epsilon) \to [0, +\infty)$, set $m([a, b])$ =b-a,
and extend by additivity:

$$m(I) = \sum_{i=1}^{n} m(I_j), \text{ if } I = I_1 \cup \cdots \cup I_n, I_j \cap I_i = \emptyset$$

$\Rightarrow$ m is a content on $(\mathbb{R}, a(\epsilon))$
m can be further extended to $(\mathbb{R}, \sigma(\epsilon))$, called Lebesgue Measure

Definition 2.4.3: countably additive

A content $\mu : \mathcal{A} \to [0, +\infty)$ is countably additive if:

$$\mu(\bigcup_{i=1}^{+\infty} A_i) = \sum_{i=1}^{+\infty} \mu(A_i) \text{ for every disjioint } A_1, A_2, \cdots \in \mathcal{A}$$

Definition 2.4.4: measure

Let (Ω, F) be an measurable space, then a content $\mu : F \to [0, +\infty)$ that is countably additive is called a measure

Lemma 2.4.2

$m : a(\epsilon) \to [0, +\infty)$ is countably additive

Proof:
Let $A_1, A_2, \cdots \in a(\epsilon), (A_k)$ disjioint , $A := \bigcup_{i=1}^{+\infty} A_i \in a(\epsilon)$
Want to show :

$$m(A) = \sum_{i=1}^{+\infty} m(A_i)$$

we can write, using $A, (A_k) \in a(\epsilon)$, $A = \bigcup_{j=1}^{n} I_j$, where $I_j \in \epsilon$ and (I_j) disjioint
$A_j = \bigcup_{k=1}^{n_i} J_{ik}$, where $J_{ik} \in \epsilon$,(J_{ik}) disjioint
Then:

$$\begin{aligned}
m(A) &= \sum_{j=1}^{n} m(I_j) \\
&= \sum_{j=1}^{n} \sum_{j=1}^{+\infty} \sum_{k=1}^{n_i} m(I_j \cap J_{ik}) \\
&= \sum_{i=1}^{+\infty} m(\underbrace{\bigcup_{j=1}^{n} I_j}_{A} \cap (\underbrace{\bigcup_{k=1}^{n_i} J_{i_{jk}}}_{A_i})) \\
&= \sum_{i=1}^{+\infty} m(A_i)
\end{aligned}$$

§2.5 Lecture 4 (02-12)–Measure and Extension

$$\Omega = \mathbb{R}, m : \epsilon \to [0, +\infty), \text{ such that } \begin{cases} m([a,b]) = b - a, \\ m((a, +\infty)) = +\infty \end{cases}$$

$$\text{extend m to } a(\epsilon) : \forall A \in a(\epsilon),$$

$$\text{if } A = \bigcup_{i=1}^{n} I_j, I_j \text{ disjioint}, \; m(A) = \sum_{j=1}^{n} m(I_j)$$

Fact:
if $I \in \epsilon$ s.t. $I = \bigcup_{i=1}^{+\infty} I_i, (I_i)$ disjoint and $I_j \in \epsilon$
Then

$$\begin{aligned} m(I) &= |I| \\ &= \sum_{i=1}^{+\infty} |I_i| = \bigcup_{i=1}^{+\infty} m(I_i) \end{aligned}$$

Lemma 2.5.1

we want to prove m is a countably additive content on $(\mathbb{R}, a(\epsilon))$

$$\text{Let } A_j \in a(\epsilon), A = \bigcup_{j=1}^{\infty} \in a(\epsilon), (A_j) \text{ disjoint}$$

$$\exists (I_i) \text{ such that } I_i \in \epsilon, \text{ disjoint } A = \bigcup_{i=1}^{n} I_i$$

$$\exists (J_{ij}) \text{ such that } J_{ij} \in \epsilon, \text{ disjoint } A_j = \bigcup_{k=1}^{j} J_{ik}$$

$$\begin{aligned} m(A) \underbrace{=}_{\text{def}} \sum_{i=1}^{n} m(I_i) &= \sum_{i=1}^{n} m(\bigcup_{j,k} \underbrace{(I_i) \cap (J_{ik})}_{\in \epsilon}) \\ &\underbrace{=}_{\text{fact}} \sum_{i=1}^{n} \sum_{j=1}^{+\infty} \sum_{k=1}^{n_j} m(I_i \cap I_{jk}) \\ &\underbrace{=}_{\text{finite add}} \sum_{j=1}^{+\infty} m(A_j) \end{aligned}$$

Theorem 2.5.1

m extends to a measure on $(\mathbb{R}, \sigma(\epsilon) = \mathcal{B}(\mathbb{R}))$ It is the unique measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$m([a,b]) = b - a$$

Definition 2.5.1

If μ is a measure on (Ω, F), then (Ω, F, μ) is said to be a measure space
If $\mu(\Omega) = 1$, then (Ω, F, μ) is said to be a probability space

Lemma 2.5.2

Let (Ω, F, μ) be a measure space, then:

① stability: Let $A_1, A_2, \cdots \in F$, then $\mu(\bigcup_{i=1}^{+\infty} A_i) \leq \sum_{i=1}^{+\infty} \mu(A_i)$

② continuity from below: Let $A_1, A_2, \cdots \in F$, $A_1 \subseteq A_2 \subseteq \cdots$, then

$$\mu(\bigcup_{i=1}^{+\infty} A_i) = \lim_{i \to +\infty} \mu(A_i) = \mu(\lim_{i \to +\infty} A_i)$$

③ continuity from above: Let $A_1, A_2, \cdots \in F$, $A_1 \supseteq A_2 \supseteq \cdots$,
$(\mu(A_i) < +\infty$ we need this for the Counterexample $A_i = [i, +\infty)$) then

$$\mu(\bigcap_{i=1}^{+\infty} A_i) = \lim_{i \to +\infty} \mu(A_i) = \mu(\lim_{i \to +\infty} A_i)$$

Proof:

① Let $B_1 = A_1, B_2 = A_2 \backslash A_1, \cdots, B_n = A_n \backslash \bigcup_{i=1}^{n-1} A_i$,
Then (B_i) disjoint, $\bigcup_{i=1}^{+\infty} A_i = \bigcup_{i=1}^{+\infty} B_i$

$$\mu(\bigcup_{i=1}^{+\infty} A_i) = \mu(\bigcup_{i=1}^{+\infty} B_i) = \sum_{i=1}^{+\infty} \mu(B_i) = \sum_{i=1}^{+\infty} \mu(A_i \backslash \bigcup_{j=1} i - 1 A_j) \leq \sum_{i=1}^{+\infty} \mu(A_i)$$

② Let $B_1 = A_1, B_2 = A_2 \backslash A_1, \cdots, B_n = A_n \backslash A_{n-1} \cdots$,

$$\begin{aligned}\mu(\bigcup_{i=1}^{+\infty} A_i) = \mu(\bigcup_{i=1}^{+\infty} B_i) = \sum_{i=1}^{+\infty} \mu(B_i) \\ &= \mu(A_1) + \sum_{i=2}^{+\infty} (\mu(A_i) - \mu A_{i-1}) \\ &= \lim_{n \to +\infty} (\mu(A_1) + \sum_{i=2}^{n} (\mu(A_i) - \mu A_{i-1})) \\ &= \lim_{n \to +\infty} \mu(A_n)\end{aligned}$$

③

$$\mu(A_1) - \mu(\bigcap_{i=1}^{+\infty} A_i) = \mu(A_1 \backslash \bigcap_{i=1}^{+\infty} A_i) = \mu(\bigcup_{i=1}^{+\infty} A_1 \backslash A_i)$$

Since $A_1 \nearrow$, by $2 := \lim_{n \to +\infty} (\mu(A_1) - \mu(A_i)) = \mu(A_1) - \lim_{n \to +\infty} \mu(A_i)$

Definition 2.5.2: σ-finite measure

Given a measure space (Ω, F, μ), μ is said to be finite if $\mu(\Omega) < +\infty$
μ is σ-finite if there exists $(E_i)_{i=1}^{+\infty}$ such that $\bigcup_{i=1}^{+\infty} E_i \in \Omega$ and $\mu(E_i) < +\infty$

Example 1

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ is σ-finite

Definition 2.5.3

if $F \in \mathcal{F}$ is such that $\mu(F) = 0$, then F is called a μ-null set

Example 2

$m(\{a\}) = 0$ because

$$m(\{a\}) = m(\bigcap_{n\geq 1}(a - \frac{1}{n}, a]) = \text{(continuous from above)} \lim_{n\to+\infty} m((a - \frac{1}{n}, a]) = 0$$

$$m(Q) = \sum_{q\in Q} m(\{q\}) = 0$$

Recall:
Start with $(R, a(\epsilon), m)$ m is content + countably additive
extention: $(R, a(\epsilon), m) \to (R, \sigma(\epsilon), m)$ m is measure

Theorem 2.5.2: Caratheodory Extention Theorem

Let F be an algebra on Ω , μ be a countably additive content on (Ω, F),
If μ is σ-finite, then μ extends to a measure on $(\Omega, \sigma(F))$

Example 3

Let $\epsilon = \begin{cases} (a, b], & -\infty \leq a < b < +\infty \\ (a, +\infty), & a \in \mathbb{R} \end{cases}$ Let $m_F : a(\epsilon) \to [0, +\infty)$ be a content, such that

$$m_F([a, b]) = F(b) - F(a), m_F((a, +\infty)) = F(+\infty) - F(a)$$

Where F is a right continuous increasing function on $\mathbb{R}$,

$$F(\pm\infty) = \lim_{x\to\pm+\infty} F(x)$$

Then m_F is a countably additive content on $(\mathbb{R}, a(\epsilon))$
By the extension theorem, m_F extends to a measure on $(\mathbb{R}, \sigma(a(\epsilon)) = \sigma(\epsilon) = \mathcal{B}(\mathbb{R}))$

(F(x)=(x) gives Lebesgue)

Definition 2.5.4: Lebesgue-Stieltjes mesures

think of $m_F(A) = \int_A \underbrace{dF(x)}_{\text{R-S integral}}$, F is the distribution function of the measure

§2.5.1 π and λ system

Definition 2.5.5: π and λ system

Let C be a collection of sets of Ω
C is a π-system if:

- $\emptyset \in C$
- $\forall A, B \in C, A \cap B \in C$

C is a λ-system if:

- $\Omega \in C$
- if $A, B \in C$, and $A \subseteq B$, then $B \backslash A \in C$
- if $A_1, A_2, \dots \in C$, and $A_1 \subseteq A_2 \subseteq \cdots$, then $\bigcup_{i=1}^{+\infty} A_i \in C$

Example 4

$$\epsilon = \begin{cases} (a, b] \\ (a, +\infty) \end{cases} \quad \text{is a } \pi\text{-system}$$

Exercise: if C is both a π-system and a λ-system, then it is a σ-algebra

Lemma 2.5.3: Dynkins Lemma

Let C be a π-system, then any λ-system containing C also contains the $\sigma(C)$
Hint: show that any such λ-system is also a π-system

Theorem 2.5.3: Uniqueness Theorem

Let C be a π-system, Let μ_1, μ_2 be two finite measures on $(\Omega, \sigma(C))$
Suppose that $\mu_1(A) = \mu_2(A)$ and $\mu_1(\Omega) = \mu_2(\Omega)$ on C, then $\mu_1(A) = \mu_2(A)$ on $\sigma(C)$

Proof.

$$D = \{A \in \sigma(C) : \mu_1(A) = \mu_2(A)\}$$
$$\text{We know } C \in D. \text{ wnat to show D is a } \lambda\text{-system}$$
$$\text{If so , by Dynkins Lemma, } \sigma(C) \subseteq D, \text{ so that } D = \sigma(C)$$

Check D is a λ-system:

- $\Omega \in D$ follows from $\mu_1(\Omega) = \mu_2(\Omega)$
- if $A, B \in D, A \subseteq B \to \mu_1(A) = \mu_2(A), \mu_1(B) = \mu_2(B)$
  $\mu_1(B \backslash A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \backslash A)$
  So, $\Rightarrow B \backslash A \in D$
- if $A_n \in D$ and A_n increasing, $\lim_{n\to+\infty} A_n = A$

$$\mu_2(A) = \mu_2 \lim_{n\to+\infty}(A_n) \underbrace{=}_{\text{cont.}} \lim_{n\to+\infty} \mu_2(A_n) = \lim_{n\to+\infty} \mu_1(A_n) = \mu_1(A)$$

■

Remark 1

Also hold for μ_1, μ_2, σ-finite

§2.6 Recitation 2 (02-14)-Exercise

EX1

Let Ω be a countable set,

$$A = \{A \subseteq \Omega : \text{ A is finite , or} A^c \text{ is finite}\}$$

① show that A is an algebra

② Let $P : A \to [0, +\infty)$ that $P(A) = \begin{cases} 0, & \text{if A is finite} \\ 1, & \text{if } A^c \text{ is finite} \end{cases}$

Is P a content/measure?
Solution: (1):

- $\emptyset \in \mathcal{A}$ because $\emptyset$ is finite
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ by defintation
- $A, B \in \mathcal{A}$
  - if A, B are finite, then $A \cup B$ is finite
  - if one of A, B is countably infinite , say B

$$(A \cup B)^c = A^c \cap B^c \text{ is finite} \Rightarrow A \cup B \in \mathcal{A}$$

(2): $\Omega = \{\omega_1, \omega_2, \cdots\}$
Let $A_i = \{\omega_i\}$ so that $P(\bigcup_i \{\omega_i\}) = P(\Omega) = +\infty$ But $\sum P(\omega_1) = 0$

EX2

Let Ω be an uncountable set, $A = \{\{\omega\}, \omega \in \Omega\}$, compute $\sigma(A)$ and justify:
Solution:

$$\sigma(A) = \{\underbrace{A \subseteq \Omega : A \text{ is countable or } A^c \text{ is countable }}_{f}\}$$

Proof.
- f is a σ-algebra
  - $\emptyset \in f$ because $\emptyset$ is countable
  - $A \in f \Rightarrow A^c \in f$ by definition
  - if $A_1, A_2, \cdots \in f$
    * if A_i are countable, then $\bigcup_{i=1}^{+\infty} A_i$ is countable
    * if one of A_i is uncountable, say A_1 thenA_1^c countable , then $(\bigcup_{i=1}^{+\infty} A_i)^c$ is uncountable
  - $\sigma(A) \subseteq f$ by $\sigma(A)$ is minimal
  - $\sigma(A) \supseteq f$
    * if $A \in f$ countable, $A = \{\omega_1, \omega_2, \cdots\} =$
    * if $A^c \in f$ countable, then

■

EX3

Let $\Omega = \mathbb{R}$

$$C_1 = \{(-\infty, b], b \in \mathbb{R}\}$$
$$C_2 = \{(a, b], -\infty \leq a < b < +\infty\}$$
$$C_3 = \{(a_1, b_1] \cup (a_2, b_2] \cup (a_3, b_3] \cdots (a_n, b_n], -\infty \leq a_1 < b_1 \leq a_2 < \cdots < b_n < +\infty\}$$

show that $\sigma(C_1) = \sigma(C_2) = \sigma(C_3)$
Solution:

Proof. $\sigma(C_1) = \sigma(C_2)$

- $C_1 \subseteq \sigma(C_2)$ because $(-\infty, b] \in \sigma(C_2) \Rightarrow \sigma(C_1) \subseteq \sigma(C_2)$ because $\sigma(C_1)$ is minimal
- $C_2 \subseteq \sigma(C_1)$ because $(a, b] = (-\infty, b] \backslash (-\infty, a] \in \sigma(C_1) \Rightarrow \sigma(C_2) \subseteq \sigma(C_1)$ because $\sigma(C_2)$ is minimal

■

§2.6.1 Special Case:

$\Omega = \{1, 2, \cdots, N\}, F = P(\Omega), P(\{1\}) = \cdots = P\{N\} = \frac{1}{N}$
For event:$E \in \Omega, P(E) = \frac{|E|}{|\Omega|}$

Definition 2.6.1: Inclusion-Exclusion Principle

$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) + \cdots + (-1)^{r-1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cap \cdots \cap E_{i_r}) + \cdots + (-1)^{n-1} P(E_1 \cap E_2 \cap \cdots \cap E_n)$

Remark 1

① Prove by induction

② $P(E_1 \cup E_2 \cup \cdots \cup E_n) \leq \sum P(E_i)$
$P(E_1 \cup E_2 \cup \cdots \cup E_n) \geq \sum P(E_i) - \sum P(E_i \cup E_j)$
$P(E_1 \cup E_2 \cup \cdots \cup E_n) \leq \sum P(E_i) - \sum P(E_i \cap E_j) + \sum P(E_i \cap E_j \cap E_k)$

Example 1: Brithday Problem

N people, what is the P[at least two people have the same birthday]?
Solution:

$$\begin{aligned}
\Omega &= \{(x_1, \cdots, x_n, x_i \in \{1, \cdots, 365\})\}|\Omega| = 365^N \\
A^c &= \{(x_1, \cdots, x_N) \in \Omega : x_i \neq x_j\}|A^c| = 365 \cdot 364 \cdots (365 - N + 1) \\
P(A) &= \frac{|A^c|}{|\Omega|} = 1 \cdots (1 - \frac{1}{365}) \cdots (1 - \frac{N-1}{365})(use 1 - x \leq e^{-x}) \\
&\leq e^{-\sum_{i=0}^{N-1} \frac{i}{365}} \\
&= e^{-\frac{N(N-1)}{730}}
\end{aligned}$$

in fact if N>23, then $P(A^c) < \frac{1}{2}$

§2.7 Lecture 5 (02-17)

Useful π-system that generates B(R):

① $\mathcal{E} = \begin{cases} = \{(a, b], -\infty \leq a < & b < +\infty\} \\ = \{(a, +\infty), a & \in R\} \end{cases}$

② $\mathcal{E}_1 = \{(a, b], -\infty \leq a < b < +\infty\}$

③ $\mathcal{E}_2 = \{(a, b), -\infty \leq a < b \leq +\infty\}$

④ $\mathcal{E}_{\text{open}} = \{A \in R, A \text{ open }\}$

⑤ $\mathcal{E}_{\text{closed}} = \{A \in R, A \text{ closed }\}$

⑥ $\mathcal{E}_{\text{half}} = \{(-\infty, b], b \in \mathbb{R}\}$

easy to check π-system
Note:

- A open iff $\forall x \in A, \exists \mathcal{E}_x > 0, \text{ s.t. } (x - \mathcal{E}_2, x + \mathcal{E}_2) \subseteq A$
- A closed iff A^c open

Proof. (2)

$$\begin{aligned}
&\mathcal{E}_1 \subseteq \sigma(\mathcal{E}) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}) \\
&\mathcal{E} \subseteq \sigma(\mathcal{E}_1), (a, +\infty) = \bigcup_{n\geq 1} \underbrace{(a, a+n]}_{\mathcal{E}_1} \in \sigma(\mathcal{E}_1) \\
\Rightarrow &\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{E}_1)
\end{aligned}$$

(3)

$$\begin{aligned}
\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2) : (a, b] = \bigcap_{n\geq 1} \underbrace{(a, b + \frac{1}{n})}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2) \\
\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1) : (a, b) = \bigcup_{n\geq 1} \underbrace{(a, b - \frac{1}{n})}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2)
\end{aligned}$$

(4)

$$\mathcal{E}_2 \subseteq \mathcal{E}_{\text{open}} \Rightarrow \sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{E}_{open})$$

Fact:
every open set $A \subseteq \mathbb{R}, A = \bigcup i = 1^{+\infty}(\underbrace{(x_i - \mathcal{E}_i, x_i + \mathcal{E}_i)}_{\mathcal{E}_2} \in \sigma(\mathcal{E}_2))$

$\Rightarrow \mathcal{E}_{open} \subseteq \sigma(\mathcal{E}_2) \Rightarrow \sigma(\mathcal{E}_{open}) \subseteq \sigma(\mathcal{E}_2)$
(e)

$$\begin{aligned}
&A \text{ open } \Leftrightarrow A^c \text{ closed implies} \\
&\mathcal{E}_{\text{closed}} \subseteq \sigma(\mathcal{E}_{\text{open}}) \Rightarrow \sigma(\mathcal{E}_{\text{closed}}) \subseteq \sigma(\mathcal{E}_{\text{open}}) \\
&\mathcal{E}_{\text{open}} = \sigma(\mathcal{E}_{\text{closed}})
\end{aligned}$$

■

Example 1: mismatch

N men and N hats
P[no one finds his own hats]=?
Solution:
$E_i = \{i^{th}$ letter in i^{th}envelope$\}$

Want to compute $P(\bigcap_{i=1}^n E_i^c) = 1 - P(\bigcup_{i=1}^n E_i)$ By inclusion-exclusion:

$$P(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \cdots + (-1)^{n-1} P(\sum_{i=1}^n E_{i1} \cap \cdots \cap E_{ir})$$

$$P(\sum_{i=1}^n E_{i1} \cap \cdots \cap E_{ir}) = \frac{|E_{i1} \cap \cdots \cap E_{ir}|}{|\Omega|} = \frac{(n-r)!}{n!}$$

$$\sum_{i_1<\cdots<i_r} P(E_{i1} \cap \cdots \cap E_{ir}) = \binom{n}{r}\frac{(n-r)!}{n!} = \frac{1}{r!}$$

$$P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N-1}\frac{1}{N!}$$

$$P(\bigcap_{i=1}^n E_i^c) \to 1 - \frac{1}{e}$$

Exercise:

- Circle: 10 couples,
  P[no couple sit next to each other]=?
- Texas Holder:
  P[Sraight]=? P[Full House]=? sol:

$$|\Omega| = \binom{52}{5}$$

$$|Srtaight| = 10 \times (4^5 - 4)$$

$$P[Straight] = \frac{10 \times (4^5 - 4)}{\binom{52}{5}} \approxeq 0.0039$$

$$|FullHouse| = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$$

$$P[FullHouse] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approxeq 0.0014$$

Conditional Probability:
"If the event B has occurred, what is the probability of event A?" $P[A \backslash B]$
N experiment: natural

$$P[A\backslash B] = \frac{\text{number of occurance of both A and B}}{\text{number of occurance of B}} = \frac{P[A \cap B]}{P[B]}$$

Definition 2.7.1

If $P[B] > 0$, then the conditional probability of A given B is $P[A\backslash B] = \frac{P[A\cap B]}{P[B]}$.

Example 2

① 2 kids

- P[two boys\at least one boy] =?

- P[two boys\younger kid is boy] =? 1/2
- P[two boys\at least one boy born on Tuesday] =?

(1)

$$A = \{BB\}, B = \{BG, GB, BB\}$$
$$P[A \backslash B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]}{P[B]} = \frac{1/4}{3/4} = \frac{1}{3}$$

(3)

$$\begin{aligned}
\Omega &= \{B_iB_j, B_iG_j, G_iB_j, G_iG_j, i, j = 1, \cdots, 7\} \\
A &= \{B_iB_j, i, j = 1, \cdots, 7\} \\
B &= \{B_2B_j, B_iB_2, B_2G_j, i, j = 1, \cdots, 7\} \text{ 13+14 elements} \\
A \cap B &= \{B_2B_j, B_iB_2, i, j = 1, \cdots, 7\} \text{ 13 elements} \\
P[A \backslash B] &= \frac{13}{27}
\end{aligned}$$

§2.8 Lecture 6 (02-19)-Conditional Probability

Definition 2.8.1: Law of total probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$
$$P[A] = P[A|B] \cdot P[B] + P[A|B^c]P[B^c]$$

More generally, Let $(B_i)_{i=1}^n$ be a partition of Ω (i.e. $B_i \cap B_j = \emptyset$ and $\bigcup_{i=1}^n B_i = \Omega$), then:

$$P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$$

Proof:

$$P[A] = P[A \cap B] + P[A \cap B^c] = P[A|B]P[B] + P[A|B^c]P[B^c]$$

Example 1

① draw balls randomly from box A(3B2W) to B(4B3W) ,then draw from B randomly,P[The second draw is Black]
By Law of total probability: P=P[2nd Black|1st Black]P[1st Black]+P[2nd Black|1st White]P[1st White]
$=\frac{5}{8} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{23}{40}$

Reverse Question:

If A happens, which B_i is the most likely? **Bayes' Formula:**

$$P[B_i|A] = \frac{P[B_i \cap A]}{P[A]} = \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{n} P[A|B_i]P[B_i]}$$

Example 2

① Covid Test:

- false negative:P[negative|infected]=0.05
- false positive:P[positive|not infected]=0.01

Suppose 5% of the population are infected, then P[infected|positive]=?

