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Some Remarks on Multivariate Chebyshev Polynomials

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Abstract—Several generalizations of the Chebyshev polynomials in one variable to Chebyshev polynomials in several variables are presented.

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1. INTRODUCTION

The Chebyshev polynomials ($T_n(x)$, $n = 0, 1, \dots,$) in one variable are extremal polynomials in differing contexts. In particular, they may be defined by this property in cases where they are the *unique* extremals. See the following examples.

(a) If $p^*_{n-1}(x)$ is the (necessarily unique) best uniform approximation to x^n on $[-1, 1]$, then

$$x^n - p^*_{n-1}(x) = 2^{1-n}T_n(x),$$

i.e., among all $p_n(x) = x^n + \cdots$, $2^{1-n}T_n(x)$ has minimal maximum norm on $[-1, 1]$.

(b) Consider the orthogonal polynomials on $[-1, 1]$ with respect to the weight function $\left(1 - x^2\right)^{-1/2}$. They satisfy the 3-term recurrence formula, $p_0 = 1$, $p_1 = x$

$$p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x), \qquad n \ge 1,$$

and the differential equation

$$\left(1 - x^2\right) p''_n(x) - xp'_n(x) + n^2x = 0.$$

But the recurrence formula is satisfied by $T_n(x)$ $(= \cos n\theta,\ x = \cos\theta)$, so $p_n(x) = T_n(x)$.

(c) Let $\mathcal{P}_n$ be the set of polynomials of degree n in one real variable. If $p \in \mathcal{P}_n$ and $|p(x)| \le 1$, $-1 \le x \le 1$, then

$$\left|p^{(j)}(t)\right| \le \left|T_n^{(j)}(t)\right|, \qquad |t| \ge 1; \qquad j = 0, 1, \dots, n.$$

Equality is possible for $j \ge 1$; and $j = 0$, $|t| > 1$, only if $p = \pm T_n$.

We further remark that there are generalizations of (a) and (c). Namely, if $p(x) = 2^{n-1}x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $\eta_j^{(n)} = \cos(j\pi)/n$, $j = 0, \dots, n$ then

(a′)

$$\max_{j=0,\dots,n} \left|p\left(\eta_j^{(n)}\right)\right| \ge \max_{j=0,\dots,n} \left|T_n\left(\eta_j^{(n)}\right)\right| = 1,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX with equality only for $p = T_n$. Since

$$\max_{-1\le x\le 1} |T_n(x)| = 1,$$

then for $p \ne T_n$,

$$\max_{-1\le x\le 1} |p(x)| > 1.$$

(c′) If $p \in \mathcal{P}_n$ and $|p(\eta_j)| \le 1, \eta_j = \cos(j\pi)/n$, $j = 0, 1, \dots, n$, then

$$\left|p^{(j)}(t)\right| \le \left|T_n^{(j)}(t)\right|, \qquad |t| \ge 1, \quad j = 0, 1, \dots, n.$$

Equality is possible for $j \ge 1$; and $j = 0$ for $|t| > 1$, only if $p = \pm T_n$.

In what follows, we wish to examine some properties of Chebyshev polynomials in more than one variable. In particular, we wish to show that a Chebyshev polynomial in several variables may provide generalizations of the same flavor as (a) and (c) above (at least for the case $j = 0$ in (c)). Or, put more colloquially, the min-max polynomial inside grows fastest outside in some configurations in dimensions higher than 1. We must also bear in mind that uniqueness is rarely a property of multivariate Chebyshev polynomials.

We have mentioned the Chebyshev polynomials as orthogonal polynomials in (b) above. There is an extensive and interesting literature about generalizations of the orthogonal Chebyshev polynomials in one variable, but we will not consider these generalizations here. The interested reader may examine [1] and [2] which also contain further useful references.

2. SOME RESULTS IN THE LITERATURE

Section 2.1.

