Application of Helgason-Ludwig Consistency Condition in Limited Angle Tomography

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Outline

I. Helgason-Ludwig Consistency Condition
   a. Chebyshev Fourier transform
   b. Connection with other theorems

II. Application in Limited Angle Tomography
   a. Regression problem
   b. Numerical data experiments
   c. “Clinical” data experiments

III. Discussion and Conclusion

I. Helgason-Ludwig Consistency Condition

II.a Chebyshev Fourier transform

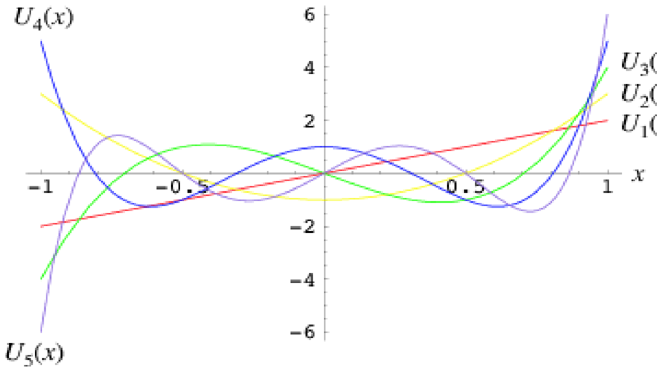
1. Chebyshev polynomials of the second kind

- Definition:

$$U_n(s) = \frac{\sin((n+1)\arccos(s))}{\sqrt{1-s^2}} \quad \left(= \frac{\sin((n+1)t)}{\sin(t)}, \text{if } s = \cos(t) \right)$$

- Orthogonal set with weight $(1-s^2)^{1/2}$:

$$\int_{-1}^{1} (1-s^2)^{1/2} U_n(s) U_m(s) ds = \begin{cases} 0, & n \neq m \\ \pi/2, & n = m \end{cases}$$



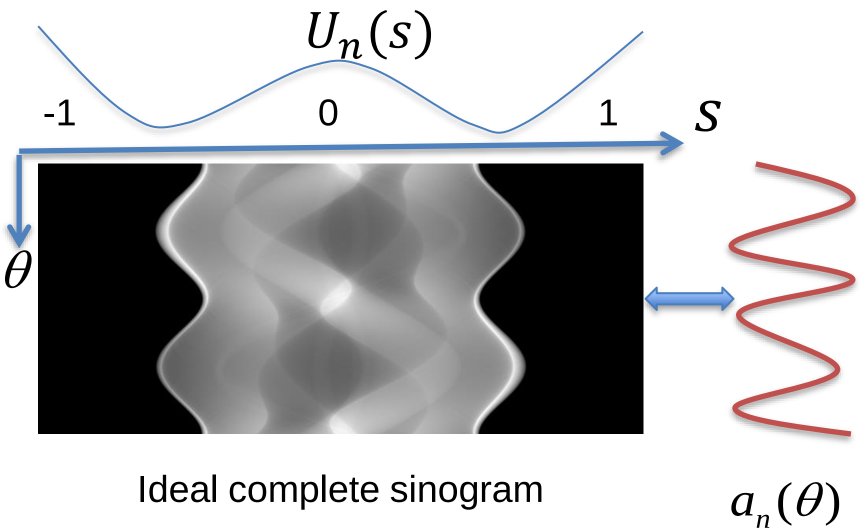
$U_n(s)$ (Wolfram MathWorld)

2. Moment curves

- n^{th} moment curve of the parallel-beam sinogram $p(s, \theta)$

$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$

- n: the order of $U_n(s)$ and the order of the moment curve



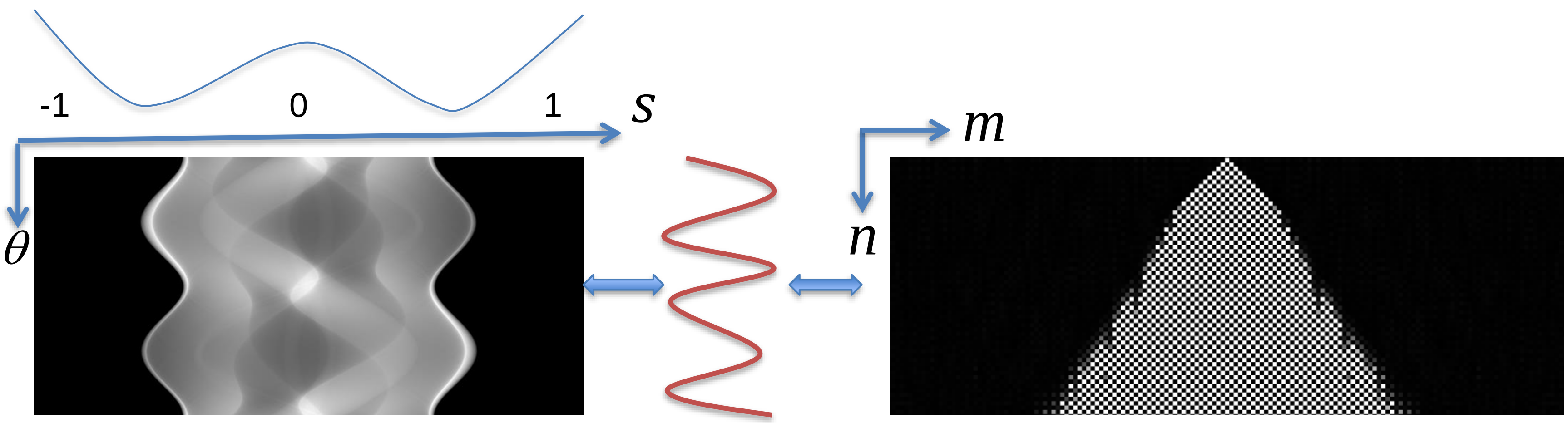
Ideal complete sinogram

3. Fourier transform of the moment curve

- Fourier transform:

$$a_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} a_n(\theta) d\theta$$

- **HLCC:** $a_{n,m} = 0 \quad |m| > n \text{ or } n + |m| \text{ is odd}$



Ideal complete sinogram

$a_n(\theta)$

Checkerboard pattern, $a_{n,m}$

4. Invertible Chebyshev Fourier transform

- Decomposition

$$a_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\int_{-1}^{1} p(s,\theta)U_n(s)dsd\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}a_n(\theta)d\theta$$

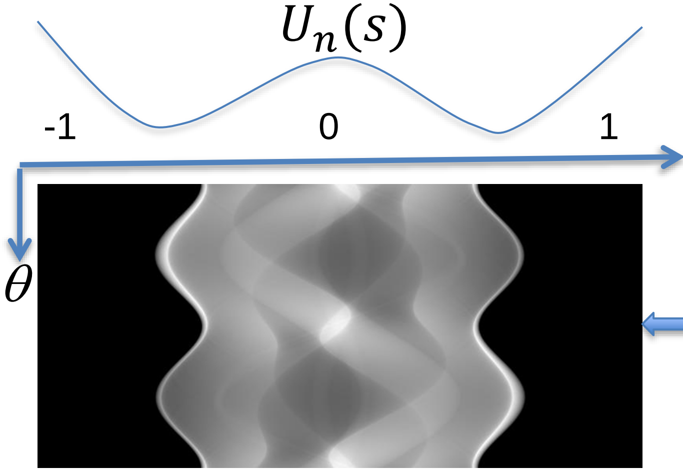
HLCC: $a_{n,m} = 0 \quad |m| > n \text{ or } n+|m| \text{ is odd}$

- Restoration

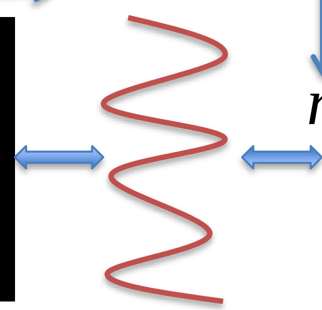
$$p(s,\theta) = \frac{2}{\pi}(1-s^2)^{1/2}\sum_{n=0}^{\infty}\boxed{\sum_{m=-n}^{n} a_{n,m}e^{im\theta}}U_n(s) = (1-s^2)^{1/2}\sum_{0}^{\infty}\boxed{a_n(\theta)}U_n(s)$$

5. Decomposition: checkerboard pattern

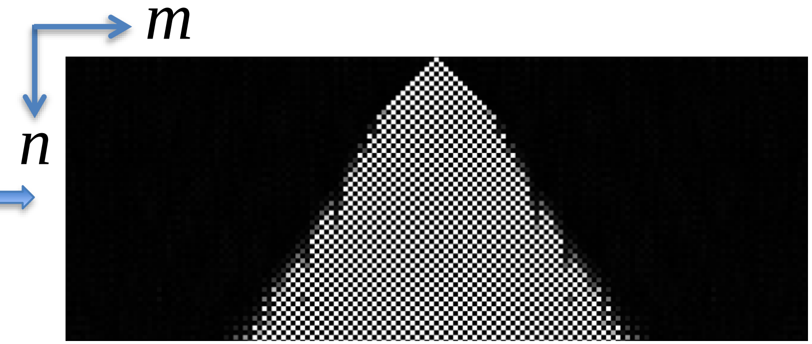
- HLCC: checkerboard pattern



Ideal complete sinogram



$a_n(\theta)$



Checkerboard pattern, zoom-in, window [0, 1]

