Math. Ann. 260, 93–99 (1982)

Mathematische Annalen

A Class of Orthogonal Polynomials in k Variables*

Richard Eier[1] and Rudolf Lidl[2]

1 Institut für Datenverarbeitung, Technische Universität Wien, A-1040 Wien, Austria
2 Department of Mathematics, University of Tasmania, Hobart, Tasmania, 7001, Australia

1. Introduction

The theory of orthogonal polynomials in one variable is very rich and well established. However, very few comparable results have been obtained for orthogonal polynomials in several variables. In this paper we introduce a family of polynomials Q_M in k complex variables with k integer parameters $\{m_1, \ldots, m_k\}$, denoted by M. Certain subclasses and special cases of these polynomials have been studied before, the case $k=2$ has been thoroughly investigated by Koornwinder [3, 4]. Other special cases and properties have been considered in [1, 2, 5, 6].

These polynomials are generalizations of Chebyshev polynomials of the first kind. It is shown that these generalized polynomials are orthogonal on a region which can be defined in terms of a simple subfamily of the polynomials Q_M. The final part of this paper indicates how to obtain generalized Chebyshev polynomials of the second kind.

2. Definition of Polynomials

In order to define generalized Chebyshev polynomials of the first kind in k complex variables $x_1, \ldots, x_k$, we first study functions $Q_{m_1, \ldots, m_k}$ described as follows. Let u_j, $1 \leqq j \leqq k+1$, be elements in the field $\mathbb{C}$ of complex numbers and let $\{n_1, \ldots, n_k\}$ be a set of k parameters with $n_i \in \mathbb{Z}$. If we consider arbitrary symmetric functions in $u_1, \ldots, u_{k+1}$, let u_{k+1} be dependent of $u_1, \ldots, u_k$ according to the relationship $u_{k+1} = \left(\prod_{j=1}^{k} u_j\right)^{-1}$. Then a function $Q_{n_1, \ldots, n_k}$, abbreviated as Q_N, $N = \{n_1, \ldots, n_k\}$, in the variables $u_1, \ldots, u_k$ is defined by

$$Q_N(u_1, \ldots, u_k) = \sum u_1^{\mu_1} u_2^{\mu_2} \ldots u_k^{\mu_k} u_{k+1}^{\mu_{k+1}} \tag{1}$$

where the sum is taken over all $(k+1)$-tuples of the exponents which are permutations of $N \cup \{0\}$. Thus there are $(k+1)!$ terms in this sum (1), each term has a permutation of $\{n_1, n_2, \ldots, n_k, 0\}$ as its vector of exponents $(\mu_1, \ldots, \mu_{k+1})$. We note

* We acknowledge support of this project by a research grant from the University of Tasmania

0025-5831/82/0260/0093/$01.40

that in this way we cannot obtain a unique representation of Q_N, since the n_i are arbitrary integers. For a unique expression of the functions in (1) we choose a set $M=\{m_1, m_2, \ldots, m_k\}$ of nonnegative integers with the property

$$m_1 \geqq m_2 \geqq \ldots \geqq m_k \geqq 0 \tag{2}$$

as parameters in Q_M instead of the set $N=\{n_1, \ldots, n_k\}$ of arbitrary integers. Such a representation of the parameters of the functions in (1) can always be achieved. Clearly, the definition of these functions yields the following property. Let $M_i=\{m_1^{(i)}, \ldots, m_k^{(i)}\}$, $m_1^{(i)} \geqq m_2^{(i)} \geqq \ldots \geqq m_k^{(i)} \geqq 0$ and $i=1,2$. Then $M_1 \neq M_2$ implies $Q_{M_1} \neq Q_{M_2}$. In the special case $k=2$ a different notation was introduced by Koornwinder [4], p. 483, which could be generalized to

$$Q_{m_1, \ldots, m_k, m_{k+1}}(u_1, \ldots, u_k) = \sum u_{i_1}^{m_1} \ldots u_{i_k}^{m_k} u_{i_{k+1}}^{m_{k+1}}$$

where the sum is taken over all permutations $(i_1, \ldots, i_{k+1})$ of $(1, 2, \ldots, k+1)$ and where $m_1 \geqq m_2 \geqq \ldots \geqq m_k \geqq m_{k+1} \geqq 0$ are integers.