In [3], Rivlin and Shapiro generalize (c) of the Introduction, for $j = 0$, as follows. Let $\mathcal{P}(n, m)$ denote the set of all real polynomials in m variables of degree $\le n$, K be a strictly convex body in m-space and P_0 a point exterior to K. They wish to determine $\max |p(P_0)|$ over all $p \in \mathcal{P}(n, m)$ satisfying $\max |p(P)| \le 1$, $P \in K$. The problem is solved by finding a pair of parallel supporting hyperplanes to K such that the points of tangency, P_1 and P_2, are collinear with P_0. Then, the absolute value of the Chebyshev polynomial (in one variable) of degree n on $\overline{P_1P_2}$ at P_0 is $[\max |p(P_0)|, \|p\| \le 1]$. An extremal polynomial is the Chebyshev polynomial of degree n on the segment of the line through P_0 perpendicular to the supporting parallel hyperplanes which pass through P_1 and P_2, respectively.

For example, if $m = 2$, $n = 2$ and K is the closed elliptical region defined by $4x^2 + 9y^2 \le 36$, then if $p(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ satisfies $|p(x, y)| \le 1$ for $(x, y) \in K$, we obtain $|p(3, 2)| \le 3$ with equality for $p^*(x, y) = x^2/9 + y^2/4 + (xy)/3 - 1$.

Section 2.2.

Newman and Xu [4] examine an analogue of (a) for the triangular planar set

$$S = \{(x, y) : x \ge 0,\ y \ge 0,\ 0 \le x + y \le 1\}. \tag{1}$$

Let $\|\cdot\|$ be the uniform norm on S. Then they show that for every $p_{m,n} \in \mathcal{P}(n + m - 1, 2)$

$$\left\|2^{2(n+m)-1} x^m y^n - p_{m,n}\right\| \ge \|T_{m,n}(x, y)\| = 1,$$

where

$$T_{m,n}(x,y) = \begin{cases} \text{(i)} & n > m \\ & T_{n-m}(2y-1)T_m(8xy-1) + 8xy(2y-1)U_{n-m-1}(2y-1)U_{m-1}(8xy-1) \\ \text{(ii)} & n = m \\ & T_n(8xy-1) \\ \text{(iii)} & n < m \\ & T_{m-n}(2x-1)T_n(8xy-1) + 8xy(2x-1)U_{m-n-1}(2x-1)U_{n-1}(8xy-1) \end{cases}$$

(and $U_r(x)$ is the Chebyshev polynomial of the second kind of degree r). The authors also remark that the extremal polynomials $T_{m,n}(x,y)$ need not be unique, in contrast to the situation on the interval.

This beautiful result appeared in 1993. The author of this survey had forgotten the result in (2.1) which appeared in 1961, and stumbled on it again only recently.

3. SOME MULTIVARIATE RESULTS

Section 3.1.

Consider the simple case, $\mathcal{P}(1,2)$ on S (defined in (1)). In this case ($n = m = 1$), the Chebyshev polynomial $8xy - 1$ is unique. Suppose

$$\|8xy - (ax + by + c)\|_S = 1,$$

so that for $y = 0,\ 0 \le x \le 1$ and $x = 0,\ 0 \le y \le 1$, we obtain $|c| \le 1$, $|a + c| \le 1$ and $|b + c| \le 1$. Suppose $x + y = 1$, $0 \le x \le 1$, let us examine $8x(1-x) - ((a-b)x + (b+c))$. The parabola $y = 8x(1-x)$, $0 \le x \le 1$ is nonnegative and $y = 2$ when $x = 1/2$. But the line $y = (a-b)x+(b+c)$ has the value $b + c$ when $x = 0$ and $a + c$ when $x = 1$ with $|b + c| \le 1$ and $|a + c| \le 1$, thus, when $x = 1/2$, $y \le 1$ with equality only if $b + c = a + c = 1$, i.e., $(a-b)x + (b+c) = 1$.

Now we examine the case that $p(x,y) = \alpha xy - (ax + by + c)$ (with $0 < \alpha$) satisfies

$$\|p(x,y)\|_S \le 1. \tag{2}$$

It is easy to see that a pointwise analogue of (c) in the case of $j = 0$ cannot be attained. For, if $p(x,y) = 4xy$ in (2), then for $x = 0.8$, $y = 0.2$ ($x + y \ne 1$), $|4xy| = 0.64$ while $|8xy - 1| = 0.28$.

