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Review Paper

A review of multivariate orthogonal polynomials☆

Mourad E.H. Ismail[a], Ruiming Zhang[b,*]

[a] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
[b] College of Science, Northwest A&F University Yangling, Shaanxi 712100, PR China

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ABSTRACT

This paper contains a brief review of orthogonal polynomials in two and several variables. It supplements the Koornwinder survey [40]. Several recently discovered systems of orthogonal polynomials have been treated in this work. We did not provide any proofs of the theorem presented here but references to the original sources are given for the benefit of the interested reader. It is hoped that collecting these scattered results in one place will make them accessible for the user.

1. Introduction

This is a survey paper of certain classes of orthogonal polynomials in two complex variables and the multivariate Hermite polynomials. The multivariate real Hermite polynomials go back to Hermite and they were studied in some detail in the book by Appell and Kampé de Feriet [4]. We felt it will be beneficial to collect scattered old and new results on 2*D* and n*D* polynomials in one place. This paper complements and updates the influential survey article [40] by Koorwinder which is now 40 years old but is still a valuable reference on the subject.

One of the earliest orthogonal polynomials are the Legendre polynomials which are orthogonal with respect to $dx =$ (Lebesque measure) on the unit interval $[-1, 1]$. They are the spherical harmonics in $\mathbb{R}^2$. The ultraspherical (or Gegenbauer) polynomials are the spherical harmonics on $\mathbb{R}^m$ and are orthogonal with respect to $\frac{\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu+1)}(1-x^2)^\nu dx$ on the unit interval $[-1, 1]$. If we replace x by $x/\sqrt{\nu}$ and let $\nu \to \infty$ the measure becomes $\pi^{-1/2}e^{-x^2}dx$ and the ultraspheical polynomials, properly scaled, tend to Hermite polynomials. The next sequence of polynomials in this hierarchy are the Jacobi polynomials which are orthogonal on the unite interval with respect to $\frac{2^{-\alpha-\beta-1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta dx$. The Laguerre polynomials arise when x is replaced by $-1+2x/\alpha$ and $\alpha \to \infty$. It is worth mentioning that none of the one variable polynomials is

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* Corresponding author.
E-mail address: ruimingzhang@yahoo.com (R. Zhang).

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named after the person who first discovered them, see the interesting paper by Askey [5].

The 2*D* history is somewhat parallel. In the late 1920s Frits Zernike (Nobel prize for physics in 1953) was working on optical problems involving telescopes and microscopes. He introduced polynomials orthogonal on $|z| \leq 1$ with respect to the area measure, so this is the 2*D* analogue of Legendre polynomials, see [58] and [59]. The more general disc or Zernike polynomials are orthogonal with respect to $(1-x^2-y^2)^{\nu}dxdy$ on the unit disc. Again if we replace (x, y) by $(x/\sqrt{\nu}, y/\sqrt{\nu})$ and let $\nu \to \infty$ this measure, properly normalized becomes $e^{-x^2-y^2}dxdy$ on $\mathbb{R}^2$. The polynomials become the 2*D*-Hermite polynomials introduced by Ito in [34] in a different way. They are defined by

$$H_{m,n}(z_1, z_2) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k}(-1)^k k! z_1^{m-k} z_2^{n-k}, \tag{1.1}$$

where $a \wedge b := \min\{a, b\}$. In 1966 Myrick [43] considered only the radial part of the polynomials orthogonal with respect to the Jacobi type measure $(x^2+y^2)^{\alpha}(1-x^2-y^2)^{\beta}dxdy$ on the unit disc. An addition theorem for the disc polynomials were found by Săpiro [48] and Koornwinder [39]. Maldonado [42] very briefly considered a class of radial functions which can be used to generate 2*D* orthogonal polynomials. Floris [15] introduced a q-analogue of the disc polynomials proved an addition theorem for them, see also Floris and Koelink [17]. Wünche [57] also studied the disc polynomials.

We shall use the notations

$$(D_{q,z}f)(z) = \frac{f(z)-f(qz)}{(1-q)z}, \quad (\delta_{q,z}f)(z) = z(D_{q,z}f)(z), \tag{1.2}$$

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, \quad m \vee n = \max\{m, n\}, \quad m \wedge n = \min\{m, n\} \tag{1.3}$$

and

$$\partial_z = \frac{\partial}{\partial_z}, \quad \delta_z = z\partial_z, \quad \delta_{q,z} = zD_{q,z}. \tag{1.4}$$

In addition we shall follow the standard notation for shifted factorials, hypergeometric functions and their q-analogues as in [3,19], and [25].

In Section 2 we treat the Complex Hermite polynomials $\{H_n(x+iy)\}_{n\geq 0}$. They are orthogonal on $\mathbb{R}^2$ and are polynomials in the two real variables x and y with complex coefficients. We state their orthogonality relations and show that they are eigenfunctions of a 2*D*-Laplace transform. In Section 3 we treat Ito's 2*D*-Hermite polynomials in some detail. This includes orthogonality, differential recurrence relations, pure recurrence relations as well as linear and multilinear generating functions. Many of the multilinear generating functions are kernels of integral operators with explicit eigenvalues and eigenfunctions. This is typical of the material treating specific sets of orthogonal polynomials in the subsequent sections. This section is followed by Section 4 which is a systematic study of their q-analogues introduced by the present authors in [33]. Section 5 contains a survey of the n-dimensional Hermite polynomials introduced by C. Hermite in 1860s. They form a biorthogonal system of polynomials. Section 5 contains recent results from [28]. In particular we mention combinatorial interpretations of the integrals of products of the polynomials times their weight functions.

In Section 6 we briefly describe a class of polynomials we introduced in [31]. This contains a general construction of bivariate orthogonal polynomials from a univariate system of polynomials orthogonal on a half line. Sections 7–11 contains important examples of this construction. In each section a specific system of orthogonal polynomials is introduced and its important properties are recorded. The paper concludes with Sections 12 where we discuss polynomial solutions to certain second order partial differential equations.

For completeness we recall the definition of the Ramanujan function $A_q(z)$ and ϑ_4,

$$A_q(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{(q;q)_n} q^{n^2}, \tag{1.5}$$

$$\vartheta_4(z;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n. \tag{1.6}$$

Note that our ϑ_4 is slightly different from ϑ_4 in Whittaker and Watson [53]. The q-Laguerre polynomials are [25, Section 21.8]

$$L_n^{(\alpha)}(z) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k+\alpha)}}{(q^{\alpha+1};q)_k}(-z)^k. \tag{1.7}$$

2. The complex Hermite polynomials

The complex Hermite polynomials are the usual Hermite polynomials $\{H_n(x+iy)\}_{n\geq 0}$, $x, y \in \mathbb{R}$ when viewed as functions of the two variables x and y. Many of the algebraic properties of Hermite polynomials $\{H_n(x)\}_{n\,\geq\,0}$ hold when x is taken as a complex variable. In particular we have the generating function [25,45,50]

$$\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n = \exp(2zt - t^2). \tag{2.1}$$

Theorem 2.1. *Assume that*

$$0 < a < b, \qquad \frac{1}{a} = 1 + \frac{1}{b}. \tag{2.2}$$

Then the complex Hermite polynomials satisfy the orthogonality relation, [36,52],

$$\int_{\mathbb{R}^2} H_m(x+iy)H_n(x-iy)\, e^{-ax^2-by^2}\, dx\, dy = \frac{\pi}{\sqrt{ab}} 2^n n! \left(\frac{a+b}{ab}\right)^n \delta_{m,n} \tag{2.3}$$

as well as the orthogonality relations [22]

$$\int_{\mathbb{R}^2} H_m(x+iy)H_n(x+iy)e^{-ax^2-by^2}\, dx\, dy$$
$$= \int_{\mathbb{R}^2} H_m(x-iy)H_n(x-iy)e^{-ax^2-by^2}\, dx\, dy = \frac{\pi}{\sqrt{ab}} 2^n n!\, \delta_{m,n}. \tag{2.4}$$

The proof follows from the generating function (2.1).

The weight function $e^{-ax^2-by^2}$, on $\mathbb{R}$ has the moments

$$\int_{\mathbb{R}^2} (x \pm iy)^{2n+1}\, e^{-ax^2-by^2}\, dx\, dy = 0,$$
$$\int_{\mathbb{R}^2} (x \pm iy)^{2n}\, e^{-ax^2-by^2}\, dx\, dy = \frac{\pi}{\sqrt{ab}}(1/2)_n, \tag{2.5}$$

provided that the condition (2.2) is satisfied.

Ismail and Simeonov [27] studied the combinatorics of the complex Hermite polynomials. One of their combinatorial results established the more general orthogonality relation

$$\int_{\mathbb{R}^2} H_m(\alpha x + \beta y)H_n(\gamma x + \delta y)e^{-ax^2-by^2}\, dx\, dy$$
$$= \frac{\pi}{\sqrt{ab}} 2^n n! \left(\frac{\alpha\gamma}{a} + \frac{\beta\delta}{b}\right)^n \delta_{m,n} \tag{2.6}$$

where

$$a > 0, \quad b > 0, \qquad \frac{\alpha^2}{a} + \frac{\beta^2}{b} = \frac{\gamma^2}{a} + \frac{\delta^2}{b} = 1. \tag{2.7}$$

Formula (2.6) implies that for

$$\begin{pmatrix} \alpha/\sqrt{a} & \beta/\sqrt{b} \\ \delta/\sqrt{a} & \delta/\sqrt{b} \end{pmatrix} \in SO(2, \mathbb{R}),$$

then $H_n(\alpha x+\beta y)$ is orthogonal to $H_m(\delta x+\gamma y)$ unless $m=n=0$. It is also clear that the orthogonality relations (2.3) and (2.4) are special cases of (2.6).

Theorem 2.2. *Suppose that we have two colored multisets. The first, of color I, has size* $\mathbf{m}=(m_1,m_2,\dots,m_k)\in\mathbb{N}_0^k$ *and the second, of color II, has size* $\mathbf{n}=(n_1,n_2,\dots,n_k)\in\mathbb{N}_0^k$. *We match elements from different sets, and to each pair of matched elements we assign weight 1 if the elements have the same color and weight* $1/a+1/b$ *if the elements have different colors.*

Then the total weight of all perfect matchings of this type is the number $B(\mathbf{m},\mathbf{n})$ *defined by*

$$B(\mathbf{m},\mathbf{n})=\frac{\sqrt{ab}}{\pi}2^{-\frac12(|\mathbf{m}|+|\mathbf{n}|)}\int_{\mathbb{R}^2}\prod_{j=1}^{k}[H_{m_j}(x+iy)H_{n_j}(x-iy)]\times e^{-ax^2-by^2}\,dx\,dy. \tag{2.8}$$

Observe that the orthogonality relation (2.3) is the special case $k=1$ of Theorem 2.2.