An important property of the functions Q_M is that they are symmetric in $u_1, \ldots, u_k$ and also in $u_1, \ldots, u_{k+1}$. Let x_j denote the j-th elementary symmetric function in $u_1, \ldots, u_{k+1}$. Since any symmetric function in $u_1, \ldots, u_{k+1}$ can be expressed in terms of the $k+1$ elementary symmetric functions x_j, $1 \leqq j \leqq k+1$, and since $x_{k+1} = u_1 \ldots u_{k+1} = 1$ according to the notation introduced above, the function Q_M can be represented as a polynomial in the complex variables $x_1, \ldots, x_k$. These polynomials $Q_M(x_1, \ldots, x_k)$ can be regarded as k-dimensional generalizations of the Chebyshev polynomials of the first kind, since the special cases $k=1$ and $k=2$ yield well known polynomials of Chebyshev type, see the examples in Sect. 4 and also Koornwinder [3] and Rivlin [7].

3. Orthogonality of the Polynomials Q_M

We wish to establish orthogonality of the polynomials $Q_M(x_1, \ldots, x_k)$ on a region of $\mathbb{C}^k$. One possibility would be to calculate the residue of $Q_{M_1} Q_{M_2}^*$, where * denotes taking the complex conjugate. This is equivalent to putting $u_j = e^{i\psi_j}$, $1 \leqq j \leqq k$, and taking the integral over $0 \leqq \psi_j \leqq 2\pi$ for $1 \leqq j \leqq k$. Then $Q_M(u_1, \ldots, u_k)$ are obviously orthogonal on $0 \leqq \psi_j \leqq 2\pi$. Indeed, let M_1 and M_2 be two sets of parameters $\{m_1, \ldots, m_k\}$ and $\{m_1', \ldots, m_k'\}$, respectively, satisfying property (2). Then

$$\frac{1}{(2\pi i)^k} \int_0^{2\pi} \ldots \int_0^{2\pi} Q_{M_1}(u_1, \ldots, u_k) Q_{M_2}^*(u_1, \ldots, u_k) d\psi_1 \ldots d\psi_k = \begin{cases} 0 & \text{if } M_1 \neq M_2 \\ \text{constant} \neq 0 & \text{if } M_1 = M_2 . \end{cases} \tag{3}$$

This can be verified immediately by using the definition of Q_{M_1} and Q_{M_2}.

Since the functions Q_M are symmetric in $u_1, \ldots, u_{k+1}$, they are also symmetric in $\psi_1, \ldots, \psi_{k+1}$, where $\psi_{k+1} = -\sum_{j=1}^{k} \psi_j$. Moreover, Q_M is periodic in ψ_j. Therefore we can reduce the interval of periodicity for the ψ_j's to the following region of orthogonality

$$\psi_{k+1} \leqq \psi_1 \leqq \ldots \leqq \psi_k \leqq 2\pi + \psi_{k+1} . \tag{4}$$

The other values of the ψ's can be obtained by permutations of the values of ψ_j in these intervals. (4) gives the basic region of orthogonality of the Q_M.

The orthogonality (3) of the $Q_M(u_1, \ldots, u_k)$ with respect to the ψ_j's can be used to prove orthogonality of the polynomials $Q_M(x_1, \ldots, x_k)$, by using the simple transformation of the variables, where x_j is expressed as the j-th elementary symmetric function of $e^{i\psi_j}$. From (3) we have

$$\int_R \cdots \int Q_{M_1}(x_1, \ldots, x_k) Q^*_{M_2}(x_1, \ldots, x_k) \left|\frac{\partial(x_1, \ldots, x_k)}{\partial(\psi_1, \ldots, \psi_k)}\right|^{-1} dx_1 \ldots dx_n$$

$$= \begin{cases} 0 & \text{if} \quad M_1 \neq M_2 \\ \text{constant} \neq 0 & \text{if} \quad M_1 = M_2, \end{cases} \tag{5}$$

where R denotes the transformed region given in (4). These integrals of polynomials in x_j's are improper integrals, since the weight function becomes infinite at the bounds of integration. The improper integrals converge, because the original integrals in the ψ_j's exist as proper integrals.

