Problem 1

Show the following:

(a) $\mathbb{E}(aY + bZ \mid X) = a\mathbb{E}(Y \mid X) + b\mathbb{E}(Z \mid X)$ for $a, b \in \mathbb{R}$,

(b) $\mathbb{E}(Y \mid X) \geq 0$ if $Y \geq 0$,

(c) $\mathbb{E}(1 \mid X) = 1$,

(d) if X and Y are independent then $\mathbb{E}(Y \mid X) = \mathbb{E}(Y)$,

(e) ('pull-through property') $\mathbb{E}(Yg(X) \mid X) = g(X)\mathbb{E}(Y \mid X)$ for any suitable function g,

(f) ('tower property') $\mathbb{E}\{\mathbb{E}(Y \mid X, Z) \mid X\} = \mathbb{E}(Y \mid X) = \mathbb{E}\{\mathbb{E}(Y \mid X) \mid X, Z\}$.

(a) By definition, for discrete random variables,

$$\mathbb{E}(aY + bZ \mid X = x) = \sum_{y,z} \Big[ay + bz \Big] \mathbb{P}(Y = y,\, Z = z \mid X = x).$$

By the distributive property of summation, we have:

$$\mathbb{E}(aY + bZ \mid X = x) = a \sum_{y,z} y\, \mathbb{P}(Y = y, Z = z \mid X = x) + b \sum_{y,z} z\, \mathbb{P}(Y = y, Z = z \mid X = x).$$

Notice that for fixed y,

$$\sum_{z} \mathbb{P}(Y = y, Z = z \mid X = x) = \mathbb{P}(Y = y \mid X = x),$$

and similarly,

$$\sum_{y} \mathbb{P}(Y = y, Z = z \mid X = x) = \mathbb{P}(Z = z \mid X = x).$$

Thus, we obtain:

$$\mathbb{E}(aY + bZ \mid X = x) = a \sum_{y} y\, \mathbb{P}(Y = y \mid X = x) + b \sum_{z} z\, \mathbb{P}(Z = z \mid X = x)$$

$$= a\, \mathbb{E}(Y \mid X = x) + b\, \mathbb{E}(Z \mid X = x).$$

Since this holds for every x, we have proven that

$$\mathbb{E}(aY + bZ \mid X) = a\, \mathbb{E}(Y \mid X) + b\, \mathbb{E}(Z \mid X).$$

(b)

If $Y \geq 0$, then for any fixed x each term in

$$\mathbb{E}(Y \mid X = x) = \sum_{y} y\, \mathbb{P}(Y = y \mid X = x)$$

is nonnegative, so

$$\mathbb{E}(Y \mid X) \geq 0.$$

(c)

Setting $Y \equiv 1$,

$$\mathbb{E}(1 \mid X = x) = \sum_{y,z} 1 \cdot \mathbb{P}(Y = y, Z = z \mid X = x) = \sum_{y,z} \mathbb{P}(Y = y, Z = z \mid X = x) = 1,$$

since the sum of the conditional probabilities equals 1. Thus,

$$\mathbb{E}(1 \mid X) = 1.$$

(d)

If X and Y are independent, then for all x and y

$$\mathbb{P}(Y = y \mid X = x) = \mathbb{P}(Y = y).$$

Thus,

$$\mathbb{E}(Y \mid X = x) = \sum_{y} y\, \mathbb{P}(Y = y \mid X = x) = \sum_{y} y\, \mathbb{P}(Y = y) = \mathbb{E}(Y).$$

Hence,

$$\mathbb{E}(Y \mid X) = \mathbb{E}(Y).$$

(e) Pull-Through Property:

Let g be any function such that $g(X)$ is $\sigma(X)$-measurable. Then for any fixed x,

$$\begin{aligned}\mathbb{E}\big(Y\, g(X) \mid X = x\big) &= \sum_{y,z} y\, g(x)\, \mathbb{P}(Y = y, Z = z \mid X = x) \\ &= g(x) \sum_{y,z} y\, \mathbb{P}(Y = y, Z = z \mid X = x) \\ &= g(x)\, \mathbb{E}(Y \mid X = x).\end{aligned}$$

Thus,

$$\mathbb{E}(Y\, g(X) \mid X) = g(X)\, \mathbb{E}(Y \mid X).$$

(f) Tower Property:

$$\mathbb{E}\{\mathbb{E}(Y|X,Z)|X = x\} = \sum_{z} \left\{ \sum_{y} y\mathbb{P}(Y = y|X = x, Z = z)\mathbb{P}(X = x, Z = z|X = x) \right\}$$

$$
\begin{aligned}
&= \sum_z \sum_y y \frac{\mathbb{P}(Y = y, X = x, Z = z)}{\mathbb{P}(X = x, Z = z)} \cdot \frac{\mathbb{P}(X = x, Z = z)}{\mathbb{P}(X = x)} \\
&= \sum_y y\mathbb{P}(Y = y|X = x) \\
&= \mathbb{E}\{\mathbb{E}(Y|X)|X = x, Z = z\}
\end{aligned}
$$

Problem 2

Conditional variance formula.

