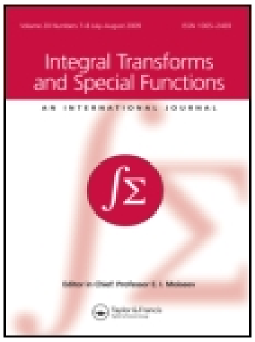

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Expansions of multivariable Bessel functions

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First kind multivariable Bessel functions $J_n^{(m)}(\{ax\}_1^m)$ are expanded in terms of $J_K^{(m)}(\{x\}_1^m)J_{n-k}^{(m)}(\{(a-1)x\}_1^m)$. Parallely we expand many variable Hermite-Bessel functions $B_n^{(m)}(\{ax\}_1^m)$ and draw further consequences.

Keywords: Multivariable Bessel functions; Hermite-Bessel functions; Jacobi–Anger generating functions

2000 Mathematics Subject Classification: 33C30

1. Introduction

First kind multivariable Bessel functions $J_n^{(m)}(\{x\}_1^m)$ Dattoli and Torre [1] are defined through the Jacobi–Anger generating function

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{in\vartheta} J_n^{(m)}(\{x\}_1^m) = \mathrm{e}^{i\sum_{s=1}^{m} x_s \sin(s\vartheta)}, \{x\}_1^m = x_1, \ldots, x_m \tag{1}$$

and by the integral representation

$$J_n^m(\{x\}_1^m) = \frac{1}{\pi}\int_0^{\pi} \cos\left(n\vartheta - \sum_{s=1}^{m} x_s \sin(s\vartheta)\right) \mathrm{d}\vartheta. \tag{2}$$

For $m = 1$ and $\mathrm{e}^{i\vartheta} = t$, equation (1) reduces to the Bessel functions of the first kind $J_n(x)$.

$$\sum_{n=-\infty}^{\infty} J_n(x)t^n = \mathrm{e}^{(x/2)(t-(t/2))}, t \neq 0, n \in \mathbb{Z} \tag{3}$$

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Kapteyn type series [2] associated with the above function have been studied in [3] and written as

$$\sum_{n=-\infty}^{\infty} J_n^{(m)}(\{nx\}_1^m) = \frac{1}{1-\sum_{s=1}^m sx_s} \sum_{s=1}^m |sx_s| < 1. \tag{4}$$

Along with multivariable Bessel functions, Hermite-Bessel functions defined by the following generating function

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{in\vartheta} B_n^{(m)}(\{x\}_1^m) = \mathrm{e}^{i\sum_{s=1}^m i^s x_s (\sin(s\vartheta))^s} \tag{5}$$

have been introduced in [4]. The functions $B_n^{(m)}(\{x\}_1^m)$ are linked to the multivariable Hermite polynomials [5] by the series expansion

$$B_n^{(m)}(\{x\}_1^m) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(\{x\}_1^m)}{2^{n+2r} r!\,(n+r)!}. \tag{6}$$

The polynomials $H_n^{(m)}(\{x\}_1^m)$ are also specified by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(\{x\}_1^m) = \mathrm{e}^{\sum_{s=1}^m x_s t^s}. \tag{7}$$

2. Expansions

In order to obtain expansions of multivariable Bessel functions, are considered the generating function (1) and written as

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{in\vartheta} J_n^{(m)}(\{ax\}_1^m) = \mathrm{e}^{i\sum_{s=1}^m a_s x_s \sin(s\vartheta)} = \mathrm{e}^{i\sum_{s=1}^m (a_s-1) x_s \sin(s\vartheta)} \mathrm{e}^{i\sum_{s=1}^m x_s \sin(s\vartheta)}$$

$$= \sum_{k=-\infty}^{\infty} \mathrm{e}^{ik\vartheta} J_k^{(m)}(\{x\}_1^m) \sum_{l=-\infty}^{\infty} \mathrm{e}^{il\vartheta} J_l^{(m)}(\{(a_s-1)x_s\}_1^m)$$

where $\{ax\}_1^m = a_1x_1, a_2x_2, \ldots, a_m x^m$.

Substitution of $l = n-k$ for fixed values of $k \in \mathbb{Z}$ gives

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{in\vartheta} J_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} J_k^{(m)}(\{x\}_1^m) \sum_{n=-\infty}^{\infty} J_{n-k}^{(m)}(\{(a_s-1)x_s\}_1^m)\mathrm{e}^{in\vartheta}$$

Equating the coefficients of powers of $\mathrm{e}^{i\vartheta}$ on each side finally gives

$$J_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} J_k^{(m)}(\{x\}_1^m) J_{n-k}^{(m)}(\{(a_s-1)x_s\}_1^m), \quad k, n \in \mathbb{Z} \tag{8}$$

In analogy to equation (1), the generating function for the multivariable extension of the modified Bessel functions of the first kind is

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{in\vartheta} I_n^{(m)}(\{x\}_1^m) = \mathrm{e}^{\sum_{s=1}^{\infty} x_s \cos(s\vartheta)}, \quad \{x\}_1^m = x_1, \ldots, x_m \tag{9}$$

With this in mind, one may adopt the same procedure to obtain the following expansion for the above functions.

$$I_n^{(m)}(\{x\}_1^m) = \sum_{k=-\infty}^{\infty} I_k^{(m)}(\{x\}_1^m)\, I_{n-k}^{(m)}(\{(a_s-1)x_s\}_1^m), \quad k, n \in \mathbb{Z} \tag{10}$$

Next we consider the generating function (5) in place of equation (1) and apply the same technique as before to get the following expansion of multivariable Hermite–Bessel functions $B_n^{(m)}$

$$B_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} B_k^{(m)}(\{x\}_1^m)\, B_{n-k}^{(m)}(\{(a_s-1)x_s\}_1^m), \quad k, n \in \mathbb{Z}. \tag{11}$$

3. Special cases

The main results (8), (10) and (11) offer many special cases. A few are given below.

Set $m = 1$ in equations (8), (10) and (11), we get the following known results of Predel [6] for ordinary Bessel and modified Bessel functions

$$J_n(ax) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_{n-k}((a-1)x), \quad k, n \in \mathbb{Z}$$

$$I_n(ax) = \sum_{k=-\infty}^{\infty} I_k(x)\, I_{n-k}((a-1)x), \quad k, n \in \mathbb{Z}$$

Setting $m = 3$ and assume that $x_2 = 0$ in equation (11). Then using [7]

$$B_n^{(3)}(x_1, x_3) = J_n^{(3)}\left(x_1 - \frac{3}{4}x_3, \frac{1}{4}x_3\right)$$

we get,

$$J_n^{(3)}\left(a_1x_1 - \frac{3}{4}a_3x_3, \frac{1}{4}a_3x_3\right) = \sum_{k=-\infty}^{\infty} J_k^{(3)}\left(x_1 - \frac{3}{4}x_3, \frac{1}{4}x_3\right) J_{n-k}^{(3)} \times \left((a_1-1)x_1 - \frac{3}{4}(a_3-1)x_3, \frac{1}{4}(a_3-1)x_3\right)$$

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