Next we evaluate the Jacobian and the region R explicitly.

Lemma 1. $\left|\frac{\partial(x_1, \ldots, x_k)}{\partial(\psi_1, \ldots, \psi_k)}\right| = \left|\prod_{1 \leq r < s \leq k+1} (u_r - u_s)\right|.$

Proof. For our purposes it is more convenient to consider an implicit version of the transformation of the ψ_j's into the x_j's. We introduce a polynomial

$$N(z) = \prod_{r=1}^{k+1} (z - u_r) = \sum_{r=0}^{k+1} (-1)^r x_r z^{k+1-r}, \quad \text{where} \quad x_0 = x_{k+1} = 1.$$

Then $N(u_s) = 0$, $1 \leq s \leq k$, describes the desired transformation in an implicit form. Taking the first partial derivatives with respect to x_r gives

$$\frac{\partial}{\partial x_r} N(u_s) = 0 = (-1)^r u_s^{k+1-r} + N'(u_s) \frac{\partial u_s}{\partial x_r}.$$

Here N' denotes the derivative of the polynomial $N(z)$ with respect to z. Because of $u_s = e^{i\psi_s}$ we obtain

$$\frac{1}{u_s} \frac{\partial u_s}{\partial x_r} = i \frac{\partial \psi_s}{\partial x_r} = -(-1)^r \frac{u_s^{k-r}}{N'(u_s)}.$$

Then the Jacobian yields

$$\frac{\partial(\psi_1, \ldots, \psi_k)}{\partial(x_1, \ldots, x_k)} = i \begin{pmatrix} N'(u_1)^{-1} & & 0 & \ldots & 0 \\ 0 & \ldots & N'(u_2)^{-1} & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & N'(u_k)^{-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} u_1^{k-1} & \ldots & u_1 & 1 \\ \vdots & & \vdots & \vdots \\ u_k^{k-1} & \ldots & u_k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (-1)^{k-1} \end{pmatrix}.$$

Since the second matrix on the right is the Vandermonde matrix, evaluating the determinant gives

$$\left|\frac{\partial(\psi_1,\dots,\psi_k)}{\partial(x_1,\dots,x_k)}\right| = \left|\prod_{s=1}^{k} N'(u_s)^{-1} \prod_{1\le r<s\le k} (u_r-u_s)\right|. \tag{6}$$

We find $N'(u_s)=\sum_{r'=1}^{s-1}(u_s-u_{r'})\sum_{r=s+1}^{k+1}(u_s-u_r)$. Then

$$\prod_{s=1}^{k} N'(u_s)=\prod_{1\le r'\le s'\le k}(u_{s'}-u_{r'})\prod_{1\le s<r\le k+1}(u_s-u_r).$$

Substitution in (6) gives the result. □

Lemma 2. *The region R of orthogonality for the polynomials $Q_M(x_1,\dots,x_k)$ is given by means of the Jacobian according to the equation*

$$\left|\frac{\partial(x_1,\dots,x_k)}{\partial(\psi_1,\dots,\psi_k)}\right|=0. \tag{7}$$

The Jacobian is explicitly given in Lemma 1.

