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**Lecture 9 Notes**

These notes correspond to Section 2.2 in the text.

## Fixed-point Iteration

A nonlinear equation of the form $f(x) = 0$ can be rewritten to obtain an equation of the form

$$g(x) = x,$$

in which case the solution is a *fixed point* of the function $g$. This formulation of the original problem $f(x) = 0$ will leads to a simple solution method known as *fixed-point iteration.* Before we describe this method, however, we must first discuss the questions of existence and uniqueness of a solution to the modified problem $g(x) = x$. The following result answers these questions.

**Theorem** *Let $g$ be a continuous function on the interval $[a, b]$. If $g(x) \in [a, b]$ for each $x \in [a, b]$, then $g$ has a fixed point in $[a, b]$. Furthermore, if $g$ is differentiable on $(a, b)$ and there exists a constant $k < 1$ such that*

$$|g'(x)| \leq k, \quad x \in (a, b),$$

*then $g$ has exactly one fixed point in $[a, b]$.*

Given a continuous function $g$ that is known to have a fixed point in an interval $[a, b]$, we can try to find this fixed point by repeatedly evaluating $g$ at points in $[a, b]$ until we find a point $x$ for which $g(x) = x$. This is the essence of the method of fixed-point iteration, the implementation of which we now describe.

**Algorithm** (Fixed-Point Iteration) Let $g$ be a continuous function defined on the interval $[a, b]$. The following algorithm computes a number $x^* \in (a, b)$ that is a solution to the equation $g(x) = x$.

Choose an initial guess $x_0$ in $[a, b]$.
**for** $k = 0, 1, 2, \ldots$ **do**
  $x_{k+1} = g(x_k)$
  **if** $|x_{k+1} - x_k|$ is sufficiently small **then**
    $x^* = x_{k+1}$
    **return** $x^*$
  **end**
**end**

Under what circumstances will fixed-point iteration converge to the exact solution $x^*$? If we denote the error in $x_k$ by $e_k = x_k - x^*$, we can see from Taylor's Theorem and the fact that $g(x^*) = x^*$ that $e_{k+1} \approx g'(x^*)e_k$. Therefore, if $|g'(x^*)| \leq k$, where $k < 1$, then fixed-point iteration is *locally convergent*; that is, it converges if $x_0$ is chosen sufficiently close to $x^*$. This leads to the following result.

**Theorem** *(Fixed-Point Theorem) Let $g$ be a continuous function on the interval $[a, b]$. If $g(x) \in [a, b]$ for each $x \in [a, b]$, and if there exists a constant $k < 1$ such that*

$$|g'(x)| \leq k, \quad x \in (a, b),$$

*then the sequence of iterates $\{x_k\}_{k=0}^{\infty}$ converges to the unique fixed point $x^*$ of $g$ in $[a, b]$, for any initial guess $x_0 \in [a, b]$.*

It can be seen from the preceding discussion why $g'(x)$ must be bounded away from 1 on $(a, b)$, as opposed to the weaker condition $|g'(x)| < 1$ on $(a, b)$. If $g'(x)$ is allowed to approach 1 as $x$ approaches a point $c \in (a, b)$, then it is possible that the error $e_k$ might not approach zero as $k$ increases, in which case fixed-point iteration would not converge.

In general, when fixed-point iteration converges, it does so at a rate that varies inversely with the constant $k$ that bounds $|g'(x)|$. In the extreme case where derivatives of $g$ are equal to zero at the solution $x^*$, the method can converge much more rapidly. We will discuss convergence behavior of various methods for solving nonlinear equations in a later lecture.

Often, there are many ways to convert an equation of the form $f(x) = 0$ to one of the form $g(x) = x$, the simplest being $g(x) = x - \phi(x)f(x)$ for any function $\phi$. However, it is important to ensure that the conversion yields a function $g$ for which fixed-point iteration will converge.

**Example** We use fixed-point iteration to compute a fixed point of $g(x) = \cos x$ in the interval $[0, 1]$. Since $|\cos x| \leq 1$ for all $x$, and $\cos x \geq 0$ on $[0, \pi/2]$, and $\pi/2 > 1$, we know that $\cos x$ maps $[0, 1]$ into $[0, 1]$. Since $\cos x$ is continuous for all $x$, we can conclude that $\cos x$ has a fixed point in $[0, 1]$. Because $g'(x) = -\sin x$, and $|-\sin x| \leq |-\sin 1| < 1$ on $[0, 1]$, we can also conclude that this fixed point is unique.

