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On calculation of generating functions of Chebyshev polynomials in several variables

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We propose a new method of calculation of generating functions of Chebyshev polynomials in several variables associated with root systems of the simple Lie algebras. We obtain the generating functions of the polynomials in two variables corresponding to the Lie algebra C_2 as an illustration. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4922997]

I. INTRODUCTION

In this paper, we propose a new method of calculation of generating functions for the Chebyshev polynomials associated with root systems of simple Lie algebras. The method is illustrated by calculation of the generating function for the bivariate Chebyshev polynomials associated with the simple Lie algebra C_2.

Chebyshev polynomials in several variables are natural generalizations of the classical Chebyshev polynomials in one variable.[1] They are used in different areas of mathematics (for example, in discrete analysis, in approximation theory,[2] in linear algebra,[3,4] and in representation theory[5–7]) and in physics.[8–12]

Generating functions are a powerful tool in the theory of classical orthogonal polynomials as well as in various applications. It is obvious that generating functions are important for the study of multivariate Chebyshev polynomials as well. However, as far as we know, up to now explicit forms were obtained only for the generating functions of the bivariate Chebyshev polynomials of both kinds associated with the simple Lie algebra A_2.[20] For that reason, the authors proposed a method of calculation of generating functions of the Chebyshev polynomials associated with a root system of any simple Lie algebra.[22]

The classical Chebyshev polynomials $T_n(x)$ of the first kind are defined by the formula

$$T_n(x) = T_n(\cos\phi) = \cos n\phi, \quad n \geq 0. \tag{1}$$

They satisfy the three-term recurrence relation,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \tag{2}$$

with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x.$$

The Chebyshev polynomials $U_n(x)$ of the second kind are defined by the formula

$$U_n(x) = \frac{\sin(n+1)\phi}{\sin\phi}, \quad n \geq 0. \tag{3}$$

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These polynomials satisfy same recurrence relation (2) as the polynomials of the first kind but with initial conditions

$$U_0(x) = 1, \quad U_1(x) = 2x.$$

It is known that the function $\cos n\phi$ can be treated as an invariant mean of the exponential function on the reflection group of the root system of Lie algebra A_1,

$$\cos n\phi = \frac{1}{2}(e^{\mathrm{i}n\phi} + e^{-\mathrm{i}n\phi}) = \frac{1}{2}\sum_{w\in W(A_1)} e^{\mathrm{i}(w\Lambda_n,\phi\alpha)}, \tag{4}$$

where α is the only simple root, $\lambda = \alpha/2$ is the only fundamental weight (we use the standard normalization $(\alpha,\alpha) = 2$), $\phi \in [0,2\pi)$, and $\Lambda_n = n\lambda$, $n \geq 0$ is the element of the related weight lattice. The Weyl group $W(A_1)$ consists of the identity e and the element $w : w\alpha = -\alpha$. At first sight, the above interpretation looks rather artificial, but its generalization allows to associate multi-dimensional analogs of polynomials (1) and (3) to the root systems of the simple Lie algebras.[13–19] Let us briefly recall their definition.

Let L be a simple Lie algebra and R be a reducible root system. A system of roots is a set of vectors in d-dimensional Euclidean space E^d supplied with a scalar product $(.,.)$. This system is completely determined by a basis of simple roots α_i, $i = 1,...,d$, and by the group of reflections of R called its Weyl group $W(R)$. Generators of the Weyl group w_i, $i = 1,\dots,d$, act on any vector $x \in E^d$ according to the formula

$$w_i\, x = x - \frac{2(x,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i. \tag{5}$$

Specifically, if $x = \alpha_i$, we obtain from (5) $w_i\,\alpha_i = -\alpha_i$. To any root, $\alpha \in R$ corresponds its coroot

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.$$

For the basis of the simple coroots $\alpha_i^\vee$, $i = 1,\dots,d$, one can define the dual basis of fundamental weights λ_i, $i = 1,\dots,d$,

$$(\lambda_i,\alpha_j^\vee) = \delta_{ij}$$

(here and in what follows the dual space E^{d^*} is identified with E^d). The bases of roots and weights are related by the linear transformation,

$$\alpha_i = \sum_j C_{ij}\lambda_j, \quad C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}, \tag{6}$$

where C is the Cartan matrix of the Lie algebra L.

A function which can be considered as a possible candidate for a d-dimensional generalization of $\cos n\phi$ must be invariant under the action of a Weyl group W and have the property of periodicity. Following Refs. 13–19, we define the periodic function in d variables (an orbit function in terminology of the work[18]) by the formula

$$\Phi_{\mathbf{n}}(\boldsymbol{\phi}) = \sum_{w\in \mathrm{W}} e^{2\pi\mathrm{i}(w\,\mathbf{n},\boldsymbol{\phi})}. \tag{7}$$