Solution.

$$P[V|P] = \frac{P[P|V]P[V]}{P[P|V]P[V] + P[P|V^c]P[V^c]} = \frac{5}{6}$$

② Prisoner parados:A,B,C. 2 executed, 1 pardoned.
A asked :"Please tell me the name of someone else who will be executed"
Guard:"B will be executed"
P[A survive|B will be executed]=?

Solution.

$$\begin{aligned} P[A|\text{Guard says B}] &= \frac{P[\text{Guard says B}|A]P[A]}{P[\text{Guard says B}|A]P[A] + P[\text{Guard says B}|B]P[B] + P[\text{Guard says B}|C]P[C]} \\ &= \frac{\frac{1}{6}}{\frac{1}{6} + 0 + \frac{1}{3}} = \frac{1}{3} \end{aligned}$$

Solution.

$$\begin{aligned} \Omega &= \{(\text{survive person,name mentioned by Guard})\} \\ &= \begin{cases} (A,B), & \frac{1}{6} \\ (A,C), & \frac{1}{6} \\ (B,C), & \frac{1}{3} \\ (C,B), & \frac{1}{3} \end{cases} P[A|B] = \frac{P(A,B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3} \end{aligned}$$

③ Two envelope problem:X,2X ,switch or not?

-

§2.9 Lecture 7 (02-21)-Independence

Definition 2.9.1: Independence

$$P[A|B] = P[A]$$

then we say A and B are independent
Two events A and B are independent iff

$$P[A \cap B] = P[A]P[B]$$

(A,B independent $\Rightarrow A^c, B$ independent)

Definition 2.9.2: Multiple events Independence

The events $A_1, A_2, \cdots, A_n$ are independent iff

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2]\cdots P[A_n]$$

Notice: Stronger than the condition$P[A \cap B] = P[A]P[B]$ (pairwise independence)

Example 1: pairwise independence<Independence

x_1, x_2, x_3 are coin flips:

$$P(x_i = 1) = P(x_i = 0) = \frac{1}{2}$$
$$A_1 = \{x_2 = x_3\}, A_2 = \{x_1 = x_3\}, A_3 = \{x_1 = x_2\}$$
$$P[A_i] = \frac{1}{2}, P[A_i \cap A_j] = P[x_1 = x_2 = x_3] = \frac{1}{4} = P[A_i]P[A_j] \text{ pairwise independent}$$

$P[A_1 \cap A_2 \cap A_3] = P[x_1 = x_2 = x_3] = \frac{1}{4}$ not independent

Example 2: Independence

Independent trials, each with success probability p, fail probability 1-p
$P_{n,m}[\underbrace{\text{first n success}}_{A_{n,m}}$ occured before first m failures]=?

Solution. (Pascal)
By Law of total probability:

$$\begin{aligned} P[A_{n,m}] &= P[A_{n,m}|\text{ 1st success }]P[\text{1st success}] + P[A_{n,m}|\text{ 1st failure }]P[\text{1st failure}] \\ &= P[A_{n-1,m}] \cdot P + P[A_{n,m-1}] \cdot (1-P) \\ \Rightarrow P_{n,m} &= P_{n-1,m} \cdot P + P_{n,m-1} \cdot (1-P) \end{aligned}$$

Boundary condition:$P_{0,m} = (1-P)^m, P_{n,0} = 0$

Or building generating function
(Fermat):
{First n success before first m failure}
$\Leftrightarrow$ {at least n success in the first m+n-1 trials} (ex)
P[exact k success in n+m-1 trials] $= \binom{n+m-1}{k} p^k (1-p)^{m+n-1-k}$ Binomial distribution
$\Rightarrow P$[at least n success in the first m+n-1 trials] $= \sum_{k=n}^{m+n-1} \binom{n+m-1}{k} p^k (1-p)^{m+n-1-k}$

Example 3

Multiple choice test, m options, p-knows the answer, 1-p random guess
P[knows the answer|correct]=$\frac{p}{p+(1-p)(\frac{1}{m})} = \frac{mp}{mp+1-p}$ Bayes

Example 4

Gambler's ruin:
bet 1 dollar each time, p-win, 1-p-lose, initial amount of money=$i \in [0, N]$
P_i[Reach N before reaching 0] $=$ win times=N-i+lose times, Method2: $p_i = pp_{i+1} + (1-p)p_{i-1}$
characteristic polynomials: take $p_i = cr^i$

$$\begin{aligned}
pr^2 - r + (1-p) &= 0 \\
r &= 1, \frac{1-p}{p} \\
&\text{if } p \neq 1-p, \text{ then } p_i = c_1 + c_2(\frac{1-p}{p})^i \\
&\text{if } p = 1-p, \text{ then } p_i = c_1 + c_2 i, c_1, c_2 \text{ determined by } p_0, p_N
\end{aligned}$$

One dimentional random walk:

$$\begin{aligned}
S_n &= S_0 + X_1 + X_2 + \cdots + X_n \\
X_i \text{ are i.i.d. } P(X_i = 1) &= P, P(X_i = -1) = 1 - P
\end{aligned}$$

Example 5

1. secretary problem:
   N candidates
   After each interview, immediately make offer or rejection
   what is the best strategy
   maximize P[best candidate is offered]

   Solution. Not making offer to first r candidate, make an offer to the next candidate that is better than {1,2,...r}
   P[Best candidate is offered]
   $= \sum_{i=0}^{N} \underbrace{P[\text{best candidate is i}]}_{\frac{1}{N}} P[\text{Best candidate is offerd}|\text{Best candidate=i}]$

§2.10 Lecture 7 (02-24)-Random walk

Definition 2.10.1: 1-dimentional random wlk

$S_n = S_0 + x_1 + x_2 + \cdots + x_n$ (x_i) i.d.d(independent+identically distributed)

$P(x_i = 1) = p, P(x_i = -1) = 1 - p$

Generalization: 2-D(simple) random walk $S_n = S_0 + x_1 + x_2 + \cdots + x_n$ (x_i) i.d.d

$$P(x_i = \pm e_1) = P(x_i = \pm e_2) = \frac{1}{4}$$

P[random walk(starting at (i,j)) exit the boundary through A]=?
By conditioning, $P_{i,j} = \frac{1}{4}P_{i+1,j}\frac{1}{4}P_{i-1,j} + \frac{1}{4}P_{i,j-1} + \frac{1}{4}P_{i,j+1}$
Boundary condition: $P_{i,j} = 1$ if $(i,j) \in A$, and $P_{i,j} = 0$ otherwise
We have

$$\begin{aligned} P_{i,j} &= \frac{1}{4}P_{i+1,j}\frac{1}{4}P_{i-1,j} + \frac{1}{4}P_{i,j-1} + \frac{1}{4}P_{i,j+1} \\ &\Leftrightarrow \frac{1}{4}(P_{i+1,j} + P_{i-1,j} - 2P_{i,j}) + \frac{1}{4}(P_{i,j-1} + P_{i,j+1} - 2P_{i,j}) \quad = 0 \\ \Leftrightarrow \Delta P = 0 \end{aligned}$$

Discrete Laplacian on Z^2,

$$\Delta P(x) = \sum_{y \approx x} \frac{1}{4}(P(y) - P(x))$$

§2.10.1 Random Variable and Measurable functions

Example 1

① 2 coinflips, $\Omega = \{HH, TT, HT, TH\}, f = P(\Omega)$ X= #heads,then we may write $X : \Omega \to \mathbb{N}, X = 2 \cdot 1_{HH} + 1_{HT} + 1_{TH}$

$$P(X \le x) = P(\{\omega \in \Omega : X(\omega \le x)\})$$

It is called the distribution function of X (right continuous increasing function)

Definition 2.10.2

Let $(\Omega_1, F_1), (\Omega_2, F_2)$ be two measurable space

- A map $X : \Omega_1 \to \Omega_2$ is called measurable iff $\forall A \in F_2, X^{-1} \in F_1$, $X^{-1} = \{\omega \in \Omega_1 : X(\omega) \in A\}$
- if (Ω_1, F_1, P) is a probability space,then a measurable funtion $X : \Omega_1 \to \Omega_2$ is called a random variable
- if $(\Omega_2, F_2) = (R, B(R))$ then a measurable function is called a Borel funtion
- if $X : (\Omega_1, F_1, P) \to (R, B(R))$ then X is a R-valued random variable and

$$F_X(x) = P(X^{-1}(-\infty, x]) = P(X \le x)$$

is called the distribution function of X

Remark: By (a), we know that for any $B \in B(R)$ we can define $P(X^{-1}(B))$

Example 2

① Let (Ω, F, P) be a probability space,$A \in F$

Then $1_A : \Omega \to \{0,1\}$ is a random variable, $1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in A^c \end{cases}$ We need to show $\forall B \in B(R), 1_A^{-1}(B) \in F$

In fact, $1_A^{-1}(B) = \begin{cases} \emptyset, & \text{if } 0,1 \notin B \\ A, & \text{if } 1 \in B, 0 \notin B \\ A^c, & \text{if } 0 \in B, 1 \notin B \\ \Omega, & \text{if } 0,1 \in B \end{cases}$

In practice, only need to check the pre-image on a smaller set:

Proposition 2.10.0

Let $(\Omega_1, F_1), (\Omega_2, F_2)$ be two measurable space
$\mathcal{E} \subseteq F_2$ such that $\sigma(\mathcal{E}) = F_2$
Then $X : \Omega_1 \to \Omega_2$ is measurable if $\forall A \in \mathcal{E}, X^{-1}(A) \in F_2$

Proof. Let $G = \{B \subseteq \Omega_2, X^{-1}(B) \in F_1\}$
Then G is a σ-algebra (ex)
Therefore if $G \supseteq \mathcal{E}$ then $G \supseteq \sigma(\mathcal{E}) = F_2$ ■

Corollary 2.10.1

et(Ω, F) be a measurable space, then $X : \Omega \to R$ is a Borel function iff the following are true.

- $\{x < a\} \in F$ for $\forall a \in R$
- $\{x \leq a\} \in F$ for $\forall a \in R$
- $\{x > a\} \in F$ for $\forall a \in R$
- $\{x \geq a\} \in F$ for $\forall a \in R$

Proof. $\{x < a, a \in \mathbb{R}\} = \{X^{-1}(-\infty, a), a \in \mathbb{R}\}$
suffices to show:$\mathcal{E} = \{(-\infty, a), a \in \mathbb{R}\}$
satisfies $\sigma(\mathcal{E}) = B(\mathbb{R})$
indeed:

$$[a, +\infty) = (-\infty, a)^c \in \sigma(\mathcal{E}) \tag{2.1}$$
$$[a, b) = [a, +\infty) \backslash (b, +\infty) = (-\infty, a)^c \backslash (-\infty, b)^c \in \sigma(\mathcal{E}) \tag{2.2}$$
$$\Rightarrow \sigma(\mathcal{E}) = B(R) \tag{2.3}$$

■

Remark 1

a R-valued random variable is a function $X : \Omega \to \mathbb{R}$ such that $\forall a \in \mathbb{R}, \{X < a\} \in F$

Lemma 2.10.1

A distribution funtion F satisfies:

① $\lim_{x\to-\infty} F(x) = 0, \lim_{x\to+\infty} F(x) = 1$

② $F(x) \leq F(y)$ if $x \leq y$

③ F is right continuous,$F(x + h) \to F(x)$ as h decreases to 0

Proof. $F(x) = P(X \leq x)$ (2) is immediate
(1): Let $A_n = \{x \leq -n\}$ then by continuous of measure:$lim_{n\to+\infty} = \lim_{n\to+\infty} = P(\bigcap_{n\geq 1} A_n) = 0$
Then by monotonicity of F, $lim_{x\to-\infty}F(x) = 0$
(3): Let $B_n = \{X \leq x + \frac{1}{n}\}$(decreaing),
then $F(x + \frac{1}{n}) = P(B_n) \underbrace{\to}_{n\to+\infty,cont.} P(\bigcap_{n\geq 1} B_n) = P(X \leq x) = F(x)$
again use F increasing to conclude ∎

Lemma 2.10.2

① $F(y) - F(x) = P(x < X \leq y), \forall x \leq y$

② $F(x) - \lim_{h\to+\infty} F(x - h) = P(X = x)$

③ $B_n = \{x - \frac{1}{n} < X \leq x\}$(decreaing),Then $\bigcap_{n\geq 1} B_n = \{X = x\}$
$\Rightarrow F(x) - F(x - \frac{1}{n}) = P(B_n) \underbrace{\to}_{n\to+\infty} P(X = x)$
again use F increasing to conclude

Naturally:

- (absolute) continuous random variable
- discrete random variable

§2.11 Lecture 8 (02-26)

properties of measurable functions(R.V.s):

- Let $\{X < Y\} = \{\omega \in \Omega_1 : X(\omega) < Y(\omega)\}$
  $\{X > Y\} = \{\omega \in \Omega_1 : X(\omega) > Y(\omega)\}$

Lemma 2.11.1

Let (Ω, F) be a measurable space, X, Y are Borel Functions

- $\{X < Y\}, \{X \leq Y\}, \{X = Y\}, \{X \neq Y\} \in F$
- $X + Y, X \cdot Y, X/Y$ are Borel functions

Proof. (1):Use that Q is dense in R

$$\{X < Y\} = \bigcup_{q \in Q} \{X < q < Y\} = \bigcup_{q \in Q} \underbrace{\{X < q\}}_{\in F} \cap \underbrace{\{q < Y\}}_{\in F} \in F$$

$$\{X = Y\} = \bigcap_{n \geq 1} \{X < Y + \frac{1}{n}\} \cap \bigcap_{n \geq 1} \{X > Y - \frac{1}{n}\} \in F$$

(2):Fact: if Y is Borel, then aY+b,$a, b \in R$ is Borel (ex)
Then

$$\forall a \in \mathbb{R}, \{X + Y < a\} = \{X < a - Y\} \in F \text{ by}(1)$$

so $X + Y$ is Borel

$$\{X^2 < a\} = \begin{cases} \emptyset, \\ \{X < \sqrt{a}\} \cap \{X > -\sqrt{a}\} \in F \end{cases}$$

so X^2 is Borel
$X \cdot Y = \frac{1}{4}[(X + Y)^2 - (X - Y)^2]$ is Borel ■

Lemma 2.11.2

Let $(X_n)_{n \geq 1}$ be a sequence of Borel functions on (Ω, F),
Then the following are Borel functions:

- $\sup_{n \geq 1} X_n, \inf_{n \geq 1} X_n, \limsup_{n \geq 1} X_n, \liminf_{n \geq 1} X_n$

In particular, if $\lim_{n \to +\infty} X_n$ exists, then $\lim_{n \to +\infty} X_n$ is Borel

Proof.

$$\{\sup_{n \geq 1} X_n < a\} = \bigcap_{n \geq 1} \{X_n < a\} \in F \forall a \in \mathbb{R} \Rightarrow \sup_{n \geq 1} X_n \text{ is a Borel function}$$

$$\{\inf_{n \geq 1} X_n > a\} = \bigcap_{n \geq 1} \{X_> < a\} \in F \forall a \in \mathbb{R} \Rightarrow \inf_{n \geq 1} X_n \text{ is a Borel function}$$

$$\limsup X_n = \inf_{m \geq 1} \sup_{n \geq m} X_n \text{ is a Borel function}$$

$$\liminf X_n = \sup_{m \geq 1} \inf_{n \geq m} X_n \text{ is a Borel function}$$

■

Lemma 2.11.3

Let $(\Omega_1, F_1), (\Omega_2, F_2), (\Omega_3, F_3)$ be two measurable space
$X : \Omega_1 \to \Omega_2, Y : \Omega_2 \to \Omega_3$ are measurable, then $Y \circ X : \Omega_1 \to \Omega_3$ is measurable

Proof.

$$\forall A \in F_3, (Y \circ X)^{-1}(A) = X^{-1}(\underbrace{Y^{-1}(A)}_{\in F_2}) \in F_1$$

■

Definition 2.11.1: σ-algebra generated by r.v.

If $X : \Omega \to \mathbb{R}$ is a random variable, then

$$\sigma(x) = \{X^{-1}(A), A \in B(\mathbb{R})\}$$

is called the σ-algebra generated by X
Let $(X_i)_{i\in I}$ be a family of r.v.s

$$\sigma(X_i, i \in I) = \sigma(\bigcup_{i\in I} \sigma(X_i))$$

is the σ-algebra generated by $(X_i)_{i\in I}$

Remark 1

$\sigma(X)$ is the smallest σ-algebra such that X is measurable

Example 1

① (Ω, F, P) ,Let $X : \Omega \to \mathbb{R}$ be a random variable, $X = b_1 1_{A_1} + b_2 1_{A_2} + \cdots b_n 1_{A_n}, b_i \in \mathbb{R}, A_i \in F, A_i \cap A_j = \emptyset$
if $b_j \neq b_j$
Then $\sigma(X) = \sigma\{A_1, A_2, \cdots, A_n\} =$"finite disjioint union of $A_1 \cdots A_n$"

Proof.
- $\supseteq$: Note that $X^{-1}(\{b_1\}) = A_1, \cdots, X^{-1}(\{b_n\}) = A_n$
$\Rightarrow A_1, \cdots, A_n \in \sigma(x), \sigma(A_1, \cdots, A_n) \in \sigma(x)$
- $\subseteq$

Lemma 2.11.4

$\sigma(X) = \sigma(\{X \leq a\}, a \in \mathbb{R})$

it suffices to show $\forall a \in \mathbb{R}, \{X \leq a\} \in \sigma(\{A_1, \cdots, A_n\})$
$\{X \leq a\} =$"finite disjioint union of A_i and $(\bigcup A_i)^c$"

■

Two specific cases: Discrete and (Absolutely)continuous random variables

Definition 2.11.2

A r.v. is discrete if it takes values in a countable set $\{X_1, X_2, \cdots\}$
prob mass function:$f(x) = P(X = x)$

Remark 2

We say $x_1, x_2, \cdots$, are atoms of F_x

Definition 2.11.3

A r.v. is (absolutely) continuous if

$$F_x(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

for some integrable function $f : R \to [0, +\infty), f(x) = F_x'(x)$ is called the probability density function of X

Remark 3

F_x is absolutely continuous
F is absolutely continuous iff

$$\forall \epsilon > 0, \exists \delta > 0$$

s.t. for any finite collecion of intervalsa_i, b_i s.t.

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$$

$$C^1 \Rightarrow \text{absolutely continuous} \Rightarrow \text{uniformly continuous}$$

Remark 4

X is a singular Continuous R.V. if F_x is continuous but F_x is not absolutely continuous (ex: Cantor funtion)

Discrete R.V.:

$$F_x(x) = P(X \leq x) = \sum_{x_i \leq x} \underbrace{f(x_i)}_{\text{prob mass function}}$$

$$f(x) = F_x(x) - \lim y \nearrow x - F_x(y)$$

Definition 2.11.4

The expectation/mean of a discrete r.v. with prob mass function f is

$$E[X] = \sum_{x:f(x)>0} x_i f(x_i)$$

whenever the sum is absolutely convergent

Remark 5

absolutely convergent $\Rightarrow$ order of the sum does not matter

Example 2

① 2 coin flips,X=#heads, $f(0) = f(2) = \frac{1}{4}, f(1) = \frac{1}{2}$
$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

Lemma 2.11.5

Let X be a r.v. taking values in N.
Then

$$E[X] = \sum_{n=1}^{+\infty} P(X \geq n)$$

Proof.

$$\begin{aligned} E[x] &= 1 \cdot P(X=1) + 2 \cdot P(X=2) + \cdots \\ &= P(X=1) + P(X=2) + P(X=3) + \cdots \\ &\qquad\qquad P(X=2) + P(X=3) + \cdots \\ &\qquad\qquad\qquad\qquad P(X=3) + \cdots \end{aligned}$$

$P(X \geq n)$ ■

§2.12 Lecture 9 (02-28)-Expectation

EX1

we have offers $X_1, X_2, \cdots$ continuous R.V. i.i.d
$T = inf\{n > 1, X_n > X_1\}$ Compute $E[T]$
Solution.

$$E[T] = \sum_{n=2}^{+\infty} nP(T=n)$$

By lemma,

$$\begin{aligned} E[T] &= \sum_{n\geq 1} P(T \geq n) = \sum_{n \geq 1} P(X_1 \geq X_2 \geq \cdots \geq X_n) = \frac{1}{n-1}, n \geq 2 \\ &= +\infty \end{aligned}$$

EX2

Independent trials, each with success probability p, fail probability 1-p
Compute P[There is n consecutive successes before m consecutive failures] Condition on the first trial:

$$P[A] = p \cdot P[A|H] + (1-p) \cdot P[A|T]$$

Multiple conditioning: $P(E_1 \cap E_2 \cdots E_n) = P(E_1)P(E_2|E_1) \cdots P(E_n|E_1 \cap \cdots E_n)$
Let $B = \{HH \cdots H\}$ (2nd–n-th)

$$P(A|H) = P(B)P(A|H \cap B) + P(B^c)P(A|H \cap B^c) = p^{n-1} \cdot 1 + (1 - p^{n-1}) \cdot P(A|T)$$

Let $C = \{TT \cdots T\}$ (2nd–m-th)

$$P(A|T) = P(C)P(A|T \cap C) + P(C^c)P(A|T \cap C^c)$$

Let q=1-p,

$$P(A|H) = \frac{p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$
$$P(A|T) = \frac{(1 - q^{m-1})p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$
$$P(A) = \frac{p^{n-1}(1 - q^{m-1})}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$

§2.13 Lecture 10 (03-02)-Var

Lemma 2.13.1: change of variable

Let $g : R \to R$ X be a r.v. with probability mass function f ,then

$$E[g(x)] = \sum_{x:f(x)>0} g(x)f(x)$$

Proof.

$$E[g(x)] = \sum_y y \cdot P(g(x) = y) = \sum_y \sum_{x:g(x)=y} y \cdot P(X = x)$$
$$= \sum_x g(x)P(X = x) = \sum_x g(x)f(x)$$

■

Definition 2.13.1

Let $k \in N$,the k-th moment of X is $mk := E[X^k]$ as long as the expectation exists
Let $k \in N$,the k-th central moment of X is $\sigma k := E[X - E[X]]^k$ as long as the expectation exists

- $\sigma_2 = Var(X) = E(X - E[X])^2$ varience,"deviation fluctuation" from the mean

- $\sigma = \sqrt{Var(X)}$ standard deviation

Fact:

- $E[aX + bY] = aE[x] + bE[Y] a, b \in R$
  $\rightarrow Var(X) = E(X - E[x])^2 = E[X^2] - (E[X])^2$

Example 1

① Bernoulli(p) $P(X = 1) = p, P(X = 0) = 1 - p$
$E[X] = 1 \cdot P(X = 1) = p, Var(X) = p - p^2$

② binomial(n,p) $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$E[X] = \sum_{k=0}^{n} kP(x = k) = \sum_k k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Recall

$$(1 + x)^n = \sum_{k=0}^{n} x^k \cdot \binom{n}{k}$$

,differentiate both sides w.r.t x, then

$$n(1 + x)^{n-1} = \sum_{k=1}^{n} k \cdot x^{k-1} \cdot \binom{n}{k}$$

Let $q = 1 - p, x = \frac{p}{q}$, then

$$\begin{aligned} E[X] &= \sum_{k=1}^{n} k \cdot \binom{n}{k} (\frac{p}{q})^{k-1} \cdot q^n \cdot \frac{p}{q} \\ &= n \cdot \frac{p}{1-p} \cdot (1 + \frac{p}{1-p})^{n-1} = np \\ Var(X) &= np(1-p) \end{aligned}$$

sol n': we have $X = Y_1 + Y_2 + \cdots + Y_n$ such that $Y_i \sim Bernoulli(p)$ and Y_i independent

$$E[X] = \sum_{i=1}^{n} E[Y_i] = np$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = E(\sum_{i=1}^{n} Y_i)^2 - (np)^2 \\ &= \sum_{i=1}^{n} E[Y_i]^2 - 2\sum_{i<j}^{n} E[Y_i]E[Y_j] - (np)^2 \\ &= np + 2p^2 \frac{n(n-1)}{2} - (np)^2 \\ &= np(1-p) \end{aligned}$$

If x,y are independent, then $E[XY] = E[X]E[Y]$
Proof:

$$E[XY] = \sum_{x,y} xyP(X = x, Y = y) = \sum_{x,y} xyP(X = x)P(Y = y)$$
$$= \sum_{x} xP(X = x) \sum_{y} yP(Y = y) = E[X]E[Y]$$

Definition 2.13.2

If E[XY]=E[X]E[Y], then we say X,Y are uncorrelated
Covariance

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E[Y])]$$

Correlation
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Lemma 2.13.2

$|\rho(X,Y)| \le 1, \rho(X,Y) = \pm 1$ iff $Y = aX + b$ for some $a, b \in \mathbb{R}$

Proof.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E[(X - E[X])(Y - E[Y])]}{\sqrt{Var(X)Var(Y)}}$$
by Cauchy-Schwarz inequality: $|\rho(X,Y)| \le 1$

■

Cauchy-Schwarz inequality:

$$|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$$

Proof.