Since (c), in the case $j = 0$, does not generalize when we consider all (x,y) satisfying $0 \le x \le s$, $x + y = s \ge 1$, we turn our attention to the following natural question. For $p \in \mathcal{P}(1,2)$ and $s \ge 0$, put

$$\|p(x,y)\|_s = \max\{|p(x,y)| \colon x + y = s, \quad 0 \le x \le s\}.$$

If $\|p\|_s \le 1$ for some suitably chosen subset of $0 \le s \le 1$, does it then follow that $\|p\|_s \le \|T_{1,1}\|_s$ for all $s \ge 1$? In other words, replace zero dimensional point evaluation by one-dimensional norms on diagonal parallels, $x + y = s \ge 1$. We shall next show that this is the case.

First, let us remark that if $\|p(x,y)\|_S \le 1$ (and $p(x,y) \ne T_{1,1}$), then $\alpha < 8$. For

$$\|p(x,y)\|_S = \|\alpha xy - (ax + by + c)\|_S \ge \left\|\alpha xy - \frac{\alpha}{8}\right\| = \frac{\alpha}{8}$$

with equality, if and only if $a = b = 0$, $c = \alpha/8$, in view of the uniqueness of the Chebyshev polynomial for $n = m = 1$. Therefore, $\|p(x,y)\|_S \le 1 \Leftrightarrow \alpha < 8$.

Suppose $p(x,y) = (\alpha xy - (ax + by - c))$, ($\alpha > 0$), satisfies

$$|p(0,0)| = |c| = 1, \qquad \|p\|_1 \le 1. \tag{3}$$

In view of (3), we have $|b+c| \le 1$ and $|a+c| \le 1$. Fix $s \ge 0$. Suppose $x+y=s$, $0 \le x \le s$, then

$$p(x,y) = -\alpha x^2 + (\alpha s - a + b)x - (bs + c) =: g_s(x).$$

Then $g_s(0) = -(bs+c)$, $g_s(s) = -(as+c)$ and $g_s'(x) = -2\alpha x + \alpha s - a + b$ implies that

$$x = \frac{s}{2} + \frac{b-a}{2\alpha}.$$

We consider two cases.

CASE I. $|b-a| \ge s\alpha$. Then we have

$$\|p\|_s = \max\{|as+c|, |bs+c|\}.$$

CASE II. $|b-a| < \alpha s$. Then we have

$$g_s\left(\frac{s}{2} + \frac{b-a}{2\alpha}\right) = \frac{\alpha}{4}s^2 - \frac{a+b}{2}s + \left(\frac{(b-a)^2}{4\alpha} - c\right),$$

and

$$\|p\|_s = \max\left\{|as+c|, |bs+c|, \left|\frac{\alpha}{4}s^2 - \frac{a+b}{2}s + \left(\frac{(b-a)^2}{4\alpha} - c\right)\right|\right\}.$$

In particular, if $p(x,y) = q(x,y) = 8xy - 1$, $\|q\|_s = 1, 0 \le s \le 1$ and

$$\|q\|_s = 2s^2 - 1, \qquad s \ge 1.$$

Now if $s = 0, 1$, $\|q\|_s = 1$ and $\|p\|_s \le 1 = \|q_s\|$, is $\|p\|_s \le \|q\|_s$ for $s > 1$? We begin by examining the size of $|as+c|$ and $|bs+c|$. Consider $w(s) = 2s^2 - 1 - (as+c)$.

$$w(1) = 1 - (a+c) \ge 0, \qquad w'(1) = 4 - a > 2.$$

Therefore, if $s > 1$, $|as+c| < 2s^2 - 1$, and similarly, $|bs+c| < 2s^2 - 1$. Thus, if $|b-a| \ge s\alpha$, $s > 1$ then

$$\|q\|_s > \|p\|_s.$$

In case II, $|b-a| \le \alpha s$, $s \ge 1$,

$$w(s) = 2s^2 - 1 - \left(\frac{\alpha}{4}s^2 - \frac{a+b}{2}s + \frac{(b-a)^2}{4\alpha} - c\right).$$

Now $w(1) \ge 0$ and

$$w'(1) = 4 - \left(\frac{\alpha}{2} - \frac{a+b}{2}\right); \qquad w''(s) = 4 - \frac{\alpha}{4} > 0.$$

It is now easy to see that

$$\frac{\alpha}{2} - \frac{a+b}{2} \le 4 - \frac{(b-a)^2}{2\alpha},$$

and so $w'(1) \ge 0 (= 0 \Leftrightarrow b = a)$. Thus,

$$\|p\|_s \le \|q\|_s, \qquad s \ge 1. \qquad \blacksquare$$

Section 3.2.

