ELSEVIER

Statistics & Probability Letters 47 (2000) 165–169

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

A simple expression for the multivariate Hermite polynomials

C.S. Withers

Industrial Research Ltd., P.O. Box 31-310, Lower Hutt, New Zealand

Received May 1999; received in revised form July 1999

Abstract

We give a simple method for obtaining the multivariate Hermite polynomials. Explicitly we give the bi- and trivariate polynomials up to order six: these are needed for the third-order Edgeworth expansions for the distribution and density of most standardised vector estimates. © 2000 Elsevier Science B.V. All rights reserved

Keywords: Multivariate; Hermite polynomials; Edgeworth expansions

1. Introduction

Suppose that $X \sim \mathcal{N}_p(0, V)$, the p-dimensional normal distribution with mean 0, covariance V positive-definite, density and distribution

$$\phi(x, V) = (2\pi)^{-p/2} \det(V)^{-1/2} \exp(-x' V^{-1} x/2)$$

and

$$\Phi(y, V) = \int_{-\infty}^{y} \phi(x, V)\, \mathrm{d}x.$$

For N the non-negative integers and v a vector integer in N^p, define

$$x^v = x_1^{v_1} \cdots x_p^{v_p}$$

and the vth Hermite polynomial with respect to V as

$$\mathrm{He}_v(x, V) = \phi(x, V)^{-1} (-\partial/\partial x)^v \phi(x, V). \tag{1.1}$$

These polynomials are the building blocks in Edgeworth and saddle-point expansions for most standardised p-dimensional estimates: the expansion of the distribution and density to magnitude $n^{-r/2}$, where $r \geqslant 2$ and n is the total sample size requires these polynomials up to order $3r-4$ for the distribution and order $3r-3$ for the density, where the *order* of v is $v_1 + \cdots + v_p$. (See, for example, Withers (1984a) for the univariate case.)

E-mail address: c.withers@irl.cri.nz (C.S. Withers)

PII: S0167-7152(99)00153-4

Hermite polynomials also play an important role in stochastic integrals, Central Limit Theorems and in L_2 theory: they are orthogonal with respect to their dual polynomials

$$\tilde{\mathrm{He}}_\nu(x,V)=\phi(x,V)^{-1}(-\partial/\partial z)^\nu\phi(Vz,V)\quad \text{at } z=V^{-1}x, \tag{1.2}$$

in the sense that

$$\int \tilde{\mathrm{He}}_j(x,V)\mathrm{He}_k(x,V)\phi(x,V)\,\mathrm{d}x=j!\quad \text{if } j=k \text{ and 0 otherwise} \tag{1.3}$$

where $j!=j_1!\cdots j_p!$. This follows by (A.15) of Takemura and Takeuchi (1988). They also show (A.7), and (A.14) that

$$\tilde{\mathrm{He}}_\nu(x,V)=\mathrm{He}_\nu(z,V^{-1})\quad \text{at } z=V^{-1}x$$

and give the recurrence relation

$$\mathrm{He}_{\nu+e_k}(x,V)=(V^{-1}x)_k\mathrm{He}_\nu(x,V)-\partial\mathrm{He}_\nu(x,V)/\partial x_k\quad \text{for}\quad k=1,\ldots,p,$$

where e_k is the kth unit vector in R^p.

Other methods for computing multivariate Hermite polynomials have been given by Withers (1984b, 1985), Section 5.8 of Barndorff-Nielsen and Cox (1989) and Holmquist (1993). However, these methods are not as suited as the present method to obtaining most particular values as they require keeping track of a large number of terms, many of which are the same unless $\nu_j\equiv 1$. Barndorff-Nielsen and Pederson (1979) gave the bivariate polynomials up to order six, but our form is much simpler.