Since $0 \le E[(aX - bY)^2] \forall a, b \in R \Rightarrow a^2E[X^2] + 2abE[XY] + b^2E[Y^2] \ge 0$
$$\Rightarrow \frac{1}{4}\Delta = E[XY]^2 - E[X^2]E[Y^2] \le 0$$

■

Example 2

① $X \sim Geometric(p), P(X = k) = (1 - p)^{k-1} \cdot p$
$E[X] = \frac{1}{p}, Var(x) = \frac{1-p}{p^2}$

§2.14 Lecture 11 (03-05)-poisson random variable

Poisson random variable:
Observe the number of customers in the past days, X_i is the number of customers on day i
How to predict the number of customers tomorrow?

- One may take $\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- n intervals during each interval, at most 1 customers
  $\Rightarrow$ number of customers in n intervals is $Bern(p)$
  Take P, s.t. $np = E[X] = \lambda$
  number of customer is Binomial$(n, \frac{\lambda}{n})$

$$\begin{aligned}P(X=k) &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k (1-\frac{\lambda}{n})^{n-k}\\ &= \frac{n!}{(n-k)!n^k}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^{n-k}\\ &\Rightarrow (n\to+\infty, k \text{ fixed})\frac{\lambda^k}{k!}e^{-\lambda}\end{aligned}$$

Definition 2.14.1

X is a Poisson(x) is given by probability mass function

$$f(k) = \frac{\lambda^k}{k!}e^{-\lambda}, k \in N$$

if $X \sim Poisson(\lambda)$, then

$$E[X] = \sum_{k\in N^+} k\cdot\frac{\lambda^k}{k!}e^{-\lambda} = \lambda\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^{\lambda-\lambda} = \lambda$$

Continuous r.v.

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du, f(x) = F_x'(x)$$

is the probability density function of X

$$P(x \le X \le X + dx) = \int_{x}^{x+dx} f(u)du \approx f(x)dx$$

Expectation: the expectation of a r.v. X is defined by

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} xdF_x(x)$$

k-th moment:

$$E[x^k] = \sum_{-\infty}^{+\infty} x^k f(x)dx$$
$$Var[x] = E[(X - E[X])^2] = \int_{-\infty}^{+\infty} (x - E[X])^2 f(x)dx$$

Recall if X is a N-valued r.v. then

$$E[x] = \sum_{n \in N} P(x \geq n)$$

Lemma 2.14.1

if X is a non-negative r,v, with density function f then

$$E[X] = \int_0^{+\infty} P(X > x)dx = \int_0^{+\infty} (1 - F_x(x))dx$$

Proof.

$$\begin{aligned}\int_0^{+\infty} P(X > x)dx &= \int_0^{+\infty} \int_x^{+\infty} f(y)dydx \\ &= \int_0^{+\infty} \int_0^{y} dx f(y)dy = \int_0^{+\infty} yf(y)fy = E[X]\end{aligned}$$

■

How to define $\int xdF_x(x)$ in general?

Definition 2.14.2: Lebesgue integral and expectation

recall Riemann integral

$$\sum f(x_i^\star)\Delta x_i \to \int f(x)dx$$

make sense if f has finitely many discontinuities (for $f = 1_Q$ R-I does not exist)
Lebesgue integral:

$$\begin{aligned}\{x \in \mathbb{R} : f(x) \in [y_i, y_i + \Delta y_i)\} &= f^{-1}([y_i, y_i + \Delta y_i)) \\ \text{idea: } \sum m(f^{-1}([y_i, y_i + \Delta y_i))) \cdot \Delta y_i &\to \int f(x)dx \\ \int 1_Q(x)dx &= 0\end{aligned}$$

More generally, given any measure space $(\Omega, \mathcal{F}, \mu)$ and Borel function f
Define $\int_\Omega f du$

Step1: If $f = 1_A$,where $A \in F$ define $\int_\Omega f d\mu = \int_\Omega 1_A d\mu = \mu(A)$
Step2: simple functions:

$$f = \sum_{i=1}^{n} a_i 1_{A_i}, A_i \in F \text{ and } a_i \geq 0, A_i \cap A_j = \emptyset$$

$$\text{Define } \int_\Omega f d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Fact: if f,g are simple function then f+g,fg,max{f,g},min{f,g} are simple functions

Proposition 2.14.0

if f,g are simple function, then

- $\int_\Omega a f d\mu = a \int_\Omega f d\mu \forall a \in \mathbb{R}$
- $\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$
- if $f \leq g$ then $\int_\Omega f d\mu \leq \int_\Omega g d\mu$

Step3: approximates non-negative Borel functions by simple functions
Let $f \geq 0$ Borel, then $f = \sup_i f_i$,where

$$f_i = \sum_{k=0}^{i \cdot 2^i} \frac{k-1}{2^i} 1_{\{\frac{k-1}{2^i} \leq f < \frac{k}{2^i}\}} + i 1_{\{f > i\}}$$

$f_i \geq 0$ are simple and $\lim_{i \to \infty} f_i = \sup_i f_i = f$

Definition 2.14.3

For every non-negative Borel function f, define

$$\int_\Omega f d\mu = \sup_i \int_\Omega f_i d\mu$$

Q:if $f = \sup f_i, f = \sup g_i$ Does $\sup_i \int f_i d\mu = \sup_i \int g_i d\mu$?

Consistency follows from:
Monotone Convergence theorem: For every increasing sequence $\{f_n\}$ of measurable functions:

$$\limsup_n \int_\Omega f_n d\mu = \int_\Omega \limsup_n f_n d\mu$$

(If (X_n)) is a sequence of r.v.s, $X_n \nearrow x$ then $\lim_{n \to +\infty} E[X_n] = E[\lim_n X_n] = E[X]$
Assume MCT:

$$\text{If } f = \sup f_i = \sup g_i$$
$$\text{Then } g_i \leq \sup_i f_i$$

§2.15 Recitation (03-07)

Problem 1

Let $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2), independent$
Compute the probability mass function of $X+Y$

$$\begin{aligned}
P(X+Y=n) &= \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\
&= \sum_{k=0}^{n} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \cdot \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n
\end{aligned}$$

$\sum_k f_x(k) f_Y(n-k) = f_X * f_Y$

Problem 2

m balls, n boxes, uniform random
Compute the E[number of empty boxes]
number of empty boxes:$1_{b_1 empty} + \cdots 1_{b_n empty}$
linearity of expectation:

$$\begin{aligned}
E[\#] &= \sum_{i=1}^{n} E1_{b_i empty} \\
&= nP(\text{box 1 is empty}) \\
&= n(\frac{n-1}{n})^m
\end{aligned}$$

Note:

$$E[X] = \int X dP$$
$$E[X] = \int 1_A dP = P(A)$$

Problem 3

Coupon collector: n types of coupons, pick one at random
T_n =time to complete the collection of n coupons
Compute $E[T_n]$ Let T_i =time to collect the i-th new coupon
$T_n = T_1 + T_2 + \cdots + T_n$
$E[T_1] = 1, T_j - T_{j-1} \sim Geo(1 - \frac{i-1}{n})$ every trial with success prob:$1 - \frac{i-1}{n}$

$$E[T_j - T_{j-1}] = \frac{1}{1 - \frac{i-1}{n}}$$
$$E[T_n] = \sum_{j=1}^{n} E[T_j - T_{j-1}] = n \sum_{j=1}^{n} \frac{1}{j}$$

Problem 4

Let X be a r.v., $E[X] = 1$
show that

$$t \in (0,1), P(X > t) > \frac{(1-t)^2}{E[X^2]}$$

(hint:Cauchy Schwarz)

$$\text{Let } Y = 1_{X>t}$$
$$E[\mathbf{1}_{X>t}] = P(X > t) = E[Y] = E[Y^2]$$
$$\text{By Cauchy-Schwarz: } E[XY] \leq \sqrt{E[X^2]E[Y^2]}$$
$$E[X^2]E[Y^2] \geq E[XY]^2 = E[X(1 - 1_{x \leq t})]^2$$
$$= (1 - E[X 1_{X \leq t}])^2$$
$$\geq (1-t)^2$$

Definition 2.15.1: Paykey-Zygmund inequality

for $t \in (0,1)$

$$P(Y > tE[Y]) \geq (1-t)^2 \frac{(E[Y])^2}{E[Y^2]}$$

second moment method

§2.16 Lecture 12 (03-10)

Midterm March 26

Definition 2.16.1: Monetone Convergence Theorem

For any increasing sequence of function:$\{f_n\}$ such that $\{f_n\}$ is bounded from below
Then

$$\lim_{n \to +\infty} \int f_n d\mu = \int \lim_{n \to +\infty} f_n d\mu$$

: **Proof:**
Since $\int f_n d\mu \leq \int f \mu \forall n \in N$ Then $\limsup \int f_n d\mu \leq f d\mu$
Fact:
if ϕ is a simple function, then $u(A) := \int_A d\mu$ defines a measure(ex.)
Take a sequence $\{\phi_k\}$ of the simple function: $\phi_k \nearrow f$
Let $\alpha \in (0,1)$ Fix a given ϕ_k
Let $A_n = \{f_n > \alpha\phi_k\} = \{\omega \in \Omega : f_n(\omega) > \alpha\phi_k(\omega)\}$

Then $(A_n) \nearrow$, and

$$\begin{aligned} \lim_{n\to+\infty} &= \bigcup_{n\geq 1} A_n = \{\exists n \in N s.t. f_n > \alpha\phi_k\} \\ &= \{sup_n f_n > \alpha\phi k\} = \Omega \end{aligned}$$

$$\text{Since } \int_\Omega f_n d\mu \geq \int_{A_n} f_n d\mu \geq \alpha \int_{A_n} \phi_k d\mu$$

$$\text{send } n \to +\infty \liminf \int_\Omega f_n d\mu \geq \alpha \lim_{n\to+\infty} \int_{A_n} \phi_k d\mu = \alpha \int_\Omega \phi_k d\mu$$

Send $\phi_k \to f$ and $\alpha \to 1$ to complete the pf

Proposition 2.16.0

Let f, g be Lebesgue measurable. Then

- $$\int_\Omega af + bg d\mu = a\int_\Omega f d\mu + b\int_\Omega g d\mu$$
- if $f \leq g$ then
$$\int_\Omega f d\mu \leq \int_\Omega g d\mu$$

Proposition 2.16.0

f is (Lebesgue) integrable i |f| is integrable.
If there exists a function Y, s.t. $|f| \leq Y$, and Y is integrable, then f is integrable.

Proof:

Note that $f = f^+ - f^-,\ |f| = f^+ + f^-$

Therefore $\int_\Omega |f| d\mu < \infty \Leftrightarrow \int_\Omega f^+ d\mu < \infty$ and $\int_\Omega f^- d\mu < \infty \Leftrightarrow \int_\Omega f d\mu < \infty$.

If $|f| \leq Y,$ then $\int_\Omega |f| d\mu \leq \int_\Omega Y d\mu < \infty \Rightarrow$ |f| integrable $\Rightarrow f$ integrable.

eg. $(R, B(R), m),\ f(x) = \sin x$. Is f Lebesgue integrable?

$|f(x)| \leq 1 \Rightarrow \int_R |f| dx \geq \sum_k \int_{k\pi+\frac{\pi}{6}}^{(k+1)\pi-\frac{\pi}{6}} |sinx| dx = +\infty$

$\Rightarrow f$ is NOT integrable

Definition 2.16.2

We say $f = g$ almost everywhere (a.e.) if $\{f \neq g\}$ has measure 0.
r.v.s. $X = Y$ almost surely (a.s.) if $\mathbb{P}(\{X \neq Y\}) = 0$

Theorem 2.16.1

f $\mu(A) = 0$, then $\int_A f\,d\mu = 0$ for any measurable function f.

Proof. We prove this in three steps:

Step 1: Simple functions.

For a simple function $s = \sum_{i=1}^n a_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of set E_i:

$$\int_A s\,d\mu = \sum_{i=1}^n a_i \mu(A \cap E_i)$$

Since $\mu(A) = 0$, we have $\mu(A \cap E_i) \leq \mu(A) = 0$ for any measurable set E_i. Therefore, $\mu(A \cap E_i) = 0$.

Hence, $\int_A s\,d\mu = \sum_{i=1}^n a_i \cdot 0 = 0$

Step 2: Non-negative measurable functions.

For any non-negative measurable function $f \geq 0$, there exists an increasing sequence of simple functions $\{s_n\}$ such that $s_n \uparrow f$ pointwise.

By the Monotone Convergence Theorem:

$$\int_A f\,d\mu = \lim_{n\to\infty} \int_A s_n\,d\mu$$

From Step 1, for each n, $\int_A s_n\,d\mu = 0$. Therefore, $\int_A f\,d\mu = \lim_{n\to\infty} 0 = 0$

Step 3: General measurable functions.

For any measurable function f, we can decompose it as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are both non-negative measurable functions.

By the linearity of the integral:

$$\int_A f\,d\mu = \int_A f^+\,d\mu - \int_A f^-\,d\mu$$

From Step 2, $\int_A f^+\,d\mu = 0$ and $\int_A f^-\,d\mu = 0$. Therefore, $\int_A f\,d\mu = 0 - 0 = 0$

Thus, if $\mu(A) = 0$, then $\int_A f\,d\mu = 0$ for any measurable function f. ∎

Corollary 2.16.1

f $f = g$ almost everywhere (a.e.), then $\int_\Omega f\,d\mu = \int_\Omega g\,d\mu$.

Proof. Let $E = \{x \in \Omega : f(x) \neq g(x)\}$. Since $f = g$ a.e., we have $\mu(E) = 0$.

Consider $h = f - g$. Then $h = 0$ on $\Omega \setminus E$, and $h \neq 0$ only on E.

Therefore:

$$\int_\Omega (f-g)\,d\mu = \int_\Omega h\,d\mu = \int_E h\,d\mu + \int_{\Omega\setminus E} h\,d\mu = \int_E h\,d\mu + 0$$

Since $\mu(E) = 0$, by our theorem, $\int_E h\,d\mu = 0$. Thus, $\int_\Omega (f-g)\,d\mu = 0$.

By the linearity of the integral:

$$\int_\Omega f\, d\mu - \int_\Omega g\, d\mu = \int_\Omega (f-g)\, d\mu = 0$$

Therefore, $\int_\Omega f\, d\mu = \int_\Omega g\, d\mu$. ∎

Proposition 2.16.0

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f : \Omega \to [0, +\infty)$ be a Borel function.

Then $\nu(A) = \int_A f\, d\mu$, $\forall A \in \mathcal{F}$, defines a measure.

Definition 2.16.3

We say f is the Radon-Nikodym derivative (or density) of ν with respect to μ. Write $f = \frac{d\nu}{d\mu}$.

Proof of Proposition.

- $\nu(\emptyset) = 0$ is obvious.
- Countable additivity: Let $(A_i)_{i=1}^{+\infty}$ be disjoint. Let $A = \cup_{i=1}^{+\infty} A_i$.

  Then:

$$\begin{aligned}
\nu(A) &= \int_A f\, d\mu \\
&= \int_\Omega f \cdot \mathbf{1}_A\, d\mu \\
&= \int_\Omega f \cdot \left(\lim_{n\to\infty} \sum_{i=1}^n \mathbf{1}_{A_i} \right) d\mu \\
&\overset{\text{MCT}}{=} \lim_{n\to\infty} \int_\Omega f \cdot \left(\sum_{i=1}^n \mathbf{1}_{A_i} \right) d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^n \int_\Omega f \cdot \mathbf{1}_{A_i}\, d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^n \int_{A_i} f\, d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^n \nu(A_i) \\
&= \sum_{i=1}^{+\infty} \nu(A_i)
\end{aligned}$$

  Therefore, ν is a measure.

∎

Definition 2.16.4

We say ν is absolutely continuous with respect to μ (denoted $\nu \ll \mu$) if for any $A \in \mathcal{F}$ such that $\mu(A) = 0$, then $\nu(A) = 0$.

Example 1

If $\nu(A) = \int_A f \, d\mu$ for some non-negative Borel function f, then $\mu(A) = 0 \Rightarrow \nu(A) = \int_A f \, d\mu = 0$. Therefore, $\nu \ll \mu$.

Example 2: Lebesgue Measure Equivalence

$M_{Leb} \ll 2M_{Leb}$ and $2M_{Leb} \ll M_{Leb}$

Definition 2.16.5: Lebesgue Measure

The Lebesgue measure, denoted by m or λ, is a complete measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$ that satisfies:

1. (Normalization) The measure of the unit cube is 1: $m([0,1]^n) = 1$.
2. (Translation invariance) For any measurable set E and any point $x \in \mathbb{R}^n$, $m(E + x) = m(E)$, where $E + x = \{y + x : y \in E\}$.
3. (Countable additivity) For any countable collection $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint measurable sets, $m(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

Theorem 2.16.2: Properties of Lebesgue Measure

The Lebesgue measure has the following properties:

1. For an interval $[a, b] \subset \mathbb{R}$, $m([a, b]) = b - a$.
2. More generally, for a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, $m([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$.
3. For any $c \in \mathbb{R}$ and any measurable set $E \subset \mathbb{R}^n$, $m(cE) = |c|^n m(E)$, where $cE = \{cx : x \in E\}$.
4. There exist subsets of $\mathbb{R}$ that are not Lebesgue measurable.

Definition 2.16.6: Equivalent Measures

If $\mu \ll \nu$ and $\nu \ll \mu$, then we say μ and ν are equivalent (denoted $\mu \sim \nu$).

Example 3: Dirac Measures and Counting Measure

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$: Dirac measure. For each $k \in \mathbb{N}$, define μ_k such that

$$\mu_k(A) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{else} \end{cases}$$

Counting measure: $\nu(A) = \sum_{k \in \mathbb{N}} \mu_k(A)$ (= "number of elements in A").
Therefore, $\mu_k \ll \nu$ for all $k \in \mathbb{N}$, but $\nu \not\ll \mu_k$.

Theorem 2.16.3: Radon-Nikodym Theorem

If μ, ν are σ-finite measures, and $\nu \ll \mu$, then there exists a Borel function f, such that

$$\forall A \in \mathcal{F}, \quad \nu(A) = \int_A f \, d\mu.$$

Proposition 2.16.0: Equivalent Characterization of Absolute Continuity

$\nu \ll \mu \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0,$ s.t. $\forall A \in \mathcal{F},$ with $\mu(A) < \delta$ we have $\nu(A) < \varepsilon$.

Proof. We will prove both directions of the equivalence.

($\Rightarrow$) **Necessity:** Suppose $\nu \ll \mu$. We need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have $\nu(A) < \varepsilon$.

We will prove this by contradiction. Suppose, contrary to our claim, that there exists some $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $A_n \in \mathcal{F}$ with $\mu(A_n) < \frac{1}{2^n}$ but $\nu(A_n) \geq \varepsilon$.

Let us define $B_k = \bigcup_{n=k}^{\infty} A_n$ for each $k \in \mathbb{N}$. Then $\{B_k\}_{k=1}^{\infty}$ forms a decreasing sequence of sets whose limit is $\lim_{k\to\infty} B_k = \bigcap_{k=1}^{\infty} B_k = \liminf_{n\to\infty} A_n$.

For each k, we can estimate:

$$\begin{aligned} \mu(B_k) &\leq \sum_{n=k}^{\infty} \mu(A_n) \\ &< \sum_{n=k}^{\infty} \frac{1}{2^n} \\ &= \frac{1}{2^{k-1}} \end{aligned}$$

Since $A_k \subseteq B_k$ for each k, and $\nu(A_k) \geq \varepsilon$, it follows that $\nu(B_k) \geq \varepsilon$ for all k.
Now, let $B = \bigcap_{k=1}^{\infty} B_k$. By the continuity of measure for decreasing sequences:

$$\begin{aligned} \mu(B) &= \lim_{k\to\infty} \mu(B_k) \\ &\leq \lim_{k\to\infty} \frac{1}{2^{k-1}} \\ &= 0 \end{aligned}$$

Similarly, by the continuity of measure:

$$\begin{aligned} \nu(B) &= \lim_{k\to\infty} \nu(B_k) \\ &\geq \varepsilon > 0 \end{aligned}$$

This gives us a set B with $\mu(B) = 0$ but $\nu(B) \geq \varepsilon > 0$, which contradicts our assumption that $\nu \ll \mu$. Therefore, our original claim must be true.

(⇐) Sufficiency: Suppose that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have $\nu(A) < \varepsilon$.

We need to show that $\nu \ll \mu$, that is, for any $A \in \mathcal{F}$ with $\mu(A) = 0$, we have $\nu(A) = 0$.

Let $A \in \mathcal{F}$ with $\mu(A) = 0$. For any $\varepsilon > 0$, by our assumption, there exists a $\delta > 0$ such that for all $E \in \mathcal{F}$ with $\mu(E) < \delta$, we have $\nu(E) < \varepsilon$.

Since $\mu(A) = 0 < \delta$, it follows that $\nu(A) < \varepsilon$. But this is true for any $\varepsilon > 0$, no matter how small. Therefore, $\nu(A) = 0$, which proves that $\nu \ll \mu$. ■

§2.17 Lecture 13 (03-12)

Definition 2.17.1: Radon-Nikodym Derivative

If $\nu(A) = \int_A f \, d\mu$ for some nonnegative Borel measurable function f, then f is the Radon-Nikodym derivative of ν with respect to μ.

Definition 2.17.2: Absolute Continuity

$\nu \ll \mu$ if and only if $\forall A \in \mathcal{F}$ such that $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Theorem 2.17.1: Radon-Nikodym Theorem

If μ, ν are σ-finite measures, then $\nu \ll \mu \Leftrightarrow \nu(A) = \int_A f \, d\mu, \forall A \in \mathcal{F}$ for some nonnegative measurable function f.

Definition 2.17.3: Equivalent Measures

$\mu \sim \nu$ if and only if $\mu \ll \nu$ and $\nu \ll \mu$.

Proposition 2.17.0: Continuous Random Variables

A continuous random variable X satisfies $\mathbb{P}_X(A) = \mathbb{P}(X \in A), \forall A \in \mathcal{B}(\mathbb{R})$.

$\mathbb{P}_X \ll m \Leftrightarrow \exists f$ such that $\mathbb{P}_X(A) = \int_A f(x) \, dm$, where f is the probability density function.

Remark 1

We often write $\int_A f \, dm = \int_A f(x) \, dx$ when m is the Lebesgue measure.

Example 1: Exponential Distribution

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where m is the Lebesgue measure. Define the function:

$$g(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0,+\infty)}(x), \quad \lambda > 0$$

Then for every $B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}_X(B) = \int_B g(x)\, dx$$

defines a measure.