Obviously $\Phi_{\mathbf{n}}(\boldsymbol{\phi})$ is a W-invariant function because $\Phi_{\tilde{w}\,\mathbf{n}}(\boldsymbol{\phi}) = \Phi_{\mathbf{n}}(\boldsymbol{\phi})$, $\forall\tilde{w} \in W$. In formula (7), $\mathbf{n}$ is expressed in the basis of fundamental weights $\{\lambda_i\}$ and $\boldsymbol{\phi}$ is expressed in the dual basis of coroots $\{\alpha_i^\vee\}$,

$$\mathbf{n} = \sum_{i=1}^{d} n_i\lambda_i, \quad n_i \in Z, \quad \boldsymbol{\phi} = \sum_{i=1}^{d}\phi_i\alpha_i^\vee, \quad \phi_i \in [0,1).$$

Then, we define the new variables x_i (generalized cosines) by the formulas

$$x_i \;=\; \Phi_{\mathbf{e}_i}(\boldsymbol{\phi}), \quad \mathbf{e}_i = (\overbrace{0,...,0}^{i-1},1,\overbrace{0,...,0}^{d-i}). \tag{8}$$

It is shown in works[13–19] that the function $\Phi_{\mathbf{n}}(\boldsymbol{\phi})$ defined by formula (7) with non-negative integer n_i from $\mathbf{n} = (n_1, \ldots, n_d)$ can be expressed in terms of x_i. This function gives us the multivariate Chebyshev polynomials $\Phi_{n_1,\ldots,n_d}$ of the first kind up to a normalization. For the functions $\Phi_{\mathbf{n}}(\boldsymbol{\phi})$, we have the following multiplication rule:

$$\Phi_{\mathbf{k}}\Phi_{\mathbf{s}} = \sum_{w\in W} \Phi_{w\,\mathbf{k}+\mathbf{s}}, \tag{9}$$

which allows us to obtain recurrence relations for corresponding polynomials of several variables.

Let us return to the classical Chebyshev polynomials and demonstrate the method of calculation of generating functions in this simplest case. There are two possible variants of calculations. The first of them starts with definition (1) and the second starts with recurrence relation (2). In the multivariate case, the starting points are orbit function (7) and recurrence relation (9), respectively.

Let us begin with cosine function (1) and represent it as a trace of the following diagonal matrix:

$$\cos n\phi = \frac{1}{2}(e^{\mathrm{i}n\phi} + e^{-\mathrm{i}n\phi}) = \frac{1}{2}\,\mathrm{tr}M^n, \quad M = \begin{pmatrix} e^{\mathrm{i}\phi} & 0 \\ 0 & e^{-\mathrm{i}\phi} \end{pmatrix}.$$

Let I_2 be the identity 2×2 matrix. Introduce the matrix

$$R_p = (I_2 - pM)^{-1}, \tag{10}$$

where p is a real parameter. Taking into account the relation

$$M^n = \frac{1}{n!}\left.\frac{\mathrm{d}^n R_p}{\mathrm{d}p^n}\right|_{p=0},$$

we obtain

$$\cos n\phi = \frac{1}{n!}\left.\frac{\mathrm{d}^n F_1(p)}{\mathrm{d}p^n}\right|_{p=0}, \quad F_1(p) = \frac{1}{2}\,\mathrm{tr}R_p.$$

Calculation of $F_1(p)$ gives us the well known generating function of the Chebyshev polynomials of the first kind,

$$F_1(p) = \frac{1 - px}{1 - 2xp + p^2}, \tag{11}$$

where

$$x = \cos\phi = \frac{1}{2}(e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi})$$

is the new variable introduced according to definition (8).

Now, consider recurrent relation (2) with arbitrary initial conditions T_0, T_1. This relation can be rewritten in the form

$$T_m = r_n T_{m-1-n} + s_n T_{m-2-n}, \quad r_0 = 2x,\ s_0 = -1. \tag{12}$$

The coefficients r_n, s_n are related by the formula

$$V_{n+1} = AV_n = A^{n+1}V_0,$$

where V_n is the vector with the components r_n, s_n and A is the following matrix:

$$V_n = \begin{pmatrix} r_n \\ s_n \end{pmatrix}, \quad A = \begin{pmatrix} 2x & 1 \\ -1 & 0 \end{pmatrix}, \quad \det A = 1.$$

Then, we can rewrite (12) in the matrix form,

$$T_m = r_{m-2}T_1 + s_{m-2}T_0 = (T_1, T_0)A^{m-2}\begin{pmatrix} r_0 \\ s_0 \end{pmatrix} = (T_1, T_0)A^m\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

substituting in it $n = m - 2$ and taking into account the invertibility of A. Replacing A^m by R_p from (10) in formula (13), we obtain the function

$$F(p) = \frac{T_0 + (T_1 - 2xT_0)p}{1 - 2xp + p^2}, \tag{14}$$

which is the generating function for the polynomials satisfying relation (2) with arbitrary initial condition T_0, T_1. Substitution of $T_0 = 1, T_1 = x$ in (14) yields generating function (11) of the first kind Chebyshev polynomials $F_1(p)$, and substitution of $T_0 = 1, T_1 = 2x$ yields the generating function of the second kind Chebyshev polynomials

$$F_2(p) = \frac{1}{1 - 2xp + p^2}.$$

Let us consider the general case of the orbit function $\Phi_{\mathbf{n}}$ defined by (7). Since the components of $\mathbf{n}$ are integer, scalar product in the function $\Phi_{\mathbf{n}}$ given by (7) can be represented in the form $(w\,\mathbf{n}, \boldsymbol{\phi}) = \sum_k (w\lambda_k, \boldsymbol{\phi}) n_k$, and the function itself can be written as

$$\Phi_{\mathbf{n}} = \sum_{w \in \mathrm{W}} \prod_k \left(e^{2\pi \mathrm{i}(w\lambda_k, \boldsymbol{\phi})}\right)^{n_k} = \operatorname{tr}\left(\prod_k M_k^{n_k}\right). \tag{15}$$