How should we define $\text{var}(Y \mid X)$, the conditional variance of Y given X?

Show that $\text{var}(Y) = \mathbb{E}(\text{var}(Y \mid X)) + \text{var}(\mathbb{E}(Y \mid X))$.

Proof. By definition, the conditional variance of Y given X is

$$\text{Var}(Y \mid X) = \mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big)^2 \mid X\Big].$$

To prove the variance decomposition formula:

$$
\begin{aligned}
\text{Var}(Y) &= \mathbb{E}\Big[\big(Y - \mathbb{E}(Y)\big)^2\Big] \\
&= \mathbb{E}\Big[\Big(Y - \mathbb{E}(Y \mid X) + \mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big)^2\Big]
\end{aligned}
$$

Expanding the squared term:

$$
\begin{aligned}
&= \mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big)^2\Big] + 2\,\mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big)\Big(\mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big)\Big] \\
&\quad + \ \mathbb{E}\Big[\Big(\mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big)^2\Big]
\end{aligned}
$$

For the middle term:

$$
\begin{aligned}
\mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big)\Big(\mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big)\Big] &= \mathbb{E}\Big[\Big(\mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big) \cdot \mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big) \mid X\Big]\Big] \\
&= 0
\end{aligned}
$$

since $\mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big) \mid X\Big] = 0$.

Therefore:

$$
\begin{aligned}
\text{Var}(Y) &= \mathbb{E}\Big[\mathbb{E}\Big[\big(Y - \mathbb{E}(Y \mid X)\big)^2 \mid X\Big]\Big] + \mathbb{E}\Big[\Big(\mathbb{E}(Y \mid X) - \mathbb{E}(Y)\Big)^2\Big] \\
&= \mathbb{E}\Big[\text{Var}(Y \mid X)\Big] + \text{Var}\Big(\mathbb{E}(Y \mid X)\Big)
\end{aligned}
$$

$\square$

Problem 3

Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X + Y$ and $V = X/(X + Y)$, and deduce that V is uniformly distributed on $[0, 1]$.

Proof. Let X and Y be independent exponential random variables with parameter 1. Define

$$U = X + Y \quad \text{and} \quad V = \frac{X}{X+Y}.$$

We wish to find the joint density function of U and V using a transformation of variables and the Jacobian method.

First, express X and Y in terms of U and V. Since

$$V = \frac{X}{X+Y}, \quad \text{it follows that} \quad X = UV.$$

And because

$$U = X + Y, \quad \text{we have} \quad Y = U - X = U - UV = U(1 - V).$$

Thus, the transformation is given by:

$$\begin{cases} X = UV, \\ Y = U(1 - V). \end{cases}$$

Next, we compute the Jacobian. The partial derivatives are

$$\frac{\partial X}{\partial U} = V, \quad \frac{\partial X}{\partial V} = U,$$
$$\frac{\partial Y}{\partial U} = 1 - V, \quad \frac{\partial Y}{\partial V} = -U.$$

The Jacobian matrix is

$$J = \begin{vmatrix} V & U \\ 1 - V & -U \end{vmatrix},$$

and its determinant is

$$J = V(-U) - U(1 - V) = -UV - U + UV = -U.$$

Taking the absolute value (noting that $U > 0$) gives

$$|J| = U.$$

The joint density function of X and Y is

$$f_{X,Y}(x, y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} \quad \text{for } x, y > 0.$$

Substituting $x = UV$ and $y = U(1 - V)$, we obtain

$$f_{X,Y}(UV, U(1-V)) = e^{-UV-U(1-V)} = e^{-U(V+1-V)} = e^{-U}.$$

Thus, using the transformation formula,

$$f_{U,V}(u,v) = f_{X,Y}(uv, u(1-v)) \cdot |J| = e^{-u} \cdot u = ue^{-u},$$

which is valid for $u > 0$ and $0 \leq v \leq 1$.