To verify this is a probability measure, we check:

$$\begin{aligned} \mathbb{P}_X(\mathbb{R}) &= \int_{-\infty}^{+\infty} \lambda e^{-\lambda x} \mathbf{1}_{[0,+\infty)}(x)\, dx \\ &= \int_0^{+\infty} \lambda e^{-\lambda x}\, dx \\ &= \left[-e^{-\lambda x}\right]_0^{+\infty} \\ &= 0 - (-1) = 1 \end{aligned}$$

Therefore, $\mathbb{P}_X$ defines a probability measure.

This is the **exponential distribution** with parameter λ, denoted as $X \sim \text{Exp}(\lambda)$.

Fact: If $X \sim \text{Exp}(\lambda)$, then:

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{+\infty} x\lambda e^{-\lambda x}\, dx \\ &= -\int_0^{+\infty} x\, d(e^{-\lambda x}) \\ &= -\left[xe^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\, dx \\ &= 0 - 0 + \frac{1}{\lambda}\left[-e^{-\lambda x}\right]_0^{+\infty} \\ &= \frac{1}{\lambda} \end{aligned}$$

Also:

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\, dx \\ &= \dots \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Lemma 2.17.1: Standardization of Normal Distribution

If $X \sim N(\mu, \sigma^2)$, then $Y = \frac{X-\mu}{\sigma}$ is $N(0,1)$ (standard normal).

Proof.

$$\mathbb{P}(Y \leq a) = \mathbb{P}(X \leq \mu + a\sigma) = \int_{-\infty}^{\mu+a\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

■

Making the substitution $y = \frac{x-\mu}{\sigma}$, we get:

$$\mathbb{P}(Y \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

which is the CDF of a standard normal distribution.

Lemma 2.17.2: Moment Generating Function of Standard Normal

If $X \sim N(0,1)$, then $\mathbb{E}[e^{tX}] = e^{\frac{1}{2}t^2}$.

Proof.

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2-2tx)}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}} e^{\frac{t^2}{2}}\, dx \\
&= e^{\frac{t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}}\, dx \\
&= e^{\frac{t^2}{2}} \cdot 1 \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

where in the fourth line we used the substitution $y = x - t$. ■

Remark 2: Applications of MGF

(1) If $\mathbb{E}[e^{tX}] < +\infty$, then:

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k\right] \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}[X^k]
\end{aligned}$$

Therefore, the moment generating function determines all moments of the random variable.

(2) Let X be a continuous random variable with density function f. Then:

$$\mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx = \mathcal{L}[f]$$

where $\mathcal{L}[f]$ denotes the Laplace transform of f.

(3) Fact: If $M_X(t) = \mathbb{E}[e^{tX}]$ converges absolutely in a neighborhood of 0, then it uniquely determines the probability density function f.

(4) If $X \sim N(\mu, \sigma^2)$, compute $\mathbb{E}[e^{tX}]$.

Proposition 2.17.0: Lack of memory

If $X \sim Exp(\lambda)$, Then

$$P(X > s + t | X > s) = P(X > t)$$

Proof.

$$\begin{aligned}
\mathbb{P}[X > t + s | X > s] &= \frac{\mathbb{P}[X > t + s, X > s]}{\mathbb{P}[X > s]} \\
&= \frac{\mathbb{P}[X > t + s]}{\mathbb{P}[X > s]} \\
&= \frac{\int_{t+s}^{+\infty} \lambda e^{-\lambda x}\, dx}{\int_{s}^{+\infty} \lambda e^{-\lambda x}\, dx} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda t} \\
&= \mathbb{P}[X > t]
\end{aligned}$$

■

Example 2: Probability Density Function of Normal Distribution

The probability density function of a normal distribution is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \mu \in \mathbb{R}, \sigma > 0$$

This is also known as the Gaussian distribution, denoted as $N(\mu, \sigma^2)$.
Special case: When $\mu = 0$ and $\sigma = 1$, we have:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

which is called the standard normal distribution, denoted as $N(0, 1)$.

Proposition 2.17.0: Properties of Normal Distribution

1. **Normalization:** Verify that $\int_{\mathbb{R}} f(x)\, dx = 1$, since

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

2. **Mean and Variance:** If $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

3. **Special Case - Standard Normal Distribution:** If $X \sim N(0,1)$, then:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 0$$

$$\begin{aligned}
\mathrm{Var}(X) = \mathbb{E}[X^2] &= \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{d\lambda} \left[\int_{-\infty}^{+\infty} e^{-\lambda x^2}\, dx \right]_{\lambda=\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{d\lambda} \left[\sqrt{\frac{\pi}{\lambda}} \right]_{\lambda=\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \left(-\frac{1}{2}\right) \cdot \frac{d}{d\lambda} \left[\lambda^{-\frac{1}{2}}\right]_{\lambda=\frac{1}{2}} \cdot \sqrt{\pi} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \cdot \lambda^{-\frac{3}{2}} \cdot \sqrt{\pi}\Big|_{\lambda=\frac{1}{2}} \\
&= \frac{1}{4} \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \cdot 2\sqrt{2} \\
&= 1
\end{aligned}$$

Example 3: Cauchy Distribution

Let $f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)\, dx &= \int_{-\infty}^{+\infty} \frac{1}{\pi(1+x^2)}\, dx \\
&= \frac{1}{\pi} [\arctan(x)]_{-\infty}^{+\infty} \\
&= \frac{1}{\pi} \cdot \pi = 1
\end{aligned}$$

This defines a random variable $Y \sim \mathrm{Cauchy}(0,1)$.
Fact: $\mathbb{E}[e^{tX}] = \int \frac{e^{tx}}{\pi(1+x^2)}\, dx$ converges only for $t = 0$.
The expected value is also divergent:

$$\mathbb{E}[X] = \int \frac{x}{\pi(1+x^2)}\, dx \quad \text{diverges}$$

Characteristic Function: While the moment generating function doesn't exist for the Cauchy distribution, the characteristic function is well-defined:

$$\phi(t) = \mathbb{E}[e^{itX}] = \int e^{itx} \frac{1}{\pi(1+x^2)}\, dx$$

§2.18 Recitation (03-14)

Problem 1

We know that $X \sim Exp(\lambda)$,Then $P(X > t + s|X > s) = P(X > t)$
Identify all distribution that satisfy the lack of memory property.

Proof. We proceed in several steps to identify all distributions that satisfy the memoryless property.

Step 1: First, we recall the definition of the memoryless property. A random variable X has the memoryless property if

$$\mathbb{P}(X > t + s|X > s) = \mathbb{P}(X > t) \quad \forall s, t \geq 0 \tag{1}$$

We know that if $X \sim \text{Exp}(\lambda)$ (i.e., X follows an exponential distribution with parameter λ), then X satisfies property (1). We want to prove that exponential distributions are the only ones with this property.

Step 2: Let $G(t) = \mathbb{P}(X > t)$ be the survival function of X. Using property (1), we have:

$$\begin{aligned}
\mathbb{P}(X > t + s|X > s) &= \mathbb{P}(X > t) \\
\frac{\mathbb{P}(X > t + s, X > s)}{\mathbb{P}(X > s)} &= \mathbb{P}(X > t) \\
\frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > s)} &= \mathbb{P}(X > t) \\
\frac{G(t + s)}{G(s)} &= G(t)
\end{aligned}$$

Rearranging, we get the functional equation:

$$G(t + s) = G(t) \cdot G(s) \quad \forall s, t \geq 0 \tag{2}$$

Step 3: Define $H(t) = -\log G(t)$. Taking the logarithm of both sides of equation (2):

$$\begin{aligned}
\log G(t + s) &= \log G(t) + \log G(s) && (2.4) \\
-H(t + s) &= -H(t) - H(s) && (2.5) \\
H(t + s) &= H(t) + H(s) \quad \forall s, t \geq 0 && (3)
\end{aligned}$$

This is the Cauchy functional equation. We will now prove that any continuous solution to this equation must have the form $H(t) = \lambda t$ for some constant $\lambda > 0$.

Step 4: We claim that there exists λ such that $H(t) = \lambda t$ for all $t \geq 0$. We prove this in several sub-steps:

(a) For integer values: Let $\lambda = H(1)$. For any integer $k \in \mathbb{Z}^+$, we have:

$$\begin{aligned}
H(k) &= H(1 + 1 + \ldots + 1) \quad (k \text{ times}) \\
&= H(1) + H(1) + \ldots + H(1) \quad (\text{using equation (3) repeatedly}) \\
&= k \cdot H(1) = \lambda k
\end{aligned}$$

(b) For rational values: For any rational number $\frac{p}{q}$ where $p, q \in \mathbb{Z}^+$, using equation

(3), we have:

$$H\left(\frac{p}{q}\cdot q\right) = q\cdot H\left(\frac{p}{q}\right)$$
$$H(p) = q\cdot H\left(\frac{p}{q}\right)$$
$$\lambda p = q\cdot H\left(\frac{p}{q}\right)$$
$$H\left(\frac{p}{q}\right) = \lambda\cdot\frac{p}{q}$$

(c) For all real values: By the continuity of H and the density of rational numbers in $\mathbb{R}$, for any real number $t \geq 0$, there exists a sequence of rational numbers $\{t_n\}$ such that $t_n \to t$. By the continuity of H, we have:

$$H(t) = \lim_{n\to\infty} H(t_n) = \lim_{n\to\infty} \lambda t_n = \lambda t$$

Therefore, $H(t) = \lambda t$ for all $t \geq 0$.
Step 5: Since $H(t) = -\log G(t) = \lambda t$, we have:

$$G(t) = e^{-\lambda t}, \quad \lambda > 0$$

This is exactly the survival function of an exponential distribution with parameter λ. Note that λ must be positive since $G(t)$ is a decreasing function of t (as t increases, the probability of surviving beyond t decreases).
Step 6: Finally, the probability density function of X is the negative derivative of the survival function:

$$f(t) = -\frac{d}{dt}G(t) = -\frac{d}{dt}e^{-\lambda t} = \lambda e^{-\lambda t}, \quad t \geq 0$$

This is the probability density function of an exponential distribution with parameter λ.

In conclusion, the only continuous probability distributions that satisfy the memoryless property are exponential distributions. ■

Problem 2

(Ω, F, μ) measure space, X is a integrable borel function, Y is an simple function
Show that (1)

$$\exists A \in F, \mu(A) < +\infty, s.t. \int_\Omega Y d\mu = \int_A Y d\mu$$

(2)

$$\exists \epsilon > 0, \exists A_\epsilon \in F, \mu(A_\epsilon) < +\infty, s.t. |\int_{A_\epsilon} X d\mu - \int_\Omega X d\mu| < \epsilon$$

Proof. **Part 1:** Let Y be a simple function, which can be written as

$$Y = \sum_{i=1}^{n} a_i 1_{A_i}$$

where $a_i \in \mathbb{R}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and $a_i \neq 0$. Let $A = \bigcup_{i=1}^{n} A_i$.

Since Y is zero outside A, we have

$$\int_\Omega Y\,d\mu = \int_A Y\,d\mu$$

Also, $\mu(A) = \sum_{i=1}^n \mu(A_i) < +\infty$ since each A_i has finite measure.

Part 2: Suppose that $X \geq 0$ is a non-negative Borel function. Since X is integrable, there exists a sequence of simple functions $\{Y_n\}$ such that $Y_n \uparrow X$ pointwise and $\lim_{n\to\infty} \int_\Omega Y_n\,d\mu = \int_\Omega X\,d\mu$.

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \int_\Omega Y_N\,d\mu - \int_\Omega X\,d\mu \right| < \epsilon$$

From Part 1, there exists $A_N \in \mathcal{F}$ with $\mu(A_N) < +\infty$ such that

$$\int_\Omega Y_N\,d\mu = \int_{A_N} Y_N\,d\mu$$

This implies

$$\left| \int_{A_N} Y_N\,d\mu - \int_\Omega X\,d\mu \right| < \epsilon$$

Since $0 \leq Y_N \leq X$ and Y_N approximates X on A_N, we have

$$\int_{A_N} X\,d\mu - \epsilon \leq \int_{A_N} Y_N\,d\mu \leq \int_{A_N} X\,d\mu$$

Which leads to

$$\left| \int_{A_N} X\,d\mu - \int_\Omega X\,d\mu \right| < \epsilon$$

Now, suppose X is an arbitrary integrable Borel function. Let $X = X^+ - X^-$, where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$ are the positive and negative parts of X, respectively.

By applying the above result to X^+ and X^-, we can find $A^+, A^- \in \mathcal{F}$ with $\mu(A^+), \mu(A^-) < +\infty$ such that

$$\left| \int_{A^+} X^+\,d\mu - \int_\Omega X^+\,d\mu \right| < \frac{\epsilon}{2}$$
$$\left| \int_{A^-} X^-\,d\mu - \int_\Omega X^-\,d\mu \right| < \frac{\epsilon}{2}$$

Let $A = A^+ \cup A^-$. Then $\mu(A) < +\infty$ and

$$\begin{aligned}
\left| \int_A X\,d\mu - \int_\Omega X\,d\mu \right| &= \left| \int_A (X^+ - X^-)\,d\mu - \int_\Omega (X^+ - X^-)\,d\mu \right| \\
&\leq \left| \int_A X^+\,d\mu - \int_\Omega X^+\,d\mu \right| + \left| \int_A X^-\,d\mu - \int_\Omega X^-\,d\mu \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, for any integrable Borel function X and any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{F}$ with $\mu(A_\epsilon) < +\infty$ such that

$$\left| \int_{A_\epsilon} X\,d\mu - \int_\Omega X\,d\mu \right| < \epsilon$$

■

Problem 3

Let X be a continuous non-negative random variable with $\mathbb{E}[X] < \infty$. Then:

1. $\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}[X > y]\, dy - \int_0^{+\infty} \mathbb{P}[X < -y]\, dy$
2. $\lim_{y \to +\infty} y \cdot \mathbb{P}[X > y] = 0$

Proof. Let's prove each part separately.

Part (1): We begin with the definition of expectation. For a continuous random variable X with probability density function $f_X(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$$

We can decompose X into its positive and negative parts:

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.

In class, we showed that for a non-negative random variable Z:

$$\mathbb{E}[Z] = \int_0^{+\infty} \mathbb{P}[Z > y]\, dy$$

Applying this to X^+ and X^-:

$$\begin{aligned}
\mathbb{E}[X^+] &= \int_0^{+\infty} \mathbb{P}[X^+ > y]\, dy = \int_0^{+\infty} \mathbb{P}[X > y]\, dy \\
\mathbb{E}[X^-] &= \int_0^{+\infty} \mathbb{P}[X^- > y]\, dy = \int_0^{+\infty} \mathbb{P}[-X > y]\, dy = \int_0^{+\infty} \mathbb{P}[X < -y]\, dy
\end{aligned}$$

Therefore:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X^+] - \mathbb{E}[X^-] \\
&= \int_0^{+\infty} \mathbb{P}[X > y]\, dy - \int_0^{+\infty} \mathbb{P}[X < -y]\, dy
\end{aligned}$$

Part (2): We need to prove that $\lim_{y \to +\infty} y \cdot \mathbb{P}[X > y] = 0$.

For any $y > 0$, we have:

$$\begin{aligned}
y \cdot \mathbb{P}[X > y] &= y \cdot \int_y^{+\infty} f_X(x)\, dx \\
&\le \int_y^{+\infty} x \cdot f_X(x)\, dx \\
&= \mathbb{E}[X \cdot \mathbf{1}_{\{X > y\}}]
\end{aligned}$$

We claim that $\mathbb{E}[X \cdot \mathbf{1}_{\{X > y\}}] \to 0$ as $y \to +\infty$.

This follows because X is integrable ($\mathbb{E}[X] < \infty$), and as y increases, the set $\{X > y\}$ becomes smaller. By the dominated convergence theorem:

$$\lim_{y \to +\infty} \mathbb{E}[X \cdot \mathbf{1}_{\{X > y\}}] = 0$$

Therefore:

$$\lim_{y \to +\infty} y \cdot \mathbb{P}[X > y] = 0$$

This result can also be approached using approximation by simple functions. Additionally, we can leverage the following fact from measure theory: In a measure space $(\Omega, \mathcal{F}, \mu)$, if f is integrable, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have $\left|\int_A f \, d\mu\right| < \varepsilon$. ■

§2.19 Lecture 14 (03-17)

Theorem 2.19.1: Chain Rule for Radon-Nikodym Derivatives

Suppose $\nu \ll \mu$ (i.e., ν is absolutely continuous with respect to μ). Then a function f is integrable with respect to ν if and only if $f \cdot \frac{d\nu}{d\mu}$ is integrable with respect to μ. Furthermore, in this case:

$$\int f \, d\nu = \int f \cdot \frac{d\nu}{d\mu} \, d\mu$$

where $\frac{d\nu}{d\mu}$ denotes the Radon-Nikodym derivative of ν with respect to μ.

Remark 1: Probabilistic Interpretation

If $Q \ll P$ are probability measures, and X is a Q-integrable random variable, then:

$$\mathbb{E}^Q[X] = \mathbb{E}^P\left[X \cdot \frac{dQ}{dP}\right]$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative (likelihood ratio) of Q with respect to P.

Proof. We proceed in several steps:

Step 1: First, we verify the theorem for indicator functions. For any measurable set A:

$$\begin{aligned}
\int 1_A \, d\nu &= \nu(A) \\
&= \int_A \frac{d\nu}{d\mu} \, d\mu \quad \text{(by definition of Radon-Nikodym derivative)} \\
&= \int 1_A \cdot \frac{d\nu}{d\mu} \, d\mu
\end{aligned}$$

Step 2: By linearity, we extend the result to simple functions. Let $\varphi = \sum_{i=1}^n a_i 1_{A_i}$, where $A_i \cap A_j = \emptyset$ for $i \neq j$. Then:

$$\begin{aligned}
\int \varphi \, d\nu &= \sum_{i=1}^{n} a_i \int 1_{A_i} \, d\nu \\
&= \sum_{i=1}^{n} a_i \int 1_{A_i} \cdot \frac{d\nu}{d\mu} \, d\mu \quad \text{(by Step 1)} \\
&= \int \sum_{i=1}^{n} a_i 1_{A_i} \cdot \frac{d\nu}{d\mu} \, d\mu \\
&= \int \varphi \cdot \frac{d\nu}{d\mu} \, d\mu
\end{aligned}$$

Step 3: For a non-negative Borel function $g \geq 0$, there exists a sequence $\{\varphi_n\}_{n\geq 1}$ of simple functions such that $\varphi_n \uparrow g$ pointwise. Applying the Monotone Convergence Theorem (MCT):

$$\begin{aligned}
\int g \, d\nu &= \lim_{n\to\infty} \int \varphi_n \, d\nu \quad \text{(by MCT)} \\
&= \lim_{n\to\infty} \int \varphi_n \cdot \frac{d\nu}{d\mu} \, d\mu \quad \text{(by Step 2)} \\
&= \int \lim_{n\to\infty} \left(\varphi_n \cdot \frac{d\nu}{d\mu} \right) d\mu \quad \text{(by MCT)} \\
&= \int g \cdot \frac{d\nu}{d\mu} \, d\mu
\end{aligned}$$

Step 4: For a general Borel function g, we decompose it as $g = g^+ - g^-$, where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$. Applying the result from Step 3 to both g^+ and g^-, we obtain:

$$\begin{aligned}
\int g \, d\nu &= \int g^+ \, d\nu - \int g^- \, d\nu \\
&= \int g^+ \cdot \frac{d\nu}{d\mu} \, d\mu - \int g^- \cdot \frac{d\nu}{d\mu} \, d\mu \\
&= \int (g^+ - g^-) \cdot \frac{d\nu}{d\mu} \, d\mu \\
&= \int g \cdot \frac{d\nu}{d\mu} \, d\mu
\end{aligned}$$

This completes the proof of the chain rule. ■

Example 1

eg : $(\Omega, \mathcal{F}, \mathbb{P})$, $\quad X \sim N(0,1)$. Let $\theta > 0$. $\quad X + \theta \sim N(\theta, 1)$
We can define a new prob. measure $\mathbb{Q}$, s.t. under $\mathbb{Q}$, $X + \theta \sim N(0,1)$.
Soln : Let $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\theta X - \frac{1}{2}\theta^2}$

$\mathbb{Q}$ is a prob. measure: $\mathbb{Q}(\Omega) = \int_\Omega d\mathbb{Q} = \int_\Omega \dfrac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} = \int_\Omega e^{-\theta X - \frac{1}{2}\theta^2} d\mathbb{P} = e^{-\frac{1}{2}\theta^2} \mathbb{E}^{\mathbb{P}}[e^{-\theta X}]$

$$= e^{-\frac{1}{2}\theta^2} \cdot e^{\frac{1}{2}\theta^2} = 1 \quad \text{MGF for } N(0,1)$$

Compute $\mathbb{E}^{\mathbb{Q}}[e^{t(X+\theta)}] = \mathbb{E}^{\mathbb{P}}\left[e^{t(X+\theta)} \dfrac{d\mathbb{Q}}{d\mathbb{P}}\right]$

$\forall t \in \mathbb{R}$ $\qquad = \mathbb{E}^{\mathbb{P}}\left[e^{t(X+\theta)} e^{-\theta X - \frac{1}{2}\theta^2}\right] = e^{t\theta - \frac{1}{2}\theta^2} \mathbb{E}^{\mathbb{P}}[e^{(t-\theta)X}]$

$$= e^{t\theta - \frac{1}{2}\theta^2} e^{\frac{1}{2}(t-\theta)^2} = e^{\frac{1}{2}t^2}$$

$\Rightarrow X + \theta \sim^{\mathbb{Q}} N(0,1)$

Fact:

if $M(t) := \mathbb{E}[e^{tX}]$ converges in $t \in (-\delta, \delta)$ for some $\delta > 0$.
Then $\{M(t), t \in (-\delta, \delta)\}$ determines the distribution of X.

§2.19.1 Joint distribution

Given probability space $(\Omega_1, F_1, P_1), (\Omega_2, F_2, P_2)$ can define the product space $(\Omega_1 \times \Omega_2, F_1 \otimes F_2, P_1 \otimes P_2)$ where

$$F_1 \otimes F_2 = \sigma(\{A_1 \times A_2 : A_1 \in F_1, A_2 \in F_2\})$$

define

$$P_1 \otimes P_2(A_1 \times A_2) = P_1(A_1)P_2(A_2)$$

and extend to $F_1 \otimes F_2$ by Caratheodory extension theorem

Definition 2.19.1: Joint distribution

Joint distribution of X, Y is $F : \mathbb{R}^2 \to [0,1]$ s.t.

$$F(x,y) = P(X \le x, Y \le y)$$

Definition 2.19.2

If X,Y are continuous random variables, then the joint density function $f : \mathbb{R}^2 \to [0, +\infty)$ is given by

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v) du dv$$
$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

Remark 2

(1)

$$P[a < X \leq b, c < Y \leq d] = \int_a^b \int_c^d f(u,v)dudv$$

(2):by uniqueness of extension,

$$B(R^2) = \sigma\{\text{left open right closed cubes in } R^2\}$$
$$P((X,Y) \in A) = \int_A f(u,v)dudv, \text{ for every } A \in B(R^2)$$

May cover individual distribution from the joint distribution

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \in R) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(u,v)dudv$$
$$f_X(x) = \int_{-\infty}^{+\infty} f(x,v)dv, f_Y(y) = \int_{-\infty}^{+\infty} f(u,y)du$$

Remark 3

if X,Y are independent$\Rightarrow$ $F(x,y) = F_X(x)F_Y(y) \Leftrightarrow (cont.) f(x,y) = f_X(x)f_Y(y)$

Example 2

①

If X, Y have joint density function

$f(x,y) = \dfrac{\alpha^x}{x!}\dfrac{\beta^y}{y!}e^{-\alpha-\beta} \quad \forall x, y \in \mathbb{N}.$

Soln: • X, Y are indep, because $f(x,y) = f_1(x)f_2(y)$.