Checkerboard pattern when the order n increases

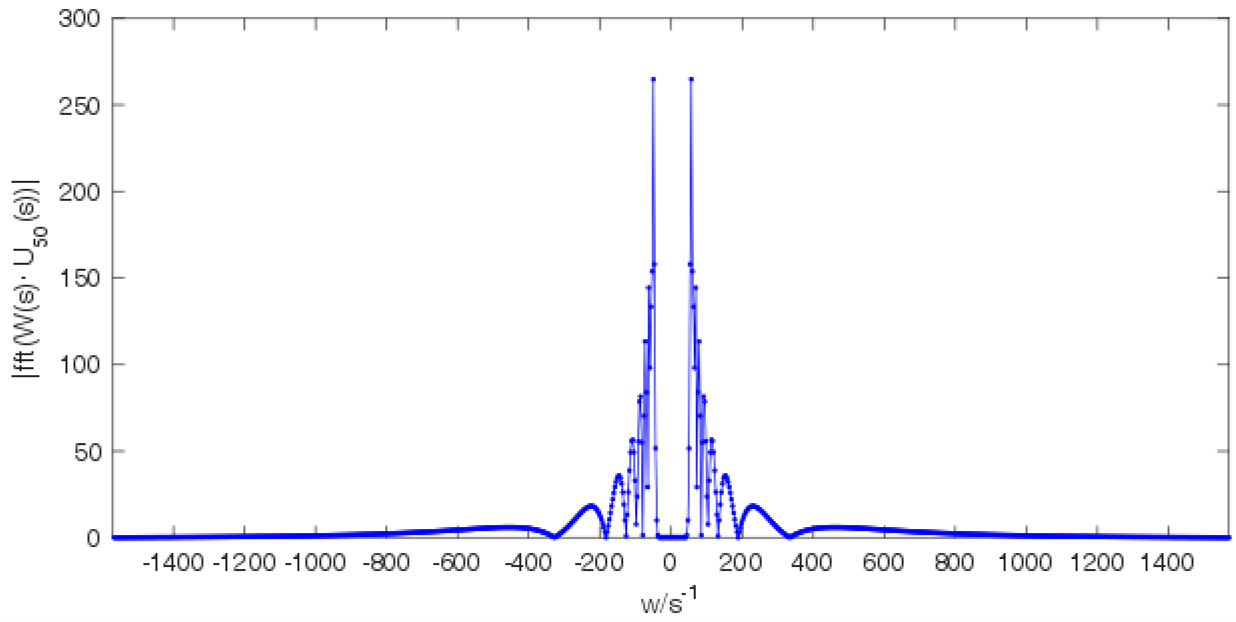
6. Restoration

- Higher orders add fine details to the restored sinogram

Restored sinograms when the order n increases

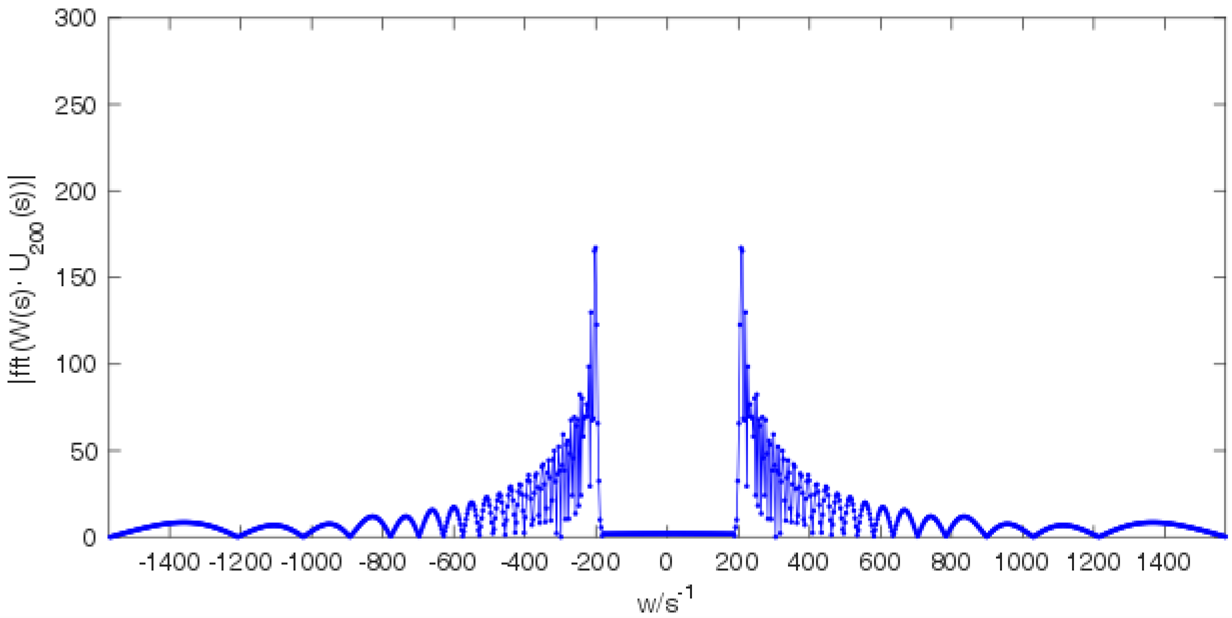
II.b Connection with other theorems

7. Fourier property of $U_n(s)$



FFT of $U_{50}(s)$

7. Fourier property of $U_n(s)$



FFT of $U_{200}(s)$

7. Fourier property of $U_n(s)$

- Each Chebyshev Polynomial $U_n(s)$ can be regarded as a band filter with the highest frequency response at n rad/s

$$F(W(s)\cdot U_n(s))(w)\begin{cases} =0, & 0\le w\le w_{L,n}, \\ >0, & w_{L,n}<w<w_{H,n}, \\ \approx 0, & w_{H,n}<w \end{cases}$$

where $w_{L,n}\approx n$ rad/s

- The Chebyshev polynomials are a compact orthogonal set, i.e., the lower orders of Chebyshev polynomials, $U_0(s) - U_{n-1}(s)$, will exactly cover the frequency band $[0, w_{L,n}]$

8. Connection 1: Fourier property of the sinogram

- Restored sinogram with orders 0 to $n-1$ cover the frequency band $[0, w_{L,n}]$ at w_s direction

Fourier transforms of the restored sinograms

9. Projection of the sinogram to Chebyshev space

- The moment curve is the orthogonal projection of the sinogram onto the nth basis of the space spanned by the Chebyshev polynomials

$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$

- Restored sinogram from a single order:

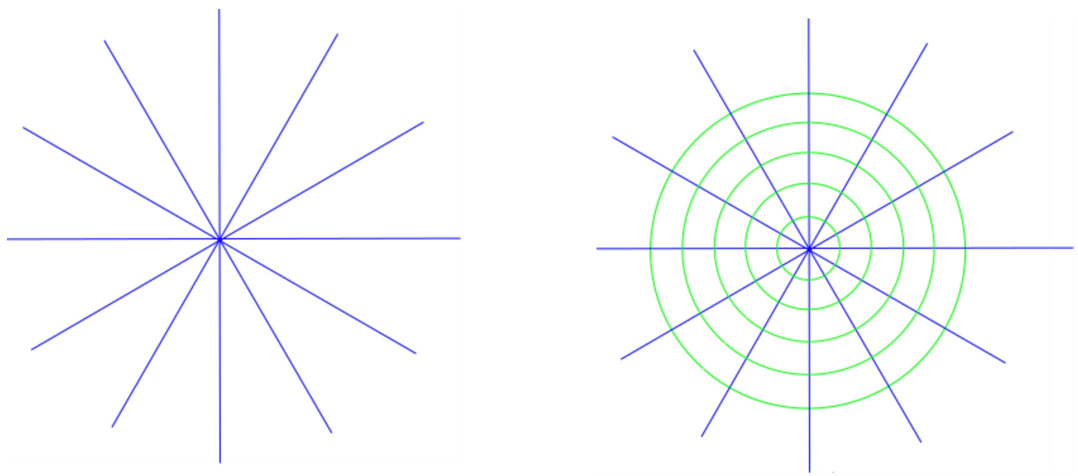
$$p_n(s,\theta) = (1 - s^2)^{1/2} a_n(\theta) U_n(s)$$

10. Connection 2: central slice theorem

- The central slice theorem:

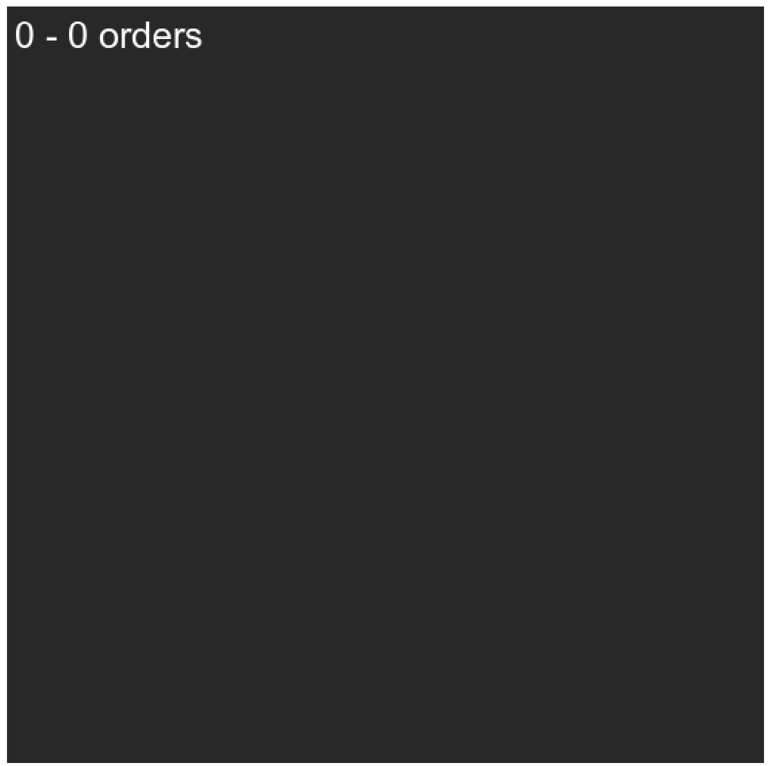
$$P(w,\theta) = F(w\cos\theta\,, w\sin\theta),$$

  where $P(w,\theta)$ is the 1-D Fourier transform of $p(s,\theta)$ w.r.t. s, $F(w_x, w_y)$ is the 2-D Fourier transform of the image $f(x,y)$
- $p_n(s,\theta)$ contains the exact frequency components at a circle of radius $w_{L,n}$



10. Connection 2: central slice theorem

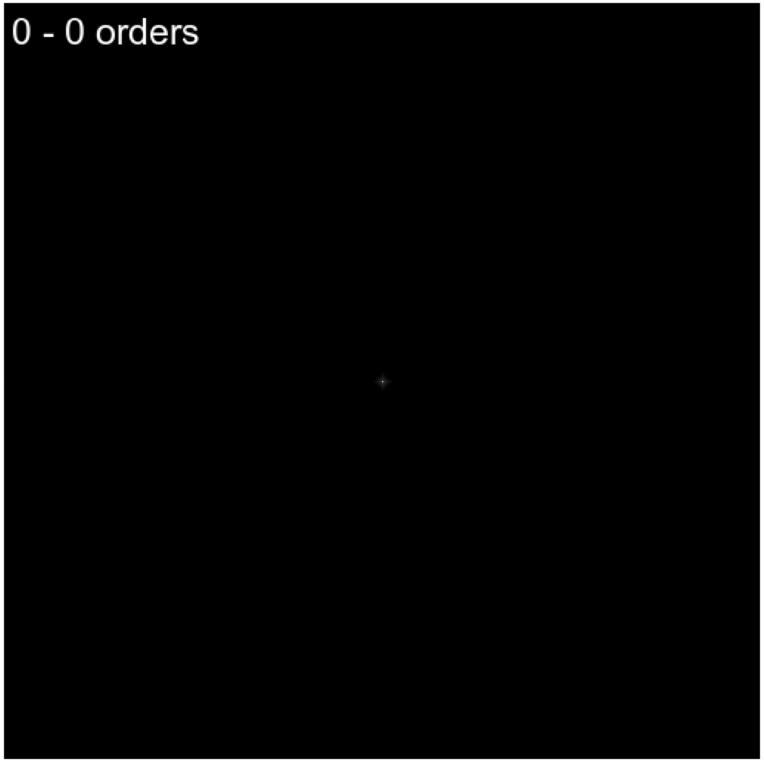
- Image reconstructions: more orders, less ringing artifacts

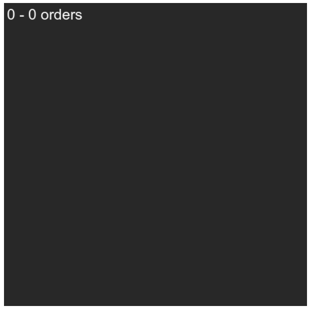


Reconstructed images when the order increases

11. Connection 2: central slice theorem

- More orders, more high frequency components



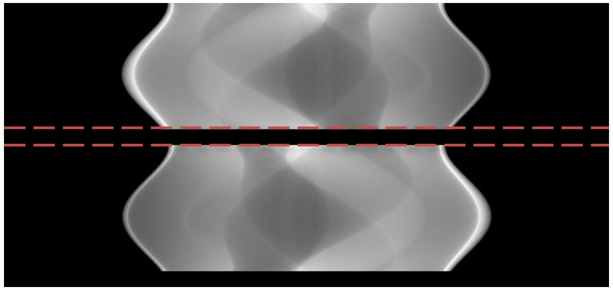


Fourier transforms of the reconstructed images

II. Application in Limited Angle Tomography

II.a Regression problem

1. Parallel-beam limited angle tomography

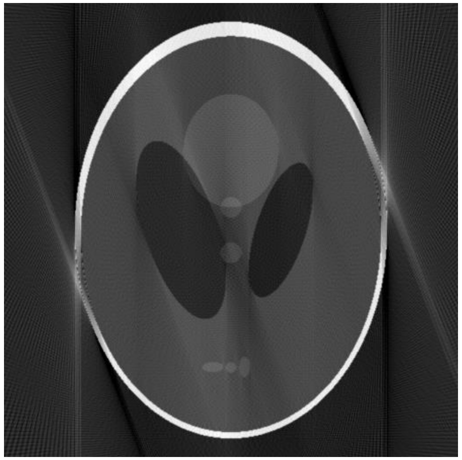


measured

Limited angle sinogram, 160º

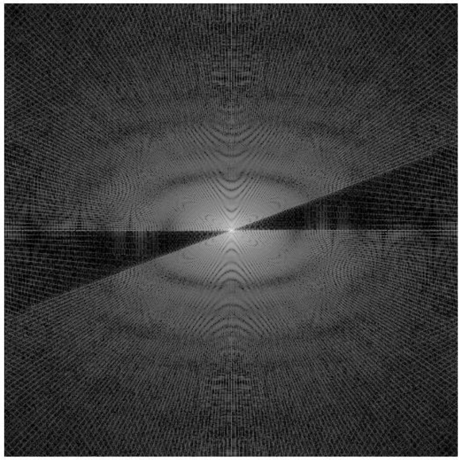
FBP

central slice theorem



2D FFT

2D IFFT



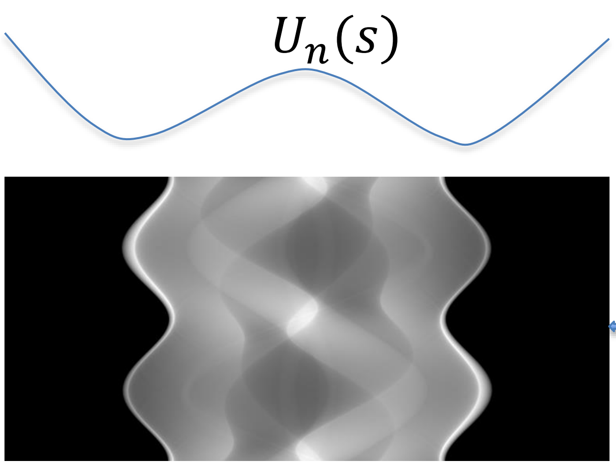
Limited angle reconstruction, [-0.2, 1.2]

Frequency components of the reconstruction

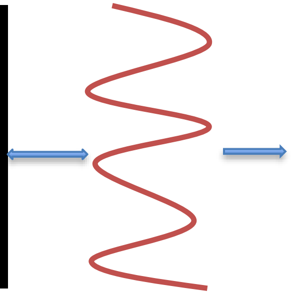
2. n^th moment curve, complete data

- Chebyshev transform, n^{th} moment curve:

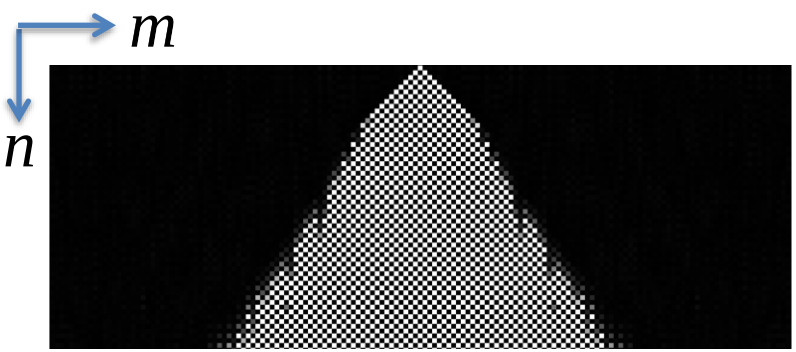
$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$



Ideal complete sinogram



$a_n(\theta)$

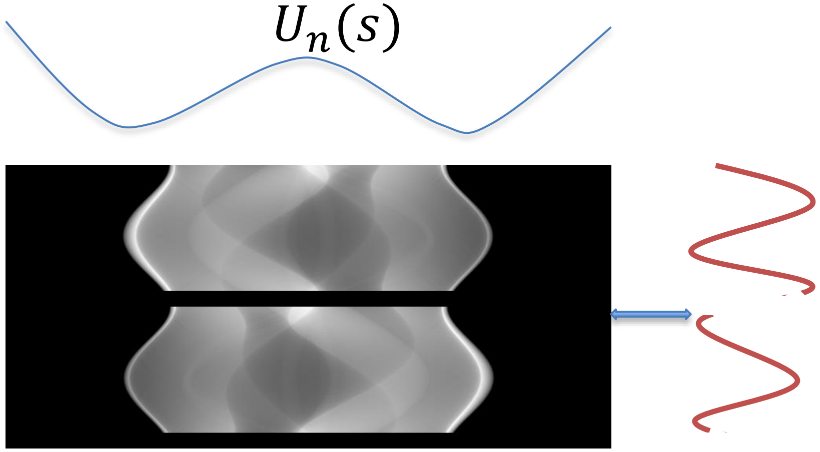


Checkerboard pattern, window [0, 1]

3. n^{th} moment curve, limited angle data

- Chebyshev transform, n^{th} moment curve:

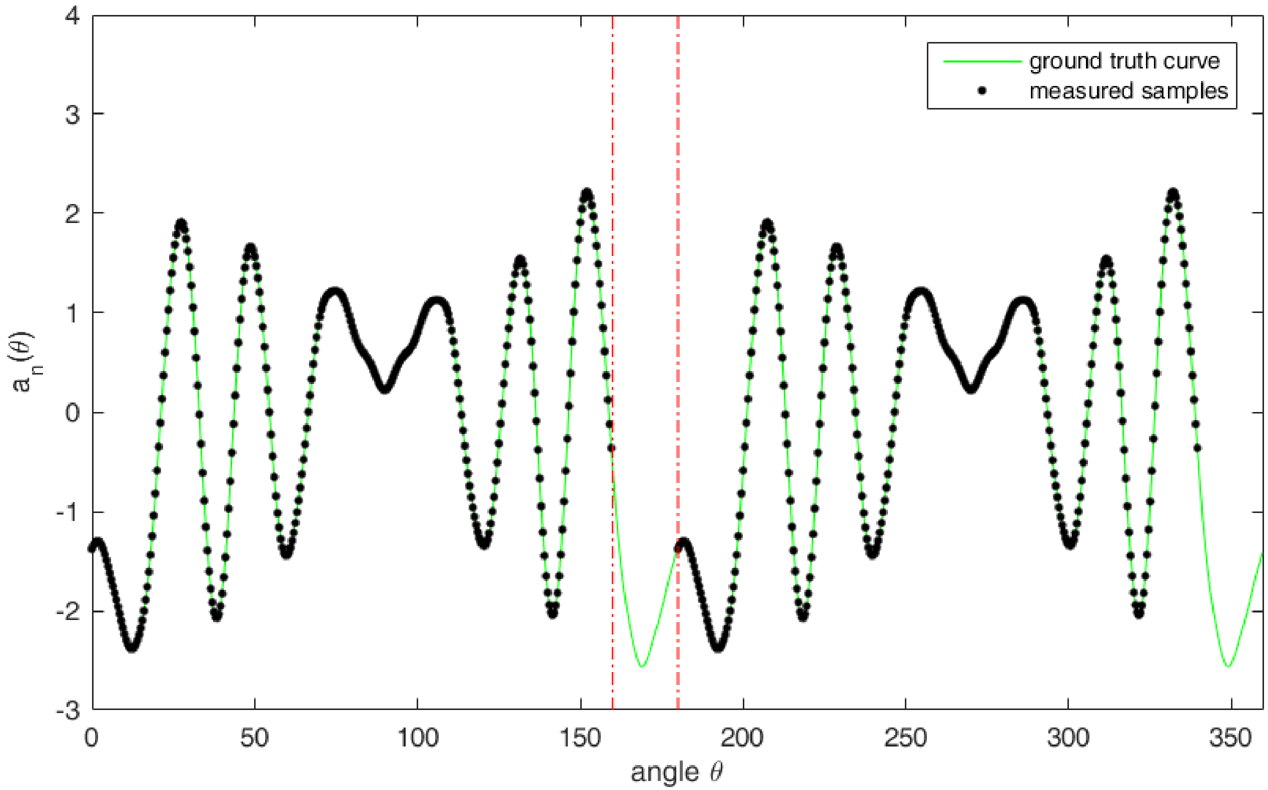
$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$



Limited angle parallel-beam sinogram,
160º measured

$a_n(\theta)$

4. n^{th} moment curve, e.g. n = 100



Curve fitting problem, n = 100

5. Analytical Form of $a_n(\theta)$

- Based on the checkerboard pattern, the analytical form of $a_n(\theta)$ is known (Fourier expansion)
- If n is even:

$$a_n(\theta) = b_0 + b_2\cos(2\theta) + c_2\sin(2\theta) + b_4\cos(4\theta) + c_4\sin(4\theta) + \ldots + b_n\cos(n\theta) + c_n\sin(n\theta)$$

- If n is odd:

$$a_n(\theta) = b_1\cos(\theta) + c_1\sin(\theta) + b_3\cos(3\theta) + c_3\sin(3\theta) + \ldots + b_n\cos(n\theta) + c_n\sin(n\theta)$$

- In both cases, $n + 1$ unknown parameters

6. Limit angle regression problem

- When n is even:

$$\begin{bmatrix} 1 & \cos(2\theta_0) & \sin(2\theta_0) & \cos(4\theta_0) & \sin(4\theta_0) & \dots & \cos(n\theta_0) & \sin(n\theta_0) \\ 1 & \cos(2\theta_1) & \sin(2\theta_1) & \cos(4\theta_1) & \sin(4\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(2\theta_2) & \sin(2\theta_2) & \cos(4\theta_2) & \sin(4\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \cdots \\ 1 & \cos(2\theta_{159}) & \sin(2\theta_{159}) & \cos(4\theta_{159}) & \sin(4\theta_{159}) & \dots & \cos(n\theta_{159}) & \sin(n\theta_{159}) \end{bmatrix} \begin{bmatrix} b_0 \\ b_2 \\ c_2 \\ b_4 \\ c_4 \\ \cdots \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_n(\theta_0) \\ a_n(\theta_1) \\ a_n(\theta_2) \\ \cdots \\ a_n(\theta_{158}) \\ a_n(\theta_{159}) \end{bmatrix}$$

- When n is odd:

$$\begin{bmatrix} \cos(\theta_0) & \sin(\theta_0) & \cos(3\theta_0) & \sin(3\theta_0) & \dots & \cos(n\theta_0) & \sin(n\theta_0) \\ \cos(\theta_1) & \sin(\theta_1) & \cos(3\theta_1) & \sin(3\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ \cos(\theta_2) & \sin(\theta_2) & \cos(3\theta_2) & \sin(3\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \cdots \\ \cos(\theta_{159}) & \sin(\theta_{159}) & \cos(3\theta_{159}) & \sin(3\theta_{159}) & \dots & \cos(n\theta_{159}) & \sin(n\theta_{159}) \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ b_3 \\ c_3 \\ \cdots \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_n(\theta_0) \\ a_n(\theta_1) \\ a_n(\theta_2) \\ \cdots \\ a_n(\theta_{158}) \\ a_n(\theta_{159}) \end{bmatrix}$$

