1. Find the equation of the plane through the points (2, 1, -1), (0, -2, 0), and (1, -1, 2).

We can get two vectors in the plane by subtracting pairs of points in the plane:

$$
\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
$$

The cross product of these two vectors will be in the unique direction orthogonal to both, and hence in the direction of the normal vector to the plane.

$$
\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ -1 \end{bmatrix}
$$

The equation of a plane is $ax + by + cz = d$, where $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the normal vector to the plane. We can plug in $a = 7$, $b = -5$, and $c = -1$ to get $7x - 5y - z = d$. Plug in the point $(0, -2, 0)$ to get $7(0) - 5(-2) - 0 = d$, or $d = 10$. The equation of the plane is hence $7x - 5y - z = 10$.

2. Solve the following system of equations:

$$\begin{aligned}
a + 3b - 2c - 2d &= 11 \\
-a - 2b + 4c + 5d &= -4 \\
-2a - 4b + 9c + 11d &= -7
\end{aligned}$$

We set up the augmented matrix, and then put the left side in reduced row echelon form.

$$\left[\begin{array}{cccc|c}
1 & 3 & -2 & -2 & 11 \\
-1 & -2 & 4 & 5 & -4 \\
-2 & -4 & 9 & 11 & -7
\end{array}\right]\begin{array}{l} . \\ +R1 \\ +2R1 \end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 3 & -2 & -2 & 11 \\
0 & 1 & 2 & 3 & 7 \\
0 & 2 & 5 & 7 & 15
\end{array}\right]\begin{array}{l} -3R2 \\ . \\ -2R2 \end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 0 & -8 & -11 & -10 \\
0 & 1 & 2 & 3 & 7 \\
0 & 0 & 1 & 1 & 1
\end{array}\right]\begin{array}{l} +8R3 \\ -2R3 \\ . \end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 0 & 0 & -3 & -2 \\
0 & 1 & 0 & 1 & 5 \\
0 & 0 & 1 & 1 & 1
\end{array}\right]$$

d is a free variable. We can now read off the answer:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

3. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ linearly independent?

We can create a matrix whose columns are the vectors given and put it in row echelon form.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{array}{l} . \\ -R1 \\ -R1 \\ -R1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 3 & -3 & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ -2R2 \\ -3R2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

This isn't quite in row echelon form yet, but we've gone far enough that it is clear that the third column will not be a pivot column. Therefore, the vectors are linearly dependent.

Incidentally, the linear combination of vectors that proves that they are linearly dependent is

$$5 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

You were not required to find this to do the problem.

4. Determine whether the following matrix is invertible, and if so, find its inverse:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ 5 & 2 & -2 \end{bmatrix}$$

We put the original matrix on the left side of an augmented matrix, and put the identity matrix on the right side. We then put the left side into reduced row echelon form.

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 3 & 7 & 4 & 0 & 1 & 0 \\ 5 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} . \\ -R1 \\ . \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 1 & 0 \\ 5 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \text{swap} \\ \text{swap} \\ . \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ 5 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} . \\ -2R1 \\ -5R1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ 0 & -8 & -7 & 5 & -5 & 1 \end{array}\right] \begin{array}{l} -2R2 \\ . \\ +8R2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 5 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 29 & -21 & 1 \end{array}\right] \begin{array}{l} +R3 \\ -R3 \\ . \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 22 & -16 & 1 \\ 0 & 1 & 0 & -26 & 19 & -1 \\ 0 & 0 & 1 & 29 & -21 & 1 \end{array}\right]$$

Thus, the matrix is invertible, with inverse

$$\begin{bmatrix} 22 & -16 & 1 \\ -26 & 19 & -1 \\ 29 & -21 & 1 \end{bmatrix}$$

5. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$. Compute $(\mathbf{u} + \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}) \cdot \mathbf{u}$.

By the definition of projection, $\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}$ is orthogonal to $\mathbf{u}$. As such, $(\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}) \cdot \mathbf{u} = 0$. We can then compute

$$
\begin{aligned}
& (\mathbf{u} + \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}) \cdot \mathbf{u} \\
=\ & \mathbf{u} \cdot \mathbf{u} + (\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}) \cdot \mathbf{u} \\
=\ & ||\mathbf{u}||^2 + 0 \\
=\ & 3^2 + 4^2 + (-1)^2 + 2^2 \\
=\ & 9 + 16 + 1 + 4 = 30
\end{aligned}
$$

6. Exactly one of these matrices is invertible. Which one? Be sure to justify your answer.

$$
\begin{bmatrix} 2 & 3 & 1 & 5 \\ 3 & 9 & 0 & -1 \\ 4 & -3 & -1 & 2 \end{bmatrix}
\qquad
\begin{bmatrix} 3 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\qquad
\begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & -5 \\ 0 & 6 & 2 \end{bmatrix}
$$

The left matrix is not square, and hence not invertible. The center matrix is given to us in row echelon form, and we can see that the right column is not a pivot column. As such, it is not invertible, either. That leaves the right matrix as the only possible invertible matrix. You weren't required to compute this, but the inverse of the right matrix is

$$
\frac{1}{32}\begin{bmatrix} 28 & -6 & -15 \\ -8 & 4 & 10 \\ 24 & -12 & -14 \end{bmatrix}
$$