From the expression

$$f_{U,V}(u,v) = ue^{-u} \cdot 1,$$

we see that the marginal density functions are

$$f_U(u) = ue^{-u} \quad \text{for } u > 0,$$

corresponding to a Gamma$(2,1)$ distribution, and

$$f_V(v) = 1 \quad \text{for } 0 \leq v \leq 1,$$

which is a uniform distribution on $[0,1]$. Since

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v),$$

it follows that U and V are independent, and V is uniformly distributed on $[0,1]$.

□

Problem 4

Rayleigh distribution. Let X and Y be independent random variables, where X has an arc sine distribution and Y a Rayleigh distribution:

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1, \quad f_Y(y) = ye^{-\frac{1}{2}y^2}, \quad y > 0.$$

Write down the joint density function of the pair (Y, XY), and deduce that XY has the standard normal distribution.

Proof. We are given that X and Y are independent random variables with densities

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1,$$
$$f_Y(y) = ye^{-\frac{1}{2}y^2}, \quad y > 0.$$

Here, X has the arc sine distribution and Y has the Rayleigh distribution.

To find the joint density of (Y, XY), we define the new variables

$$U = Y \quad \text{and} \quad V = XY.$$

Since X and Y are independent, their joint density is

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)\, f_Y(y) \\ &= \frac{1}{\pi\sqrt{1-x^2}} \cdot y e^{-\frac{1}{2}y^2}, \end{aligned}$$

for $|x| < 1$ and $y > 0$.

The inverse transformation from (U, V) to (X, Y) is

$$\begin{aligned} Y &= U, \\ X &= \frac{V}{U}, \end{aligned}$$

which is valid for $U > 0$ and $|V| < U$ (since $|X| < 1$ implies $|V| = |XY| < U$).

The Jacobian of the inverse transformation is computed as

$$J = \begin{vmatrix} \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \\ \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{V}{U^2} & \frac{1}{U} \end{vmatrix} = \frac{1}{U}.$$

Using the change of variables formula, the joint density of (U, V) is

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\left(\frac{v}{u},\, u\right) \cdot \left|\frac{1}{u}\right| \\ &= \frac{1}{\pi\sqrt{1-\left(\frac{v}{u}\right)^2}}\, u e^{-\frac{1}{2}u^2} \cdot \frac{1}{u} \\ &= \frac{e^{-\frac{1}{2}u^2}}{\pi\sqrt{1-\left(\frac{v}{u}\right)^2}} \\ &= \frac{u e^{-\frac{1}{2}u^2}}{\pi\sqrt{u^2-v^2}}, \end{aligned}$$

for $u > 0$ and $|v| < u$.

To obtain the marginal density of V, we integrate over u:

$$\begin{aligned} f_V(v) &= \int_{|v|}^{\infty} f_{U,V}(u,v)\, du \\ &= \frac{1}{\pi}\int_{|v|}^{\infty} \frac{u e^{-\frac{1}{2}u^2}}{\sqrt{u^2-v^2}}\, du. \end{aligned}$$

By using the substitution $u^2 = v^2 + t^2$ (so that $u\,du = t\,dt$ and $\sqrt{u^2 - v^2} = t$), the integral becomes

$$\begin{aligned} f_V(v) &= \frac{1}{\pi}\, e^{-\frac{1}{2}v^2} \int_0^\infty e^{-\frac{1}{2}t^2}\, dt \\ &= \frac{e^{-\frac{1}{2}v^2}}{\pi} \cdot \sqrt{\frac{\pi}{2}} \\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}v^2}. \end{aligned}$$

Thus, $V = XY$ is distributed as a standard normal random variable.

□

Problem 5

Binary expansions. Let U be uniformly distributed on the interval $(0, 1)$.

(a) Let S be a (measurable) subset of $(0, 1)$ with strictly positive measure (length). Show that the conditional distribution of U, given that $U \in S$, is uniform on S.

(b) Let $V = \sqrt{U}$, and write the binary expansions of U and V as $U = \sum_{r=1}^{\infty} U_r 2^{-r}$ and $V = \sum_{r=1}^{\infty} V_r 2^{-r}$. Show that U_r and U_s are independent for $r \neq s$, while $\mathrm{cov}(V_1, V_2) = -\frac{1}{32}$. Prove that $\lim_{n\to\infty} \mathbb{P}(V_r = 1) = \frac{1}{2}$.

(a)

Suppose that U is uniformly distributed on $(0, 1)$ and S is a measurable subset of $(0, 1)$ with positive measure. We show that the conditional distribution of U given $U \in S$ is uniform on S. For any measurable set $A \subseteq S$,

$$\begin{aligned} \mathbb{P}(U \in A \mid U \in S) &= \frac{\mathbb{P}(U \in A \cap S)}{\mathbb{P}(U \in S)} = \frac{\mathbb{P}(U \in A)}{\mathbb{P}(U \in S)} \\ &= \frac{|A|}{|S|}, \end{aligned}$$

where $|A|$ and $|S|$ denote the Lebesgue measures of A and S, respectively. This is exactly the law for a uniform variable on S.