To use fixed-point iteration, we first choose an initial guess $x_0$ in $[0, 1]$. As discussed above, fixed-point iteration will converge for any initial guess, so we choose $x_0 = 0.5$. The table on page 4 shows the outcome of several iterations, in which we compute $x_{k+1} = \cos x_k$ for $k = 0, 1, 2, \ldots$. As the table shows, it takes nearly 30 iterations to obtain the fixed point to five decimal places, and there is considerable oscillation in the first iterations before a reasonable approximate solution is obtained. This oscillation is shown in Figure 1.

As $x_k$ converges, it can be seen from the table that the error is reduced by a factor of roughly 2/3 from iteration to iteration. In other words, fixed-point iteration, in this case, converges even more slowly than the bisection method. This suggests that $\cos x$ is a relatively poor choice for the iteration function $g(x)$ to solve the equation $g(x) = \cos x$. $\square$

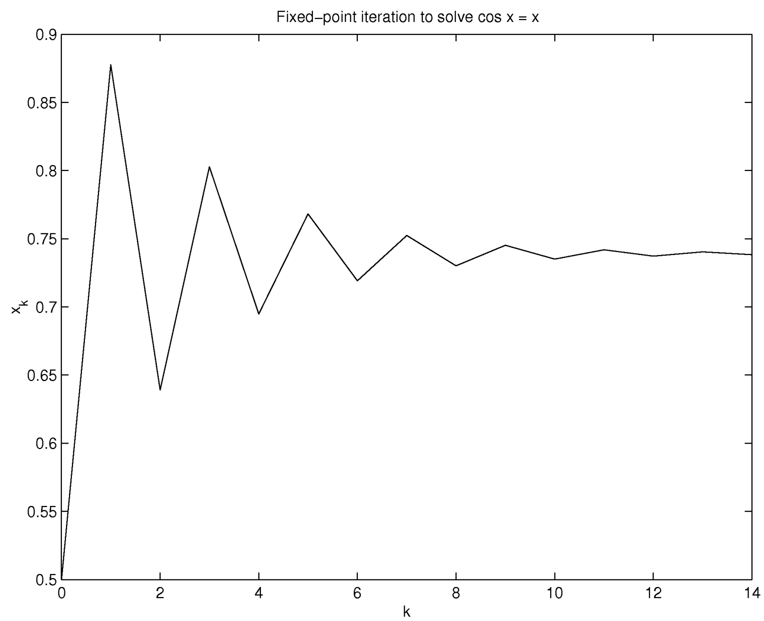


Figure 1: Fixed-point iteration applied to the equation $\cos x = x$, with $x_0 = 0.5$.

| $k$ | $x_k$ | $g(x_k)$ |
|---|---|---|
| 0 | 0.50000000000000 | 0.87758256189037 |
| 1 | 0.87758256189037 | 0.63901249416526 |
| 2 | 0.63901249416526 | 0.80268510068233 |
| 3 | 0.80268510068233 | 0.69477802678801 |
| 4 | 0.69477802678801 | 0.76819583128202 |
| 5 | 0.76819583128202 | 0.71916544594242 |
| 6 | 0.71916544594242 | 0.75235575942153 |
| 7 | 0.75235575942153 | 0.73008106313782 |
| 8 | 0.73008106313782 | 0.74512034135144 |
| 9 | 0.74512034135144 | 0.73500630901484 |
| 10 | 0.73500630901484 | 0.74182652264325 |
| 11 | 0.74182652264325 | 0.73723572544223 |
| 12 | 0.73723572544223 | 0.74032965187826 |
| 13 | 0.74032965187826 | 0.73824623833223 |
| 14 | 0.73824623833223 | 0.73964996276966 |
| 15 | 0.73964996276966 | 0.73870453935698 |
| 16 | 0.73870453935698 | 0.73934145228121 |
| 17 | 0.73934145228121 | 0.73891244933210 |
| 18 | 0.73891244933210 | 0.73920144413580 |
| 19 | 0.73920144413580 | 0.73900677978081 |
| 20 | 0.73900677978081 | 0.73913791076229 |
| 21 | 0.73913791076229 | 0.73904958059521 |
| 22 | 0.73904958059521 | 0.73910908142053 |
| 23 | 0.73910908142053 | 0.73906900120401 |
| 24 | 0.73906900120401 | 0.73909599983575 |
| 25 | 0.73909599983575 | 0.73907781328518 |
| 26 | 0.73907781328518 | 0.73909006398825 |
| 27 | 0.73909006398825 | 0.73908181177811 |
| 28 | 0.73908181177811 | 0.73908737057104 |
| 29 | 0.73908737057104 | 0.73908362610348 |
| 30 | 0.73908362610348 | 0.73908614842288 |