Problem 1

Uniform distribution. A random variable that is equally likely to take any value in a finite set S is said to have the uniform distribution on S. If U is such a random variable and $\emptyset \neq R \subseteq S$, show that the distribution of U conditional on $\{U \in R\}$ is uniform on R.

Proof. Since U is uniformly distributed on set S, for any $s \in S$, we have:

$$P(U = s) = \frac{1}{|S|}$$

For any element $r \in R$, we need to compute the conditional probability $P(U = r \mid U \in R)$.

By the definition of conditional probability:

$$P(U = r \mid U \in R) = \frac{P(U = r \text{ and } U \in R)}{P(U \in R)}$$

Since $r \in R$, the event $\{U = r \text{ and } U \in R\}$ is equivalent to $\{U = r\}$. Therefore:

$$P(U = r \mid U \in R) = \frac{P(U = r)}{P(U \in R)}$$

We know that $P(U = r) = \frac{1}{|S|}$.

For $P(U \in R)$, we use the fact that U has a uniform distribution on S:

$$P(U \in R) = \sum_{t \in R} P(U = t) = \sum_{t \in R} \frac{1}{|S|} = \frac{|R|}{|S|}$$

Substituting these values:

$$P(U = r \mid U \in R) = \frac{\frac{1}{|S|}}{\frac{|R|}{|S|}} = \frac{1}{|R|}$$

Since this probability equals $\frac{1}{|R|}$ for every $r \in R$, the conditional distribution assigns equal probability to all elements in R. This is precisely the definition of a uniform distribution on R.

Therefore, the distribution of U conditional on $\{U \in R\}$ is uniform on R. $\square$

Problem 2

3. Let X be a random variable with distribution function

$$\mathbb{P}\{X \leq x\} = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Let F be a distribution function which is continuous and strictly increasing. Show that $Y = F^{-1}(X)$ is a random variable having distribution function F.

Is it necessary that F be continuous and/or strictly increasing?

Proof. We need to show that if X has the given distribution function and F is continuous and strictly increasing, then $Y = F^{-1}(X)$ has distribution function F.

For any real number y, we have:

$$\begin{aligned} \mathbb{P}\{Y \leq y\} &= \mathbb{P}\{F^{-1}(X) \leq y\} \\ &= \mathbb{P}\{X \leq F(y)\} \end{aligned}$$

This equivalence holds because F is strictly increasing, so the inequality is preserved when applying F to both sides.

Since X has the distribution function:

$$\mathbb{P}\{X \leq x\} = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Therefore:

$$\mathbb{P}\{X \leq F(y)\} = \begin{cases} 0 & \text{if } F(y) \leq 0 \\ F(y) & \text{if } 0 < F(y) \leq 1 \\ 1 & \text{if } F(y) > 1 \end{cases}$$

Since F is a distribution function, we know $0 \leq F(y) \leq 1$ for all y, which means we're always in the middle case where $\mathbb{P}\{X \leq F(y)\} = F(y)$. Thus, $\mathbb{P}\{Y \leq y\} = F(y)$, which proves that Y has distribution function F.

Regarding the necessity of the conditions:

Strict monotonicity: If F is not strictly increasing but merely non-decreasing, then for some values $a < b$ with $F(a) = F(b)$, the inverse function F^{-1} is not uniquely defined at $F(a)$. This makes $Y = F^{-1}(X)$ ambiguous.

Continuity: If F has jump discontinuities, then there exist values c in the range of F for which $F^{-1}(c)$ is not defined. If X takes such values with positive probability, then $Y = F^{-1}(X)$ is not properly defined as a random variable.

Both issues can be addressed by using the generalized inverse function:

$$F^{-1}(u) = \inf\{y : F(y) \geq u\}$$

With this definition, $Y = F^{-1}(X)$ will have distribution function F even when F is not continuous or strictly increasing, provided that F is at least right-continuous and non-decreasing. □

Problem 3

(a) Show that any discrete random variable may be written as a linear combination of indicator variables.
(b) Show that any random variable may be expressed as the limit of an increasing sequence of discrete random variables.
(c) Show that the limit of any increasing convergent sequence of random variables is a random variable.