• $f_X(x) = \sum_{y \in \mathbb{N}} \dfrac{\alpha^x}{x!}\dfrac{\beta^y}{y!}e^{-\alpha-\beta} = \dfrac{\alpha^x}{x!}e^{-\alpha-\beta}\sum_{y \in \mathbb{N}}\dfrac{\beta^y}{y!} = \dfrac{\alpha^x}{x!}e^{-\alpha}$

$\Rightarrow X \sim \text{Poisson}(\alpha)$

Similarly, $f_Y(y) = \sum_{x \in \mathbb{N}} \dfrac{\alpha^x}{x!}\dfrac{\beta^y}{y!}e^{-\alpha-\beta} = \ldots \quad Y \sim \text{Poisson}(\beta)$.

②

The joint density function of X, Y is $f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0 \le x, y < \infty \\ 0 & \text{else} \end{cases}$

Find the density funct. of $\dfrac{X}{Y}$.

$$\underline{\text{Soln}} : F_{\frac{X}{Y}}(a) = P\left(\frac{X}{Y} \le a\right) = \iint_{[x \le ay]} f(x,y)\,dx\,dy$$

$$= \int_0^{+\infty} \int_0^{ay} e^{-(x+y)}\,dx\,dy = \int_0^{+\infty} [1 - e^{-ay}]e^{-y}\,dy$$

$$= \int_0^{+\infty} (1 - e^{-ay})e^{-y}\,dy = 1 - \frac{1}{a+1}.$$

$$f_{\frac{X}{Y}}(a) = \frac{1}{(a+1)^2}, \quad a \ge 0$$

§2.20 Lecture 15 (03-19)

§2.20.1 Joint Distribution Function and Density Function

Definition 2.20.1: Joint Distribution Function

For random variables X and Y, the joint distribution function is defined as:

$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$

Definition 2.20.2: Joint Density Function

For continuous random variables X and Y, if the second-order partial derivative of their joint distribution function $F(x,y)$ exists, the joint density function is defined as:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

For any measurable set $A \subset \mathbb{R}^2$, we have:

$$\mathbb{P}((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

§2.20.2 Distribution of Sum of Random Variables

Proposition 2.20.0: Density Function of Sum

Let X and Y be continuous random variables with joint density function $f(x,y)$.

The density function of $X + Y$ is:

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f(x, z - x)\, dx$$

. First, the distribution function of $X + Y$ can be represented as:

$$\begin{aligned} F_{X+Y}(z) &= \mathbb{P}(X + Y \le z) \\ &= \iint_{x+y\le z} f(x, y)\, dx\, dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f(x, y)\, dy\, dx \end{aligned}$$

Differentiating with respect to z, we obtain the density function:

$$\begin{aligned} f_{X+Y}(z) &= \frac{d}{dz} F_{X+Y}(z) \\ &= \frac{d}{dz} \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f(x, y)\, dy\, dx \\ &= \int_{-\infty}^{+\infty} f(x, z - x)\, dx \end{aligned}$$

where we applied Leibniz's integral rule to exchange the order of integration and differentiation. ■

Example 1: Sum of Independent Standard Normal Random Variables

Let $X, Y \sim N(0, 1)$ be independent. Prove that $X + Y \sim N(0, 2)$.

. Since X and Y are independent, their joint density function is the product of their individual density functions:

$$f(x, y) = f_X(x) \cdot f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Computing the density function of $X + Y$:

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{+\infty} f(x, z - x)\, dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}}\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+(z-x)^2)}\, dx \end{aligned}$$

Computing the exponent term:

$$\begin{aligned} x^2 + (z - x)^2 &= x^2 + z^2 - 2zx + x^2 \\ &= 2x^2 - 2zx + z^2 \\ &= 2(x^2 - zx + \frac{z^2}{2}) \\ &= 2(x - \frac{z}{2})^2 + \frac{z^2}{2} - \frac{z^2}{2} \\ &= 2(x - \frac{z}{2})^2 + \frac{z^2}{2} \end{aligned}$$

Substituting back into the original integral:

$$\begin{aligned} f_{X+Y}(z) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[2(x-\frac{z}{2})^2+\frac{z^2}{2}]}\,dx \\ &= \frac{1}{2\pi}e^{-\frac{z^2}{4}}\int_{-\infty}^{+\infty} e^{-(x-\frac{z}{2})^2}\,dx \end{aligned}$$

Letting $u = x - \frac{z}{2}$, we have $dx = du$, and the integration limits transform:

$$\begin{aligned} f_{X+Y}(z) &= \frac{1}{2\pi}e^{-\frac{z^2}{4}}\int_{-\infty}^{+\infty} e^{-u^2}\,du \\ &= \frac{1}{2\pi}e^{-\frac{z^2}{4}}\cdot\sqrt{\pi} \\ &= \frac{1}{\sqrt{2\pi\cdot 2}}e^{-\frac{z^2}{4}} \end{aligned}$$

This is precisely the density function of $N(0,2)$, therefore $X+Y \sim N(0,2)$. ■

Fact:Sum of Independent Normal Random Variables
If $X \sim N(\mu_1,\sigma_1^2)$, $Y \sim N(\mu_2,\sigma_2^2)$ and X and Y are independent, then $X+Y \sim N(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$.

Soln2: Use that moment generating funct. $M(t) = \mathbb{E}[e^{tX}]$ determines the distr.

$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] \overset{\text{ind.}}{=} \mathbb{E}[e^{tX}\cdot\mathbb{E}[e^{tY}]] = e^{\frac{1}{2}t^2}\cdot e^{\frac{1}{2}t^2} = e^{t^2}, \quad X+Y \sim N(0,2)$

Here we used if $X \sim N(\mu,\sigma^2), \mathbb{E}[e^{tX}] = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$.

Example 2: Problems with Uniform Distributions

Let X, Y be independent random variables, each uniformly distributed on $[0,1]$ (i.e., $X, Y \sim \text{Uniform}[0,1]$, with $f_X(x) = \mathbf{1}_{[0,1]}(x)$).

Problem 1. *Compute the joint density function of $X+Y$.*

Solution. For two independent random variables X and Y, the density function of their sum $Z = X+Y$ can be calculated using the convolution formula:

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)\,dx \\ &= \int_0^1 \mathbf{1}_{[0,1]}(x)\mathbf{1}_{[0,1]}(z-x)\,dx \end{aligned}$$

Since $f_X(x) = \mathbf{1}_{[0,1]}(x)$ and $f_Y(y) = \mathbf{1}_{[0,1]}(y)$, the integration region must satisfy both $0 \le x \le 1$ and $0 \le z-x \le 1$, which means $0 \le x \le 1$ and $z-1 \le x \le z$.

This can be divided into three cases:

$$f_{X+Y}(z) = \int_0^1 \mathbf{1}_{0\le z-x\le 1}\,dx = \begin{cases} \int_0^z 1\,dx = z, & \text{if } z \in [0,1] \\ \int_{z-1}^1 1\,dx = 2-z, & \text{if } z \in (1,2] \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the density function of $X+Y$ is:

$$f_{X+Y}(z) = \begin{cases} z, & \text{if } z \in [0,1] \\ 2-z, & \text{if } z \in (1,2] \\ 0, & \text{otherwise} \end{cases}$$

This is a triangular distribution, reaching its maximum value of 1 at $z=1$.

Problem 2. *Let $X_1, \ldots, X_n$ be independent random variables, each uniformly distributed on $[0,1]$. Compute $F_{X_1+\ldots+X_n}(z)$ for $z \in [0,1]$.*

Solution. We will use induction to prove that $F_{X_1+\ldots+X_n}(z) = \frac{z^n}{n!}$ for $z \in [0,1]$.

Base case: When $n=1$, $F_{X_1}(z) = z$ for $z \in [0,1]$, which is the distribution function of a uniform distribution on $[0,1]$.

Induction hypothesis: Assume that for $n-1$, we have $F_{X_1+\ldots+X_{n-1}}(z) = \frac{z^{n-1}}{(n-1)!}$ for $z \in [0,1]$.

Induction step: We need to prove that $F_{X_1+\ldots+X_n}(z) = \frac{z^n}{n!}$ for $z \in [0,1]$.

Using the convolution formula and the induction hypothesis, we can calculate:

$$\begin{aligned} F_n(z) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f(x) f_{n-1}(y)\, dy\, dx \\ &= \int_{-\infty}^{+\infty} f(x) F_{n-1}(z-x)\, dx \end{aligned}$$

For $z \in [0,1]$, substituting the known conditions:

$$\begin{aligned} F_n(z) &= \int_0^z \frac{(z-x)^{n-1}}{(n-1)!}\, dx \\ &= \frac{1}{(n-1)!} \int_0^z (z-x)^{n-1}\, dx \end{aligned}$$

Using the substitution $u = z-x$, $dx = -du$, when $x=0$ we have $u=z$, and when $x=z$ we have $u=0$:

$$\begin{aligned} F_n(z) &= \frac{1}{(n-1)!} \int_z^0 u^{n-1}(-du) \\ &= \frac{1}{(n-1)!} \int_0^z u^{n-1}\, du \\ &= \frac{1}{(n-1)!} \cdot \frac{z^n}{n} \\ &= \frac{z^n}{n!} \end{aligned}$$

Therefore, we have proven that for $z \in [0,1]$, $F_{X_1+\ldots+X_n}(z) = \frac{z^n}{n!}$.

Note: This result is only valid for $z \in [0,1]$. For $z > 1$, the distribution function expression becomes more complex.

Problem 3. *Let $X_1, X_2, \ldots$ be independent random variables, each uniformly distributed on $[0,1]$. Define $N = \min\{n \in \mathbb{N} : X_1 + X_2 + \ldots + X_n > 1\}$. Compute $\mathbb{E}[N]$.*

Solution. We define $N = \min\{n \in \mathbb{N} : X_1 + X_2 + \ldots + X_n > 1\}$.

First, observe that the event $\{N \geq n\}$ is equivalent to the event $\{X_1 + X_2 + \ldots + X_{n-1} \leq 1\}$. This is because $N \geq n$ means that the sum of the first $n-1$ random variables is not yet sufficient to exceed 1.

Therefore:

$$\begin{aligned} P(N \geq n) &= P(X_1 + X_2 + \ldots + X_{n-1} \leq 1) \\ &= F_{X_1 + \ldots + X_{n-1}}(1) \\ &= \frac{1^{n-1}}{(n-1)!} \\ &= \frac{1}{(n-1)!} \end{aligned}$$

Using the formula for the expectation of a discrete random variable $E[N] = \sum_{n=1}^{\infty} P(N \geq n)$, we have:

$$\begin{aligned} E[N] &= \sum_{n=1}^{\infty} P(N \geq n) \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= e \end{aligned}$$

The last step follows from the fact that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$, which is the series expansion of the natural constant e.

Therefore, $E[N] = e \approx 2.71828...$

§2.21 Recitation (03-21)

Problem 1

Is is possible to have 2 biased dice, such that the sum is uniformly distributed in {2,3,...12}?

Problem 2

Flip a fair coin, What is the expected time to see the 1-st occur of HHT?

Problem 3

Let X be a non-negative r.v. show that

$$E[X^r] = \int_0^{+\infty} r x^{r-1} P(X > x) dx, r > 0$$

Problem 4

Gamma $(n, \lambda), f_{n,\lambda} = \frac{\lambda^n x^{n-1}}{P(n)} e^{-\lambda x}$,where $P(n) = (n-1)!$

Show that if $X_1, X_2, ..., X_n$ is independent $Exp(\lambda)$ then $X_1 + X_2 + \cdots + X_n \sim Gamma(n, \lambda)$

§2.22 Lecture 16 (03-24)

+5 Questions; Grade the best 4.
Bookwork Content ($\star$ - proofs are examinable)

1. Def. of algebra: algebra generated by class of subsets.
   Important examples from real line & Discrete sets. (identify the algebra/σ-algebra generated by given class of sets)*
2. $\cdots$ σ-algebra, σ-algebra $-----$
   $----------$
3. Def. of Content. Subadditivity prop. Important examples from $\mathbb{R}$.
4. $\cdots$ Measure. Subadditivity*. Continuity from above/below.*
5. Lebesgue measure. Borel sets. examples of $\mathcal{B}(\mathbb{R})$. Lebesgue-Stieltjes measure.
6. Extension Thm.
7. Def. of π-system. Example of π-system that generates $\mathcal{B}(\mathbb{R})$. Uniqueness Thm. Application of Uniqueness Thm. to show the uniqueness of Lebesgue measure.*
8. Def. of mble funct./r.v., Borel funct. σ-algebra generated by r.v.
9. Equivalent cond. for a funct. being Borel.* Be able to prove certain funct. are m'ble.
   Operation of m'ble funct.*
10. Construction of Lebesgue integral (simple $\to$ non-negative Borel $\to$ general) $\{X < a, a \in \mathbb{R}\}$
11. Monotone Conv. Thm.
12. absolute Cont. and Radon-Nikodym Derivative.

Common distributions:
Bernoulli, Binomial, Poisson, Geometric, Uniform, Exponential, Gaussian

§2.23 Lecture 17 (04-07)

Random walk
Markov property
$$P[S_{n+m}|S_0, S_1, \cdots, S_m] = P[S_{n+m} = j|S_m]$$
"position after m-th step does not depend on the info before m"
Let
$$T_y^0 = 0$$
$$T_y^k = \inf\{n \geq T_y^{k-1} : S_n = y\}$$

Definition 2.23.1

y is recurrent if $P[T_y^k < \infty] = 1$ for all k
y is transient if $P[T_y^k < \infty] < 1$ for some k

Remark 1

If y is recurrent

$$P[T_y^k < \infty] = P[T_y^k < \infty | T_y^{k-1} < \infty] P[T_y^k < \infty] + P[T_y^k < \infty | T_y^{k-1} = \infty] P[T_y^k < \infty]$$

Chapter 3

Partial Differential Equations

§3.1 Lecture 1 (02-03)Initial and Boundary Conditions

§3.1.1 What are the PDEs?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u \tag{3.1}$$
$$u = u(x, y, t) \tag{3.2}$$

(3.1) is a second order PDE

§3.1.2 Classical solutions

To simplify, we consider here equations of one unknown function u of one or more variable
By a solution we mean a sufficiently smooth function u that satisfies the PDE at every point of the domain of its definition(D).
Sometimes, we will have to impose some boundary conditions on the domain D.
$C^n = \{u : D \to R :$ n-times continuous differentiable w.r.t all variables $\}$ n is the order of the equation

Example 1

- Heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$u = u(x, t)$$
$$(t, x) \in D \subset R^2$$
$$u(t, x), \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}$$
$$n = 2$$
$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} \in C^2$$
$$u \in C^2(D)$$

- Some Solution:

$$u(t, x) = t + \frac{x^2}{2}, D = R^2$$

$$u(t,x) = \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}, D = \{(t,x) : t > 0\}$$

$$u(t,x) = e^{-t+ix} = e^{-t}cos(x) + ie^{-t}sin(x), D = R^2$$

§3.1.3 Initial and Boundary Conditions

What is Initial data?
$u(0,x) = u_0(x)$

What are Boundary Conditions? rigurous definition of Boundary

- Dirichlet Boundary Condition: $u(t,x)|_{\partial D} = g(t,x)$
- Neumann Boundary Condition: When the normal derivative of u is specified $\frac{\partial u}{\partial n}$ where n is the normal to the boundary $\partial(D)$

$$\frac{\partial u}{\partial n}(t,x) = g(t,x)$$

- Mixed Boundary Condition: Combination of Dirichlet and Neumann

§3.1.4 Linear and Nonlinear waves

Stationary waves

$$\frac{\partial u}{\partial t} = 0 \tag{3.3}$$

$$u = u(t,x) \tag{3.4}$$

This is a first order linear homogeneuous PDE
Solution of the form $u = u(t) = c, \forall c \in R$
Let's integral (3.3) :

$$0 = \int_0^t \frac{\partial u}{\partial t}(s,x)ds = u(t,x) - u(0,x)$$

Thus if an initial conditional

$$u(0,x) = f(x)$$

$$u(t,x) = f(x), \forall t, x$$

Thus for any given $f \in C^1(R)$, we have a solution of (3.3) and this is a stationary wave
The domain of this solution is R if f is defined everywhere on R
Power of domain to change the solution:
Consider the ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0, D = (-\infty, 0) \cup (0, \infty)$$

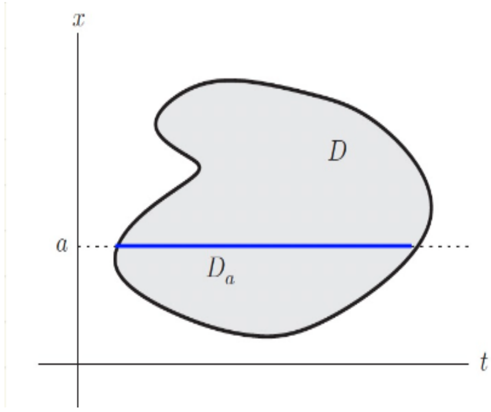
The solution is:

$$u(t) = \begin{cases} c_1, t < 0 \\ c_2, t > 0 \end{cases} \quad \forall c_1, c_2 \in R$$

similarly, for the PDE in example (3.3):

$$u(t,x) = \begin{cases} 0, x > 0 \\ x^2, x \le 0, t > 0 \\ -x^2, x \le 0, t < 0 \end{cases}$$

it is a $\mathbb{C}^1$ solution of (3.3) on $\mathbb{R}^2 \backslash \{(0,x) : x \leq 0\}$, so it is not a function of x alone Actually, it is not hard to check that if u is a clasical solution to (3.3), defined on domain $D \subset \mathbb{R}^2$ whose intersection with any horizontal line $D_a = D \cap \{(t,a) : t \in \mathbb{R}\}, \forall a \in \mathbb{R}$ is either empty or a connected interval, then $u(t,x) = f(x)$ is only a function of x.



§3.2 Lecture 1 (02-05)Transport and Traveling waves

§3.2.1 uniform transport

Let us consider the linear, homogeneuous first order PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3.5}$$

$$u = u(0,x) = f(x) \tag{3.6}$$

$$x \in \mathbb{R}, t_0 = 0, t - t_0 = t, c \in \mathbb{R}$$

use change of variable:

$$\xi = x - ct$$

ξ is called characteristic variable
let's look for a solution of (3.5) under the form

$$u(t,x) = v(t,\xi) = v(t, x - ct)$$

use chain rule:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}$$

Thus:

$$0 = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial t}$$

Conclusion:

$$\frac{\partial v}{\partial t} = 0$$

Solution is constant $v(\xi)$

$$u(t,x) = v(\xi) = v(x - ct)$$

Proposition 3.2.0

if u solves (3.5), then u has the form $u(t,x) = v(x-ct)$ where v is a function of one variable that is $\mathbb{C}^1$-smooth.
Acutually, v is satisfying

$$u(0,x) = v(x) = f(x)$$

Conclusion:

$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \\ u = u(0,x) = f(x) \end{cases}$$

The solution is

$$u(t,x) = f(x-ct)$$

$f(x-ct)$ is a traveling wave solution moving at velocity c:
if c>0, the wave moves to the right
if c<0, the wave moves to the left
if c=0, the wave is stationary

Example 1

$$\begin{cases} \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0 \\ u = u(0,x) = \frac{1}{1+x^2} \end{cases}$$

The solution is

$$u(t,x) = \frac{1}{1+(x-2t)^2}$$

The lines $x = ct + k$ are called the characteristic lines for the equation (3.5)
on the characteristic lines, the solution is always constant
on geometric interpretation, $u(t,x) = f(x-ct) = f(k)$

Consider the equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0 \tag{3.7}$$

where a and c are given numbers.
Use change of variable:

$$u(t,x) = v(t,\xi) = v(t, x-ct)$$

we will get by the same computation

$$\frac{\partial v}{\partial t} + av = 0$$

The solution will be

$$v(\xi) = e^{-at}f(\xi)$$
$$u(t,x) = v(t,\xi) = v(x-ct) = e^{-at}f(x-ct)$$

So,

$$\begin{cases} u(t,x) = e^{-at}f(x-ct) \\ u(0,x) = f(x) \end{cases}$$

§3.2.2 Nonuniform transport

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0 \tag{3.8}$$

In the uniform case(c constant), characteristic line is x=ct+k (the solution of ODE below)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c$$

In the general non-uniform case, we can still define characteristic curves

$$\frac{d}{dt}x(t) = c(x(t))$$

Let's show that the solutions of (3.8) are constants along the characteristic curves (t,x(t))
We need to check that $u(t, x(t)) = h(t)$ is constant of t
compute the derivative of $h(t)$

$$\begin{aligned}\frac{d}{dt}h(t) = \frac{d}{dt}u(t, x(t)) &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\partial u}{\partial t} + c(x(t)) \cdot \frac{\partial u}{\partial x} = 0\end{aligned}$$

h(t) and thus u(t,x(t)) is constant along the characteristic curves

Definition 3.2.1

The graph of a solution of the ODE

$$\frac{d}{dt}x(t) = c(x(t))$$

is called a characteristic curve for the transport equation (3.8) with speed c(x)

Thus at each point (t,x) the slope of the characteristic curve is c(x) (the wave speed)
Thus, we proved:

Proposition 3.2.0

The solutions to the linear transport equation (3.8) are constant along the characteristic curves

In the particular case c(x)=c, the characteristic curves are straight lines of slope c
Moreover,
(3.8) is an autonomous first order ODE which can be "solved" using the method of separation of variables
Assume that c(x)≠0

$$\begin{aligned}\frac{dx(t)}{dt} &= c(x(t)) \\ \frac{dx}{c(X)} &= dt \rightarrow \\ \beta(x) &= \int \frac{dx}{c(x)} = t + k\end{aligned}$$

Thus characteristic curves satisfies the implicit equatioin

$$\beta(x) = t + k$$

We can find x from the equality using β^{-1}(the inverse of β)

$$\begin{aligned} \beta(x) &= t + k \\ \beta^{-1}\beta(y) &= y \\ x(t) &= \beta^{-1}(t+k) \end{aligned}$$

Remark 1

How to determine where $c(x_0) = 0$?
Solve this ODE

$$\begin{cases} \frac{dx}{dt} = c(x) \\ x(t_0) = x_0 \end{cases}$$

and $x(t) = x_0$ wil be the characteristic curve going through (t_0, x_0)

From the proposition, we know that $u(t, x(t))$=constant

$$\begin{aligned} u(t, x(t)) &= v(k) \\ &= v(\beta(x(t)) - t) \end{aligned}$$

where k is the constant determining the characteristic curve (t,x(t))

$$\begin{aligned} u(t, x) &= v(\xi) \\ \text{where } \xi &= \beta(x) - t \end{aligned}$$

which we will call characteristic variable

$$u(t, x) = v(\beta(x) - t) \tag{3.9}$$

This formula way fail at points where c vanishes

$$\begin{aligned} t &= 0 \\ u(0, x) = f(x) &= v(\beta(x) - 0) \\ f(x) &= v(\beta(x)) \\ v(\xi) &= f(\beta^{-1}(\xi)) \\ &= f \cdot \beta^{-1}(\xi) \\ u(t, x) &= f \cdot \beta^{-1}(\beta(x) - t) \end{aligned}$$

This is the formula for the solution of (3.8) in the general case
which is:

$$\begin{cases} \frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0 \\ u(0, x) = f(x) \end{cases}$$

Example 2

Let us apply the method of characteristics to solve the nonuniform transport equation:

$$\frac{\partial u}{\partial t} + \frac{1}{x^2+1}\frac{\partial u}{\partial x} = 0$$

The characteristic curves are the graphs of the solutions of the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x^2+1}$$

$$\begin{aligned} \Rightarrow \beta(x) &= \int (x^2+1)dx \\ &= \frac{x^3}{3} + x \\ &= t + k \end{aligned}$$

It is not particularly helpful to find x(t) form this equation.
characteristic variable ξ is given by $\xi = \frac{x^3}{3} + x - t$ and the general solution is of the form

$$u = v(\frac{x^3}{3} + x - t)$$

where v is any $\mathbb{C}^1$ function.