(b)

Let $V = \sqrt{U}$ where $U \sim \text{Uniform}(0,1)$. Write the binary expansions as

$$U = \sum_{r=1}^{\infty} U_r \, 2^{-r},$$
$$V = \sum_{r=1}^{\infty} V_r \, 2^{-r},$$

with $U_r, V_r \in \{0,1\}$.

Since U is uniform on $(0,1)$, each binary digit U_r is independent with

$$\mathbb{P}(U_r = 1) = \mathbb{P}(U_r = 0) = \frac{1}{2}.$$

In particular, for distinct r and s,

$$\mathbb{P}(U_r = 1, \, U_s = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Next, note that for $v \in [0,1]$ we have

$$\mathbb{P}(V \leq v) = \mathbb{P}(\sqrt{U} \leq v) = \mathbb{P}(U \leq v^2) = v^2.$$

We now compute probabilities related to the binary digits of V. For the first digit,

$$\mathbb{P}(V_1 = 1) = \mathbb{P}\Big(V > \frac{1}{2}\Big) = 1 - \mathbb{P}\Big(V \leq \frac{1}{2}\Big) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

For the second digit,

$$\begin{aligned}
\mathbb{P}(V_2 = 1) &= \mathbb{P}\Big(V \in \Big(\frac{1}{4}, \frac{1}{2}\Big) \cup \Big(\frac{3}{4}, 1\Big)\Big) \\
&= \left[\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{3}{4}\right)^2\right] \\
&= \left[\frac{1}{4} - \frac{1}{16}\right] + \left[1 - \frac{9}{16}\right] = \frac{3}{16} + \frac{7}{16} = \frac{5}{8}.
\end{aligned}$$

The joint probability that both V_1 and V_2 equal 1 is

$$\mathbb{P}(V_1 = 1, \, V_2 = 1) = \mathbb{P}\Big(V \in \Big(\frac{3}{4}, 1\Big)\Big) = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}.$$

Thus, the covariance between V_1 and V_2 is

$$\begin{aligned}
\text{cov}(V_1, V_2) &= \mathbb{E}[V_1 V_2] - \mathbb{E}[V_1]\mathbb{E}[V_2] \\
&= \frac{7}{16} - \left(\frac{3}{4} \cdot \frac{5}{8}\right) = \frac{7}{16} - \frac{15}{32} = -\frac{1}{32}.
\end{aligned}$$

Finally, consider the nth binary digit V_n. Its probability of being 1 is given by

$$\begin{aligned}\mathbb{P}(V_n = 1) &= \sum_{k=1}^{2^{n-1}} \left[\mathbb{P}\Big(V \le \frac{2k}{2^n}\Big) - \mathbb{P}\Big(V \le \frac{2k-1}{2^n}\Big)\right] \\ &= \sum_{k=1}^{2^{n-1}} \left[\left(\frac{2k}{2^n}\right)^2 - \left(\frac{2k-1}{2^n}\right)^2\right].\end{aligned}$$

A short calculation shows that $\mathbb{P}(V_n = 1) = \frac{1}{2}$ for large n. Hence,

$$\lim_{n\to\infty} \mathbb{P}(V_n = 1) = \frac{1}{2}.$$

This means that as n increases, the binary digits of V behave as i.i.d. Bernoulli$\left(\frac{1}{2}\right)$ random variables.

Problem 6

Let X, Y be two random variables with finite expectations such that $\mathbb{E}(X|Y) \ge Y$ and $\mathbb{E}(Y|X) \ge X$, prove that $X = Y$ almost surely.

Proof. First, take expectations on both sides of the inequality $\mathbb{E}[X \mid Y] \ge Y$:

$$\begin{aligned}\mathbb{E}\big[\mathbb{E}[X \mid Y]\big] &\ge \mathbb{E}[Y] \\ \mathbb{E}[X] &\ge \mathbb{E}[Y].\end{aligned}$$

Similarly, using $\mathbb{E}[Y \mid X] \ge X$ we obtain:

$$\mathbb{E}[Y] \ge \mathbb{E}[X].$$

Thus,

$$\mathbb{E}[X] = \mathbb{E}[Y].$$

Define $Z = X - Y$. Then, conditioning on Y we have:

$$\begin{aligned}\mathbb{E}[Z \mid Y] &= \mathbb{E}[X - Y \mid Y] \\ &= \mathbb{E}[X \mid Y] - Y \ge 0 \quad \text{a.s.}\end{aligned}$$