In formula (15), M_k are the following diagonal matrices:

$$M_k = \operatorname{diag}(e^{2\pi \mathrm{i}(w_1\lambda_k, \boldsymbol{\phi})}, e^{2\pi \mathrm{i}(w_2\lambda_k, \boldsymbol{\phi})}, \ldots, e^{2\pi \mathrm{i}(w_{|W|}\lambda_k, \boldsymbol{\phi})}),$$

where w_i are the elements of the Weyl group W and $|W|$ is the number of elements in this group.

Define the matrices $R_k = (I_{|W|} - p_k M_k)^{-1}$, where $I_{|W|}$ is the identity matrix of size $|W|$ and p_k are real parameters. In this notation, $\Phi_{\mathbf{n}}$ has the form

$$\Phi_{\mathbf{n}}(\boldsymbol{\phi}) = \Phi_{n_1,\ldots,n_d}(\boldsymbol{\phi}) = \frac{1}{n_1!\ldots n_d!} \frac{\partial^{n_1+\ldots+n_d}}{\partial^{n_1} p_1 \ldots \partial^{n_d} p_n} \left(\operatorname{tr}(R_{p_1} \ldots R_{p_d})\right)\Big|_{p_1=\ldots=p_d=0}. \tag{16}$$

Introduce new independent variables defined by formulas (8),

$$x_i = \Phi_{\mathbf{e}_i}(\boldsymbol{\phi}), \quad \mathbf{e}_i = (\overbrace{0,\ldots,0}^{i-1}, 1, \overbrace{0,\ldots,0}^{d-i}).$$

The simple structure of the matrices R_{p_k} allows us to express the coefficients $\Phi_{n_1,\ldots,n_d}(\boldsymbol{\phi})$ of the function

$$F^I_{p_1,\ldots,p_d} = \operatorname{tr}(R_{p_1} \ldots R_{p_d}) = \sum_{n_1 \ldots n_d \geq 0} \Phi_{n_1,\ldots,n_d}(\boldsymbol{\phi}) p_1^{n_1} \ldots p_d^{n_d} \tag{17}$$

in terms of x_i. It is evident that the function $F^I_{p_1,\ldots,p_d}$ can be considered as a generating function of the multivariate Chebyshev polynomials of the first kind.

Generalization of polynomials (3) for the case of several variables is based on the Weyl character formula,[21] from which it follows that

$$U_{\mathbf{n}}(\boldsymbol{\phi}) = \frac{\sum\limits_{w \in \mathrm{W}} \det w \;\; e^{2\pi \mathrm{i}(w\,(\mathbf{n}+\boldsymbol{\rho}), \boldsymbol{\phi})}}{\sum\limits_{w \in \mathrm{W}} \det w \;\; e^{2\pi \mathrm{i}(w\,\boldsymbol{\rho}, \boldsymbol{\phi})}}, \tag{18}$$

where $\det w = (-1)^{\ell(w)}$ and $\ell(w)$ is the minimal number of generating elements w_i of the Weyl group required for expressing w as a product of w_i. Here, $\boldsymbol{\rho}$ is the Weyl vector equal to the half-sum of the positive roots. Using formula (6), the Weyl vector can be expressed in the basis of fundamental weights.

Calculation of the generating function of Chebyshev polynomials of the second kind in several variables requires only a minor modification of the above method. The function

$$\Phi^{as}_{\mathbf{n}+\boldsymbol{\rho}} = \sum_{w \in \mathrm{W}} \det w \;\; e^{2\pi \mathrm{i}(w\,(\mathbf{n}+\boldsymbol{\rho}), \boldsymbol{\phi})},$$

which stands in the numerator of (18), can be represented as the difference between the expressions $\Phi^{as+}_{\mathbf{n}+\boldsymbol{\rho}}$ and $\Phi^{as-}_{\mathbf{n}+\boldsymbol{\rho}}$, i.e.,

$$\Phi^{as}_{\mathbf{n}+\boldsymbol{\rho}} = \Phi^{as+}_{\mathbf{n}+\boldsymbol{\rho}} - \Phi^{as-}_{\mathbf{n}+\boldsymbol{\rho}} = \sum_{w\in \mathrm{W},\ \det w=1} e^{2\pi \mathrm{i}(w\,(\mathbf{n}+\boldsymbol{\rho}),\boldsymbol{\phi})} - \sum_{w\in \mathrm{W},\ \det w=-1} e^{2\pi \mathrm{i}(w\,(\mathbf{n}+\boldsymbol{\rho}),\boldsymbol{\phi})}.$$

