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Chi distribution

In probability theory and statistics, the **chi distribution** is a continuous probability distribution. It is the distribution of the positive square root of the sum of squares of a set of independent random variables each following a standard normal distribution, or equivalently, the distribution of the Euclidean distance of the random variables from the origin. It is thus related to the chi-squared distribution by describing the distribution of the positive square roots of a variable obeying a chi-squared distribution.

The most familiar examples are the Rayleigh distribution (chi distribution with two degrees of freedom) and the Maxwell–Boltzmann distribution of the molecular speeds in an ideal gas (chi distribution with three degrees of freedom).

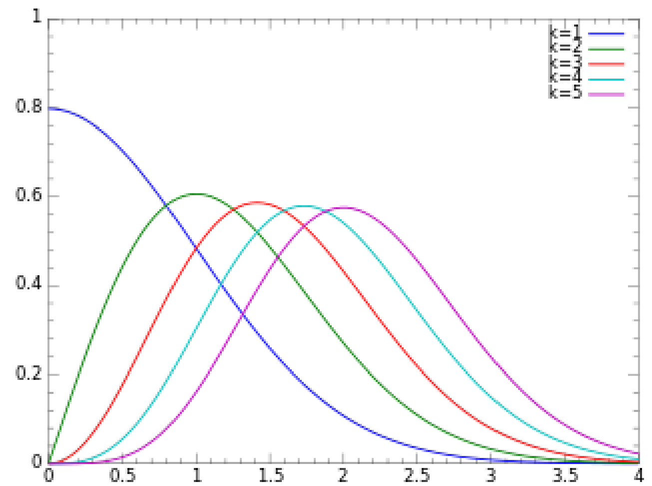
If Z_i are k independent, normally distributed random variables with mean 0 and standard deviation 1, then the statistic

$$Y = \sqrt{\sum_{i=1}^{k} Z_i^2}$$

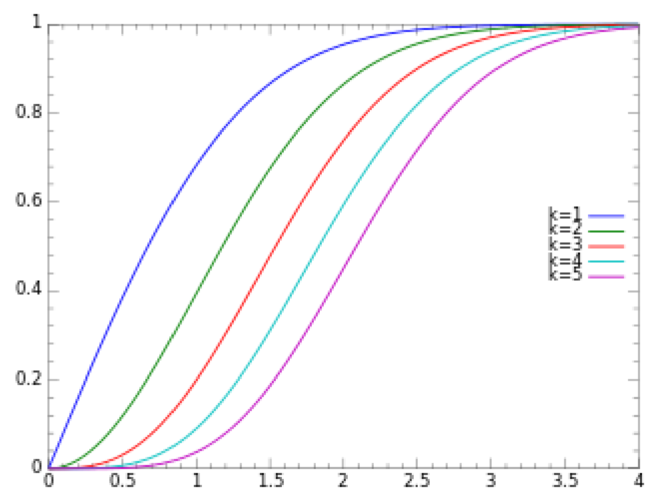
is distributed according to the chi distribution. Accordingly, dividing by the mean of the chi distribution (scaled by the square root of n − 1) yields the correction factor in the unbiased estimation of the standard deviation of the normal distribution. The chi distribution has one parameter: k which specifies the number of degrees of freedom (i.e. the number of X_i).

chi

Probability density function



Cumulative distribution function



Parameters	$k > 0$ (degrees of freedom)
Support	$x \in [0, \infty)$
PDF	$\frac{1}{2^{(k/2)-1}\Gamma(k/2)}\ x^{k-1}e^{-x^2/2}$
CDF	$P(k/2, x^2/2)$
Mean	$\mu = \sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}$
Median	$\approx \sqrt{k\left(1-\frac{2}{9k}\right)^3}$
Mode	$\sqrt{k-1}$ for $k \geq 1$
Variance	$\sigma^2 = k - \mu^2$

Contents

Skewness	$\gamma_1 = \frac{\mu}{\sigma^3}\,(1-2\sigma^2)$
Ex. kurtosis	$\frac{2}{\sigma^2}(1-\mu\sigma\gamma_1-\sigma^2)$
Entropy	$\ln(\Gamma(k/2)) +$ $\frac{1}{2}(k-\ln(2)-(k-1)\psi_0(k/2))$
MGF	Complicated (see text)
CF	Complicated (see text)

Characterization

Probability density function

The probability density function (pdf) of the chi-distribution is

$$f(x;k) = \begin{cases} \dfrac{x^{k-1}e^{-x^2/2}}{2^{k/2-1}\Gamma\left(\frac{k}{2}\right)}, & x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

where $\Gamma(z)$ is the gamma function.