Theorem 2.3 [27]**.** *The following integral relations hold*

$$\frac{\sqrt{ab}}{\pi\sqrt2}\int_{\mathbb{R}^2}e^{iz(\xi-i\eta)+(z^2+(\xi-i\eta)^2)/2}H_n(\xi+i\eta)\,e^{-a\xi^2-b\eta^2}\,d\xi\,d\eta=\left(\frac{b+a}{ab}i\right)^n H_n(z), \tag{2.9}$$

$$\frac{\sqrt{ab}}{\pi\sqrt2}\int_{\mathbb{R}^2}e^{iz(\xi-i\eta)+(z^2+(\xi-i\eta)^2)/2}H_n(\xi-i\eta)\,e^{-a\xi^2-b\eta^2}\,d\xi\,d\eta=i^nH_n(z), \tag{2.10}$$

$n\in\mathbb{N}_0$, *where a and b satisfy conditions (2.2).*

This theorem indicates that the complex Hermite polynomials are eigenfunctions of a 2-dimensional Fourier transform. However, we could not use the orthogonal system $S=\{H_n(x+iy)\}_{n=1}^\infty\cup\{1\}\cup\{H_n(x-iy)\}_{n=1}^\infty$ to define a 2d Fourier transform. First we observe that $H_n(x-iy)$ is not an eigenfunction. Furthermore, the system S is not complete. It has the same $L^2(R^2,\frac{\sqrt{ab}}{\pi}e^{-ax^2-by^2}dxdy)$ closure as $\{(x+iy)^n\}_{n=1}^\infty\cup\{1\}\cup\{(x-iy)^n\}_{n=1}^\infty$, which does not contain all of $\{x^ny^m\}_{n,m=0}^\infty$. For example, x^2, y^2 is not in the closure. Moreover, in order to do Fourier analysis we also need both 2.3 and 2.4 under the condition 2.2 to ensure the orthogonality of system S. To require they are eigenfunctions correspond eigenvalue $\pm i$, ±1, from Theorem 2.3 we also need to have the condition $\frac1a+\frac1b=1$. But the two conditions imply that $a=1,\ b=\infty$, which is impossible.

3. Ito's 2D-Hermite polynomials

The complex Hermite polynomials $\{H_{m,n}(z_1,\ z_2)\}$ defined by (1.1) were introduced by Ito [34] in his study of complex multiple Wiener integral and were used in [1] to study Landau levels. They were applied in [51] to coherent states, and in [55,56] to quantum optics and quasi probabilities, respectively. See also [10,20,21]. They are essentially the same as the polynomials in [11, (2.6.6)]. They also appeared in combinatorics in the work of Novik et al. [44].

Their exponential generating function is

$$\sum_{m,n=0}^{\infty}H_{m,n}(z_1,z_2)\frac{u^m\,v^n}{m!\,n!}=e^{uz_1+vz_2-uv}, \tag{3.1}$$

and they satisfy the orthogonality relation, [24] and [20]

$$\frac1\pi\int_{\mathbb{R}^2}H_{m,n}(x+iy,x-iy)\overline{H_{p,q}(x+iy,x-iy)}\,e^{-x^2-y^2}\,dx\,dy=m!\,n!\,\delta_{m,p}\,\delta_{n,q}. \tag{3.2}$$

By 3.1 we also get

$$e^{uz_1+vz_2}=e^{uv}\sum_{m,n\ge0}\frac{u^mv^n}{m!n!}H_{m,n}(z_1,z_2)=\sum_{m,n,\ell\ge0}\frac{u^{m+\ell}v^{n+\ell}}{m!n!\ell!}H_{m,n}(z_1,z_2),$$

then,

$$z_1^jz_2^k=\frac{\partial^{j+k}e^{uz_1+vz_2}}{\partial z_1^j\partial z_2^k}\Big|_{z_1=z_2=0}=\sum_{m,n,\ell\ge0,\,m+\ell=j,\,n+\ell=k}\frac{j!k!}{m!n!\ell!}H_{m,n}(z_1,z_2),$$

which gives

$$\begin{aligned}z_1^jz_2^k&=\sum_{j\ge m\ge0,\,k\ge n\ge0}\binom{j}{m}\binom{k}{n}\times\left(\frac{j+k-m-n}{2}\right)!H_{m,n}(z_1,\ z_2)\\&=\sum_{j\ge m\ge0,\,k\ge n\ge0}\binom{j}{m}\binom{k}{n}(j-m)!H_{m,n}(z_1,\ z_2)\\&=\sum_{j\ge m\ge0,\,k\ge n\ge0}\binom{j}{m}\binom{k}{n}(k-n)!H_{m,n}(z_1,\ z_2).\end{aligned}$$

The combinatorics of these polynomials have been studied in the recent work [27]. We note that Carlitz rediscovered these polynomials in the more recent work and found their generating functions [8].

It is important to think of the 2D-Hermite polynomials as polynomials in two complex variables or four real variables but their orthogonality is on the complex plane or on $\mathbb{R}^2$.

From the generating function it is easy to see that the polynomials satisfy the three term recurrence relations

$$\begin{aligned}z_1H_{p,q}(z_1,z_2)&=qH_{p,q-1}(z_1,z_2)+H_{p+1,q}(z_1,z_2)\\z_2H_{p,q}(z_1,z_2)&=pH_{p-1,q}(z_1,z_2)+H_{p,q+1}(z_1,z_2).\end{aligned} \tag{3.3}$$

They also satisfy the connection relation

$$H_{m,n}(z,\bar z)=m!n!\left(\frac i2\right)^{m+n}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{i^{j+k}}{j!k!}(-1)^{j+n}\times\frac{H_{j+k}(x)H_{m+n-j-k}(y)}{(m-j)!\,(n-k)!}, \tag{3.4}$$

and its inverse

$$H_r(x)H_s(y)=\sum_{j,m\ge0}\frac{r!\ s!\ i^s\ (-1)^{m-j}}{j!(r-j)!(m-j)!(s+j-m)!}H_{m,r+s-m}(z,\bar z). \tag{3.5}$$

Also note the symmetry relation

$$\overline{H_{m,n}(z,\bar z)}=H_{n,m}(z,\bar z). \tag{3.6}$$

The linearization of products is given by

$$H_{m_1,n_1}(z_1,z_2)H_{m_2,n_2}(z_1,z_2)=\sum_{j,k}\frac{n_1!n_2!H_{m_1+m_2-j-k,n_1+n_2-j-k}(z_1,z_2)}{j!k!(m_1-j)!(n_1-k)!(m_2-k)!(n_2-j)!}, \tag{3.7}$$

and its inverse is, [21],

$$\frac{H_{p+m,q+n}(z_1,z_2)}{p!\,q!\,m!\,n!}=\sum_{j=0}^{p\wedge n}\sum_{k=0}^{q\wedge m}\frac{(-1)^{j+k}}{j!k!}\frac{H_{p-j,q-k}(z_1,z_2)}{(p-j)!(q-k)!}\frac{H_{m-k,n-j}(z_1,z_2)}{(m-k)!(n-j)!}. \tag{3.8}$$

When $m = n + s \geq n$ the polynomials $H_{m,n}(z_1, z_2)$ are related to Laguerre polynomials via

$$H_{n+s,n}(z_1, z_2) = (-1)^n n!\, z_1^s L_n^{(s)}(z_1 z_2). \tag{3.9}$$

The symmetry relation (3.6) shows that $H_{n,n}(z, \bar{z})$ is real. The Rodrigues formulas are

$$\begin{aligned} H_{m,n}(z_1, z_2) &= (-1)^{m+n} e^{z_1 z_2} \partial_{z_2}^m \partial_{z_1}^n e^{-z_1 z_2}, \\ H_{m,n}(z_1, z_2) &= (-1)^{m+n} e^{z_1 z_2} \partial_{z_1}^n \partial_{z_2}^m e^{-z_1 z_2}, \end{aligned} \tag{3.10}$$

where $\partial_z := \frac{\partial}{\partial_z}$. We note that

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{3.11}$$

Note that (3.10) is equivalent to the formulas

$$\begin{aligned} H_{m,n}(z_1, z_2) &= (-1)^m e^{z_1 z_2} \partial_{z_2}^m \left(z_2^n e^{-z_1 z_2}\right), \\ H_{m,n}(z_1, z_2) &= (-1)^n e^{z_1 z_2} \partial_{z_1}^n \left(z_1^m e^{-z_1 z_2}\right). \end{aligned} \tag{3.12}$$

The operational formula

$$\begin{aligned} &(-\partial_{z_1} + z_2)^n f(z_1, z_2) \\ &= n! \sum_{k=0}^n \frac{(-1)^k}{k!} \frac{H_{m,n-k}(z_1, z_2)}{(n-k)!} \partial_{z_1}^k \left(z_1^{-m} f(z_1, z_2)\right), \end{aligned} \tag{3.13}$$

is from [21] and extends a one variable result due to Burchnall [7]. Another operational formula is

$$H_{m,n}(z_1, z_2) = \exp(-\partial_{z_1} \partial_{z_2})(z_1^m z_2^n). \tag{3.14}$$

The Kibble–Slepian formula is a multilinear generating function for Hermite polynomials and is due to Kibble [37] and later was proved using Fourier transforms by Slepian [49]. Foata [18] gave a nice combinatorial proof while Louck [41] used Boson operator calculus to give a new proof of this remarkable result. The norm of a matrix $A = \{a_{j,k} : 1 \leq j, k \leq N\}$ is the Frobenius or the Hilbert–Schmidt norm $\sqrt{\sum_{j,k=1}^N |a_{j,k}|^2}$, [23]. Ismail [26] proved the following

Theorem 3.1. *Let $Z = (z_1, z_2, \ldots, z_N)$, and H be an $N \times N$ Hermitian matrix with $\|H\| < 1$, and I_N is an $N \times N$ identity matrix. Then*

$$\begin{aligned} &\mathrm{Det}[(I_N + H)^{-1}] \exp\left(ZH(I_N + H)^{-1} Z^*\right) \\ &= \textstyle\sum_K \prod_{1 \leq j,k \leq N} \frac{(h_{j,k})^{n_{j,k}}}{n_{j,k}!} H_{r_1,c_1}(z_1, \overline{z_1}), \ldots, H_{r_N,c_N}(z_N, \overline{z_N}), \end{aligned} \tag{3.15}$$

where $K = (n_{j,k} : 1 \leq j, k \leq N)$ is a general matrix with nonnegative integer entries, c_k is the sum of the elements of K in column k and r_j is the sum of the elements of K in row j, that is

$$c_k = \sum_{j=1}^N n_{j,k}, \qquad r_j = \sum_{k=1}^N n_{j,k}. \tag{3.16}$$

It is important to note that the multilinear kernels given by Theorem 3.1 will always be positive.