Proof. **(a)**

Let X be a discrete random variable taking values in the countable set $\{x_1, x_2, x_3, \ldots\}$. We can express X as:

$$X = \sum_{i=1}^{\infty} x_i \mathbf{1}_{\{X=x_i\}}$$

where $\mathbf{1}_{\{X=x_i\}}$ is the indicator variable for the event $\{X = x_i\}$, defined as:

$$\mathbf{1}_{\{X=x_i\}}(\omega) = \begin{cases} 1 & \text{if } X(\omega) = x_i \\ 0 & \text{otherwise} \end{cases}$$

To verify this representation, consider any outcome ω in the sample space. If $X(\omega) = x_j$ for some j, then $\mathbf{1}_{\{X=x_i\}}(\omega) = 0$ for all $i \neq j$ and $\mathbf{1}_{\{X=x_j\}}(\omega) = 1$. Therefore:

$$\sum_{i=1}^{\infty} x_i \mathbf{1}_{\{X=x_i\}}(\omega) = x_j \cdot 1 + \sum_{i \neq j} x_i \cdot 0 = x_j = X(\omega)$$

Thus, X is indeed a linear combination of indicator variables.

(b)

Let X be any random variable. For each positive integer m, we partition the real line into intervals of the form $[k2^{-m}, (k+1)2^{-m})$ for $k \in \mathbb{Z}$ (i.e., all integers from $-\infty$ to ∞).

We define the discrete random variable X_m as:

$$X_m = \sum_{k=-\infty}^{\infty} k2^{-m} I_{k,m}$$

where $I_{k,m}$ is the indicator function of the event $\{k2^{-m} \leq X < (k+1)2^{-m}\}$.

This construction has the following properties:

1. X_m is a discrete random variable because it takes values in the countable set $\{k2^{-m} : k \in \mathbb{Z}\}$.

2. For any outcome ω, if $X(\omega) \in [k2^{-m}, (k+1)2^{-m})$ for some integer k, then $X_m(\omega) = k2^{-m}$, which is the left endpoint of the interval containing $X(\omega)$.

3. For each fixed ω, $X_m(\omega) \leq X(\omega)$ since $X_m(\omega) = k2^{-m}$ when $X(\omega) \in [k2^{-m}, (k+1)2^{-m})$.

4. As m increases, the partition becomes finer and $X_m(\omega)$ becomes a better approximation of $X(\omega)$. Specifically, if $X(\omega) \in [k2^{-m}, (k+1)2^{-m})$, then:

$$0 \leq X(\omega) - X_m(\omega) < (k+1)2^{-m} - k2^{-m} = 2^{-m}$$

5. The sequence $\{X_m(\omega)\}_{m=1}^{\infty}$ is increasing for each fixed ω. This is because as m increases, the left endpoints of the intervals in the partition either stay the same or increase to a value closer to $X(\omega)$.

Since $2^{-m} \to 0$ as $m \to \infty$, we have $X_m(\omega) \uparrow X(\omega)$ as $m \to \infty$ for all ω.

Therefore, any random variable X can be expressed as the limit of an increasing sequence of discrete random variables $\{X_m\}_{m=1}^{\infty}$.

(c)

Let $\{Y_n\}$ be an increasing sequence of random variables converging to Y. We need to show that Y is a random variable, which means we need to prove that for any Borel set B, the preimage $Y^{-1}(B) = \{\omega : Y(\omega) \in B\}$ is measurable.

Consider any $a \in \mathbb{R}$. We need to show that the set $\{\omega : Y(\omega) \leq a\}$ is measurable.

Since $\{Y_n\}$ is increasing and converges to Y, we have:

$$\{\omega : Y(\omega) \leq a\} = \{\omega : \lim_{n\to\infty} Y_n(\omega) \leq a\}$$

For an increasing sequence, $\lim_{n\to\infty} Y_n(\omega) \leq a$ if and only if $Y_n(\omega) \leq a$ for all n. Therefore:

$$\{\omega : Y(\omega) \leq a\} = \bigcap_{n=1}^{\infty} \{\omega : Y_n(\omega) \leq a\}$$

Since each Y_n is a random variable, each set $\{\omega : Y_n(\omega) \leq a\}$ is measurable. The countable intersection of measurable sets is measurable, so $\{\omega : Y(\omega) \leq a\}$ is measurable.

Since the preimage of any interval $(-\infty, a]$ under Y is measurable, and the Borel σ-algebra is generated by such intervals, Y is a random variable. □

Problem 4

Express the distribution functions of

$$X^+ = \max\{0, X\}, \quad X^- = -\min\{0, X\}, \quad |X| = X^+ + X^-, \quad -X,$$

in terms of the distribution function F of the random variable X.

Proof. Let X be a random variable with distribution function $F(x) = \mathbb{P}(X \leq x)$. We'll express the distribution functions of the derived random variables in terms of F.

