Chi-Square Distributions

Chi-squared distributions are very important distributions in the field of statistics. As such, if you go on to take the sequel course, Stat 415, you will encounter the chi-squared distributions quite regularly. In this course, we'll focus just on introducing the basics of the distributions to you. In Stat 415, you'll see its many applications.

As it turns out, the chi-square distribution is just a special case of the gamma distribution! Let's take a look.

Definition. Let X follow a gamma distribution with $\theta = 2$ and $\alpha = r/2$, where r is a positive integer. Then the probability density function of X is:

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}$$

for $x > 0$. We say that X follows a **chi-square distribution with** r **degrees of freedom**, denoted $\chi^2(r)$ and read "chi-square-r."

There are, of course, an infinite number of possible values for r, the degrees of freedom. Therefore, there are an infinite number of possible chi-square distributions. That is why (again!) the title of this page is called Chi-Square Distributions (with an s!), rather than Chi-Square Distribution (with no s)!

As the following theorems illustrate, the moment generating function, mean and variance of the chi-square distributions are just straightforward extensions of those for the gamma distributions.

Theorem. Let X be a chi-square random variable with r degrees of freedom. Then, the **moment generating function of** X is:

$$M(t) = \frac{1}{(1-2t)^{r/2}}$$

for $t < ½$.

Proof. The moment generating function of a gamma random variable is:

$$M(t)=\frac{1}{(1-\theta t)^{\alpha}}$$

The proof is therefore straightforward by substituting 2 in for θ and r/2 in for α.

Theorem. Let X be a chi-square random variable with r degrees of freedom. Then, the **mean of** X is:

$$\mu=E(X)=r$$

That is, the mean of X is the number of degrees of freedom.

Proof. The mean of a gamma random variable is:

$$\mu=E(X)=\alpha\theta$$

The proof is again straightforward by substituting 2 in for θ and r/2 in for α.

Theorem. Let X be a chi-square random variable with r degrees of freedom. Then, the **variance of** X is:

$$\sigma^2=Var(X)=2r$$

That is, the variance of X is twice the number of degrees of freedom.

Proof. The variance of a gamma random variable is:

$$\sigma^2=Var(X)=\alpha\theta^2$$

The proof is again straightforward by substituting 2 in for θ and r/2 in for α.

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