The following results, namely Theorem 3.2 and Corollary 3.3, are due to Ismail and Zhang [32].

Theorem 3.2. *Following the notation in Theorem 3.1 we let δ be a fixed real number such that $0 < \delta < \frac{1}{3N}$, then (3.15) is also true for*

$$\left|h_{j,k}\right| \leq \delta, \qquad 1 \leq j, k \leq N \tag{3.17}$$

and H is not necessarily Hermitian.

It is known that a multivariate analytic function $f(z_1, \ldots, z_n)$ at $(0, \ldots, 0)$ can be expanded in a convergent polynomial series, [6]. Then one may prove Theorem 3.2 on a domain, possibly all of $||H|| < 1$, in the following steps: first prove the polynomial series is exactly the right hand side of 3.15, then prove both sides of 3.15 are analytic inside the domain by applying the estimate in Theorem 3.5, finally prove Theorem 3.2 by analytic continuation. It is interesting to see this can be carried out for a larger domain.

Corollary 3.3. *For $N \in \mathbb{N}$, let $W = \left(\rho_1 e^{i\theta_1}, \ldots, \rho_N e^{i\theta_N}\right)^T$ that $\rho_j > 0$, $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, N$ in (3.15), H, I_N K, c_j and r_j are the same as in Theorem 3.1 but H is not necessarily Hermitian, then*

$$\begin{aligned} &\det\,(I_N + H)^{-1} \exp\left(W^* H (I_N + H)^{-1} W\right) \\ &\quad = \textstyle\sum_K \prod_{j=1}^N \prod_{k=1}^N \left(-h_{j,k}\right)^{k_{jk}} \binom{c_j}{k_{1,j}, \ldots, k_{N,j}} \left(\rho_j e^{i\theta_j}\right)^{r_j - c_j} L_{c_j}^{(r_j - c_j)}\left(\rho_j^2\right), \end{aligned} \tag{3.18}$$

where $\{L_n^{(\alpha)}(x)\}$ are Laguerre polynomials.

This follows from Theorem 3.2 and (3.9).

It must be noted that Ismail and Zeng [30] gave a purely combinatorial proof of (3.15) but as a formal power series.

Example 1 (Poisson Kernel)**.** As an example consider the case $N = 2$ with

$$H = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

In this case Theorem 3.1 gives the Poisson kernel. Indeed in the sum in (3.15) we must have $n_{1,1} = n_{2,2} = 0$ and we get

$$\begin{aligned} &\sum_{n_{1,2}, n_{2,1}=0}^\infty \frac{u^{n_{1,2}}}{n_{1,2}!} \frac{v^{n_{2,1}}}{n_{2,1}!} H_{n_{1,2}, n_{2,1}}(z_1, \overline{z_1}) H_{n_{2,1}, n_{1,2}}(z_2, \overline{z_2}) \\ &= (1 - uv)^{-1} \exp\left(\frac{-uv z_1 \overline{z_1} + u z_1 \overline{z_2} + v \overline{z_1} z_2 - uv z_2 \overline{z_2}}{1 - uv}\right). \end{aligned} \tag{3.19}$$

The Poisson kernel (3.19) is stated in [56] without proof. Carlitz discovered the 2D-Hermite polynomials independtly of Ito's work and established (3.19). He did not prove the orthogonality relation.

Example 2 (General2 × 2 Case)**.**

Ismail [26] considered the general 2 × 2 case

$$H = \begin{pmatrix} a & u \\ v & b \end{pmatrix}, \quad a, b \in \mathbb{R}$$

This leads to the bilinear generating function

$$\begin{aligned} &\textstyle\sum_{n_{j,k} \geq 0, 1 \leq j,k \leq 2} \frac{a^{n_{1,1}}}{n_{1,1}!} \frac{b^{n_{2,2}}}{n_{2,2}!} \frac{u^{n_{1,2}}}{n_{1,2}!} \frac{v^{n_{2,1}}}{n_{2,1}!} H_{n_{1,1}+n_{1,2}, n_{1,1}+n_{2,1}}(z_1, \overline{z_1}) \\ &\quad \times H_{n_{2,1}+n_{2,2}, n_{1,2}+n_{2,2}}(z_2, \overline{z_2}) \\ &\quad = [(1 + a)(1 + b) - uv]^{-1} \times \exp \\ &\left(\frac{(a(1+b) - uv) z_1 \overline{z_1} + u z_1 \overline{z_2} + v \overline{z_1} z_2 + (-uv + b(1 + a)) z_2 \overline{z_2}}{(1 + a)(1 + b) - uv}\right). \end{aligned} \tag{3.20}$$

An important observation is that (3.19) is essentially a generating function for Laguerre polynomials. Write the series in (3.19) as a sum over $n_{1,2} \geq n_{2,1}$ plus a sum over $n_{1,2} \leq n_{2,1}$ minus the sum over $n_{1,2} = n_{2,1}$. The result is

$$\begin{aligned} &\sum_{s,n=0}^\infty \frac{n!}{(n+s)!} u^{n+s} v^n (z_1 \bar{z}_2)^n L_n^{(s)}(|z_1|^2) L_n^{(s)}(|z_2|^2) \\ &\quad + \sum_{s,n=0}^\infty \frac{n!}{(n+s)!} u^n v^{n+s} (\bar{z}_1 z_2)^n L_n^{(s)}(|z_1|^2) L_n^{(s)}(|z_2|^2) \\ &\quad - \sum_{n=0}^\infty (uv)^n L_n(|z_1|^2) L_n(|z_2|^2) \\ &\quad = (1 - uv)^{-1} \exp\left(\frac{-uv z_1 \overline{z_1} + u z_1 \overline{z_2} + v \overline{z_1} z_2 - uv z_2 \overline{z_2}}{1 - uv}\right). \end{aligned} \tag{3.21}$$

The third series can be summed by the case $\alpha = 0$ of Hille–Hardy formula, [25, (4.7.20)]

$$\sum_{n=0}^{\infty} \frac{n!\, r^n}{(\alpha+1)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = (1-r)^{-\alpha-1} \exp\left(-\frac{r(x+y)}{1-r}\right) {}_0F_1(-;\alpha+1;xyr(1-r)^{-2}). \tag{3.22}$$

It will be of interest to give a direct proof of (3.21) using properties of Laguerre polynomials.

We call a sequence of polynomials $\{p_{m,\,n}(z_1, z_2)\}$ a bivariate Appell sequence if its exponential generating function is of the type

$$\sum_{m,n=0}^{\infty} p_{m,n}(z_1, z_2) \frac{u^m\, v^n}{m!n!} = A(u, v) \exp(uz_1 + vz_2), \tag{3.23}$$

where $A(u, v)$ is analytic in u and v and we normalize our formulas by $A(0, 0) = 1$.

Theorem 3.4 [26]**.** *The only bivariate Appell sequence $p_n(z_1, z_2)$ such that $p_n(z, \bar{z})$ is orthogonal on $\mathbb{R}^2$ is $\{H_{m,n}(z, \bar{z})\}$.*

For properties of Appell polynomials in one variable we refer the interested reader to [45].

Theorem 3.5 [32]**.** *For all $m, n \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ we have*
$|H_{m,n}(\bar{z}, z)| \le e^{|z|^2}\sqrt{m! \cdot n!}$.

Theorem 3.5 is very useful in justifying interchanging sums, or sums and integrals.

We now construct some integral operators. Let

$$K(z_1.\bar{z}_1, z_2, \bar{z}_2; u, v) = \frac{1}{\pi}(1-uv)^{-1} \exp\left(\frac{-uvz_1\overline{z_1} + uz_1\overline{z_2} + v\overline{z_1}z_2 - uvz_2\overline{z_2}}{1-uv}\right), \tag{3.24}$$

and consider the integral operator

$$(Tf)(z_1, \bar{z_1}) = \int_{\mathbb{R}^2} K(z_1, \bar{z}_1, x+iy, x-iy, u, v) f(x+iy, x-iy) \times e^{-x^2-y^2} dxdy, \tag{3.25}$$

defined on $L_2(e^{-x^2-y^2}dxdy, \mathbb{R}^2)$. We know that the polynomials $\{H_r(x)H_s(y)\}_{r,\,s\,\ge\,0}$ are dense in $L_2(e^{-x^2-y^2}dxdy, \mathbb{R}^2)$. Hence formula (3.5) shows that $\{H_{m,n}(z, \bar{z})\}_{m,n\ge 0}$ are also dense in the same space, with $z = x + iy$. Therefore the polynomial system $\{H_{m,n}(z, \bar{z})\}_{m,n\ge 0}$ is complete in $L_2(e^{-x^2-y^2}dxdy, \mathbb{R}^2)$. Then the orthogonality relation (3.2) and the symmetry (3.6) imply

$$\int_{\mathbb{R}^2} K(z_1, \bar{z}_1, x+iy, x-iy, u, v) H_{m,n}(x+iy, x-iy) e^{-x^2-y^2} dxdy = u^m v^n H_{m,n}(z_1, \bar{z_1}). \tag{3.26}$$

This shows that $\{H_{m,n}(z_1, \bar{z_1})\}$ are the eigenfunctions of T with eigenvalues $u^m v^n$ and the completeness of the system implies that there are no other eigenfunctions. For $0 < u, v < 1$ we observe that T is self-adjoint, compact and positive definite. But the series expansion for K cannot be obtained by applying the spectral theorem directly. The expansion obtained from spectral theorem is a re-ordering of $\{u^m v^n\}_{m,n=0}^{\infty}$ so that the obtained 1d sequence decreases to 0. It is not hard to see the expansion also provides an example for the generalized Mercer's theorem, [14].

The Fourier transform in two dimensions corresponds to the special value $u = v = i$ in (3.26). This case was stated explicitly in [55] and can be proved by imitating the classical proof for the one variable Hermite polynomial case in [54]. One cannot however simply put $u = v = i$ in (3.26) because the Poisson kernel formula (3.19) is valid for $\max\{|u|, |v|\} < 1$, since the right hand side of (3.26) can be expanded in power series in u and v when $|u| < 1$ and $|v| < 1$.

We next consider linearization of products of 2*D*-Hermite polynomials.