In Section 2 we prove our main result:

$$\mathrm{He}_\nu(x,V)=E(z+\mathrm{i}Z)^\nu\quad \text{where}\quad z=V^{-1}x,\ Z\sim\mathcal{N}_p(0,V^{-1}) \tag{1.4}$$

and $\mathrm{i}=\sqrt{-1}$. This simple formula does not appear to have been noticed even for the univariate case: for $\phi(x)$ the density of $X\sim\mathcal{N}_1(0,1)$, then

$$\mathrm{He}_n(x)=\phi(x)^{-1}(-\mathrm{d}/\mathrm{d}x)^n\phi(x)=E(x+\mathrm{i}X)^n. \tag{1.5}$$

In Section 3 we apply it to give all the Hermite polynomials up to order three. Note that (1.4) implies

$$\tilde{\mathrm{He}}_\nu(x,V)=E(x+\mathrm{i}X)^\nu, \tag{1.6}$$

$$\mathrm{He}_\nu(x,V)=\sum_{0\leqslant k\leqslant(\nu_1+\cdots+\nu_p)/2}(-1)^k h_{\nu,k}(z)\quad \text{at}\quad z=V^{-1}x \tag{1.7}$$

where

$$h_{\nu,k}(z)=\sum_{n_1+\cdots+n_p=2k}(EZ_1^{n_1}\cdots Z_p^{n_p})\prod_{j=1}^p\binom{\nu_j}{n_j}z_j^{\nu_j-n_j},$$

and

$$\partial\mathrm{He}_\nu(x,V)/\partial x_k=\sum_{j=1}^p V^{kj}\nu_j\mathrm{He}_{\nu-e_j}(x,V)\quad \text{for}\quad k=1,\ldots,p \tag{1.8}$$

for e_k as above, and $(V^{kj})=V^{-1}$.

2. Proof of main result

Here we prove our main result (1.4).

For, taking t in C^p, by Taylor's expansion in C^p,

$$\begin{aligned}\sum_{v\in N^p} \mathrm{RHS}(1.4) t^v/v! &= E\exp(t'(z+\mathrm{i}Z))\\ &= \exp(t'z - t'V^{-1}t/2)\\ &= \phi(x,V)^{-1}\phi(x-t,V)\\ &= \sum_{v\in N^p} \mathrm{LHS}(1.4)\ t^v/v!.\end{aligned}$$

As well as being the generating function with respect to t for $\{\mathrm{He}_v(x,V), v\in N^p\}$, this is also the generating function with respect to $s=V^{-1}t$ for $\{\tilde{\mathrm{He}}_v(x,V), v\in N^p\}$.

3. Bi- and trivariate polynomials up to order six

Here we use our basic result (1.4) to give the bi- and trivariate Hermite polynomials up to order six in terms of the moments of $Z\sim\mathcal{N}_p(0,V^{-1})$ where V is $p\times p$ and non-singular. Those up to order five and six are the building blocks needed for the expansions of the distribution and density up to magnitude $n^{-3/2}$ of most asymptotically normal vector estimates where n is the total sample size. A vector estimate of dimension q requires the p-variate Hermite polynomials for $p\leqslant q$.

Set $z=V^{-1}x$,

$$H_v = \mathrm{He}_v(x,V)$$

and

$$\mu_v = EZ^v = EZ_1^{v_1}\cdots Z_p^{v_p}.$$

For $p=2r$ these are given by changing the indices in

$$\mu_{1\cdots 1} = EZ_1\cdots Z_{2r} = \sum_{1\cdots p}^{m} V^{1,2}\cdots V^{r-1,r} \quad \text{with } m=1.3\cdots(2r-1),$$

where

$$(V^{ij}) = V^{-1} \quad \text{and} \quad \sum_{1\cdots r}^{m} a_{1\cdots r} = \sum a_{p_1\cdots p_r}$$

summed over all m say, permutations $p_1\cdots p_r$ of $1\cdots r$ giving distinct terms.