Further calculation repeats the scheme given above (15)–(17) for each of the functions $\Phi^{as\pm}_{\mathbf{n}+\boldsymbol{\rho}}$ separately. In the considered case, we must use the relation

$$\Phi^{as}_{\mathbf{n}+\boldsymbol{\rho}} = \frac{1}{n_1!\ldots n_d!}\frac{\partial^{n_1+\ldots+n_d}}{\partial^{n_1}p_1\ldots\partial^{n_d}p_n}\left(\mathrm{tr}(R^+_{p_1}\ldots R^+_{p_d} - R^-_{p_1}\ldots R^-_{p_d})\right)\Big|_{p_1=\ldots=p_d=0}, \tag{19}$$

instead of $\Phi_{\mathbf{n}}$ given by (16). In formula (19), the matrices $R^{\pm}_{p_i}$ are used to represent functions $\Phi^{as\pm}_{\mathbf{n}+\boldsymbol{\rho}}$. More details are presented in the example below.

To obtain Chebyshev polynomials of the second kind, we must in accordance with formula (18) divide the function $\Phi^{as}_{\mathbf{n}+\boldsymbol{\rho}}$ by the singular element $\Phi^{as}_{\boldsymbol{\rho}}$, where $\boldsymbol{\rho} = \sum_{i=1}^{d}\lambda_i$. Thus, the generating function of Chebyshev polynomials of the second kind has the form

$$F^{II}_{p_1,\ldots,p_d} = \frac{\mathrm{tr}(R^+_{p_1}\ldots R^+_{p_d} - R^-_{p_1}\ldots R^-_{p_d})}{\Phi^{as}_{\boldsymbol{\rho}}}.$$

As in the case of the first kind polynomials, transition to the variables $x_i = U_{\mathbf{e}_i}(\boldsymbol{\phi})$ in $F^{II}_{p_1,\ldots,p_d}$ does not require cumbersome calculations.

To illustrate the above method, we calculate the generating functions of the bivariate Chebyshev polynomials associated with the system of simple roots of the Lie algebra C_2. The generating functions of the Chebyshev polynomials associated with the Lie algebras G_2 and A_3 can be found in Refs. 22 and 23, respectively.

II. POLYNOMIALS ASSOCIATED WITH THE LIE ALGEBRA C_2

A. Polynomials of the first kind

The root system of the Lie algebra C_2 has two fundamental roots α_1, α_2 and includes the positive roots $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$ together with their reflections.[21] Using formula (5) and the Cartan matrix of the algebra C_2 given by

$$C_{C_2} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

we obtain the following action of generating elements w_1, w_2 of the Weyl group $W(C_2)$ on the fundamental roots,

$$w_1\alpha_1 = -\alpha_1,\quad w_1\alpha_2 = 2\alpha_1+\alpha_2,\quad w_2\alpha_1 = \alpha_1+\alpha_2,\quad w_2\alpha_2 = -\alpha_2.$$

Next, we rewrite these relations for the fundamental weights by means of formula (6),

$$w_1\lambda_1 = \lambda_2-\lambda_1,\quad w_1\lambda_2 = \lambda_2,\quad w_2\lambda_1 = \lambda_1,\quad w_2\lambda_2 = 2\lambda_1-\lambda_2.$$

The action of the other group elements on the fundamental weights is determined by their representation in terms of the generating elements,

$$w_3 = w_1w_2,\quad w_4 = w_2w_3,\quad w_5 = w_1w_2w_1,\quad w_6 = w_2w_1w_2,\quad w_7 = (w_1w_2)^2,\quad e = w_0.$$

The determinants of the Weyl group elements are given by

$$\det w_1 = \det w_2 = \det w_5 = \det w_6 = -1,$$

and the others are equal to the unity. The Weyl vector has the form

$$\boldsymbol{\rho} = \lambda_1 + \lambda_2.$$

Using these formulas and the notation $\mathbf{n} = m\lambda_1 + n\lambda_2$, $\boldsymbol{\phi} = \phi\alpha_1^\vee + \psi\alpha_2^\vee$, we find the following $W(C_2)$-invariant function of two variables:

$$\begin{aligned} \Phi_{m,n} = {} & e^{2\pi\mathrm{i}(m\phi+n\psi)} + e^{2\pi\mathrm{i}(m(\psi-\phi)+n\psi)} + e^{2\pi\mathrm{i}(m\phi+n(2\phi-\psi))} + e^{2\pi\mathrm{i}(m(\psi-\phi)+n(-2\phi+\psi))} + \\ & + e^{2\pi\mathrm{i}(m(\phi-\psi)+n(2\phi-\psi))} + e^{2\pi\mathrm{i}(-m\phi+n(-2\phi+\psi))} + e^{2\pi\mathrm{i}(m(\phi-\psi)-n\psi)} + e^{2\pi\mathrm{i}(-m\phi-n\psi)}. \end{aligned} \tag{20}$$

Note that the function $\Phi_{m,n}$ given by (20) is real-valued.