Note that (3.7) can be written in the form

$$H_{m_1,n_1}(z, \bar{z}) H_{m_2,n_2}(z, \bar{z}) = \sum_s \frac{n_1!\, H_{m_1+m_2-s,n_1+n_2-s}(z,\bar{z})}{m_1!\, s!\, (n_1-s)!\, (m_2-s)!} {}_3F_2\left(\left.\begin{matrix} -s, -m_1, -n_2 \\ n_1-s+1, m_2-s+1 \end{matrix}\right| -1\right). \tag{3.27}$$

It is clear that (3.27) and the orthogonality relation (3.2) give the value of the integral

$$\int_{\mathbb{R}^2} \prod_{j=1}^{k} H_{m_j,n_j}(z, \bar{z}) e^{-x^2-y^2} dxdy, \tag{3.28}$$

when $k = 3$. This raises the question of the evaluation of a product of k polynomials. For $k > 3$ this integral does not have a nice closed form but has an interesting combinatorial interpretation; see [27].

Formulas (3.7) and (3.8) are inverse relations in a combinatorial sense, see Riordan [46,47]. Another result which immediately follows from the generating function is the addition formula

$$\frac{H_{m,n}(z+w, \bar{z}+\bar{w})}{m!\, n!} = 2^{-(p+q)/2} \sum_{j=0}^{p} \sum_{k=0}^{q} \frac{H_{j,k}(\sqrt{2}z, \sqrt{2}\bar{z})}{j!\, k!} \frac{H_{p-j,q-k}(\sqrt{2}w, \sqrt{2}\bar{w})}{(p-j)!\, (q-k)!}. \tag{3.29}$$

The above formula was proved in [21] but also follows from the exponential generating functions [26].

In [26] Ismail derived analogues of

$$\sum_{n=0}^{\infty} H_{n+k}(x) \frac{t^n}{n!} = e^{2xt-t^2} H_k(x-t),$$

$$H_n(cx) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n! c^{n-2k}}{k!(n-2k)!} (1-c^2)^k H_{n-2k}(x),$$

[25, (4.6.29)] and [25, (4.6.33)]. His results are

$$\sum_{m,n=0}^{\infty} H_{m+j,n+k}(z, \bar{z}) \frac{u^m\, v^n}{m!n!} = (-1)^{j+k} e^{uz+v\bar{z}-uv} H_{j,k}(z-v, \bar{z}-\bar{v}), \tag{3.30}$$

and

$$H_{m,n}(cz, \bar{c}\bar{z}) = \sum_{j=0}^{m\wedge n} H_{m-j,n-j}(z, \bar{z}) \frac{m!\, n! c^{m-j} \bar{c}^{n-j}}{j!\, (m-j)!\, (n-j)!} (c\bar{c}-1)^j. \tag{3.31}$$

Notice that by setting $z = 1$, formula 3.31 gives yet another representation of $H_{m,n}(c, \bar{c})$ in terms of $c^{m-j}\bar{c}^{n-j}(c\bar{c}-1)^j$. He also extended

$$\sum_{m_1+m_2+\cdots+m_r=n} \left[\prod_{j=1}^{r} \frac{a_j^{m_j}}{m_j!} H_{m_j}(x_j)\right] = \frac{1}{n!} H_n\left(\sum_{j=1}^{r} a_j x_j\right), \tag{3.32}$$

provided that $\sum_{j=1}^{r} a_j^2 = 1$, [12, (10.13.40)], to

$$\sum_{\sum_{j=1}^{r} m_j=m, \sum_{j=1}^{r} n_j=n} \prod_{j=1}^{r} H_{m_j,n_j}(z_j, \bar{z}_j) \frac{a_j^{m_j} \bar{a}_j^{\,n_j}}{m_j!\, n_j!} = H_{m,n}\left(\sum_{j=1}^{r} a_j z_j, \sum_{j=1}^{r} \bar{a}_j \bar{z}_j\right), \tag{3.33}$$

when $\sum_{j=1}^{r} |a_j|^2 = 1$.

The Hermite polynomials have the generating function [45, Section 106]

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n H_n(x) = (1-2xt)^{-c} {}_2F_0\left(\left.\begin{matrix} c/2, (c+1)/2 \\ - \end{matrix}\right| \frac{-4t^2}{(1-2xt)^2}\right). \tag{3.34}$$

In general (3.34) holds as a formal power series since the ${}_2F_0$ diverges, unless c is a negative integer, in which case (3.34) holds as equality between functions. Rainville [45] attributes (3.34) to Brafman for general c and to Truesdell when $c = 1$. We now extend (3.34) to the complex Hermite polynomials and again our result will be a formal power series identity unless a or b is a negative integer when it becomes an equality. The generating function is

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n}{m!\, n!} u^m v^n H_{m,n}(z,\bar{z}) \\ &\quad = (1-uz)^{-a}(1-v\bar{z})^{-b} {}_2F_0\left(\left.\begin{matrix} a, b \\ - \end{matrix}\right| \frac{uv}{(1-uz)(1-v\bar{z})}\right). \end{aligned} \tag{3.35}$$

We now give a moment integral representation of the $H_{m,\,n}$'s.

Theorem 3.6. *Let $z = x+iy$ with $x, y \in \mathbb{R}$. The complex Hermite polynomials have the moment representation*

$$\begin{aligned} H_{m,n}(iz, i\bar{z}) = \frac{i^{m+n}}{\pi} \int_{\mathbb{R}^2} (r+is)^m (r-is)^n \\ \times \exp\left(-(r-x)^2 - (s-y)^2\right) dr ds. \end{aligned} \tag{3.36}$$

A two complex variable case is in [32].

Theorem 3.7. *Let $w = r+is$ with $r, s \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$, then we have the moment integral representation*

$$e^{-z_1 z_2} H_{m,n}(z_1, z_2) = \frac{1}{\pi\, i^{m+n}} \int_{\mathbb{R}^2} \overline{w}^m w^n \exp\{-w\overline{w} + iz_1 w + iz_2 \overline{w}\} dr ds. \tag{3.37}$$

Note that Theorems 3.6 and 3.7 are the 2D analogues of

$$\frac{H_n(ix)}{(2i)^n} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(y-x)^2} y^n dy, \tag{3.38}$$

see [25, (4.6.41)]. Another integral is

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} H_{m,n}(cz, \bar{c}\bar{z}) dx dy = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{(c\bar{c}-1)^n}{n!}\pi & \text{if } m = n. \end{cases} \tag{3.39}$$

One more integral from [26] is

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} H_{m,n}(\zeta, \bar{\zeta})\, dx dy = \begin{cases} 0 & \text{if } n-m \text{ is odd}, \\ C(m,s) & \text{if } n-m = 2s, \end{cases} \tag{3.40}$$

where

$$\begin{aligned} &\zeta = ax + iby,\, a, b \in \mathbb{R}, \quad m = n+2s, \\ &C(m,s) = \frac{2^{-2s}\pi}{m!\, s!} {}_2F_1\left(\left.\begin{matrix} -m/2, (1-m)/2 \\ s+1 \end{matrix}\right| \frac{a^2-b^2}{4}\right). \end{aligned} \tag{3.41}$$

Several authors considered sign regularity of determinants whose entries are functions. This includes Wronskians and Turánians. One classic paper in the subject is the monumental work of Karlin and Szegő [35]. Theorem 3.6 is the key to proving Theorem 3.8 below.

Theorem 3.8 [26]. *Set $z = x+iy$ and let Δ_N be the determinant whose elements are*

$$\{(-i)^{m+n+2s}\pi H_{m+s,n+s}(iz, i\bar{z}) : 0 \leq m, n < N\}.$$

Then Δ_N is given by

$$\begin{aligned} \Delta_N = \frac{1}{N!} \int_{\mathbb{R}^{2N}} \prod_{k=1}^{N} |r_k + is_k|^{2s} \left[\prod_{1 \leq j < k \leq N} \left| r_j + is_j - r_k - is_k \right|^2 \right] \\ \times \prod_{j=1}^{N} e^{-(r_j-x)^2-(s_j-y)^2} dr_j\, ds_j. \end{aligned} \tag{3.42}$$

Hence the determinant formed by $(-i)^{m+n}(-1)^s H_{m+s,n+s}(iz, i\bar{z}) : 0 \leq m, n \leq N$ is positive for $N \geq 0$.

Ismail [26] gave two extensions of the integral [45, Section 109, (4)],

$$P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n H_n(xt) dt. \tag{3.43}$$

His two extensions of the above integral evaluation are in the following theorem.

Theorem 3.9. *With $z = x+iy$ we have the integral representation*

$$\begin{aligned} &\frac{4}{\pi} \int_0^\infty \int_0^\infty e^{-x^2-y^2} z^n (\bar{z})^m H_{m,n}(cz, c\bar{z}) dx dy \\ &\quad = m!\, n!\, c^m \bar{c}^n {}_2F_1\left(\left.\begin{matrix} -m, -n \\ 1 \end{matrix}\right| \frac{c\bar{c}-1}{c\bar{c}}\right), \end{aligned} \tag{3.44}$$

and the integral evaluation

$$\begin{aligned} &\frac{1}{\pi} \int_{\mathbb{R}^2} e^{-x^2-y^2} z^{m_1} \bar{z}^{n_1} H_{m_2,n_2}(z, \bar{z}) dx dy \\ &\quad = \begin{cases} 0 & \text{if } m_1 + m_2 \neq n_1 + n_2, \text{ or } n_2 > m_1 \\ \dfrac{m_1!\, n_1!!}{(m_1-n_2)!} & \text{if } m_1 + m_2 = n_1 + n_2. \end{cases} \end{aligned} \tag{3.45}$$

Theorem 3.9 suggests considering the more general integrals

$$\begin{aligned} &I(m_0, m_1, \ldots, m_k; n_0, n_1, \ldots, n_k) \\ &\quad := \tfrac{1}{\pi} \int_{\mathbb{R}^2} e^{-x^2-y^2} z^{m_0} \bar{z}^{n_0} \prod_{j=1}^{s} H_{m_j,n_j}(z, \bar{z}) dx dy, \quad z := x+iy. \end{aligned} \tag{3.46}$$

Of course this integral does not have a closed form but has a combinatorial interpretation. Consider $k+1$ two groups of disjoint sets $S_0, S_1, \ldots, S_k, T_0, T_1, \ldots, T_k$. The S sets are of a certain type, say type 1, and the T sets are of type 2. A perfect matching is a one to one mapping from $\cup_{j=0}^k S_j$ onto $\cup_{j=0}^k T_j$. Further we allow the elements of S_0 to be matched with elements of any T set, however elements of S_j, $j \neq 0$ are matched only with elements of T_r for $r \neq j$. Of course each element (vertex) is matched with a unique object.