Suppose now that

$$S = \{(x, y) : xy \geq 0,\ x + y = s,\ -1 \leq s \leq 1\}, \tag{4}$$

and

$$\mathcal{P}(n, 2) = \left\{ \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j \right\}.$$

Consider the $\mathcal{P}(x, y) \in P(n, 2)$ for which

$$\sum_{i+j=n} a_{i,j} x^i y^j = 2^{n-1}(x + y)^n,$$

so that

$$a_{i,n-i} = 2^{n-1}\binom{n}{i}, \qquad i = 0, \ldots, n,$$

and let the set of such $p(x, y)$ be $\Pi(n, 2)$. Then we have the following theorem.

THEOREM. *For every $p \in \Pi(n, 2)$,*

$$\|p\|_S \geq \|T_n(x + y)\|_S \tag{5}$$

with equality, if and only if $p(x, y) = \pm T_n(x + y)$, where T_n is the Chebyshev polynomial of degree n. Moreover, if $p(x, y) \in \mathcal{P}(n, 2)$ and $\|p\|_S \leq 1$, then if $xy \geq 0$ and $|x + y| \geq 1$,

$$|p(x, y)| \leq |T_n(x + y)|. \tag{6}$$

PROOF. Suppose $p(x, y) \in \Pi(n, 2)$.

(i) If $x = 0$ and $-1 \leq y \leq 1$, put

$$p(x, y) = q(y) = 2^{n-1} y^n + \sum_{i=0}^{n-1} a_i y^i.$$

Then

$$\max_{-1 \leq y \leq 1} |q(y)| \geq \max_{-1 \leq y \leq 1} |T_n(y)| = 1$$

with equality $\Leftrightarrow q(y) = \pm T_n(y)$.

(ii) Suppose $y = \lambda x$, $0 \leq \lambda < \infty$. For $-1 \leq (1 + \lambda)x \leq 1$,

$$p(x, y) = p(x, \lambda x) = 2^{n-1}((1 + \lambda)x)^n + \sum_{i=0}^{n-1} b_i((1 + \lambda)x)^i.$$

Thus

$$\max_{-1 \leq (1+\lambda)x \leq 1} |p(x, \lambda x)| \geq \max_{-1 \leq (1+\lambda)x \leq 1} |T_n(1 + \lambda)x| = 1,$$

with equality $\Leftrightarrow p(x, \lambda x) = \pm T_n((1 + \lambda)x)$. (5) now follows.

We turn next to (6). If $\|p\|_S \leq 1$, then we have the following.

(α) If $x = 0$ and $|y| \geq 1$, $|p(0, y)| \leq 1$, $-1 \leq y \leq 1$ and

$$|p(0, y)| \leq |T_n(y)|, \qquad |y| \geq 1.$$

(β) Suppose $0 \leq \lambda < \infty$ and $y = \lambda x$. Since $\|p\|_s \leq 1$, $|p(x, \lambda x)| \leq 1$ for $-1 \leq (1 + \lambda)x \leq 1$. The Chebyshev polynomial on this interval is $T_n((1 + \lambda)x)$, and so since the one-dimensional Chebyshev polynomial majorizes one-dimensional polynomials which are bounded by 1 on this interval (such as $p(x, \lambda x)$) at points of the line $y = \lambda x$ for which $(1 + \lambda)|x| \geq 1$, we obtain (6), since $x + y = (1 + \lambda)x$. ∎

REMARK 1. It does not seem plausible that (6) continues to hold for $xy < 0$, $|x - y| > 1$.

REMARK 2. The choice of $2^{n-1}(x + y)^n$ as leading term for a Chebyshev polynomial is rather natural. I would guess that in k dimensions the choice should be $2^{n-1}(x_1 + \cdots + x_k)^n$ and a Chebyshev polynomial on the corresponding domain is $T_n(x_1 + \cdots + x_k)$.

REMARK 3. It seems as if the Chebyshev polynomial, as defined above, is unique. That is surprising.

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