If we introduce the diagonal matrices A_i, $i = 1,2$,

$$A_1 = \mathrm{diag}(\phi, \psi-\phi, \phi, \psi-\phi, \phi-\psi, -\phi, \phi-\psi, -\phi)$$

and

$$A_2 = \mathrm{diag}(\psi, \psi, 2\phi-\psi, -2\phi+\psi, 2\phi-\psi, -2\phi+\psi, -\psi, -\psi),$$

then $\Phi_{m,n}$ can be expressed using the matrices $M_k = e^{2\pi\mathrm{i}A_k}$, $k = 1,2$, in the trace form,

$$\Phi_{m,n} = \mathrm{tr}(M_1^m M_2^n).$$

To define the Chebyshev polynomials of the first kind, we take the following normalization:[18]

$$T_{0,0} = \frac{1}{8}\Phi_{0,0}, \quad T_{m,0} = \frac{1}{2}\Phi_{m,0}, \quad T_{0,n} = \frac{1}{2}\Phi_{0,n}, \quad T_{m,n} = \Phi_{m,n}, \quad m\cdot n \neq 0. \tag{21}$$

With this definition, $T_{0,0} = 1$ and the new real variables x, y are given by the formulas

$$\begin{aligned} x &= T_{1,0} = e^{2\pi\mathrm{i}\phi} + e^{-2\pi\mathrm{i}\phi} + e^{2\pi\mathrm{i}(\phi-\psi)} + e^{-2\pi\mathrm{i}(\phi-\psi)}, \\ y &= T_{0,1} = e^{2\pi\mathrm{i}\psi} + e^{-2\pi\mathrm{i}\psi} + e^{2\pi\mathrm{i}(2\phi-\psi)} + e^{-2\pi\mathrm{i}(2\phi-\psi)}. \end{aligned} \tag{22}$$

Turning to the calculation of the generating function, we introduce two diagonal matrices,

$$R_p = (I_8 - pM_1)^{-1}, \quad R_q = (I_8 - qM_2)^{-1},$$

where I_8 is the unit 8×8 matrix. Substituting R_p, R_q in formula (16) and expressing the coefficients of p, q as functions of x and y ((22)), we obtain from (17),

$$F^I_{p,q} = \frac{\sum\limits_{i,j=0}^{3} K_{ij}p^i q^j}{(1 - xp + (2+y)p^2 - xp^3 + p^4)(1 - yq + (x^2 - 2y - 2)q^2 - yq^3 + q^4)}. \tag{23}$$

The coefficients K_{ij} from the numerator of the above formula as well some polynomials calculated using (23) with normalization (21) are listed in the Appendix.

The recurrence relations for the polynomials under consideration can be obtained by multiplication rule (9) by putting $\mathbf{s} = (m,n)$ and $\mathbf{k} = (1,0) = \lambda_1$ in the first relation and $\mathbf{k} = (0,1) = \lambda_2$ in the second one. As a result, we obtain

$$\begin{aligned} x\Phi_{m,n} &= \Phi_{m+1,n} + \Phi_{m-1,n} + \Phi_{m+1,n-1} + \Phi_{m-1,n+1}, \\ y\Phi_{m,n} &= \Phi_{m,n+1} + \Phi_{m,n-1} + \Phi_{m+2,n-1} + \Phi_{m-2,n+1}, \end{aligned} \tag{24}$$

where normalization (21) and definition (22) were taken into account. For each weight λ_i, $i = 1,2$, this factor is the order of the subgroup of W consisting of all elements $w \in W$, such that $w\lambda_i = \lambda_i$.

Let us reduce relations (24) to a form which is convenient for the calculation of generating function (23) in different way. The determinant of the matrix $(pI_8 - M_1)$ gives us the characteristic equation of M_1. Since all the eigenvalues of this matrix have multiplicity 2, the minimal polynomial of M_1 (after transition to the variables x, y) takes the form

$$P_1 = 1 - xp + (2+y)p^2 - xp^3 + p^4.$$

Therefore, the matrix M_1 satisfies the equation

$$M_1^4 - xM_1^3 + (2+y)M_1^2 - xM_1 + I_8 = 0.$$

Multiplying this equation from the right by $M_1^{m-4}M_2^n$ and taking the trace, we obtain

$$\Phi_{m,n} = x\Phi_{m-1,n} - (2+y)\Phi_{m-2,n} + x\Phi_{m-3,n} - \Phi_{m-4,n}. \tag{25}$$

The useful feature of this recurrence relation is that we change only one of the subscripts of $\Phi_{m,n}$.

Acting in the same way by the matrix M_2, we obtain the characteristic polynomial

$$P_2 = 1 - yq + (x^2 - 2y - 2)q^2 - yq^3 + q^4$$

and the second recurrent relation

$$\Phi_{m,n} = y\Phi_{m,n-1} - (x^2 - 2y - 2)\Phi_{m,n-2} + y\Phi_{m,n-3} - \Phi_{m,n-4}. \tag{26}$$

Relations (25) and (26) can be checked by a direct substitution of (24).