Theorem 3.10. *The number of the above mentioned matchings is the integral in (3.46).*

4. 2D – q-Hermite polynomials

Ismail and Zhang [33] introduced two q-analogues of the Ito polynomials. The first q-analogue is defined by

$$H_{m,n}(z_1, z_2|q) := \sum_{k=0}^{m\wedge n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (q;q)_k z_1^{m-k} z_2^{n-k}. \tag{4.1}$$

It is clear that

$$H_{m,n}\left(z_2, z_1 \middle| q\right) = H_{n,m}\left(z_1, z_2 \middle| q\right). \tag{4.2}$$

The second q-analogue is defined by the explicit representation

$$h_{m,n}(z_1, z_2|q) := \sum_{j=0}^{m\wedge n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(m-j)(n-j)} (-1)^j (q;q)_j z_1^{m-j} z_2^{n-j}. \tag{4.3}$$

As a basic hypergeometric function (4.3) takes the form

$$h_{m,n}(z_1,z_2|q) = (-1)^n\left(q^{m-n+1};q\right)_n z_1^{m-n}{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{m-n+1}\end{matrix}\middle| q; -q^{m+1}z_1z_2\right). \quad (4.4)$$

Note the $(m,n)-(z_1,z_2)$ symmetry

$$h_{m,n}(z_1,z_2|q) = h_{n,m}(z_2,z_1|q), \quad (4.5)$$

and the fact

$$h_{m,n}(z_1,z_2|1/q) = q^{-mn}i^{-m-n}H_{m,n}(iz_1,iz_2|q) \quad (4.6)$$

Theorem 4.1. *The polynomials* $\{H_{m,n}(z_1,z_2|q)\}$ *and* $\{h_{m,n}(z_1,z_2|q)\}$ *have the generating functions*

$$\frac{(uv;q)_\infty}{(uz_1,vz_2;q)_\infty} = \sum_{m,n=0}^\infty H_{m,n}(z_1,z_2|q)\frac{u^m\,v^n}{(q;q)_m(q;q)_n}, \quad (4.7)$$

$$\frac{\left(-q^{1/2}uz_1,-q^{1/2}vz_2;q\right)_\infty}{(-uv;q)_\infty} = \sum_{m,n=0}^\infty \frac{h_{m,n}(z_1,z_2|q)}{(q;q)_m(q;q)_n}q^{(m-n)^2/2}u^mv^n, \quad (4.8)$$

and satisfy the functional relations

$$H_{m,n}(qz_1,z_2|q) = H_{m,n}(z_1,z_2|q) - z_1(1-q^m)H_{m-1,n}(z_1,z_2|q), \quad (4.9)$$

$$H_{m,n}(z_1,qz_2|q) = H_{m,n}(z_1,z_2|q) - z_2(1-q^n)H_{m,n-1}(z_1,z_2|q), \quad (4.10)$$

$$H_{m,n}(qz_1,z_2|q)q^{-m} = H_{m,n}(z_1,z_2|q) - q^{-1}(1-q^m)(1-q^n)H_{m-1,n-1}(z_1,z_2|q), \quad (4.11)$$

$$H_{m,n}(z_1,qz_2|q)q^{-n} = H_{m,n}(z_1,z_2|q) - q^{-1}(1-q^m)(1-q^n)H_{m-1,n-1}(z_1,z_2|q). \quad (4.12)$$

$$h_{m,n}\left(z_1q^{-1},z_2|q\right) = h_{m,n}(z_1,z_2|q) + z_1(1-q^m)q^{-m}h_{m-1,n}(z_1,z_2|q), \quad (4.13)$$

$$h_{m,n}\left(z_1,z_2q^{-1}|q\right) = h_{m,n}(z_1,z_2|q) + z_2(1-q^n)q^{-n}h_{m,n-1}(z_1,z_2|q), \quad (4.14)$$

$$q^mh_{m,n}(z_1/q,z_2|q) = h_{m,n}(z_1,z_2|q) + (1-q^m)(1-q^n)q^{1-m-n}h_{m-1,n-1}(z_1,z_2|q), \quad (4.15)$$

$$q^nh_{m,n}(z_1,z_2/q|q) = h_{m,n}(z_1,z_2|q) + (1-q^m)(1-q^n)q^{1-m-n}h_{m-1,n-1}(z_1,z_2|q), \quad (4.16)$$

They also satisfy the three term recurrence relations

$$\begin{aligned} z_1H_{m,n}(z_1,z_2|q) &= q^m(1-q^n)H_{m,n-1}(z_1,z_2|q) + H_{m+1,n}(z_1,z_2|q),\\ z_2H_{m,n}(z_1,z_2|q) &= q^n(1-q^m)H_{m-1,n}(z_1,z_2|q) + H_{m,n+1}(z_1,z_2|q).\end{aligned} \quad (4.17)$$

and

$$\begin{aligned} q^nz_1h_{m,n}(z_1,z_2|q) &= h_{m+1,n}(z_1,z_2|q) + (1-q^n)h_{m,n-1}(z_1,z_2|q),\\ q^mz_2h_{m,n}(z_1,z_2|q) &= h_{m,n+1}(z_1,z_2|q) + (1-q^m)h_{m-1,n}(z_1,z_2|q).\end{aligned} \quad (4.18)$$

Moreover they have the Rodrigues type formulas

$$H_{m,n}(z_1,z_2|q) = \frac{(1-1/q)^{m+n}q^{mn}}{(qz_1z_2;q)_\infty}D^m_{q^{-1},z_2}D^n_{q^{-1},z_1}((qz_1z_2;q)_\infty), \quad (4.19)$$

$$h_{m,n}(z_1,z_2|q) = (q-1)^{m+n}(-z_1z_2;q)_\infty D^m_{q,z_2}D^n_{q,z_1}\frac{1}{(-z_1z_2;q)_\infty}. \quad (4.20)$$

Furthermore we also have the operational formula

$$\begin{aligned} H_{m,n}(z_1,z_2|q) &= ((1-q)^2D_{q,z_1}D_{q,z_2};q)_\infty z_1^mz_2^n,\\ h_{m,n}(z_1,z_2|q) &= \frac{q^{mn}}{(-q^{-1}(1-q)^2D_{q^{-1},z_1}D_{q^{-1},z_2};q)_\infty}z_1^mz_2^n.\end{aligned} \quad (4.21)$$

and the lowering relations

$$\begin{aligned} D_{q,z_1}H_{m,n}(z_1,z_2|q) &= \frac{1-q^m}{1-q}H_{m-1,n}(z_1,z_2|q),\\ D_{q,z_2}H_{m,n}(z_1,z_2|q) &= \frac{1-q^n}{1-q}H_{m,n-1}(z_1,z_2|q),\end{aligned} \quad (4.22)$$

and

$$\begin{aligned} D_{q^{-1},z_1}h_{m,n}(z_1,z_2|q) &= \frac{q^{n-m+1}(1-q^m)}{1-q}h_{m-1,n}(z_1,z_2|q),\\ D_{q^{-1},z_2}h_{m,n}(z_1,z_2|q) &= \frac{q^{m-n+1}(1-q^n)}{1-q}h_{m,n-1}(z_1,z_2|q).\end{aligned} \quad (4.23)$$

We note that (4.11) and (4.12) imply the symmetry relation

$$H_{m,n}(qz_1,z_2|q)q^{-m} = H_{m,n}(z_1,qz_2|q)q^{-n}. \quad (4.24)$$

Theorem 4.2. *The 2D – q-Hermite polynomials have the raising relations*

$$\begin{aligned} H_{m+1,n}(z_1,z_2|q) &= q^n\frac{1-1/q}{(qz_1z_2;q)_\infty}D_{q^{-1},z_2}((qz_1z_2;q)_\infty H_{m,n}(z_1,z_2|q)),\\ H_{m,n+1}(z_1,z_2|q) &= q^m\frac{1-1/q}{(qz_1z_2;q)_\infty}D_{q^{-1},z_1}((qz_1z_2;q)_\infty H_{m,n}(z_1,z_2|q)).\end{aligned} \quad (4.25)$$

and

$$\begin{aligned} h_{m+1,n}(z_1,z_2|q) &= (q-1)(-z_1z_2;q)_\infty D_{q,z_2}\left(\frac{h_{m,n}(z_1,z_2|q)}{(-z_1z_2;q)_\infty}\right),\\ h_{m,n+1}(z_1,z_2|q) &= (q-1)(-z_1z_2;q)_\infty D_{q,z_1}\left(\frac{h_{m,n}(z_1,z_2|q)}{(-z_1z_2;q)_\infty}\right).\end{aligned} \quad (4.26)$$

Moreover the same polynomials have the multiplication formulas

$$H_{m,n}(az_1,bz_2|q) = \sum_{j=0}^{m\wedge n}\begin{bmatrix} m \\ j\end{bmatrix}_q\begin{bmatrix} n \\ j\end{bmatrix}_q\frac{H_{m-j,n-j}(z_1,z_2|q)}{a^{j-m}b^{j-n}}(q,1/ab;q)_j(q;q)_j, \quad (4.27)$$

$$h_{m,n}(az_1,bz_2;q) = \sum_{j=0}^{m\wedge n}\begin{bmatrix} m \\ j\end{bmatrix}_q\begin{bmatrix} n \\ j\end{bmatrix}_q(q,ab;q)_ja^{m-j}b^{n-j}h_{m-j,n-j}(z_1,z_2;q). \quad (4.28)$$

This theorem can be proved using the generating functions of Theorem 4.1.

The polynomials $H_{m,n}$ have the representation

$$H_{m,n}\left(z_1,z_2\middle|q\right) = (-1)^n\frac{(q;q)_mq^{\binom{n}{2}}}{(q;q)_{m-n}}z_1^{m-n}p_n\left(z_1z_2,q^{m-n}\middle|q\right), \quad (4.29)$$

where $p_n(x;q^\alpha|q)$ is the little q-Laguerre or Wall's polynomials, [38], defined by

$$p_n(x;a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n},0 \\ aq\end{matrix}\middle| q;qx\right).$$

Theorem 4.3. *The polynomials $\left\{H_{m,n}\left(z,\bar{z}\middle|q\right)\right\}$ satisfy the following orthogonality*

$$\int_{\mathbb{C}} H_{m,n}\left(z,\bar{z}\middle|q\right)\overline{H_{s,t}\left(z,\bar{z}\middle|q\right)}d\mu(z,\bar{z}) = \frac{q^{mn}(q;q)_m(q;q)_n}{(q;q)_\infty}\delta_{m,s}\delta_{n,t}, \tag{4.30}$$

where

$$d\mu(z,\bar{z}) = \sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\delta(r-q^{k/2})\,\frac{d\theta}{2\pi},$$

and $z = re^{i\theta}, r\in\mathbb{R}^+, \theta\in[0,2\pi],\ m,n,s,t\in\mathbb{N}_0$.

It is clear that the orthogonality relation (4.30) and the generating function (4.7) imply the q-beta integral evaluation

$$\int_{\mathbb{C}}\frac{d\mu(z,\bar{z})}{(u_1z, v_1\bar{z}, v_2z, u_2\bar{z};q)_\infty} = \frac{(u_1u_2v_1v_2;q)_\infty}{(q,u_1u_2,v_1v_2,u_1v_1,u_2v_2;q)_\infty}. \tag{4.31}$$

The orthogonality relation of the h's is given in the following theorem.