7. Limited angle regression problem

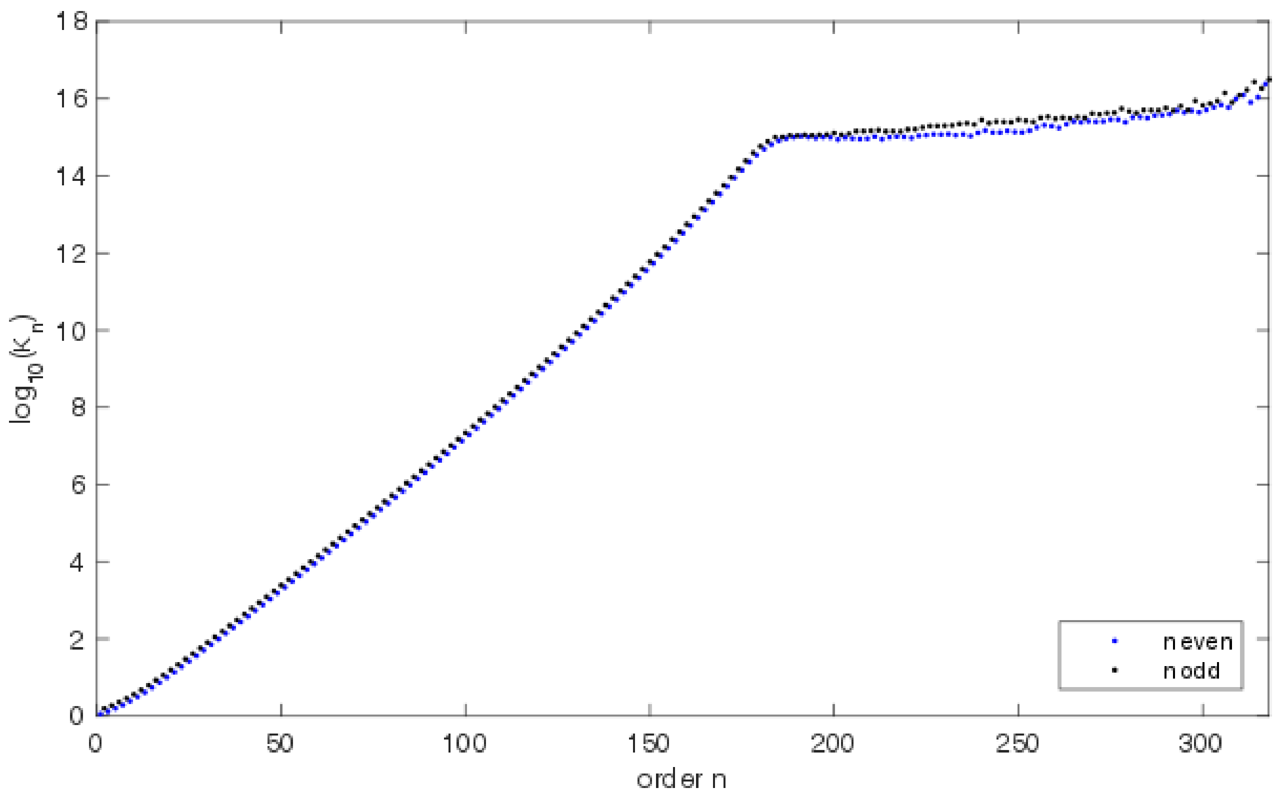
- Regression problem:

$$\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{y}$$

- Advantages of this regression form:
  - Matrix $\boldsymbol{X}$ depends only on the scan trajectory with fixed n
  - The condition number of $\boldsymbol{X}$ tells whether it is easy to estimate $\boldsymbol{\beta}$
  - Ill-conditioned regession problem

8. Ill-conditioned regression problem

- Condition number increases exponentially



8. Ill-conditioned regression problem

- Lasso regression:

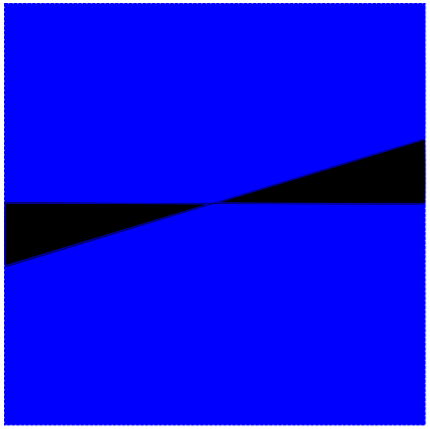
$$\boldsymbol{\beta} = \arg\min \tfrac{1}{2} \| \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{y} \|_2^2 + \lambda \| \boldsymbol{\beta} \|_1$$

- Iterative soft thresholding algorithm[1]

[1] Ingrid Daubechies, Michel Defrise, and Christine De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. Communications on pure and applied mathematics, 57(11):1413–1457, 2004.

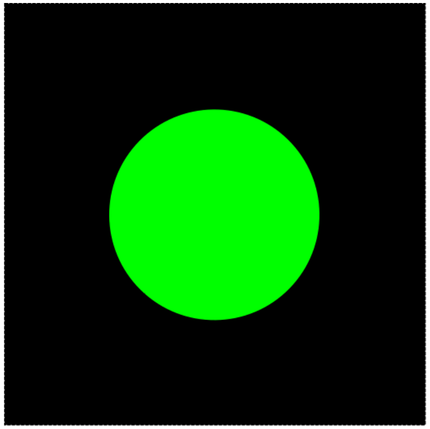
9. Image fusion at frequency domain

- Two images with different frequency components missing
- A double wedge region is missing at the Fourier domain of limited angle reconstruction
- Low frequency components inside a circular area are available at the Fourier domain of HLCC reconstruction with certain orders



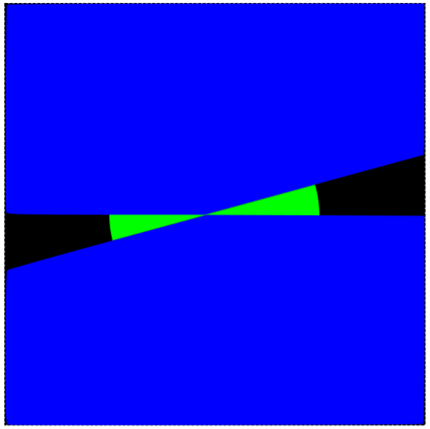
$F_{limited}$

+



F_{HLCC}

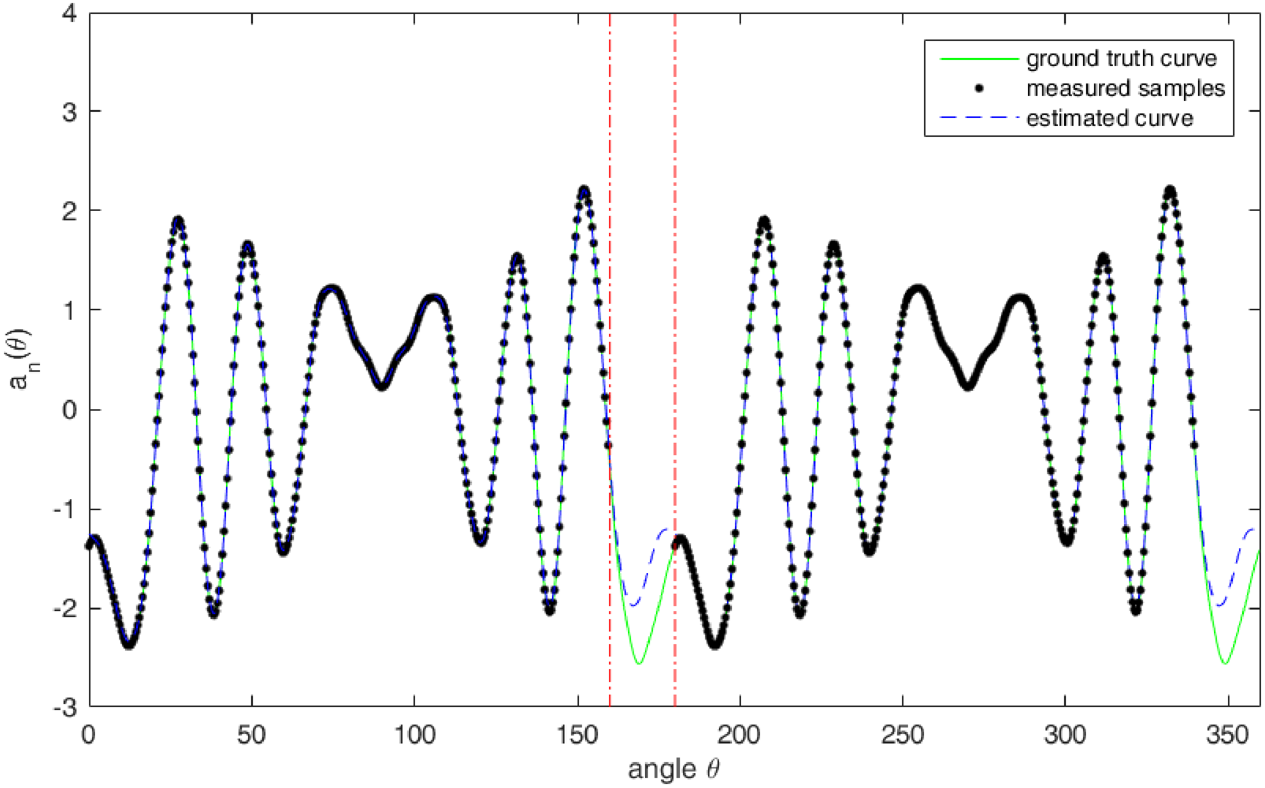
=



F_{fused}

II.b Numerical experiments (Shepp-Logan phantom)