Relations (25) and (26) allow us to derive generating function (23) in the same way as generating function (14) was derived in the classical case. To this end, let us rewrite them in the form

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{m-1,n}\,; & -\Phi_{m-2,n}\,; & \Phi_{m-3,n}\,; & -\Phi_{m-4,n}\end{pmatrix}\begin{pmatrix} x \\ y+2 \\ x \\ 1\end{pmatrix}, \tag{27}$$

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{m,n-1}\,; & -\Phi_{m,n-2}\,; & \Phi_{m,n-3}\,; & -\Phi_{m,n-4}\end{pmatrix}\begin{pmatrix} y \\ x^2-2y-2 \\ y \\ 1\end{pmatrix}. \tag{28}$$

By means of the iteration of relations (27) and (28), it is not difficult to transform them in the form

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{m-1-k,n}\,; & -\Phi_{m-2-k,n}\,; & \Phi_{m-3-k,n}\,; & -\Phi_{m-4-k,n}\end{pmatrix} M_x^k \begin{pmatrix} x \\ y+2 \\ x \\ 1\end{pmatrix}, \tag{29}$$

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{m,n-1-k}\,; & -\Phi_{m,n-2-k}\,; & \Phi_{m,n-3-k}\,; & -\Phi_{m,n-4-k}\end{pmatrix} M_y^k \begin{pmatrix} y \\ x^2-2y-2 \\ y \\ 1\end{pmatrix}, \tag{30}$$

where

$$M_x = \begin{pmatrix} x & -1 & 0 & 0 \\ y+2 & 0 & -1 & 0 \\ y & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad M_y = \begin{pmatrix} y & -1 & 0 & 0 \\ x^2-2y-2 & 0 & -1 & 0 \\ y & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Substituting $k = m-4$ in (29) and $k = n-4$ in (30), we find

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{3,n}\,; & -\Phi_{2,n}\,; & \Phi_{1,n}\,; & -\Phi_{0,n}\end{pmatrix} M_x^{m-4} \begin{pmatrix} x \\ y+2 \\ x \\ 1\end{pmatrix},$$

$$\Phi_{m,n} = \begin{pmatrix}\Phi_{m,3}\,; & -\Phi_{m,2}\,; & \Phi_{m,1}\,; & -\Phi_{m,0}\end{pmatrix} M_y^{n-4} \begin{pmatrix} y \\ x^2-2y-2 \\ y \\ 1\end{pmatrix}.$$

The matrices M_x, M_y are invertible since $\det(M_x) = \det(M_y) = 1$, so we have

$$M_x^{-4}\begin{pmatrix} x \\ y+2 \\ x \\ 1 \end{pmatrix} = M_y^{-4}\begin{pmatrix} y \\ x^2-2y-2 \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Therefore, relations (29) and (30) can be converted to the final form

$$\Phi_{m,n} = \Big(\Phi_{3,n}\,; \quad -\Phi_{2,n}\,; \quad \Phi_{1,n}\,; \quad -\Phi_{0,n}\Big)\, M_x^m \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \tag{31}$$

$$\Phi_{m,n} = \Big(\Phi_{m,3}\,; \quad -\Phi_{m,2}\,; \quad \Phi_{m,1}\,; \quad -\Phi_{m,0}\Big)\, M_y^n \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{32}$$

The obtained formulas allow us to represent any Chebyshev polynomial $\Phi_{m,n}$ with subscripts $m,n > 3$ by the polynomials with $0 \le m,n \le 3$. Indeed, using relation (31) with the polynomials $\Phi_{m,n},\ 0 \le m,n \le 3$, one can find the polynomials $\Phi_{m,0}, \Phi_{m,1}, \Phi_{m,2}, \Phi_{m,3}$ with arbitrary subscript m. Then, using relation (32) with the calculated polynomials, one can find a polynomial $\Phi_{m,n}$ with an arbitrary subscript n.

The considered modification of the method has the advantage of the arbitrariness of initial polynomials, which in turn allows us to construct different series of polynomials which satisfy recurrence relations (25) and (26), but generally speaking not original relations (24). In other words, the arbitrariness in the assignment of initial polynomials is limited.

B. Polynomials of the second kind

The generating function of the Chebyshev polynomials of the second kind has the form

$$U_{\mathbf{n}}(\boldsymbol{\phi}) = \frac{\sum\limits_{w\in \mathrm{W}} \det w\; e^{2\pi \mathrm{i}(w\,(\mathbf{n}+\boldsymbol{\rho}),\boldsymbol{\phi})}}{\sum\limits_{w\in \mathrm{W}} \det w\; e^{2\pi \mathrm{i}(w\,\boldsymbol{\rho},\boldsymbol{\phi})}} = \frac{\Phi^{as}_{\mathbf{n+1}}}{\Phi^{as}_{\mathbf{1}}},$$

where we take into account the form of the Weyl vector $\boldsymbol{\rho} = \lambda_1 + \lambda_2$. Using notation from Sec. II A and the presentation of the elements of Weyl group $W(C_2)$ by means of the generating ones, we obtain for $\Phi^{as}_{m,n}$ and the singular element $\Phi^{as}_{1,1}$,