Theorem 4.4. *The polynomials $\{h_{m,n}(z,\bar{z}|q)\}$ satisfy the orthogonality relation*

$$\int_{\mathbb{R}^2} h_{m,n}(z,\bar{z}|q)\overline{h_{s,t}(z,\bar{z}|q)}\frac{dxdy}{(-z\bar{z};q)_\infty} = \frac{\pi\log q^{-1}(q;q)_m(q;q)_n}{q^{(m-n)^2/2+(m+n)/2}}\delta_{m,s}\delta_{n,t}, \tag{4.32}$$

where $x = x+iy, x,y\in\mathbb{R},\ m,n,s,t\in\mathbb{N}_0$.

As it was the case with the $H_{m,\,n}$'s the generating function (4.8) show that the orthogonality relation (4.32) is equivalent to the q-beta type integral

$$\int_{\mathbb{R}^2}\frac{(-q^{1/2}u_1z, -q^{1/2}v_1\bar{z}, -q^{1/2}v_2z, -q^{1/2}u_2\bar{z};q)_\infty}{(-z\bar{z};q)_\infty}dxdy = \pi\ln q^{-1}\frac{(-u_1v_1,-u_2v_2,-u_1u_2,-v_1v_2;q)_\infty}{(u_1u_2v_1v_2;q)_\infty}. \tag{4.33}$$

Theorem 4.5. *The polynomials $\{H_{m,\,n}(z_1,\ z_2)\}$ have the generating function*

$$\sum_{m,n=0}^{\infty}H_{m,n}\left(z_1,z_2\middle|q\right)\frac{u^m(a/u;q)_nv^n(b/v;q)_n}{(q;q)_m(q;q)_n} = \frac{(az_1.bz_2;q)_\infty}{(uz_1vz_2;q)_\infty}{}_2\phi_2\left(\begin{matrix}a/u, b/v\\ az_1, bz_2\end{matrix}\middle|q;uv\right). \tag{4.34}$$

As we saw in Sections 3 the Ito polynomials are orthogonal with respect to two different measures. The next theorem show that our $H_{m,\,n}(z_1,\,z_2|q)$ share this property.

Theorem 4.6. *We have the orthogonality relation*

$$\sum_{j=0}^{p}\sum_{k=0}^{s}\int_0^\pi (q,e^{2i\theta},e^{-2i\theta};q)_\infty \times\frac{H_{j,k}(re^{i\theta},re^{-i\theta}|q)H_{s-k,p-j}(re^{i\theta},re^{-i\theta}|q)}{(q;q)j(q;q)_k(q;q)_{s-k}(q;q)_{p-j}}\frac{d\theta}{\pi} = \frac{r^{2p}(1/r^2;q)_p}{(q;q)_p}{}_1\phi_1\left(\begin{matrix}q^{-s}\\ q^{1-p}r^2\end{matrix}\middle|q,q\right)\delta_{s,p}. \tag{4.35}$$

The large degree asymptotics of $H_{m,n}\left(z,\bar{z}\middle|q\right)$ are straightforward. Indeed (4.1), Tannery's theorem, and the q binomial theorem show that

$$\lim_{m\to\infty}z_1^{-m}H_{m,n}\left(z_1,z_2\middle|q\right) = z_2^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}(-z_1z_2)^{-k} = z_2^n(1/z_1z_2;q)_n, \tag{4.36}$$

Similarly we establish the limiting relations

$$\lim_{n\to\infty}z_2^{-n}H_{m,n}\left(z_1,z_2\middle|q\right) = z_1^m(1/z_1z_2;q)_m, \tag{4.37}$$

$$\lim_{m,n\to\infty}z_1^{-m}z_2^{-n}H_{m,n}\left(z_1,z_2\middle|q\right) = (1/z_1z_2;q)_\infty. \tag{4.38}$$

The convergence in (4.36)–(4.38) is uniform on compact subsets $\mathbb{C}\times\mathbb{C}$. The next two theorems give the Plancherel–Rotach asymptotics of the polynomials $\{H_{m,\,n}(z_1,\,z_2|q)\}$ and $\{h_{m,\,n}(z_1,\,z_2|q)\}$.

Theorem 4.7. *Let $a, b, c, d \geq 0$, $\tau = \tau(m,n;a,b,c,d) = \lfloor(a+c)m+(b+d)n\rfloor$ and $\chi = \chi(m,n;a,b,c,d) = \{(a+c)m+(b+d)n\}$, for $0 < \tau(m,\ n;\ a,\ b,\ c,\ d) < m\wedge n$, then*

$$\lim_{m,n\to\infty}\frac{(q;q)_\infty H_{m,n}\left(z_1q^{am+bn-\frac{1}{4}}, z_2q^{cm+dn-\frac{1}{4}}|q\right)(-z_1z_2)^\tau}{z_1^mz_2^nq^{am^2+(b+c)mn+dn^2-(m+n)/4-\tau^2/2-\tau\chi}} = \theta_4\left(z_1z_2q^\chi;q^{1/2}\right),$$

holds uniformly on compact subsets of the z_1 and z_2 planes, where ϑ_4 is defined in (1.6).

Theorem 4.8. *For $a, b\in\mathbb{C}$ and $0<\epsilon<1$ the following asymptotic result holds uniformly when w_1, w_2 are in compact subsets of the complex plane*

$$\lim_{m,n\to\infty}\frac{h_{m,n}\left(w_1q^{-am-bn}, w_2q^{-(1-a)m-(1-b)n}|q\right)(q;q)_\infty^2}{w_1^mw_2^nq^{(a-b)mn-am^2+(b-1)n^2}} = A_q\left(\frac{1}{w_1w_2}\right). \tag{4.39}$$

Since the polynomial $H_{m,n}(z,\bar{z})$ factors as a function of θ times a radial function it is clear that with $z_1 = z, z_2 = \bar{z}$ the zeros of the polynomials investigated here as functions of z lie on circles. Now (4.38) implies the following asymptotic result.

Theorem 4.9. *Assume that the zeros of $H_{m,n}(z,\bar{z}|q)$ lie on the circles with radii*

$$r_1(H,m,n) > r_2(H,m,n) > \cdots. \tag{4.40}$$

Then

$$\lim_{m,n\to\infty}r_j(H,m,n) = q^{j/2},\quad j=1,2,\ldots. \tag{4.41}$$

Note that the limiting distribution of the radii of the circles of zeros coincides with the location of the masses of the orthogonality measure of the polynomials.

Theorem 4.7 implies the following results about zeros of $H_{m,\,n}$.

Theorem 4.10. *The zeros of $H_{m,n}\left(z_1q^{(am+bn-1/4)}, z_2q^{(cm+dn-1/4)}|q\right)$ asymptotically lie on the the curves $z_1z_2q^\chi = q^{\pm(n+1/2)}, n = 0,1,2,\ldots$. These curves become circles when $z_1 = z, z_2 = \bar{z}$.*

The function $A_q(z)$ has infinity many zeros and they are all positive and simple. We denote them by

$$0 < i_1(q) < i_2(q) < \cdots < i_n(q) < \cdots. \tag{4.42}$$

Theorem 4.11. *Assume that the zeros of $h_{m,n}(z,\bar{z}|q)$ lie on the circles with radii*

$$r_1(h,m,n) > r_2(h,m,n) > \cdots. \tag{4.43}$$

Then

$$\lim_{m,n\to\infty}q^{(m+n)/2}r_j(h,m,n) = 1/\sqrt{i_j(q)},\quad j=1,2,\ldots, \tag{4.44}$$

where $\{i_j(q)\}$ are the zeros of the Ramanujan function ordered as in (4.42).

We now identify the q-Sturm–Liouville problems whose eigenfunctions are either $\{H_{m,\,n}(z_1,\,z_2|q)\}$ or $\{h_{m,\,n}(z_1,\,z_2|q)\}$. Using the first equations in (4.22) and (4.25) we find that $H_{m,\,n}$ satisfy the eigenvalue equation

$$(1-z_1z_2)D_{q^{-1},z_2}D_{q,z_1}f(z_1,z_2)-\frac{qz_1}{1-q}D_{q,z_1}f(z_1,z_2)=\lambda f(z_1,z_2), \tag{4.45}$$

with

$$\lambda=\lambda_{m,n}=-q^{1-n}\frac{1-q^m}{(1-q)^2}. \tag{4.46}$$

Similarly we show that $H_{m,n}$ solves the eigenvalue equation

$$(1-z_1z_2)D_{q^{-1},z_2}D_{q,z_1}f(z_1,z_2)-\frac{qz_2}{1-q}D_{q,z_1}f(z_1,z_2)=\lambda f(z_1,z_2), \tag{4.47}$$

with

$$\lambda=\lambda_{m,n}=-q^{1-m}\frac{1-q^n}{(1-q)^2}. \tag{4.48}$$

Combine (4.23) and (4.26) to see that $h_{m,\,n}(z_1,\,z_2|q)$ satisfies the equation

$$(1+z_1z_2)D_{q,z_2}D_{q^{-1},z_1}f(z_1,z_2)-\frac{z_1}{1-q}D_{q^{-1},z_1}f(z_1,z_2)=\lambda f(z_1,z_2), \tag{4.49}$$

where

$$\lambda=\lambda_{m,n}=\frac{1-q^m}{(1-q)^2}q^{n-m+1}. \tag{4.50}$$

Similarly we find that

$$(1+z_1z_2)D_{q,z_1}D_{q^{-1},z_2}f(z_1,z_2)-\frac{z_2}{1-q}D_{q^{-1},z_1}f(z_1,z_2)=\lambda f(z_1,z_2), \tag{4.51}$$

with

$$\lambda=\lambda_{m,n}=\frac{1-q^n}{(1-q)^2}q^{m-n+1}. \tag{4.52}$$

We note the relation

$$h_{m,n}(z_1,z_2|q)=(-1)^n(q;q)_nz_1^{m-n}L_n^{(m-n)}(z_1z_2;q), \tag{4.53}$$

where $L_n^{(\alpha)}(z;q)$ is a q-Laguerre polynomial, see (1.7).