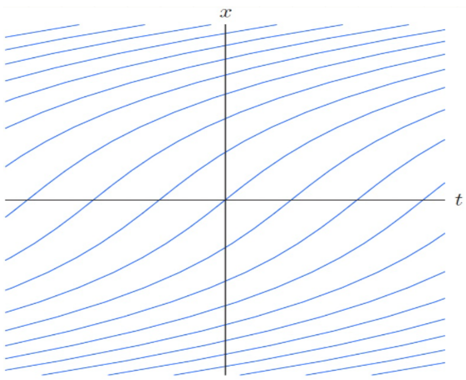
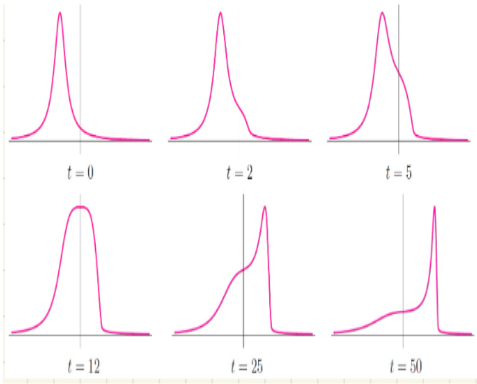


Figure 3.1: Characteristic curves for $u_t + (x^2+1)^{-1}u_x = 0$

In the case of the initial condition:

$$u(0,x) = \frac{1}{1+(x+3)^2}$$

The solution looks as follows:



§3.3 Lecture 3 (02-10)-Nonlinear transport and shocks

Nonlinear transport equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3.10}$$

characteristic equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(t, x(t)), \text{ x(t) is the characteristic curve} \tag{3.11}$$

Let's show that u the solution of (3.10) is constant along the characteristic curves x(t) we need to check that $h(t) = u(t, x(t))$ is constant of t

$$\begin{aligned}
\frac{d}{dt}h(t) &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot u(t, x(t)) \\
&= 0(\text{ by } (3.10))
\end{aligned}$$

As u is constant along the characteristic curves, it must be of the form function of characteristic variable:

$$\xi = x - ut$$

$$u(t, x) = f(x - tu(t, x)) \tag{3.12}$$

Example 1

Suppose the initial data is:

$$f(\xi) = \alpha\xi + \beta$$

Then (3.12) becomes:

$$u = \alpha(x - tu) + \beta$$

$$u(t, x) = \frac{\alpha x + \beta}{1 + \alpha t} \tag{3.13}$$

As far as $1 + dt \neq 0$, the graoh of the solution is line

If $\alpha > 0$,

$$\begin{aligned} u(t,x) &\to 0 \\ t &\to +\infty \end{aligned}$$

If $\alpha < 0$, then the solution blows up as $t \to -\frac{1}{\alpha}$ since $\dot{x} = u(t, x(t))$
As u(t,x(t))is a constant ,

$$\begin{aligned} u(t,x(t)) &= u(0,x(0)) \\ &= u(0,y) \\ &= f(y) \end{aligned}$$

Our characteristic ODE is:

$$\begin{aligned} \dot{x} &= f(x) \\ \Rightarrow x(t) &= tf(y) + y \\ \Rightarrow u(t,x(t)) &= u(t,tf(y)+y) \\ &= u(0,y) \\ &= f(y), \forall t \\ u(t,tf(y)+y) &= f(y)\forall t,y \\ u(t,x) &= f(y) \\ x &= tf(y) + y \end{aligned}$$

I find $y \in \mathbb{R}$ Suppose let $f(y) = y$

$$\begin{aligned} x = ty + y, y &= \frac{x}{t+1} \\ \Rightarrow u(t,x) = f(y) = y &= \frac{x}{t+1} \\ \Rightarrow u(t,x) &= \frac{x}{t+1} \end{aligned}$$

which agrees with (3.13) for $\alpha = 1, \beta = 0$
There is a problem with this construction that becomes clear when considering the following example:

$$u(0,x) = f(x) = \frac{1}{2}\pi - tan^{-1}(x)$$

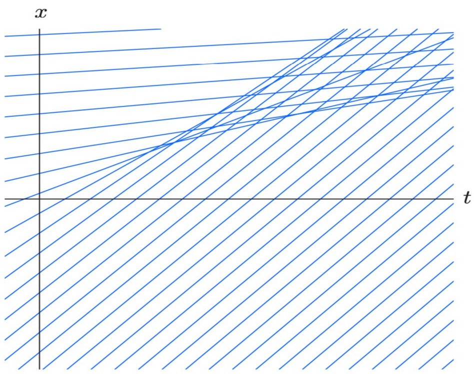


Figure 3.2: Characteristic lines for $u(0,x) = \frac{1}{2}\pi - tan^{-1}(x)$

Two characteristic lines that are not parallel must cross and the value of the solution is supposed to be equal the slope of the characteristic line passing through the point.
At a crossing point, the solution will have to be equal to two different values, one corresoponding to each characteristic line.
While mathematically, such a multiply valued solutions are possible, they are physically untenbale.
One needs to decide which (if any) of the possible values is the physically appropriate.

§3.3.1 The wave equation:d'Alembert's formula

Let us consider the 1D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.14}$$

where c>0 is a given constant
the wave speed, the initial data are of the form:

$$u(0,x) = f(x), \frac{\partial u}{\partial t} = g(x) \tag{3.15}$$

The initial value problem is to find a $\mathbb{C}^2$ function that solves (3.14) and satisfies (3.15)
In this section, we will consider only the 1D case posed on the whole line R
Let us define the wave operator:

$$\Box = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$$

Thus the wave equation becomes:

$$\Box u = 0$$

In analogy with the elementary factorization

$$t^2 - c^2x^2 = (t - cx)(t + cx)$$

We can write

$$\Box = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})$$

Now, if

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0 \tag{3.16}$$

then u is automatically a solution of the wave equation, because

$$\Box u = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0$$

Clearly, (3.16) is a transport equation with constant wave speed c, and its solutions are traveling waves with speed c:

$$u(t,x) = p(\xi) = p(x - ct)$$

where p is an arbitrary function.
If $p \in \mathbb{C}^2, u(t,x)$ is a classical solution of the wave equation (3.14).
The factorization of $\Box$ can be written in the reverse order:

$$\Box = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})$$

Solving the transport equation:

$$(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u = 0$$

with -c instead of c, we get:

$$u(t,x) = q(x + ct)$$

where q is an arbitrary function.
the solution represent traveling waves moving to the left with constant speed c>0
If $q \in \mathbb{C}^2$, we get a second family of solutions to the wave equation.
Thus there are both left and right traveling-wave solution,
Thus, we have two families of solutions of the wave equation:

$$u(t,x) = p(x - ct)$$
$$u(t,x) = q(x + ct), \forall p, q \in \mathbb{C}^2$$

Any linear combination of these solutions is still a solution of (3.14):

$$u(t,x) = p(x - ct) + q(x + ct), c > 0$$

Theorem 3.3.1

Every solution to (3.14) can be written in the form

$$u(t,x) = p(x - ct) + q(x + ct), \forall p, q \in \mathbb{C}^2 \tag{3.17}$$

Proof:

$$\xi = x - ct$$
$$\eta = x + ct$$
$$x = \frac{\xi + \eta}{2}, t = \frac{\eta - \xi}{2c}$$
$$u(t,x) = u(\frac{\eta - \xi}{2c}, \frac{\xi + \eta}{2})$$
$$\text{so, } u(t,x) = v(\xi, \eta)$$

Lets find a PED for v:

$$\frac{\partial u}{\partial t} = c(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta})$$
$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2})$$
$$\frac{\partial^2 u}{\partial x^2} = c^2(\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2})$$
$$0 = \Box u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$$
$$= -4c^2 \frac{\partial^2 v}{\partial \xi \partial \eta} = 0$$
$$\Leftrightarrow \frac{\partial^2 v}{\partial \xi \partial \eta} = 0$$
$$\frac{\partial}{\partial \xi} \frac{\partial v}{\partial \eta} v = 0$$
$$\Leftrightarrow \frac{\partial w}{\partial \xi} = 0$$
$$\Leftrightarrow w(\xi, \eta) = w(\eta)$$
$$\Leftrightarrow w = r(\zeta)$$

where ζ is an arbitary function of η

$$w = \frac{\partial v}{\partial \eta} = r(\zeta)$$

Integrating we get:

Now we want to solve the initial value problem (3.17):

$$\begin{aligned}
u(t,x) &= p(x-ct)+q(x+ct)\\
u(0,x) &= p(x)+q(x)=f(x)\\
\frac{\partial u}{\partial t} &= -cp'(x)+cq'(x)=g(x)\\
&\begin{cases} p'(x)+q'(x) & = f(x)\\ -cp'(x)+cq'(x) & = g(x)\end{cases}\\
2p'(x) &= f(x)+\frac{g(x)}{c}\\
p(x) &= \frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(z)dz+a\\
&\text{where a is an integration constant}\\
q(x) &= f(x)-p(x)\\
&= \frac{1}{2}+\frac{1}{2c}\int_0^x g(z)dz-a\\
u(t,x) &= p(\xi)+q(\eta)\\
&= \frac{f(\xi)+f(\eta)}{2}-\frac{1}{2c}\int_0^\xi g(z)dz+\frac{1}{2c}\int_0^\eta g(z)dz\\
&= \frac{f(\xi)+f(\eta)}{2}+\frac{1}{2c}\int_\eta^\xi g(z)dz
\end{aligned}$$

This is called d'Alembert's solution to the IVP (3.15)

Theorem 3.3.2: d'Alembert's solution

The solution of the problem:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}\\
u(0,x) &= f(x)\\
\frac{\partial u}{\partial t} &= g(x), x\in\mathbb{R}
\end{aligned}$$

is given by:

$$u(t,x)=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz, \forall f\in\mathbb{C}^2, g\in\mathbb{C}^1$$

Note that this formula provides a classical solution if $f\in\mathbb{C}^2$ and $g\in\mathbb{C}^1$

External forcing:
If a homogeneous medium is subject to an external force F, then the system is described as:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}+F(x,t) \tag{3.18}$$

We apply a generalized version of d'Alembert's technique. To simplify,we first impose homogeneous initial data:

$$\begin{aligned} u(0,x) &= 0 \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}$$

We search for a solution in the form $u(t,x) = v(\xi,\eta)$ and from the chain rule,we get

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{1}{-4c^2} F(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2})$$

Intergrating with respect to η, we get

$$\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \xi} = \frac{1}{-4c^2}\int_{\xi}^{\eta} F(\frac{\zeta-\xi}{2c}, \frac{\zeta+\xi}{2})d\zeta$$

From the identities:

$$\frac{\partial u}{\partial t} = c(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}), \frac{\partial u}{\partial x} = c(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta})$$

We get:

$$\frac{\partial v}{\partial \xi} = -\frac{1}{2c}\frac{\partial u}{\partial t} + \frac{1}{2c}\frac{\partial u}{\partial x}$$

Taking $\xi = \eta$:

$$\frac{\partial v}{\partial \xi} = -\frac{1}{2c}\frac{\partial u}{\partial t} + \frac{1}{2c}\frac{\partial u}{\partial x} = 0$$

because of homogeneuous initial conditions. Thus (3.18) becomes:

$$\frac{\partial v}{\partial \xi} = \frac{1}{-4c^2}\int_{\xi}^{\eta} F(\frac{\zeta-\xi}{2c}, \frac{\zeta+\xi}{2})d\zeta$$

Now, we integrate this with respect to ξ:

$$-v(\xi,\eta) = v(\eta,\eta) - v(\xi,\eta) = \frac{1}{-4c^2}\int_{\xi}^{\eta}\int_{\xi}^{\zeta} F(\frac{\zeta-\xi}{2c}, \frac{\zeta+\xi}{2})d\zeta d\xi$$

since $v(\xi,\xi) = 0$, Thus:

$$\begin{aligned} v(\xi,\eta) &= \frac{1}{4c^2}\int_{\xi}^{\eta}\int_{\phi}^{\zeta} F(\frac{\zeta-\phi}{2c}, \frac{\zeta+\phi}{2})d\zeta d\phi \\ &= \frac{1}{4c^2}\iint_{T(\xi,\eta)} F(\frac{\zeta-\phi}{2c}, \frac{\zeta+\phi}{2})d\zeta d\phi \end{aligned}$$

That is the double integral takes place over the triangle

$$T(\xi,\eta) = \{(\phi,\zeta) : \xi \le \phi \le \zeta \le \eta\}$$

Recalling that $\xi = x - ct$ and $\eta = x + ct$ and setting $\phi = y - cs$ and $\zeta = y + cs$, the inequality $\xi \leq \phi \leq \zeta \leq \eta$ becomes $x - ct \leq y - cs \leq y + cs \leq x + ct$,
So T(ξ, η) cna be rewritten as $D(t, x) = \{(s, y)|x - c(t-s) \leq y \leq x + c(t-s), 0 \leq s \leq t\}$

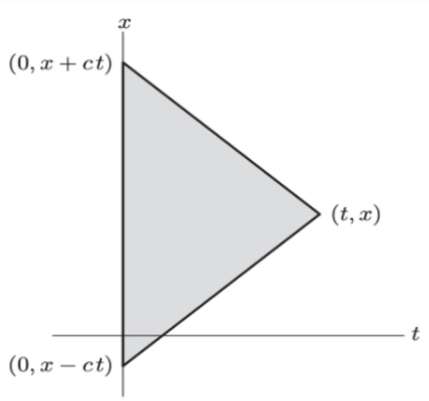


Figure 3.3: The Domain of integration $D(t, x)$

Using the change of variables $\frac{\zeta-\phi}{2c} = s$ and $\frac{\zeta+\phi}{2} = y$ for the double integrals and computing the Jacobian, we get: $\det \begin{pmatrix} \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial s} \\ \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial s} \end{pmatrix} = \det \begin{pmatrix} 1 & -c \\ 1 & c \end{pmatrix} = 2c$
Thus,

$$\begin{aligned} u(t,x) &= \frac{1}{2c} \iint_D (t,x) F(s,y) ds dy \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y) dy ds \end{aligned}$$

is the formula for the solution to the forced wave equation with homogeneuous initial conditioni.
To solve the case of general initial condition, one needs to take the sum of the solution to the forced equation with homogeneuous initial conditions plus the d'Alembert solution to the unforced equation subject to inhomogeneuous boundary conditions.

§3.4 Lecture 4 (02-12)

Theorem 3.4.1

The solution of the problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t) \\ u(0,x) = f(x) \\ \frac{\partial u}{\partial t} = g(x) \end{cases}$$

is given by

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y)dyds$$

The triangle D(t,x) is called the domain of dependence of the point (t,x)

Example 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + sin\omega t sinx, u(0,x) = 0, \frac{\partial u}{\partial t} = 0$$

So:

$$\begin{aligned} F(t,x) &= sin\omega t sinx \\ u(t,x) &= \frac{1}{2c}\int_0^t \int_{x-t+s}^{x+t-s} sin\omega s sinydyds \\ &= \frac{1}{2}\int_0^t sin\omega s[cos(x-t+s) - cos(x+t-s)]ds \\ &= \begin{cases} \frac{sin\omega t - \omega sint}{1-\omega^2} sinx, \omega \neq 1 \\ \frac{sint - tcost}{2} sinx, \omega = 1 \end{cases} \end{aligned}$$

When $\omega \neq 1$,the solution is bounded.
If $\omega \in Q$,then the solution u is periodic in time.
If $\omega \in I$,then u is quasi-periodic(it never exactly repeats) in time.
When $\omega = 1$ the solution is unbounded.

§3.4.1 Exercises

① Supposed u(t,x) is defined for all $(t,x) \in R^2$ and solves $\frac{\partial u}{\partial t} + 2u = 0$.
Prove that $\lim_{t\to+\infty} u(t,x) = 0$ for all x
SOL:

$$\begin{aligned} u(0,x) &= f(x) \\ u(t,x) &= e^{-2t} f(x) \\ \lim_{t\to+\infty} u(t,x) &= 0 \end{aligned}$$

② Let u(t,x) solve the initial value problem $\frac{\partial u}{\partial t} + u^c = 0, u(0,x) = f(x)$, where f(x) is a bounded $\mathbb{C}^1$ function of $x \in \mathbb{R}$
(a): show that if $f(x) \geq 0$ for all x, then u(t,x) is defined for all $t > 0$, and $\lim_{t\to+\infty} u(t,x) = 0$
(b): On the other hand , if $f(x) < 0$, then the solution u(t,x) is not defined for all t>0, but in fact, $\lim_{t\to r^-} u(t,x) = -\infty$, for some $0 < r < \infty$ Given x , what is the corresoponding value of r?
(c):Given f(x) as in part (b), what is the longest time interval $0 < t < t_\star$ on which u(t,x) is defined for all $x \in \mathbb{R}$?

SOL:

$$\frac{\partial u}{\partial t} = -u^2, u(0,x) = f(x)$$
$$\frac{du}{u^2} = -dt \Rightarrow -\frac{1}{u} = -t - c(x)$$
$$\frac{1}{u} = t + c(x) \Rightarrow u(t,x) = \frac{1}{t + c(x)}$$
$$u(0,x) = f(x) = \frac{1}{c(x)}$$
$$u(t,x) = \frac{1}{t + \frac{1}{f(x)}} = \frac{f(x)}{tf(x) + 1}$$
$$(a) : f(x) \geq 0 \Rightarrow u(t,x) \text{ is defined for any } t \geq 0 \text{ and } \lim_{t \to +\infty} u(t,x) = 0$$
$$(b) : f(x) < 0 \Rightarrow \tau(x) = \frac{1}{f(x)} \Rightarrow u(t,x) = \frac{1}{t + \tau(x)}$$
$$(c) : If f(x) < 0, [0, \tau(x))$$
$$t_\star = \inf\{\tau(x) : x \in \mathbb{R}\} = inf_{x \in \mathbb{R}} - \frac{1}{f(x)} = -\frac{1}{inf_{x \in \mathbb{R}} f(x)}$$

So we can have $f(x) = tan^{-1}(x)$

③ Solve the following initial value problem and graph the solution at times t=1,2,and 3:
(a) $u_t - 3u_x = 0, u(0,x) = e^{-x^2}$
Recall:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$
$$\text{We knnow that } u(t,x) = v(\xi) = (x - ct)$$
$$\forall v \in \mathbb{C}^1$$
$$u(0,x) = f(x) = v(x)$$
$$\text{Thus } u(t,x) = f(x - ct)$$

In our case:

$$u(t,x) = f(x + 3t) = e^{-(x+3t)^2}$$

④ Graph some of the characteristic lines for the following equations and write down a formula for the general solution:
(a)$u_t - 3u_x = 0$,
SOL: The characteristic lines of the equation $\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$ are the solution of the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(x)$$

In our case $\frac{\mathrm{d}x}{\mathrm{d}t} = -3 \to x(t) = -3t + k$

⑤ (a)Prove that if the initial data is bounded,$|f(x)| \leq M$ for all $x \in \mathbb{R}$,then the solution to the damped transport equation $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$ with a>0 satisfies

$u(t,x) \to 0$ as $t \to \infty$
(b) Find a solution to it that is defined for all (t,x) but does not satisfy $u(t,x) \to 0$ as $t \to +\infty$
SOL: Recall the solution to it with initial condition u(0,x)=f(x) is given by

$$\begin{aligned} u(t,x) &= f(x-ct)e^{-at} \\ |u(t,x)| &= |f(x-ct)|e^{-at} \leq Me^{-at} \\ \text{(b) } f(x) &= e^{Ax} \\ u(t,x) &= e^{A(x-ct)-at} = e^{-(Ac+a)t+Ax} \\ \text{Choose } A &\leq -\frac{a}{c} \end{aligned}$$

⑥ (a) Write down a formula for the general solution to the nonlinear partial differential equation $u_t + u_x + u^2 = 0$.
(b) Show that if the initial data is positive and bounded, $0 \leq u(0,x) = f(x) \leq M$, then the solution exists for all $t > 0$, and $u(t,x) \to 0$ as $t \to \infty$.
(c) On the other hand, if the initial data is negative somewhere, so $f(x) < 0$ at some $x \in \mathbb{R}$, then the solution blows up in finite time: $\lim_{t\to\tau^-} u(t,y) = -\infty$ for some $\tau > 0$ and some $y \in \mathbb{R}$.
(d) Find a formula for the earliest blow-up time $\tau_* > 0$.
SOL: Writing u(t,x) in terms of the characteristic variable $\xi = x - t$ we get:

$$u(t,x) = v(t,\xi) = u(t,\xi + t)$$

Let us find the equation for v:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u^2 = -v^2 \\ \Rightarrow \frac{dv}{v^2} &= -dt \Rightarrow -\frac{1}{v} = -t - c(\xi) \\ \Rightarrow v(t,\xi) &= \frac{1}{t + c(\xi)} \\ \Rightarrow u(t,x) &= \frac{1}{t + c(x-t)} \end{aligned}$$

Where c is an arbitrary $\mathbb{C}^1$ function.
(b): if there is an initial condition $u(0,x) = f(X)$, then we can determinec(x):

$$\begin{aligned} u(0,x) = f(x) &= \frac{1}{c(x)} \\ \Rightarrow u(t,x) &= \frac{1}{t + \frac{1}{f(x-t)}} = \frac{f(x-t)}{tf(x-t)+1} \end{aligned}$$

If $0 \leq f(x)$, then $tf(x-t) + 1 \neq 0$,so u(t,x) id defined for any t>0.
The function $\frac{s}{ts+1}$ is increasing in s
Therefore:

$$0 \leq u(t,x) \leq \frac{M}{Mt+1} \underbrace{\to}_{(\text{as } t\to\infty)} 0 \text{ as } f(x) \leq M$$

(c): The solution formula implies that, if $f(x) < 0$ then

$$\begin{aligned} u(t, x_\star) &\to -\infty \\ \text{as } t \to \tau &= -\frac{1}{f(x)} \\ where x_\star &= x - \frac{1}{f(x)} \end{aligned}$$

(d):

$$\tau_* = -\frac{1}{\inf_{x\in\mathbb{R}} f(x)}$$

§3.5 Lecture 5 (02-17)-Exercise

Problem 1

2.2.17. (a) Solve the initial value problem $u_t - xu_x = 0$, $u(0, x) = (x^2+1)^{-1}$. (b) Graph the solution at times $t = 0, 1, 2, 3$. (c) What is $\lim_{t\to\infty} u(t, x)$?
Solution. (a)
(1) find characteristic curves by solving

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x$$

$$x(t) = x_0 e^{-t}$$

(2) find u from the information that u(t,x) is constant along the characteristics:

$$u(t, x(t)) = u(0, x(0)) = f(x(0))$$

$$\begin{aligned} u(t, x_0 e^{-t}) = f(x_0) &= \frac{1}{x_0^2 + 1} \\ \text{Denote } x = x_0 e^{-t} &\Rightarrow x_0 = e^t x \\ u(t, x) &= \frac{1}{e^{2t}x^2 + 1} \\ &= \frac{e^{-2t}}{x^2 + e^{-2t}} \end{aligned}$$

(b)t=0,1,2,3

(c) $\lim_{t\to+\infty} u(t, x) = \begin{cases} 0, x \neq 0 \\ 1, x = 0 \end{cases}$

Problem 2

2.2.25. Suppose that $c(x) \in C^1$ is continuously differentiable for all $x \in \mathbb{R}$. (a) Prove that the characteristic curves of the transport equation (2.16) cannot cross each other. (b) A point where $c(x_*) = 0$ is known as a fixed point for the characteristic equation $\frac{dx}{dt} = c(x)$. Explain why the characteristic curve passing through a fixed point (t, x_*) is a horizontal straight line. (c) Prove that if $x = g(t)$ is a characteristic curve, then so are all the horizontally translated curves $x = g(t + \delta)$ for any δ. (d) True or false:

Every characteristic curve has the form $x = g(t + \delta)$, for some fixed function $g(t)$. (e) Prove that each non-horizontal characteristic curve is the graph $x = g(t)$ of a strictly monotone function. (f) Explain why a wave cannot reverse its direction. (g) Show that a non-horizontal characteristic curve starts, in the distant past, $t \to -\infty$, at either a fixed point or at $-\infty$ and ends, as $t \to +\infty$, at either the next-larger fixed point or at $+\infty$.
Solution. (a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(x), x(t_0) = x_0$$

If $x_1(t), x_2(t)$ are characteristic curves that cross each other at (t_0, x_0) then they both solve the same Cauchy problem.
By uniqueness, we have $x_1(t) = x_2(t)$ for all t
(b)
We know that $x_\star$ is a zero of C:$C(x_\star) = 0$ Let's consider the following problem:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = c(x) \\ x(0) = x_\star \end{cases}$$

So, $x(t) = x_\star$
(c)
If g(t) is a characteristic curve

$$\frac{d}{dt}g(t) = c(g(t))$$

Then

$$g(t + \delta) = g_\delta(t) \text{ is also characteristic}$$

since it satisfies

$$\frac{d}{dt}g_\delta(t) = c(g_\delta(t))$$

since c does not depend on t
(d)
False: If c admit a zero and $c \neq 0$,then $\exists$ single g such that any characteristicis of the form $g(t + \delta)$.
In deed, the fixed point solution and non-fixed point solutioins are not given with the same g.
(e)
To determine the monotonicity of g, we need to check the sign of g'(t).
it is always >0 or<0 in between the horizontal lines.
Then g'(t)=c(g(t)).
If $\exists t_1$ and t_2 so that $g'(t_1) > 0$ and $g'(t_2) < 0$ then $\exists t_\star$ between t_1, t_2 s,t, $c(g(t_\star))$=0, therefore x(t) is a fixed point solution and the graph should be a horizontal line
(f)
c(x(t)) is the soeed of the wave
Reformultion: Can the function c(x(t)) change its sign along a fixde characteristic curve?
If c(x(t1)>0),c(x(t2))<0, then exists $c(x(t_\star))$=0, therefore x(t) is a fixed point solution and the graph should be a horizontal line
(g)

Problem 3

2.4.11.(a) Write down an explicit formula for the solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \quad u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = \cos x, \quad -\infty < x < \infty, \quad t \geq 0.$$

(b) True or false: The solution is a periodic function of t.
(c) Now solve the forced initial value problem

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = \cos 2t, \quad u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = \cos x, \quad -\infty < x < \infty, \quad t \geq 0.$$

(d) True or false: The forced equation exhibits resonance. Explain.
(e) Does the answer to part (d) change if the forcing function is $\sin 2t$?