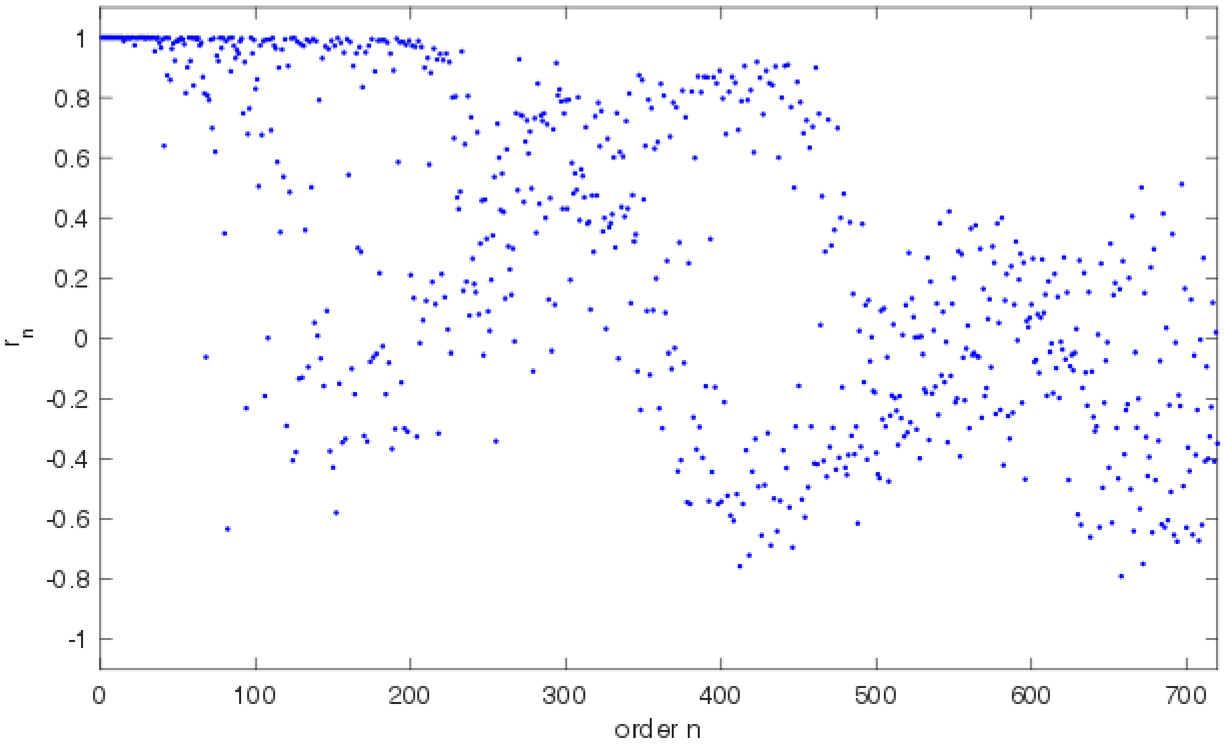
1. Estimation of moment curves



Estimated moment curve, e.g. n = 100, r = 0.86

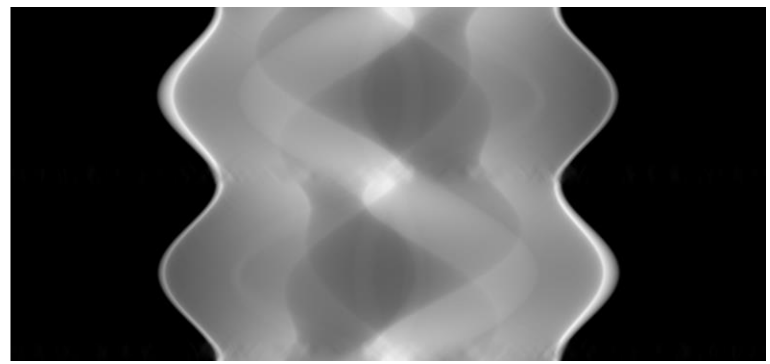
1. Estimation of moment curves

- Linear correlation coefficients, 720 orders



2. Restored sinogram

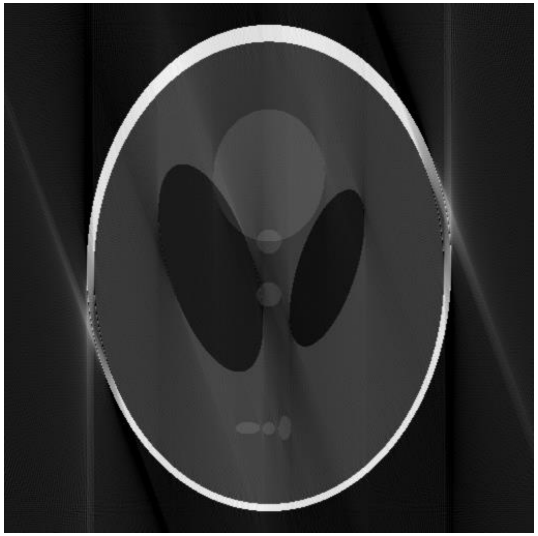
Limited angle sinogram, 160º



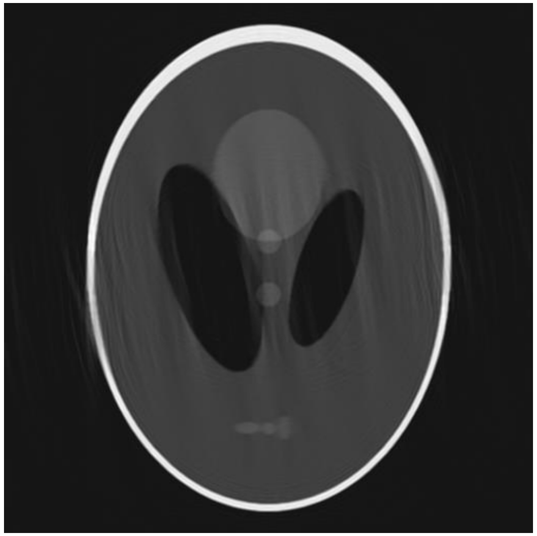
Restored sinogram, 360º

3. Image reconstruction from the restored sinogram

- Different reconstructed images with different types of artifacts
  - reconstruction from limited angle sinogram, streak artifacts
  - reconstruction from restored sinogram with HLCC, “regression artifacts”



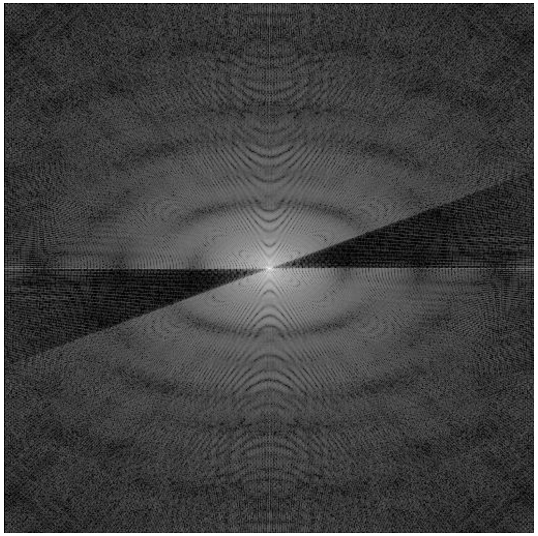
Limited angle reconstruction, $f_{limited}$



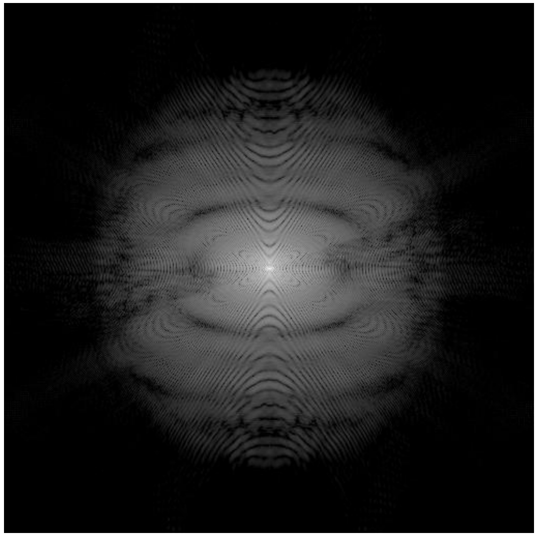
Recondtrution of restored sinogram, f_{HLCC}

4. Fourier transform of different images

- A double wedge region is missing at the Fourier domain of limited angle reconstruction
- Low frequency components inside a circular area are available at the Fourier domain of HLCC reconstruction with certain orders