Since the moment problem associated with $L_n^{(\alpha)}(x;q)$ is indeterminate [25, Section 21.8], they have infinitely many orthogonal measures. Let $x^\alpha d\mu(x)$ be such a measure, for example,

$$d\mu(x)=x^{-\alpha}w_{QL}(x;\alpha,c,\lambda)dx,\quad \alpha,\lambda,c>0,$$

$$d\mu(x)=\frac{x}{(-x;q)_\infty}\delta(x-cq^y),\quad y\in\mathbb{Z},c,\alpha+1>0,$$

etc. it is clear that our proof shows that

$$d\sigma\left(re^{i\theta},re^{-i\theta}\right)=\frac{1}{2}d\theta d\mu(r^2),\quad r\in\mathbb{R}^+,\theta\in[0,2\pi]$$

is also an orthogonal measure for $h_{m,n}\left(re^{i\theta},re^{-i\theta}|q\right)$ where $r\in\mathbb{R}^+,\theta\in[0,2\pi]$.

Theorem 4.12. *Assume that $q=e^{-2k^2}$ and $|q|<1$, then we have*

$$\begin{aligned}&\left(-z_1qe^{2imk},-z_2qe^{-2imk};q\right)_\infty\\&=\sum_{s,t=0}^{\infty}\frac{H_{s,t}(z_1,z_2|q)}{(q;q)_s(q;q)_t}q^{\frac{(s-t)^2}{2}}\left(q^{\frac12}e^{2imk}\right)^s\left(q^{\frac12}e^{-2imk}\right)^t,\end{aligned} \tag{4.54}$$

$$\begin{aligned}&\frac{(z_1z_2;q)_\infty}{\left(-z_1z_2;z_1e^{2mk},z_2e^{-2mk};q\right)_\infty}\\&=\sum_{s,t=0}^{\infty}\frac{h_{s,t}(z_1,z_2|q)}{(q;q)_s(q;q)_t}\left(q^{\frac12}e^{mk}\right)^s\left(q^{\frac12}e^{-mk}\right)^t\end{aligned} \tag{4.55}$$

and when $|z_1z_3|<1$, $|z_2/z_3|<1$, we have

$$\frac{(z_1z_2;q)_\infty}{(-z_1z_2,z_1z_3,z_2/z_3;q)_\infty}=\sum_{m,n=0}^{\infty}\frac{h_{m,n}(z_1,z_2|q)}{(q;q)_m(q;q)_n}q^{(m+n)/2}z_3^{(m-n)/2}. \tag{4.56}$$

5. The Hermite polynomials on $\mathbb{R}^n$

We follow the notation in [12, Section 12.8]. First we define

$$\phi(\mathbf{x})=(C\mathbf{x},\mathbf{x}),\qquad \psi(\mathbf{x})=(C^{-1}\mathbf{x},\mathbf{x})=\phi(C^{-1}\mathbf{x}), \tag{5.1}$$

where C is a positive definite $n\times n$ matrix, $\mathbf{x}^t=(x_1,\ldots,x_n)\in\mathbb{R}^n$, and $(\mathbf{x},\mathbf{y})=\sum_{j=1}^n x_jy_j$. The integral evaluation [12, §12.8]

$$\int_{\mathbb{R}^n}e^{-\phi(\mathbf{x})/2+(\mathbf{a},\mathbf{x})}\,d\mathbf{x}=\kappa_ne^{\psi(\mathbf{a})/2},\qquad \kappa_n:=(2\pi)^{n/2}\Delta^{-1/2} \tag{5.2}$$

where $\mathbf{a}^t=(a_1,\ldots,a_n)\in\mathbb{R}^n$, $d\mathbf{x}=dx_1,\ldots,dx_n$, and Δ is the determinant of C, follows from diagonalizing ϕ and a change of variable on the normal distribution.

Let $\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}_0^n$. The polynomials $\{H_{\mathbf{m}}(\mathbf{x})\}$ and $\{G_{\mathbf{m}}(\mathbf{x})\}$ are defined through the generating functions [12, §12.8]

$$\sum_{\mathbf{m}\in\mathbb{N}_0^n}H_{\mathbf{m}}(\mathbf{x})\prod_{j=1}^n\frac{u_j^{m_j}}{m_j!}=e^{(C\mathbf{x},\mathbf{u})-\phi(\mathbf{u})/2}=e^{\phi(\mathbf{x})/2-\phi(\mathbf{x}-\mathbf{u})/2}, \tag{5.3}$$

$$\sum_{\mathbf{m}\in\mathbb{N}_0^n}G_{\mathbf{m}}(\mathbf{x})\prod_{j=1}^n\frac{u_j^{m_j}}{m_j!}=e^{(\mathbf{x},\mathbf{u})-\psi(\mathbf{u})/2}=e^{\phi(\mathbf{x})/2-\phi(\mathbf{x}-C^{-1}\mathbf{u})/2}. \tag{5.4}$$

The second equality in (5.3) follows from the identity $\phi(\mathbf{x})-2(\mathbf{x},C\mathbf{u})+\phi(\mathbf{u})=\phi(\mathbf{x}-\mathbf{u})$. The second equality in (5.4) is obtained by replacing $\mathbf{u}$ by $C^{-1}\mathbf{u}$ in (5.3).

The generating function (5.3) leads to the Rodrigues formula [12, Section 12.8 (20)]

$$H_{\mathbf{m}}(\mathbf{x})=(-1)^{|\mathbf{m}|}e^{\phi(\mathbf{x})/2}\frac{\partial^{\mathbf{m}}}{\partial\mathbf{x}}e^{-\phi(\mathbf{x})/2},\qquad \mathbf{m}\in\mathbb{N}_0^n, \tag{5.5}$$

where $|\mathbf{m}|=\sum_{j=1}^n m_j$ and $\frac{\partial^{\mathbf{m}}}{\partial\mathbf{x}}:=\prod_{j=1}^n\frac{\partial^{m_j}}{\partial x_j^{m_j}}$. If $\mathbf{m}\in\mathbb{Z}\setminus\mathbb{N}_0$, we set $H_{\mathbf{m}}(\mathbf{x})=G_{\mathbf{m}}(\mathbf{x})=0$.

The following biorthogonality relation holds [12, Section 12.8]:

$$I_{\mathbf{m},\mathbf{k}}:=\kappa_n^{-1}\int_{\mathbb{R}^n}H_{\mathbf{m}}(\mathbf{x})G_{\mathbf{k}}(\mathbf{x})e^{-\phi(\mathbf{x})/2}\,d\mathbf{x}=\delta_{\mathbf{m},\mathbf{k}}\prod_{j=1}^n m_j!. \tag{5.6}$$

More generally, Ismail and Simeonov [28] considered the integrals

$$I(\{\mathbf{m}_i\}_{i=1}^r,\{\mathbf{k}_i\}_{i=1}^s):=\kappa_n^{-1}\int_{\mathbb{R}^n}\prod_{i=1}^rH_{\mathbf{m}_i}(\mathbf{x})\prod_{i=1}^sG_{\mathbf{k}_i}(\mathbf{x})e^{-\phi(\mathbf{x})/2}\,d\mathbf{x}, \tag{5.7}$$

where $\{\mathbf{m}_i\}_{i=1}^r\subset\mathbb{N}_0$ and $\{\mathbf{k}_i\}_{i=1}^s\subset\mathbb{N}_0$. If $F(U,\,V)$ is the exponential generating function

$$\begin{aligned}F(U,V):=&\sum_{\{\mathbf{m}_i\}_{i=1}^r\subset\mathbb{N}_0}\sum_{\{\mathbf{k}_i\}_{i=1}^s\subset\mathbb{N}_0}I(\{\mathbf{m}_i\}_{i=1}^r,\{\mathbf{k}_i\}_{i=1}^s)\prod_{i=1}^r\prod_{j=1}^n\frac{u_{i,j}^{m_{i,j}}}{m_{i,j}!}\\&\times\prod_{i=1}^s\prod_{j=1}^n\frac{v_{i,j}^{m_{i,j}}}{m_{i,j}!},\end{aligned} \tag{5.8}$$

where $U=[u_{i,j}]$ is a $r\times n$ matrix and $V=[v_{i,j}]$ is an $s\times n$ matrix.

A formula of Feldheim [13] is

$$i^{-|\mathbf{m}|}H_{\mathbf{m}}(i\mathbf{x}) = \kappa_n^{-1}\int_{\mathbb{R}^n}\left\{\prod_{j=1}^n (C\mathbf{y})_j^{m_j}\right\}e^{-\phi(\mathbf{x}-\mathbf{y})/2}\,d\mathbf{y}, \qquad \mathbf{m}\in\mathbb{N}_0^n. \tag{5.9}$$

Burchnal [7] proved that

$$\left(-\frac{d}{dx}+2x\right)^m f(x) = \sum_{k=0}^m (-1)^k\binom{m}{k}H_{m-k}(x)f^{(k)}(x).$$

Ismail and Simeonov [28] extended proved the following multivariate version of Burchnal's formula.

Theorem 5.1.

$$\prod_{j=1}^n\left(-\frac{\partial}{\partial x_j}+(C\mathbf{x})_j\right)^{m_j} f(\mathbf{x}) = \sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n}(-1)^{|\mathbf{k}|}\prod_{j=1}^n\binom{m_j}{k_j}H_{\mathbf{m}-\mathbf{k}}(\mathbf{x})\frac{\partial^{\mathbf{k}}}{\partial\mathbf{x}}f(\mathbf{x}). \tag{5.10}$$

Using a technique similar to Louck's proof of the Kibble–Slepian formula, [41], Ismail and SImeonov also proved the following generating function.

Theorem 5.2. *Let S and C be positive definite $n\times n$ matrices such that $I+\frac{1}{2}C^{1/2}SC^{1/2}$ is invertible, and let $\{H_{\mathbf{m}}(\mathbf{x})\}$ be the Hermite polynomials associated with C. Then*

$$\det\left(I+C^{1/2}SC^{1/2}/2\right)^{-1/2} \times \exp\left[(1/4)\left(CSC^{1/2}\left(I+C^{1/2}SC^{1/2}/2\right)^{-1}C^{1/2}\mathbf{x},\mathbf{x}\right)\right] = \sum_{\substack{M\in M_n(\mathbb{N}_0)\\ M=M^t}} 2^{-\operatorname{tr}(M)}\left\{\prod_{1\le j\le l\le n}\frac{(s_{j,l}/2)^{m_{j,l}}}{m_{j,l}!}\right\}H_{\mathbf{m}}(\mathbf{x}), \tag{5.11}$$

where $M_n(\mathbb{N}_0)$ is the set of all $n\times n$ matrices with entries from $\mathbb{N}_0$, $M=[m_{j,l}]_{j,l=1}^n$, $\operatorname{tr}(M)=\sum_{j=1}^n m_{j,j}$ is the trace of M, and $\mathbf{m}=(m_1,\dots,m_n)$ with $m_j = m_{j,j}+\sum_{l=1}^n m_{j,l}$, $j=1,\dots,n$.