Solution. (a)

$$\begin{aligned} u(t,x) &= \frac{sin(x-2t)+sin(x+2t)}{2} + \frac{1}{4}(sin(x+2t) - sin(x-2t)) \\ &= \frac{1}{4}sin(x-2t) + \frac{3}{4}sin(x+2t) \end{aligned}$$

(b)
T s.t. u(t+T,x)=u(t,x)
$T = \pi$
(c)

$$\begin{aligned} u(t,x) &= \frac{1}{4}sin(x-2t) + \frac{3}{4}sin(x+2t) + \frac{1}{4}\sum_{0}^{t}\sum_{x-2(t-s)}^{x+2(t-s)} cos2sdyds \\ &= \frac{1}{4}sin(x-2t) + \frac{3}{4}sin(x+2t) + \frac{1}{4}(1-cos2t) \end{aligned}$$

(d)
resonance:unbounded
False, no resonance since the solution is bounded

Problem 4

2.4.13. Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$. The total energy

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right] dx$$

represents the sum of kinetic and potential energies of the displacement $u(t,x)$ at time t. Suppose that $\nabla u \to 0$ sufficiently rapidly as $x \to \pm\infty$; more precisely, one can find $\alpha > \frac{1}{2}$ and $C(t) > 0$ such that $|u_t(t,x)|, |u_x(t,x)| \leq C(t)/|x|^\alpha$ for each fixed t and all sufficiently large $|x| \gg 0$. For such solutions, establish the Law of Conservation of Energy by showing that $E(t)$ is finite and constant. Hint: You do not need the formula for the solution.

Solution. We need to show that $\frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0$

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}[(\frac{\partial u}{\partial t})^2 + c^2(\frac{\partial u}{\partial x})^2]dx \\
&= \int_{-\infty}^{+\infty}[(\frac{\partial u}{\partial t})\cdot\frac{\partial^2 u}{\partial t^2} + c^2(\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x\partial t}]dx \\
&= c^2\int_{-\infty}^{+\infty}[(\frac{\partial u}{\partial t})\cdot\frac{\partial^2 u}{\partial x^2} + (\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x\partial t}]dx \\
&= c^2\int_{-\infty}^{+\infty}1\cdot\frac{\partial}{\partial x}(\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x})dx \text{ integration by parts} \\
&= -c^2\int_{-\infty}^{+\infty}\underbrace{\frac{\partial}{\partial x}1}_{=0}\cdot(\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x})dx + 0(\text{boundary terms}) \\
&= 0
\end{aligned}$$

§3.6 Lecture 6 (02-19)–Fourier Series

Problem 5

2.4.15
Solution.

$$\begin{aligned}
E(t) &= \int_{-\infty}^{+\infty}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx \\
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_{-\infty}^{+\infty}[\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t}]dx \\
&= \int_{-\infty}^{+\infty}[c^2(\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t}) - a(\frac{\partial u}{\partial t})^2]dx \\
&= c^2\int_{-\infty}^{+\infty}1\cdot\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t})dx - a\sum_{-\infty}^{+\infty}(\frac{\partial u}{\partial t})^2 dx \\
&\leq 0 \Leftrightarrow \frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 0
\end{aligned}$$

So, $E(t)$ is decreasing

We need to prove that there is a unique solution to the problem $\begin{cases} \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \\ u(0,x) = f(x) \\ \frac{\partial u}{\partial t} = g(x) \end{cases}$

Suppose $\exists u_1(t,x), u_2(t,x)$

$$u(t,x) = u_1(t,x) - u_2(t,x)$$

u satisfies the equation with initial data

$$u(0,x) = \frac{\partial u}{\partial t} = 0$$

We need to show that $u(t,x) = 0 \forall t0, x$

$$E(0) = \int_{-\infty}^{+\infty}(\frac{\partial u}{\partial t})^2 + c^2(\frac{\partial u}{\partial x})^2 dx = 0$$

Since E(0)=0, $\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq 0, E(t \geq 0) \Rightarrow E(t) = 0 \Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} = E, \forall x, t \Rightarrow u$ is constant $\Rightarrow u(t,x) = 0$

Problem 6

2.4.17
Solution.

$$\Box = \frac{\partial^2}{\partial t^2} - c^2(x)\frac{\partial^2}{\partial x^2}$$
$$(\frac{\partial}{\partial t} + c(x)\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c(x)\frac{\partial}{\partial x}) = \frac{\partial^2}{\partial t^2} - c^2(x)\frac{\partial^2}{\partial x^2} - c(x)c'(x)\frac{\partial}{\partial x}$$

since $c'(x) \neq 0$, the d'Alembert formula does not apply

§3.6.1 Fourier Series

The linear heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial t} = L[u] \end{cases} \tag{3.19}$$

$$\begin{aligned} L[u] &= \frac{\partial^2 u}{\partial x^2} \\ L[u+v] &= L[u] + L[v] \\ L[\alpha u] &= \alpha L[u] \end{aligned}$$

These are true for any functions u and v in $\mathbb{C}^2$
Let us find solutions to (3.19) in the form.

$$u(t,x) = e^{\lambda t} v(x)$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}[e^{\lambda t} v(x)] \\ &= \lambda e^{\lambda t} \\ L[u] &= L[e^{\lambda t} v] \\ &= \frac{\partial^2}{\partial x^2}[e^{\lambda t} v] \\ &= e^{\lambda t}\frac{\partial^2}{\partial x^2} v \\ &= e^{\lambda t} L[v] \\ \frac{\partial u}{\partial t} &= L[u] \end{aligned}$$

Thus $u = e^{\lambda t} v$ will be a solution to (3.19) iff

$$L[v] = v'' = \lambda v$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x \in \mathbb{R}$$

Any nonzero linear combination of these particular solutions is still a solution of (3.19) For example

$$u(t,x) = c_1 e^{-t}\cos x + c_2 e^{-4t}\sin 2x + c_3 x + c_4$$

The heated ring

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}\\
u(t,-\pi) &= u(t,\pi)\\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x}\\
u(0,x) &= f(x)\\
u(t,x) &= e^{\lambda t}v(x)\\
v'' &= \lambda v\\
v(-\pi) &= v(\pi), v'(-\pi) = v'(\pi)
\end{aligned}$$

find λ and v(x)
case 1: $\lambda = \omega^2 > 0$

$$\begin{aligned}
v(x) &= ae^{\omega x} + be^{-\omega x}, \forall a,b \in \mathbb{R}\\
ae^{\omega\pi} + be^{-\omega\pi} &= ae^{-\omega\pi} + be^{\omega\pi}\\
a\omega e^{-\omega\pi} - b\omega e^{\omega\pi} &= a\omega e^{\omega\pi} - b\omega e^{-\omega\pi}\\
\omega &\neq 0\\
e^{\omega\pi}(b-a) + (a-b)e^{-\omega\pi} &= 0\\
(e^{\omega\pi} - e^{-\omega\pi})(b-a) &= 0\\
\Leftrightarrow \underbrace{(e^{2\omega\pi} - 1)}_{\neq 0}(b-a) &= 0\\
\Rightarrow a = b \text{ and } a &= -b\\
\Rightarrow a = b = 0 \Rightarrow v(x) &= 0
\end{aligned}$$

case 2: $\lambda = 0$

$$\begin{aligned}
v'' &= 0\\
v(x) = a + bx, \forall a,b \in \mathbb{R}\\
a - b\pi &= a + b\pi\\
\Rightarrow b &= 0, \forall a \in \mathbb{R}\\
\text{Thus } v(x) &= a\\
u(t,x) &= e^{\lambda t}v(x) = a
\end{aligned}$$

case 3: $\lambda = -\omega^2 < 0$

$$\begin{aligned}
v'' &= \lambda v \\
v(x) &= a\cos\omega x + b\sin\omega x, \forall a, b \in \mathbb{R} \\
a\cos\omega\pi - b\sin\omega\pi &= a\cos\omega\pi + b\sin\omega\pi \\
a\sin\omega\pi + b\cos\omega\pi &= -a\sin\omega\pi + b\cos\omega\pi \\
2b\sin\omega\pi &= 0, 2a\sin\omega\pi = 0 \\
\text{If } \sin\omega\pi &\neq 0 \\
\text{then } a &= b = 0 \text{ If } \sin\omega\pi = 0 \\
\Rightarrow \forall a, b &\in \mathbb{R}, \omega = 1, 2, \cdots \\
v(x) &= a\cos kx + b\sin kx \forall a, b \in \mathbb{R} \\
\text{With } \lambda &= -k^2 \\
u_k(t,x) &= e^{-k^2 t}(\cos kx) \\
u'_k(t,x) &= e^{-k^2 t}(\sin kx)
\end{aligned}$$

By Fourier:

$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k e^{-k^2 t}\cos kx + b_k e^{-k^2 t}\sin kx]$$

With coefficients a_k, b_k,the parameters a_k, b_k are fixed such that

$$u(0,x) = f(x)$$

§3.6.2 Fourier Series 2

We introduce L^2-inner product for continuous functions on the interval $[-\pi.\pi]$

$$< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

Basic properties of the inner product:

- symmetry: $< f, g >=< g, f >$
- bilinearity: $< \alpha f_1 + \beta f_2, g >= \alpha < f_1, g > +\beta < f_2, g >$
- positivity: $< f, f >> 0$ if $f \neq 0$, and $< 0, 0 >= 0$

The associated norm (L^2-norm) is defined by

$$||f|| = \sqrt{< f, f >}$$

Lemma 3.6.1

The trigonometric functions $1, \cos x, \sin x, \cos(2x), \sin(2x), \cdots$ satisfies the relations

$$
\begin{aligned}
&< \cos(kx), \cos(lx) > =< \sin(kx), \sin(lx) >= 0, \forall k \neq l \text{ integers} \\
&< \cos(ux), \sin(lx) > = 0, \forall u, l \\
&||1|| = \sqrt{2} \\
&|| \cos(kx)|| = || \sin(kx)|| = 1, \forall k \neq 0 \\
&f \perp g \Leftrightarrow < f, g >= 0
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx] \\
< f, \cos(lx) > &= \frac{a_0}{2} < 1, \cos(lx) > + \sum_{k=1}^{\infty}[a_k < \cos kx, \cos lx > + b_k < \sin kx, \cos lx >] \\
&= a_l < \cos lx, \cos lx > \\
&= a_l \\
b_l &=< f, \sin(lx) >
\end{aligned}
$$

§3.7 Lecture 7 (02-24)

Definition 3.7.1

The Fourier series of a function $f : [-\pi, \pi] \to \mathbb{R}$ is

$$f \approx \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx]$$

where $a_k =< f, \cos(kx) >, b_k =< f, \sin(kx) >, k = 1, 2, \cdots$

Example 1

① f(x)=x

$$a_k = 0, b_k = \frac{2}{k}(-1)^{k+1}$$

$$\text{Therefore }, x \approx 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots)$$

Definition 3.7.2: Piecewise continuous function

A function $f : [a, b] \to \mathcal{R}$ is piecewise continuous if it is defined and countinuous except possibily at finite number of points

Proposition 3.7.0: even and odd function

If f is even,then $b_k = 0$

If f is odd, then $a_k = 0$

Example 2

①

$$f(x) = |x|, \text{ even}$$
$$a_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ -\frac{4}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}$$

by theorem, this series converges to the 2π periodic extension of |x|

$$\text{Let } x = 0, \text{ we have}$$
$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

② computing $\sum_{k=1}^{\infty} \frac{1}{k^2} = S$

$$\frac{S}{4} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$
$$\frac{3}{4}S = S - \frac{S}{4} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$
$$S = \frac{\pi^2}{6}$$

§3.7.1 Complex Fourier series

Euler formulas:

$$e^{-ikx} = \cos(kx) - i\sin(kx)$$
$$\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx})$$
$$\sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx})$$

For any $f, g : [-\pi, \pi] \to C$
L^2-Hermitian inner product defined by

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

where $\overline{g(x)}$ is the complex conjugate of g(x)
L^2-Hermitian norm is defined by

$$||f|| = \sqrt{< f, f >}$$

Direct computation shows that

$$< e^{ikx}, e^{iex} >= \begin{cases} 1, & \text{if } k = e \\ 0, & \text{if } k \neq e \end{cases}$$

and

$$||e^{ikx}|| = 1$$

If f is a given continuous (or piecewise continuous) complex function then

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

with $c_k =< f, e^{ikx} >$
It is easy to see that

$$\begin{aligned} a_k &= c_k + c_{-k} \\ b_k &= i(c_k - c_{-k}) \\ c_k &= \frac{1}{2}(a_k - ib_k) \\ c_{-k} &= \frac{1}{2}(a_k + ib_k) \\ k &= 0, 1, 2, \cdots \end{aligned}$$

Example 3

① Let σ be the step function

$$\sigma(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

then

$$c_k = \begin{cases} \frac{1}{\pi i k}, & \text{if k odd} \\ \frac{1}{2}, & \text{if } k = 0 \\ 0, & \text{if k even} \end{cases}$$

$$\sigma(x) \approx \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{\pi i k} e^{ikx}$$

§3.8 Lecture 8 (02-26)–Fourier Series

Problem 1

2.2.13
Solution. Hint:

$$D = \{(x,t) \in R^2 : x, t \geq 0\}$$

find $u = u(t, \xi)$

§3.8.1 Differentiation and integration of Fourier series

In general the integral of periodic functions is not periodic,eg:f(x)=1

Lemma 3.8.1

if f is 2π-periodic and piecewise continuous function, then

$$g(x) = \int_0^x f(y)dy$$

is also 2π-periodic iff

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

Theorem 3.8.1

if f is piecewise continuous and has zero mean on $[-\pi, \pi]$,then its fourier series:

$$f(x) \approx \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

can be integrated term by term

$$g(x) = \int_0^x f(y)dy \approx m + \sum_{k=1}^{\infty} \frac{1}{k}[a_k \sin kx - b_k \cos kx]$$

$$\text{where } m = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx(\text{ integration by parts})$$

Example 1

① $f(x) = x$

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

$$x \approx 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

$$\frac{x^2}{2} \approx \frac{\pi^2}{6} - 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos kx$$

$$\bar{f} \text{ theorem at } x = 0$$

$$0 = \frac{\pi^2}{6} - 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12}$$

Theorem 3.8.2

if f is piecewise C^2 and 2π-periodic extension and

$$f(x) \approx \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx]$$

then

$$f'(x) \approx \sum_{k=1}^{\infty} kb_k \cos kx - ka_k \sin kx = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$$

(computing <f'(x),cos(kx)> using integration by parts)

Example 2

① $f(x) = |x|$

$$\begin{aligned}\frac{d}{dx}|x| &= \mathrm{sign}(x) \\ &= \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}\end{aligned}$$

We know the fourier series of |x|,

$$sign(x) \approx \frac{4}{\pi}(\sin(x) + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$$

§3.8.2 change of scale

$$\begin{aligned}2\pi &\to 2l \\ y \in [-\pi, \pi] &\to x \in [-l, l] \\ x &= \frac{l}{\pi}y \\ f : [-l, l] \to \mathbb{R} &\text{ piecewise continuous}\end{aligned}$$

Definition 3.8.1

$$f \approx \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos \frac{k\pi}{l}x + b_k \sin \frac{k\pi}{l}x]$$

Fourier series of f where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{k\pi}{l} x dx$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}xdx$$

All results seen before remain valid in this more general setting

$$f : [a, b] \to \mathbb{R}$$

$$l = \frac{b-a}{2}$$
$$\bar{f} : \mathbb{R} \to \mathbb{R}$$
$$b - a \text{ periodic}$$

§3.8.3 Convergence of Fourier series

A sequence of functions v_n converges pointwise to a function $v_\star$ on a set $I \subset R$ if

$$\lim_{n\to\infty} v_n(x) = v_\star(x), \forall x \in I$$

$$\Leftrightarrow \text{For any } \epsilon > 0, \forall x \in I, \exists N = N(\epsilon, x) \text{ such that } |v_n(x) - v_\star(x)| < \epsilon \forall n \geq N$$

A sequence of functions v_n converges uniformly to a function $v_\star$ on a set $I \subset R$ if for any $\epsilon > 0, \exists N = N(\epsilon)$ such that

$$|v_n(x) - v_\star(x)| < \epsilon, \forall x \in I$$

uniform convergence eg:

$$v_n(x) = \frac{1}{n}\sin(x)$$
$$v_n \to 0 \text{ uniformly}$$
$$|v_n(x) - 0| = \frac{1}{n}|\sin(x)|$$
$$n > N(\epsilon) = [\frac{1}{\epsilon}] + 1$$

uniform convergence $\Leftrightarrow \sup_{x\in I}$
Recall that if a sequence of continuous functions v_n on I converges to $v_\star$ uniformly, then $v_\star : I \to R$ is continuous on I

If $U_N(x) \to U_\star(x)$ pointwise on some set I, we say that the series converges pointwise if
$U_N(x) \to U_\star(x)$ uniformly on I

We say that the series converges uniformly
In particular if $U_n(x)$ are continuous on a set I and the series converges uniformly , then the sum is continuous function

Theorem 3.8.3: Weiezstrass m-test

Let $I \subset \mathbb{R}$

$$|u_k(x)| \leq M_k, \forall x \in I, k \geq 1$$

where $M_k \geq 0$ are numbers such that $\sum_{k=1}^{\infty} m_k < \infty$

Then the series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent to a function on I.
In particular, if $u_k(x)$ are continuous on I, then $f : I \to \mathbb{R}$ is continuous on I

Proposition 3.8.0

If $\sum_{k=1}^{\infty} u_k$ converges uniformly on an interval $I \subset \mathbb{R}$ then,

$$\int_0^x (\sum_{k=1}^{\infty} u_k(y))dy = \sum_{k=1}^{\infty} \int_0^x u_k(y)dy$$

Proposition 3.8.0

Suppose $\sum_{k=1}^{\infty} u_k(x)$ converges pointwise on an interval $I \subset R$ and u_k are all C^1 functions on I with $\sum_{k=1}^{\infty} u_k'$ convergent uniformly.
Then$(\sum_{k=1}^{\infty} u_k)'$ exists and we have the equality

$$(\sum_{k=1}^{\infty} u_k)' = \sum_{k=1}^{\infty} u_k'$$

For

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

x-real variable

$$|e^{ikx}| = 1$$
$$|c_k e^{ikx}| = |c_k| = M_k, \forall x \in \mathbb{R}$$

Theorem 3.8.4

Let the Fourier coefficients of f satisfies the condition

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty$$

then the Fourier series of f converges uniformly to a continuous function $\bar{f}$ with same coefficients $c_k =< f, e^{ikx} >=< \bar{f}, e^{ikx} >$

Proof. The Fourier series of

$$f \approx \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

as $\sum_{k=-\infty}^{\infty} |c_k| < \infty$
This series satisfies the conditions of the Weierstrass m-test with $M_k = |c_k|, u_k(x) = c_k e^{ikx}$

$$\Rightarrow \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \bar{f}(x)$$

converges uniformly to a function $\bar{f}:\mathbb{R}\to\mathbb{R}$ periodic continuous
Warning: f is not necessarily equal to $\bar{f}$ ■

Corollary 3.8.1

f f is 2π periodic and piecewise C^1 and f is continuous on the interval

$$[a,b]\subset[-\pi,\pi]$$

,then the Fourier series of f converge uniformly to f on any closed subinterval $(a+\delta,b-\delta)$ for $0<\delta<\frac{b-a}{2}$

§3.9 Lecture 9 (03-02)–Exercises

Problem 1

3.1.4
eigensoltuion: $u(t,x)=\phi(t)v(x)$

$$\begin{aligned}
\frac{\partial u}{\partial t}&=\phi'(t)v(x)\\
\frac{\partial u}{\partial x}&=\phi(t)v'(x)\\
\Rightarrow \phi'(t)v(x)&=\phi(t)v'(x)\\
\Rightarrow \frac{\phi'(t)}{\phi(t)}&=\frac{v'(x)}{v(x)}=\lambda\\
\frac{\phi'(t)}{\phi(t)}&=\lambda\Rightarrow\phi(t)=c_1e^{\lambda t}\\
\frac{v'(x)}{v(x)}&=\lambda\Rightarrow v(x)=c_2e^{\lambda x}\\
u(t,x)=c_1c_2e^{\lambda(t+x)}&=ce^{\lambda(t+x)},\forall c\in\mathbb{R}
\end{aligned}$$

Problem 2

3.1.7
(a)

$$\begin{aligned}
\frac{\partial u}{\partial t}&=\phi'(t)v(x)\\
\phi'(t)v(x)&=\phi(t)(x^2v'(x))'\\
\frac{\phi'(t)}{\phi(t)}&=\frac{(x^2v'(x))'}{v(x)}=\lambda\\
\frac{\phi'(t)}{\phi(t)}&=\lambda\Rightarrow\phi(t)=c_1e^{\lambda t}\\
\frac{(x^2v'(x))'}{v(x)}&=\lambda\Rightarrow x^2v''(x)+2xv'(x)=\lambda v(x)
\end{aligned}$$