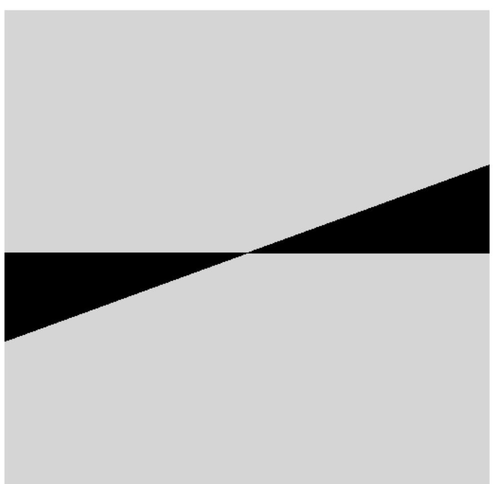
$F_{limited}$, Fourier transform of $f_{limited}$



F_{HLCC} , Fourier transform of t f_{HLCC}

5.Double wedge mask

- A double wedge mask can be computed based on the trajectory
- Smooth transition at the boundaries is necessary



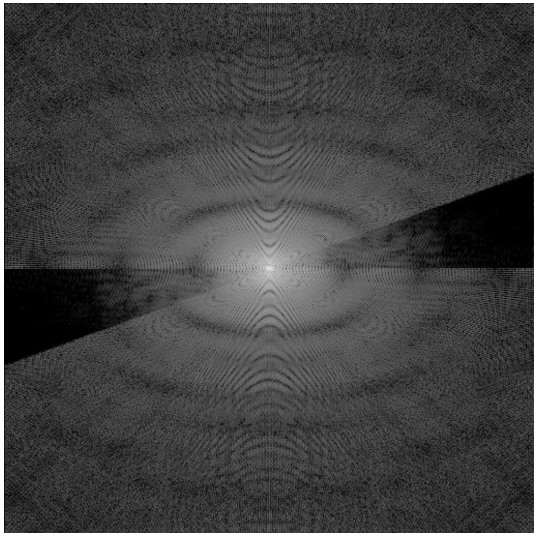
Binary double wedge mask

Smoothed double wedge mask

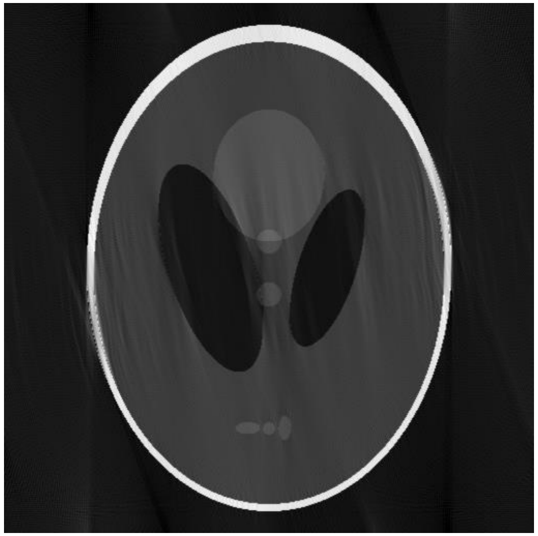
6. Fused image

- Combine two Fourier components:

$$F_{fused} = F_{limited} \ .^{*} \ mask + F_{HLCC} \ .^{*}(\mathbf{1} - mask)$$



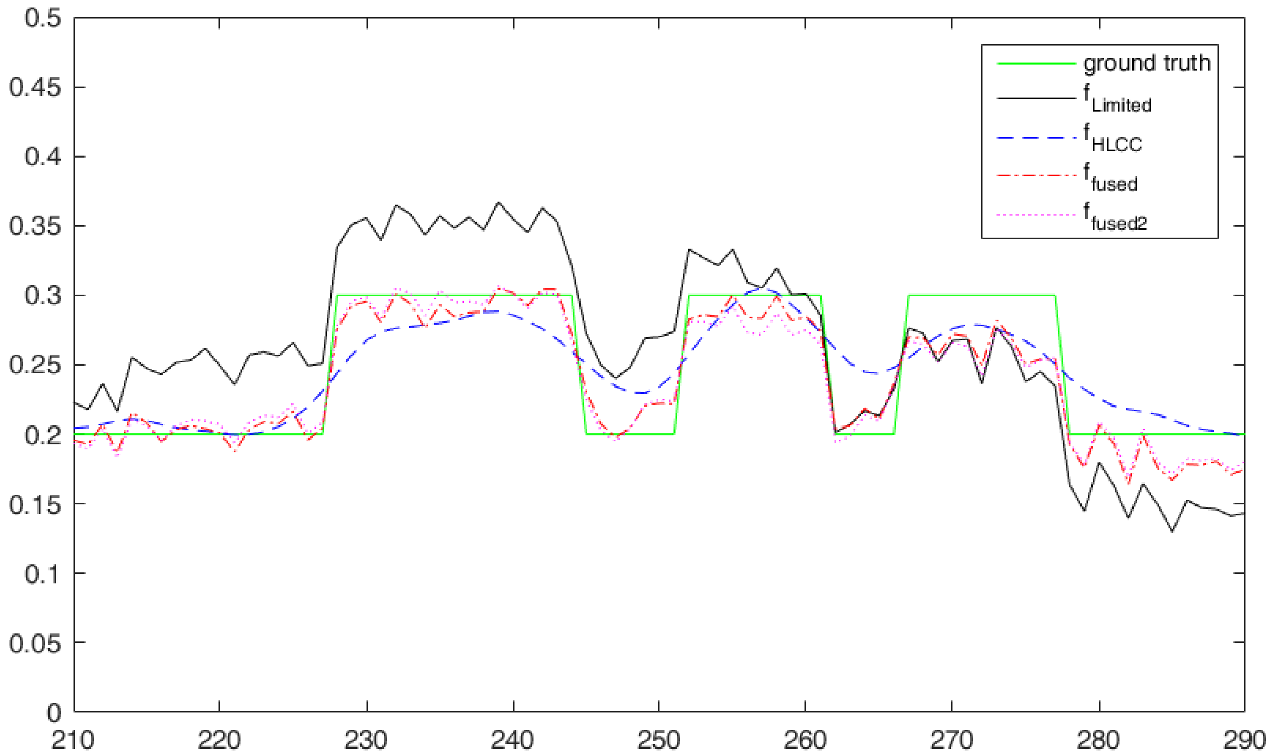
Combined Fourier components F_{fused}

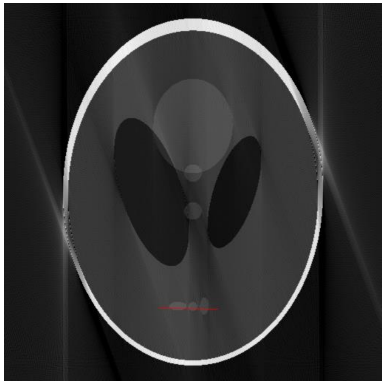


Fused image $f_{fused},$

7. Line profile

- Fused image has better edge sharpness at upper and lower area





8. Image fusion with bilateral filter

- Apply strong bilateral filter to remove regression artifacts before image fusion
- Only minor artifacts remain

Bilateral filter of f_{HLCC}

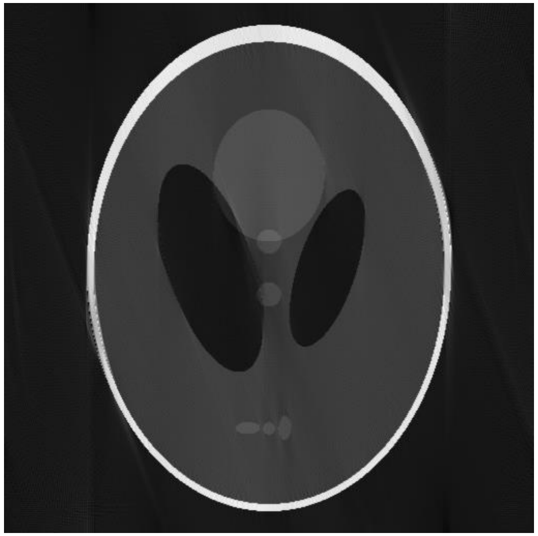
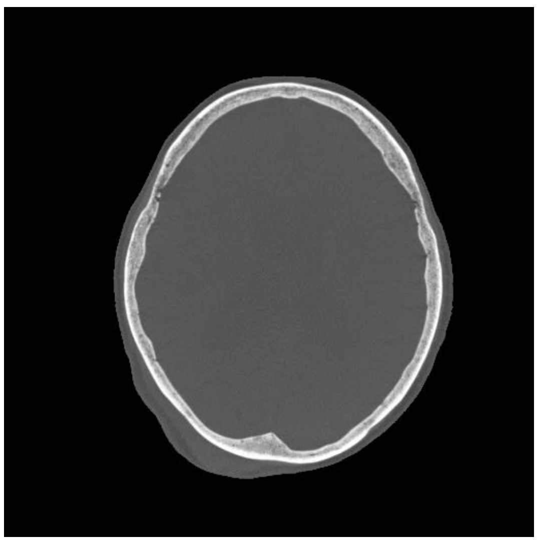