Distribution function of $X^+ = \max\{0, X\}$:

For $x < 0$, since $\max\{0, X\} \geq 0 > x$ for all values of X:

$$\mathbb{P}(X^+ \leq x) = 0$$

For $x \geq 0$:

$$\begin{aligned} \mathbb{P}(X^+ \leq x) &= \mathbb{P}(\max\{0, X\} \leq x) \\ &= \mathbb{P}(X \leq x) \\ &= F(x) \end{aligned}$$

Therefore:

$$\mathbb{P}(X^+ \leq x) = \begin{cases} 0 & \text{if } x < 0, \\ F(x) & \text{if } x \geq 0. \end{cases}$$

Distribution function of $X^- = -\min\{0, X\}$:

For $x < 0$, since $X^- \geq 0$ by definition:

$$\mathbb{P}(X^- \leq x) = 0$$

For $x \geq 0$, noting that $X^- = 0$ when $X \geq 0$ and $X^- = -X$ when $X < 0$:

$$\begin{aligned} \mathbb{P}(X^- \leq x) &= \mathbb{P}((X \geq 0) \cup (X < 0 \text{ and } -X \leq x)) \\ &= \mathbb{P}(X \geq 0) + \mathbb{P}(-x \leq X < 0) \end{aligned}$$

$$
\begin{aligned}
&= 1 - \lim_{y\uparrow 0} F(y) + F(0) - F(-x) \\
&= 1 - \lim_{y\uparrow -x} F(y)
\end{aligned}
$$

Therefore:

$$
\mathbb{P}(X^- \le x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \lim_{y\uparrow -x} F(y) & \text{if } x \ge 0. \end{cases}
$$

Distribution function of $|X| = X^+ + X^-$:

For $x < 0$, since $|X| \ge 0$:

$$
\mathbb{P}(|X| \le x) = 0
$$

For $x \ge 0$, using $|X| = X$ when $X \ge 0$ and $|X| = -X$ when $X < 0$:

$$
\begin{aligned}
\mathbb{P}(|X| \le x) &= \mathbb{P}(0 \le X \le x) + \mathbb{P}(-x \le X < 0) \\
&= F(x) - F(0) + F(0) - F(-x) \\
&= F(x) - \lim_{y\uparrow -x} F(y)
\end{aligned}
$$

Therefore:

$$
\mathbb{P}(|X| \le x) = \begin{cases} 0 & \text{if } x < 0, \\ F(x) - \lim_{y\uparrow -x} F(y) & \text{if } x \ge 0. \end{cases}
$$

Distribution function of $-X$:

For any real x:

$$
\begin{aligned}
\mathbb{P}(-X \le x) &= \mathbb{P}(X \ge -x) \\
&= 1 - \mathbb{P}(X < -x) \\
&= 1 - \lim_{y\uparrow -x} F(y)
\end{aligned}
$$

$\square$

Problem 5

The real number m is called a median of the distribution function F whenever $\lim_{y\to m} F(y) \le \frac{1}{2} \le F(m)$.

(a) Show that every distribution function F has at least one median, and that the set of medians of F is a closed interval of $\mathbb{R}$.

(b) Show, if F is continuous, that $F(m) = \frac{1}{2}$ for any median m.

Proof. **(a)**

Define $m = \sup\{x : F(x) < \frac{1}{2}\}$. We will show that m is a median.

For any $y < m$, by the definition of supremum, we have $F(y) < \frac{1}{2}$. This implies that $\lim_{y\uparrow m} F(y) \leq \frac{1}{2}$.

Now we need to show that $F(m) \geq \frac{1}{2}$. Suppose, for contradiction, that $F(m) < \frac{1}{2}$. Since F is right-continuous, there exists some $\delta > 0$ such that $F(m + \epsilon) < \frac{1}{2}$ for all $0 < \epsilon < \delta$. But this means that $m + \frac{\delta}{2}$ belongs to the set $\{x : F(x) < \frac{1}{2}\}$, which contradicts m being the supremum of this set. Therefore, $F(m) \geq \frac{1}{2}$.

Thus, m satisfies $\lim_{y\uparrow m} F(y) \leq \frac{1}{2} \leq F(m)$, making it a median.

Similarly, define $M = \sup\{x : F(x) \leq \frac{1}{2}\}$. Using a similar argument, we can show that M is also a median.

For any $x \in [m, M]$, we have:

$$\lim_{y\uparrow x} F(y) \leq \lim_{y\uparrow m} F(y) \leq \frac{1}{2}$$

because F is non-decreasing. Also:

$$F(x) \geq \frac{1}{2}$$

because $x \geq m$ and F is non-decreasing. Therefore, every $x \in [m, M]$ is a median.