Cauchy-Euler equation

$$
\begin{aligned}
a(a-1)+2a &= \lambda \\
a^2+a &= \lambda \\
\Delta &= 1+4\lambda \\
(1): \lambda > -\frac{1}{4} \Rightarrow a_{1,2} &= \frac{-1 \pm \sqrt{1+4\lambda}}{2} \\
v(x) &= c_1 x^{a_1} + c_2 x^{a_2} \\
u(t,x) &= e^{\lambda t}(c_1 x^{a_1} + c_2 x^{a_2}), \forall c_1, c_2 \\
(2): \lambda = -\frac{1}{4} \Rightarrow a &= -\frac{1}{2} \\
v(x) &= c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x \\
u(t,x) &= e^{-\frac{1}{4}t}(c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x), \forall c_1, c_2 \\
(3), \lambda < -\frac{1}{4} \Rightarrow a_{1,2} &= \frac{-1 \pm i\sqrt{4\lambda+1}}{2} \\
v(x) &= c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x \\
u(t,x) &= e^{\lambda t}(c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x), \forall c_1, c_2 \\
(b): \text{ case 1: } u(t,x) &= e^{\lambda t}(c_1 x^{a_1} + c_2 x^{a_2}) \\
u(t,1) = 0 \Rightarrow c_1 + c_2 &= 0 \\
u(t,2) = 0 \Rightarrow c_1 \cdot 2^{a_1} + c_2 \cdot 2^{a_2} &= 0 \\
det = 2^{a_2} - 2^{a_1} &\neq 0 \\
\Rightarrow c_1 = c_2 &= 0 \\
\text{case 2: } u(t,x) &= e^{\lambda t}(c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x) \\
u(t,1) = 0 \Rightarrow c_1 + c_2 ln1 &= 0 \to c_1 = 0 \\
u(t,2) = 0 \Rightarrow c_2 2^{-\frac{1}{2}} \ln 2 &= 0 \to c_2 = 0 \\
\text{case 3: } u(t,x) &= e^{\lambda t}(c_1 x^{-\frac{1}{2}} \cos \frac{\sqrt{4\lambda+1}}{2} \ln x + c_2 x^{-\frac{1}{2}} \sin \frac{\sqrt{4\lambda+1}}{2} \ln x) \\
u(t,1) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 &= 0 \to c_1 = 0 \\
u(t,2) = 0 \Rightarrow c_2 \sin \frac{\sqrt{4\lambda+1}}{2} \ln 2 &= 0 \\
c_2 \neq 0, \frac{1+4\lambda}{2} ln2 &= k\pi \\
\lambda_k &= -\frac{1}{4} - \frac{k^2\pi^2}{(ln2)^2} \\
u_k(x,t) &= x^{-\frac{1}{2}} \sin k\pi \frac{lnx}{\ln 2} e^{\lambda_k t}, \forall k \geq 1
\end{aligned}
$$

Problem 3

3.2.4

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$

$$\begin{aligned}
x \sim e^{ikx} &= cos(kx) + isin(kx) \\
a_k =< g, \cos(kx) > &= \frac{1}{2}p_0 < 1, \cos(kx) > + \sum_{j=1}^{\infty} p_j < \cos(jx), \cos(jx) + 0 > \\
&= \begin{cases} 0, & \text{if } k > n \\ p_k, & \text{if } k \in [0, n] \end{cases} \\
b_k =< g, \sin(kx) > &= \frac{1}{2}p_0 < 1, \sin(kx) > + \sum_{j=1}^{\infty} p_j < \cos(jx), \sin(jx) > + q_j < \sin jx, \cos kx > \\
&= \begin{cases} 0, & \text{if } k > n \\ q_k, & \text{if } k \in (0, n] \end{cases}
\end{aligned}$$

Problem 4

3.2.9

$$\begin{aligned}
(a) \int_a^{a+l} &= \int_l^{a+l} + \int_0^l - \int_0^a \\
\text{we need to prove } \int_l^{a+l} &= \int_0^a \\
\int_l^{a+l} f(x)dx &= \int_0^a f(s+l)ds \\
\text{change of variable } x = s + l & \\
\int_l^{a+l} f(x)dx &= \int_0^a f(s)ds \\
(b) \int_0^l f(x+a)dx &= \int_a^{a+l} f(s)ds(changex + a = s) \\
&= \int_0^l f(s)ds
\end{aligned}$$

§3.9.1 Hilbert space

$$L^2 = L^2[-\pi,\pi] \text{ is the space of square integrable functions}$$
$$f:[-\pi,\pi]\to\mathbb{R}$$
$$||f|| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx} \text{ is called } L^2 \text{ norm}$$
$$||1|| = 1$$
$$\text{The triangle inequality } ||f+g|| \le ||f|| + ||g|| \forall f,s \in L^2$$
$$||cf|| = |c|||f||, \forall c\in\mathbb{R}$$
$$< f,g >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\bar{g(x)}dx \text{ is the } L^2\text{-Hermitian inner product of } f,g\in L^2$$
$$\text{Cauchy-Schwarz inequality } |< f,g >| \le ||f||\cdot||g||$$
$$c_k =< f,e^{ikx} > = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-iux}dx$$

Every piecewise continuous function belongs to L^2

$$\frac{1}{|x|^\alpha}\in L^2 \text{ if } \alpha<\frac{1}{2}$$
$$\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi} + \int_{-\pi}^{-\epsilon}\frac{1}{|x|^{2\alpha}} \text{ limit exists iff } \alpha<\frac{1}{2}$$

Definition 3.9.1

Let v_n be a sequence in L^2 we say that $v_n\to f$ in L^2 if

$$\lim_{n\to\infty}||v_n - f|| = 0$$

Definition 3.9.2

Let $\{\phi_n\}$ be a sequence in L^2 we say that $\{\phi_n\}$ is a an orthonormal system if

$$< \phi_j,\phi_k >= \begin{cases} 1, & \text{if } j=k \\ 0, & \text{if } j\ne k \end{cases}$$

$$f\sim\sum_{k=1}^{\infty} c_k\phi_k$$

Theorem 3.9.1

Let $\{\phi_n\}$ be an orthonormal system in L^2 and

$$\begin{aligned} V_n &= \text{span}\{\phi_1,\phi_2,\cdots,\phi_n\} \\ &= \{\alpha_1\phi_1,\cdots,\alpha_n\phi_n\} \end{aligned}$$

is the n-dimentional space spanned by the independent elements $\phi_1, \phi_2, \cdots, \phi_n$
Then for any f we have

$$||f - s_n|| = \inf_{p \in V_n} ||f - p||$$

where

$$s_n = \sum_{k=1}^{n} c_k \phi_k$$
$$c_n =< f, \phi_k >$$

§3.10 Lecture 10 (03-10)

Midterm:3/19 $1 \to 3.4 change of scale$

Problem 1

3.2.18

$$x^{\frac{1}{3}} : R \to R$$

This is a continuous function $\Rightarrow$ it is piecewise cont.

piecewise C^1: (limit from right and left are finite)

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}, x \neq 0$$
$$\lim_{x \to 0^+} f'(x) = +\infty \to \text{ not } C^1, C^2$$

Problem 2

3.2.27

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx \\ &= \frac{(-1)^n (e^\pi - e6 - \pi)}{\pi(n^2+1)} \\ &= \frac{2(-1)^n \sinh(\pi)}{\pi(n^2+1)} \\ b_n &= \frac{2(-1)^{n+1} n \sinh(\pi)}{n^2+1} \end{aligned}$$

(b): As $\bar{f}$ (the periodic extension of f) is continuous, each point converges uniformly to $\bar{f}$

Problem 3

3.2.34/35

$$
\begin{aligned}
f(x) &\text{ is odd} \\
f(x) &= -f(-x) \\
f'(x) &= f'(-x) \rightarrow \text{even} \\
f(x) &\text{ is even} \\
f(x) &= f(-x) \\
\text{constant } f(x) &\text{ can be even or odd} \\
(1+\sin(x))' &= \cos(x) \rightarrow \text{ neither}
\end{aligned}
$$

Problem 4

3.2.45

$$
\begin{aligned}
\int_{-\pi}^{0} f(x)\cos(kx)dx &= \int_{0}^{\pi} f(x-\pi)\cos(k(x-\pi))dx \\
&= \int_{0}^{\pi} f(x)\cos(kx)dx
\end{aligned}
$$

(b):

$$
\begin{aligned}
f(x+\frac{\pi}{2}) &= f(x) \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{2})\cos\Big(k(x+\frac{\pi}{2})\Big)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(kx+\frac{k\pi}{2}\right)dx
\end{aligned}
$$

$$
\begin{aligned}
\text{If k is odd } b_k &= -b_k = 0, a_k = 0 \\
\text{If k is even and not divisible by 4 } a_k &= -a_k = 0, b_k = 0 \\
\text{If k is divisible by 4 } a_k, b_k &\text{ can be non-zero}
\end{aligned}
$$

Problem 5

3.2.51

$$
\begin{aligned}
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} \\
\sin^3(x) = (\frac{e^{ix}-e^{-ix}}{2i})^3 &= \frac{e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}}{-8}
\end{aligned}
$$

Problem 6

3.2.55

§3.11 Lecture 11 (03-12)

Problem 1

3.3.2

Problem 2

3.3.6
Fubini's theorem

Problem 3

3.4.3

Problem 4

3.4.9

Problem 5

3.4.1

Corollary 3.11.1

f $\{\phi_n\}$ is a complete orthonormal fmaily and all generalized Fourier coefficients c_k are zero, then f=0
More generally, $f, g \in L^2$

Proof of BIG Theorem:
1. geometric series,2.the kernel, intergral is 1.3. consider the difference goes to 0 by bessel's inequality.

§3.12 Lecture 12 (03-17)–Separation of variables

§3.12.1 The heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{3.20}$$

u=u(t,x) provides the value of the temperature at time t and position x, $\alpha > 0$ is a constant called the thermal diffusivity.
u(0,x)=f(x) (the initial condition) + boundary solutions

$$u(t, x) = e^{-\lambda t} v(x) \tag{3.21}$$

eigensoltuion = separable solution
substituting 3.21 into 3.20 we get

$$-\alpha v''(x) = \lambda v(x)$$

λ is called eigenvalue, v(x) is called eigenfunction
Dirichlet boundary conditions: $u(t, 0) = u(t, l) = 0$

It is not hard to see that if λ is complex or $\lambda \leq 0$, then the only solution to it is v=0
Non-trivial solutions exist only if $\lambda > 0$

$$v(x) = a\cos(\omega x) + b\sin(\omega x)$$

where $\omega = \sqrt{\lambda/\alpha}$ and $a, b \in \mathbb{R}$
The boundary condition $v(0) = 0 \to a = 0$
$v(l) = 0 \to v(l) = b\sin(\omega l) = 0$, v will be non-trivial if $b \neq 0$ and $\sin(\omega l) = 0$
To solve the heat equation, we introduce the series

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} b_n u_n(t,x) \\ &= \sum_{n=1}^{\infty} b_n exp - \frac{\alpha}{e} v_n(x) \end{aligned}$$

§3.12.2 The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.22}$$

1-D wave equation on a bounded interval $[0, l]$

§3.13 Lecture 13 (04-07)

$$\sum_{k=1}^{+\infty} \frac{\cos(kx)}{k^2}$$

§3.14 Lecture 14 (04-09)

3.5.43

Proof. ■

4.1.16

Proof.

$$\begin{aligned} v_t &= -\alpha e^{-\alpha t} u_t \\ v_{xx} &= \alpha^2 e^{-\alpha t} u_{xx} \\ -\alpha e^{-\alpha t} u_t &= \zeta \alpha^2 e^{-\alpha t} u_{xx} + \alpha e^{-\alpha t} u \end{aligned}$$

■

Chapter 4

Algorithm

§4.1 Lecture 1 (02-03)Asymptotic bounds and big O

Definition 4.1.1: big-Theta notation(=)

Let $f, g : N \to R^+$, we say that $f(n) = \Theta(g(n))$ if there $\exists$ $a, b > 0$ and $n_0 \in \mathbb{N}$such that $a \cdot g(n) \leq f(n) \leq b \cdot g(n)$ for all $n > n_0$.

Intuitively, $f(n) = \Theta(g(n))$ means that $f(n)$ grows as fast as $g(n)$ up to a constant when n tends to infinity.

Examples: $3n = \Theta(n)$, $2n^2 + 7n = \Theta(n^2)$, $\ln n^2 = \Theta(\log n)$

Definition 4.1.2: big-O notation(≦)

Let $f, g : N \to R^+$, we say that $f(n) = O(g(n))$ if $\exists$ $n_0 \in \mathbb{N}$ and $b > 0$ such that $f(n) \leq b \cdot g(n)$ for all $n > n_0$.
$\lim_{n\to\infty} sup \frac{f(n)}{g(n)} < +\infty$

Definition 4.1.3: big-Omega notation(≥)

Let $f, g : N \to R^+$, we say that $f(n) = \Omega(g(n))$ if $\exists$ $n_0 \in \mathbb{N}$ and $a > 0$ such that $a \cdot g(n) \leq f(n)$ for all $n > n_0$.
$\lim_{n\to\infty} inf \frac{f(n)}{g(n)} > 0$

Remark 1

$$f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$
$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

Definition 4.1.4: ittle-o notation(<)

We write $f(n) = o(g(n))$ if for any $b > 0$ there $\exists$ $n_0 \in \mathbb{N}$ such that $f(n) < b \cdot g(n)$ for all $n > n_0$.

Definition 4.1.5: little-omega notation(>)

We write $f(n) = \omega(g(n))$ if for any $a > 0$ there $\exists n_0 \in \mathbb{N}$ such that $a \cdot g(n) < f(n)$ for all $n > n_0$.

Remark 2

$f(n) = o(g(n)) \Rightarrow f(n) = O(g(n))$ and $f(n) \neq \Theta(g(n))$
$f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n))$ and $f(n) \neq \Theta(g(n))$
$f(n) = \omega(g(n)) \Leftrightarrow g(n) = \omega(f(n))$

$$n^{1+e}, nlog(n), e \to 0$$

Theorem 4.1.1

Any Algorithm has to check at least log(n+1) entries of the sorted array for searching an element in the worst case

§4.2 Recitation 1 (02-07)–Problem Solving

Notation	Bounding Conditions	Relation	Type
$T(n) = O(f(n))$	$\exists c > 0, \exists n_0, \forall n \geq n_0, T(n) \leq cf(n)$	$\leq$	O
$T(n) = \Omega(f(n))$	$\exists c > 0, \exists n_0, \forall n \geq n_0, T(n) \geq cf(n)$	$\geq$	Ω
$T(n) = \Theta(f(n))$	$\exists c_1, c_2 > 0, \exists n_0, \forall n \geq n_0, c_1 f(n) \leq T(n) \leq c_2 f(n)$	$=$	Θ
$T(n) = o(f(n))$	$\forall c > 0, \exists n_0, \forall n \geq n_0, T(n) < cf(n)$	$<$	o
$T(n) = \omega(f(n))$	$\forall c > 0, \exists n_0, \forall n \geq n_0, T(n) > cf(n)$	$>$	ω

Possible useful formula:

$$\int_1^n x^k dx = \frac{n^{k+1}}{k+1} - \frac{1}{k+1}$$

1. Multiplication Rule. Prove that for $h(n) = \Omega(1)$, if $f(n) = \Theta(g(n))$ then $f(n) \cdot h(n) = \Theta(g(n) \cdot h(n))$ (The same also holds for $O, \Omega, \omega, \omega$)
   Solution:

   Since $f(n) = \Theta(g(n)), \exists c_1, c_2 > 0, \exists n_0$ such that
   $c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0$
   Since $h(n) = \Omega(1)$, there exist constants $c_3 > 0$ and n_1 such that
   $h(n) \geq c_3, \forall n \geq n_1$
   Combining the two, we have:
   $c_1 g(n)h(n) \leq f(n)h(n) \leq c_2 g(n)h(n)$
   Therefore $f(n)h(n) = \Theta(g(n)h(n))$

2. Addition Rule.Let $f_1, ..., f_k : N \to R_{\geq 0}$ for a fixed number k and $f(n) = \sum_{i=1}^{k} f_i(n)$ .Prove that if $f_i(n) = O(f_1(n))$ for all $i \in \{1, \ldots, k\}$,then $f(n) = \Theta(f_1(n))$ .Here

f_1 can be replaced with any one of $f_1, ..., f_k$
Solution:

Since $f_i(n) = O(f_1(n))$, there exist constants c_i and n_0 such that
$f_i(n) \leq c_i f_1(n), \forall n \geq n_0$
Therefore,
$\sum_{i=1}^{k} f_i(n) \leq \sum_{i=1}^{k} c_i f_1(n) = C f_1(n)$ where $C = \sum_{i=1}^{k} c_i$

Additionally, there exists at least one $f_j(n) = \Omega(f_1(n))$, so $\sum_{i=1}^{k} f_i(n) = \Theta(f_1(n))$

3. Let f and g be non-negative functions. Define $T(n) = max(f(n), g(n))$ .Prove that $T(n) = \Theta(f(n) + g(n))$
Solution:

Since $max(f(n), g(n)) \leq f(n) + g(n)$, we have $T(n) = O(f(n) + g(n))$
Also, $f(n) + g(n) \leq 2max(f(n), g(n))$, so $T(n) = \Omega(f(n) + g(n))$
Therefore, $T(n) = \Theta(f(n) + g(n))$

4. Let f and g be non-decreasing real-valued functions defined on positive integers and $f(1) \geq 1$ and $g(1) \geq 1$ .Prove or disprove that $f(n) = O(g(n)) \Rightarrow 2^{f(n)} = O(2^{g(n)})$
Solution:

Since $f(n) = O(g(n))$, there exists a constant $c > 0$ and n_0 such that
$f(n) \leq c \cdot g(n), \forall n \geq n_0$
Therefore,
$2^{f(n)} \leq 2^{c \cdot g(n)}$
Since 2^c is constant, we conclude that $2^{f(n)} = O(2^{g(n)})$

5. Prove that $log(n!) = \Theta(nlogn)$ .Hint:To show an upper bound,compare $n!$ with n^n .To show a lower bound, compare it with $\left(\frac{n}{2}\right)^{\frac{n}{2}}$
Solution:

We know that:
$log(n!) = log(1) + log(2) + \ldots + log(n)$
Since $log(n)$ is increasing, the sum can be bounded by:
$log(n!) \leq \sum_{k=1}^{n} log(n) = nlog(n)$
From the bound, we can use the fact that $log(k) \geq log(\frac{n}{2})$ for all $k \geq 2$
Hence, we can sum the lower half of the terms to get:
$log(n!) \geq \sum_{k=\frac{n}{2}}^{n} log(k) \geq \sum_{k=\frac{n}{2}}^{n} log(\frac{n}{2}) = \frac{n}{2} log(\frac{n}{2})$

6. Show that $c^n = o(n!)$ for any constant C
Solution:

We need to prove that

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

Using the recursive relation:

$$\frac{c^n}{n!} = \frac{c^{n-1}}{(n-1)!} \times \frac{c}{n}$$

As $n \to \infty, \frac{c}{n} \to 0,$ Thus,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

7. Show that $c^{n^{1.001}} = \omega(n!)$ for any constant C .Hint: For any fixed numbers $p, q > 0$,we have $n^p = \omega(log^q n)$
Solution:

We need to prove that

$$\lim_{n\to\infty} \frac{c^{n^{1.001}}}{n!} = \infty$$

Taking the algorithm, we have:

$$log(\frac{c^{n^{1.001}}}{n!}) = n^{1.001} log(c) - log(n!)$$

Since $log(n!) \approxeq nlogn,$ and $n^1.001$ grows faster than $nlogn,$ we conclude that

$$\lim_{n\to\infty} log(\frac{c^{n^{1.001}}}{n!}) = \infty$$

So, $\lim_{n\to\infty} \frac{c^{n^{1.001}}}{n!} = \infty$

Therefore, $c^{n^{1.001}} = \omega(n!)$

8. Show $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ .Hint: Show that $n^{k+1} - (n-1)^{k+1} = \Theta(n^k)$
Solution:

Upperbound: Since $i^k \leq n^k$, we have

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n^{k+1}$$

Lowerbound: Using the integral approximation

$$\sum_{i=1}^{n} i^k \geq \int_1^n x^k dx = \frac{n^{k+1}}{k+1} - \frac{1}{k+1}$$

Therefore,

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

9. Search for the peak. Suppose we have an array $A[1...n]$ and we know that there exists some k (value unknown) such that $A[1] < A[2] < ... < A[k]$ and $A[k] > A[k+1] > \ldots > A[n]$ .We call $A[k]$ the peak of A .Design an algorithm to find the peak and analyze it in $O(logn)$ time.
Solution:

   Use Binary Search Algorithm:
   1.Set $l = 1, r = n$
   2.While $l < r$ do
   3.Set $m = \left\lfloor \frac{l+r}{2} \right\rfloor$
   4.If $A[m] < A[m+1]$ then $l = m+1$
   5.Else $r = m$
   6.Return l
   Analysis:
   Let $T(n)$ be the time complexity of the algorithm
   Let $n_0 = 1$
   Let $c = 1$
   Let $T(n) = T\left(\frac{n}{2}\right) + c$
   Let $T(n) = O(logn)$

10. 3SUM in a sorted array.Let $A[1\ldots n]$ be a sorted array. We want to see $A[i] + A[j] + A[k] = x$ for some different i, j, k ,where x is a given number. Design an algorithm $3SUM(n, a, x)$ which can return (i, j, k) or 0 if no such three elements exist.How fast can this problem be solved?

Solution:

fix one element A[i], then use two pointers on the remaining array to find $A[j] + A[k] = x - A[i]$
The time complexity is $O(n^2)$
Algorithm 3SUM(n,a,x):
Input: Array A of n intergers, integer x
Output: indices i,j,k such that A[i]+A[j]+A[k]=x or return "No solution" if no such triplet exists
for i=0 to n-1 do
 //Avoid duplicates by skipping repeated elements
 if i>0 and A[i]==A[i-1]
 continue
 left=i+1,right=n-1
 while left<right do
 sum=A[i]+A[left]+A[right]
 if sum==x
 return i,left,right //Found the triplet
 if sum<x
 left=left+1 //increase the sum by moving left pointer to the right
 else
 right=right-1 //decrease the sum by moving right pointer to the left
return "No solution" //no such triplet exists

§4.3 Recitation 02

Find tight big-O bounds for the following recurrences (and show your bounds are correct).

Example 1. Substitution. $T(n) = \sqrt{2n} \cdot T(\sqrt{2n}) + \sqrt{n}$ with base case $T(n) = 1$ for $n \in [0, 1]$.

Example 2. Recursion Tree. $T(n) = T(n/3) + T(n/4) + T(n/5) + n$

Example 3. Recall The Master Theorem. Let $a \geq 1$, $b > 1$, and $f(n)$ be a function. If $T(n) = aT(n/b) + f(n)$, then we can solve $T(n)$ as:

$$T(n) = \begin{cases} \Theta(n^{log_b a}) & \text{if } f(n) = O(n^{log_b(a-\epsilon)}) \text{ for some } \epsilon > 0 \\ \Theta(n^{log_b a} logn) & \text{if } f(n) = \Theta(n^{log_b a}) \\ \Theta(f(n)) & \text{if } f(n) = \Omega(n^{log_b(a+\epsilon)}) \text{ for some } \epsilon > 0 \end{cases}$$

Then what is the asymptotic bound for: $T(n) = 4T(\frac{n}{2}) + n^2\sqrt{2}$.

Find tight big-O bounds for the following recurrences (and show your bounds are correct).

Problem 1. $T(n) = T(bn) + T(n - bn) + n$, where $b \in (0, 1)$ is a constant.

Solution. by recursion tree

$$H = log(n)$$
$$T(n) = O(n) \cdot H = O(nlog(n))$$

Problem 2. $T(n) = n^{\frac{3}{4}}T(n^{\frac{1}{4}}) + n$.

Solution. Suppose $T(n) = C \cdot n^k$

$$\begin{aligned} T(m^4) &= m^3 T(1) + m^4 \\ T(n) &= O(n) \end{aligned}$$

Problem 3. $T(n) = 2T(2n/3) + T(n/3) + n^2$
Solution. by recursion tree

$$T(n) = O(n^2 logn)$$

Problem 4. Let $T(n) = 4T(n/3) + n$. Use induction to prove $T(n) = O(n^{log_3 4})$.
Problem 5. $T(n) = T(n-2) + n^2$
Solution.

$$\begin{aligned} T(n) - T(n-1) &= T(n-2) - T(n-1) + n^2 \\ \text{Let } S(n) &= T(n) - T(n-1) \\ S(n) &= S(n-1) + n^2 \\ S(n) &= O(n^3) \\ T(n) &= T(n-1) + O(n^3) \\ T(n) &= O(n^3) \end{aligned}$$

§4.4 Lecture 2 (02-19) – Divide and Conquer