$$\begin{aligned}\Phi^{as}_{m,n} = &\Big(e^{2\pi \mathrm{i}(m\phi+n\psi)} + e^{2\pi \mathrm{i}(m(\psi-\phi)+n(-2\phi+\psi))} + e^{2\pi \mathrm{i}(-m\phi-n\psi)} + e^{2\pi \mathrm{i}(m(\phi-\psi)+n(2\phi-\psi))}\Big) - \\ &- \Big(e^{2\pi \mathrm{i}(m(\psi-\phi)+n\psi)} + e^{2\pi \mathrm{i}(m\phi+n(2\phi-\psi))} + e^{2\pi \mathrm{i}(-m\phi+n(-2\phi+\psi))} + e^{2\pi \mathrm{i}(m(\phi-\psi)-n\psi)}\Big),\end{aligned} \tag{33}$$

$$\begin{aligned}\Phi^{as}_{1,1} = &(e^{2\pi \mathrm{i}(\phi+\psi)} + e^{2\pi \mathrm{i}(2\psi-3\phi)} + e^{2\pi \mathrm{i}(-\phi-\psi)} + e^{2\pi \mathrm{i}(-2\psi+3\phi)}) - \\ &- (e^{2\pi \mathrm{i}(2\psi-\phi)} + e^{2\pi \mathrm{i}(3\phi-\psi)} + e^{-2\pi \mathrm{i}(-3\phi+\psi)} + e^{2\pi \mathrm{i}(\phi-2\psi)}),\end{aligned} \tag{34}$$

respectively. We do not introduce at this stage new variables by the formulas

$$U_{10} = \frac{\Phi^{as}_{2,1}}{\Phi^{as}_{1,1}}, \quad U_{01} = \frac{\Phi^{as}_{1,2}}{\Phi^{as}_{1,1}},$$

because it will be more suitable to do this later. Using formulas (33) and (34), we define the following four diagonal matrices:

$$M_{1+} = \mathrm{diag}(e^{2\pi\mathrm{i}\phi}, e^{2\pi\mathrm{i}(\psi-\phi)}, e^{-2\pi\mathrm{i}\phi}, e^{2\pi\mathrm{i}(\phi-\psi)}), \quad M_{2+} = \mathrm{diag}(e^{2\pi\mathrm{i}\psi}, e^{2\pi\mathrm{i}(-2\phi+\psi)}, e^{-2\pi\mathrm{i}\psi}, e^{2\pi\mathrm{i}(2\phi-\psi)}),$$
$$M_{1-} = \mathrm{diag}(e^{2\pi\mathrm{i}(\psi-\phi)}, e^{2\pi\mathrm{i}\phi}, e^{-2\pi\mathrm{i}\phi}, e^{2\pi\mathrm{i}(\phi-\psi)}), \quad M_{2-} = \mathrm{diag}(e^{2\pi\mathrm{i}\psi}, e^{2\pi\mathrm{i}(2\phi-\psi)}, e^{2\pi\mathrm{i}(-2\phi+\psi)}, e^{-2\pi\mathrm{i}\psi}).$$

In terms of these matrices, the function $\Phi^{as}_{m,n}$ has the form

$$\Phi^{as}_{m,n} = \mathrm{tr}(M^m_{1+}M^n_{2+} - M^m_{1-}M^n_{2-}).$$

Introducing the matrices $R_{1\pm} = (I_4 - pM_{1\pm})^{-1}$, $R_{2\pm} = (I_4 - qM_{2\pm})^{-1}$, one can say that the function,

$$F^{II}_r(p,q) = \frac{\mathrm{tr}(R_{1+}R_{2+} - R_{1-}R_{2-})}{\Phi^{as}_{1,1}},$$

is the generating function of the bivariate Chebyshev polynomials of the second kind,

$$U_{m,n} = \frac{1}{(m+1)!(n+1)!}\left.\frac{\partial^{m+n+2}F^{II}_r(p,q)}{\partial p^{m+1}\partial q^{n+1}}\right|_{p=0,q=0}.$$

As a result of simple calculations, we obtain

$$\mathrm{tr}(R_{1+}R_{2+} - R_{1-}R_{2-}) = \frac{pq(1+q+p^2q+p^2q^2-\tilde{x}pq)\Phi^{as}_{1,1}}{(1-\tilde{x}p+(\tilde{y}+2)p^2-\tilde{x}p^3+p^4)(1-\tilde{y}q+(\tilde{x}^2-2\tilde{y}-2)q^2-\tilde{y}q^3+q^4)},$$

where notations (22) were used as an intermediate one,

$$\tilde{x} = \mathrm{tr}(M_{1+}) = \mathrm{tr}(M_{1-}) = e^{2\pi\mathrm{i}\phi} + e^{2\pi\mathrm{i}(\psi-\phi)} + e^{-2\pi\mathrm{i}\phi} + e^{2\pi\mathrm{i}(\phi-\psi)},$$

$$\tilde{y} = \mathrm{tr}(M_{2+}) = \mathrm{tr}(M_{2-}) = e^{2\pi\mathrm{i}\psi} + e^{2\pi\mathrm{i}(-2\phi+\psi)} + e^{-2\pi\mathrm{i}\psi} + e^{2\pi\mathrm{i}(2\phi-\psi)}.$$