Conversely, if $x < m$, then $F(x) < \frac{1}{2}$, so x cannot be a median. If $x > M$, then by definition of M, we have $F(x) > \frac{1}{2}$, and by right-continuity of F, we get $\lim_{y\uparrow x} F(y) > \frac{1}{2}$, so x cannot be a median.

Therefore, the set of all medians is exactly the closed interval $[m, M]$.

(b)

Let m be any median of the continuous distribution function F. By definition of median:

$$\lim_{y\uparrow m} F(y) \leq \frac{1}{2} \leq F(m)$$

Since F is continuous at m, we have $\lim_{y\uparrow m} F(y) = F(m)$. The inequality above becomes:

$$F(m) \leq \frac{1}{2} \leq F(m)$$

This can only be satisfied when $F(m) = \frac{1}{2}$.

Therefore, when F is continuous, every median m satisfies $F(m) = \frac{1}{2}$. □

Problem 6

Let $(\Omega, \mathcal{F})$ be a measurable space, $A_1, \cdots A_n \in \mathcal{F}$ and $b_1, \cdots b_n \in \mathbb{R}$. Set $X = b_1 1_{A_1} + \cdots + b_n 1_{A_n}$. Suppose that for all $i \neq j$, $A_i \cap A_j = \emptyset$ and $b_i \neq b_j$. Show that $\sigma(X) = \sigma(\{A_1, \cdots, A_n\})$.

Proof. Let $(\Omega, \mathcal{F})$ be a measurable space, $A_1, \ldots, A_n \in \mathcal{F}$ be disjoint sets and $b_1, \ldots, b_n \in \mathbb{R}$ be distinct real numbers. We define $X = \sum_{i=1}^n b_i 1_{A_i}$.

We want to show that $\sigma(X) = \sigma(\{A_1, \ldots, A_n\})$.

$\Rightarrow \sigma(X) \subseteq \sigma(\{A_1, \ldots, A_n\})$:

For any Borel set $B \in \mathcal{B}(\mathbb{R})$, we need to show that $X^{-1}(B) \in \sigma(\{A_1, \ldots, A_n\})$.

Note that X takes only the values from the set $\{b_1, \ldots, b_n, 0\}$, where 0 is taken when $\omega \notin \bigcup_{i=1}^n A_i$. Let's define $A_0 = \Omega \setminus \bigcup_{i=1}^n A_i$ and $b_0 = 0$ for convenience.

For any $\omega \in \Omega$, there is a unique index $j \in \{0, 1, \ldots, n\}$ such that $\omega \in A_j$, and thus $X(\omega) = b_j$.

Therefore,

$$\begin{aligned} X^{-1}(B) &= \{\omega \in \Omega : X(\omega) \in B\} \\ &= \{\omega \in \Omega : \exists j \in \{0, 1, \ldots, n\} \text{ such that } \omega \in A_j \text{ and } b_j \in B\} \\ &= \bigcup_{j=0}^n \{A_j : b_j \in B\} \end{aligned}$$

Since each $A_j \in \sigma(\{A_1, \ldots, A_n\})$ (note that A_0 can be expressed as $\Omega \setminus \bigcup_{i=1}^n A_i$, which is in $\sigma(\{A_1, \ldots, A_n\})$), we have $X^{-1}(B) \in \sigma(\{A_1, \ldots, A_n\})$.

Thus, $\sigma(X) \subseteq \sigma(\{A_1, \ldots, A_n\})$.

$\Leftarrow \sigma(\{A_1, \ldots, A_n\}) \subseteq \sigma(X)$:

We need to show that for each $i \in \{1, \ldots, n\}$, $A_i \in \sigma(X)$.

Since the values $b_1, \ldots, b_n$ are distinct, for each $i \in \{1, \ldots, n\}$, the set $\{b_i\}$ is a Borel set in $\mathbb{R}$. We have:

$$\begin{aligned} X^{-1}(\{b_i\}) &= \{\omega \in \Omega : X(\omega) = b_i\} \\ &= \{\omega \in \Omega : \sum_{j=1}^n b_j 1_{A_j}(\omega) = b_i\} \end{aligned}$$

For any $\omega \in A_i$, we have $X(\omega) = b_i$ because $A_i \cap A_j = \emptyset$ for $i \neq j$. For any $\omega \notin A_i$, we have $X(\omega) \neq b_i$ because the values $b_1, \ldots, b_n$ are distinct and the sets $A_1, \ldots, A_n$ are disjoint.

Therefore, $X^{-1}(\{b_i\}) = A_i$, which implies $A_i \in \sigma(X)$ for each $i \in \{1, \ldots, n\}$.