Proof. The region of orthogonality in ψ_j's given by (4) is bounded by $\psi_{k+1}=\psi_1$, $\psi_1=\psi_2,\dots,\psi_k=2\pi+\psi_{k+1}$. In terms of the u_j's this means $u_{k+1}=u_1$, $u_1=u_2,\dots,u_k=u_{k+1}$. Thus the result follows from Lemma 1. □

In order to express the Jacobian in terms of the x_j's we use polynomials $g_m(x_1,\dots,x_k)$, or g_m for short, defined by

$$g_m(x_1,\dots,x_k)=\sum_{j=1}^{k+1} u_j^m \tag{8}$$

where x_j and u_j are as before. These polynomials g_m can be regarded as special cases of the polynomials $Q_M(x_1,\dots,x_k)$; we have

$$g_m(x_1,\dots,x_k)=\frac{1}{k!}Q_{M_0}(x_1,\dots,x_k),\quad \text{with}\quad M_0=\{m,0,\dots,0\}.$$

The polynomials g_m were studied in [5, 6].

Lemma 3.

$$\left|\frac{\partial(x_1,\dots,x_k)}{\partial(\psi_1,\dots,\psi_k)}\right|^2=\det\begin{pmatrix} g_0 & g_{-1} & \cdots & g_{-k} \\ g_1 & g_0 & \cdots & g_{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_k & g_{k-1} & \cdots & g_0 \end{pmatrix} =: \det G.$$

Proof. The product in Lemma 1 can be seen as the determinant of a $(k+1)\times(k+1)$ Vandermonde matrix V,

$$V=\begin{pmatrix} u_1^k & \dots & u_1 & 1 \\ \vdots & & \vdots & \vdots \\ u_k^k & \dots & u_k & 1 \\ u_{k+1}^k & \dots & u_{k+1} & 1 \end{pmatrix}$$

Let $V^\dagger$ denote the Hermitian conjugate of V. We form the product $V^\dagger V$ and obtain the matrix G using the definition of the polynomials g_m. Forming the determinant of this matrix the result follows from Lemma 1 and the fact that $\prod_{j=1}^{k+1} u_j = 1$. □

Lemma 3 makes it possible to calculate the Jacobian quite simply in terms of the polynomials $g_m(x_1, \dots, x_k)$. These polynomials are easily obtained from the generating function (see [5], p. 185).

$$\sum_{m=0}^{\infty} g_m z^m = \frac{\sum_{i=0}^{k} (k+1-i)(-1)^i x_i z^i}{\sum_{i=0}^{k+1} (-1)^i x_i z^i}$$

and

$$\sum_{m=0}^{\infty} g_{-m} z^m = \frac{\sum_{i=0}^{k} (k+1-i)(-1)^i x_{k+1-i} z^i}{\sum_{i=0}^{k+1} (-1)^i x_{k+1-i} z^i}, \quad \text{where we set} \quad x_0 = x_{k+1} = 1\,.$$

We have $g_{-m} = g_m^*$ and $x_i = x_{k+1-i}^*$, i.e. the x_i's are complex conjugates. We summarize the results in the following theorem.

Theorem. *The polynomials $Q_M(x_1, \dots, x_k)$ in the complex variables $x_1, \dots, x_k$, $M = \{m_1, \dots, m_k\}$, $m_1 \geqq m_2 \geqq \dots \geqq m_k \geqq 0$, $m_i \in \mathbb{Z}$, are orthogonal on the region R whose bounds are determined by setting the Jacobian of Lemma 3 equal to 0. The reciprocal Jacobian is the weight function for this orthogonality.*

4. Examples

We specialize the definition of Q_M and the Theorem to the cases $k=1$, $k=2$, and $k=3$.

For $k=1$ the polynomials $Q_m(x)$ are defined by $Q_m(x) = u_1^k + u_2^k = u_1^k + u_1^{-k}$, where $x = u_1 + u_1^{-1}$. For $u_1 = e^{i\varphi}$ we obtain the classical Chebyshev polynomial of the first kind: $Q_m(2\cos\varphi) = 2\cos m\varphi$ in the real variable $x = 2\cos\varphi$. We know (see [5], p. 182) that $g_0(x) = 2$, $g_1(x) = g_{-1}(x) = x$. The theorem verifies that the weight function for the orthogonal polynomials $Q_m(x)$ is given by

$$\left|\det\begin{pmatrix} 2 & x \\ x & 2 \end{pmatrix}\right|^{-1/2} = (4 - x^2)^{-1/2}.$$

The interval of orthogonality is, of course, $[-2, 2]$. For further properties see Rivlin [7].