Image fusion f_{fused2}

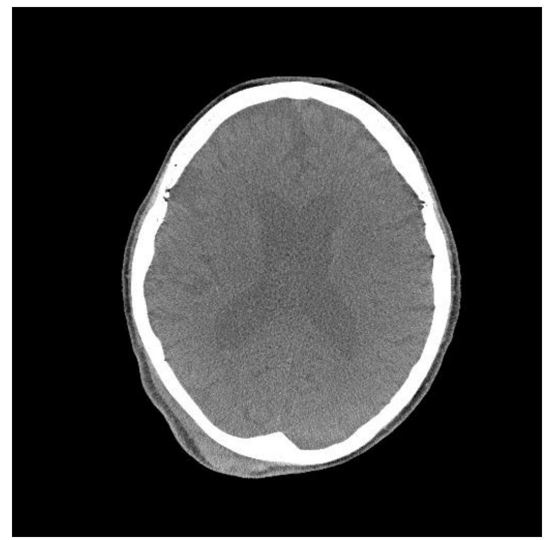
II.c “Clinical” experiments

1. Clinical data

- Choose one slice from a clinical 3-D dataset and reproject it to get the limited sinogram



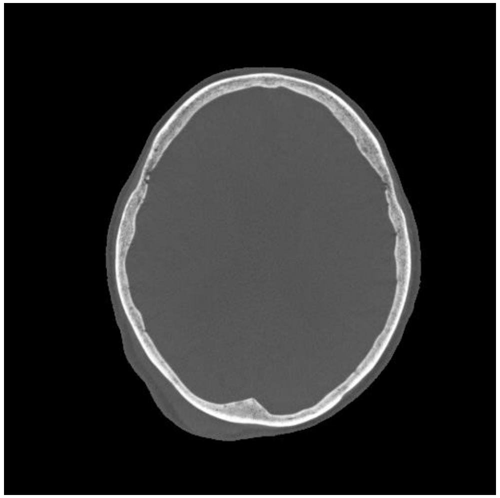
Ground truth, window [-1000, 2048] HU



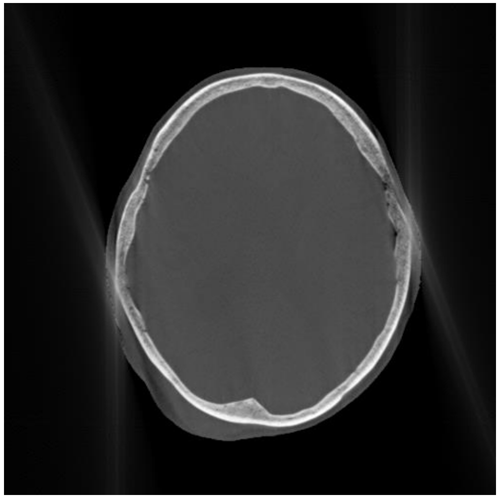
Ground truth, window [-204, 307] HU

2. Reconstruction results

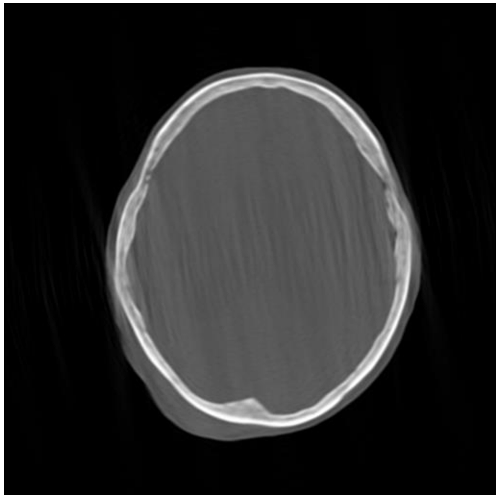
- Window [-1000, 2048] HU



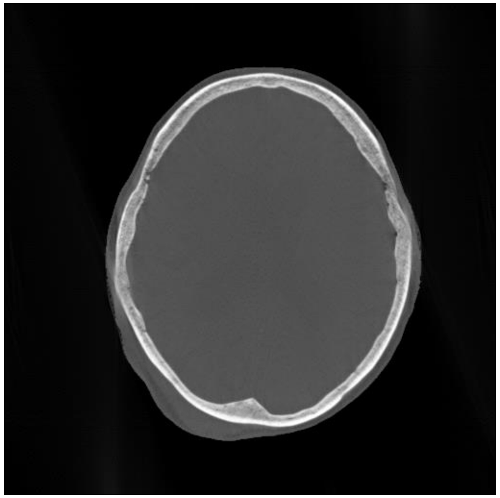
f_{GT}



$f_{limited}$



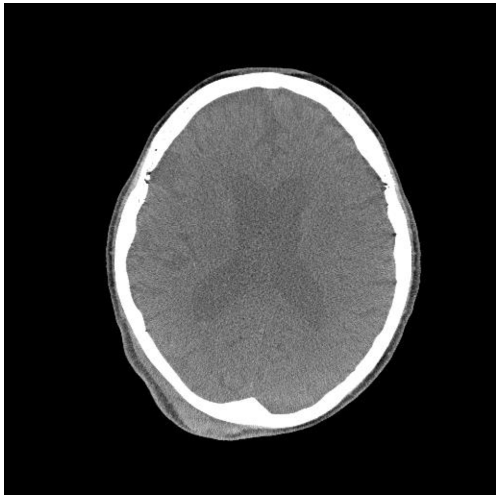
f_{HLCC}



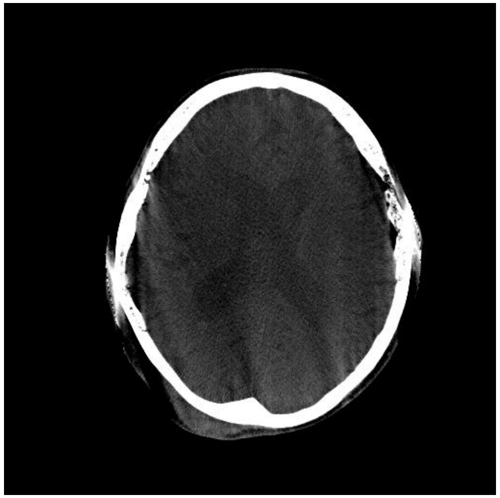
f_{fused2}

3. Reconstruction results

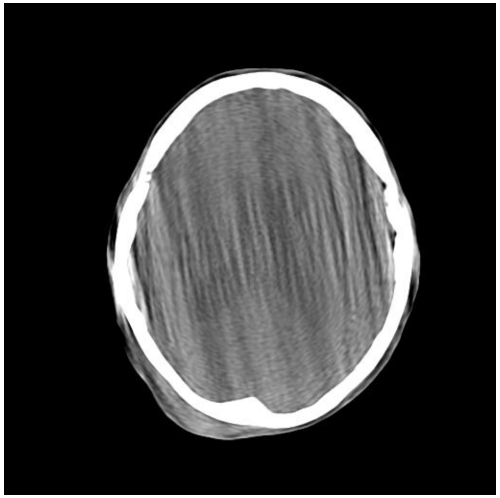
- Window [-204, 307] HU



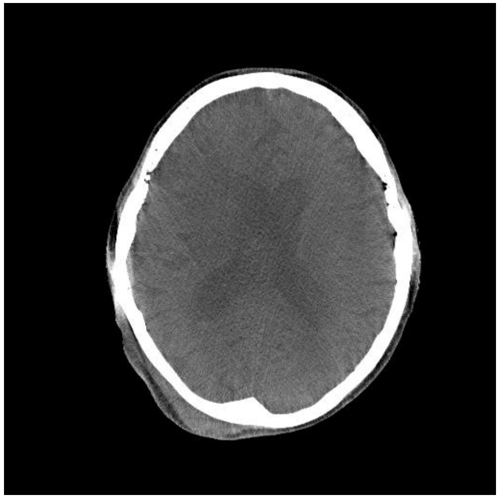
f_{GT}



$f_{limited}$



f_{HLCC}



f_{fused2}

III. Conclusion

Conclusion

- Chebyshev Fourier transform links the connections among domains of sinogram, Fourier of sinogram, image, and Fourier of image
- The limited angle sinogram restoration problem is converted to an ill-conditioned regression problem
- With HLCC, streaks are reduced but still regression artifacts occur
- Image fusion at frequency domain with bilateral filter can reduce streaks and regression artifacts both, although still some artifacts remain

Thank you for your attention!

Questions and suggestions?