This summation convention is clearer in the dual notation

$$\mu_v = \mu^{r_1 r_2\cdots} = EZ_{r_1}Z_{r_2}\cdots \quad \text{for} \quad Z_1^{v_1}\cdots Z_p^{v_p} = Z_{r_1}Z_{r_2}\cdots$$

For example, the third term in $H_{33}=H_{330\cdots 0}$ below is

$$\sum_{12}^{2} z_1^3 z_2 \mu_{02} = \sum_{12}^{2} z_1^3 z_2 \mu^{22} = z_1^3 z_2 \mu^{22} + z_2^3 z_1 \mu^{11} = z_1^3 z_2 \mu_{02} + z_2^3 z_1 \mu_2,$$

where we suppress trailing zeros in μ_v, H_v: $\mu_2=\mu_{20\cdots 0}$, $\mu_{02}=\mu_{020\cdots 0}$ and so on.

The univariate polynomials: Taking $v=(n,0,\ldots,0)$ gives

$$H_n = H_{n0\cdots 0} = \sum_{0\leqslant j\leqslant n/2} \binom{n}{2j} z_1^{n-2j}(-1)^j m_{2j} = \sum_{0\leqslant j\leqslant n/2} (n)_{2j} z_1^{n-2j}(-V^{11}/2)^j/j!$$

where $m_{2j}=EZ_1^{2j}=(V^{11})^j 1.3.5.\cdots(2j-1)$, and $(n)_k=n!/(n-k)!=n(n-1)\cdots(n-k+1)$.

For V the identity matrix, this result is in Abramowitz and Stegun (1964, 23.2.11, p. 775).

The bivariate polynomials:

$$H_{11}=z_1z_2-\mu_{11},$$

$$H_{21}=z_1^2z_2-z_2\mu_2-2z_1\mu_{11},$$

$$H_{31}=z_1^3z_2-3z_1z_2\mu_2-3z_1^2\mu_{11}+\mu_{31},$$

$$H_{22}=z_1^2z_2^2-z_1^2\mu_{02}-2z_1z_2\mu_{11}-z_2^2\mu_2+\mu_{22},$$

$$H_{41}=z_1^4z_2-4z_1^3\mu_{11}-6z_1^2z_2\mu_2+4z_1\mu_{31}+z_2\mu_4,$$

$$H_{32}=z_1^3z_2^2-z_1^3\mu_{02}-6z_1^2z_2\mu_{11}-3z_1z_2^2\mu_2+3z_1\mu_{22}+2z_2\mu_{31},$$

$$H_{51}=z_1^5z_2-5z_1^4\mu_{11}-10z_1^2z_2\mu_2+10z_1^2\mu_{31}+5z_1z_2\mu_4-\mu_{51},$$

$$H_{42}=z_1^4z_2^2-\mu_{02}-6z_1^2z_2^2\mu_2+z_2^4\mu_4-8z_1^3z_2\mu_{11}+6z_1^2\mu_{22}+8z_1z_2\mu_{31}-\mu_{42},$$

$$H_{33}=z_1^3z_2^3-9z_1^2z_2^2\mu_{11}-3\sum_{12}^{2} z_1^3z_2\mu_{02}+9z_1z_2\mu_{22}+3\sum_{12}^{2} z_1^2\mu_{13}-\mu_{33},$$

where

$$\mu_{11}=EZ_1Z_2=V^{12},$$

$$\mu_{31}=3V^{11}V^{12},$$

$$\mu_{22}=V^{11}V^{22}+2(V^{12})^2,$$

$$\mu_{51}=15(V^{11})^2V^{12},$$

$$\mu_{42}=3V^{11}V^{22}+12V^{11}(V^{12})^2,$$

$$\mu_{33}=9V^{11}V^{12}V^{22}+6(V^{12})^3.$$

The trivariate polynomials:

$$H_{111}=z_1z_2z_3-\sum_{123}^{3} z_1V^{23},$$

$$H_{211}=z_1^2z_2z_3-z_1^2\mu_{011}-z_2z_3V^{11}-2z_1\sum_{23}^{2} z_2V^{13}+\mu_{211},$$

$$H_{311}=z_1^3z_2z_3-z_1^3V^{23}-3z_1(z_2z_3V^{11}+z_1z_2V^{13}+z_1z_3V^{12})+3z_1\mu_{211}+3(z_2V^{13}+z_3V^{12})V^{11},$$