For $k=2$ the polynomials $Q_{m_1, m_2}(x_1, x_2)$ are defined by

$$Q_{m_1, m_2}(x_1, x_2) = \sum u_1^{\mu_1} u_2^{\mu_2} u_3^{\mu_3}$$

where (μ_1, μ_2, μ_3) runs over of all permutations of $\{m_1, m_2, 0\}$, i.e. there are 6 such triples. We note that $u_3=(u_1u_2)^{-1}$, therefore

$$Q_{m_1,m_2}(x_1,x_2)=u_1^{m_1}u_2^{m_2}+u_1^{m_1-m_2}u_2^{-m_2}+u_1^{-m_2}u_2^{m_1-m_2}+u_1^{m_2}u_2^{m_1}$$
$$+u_1^{m_2-m_1}u_2^{-m_1}+u_1^{-m_1}u_2^{m_2-m_1}.$$

Here the variables x_1 and x_2 are complex conjugates.

These are essentially the polynomials $p_{m,n}^{(-1/2)}$ investigated by Koornwinder [3], Eier et al. [2] and others. From the generating function for the g_m's we obtain

$$g_0=3,\quad g_1=x_1,\quad g_{-1}=x_2,\quad g_2=x_1^2-2x_2,\quad g_{-2}=x_2^2-2x_1.$$

Then the theorem shows that the $Q_{m_1,m_2}(x_1,x_2)$ are orthogonal on the region R defined by

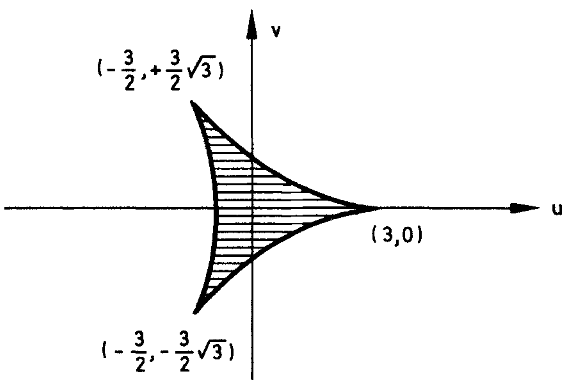
$$\det G=-x_1^2x_2^2+4x_1^3+4x_2^3-18x_1x_2+27=0.$$

The resulting curve is a closed three-cusped curve of fourth degree which is known as Steiner's hypocycloid. The polynomials $Q_{m_1,m_2}(x_1,x_2)$ are orthogonal on R with respect to the weight function $(\det G)^{-1/2}$. □

The region of orthogonality can be described by the equation

$$-(u^2+v^2+9)^2+8(u^3-3uv^2)+108=0$$

in the real variables u and v and is of the form



See [4], p. 451.

For $k=3$ the polynomials $Q_{m_1,m_2,m_3}(x_1,x_2,x_3)$ are defined by

$$Q_{m_1,m_2,m_3}(x_1,x_2,x_3)=\sum u_1^{\mu_1}u_2^{\mu_2}u_3^{\mu_3}u_4^{\mu_4}$$

where $(\mu_1,\mu_2,\mu_3,\mu_4)$ runs over all permutations of $\{m_1,m_2,m_3,0\}$, $m_1\geqq m_2\geqq m_3\geqq 0$ and x_i is the i-th elementary symmetric function in u_1, u_2, u_3, u_4 with $u_1u_2u_3u_4=1$. The generating function for the g_m's yields $g_0=4$, $g_1=x_1$, $g_2=x_1^2-2x_2$, $g_3=x_1^3-3x_1x_2+3x_3$, $g_{-1}=x_3$, $g_{-2}=x_3^2-2x_2$, $g_{-3}=x_3^3-3x_2x_3+3x_1$. Then the theorem shows that the polynomials $Q_{m_1,m_2,m_3}(x_1,x_2,x_3)$ are orthogonal on the region R defined by