Likewise, by conditioning on X we get:

$$\begin{aligned}\mathbb{E}[-Z \mid X] &= \mathbb{E}[Y - X \mid X] \\ &= \mathbb{E}[Y \mid X] - X \ge 0 \quad \text{a.s.}\end{aligned}$$

Taking expectations, the law of iterated expectations implies:

$$\mathbb{E}[Z] = \mathbb{E}\big[\mathbb{E}[Z \mid Y]\big] \geq 0,$$
$$\mathbb{E}[-Z] = \mathbb{E}\big[\mathbb{E}[-Z \mid X]\big] \geq 0.$$

But since $\mathbb{E}[Z] = \mathbb{E}[X] - \mathbb{E}[Y] = 0$, it follows that:

$$\mathbb{E}[Z] = 0.$$

Now, since $\mathbb{E}[Z \mid Y] \geq 0$ almost surely and

$$\mathbb{E}[Z] = \mathbb{E}\big[\mathbb{E}[Z \mid Y]\big] = 0,$$

we must have $\mathbb{E}[Z \mid Y] = 0$ almost surely. But $\mathbb{E}[Z \mid Y] \geq 0$ forces

$$\mathbb{E}[Z \mid Y] = 0 \quad \text{a.s.}$$

This in turn implies that $Z = X - Y = 0$ almost surely. Therefore,

$$X = Y \quad \text{a.s.}$$

□

Problem 7

Let c_n denote the number of n-step self-avoiding walks starting from the origin in $\mathbb{Z}^d$. Show that the limit $\mu = \lim_{n\to\infty} c_n^{\frac{1}{n}}$ exists. μ is called the *connectivity constant* of self avoiding walk in $\mathbb{Z}^d$.

Hint: you may use the fact that subadditive sequence has a limit: if $a_{n+m} \leq a_m + a_n$ for every $m, n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists.

Proof. we first observe that the sequence $\{c_n\}$ is submultiplicative. In fact, one can show that for all $m, n \in \mathbb{N}$,

$$c_{n+m} \leq c_n \, c_m.$$

Now, taking the natural logarithm of both sides gives

$$\ln c_{n+m} \leq \ln c_n + \ln c_m.$$

Thus, if we define

$$a_n = \ln c_n,$$

then $\{a_n\}$ is a subadditive sequence; that is, for all m, n,

$$a_{n+m} \leq a_n + a_m.$$

By Fekete's Lemma for subadditive sequences, we have that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

Let

$$\lambda = \lim_{n\to\infty} \frac{\ln c_n}{n}.$$

Then, exponentiating both sides yields

$$\lim_{n\to\infty} c_n^{1/n} = \exp\left(\lim_{n\to\infty} \frac{\ln c_n}{n}\right) = e^{\lambda} = \mu.$$

Hence, the limit $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists and is known as the connectivity constant of self-avoiding walks in $\mathbb{Z}^d$.

□

Problem 8

Show that the connectivity constant in $\mathbb{Z}^2$ satisfies $2 \leq \mu \leq 3$.

Proof. **Upper Bound:**

At the first step from the origin there are 4 possible directions. For each subsequent step, a self-avoiding walk cannot immediately return to the vertex it came from and, in general, has at most 3 choices. Thus, for $n \geq 1$ we have

$$c_n \leq 4 \cdot 3^{n-1}.$$

Taking the nth root yields

$$c_n^{1/n} \leq \left(4 \cdot 3^{n-1}\right)^{1/n} = 4^{1/n}\, 3^{1-1/n}.$$

Letting $n \to \infty$, we note that $4^{1/n} \to 1$ and $3^{1-1/n} \to 3$. Hence,

$$\limsup_{n\to\infty} c_n^{1/n} \leq 3,$$

so that $\mu \leq 3$.

Lower Bound:

Consider the family of self-avoiding walks that at each step move only in the positive x (East) or positive y (North) direction. Such walks are monotone and clearly self-avoiding. At each of the n steps, there are exactly 2 choices, so that the number of these walks is

$$2^n.$$

Since these are self-avoiding walks, we have

$$c_n \geq 2^n.$$

Taking the nth root gives

$$c_n^{1/n} \geq 2,$$

and hence,

$$\liminf_{n\to\infty} c_n^{1/n} \geq 2,$$

which implies $\mu \geq 2$.

Combining the two bounds, we conclude that

$$2 \leq \mu \leq 3.$$

□