Dividing this expression by the singular element $\Phi^{as}_{1,1}$ and by the product pq (to ensure that polynomial subscripts are in agreement with the order of differentiation), we obtain the generating function of the Chebyshev polynomials of the second kind $F^{II}_r(p,q;\tilde{x},\tilde{y})$ in terms of $\tilde{x},\tilde{y}$ variables. Calculation of derivatives gives us

$$U_{10} = \left.\frac{\partial F^{II}_r(p,q;\tilde{x},\tilde{y})}{\partial p}\right|_{p=0,q=0} = \tilde{x}, \quad U_{01} = \left.\frac{\partial F^{II}_r(p,q;\tilde{x},\tilde{y})}{\partial q}\right|_{p=0,q=0} = \tilde{y}+1.$$

Here, it is convenient to introduce the new variables,

$$x = \tilde{x}, \quad y = \tilde{y}+1.$$

In terms of these variables, the generating function $F^{II}_{p,q}$ has the final form

$$F^{II}_{p,q} = \frac{1+q+p^2q+p^2q^2-xpq}{(1-xp+(y+1)p^2-xp^3+p^4)(1-(y-1)q+(x^2-2y)q^2-(y-1)q^3+q^4)}. \tag{35}$$

A few polynomials of the second kind calculated by the formula

$$U_{m,n} = \frac{1}{m!n!}\left.\frac{\partial^{m+n}F^{II}_{p,q}}{\partial p^m\partial q^n}\right|_{p=0,q=0}$$

are listed in the Appendix. These polynomials coincide with the ones which were found in Ref. 19 from the recurrence relations if one interchanges the polynomial subscripts and performs the substitution $x \to X_2,\ y \to X_1$.

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APPENDIX: THE COEFFICIENTS FOR GENERATING FUNCTION (23) AND A FEW BIVARIATE CHEBYSHEV POLYNOMIALS

The list of the coefficients which define generating function (23),

$$
\begin{aligned}
K_{00} &= 8,\\
K_{10} &= -6x,\\
K_{20} &= 4y + 8,\\
K_{30} &= -2x,\\
K_{01} &= -6y,\\
K_{11} &= 5xy - 2x,\\
K_{21} &= -4y^2 + 2x^2 - 10y,\\
K_{31} &= 2xy - 2x,\\
K_{02} &= 4x^2 - 8y - 8,\\
K_{12} &= -4x^3 + 9xy + 10x,\\
K_{22} &= 3x^2y - 6y^2 + 4x^2 - 20y - 8,\\
K_{32} &= -2x^3 + 5xy + 6x,\\
K_{03} &= -2y,\\
K_{13} &= 2xy - 2x,\\
K_{23} &= -2y^2 + 2x^2 - 6y,\\
K_{33} &= xy - 2x.
\end{aligned}
$$

The list of a few bivariate Chebyshev polynomials of the first kind associated with the Lie algebra C_2 generated by function (23) with normalization (21),

$$
\begin{aligned}
T_{0,0} &= 1,\\
T_{1,0} &= x,\\
T_{2,0} &= x^2 - 2y - 4,\\
T_{3,0} &= x^3 - 3xy - 3x,\\
T_{0,1} &= y,\\
T_{1,1} &= xy - 2x,\\
T_{2,1} &= x^2y - 2y^2 - 6y,\\
T_{3,1} &= x^3y - 3xy^2 - 4xy + 2x,\\
T_{0,2} &= y^2 - 2x^2 + 4y + 4,\\
T_{1,2} &= xy^2 - 2x^3 + 3xy + 6x,\\
T_{2,2} &= x^2y^2 - 2x^4 - 2y^3 - 12y^2 + 8x^2y + 10x^2 - 20y - 8,\\
T_{3,2} &= x^3y^2 - 2x^5 - 3xy^3 + 10x^3y - 15y^2x + 10x^3 - 25xy - 10x,\\
T_{0,3} &= y^3 - 3x^2y + 6y^2 + 9y,\\
T_{1,3} &= xy^3 - 3x^3y + 5y^2x + 2x^3 + 6xy - 6x,\\
T_{2,3} &= x^2y^3 - 2y^4 - 3x^4y - 16y^3 + 12x^2y^2 + 20x^2y - 40y^2 - 30y,\\
T_{3,3} &= x^3y^3 - 3x^5y - 3xy^4 + 15x^3y^2 - 21xy^3 + 18x^3y - 45y^2x - 2x^3 - 21xy + 6x.
\end{aligned}
$$

The list of a few bivariate Chebyshev polynomials of the second kind associated with the Lie algebra C_2 generated by function (16),

$$
\begin{aligned}
U_{0,0} &= 1,\\
U_{1,0} &= x,\\
U_{0,1} &= y,\\
U_{2,0} &= x^2 - y - 1,\\
U_{1,1} &= xy - x,\\
U_{0,2} &= -x^2 + y^2 + y,\\
U_{3,0} &= x^3 - 2xy - x,
\end{aligned}
$$

$$U_{2,1} = x^2 y - x^2 - y^2 - y + 1,$$
$$U_{1,2} = -x^3 + x y^2 + x,$$
$$U_{0,3} = y^3 - 2x^2 y + x^2 + 2y^2 - 1.$$

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