Cumulative distribution function

The cumulative distribution function is given by:

$$F(x;k) = P(k/2, x^2/2)$$

where $P(k,x)$ is the regularized gamma function.

Generating functions

The moment-generating function is given by:

$$M(t) = M\left(\frac{k}{2},\frac{1}{2},\frac{t^2}{2}\right) + t\sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}M\left(\frac{k+1}{2},\frac{3}{2},\frac{t^2}{2}\right),$$

where $M(a,b,z)$ is Kummer's confluent hypergeometric function. The characteristic function is given by:

$$\varphi(t;k) = M\left(\frac{k}{2},\frac{1}{2},\frac{-t^2}{2}\right) + it\sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}M\left(\frac{k+1}{2},\frac{3}{2},\frac{-t^2}{2}\right).$$

Properties

Moments

The raw moments are then given by:

$$\mu_j = 2^{j/2}\frac{\Gamma((k+j)/2)}{\Gamma(k/2)}$$

where $\Gamma(z)$ is the gamma function. The first few raw moments are:

$$\mu_1 = \sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}$$
$$\mu_2 = k$$
$$\mu_3 = 2\sqrt{2}\,\frac{\Gamma((k+3)/2)}{\Gamma(k/2)} = (k+1)\mu_1$$
$$\mu_4 = (k)(k+2)$$
$$\mu_5 = 4\sqrt{2}\,\frac{\Gamma((k+5)/2)}{\Gamma(k/2)} = (k+1)(k+3)\mu_1$$
$$\mu_6 = (k)(k+2)(k+4)$$

where the rightmost expressions are derived using the recurrence relationship for the gamma function:

$$\Gamma(x+1) = x\Gamma(x)$$

From these expressions we may derive the following relationships:

Mean: $\mu = \sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}$

Variance: $\sigma^2 = k - \mu^2$

Skewness: $\gamma_1 = \frac{\mu}{\sigma^3}\,(1-2\sigma^2)$

Kurtosis excess: $\gamma_2 = \frac{2}{\sigma^2}(1-\mu\sigma\gamma_1-\sigma^2)$

Entropy

The entropy is given by:

$$S = \ln(\Gamma(k/2)) + \frac{1}{2}(k - \ln(2) - (k-1)\psi^0(k/2))$$

where $\psi^0(z)$ is the polygamma function.

Related distributions

- If $X \sim \chi_k(x)$ then $X^2 \sim \chi^2_k$ (chi-squared distribution)
- $\lim_{k\to\infty} \frac{\chi_k(x)-\mu_k}{\sigma_k} \xrightarrow{d} N(0,1)$ (Normal distribution)
- If $X \sim N(0,1)$ then $|X| \sim \chi_1(x)$
- If $X \sim \chi_1(x)$ then $\sigma X \sim HN(\sigma)$ (half-normal distribution) for any $\sigma > 0$
- $\chi_2(x) \sim \mathrm{Rayleigh}(1)$ (Rayleigh distribution)
- $\chi_3(x) \sim \mathrm{Maxwell}(1)$ (Maxwell distribution)
- $\|\boldsymbol{N}_{i=1,\ldots,k}(0,1)\|_2 \sim \chi_k(x)$ (The 2-norm of k standard normally distributed variables is a chi distribution with k degrees of freedom)

- chi distribution is a special case of the generalized gamma distribution or the Nakagami distribution or the noncentral chi distribution

Various chi and chi-squared distributions

Name	Statistic
chi-squared distribution	$\sum_{i=1}^{k}\left(\frac{X_i-\mu_i}{\sigma_i}\right)^2$
noncentral chi-squared distribution	$\sum_{i=1}^{k}\left(\frac{X_i}{\sigma_i}\right)^2$
chi distribution	$\sqrt{\sum_{i=1}^{k}\left(\frac{X_i-\mu_i}{\sigma_i}\right)^2}$
noncentral chi distribution	$\sqrt{\sum_{i=1}^{k}\left(\frac{X_i}{\sigma_i}\right)^2}$

See also

- Nakagami distribution

References

- Martha L. Abell, James P. Braselton, John Arthur Rafter, John A. Rafter, Statistics with Mathematica (1999), 237f. (https://books.google.co.uk/books?id=k3rkxOURuOMC&pg=PA237)
- Jan W. Gooch, Encyclopedic Dictionary of Polymers vol. 1 (2010), Appendix E, p. 972 (https://books.google.co.uk/books?id=HRgy8iHQtdwC&pg=PA972).

External links

- http://mathworld.wolfram.com/ChiDistribution.html