$$\det(G)=256-27(x_1^4+x_3^4)+144x_2(x_3^2+x_1^2)-4x_2^3(x_1^2+x_3^2)+18x_1x_2x_3(x_1^2+x_3^2)$$
$$-80x_1x_2^2x_3+x_1^2x_2^2x_3^2-192x_1x_3-10x_1^3x_3^3-128x_2^2+16x_2^4=0,$$

with respect to the weight function $(\det G)^{-1/2}$.

5. Chebyshev Polynomials of the Second Kind

In this final section we indicate very briefly how the classical Chebyshev polynomials of the second kind might be generalized to a class of polynomials in k variables with k parameters. We use the notation introduced in Sect. 2 and define

$$P_M(x_1, \dots, x_k) = \frac{\sum(-1)^r u_1^{\mu_1} \dots u_{k+1}^{\mu_{k+1}}}{\prod\limits_{1 \leq i < j \leq k+1} (u_i - u_j)} \tag{9}$$

as Chebyshev polynomials in k complex variables $x_1, \dots, x_k$ and k parameters $\{m_1, \dots, m_k\} = M$ satisfying (2). Here the summation runs over all $(k+1)$-tuples $(\mu_1, \dots, \mu_{k+1})$ which are permutations of $\{m_1+1, \dots, m_k+1, 0\}$. The exponent r of -1 indicates the number of inversions in the exponents $(\mu_1, \dots, \mu_{k+1})$. We recall that $u_{k+1} = \prod\limits_{i=1}^{k} u_i$. Special cases of these polynomials have been studied before. For $k=1$ we obtain from (9) the classical Chebyshev polynomials of the second kind, given in the form

$$P_m(x) = \frac{u^{m+1} - u^{-(m+1)}}{u - u^{-1}}$$

where $x = x_1$, $u = u_1$, $u^{-1} = u_2$ and $\{m\} = \{m_1\} = M$. Let $u = e^{i\psi}$ and $x = u + u^{-1}$. Then $P_m(2\cos\psi) = (\sin\psi)^{-1}(\sin(m+1)\psi)$. See Rivlin [7] for further properties and [1, 5] for generalizations.

In the case $k=2$ formula (9) essentially gives the polynomials $p_{m_1,m_2}^{(1/2)}(x_1, x_2)$ studied in [2–4].

Arguments similar to those in Sect. 3 can establish orthogonality of the general polynomials $P_M(x_1, \dots, x_k)$ on the region R defined in the Theorem of Sect. 3. Details of this and other properties of these generalized polynomials especially a second order partial differential equation will be given elsewhere.

References

1. Dunn, K.B., Lidl, R.: Multi-dimensional generalizations of the Chebyshev polynomials, I, II. Proc. Japan Acad. **56**, 154–165 (1980)
2. Eier, R., Lidl, R., Dunn, K.B.: Differential equations for generalized Chebyshev polynomials. Rend. Mat. **14**, no. 3 (1981)
3. Koornwinder, T.H.: Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators I–IV. Jndag. Math. **36**, 48–66, 357–381 (1974)
4. Koornwinder, T.: Two-variable analogues of the classical orthogonal polynomials. In: Theory and applications of special functions, pp. 435–495, ed.: Askey, R. New York: Academic Press 1975
5. Lidl, R.: Tschebyscheffpolynome in mehreren Variablen. J. reine angew. Math. **273**, 178–198 (1975)
6. Ricci, P.E.: I polinomi di Tchebycheff in più variabili. Rend. Mat. **11**, 295–327 (1978)
7. Rivlin, T.J.: The Chebyshev polynomials. New York: Wiley 1974

Received December 17, 1981