$$H_{221}=z_1^2z_2^2z_3-4z_1z_2z_3V^{12}-\sum_{12}^{2}(z_1^2z_3V^{22}+2z_1z_2^2V^{13})+z_2\mu_{22}+2\sum_{12}^{2} z_2\mu_{211},$$

$$H_{411}=z_1^4z_2z_3-z_1^4V^{23}-6z_1^2z_2z_3V^{11}-4z_1^3\sum_{23}^{2} z_2V^{13}+6z_1^2\mu_{211}+z_2z_3\mu_4+4z_1\sum_{23}^{2} z_3\mu_{31}-\mu_{411},$$

$$H_{321} = z_1^3 z_2^2 z_3 - z_1^3 z_3 V^{22} - 2z_1^3 z_2 V^{23} - 3z_1 z_2^2 z_3 V^{11} - 3z_1^2(2z_2 z_3 V^{12} + z_2^2 V^{13})$$
$$+ 3z_1^2 \mu_{121} + 3z_1 z_3 \mu_{22} + 6z_1 z_2 \mu_{211} + 2z_2 z_3 \mu_{31} + z_2^2 \mu_{301} - \mu_{321},$$

$$H_{222} = z_1^2 z_2^2 z_3^2 - \sum_{123}^{3} z_1^2 z_2^2 V^{33} + \sum_{123}^{3} z_3^2 \mu_{22} - \mu_{222},$$

where

$$\mu_{211} = V^{11}V^{23} + 2V^{12}V^{13},$$
$$\mu_{321} = 3V^{11}V^{13}V^{22} + 6V^{11}V^{12}V^{23} + 6(V^{12})^2 V^{13},$$
$$\mu_{411} = 3(V^{11})^2 V^{23} + 12V^{11}V^{12}V^{33},$$
$$\mu_{222} = V^{11}V^{22}V^{33} + 2\sum_{123}^{3} V^{11}(V^{23})^2 + 8V^{12}V^{13}V^{23}.$$

Note 3.1. Grad (1949) gave the multivariate Hermite polynomials for V the identity matrix. This was extended to general V by (2.6) of Skovgaard (1986). Barndorff-Nielsen and Pederson (1979) gave the bivariate polynomials up to order six, but our form is much simpler as he substitutes for z. For example, the simplest he gives is

$$H_{11} = V^{11}V^{12}x_1^2 + ((V^{12})^2 + V^{11}V^{22})x_1 x_2 + V^{12}V^{22}x_2^2 - V^{12},$$

as compared with our

$$H_{11} = z_1 z_2 - \mu_{11}.$$

References

Abramowitz, M., Stegun, I.A., 1964. Handbook of Mathematical Functions. National Bureau of Standards, US Government Printing Office, Washington DC.

Barndorff-Nielsen, O.E., Pederson, B.V., 1979. The bivariate Hermite polynomials up to order six. Scand. J. Statist. 6, 127–128.

Barndorff-Nielsen, O.E., Cox, D.R., 1989. Asymptotic Techniques for use in Statistics. Chapman & Hall, London.

Grad, H., 1949. Note on N-dimensional Hermite polynomials. Commun. Pure Appl. Math. 3, 325–330.

Holmquist, B., 1993. The d-variate vector Hermite polynomial of order k. Manuscript.

Skovgaard, I.M., 1986. On multivariate Edgeworth expansions. Int. Statist. Rev. 54, 169–186.

Takemura, A., Takeuchi, K., 1988. Some results on univariate and multivariate Cornish-Fisher expansions: algebraic properties and validity. Sankhya Ser. A 50, 111–136.

Withers, C.S., 1984a. Asymptotic expansions for distributions and quantiles with power series cumulants. J. Roy. Statist. Soc. B 46, 389–396.

Withers, C.S., 1984b. A chain rule for differentiation with applications to multivariate Hermite polynomials. Bull. Austral. Math. Soc. 30, 247–250.

Withers, C.S., 1985. The moments of the multivariate normal. Bull. Austral. Math. Soc. 32, 103–108.