Hence, $\sigma(\{A_1, \ldots, A_n\}) \subseteq \sigma(X)$.

So we conclude that $\sigma(X) = \sigma(\{A_1, \ldots, A_n\})$. □

Problem 7

Consider the random walk on $\{0, 1, \cdots N\}$ that reflects at 0, introduced in the class. More precisely, let $S_0 = k, S_{j+1} = S_j + X_j$, where $X_j = 1$ if $S_j = 0$, otherwise $\mathbb{P}[X_j = 1] = p$ and $\mathbb{P}[X_j = -1] = q$. Compute the expected number of steps to frst reach state N, as a function of k. Distinguish the cases $p = q$ and $p \neq q$.

Proof. Random walk on $\{0, 1, \ldots, N\}$ with:

$$\mathbb{P}[X_j = 1 \mid S_j = 0] = 1 \text{ (reflection at 0)}$$
$$\mathbb{P}[X_j = 1 \mid S_j \neq 0] = p$$
$$\mathbb{P}[X_j = -1 \mid S_j \neq 0] = q = 1 - p$$

Let h_k be the expected number of steps to first reach state N from initial state k.

Recursive equations:

$$h_k = 1 + ph_{k+1} + qh_{k-1}, \quad 0 < k < N$$
$$h_0 = 1 + h_1$$
$$h_N = 0$$

Case 1: ($p = q = 1/2$)

For $0 < k < N$, the recursive equation becomes:

$$h_k = 1 + \frac{1}{2}h_{k+1} + \frac{1}{2}h_{k-1}$$
$$\Rightarrow h_{k+1} - 2h_k + h_{k-1} = -2$$

This second-order linear difference equation has general solution:

$$h_k = A + Bk + k(N - k)$$

From boundary conditions:

$$h_N = 0 \Rightarrow A + BN = 0$$
$$h_0 = 1 + h_1 \Rightarrow B = -1 \Rightarrow A = N$$

Therefore:

$$h_k = (N - k)(k + 1)$$

Case 2: ($p \neq q$)

Rearranging the recursive equation:

$$ph_{k+1} - h_k + qh_{k-1} = -1$$

Let $\rho = \frac{q}{p}$. The general solution has form:

$$h_k = A + B\rho^k + \frac{k}{q-p} + \frac{q}{q-p}$$

Applying boundary conditions:

$$\begin{aligned} h_N &= 0 \\ h_0 &= 1 + h_1 \end{aligned}$$

Solving for constants A and B:

$$\begin{aligned} A &= \frac{1}{p-q}\left(\frac{1-\rho^N}{1-\rho} - N\right) \\ B &= \frac{1}{p-q} \cdot \frac{1}{1-\rho^N} \end{aligned}$$

Final solution when $p \neq q$:

$$h_k = \begin{cases} \frac{N-k}{p-q} & \text{if } p > q \\ \frac{1}{p-q}\left[\frac{1-\rho^N}{1-\rho} - N - \frac{\rho^k - \rho^N}{1-\rho^N}\right] & \text{if } p < q \end{cases}$$

□

Problem 8

If $F : \mathbb{R} \to [0,1]$ satisfes monotone increasing, right continuity, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$, show that F is the distribution function of some random variable on some probability space. (Hint: vou may want to revisit Section 2.3, excercise 3)

Proof. Let $F : \mathbb{R} \to [0,1]$ satisfy:

- Monotone increasing
- Right continuous
- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to+\infty} F(x) = 1$

We construct a random variable with distribution function F using the hint that the limit of any increasing sequence of random variables is a random variable.

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$ where λ is Lebesgue measure.

For each $n \in \mathbb{N}$, define:

$$X_n(\omega) = \inf\{x = k2^{-n} : F(x) \geq \omega, k \in \mathbb{Z}\}$$

The sequence $\{X_n\}$ is decreasing: $X_n(\omega) \geq X_{n+1}(\omega)$ for all $\omega \in \Omega$.

Define $X(\omega) = \lim_{n\to\infty} X_n(\omega)$. By the hint, X is a random variable since it's the limit of a monotone sequence of random variables.

From the construction and right continuity of F:

$$X(\omega) = \inf\{x \in \mathbb{R} : F(x) \geq \omega\}$$

For any $a \in \mathbb{R}$:

$$\begin{aligned}\mathbb{P}(X \leq a) &= \mathbb{P}(\{\omega \in \Omega : F(a) \geq \omega\}) \\ &= F(a)\end{aligned}$$

Therefore, F is the distribution function of the random variable X. $\square$