Moreover Ismail and Simeonov [28] proved that

Proposition 5.3. *Let $S=\{S_j\}_{j=1}^n$ be a multiset of n sets, where each set S_j contains $m_j=|S_j|$ distinct objects, $j=1,\dots,n$. Let $\mathcal{M}(\mathbf{m})$ be the set of all inhomogeneous perfect matchings of this multiset. Then*

$$H_{\mathbf{m}}(\mathbf{x}) = \sum_{\mu\in\mathcal{M}(\mathbf{m})}\prod_{\text{edge}\in E(\mu)}(-c_{\text{edge}})\prod_{\text{fix}\in\text{Fix}(\mu)}(C\mathbf{x})_{i(\text{fix})}, \tag{5.12}$$

where $E(\mu)$ denotes the set of all edges and $\text{Fix}(\mu)$ denotes the set of all fixed points (unmatched elements) in a matching $\mu\in\mathcal{M}(\mathbf{m})$, $c_{\text{edge}}=c_{j_1,j_2}$, if edge $\in E(\mu)$ connects elements from sets S_{j_1} and S_{j_2}, and $i(\text{fix})=j$ if fix $\in S_j$.

With the notation $m=\sum_{j=1}^n m_j$, the polynomials $\{H_{\mathbf{m}}(\mathbf{x})$ and $\{G_{\mathbf{m}}(\mathbf{x})$ have the Rodrigues type representations

$$H_{\mathbf{m}}(\mathbf{x}) = (-1)^m\exp(\phi(\mathbf{x}/2))\,\frac{\partial^m}{\partial x_1^{m_1},\dots,\partial x_n^{m_n}}\exp(-\phi(\mathbf{x}/2)), \tag{5.13}$$

$$G_{\mathbf{m}}(\mathbf{x}) = (-1)^m\exp(\psi(\mathbf{x}/2))\,\frac{\partial^m}{\partial x_1^{m_1},\dots,\partial x_n^{m_n}}\exp(-\psi(\mathbf{x}/2). \tag{5.14}$$

One curious property of the Hermite polynomials is the following

$$\sum_{m_1+m_2+\cdots+m_n=m}\prod_{j=1}^m\frac{a_j^{m_j}}{m_j!}H_{m_1,m_2,\dots,m_n}(x_1,\dots,x_n) = \frac{[\phi(\mathbf{a})/2]^{m/2}}{m!}H_m\left(\frac{(C\mathbf{a},\mathbf{x})}{[2\phi(\mathbf{a})]^{1/2}}\right), \tag{5.15}$$

$$\sum_{m_1+m_2+\cdots+m_n=m}\prod_{j=1}^m\frac{a_j^{m_j}}{m_j!}G_{m_1,m_2,\cdots,m_n}(x_1,\dots,x_n) = \frac{[\psi(\mathbf{a})/2]^{m/2}}{m!}H_m\left(\frac{(\mathbf{a},\mathbf{x})}{[2\psi(\mathbf{a})]^{1/2}}\right). \tag{5.16}$$

6. Complex 2*D*-systems

We introduced the model of this section in [31]. Let $\{\phi_n(r;\alpha)\}$ be a system of orthogonal polynomials satisfying the orthogonality relation

$$\int_0^\infty \phi_m(r;\alpha)\phi_n(r;\alpha)r^\alpha d\mu(r,\beta) = \zeta_n(\alpha)\delta_{m,n}, \quad \alpha\ge 0. \tag{6.1}$$

It is assumed the measure μ does not depend on α but may depend on another parameter β. Let

$$\phi_n(r;\alpha) = \sum_{j=0}^n c_j(n,\alpha)r^{n-j}, \quad c_j(n,\alpha)\in\mathbb{R}, \tag{6.2}$$

and define bivariate polynomials $f_{m,n}$ by

$$f_{m,n}(z_1,z_2;\beta) = \begin{cases} z_1^{m-n}\phi_n(z_1z_2;m-n+\beta), & m\ge n,\\ f_{n,m}(z_2,z_1;\beta), & m<n.\end{cases} \tag{6.3}$$

From (6.3) it is clear that

$$-i\frac{\partial}{\partial\theta}f_{m,n}(z,\bar z;\beta) = (m-n)f_{m,n}(z,\bar z;\beta), \quad m\ge n.$$

Therefore the functional relations

$$(\delta_{z_1}-\delta_{z_2})f_{m,n}(z_1,z_2;\beta) = (m-n)f_{m,n}(z_1,z_2;\beta), \quad m\ge n, \tag{6.4}$$

$$\left(\delta_{q,z_1}-q^{m-n}\delta_{q,z_2}\right)f_{m,n}(z_1,z_2;\beta) = \frac{1-q^{m-n}}{1-q}f_{m,n}(z_1,z_2;\beta), \quad m\ge n, \tag{6.5}$$

hold.

Theorem 6.1 [31]. *The polynomials $\{f_{m,n}(z_1,z_2;\beta)\}$ satisfy the orthogonality relation*

$$\int_{\mathbb{R}^2} f_{m,n}(z,\bar z;\beta)\,\overline{f_{s,t}(z,\bar z;\beta)}\,\frac{d\theta}{2\pi}\,d\mu(r^2;\beta) = \zeta_{m\wedge n}(|m-n|+\beta)\delta_{m,s}\delta_{n,t}, \tag{6.6}$$

for all nonnegative integers m, n, s, t, where $d\mu(r;\beta)=r^\beta d\mu(r)$.

The proof follows from (6.1).

Unlike general 2*D* orthogonal polynomials [11] the $f_{m,n}$'s satisfy three term recurrence relations. Indeed we have

$$z_2f_{m+1,n}(z_1,z_2;\beta) = \frac{c_0(n,m-n+1+\beta)}{c_0(n+1,m-n+\beta)}f_{m+1,n+1}(z_1,z_2;\beta) - \frac{c_0(n,m-n+1+\beta)c_{n+1}(n+1,m-n+\beta)}{c_0(n+1,m-n+\beta)c_n(n,m-n+\beta)}f_{m,n}(z_1,z_2;\beta), \quad m\ge n, \tag{6.7}$$

and

$$z_1f_{m,n}(z_1,z_2;\beta) - \frac{c_0(n,m-n+\beta)}{c_0(n,m+1-n+\beta)}f_{m+1,n}(z_1,z_2;\beta) = u_{m,n}f_{m,n-1}(z_1,z_2;\beta), \tag{6.8}$$

with

$$u_{m,n} = \frac{c_0(n,\alpha+1)c_1(n,\alpha)-c_0(n,\alpha)c_1(n,1+\alpha)}{c_0(n-1,1+\alpha)c_0(n,1+\alpha)} \tag{6.9}$$

and $\alpha = m - n + \beta$. Assume that the three term recurrence relation satisfied by the ϕ_n's is

$$r\phi_n(r;\alpha) = a_n(\alpha)\phi_{n+1}(r;\alpha) + c_n(\alpha)\phi_n(r;\alpha) + b_n(\alpha)\phi_{n-1}(r;\alpha) \tag{6.10}$$

By comparing like powers of r we find that

$$a_n(\alpha) = \frac{c_0(n,\alpha)}{c_0(n+1,\alpha)}, \qquad c_n(\alpha) = \frac{c_1(n,\alpha)}{c_0(n,\alpha)} - \frac{c_1(n+1,\alpha)}{c_0(n+1,\alpha)}, \tag{6.11}$$

$$b_n(\alpha) = \frac{c_0(n,\alpha)c_2(n,\alpha) - c_1(n,\alpha)^2}{c_0(n-1,\alpha)c_0(n,\alpha)} - \frac{c_0(n,\alpha)c_2(n+1,\alpha) - c_1(n,\alpha)c_1(n+1,\alpha)}{c_0(n-1,\alpha)c_0(n+1,\alpha)} \tag{6.12}$$

The following recurrence relation follows from (6.2) and (6.3).

$$\begin{aligned}(z_1z_2 - c_n(m-n+\beta))f_{m,n}(z_1,z_2;\beta) &= a_n(m-n+\beta)f_{m+1,n+1}(z_1,z_2;\beta)\\ &\quad + b_n(m-n+\beta)f_{m-1,n-1}(z_1,z_2;\beta).\end{aligned} \tag{6.13}$$

We note that the definition of the polynomials $f_{m,n}$ indicates that for $m \ge n$ it may have a trivial zero at $z_1 = 0$ but will vanish on the curves $z_1z_2 = r$ where r is a zero of $\phi_n(r; m-n+\beta)$. The zeros of $\phi_n(r; m-n+\beta)$ are positive, real and simple when $m \ge n$.

The relationships (3.9), (4.29), and (4.53) show that the polynomials $H_{m,n}$ and their q-analogues are examples of $f_{m,n}$ polynomials.

Remark 6.2. In view of the defining Eq. (6.3) the question of finding the large m, n asymptotics of the two variate polynomials $f_{m,n}$ is equivalent to finding the large m, n behavior of $\phi_n(r; m-n+\beta)$. It will be of interest to carry out this program at least for some special systems, including the Freud type polynomials orthogonal with respect to $x^\alpha \exp(-p(x))$, where p is a polynomial with positive leading term.

Theorem 6.3. *Let $\phi_n(x;\alpha)$, $a_n(\alpha)$, $b_n(\alpha)$ as in (6.11) and (6.12), for $m \ge n-1$ we have*

$$z_1 f_{m,n}(z_1,z_2) - v_{m,n}f_{m+1,n}(z_1,z_2) = u_{m,n}f_{m,n-1}(z_1,z_2), \tag{6.14}$$

where

$$u_{m,n} = b_n(m-n), \quad v_{m,n} = a_n(m-n). \tag{6.15}$$

We next consider differential or q-difference equations satisfied by $f_{m,n}(z_1,z_2)$.

Theorem 6.4. *Assume that $\phi_n(r;\alpha)$ satisfies the second order differential equation*

$$A_n(r,\alpha)\delta_r^2 f + B_n(r,\alpha)\delta_r f + C_n(r,\alpha)f = 0, \tag{6.16}$$

where

$$\delta_r f = r\frac{\partial}{\partial r}f. \tag{6.17}$$

Then, $f_{m,n}(z_1,z_2;\beta)$, for $\alpha = m-n+\beta$ and $m \ge n$, satisfies the second order partial differential equations

$$A_n(z_1z_2,\alpha)\delta_{z_2}^2 f + B_n(z_1z_2,\alpha)\delta_{z_2} f + C_n(z_1z_2,\alpha)f = 0, \tag{6.18}$$

$$\begin{aligned}&A_n(z_1z_2,\alpha)\delta_{z_1}^2 f + \{B_n(z_1z_2,\alpha) - 2\alpha A_n(z_1z_2,\alpha)\}\delta_{z_1} f\\ &+ \left\{\alpha^2 A_n(z_1z_2,\alpha) - \alpha B_n(z_1z_2,\alpha) + C_n(z_1z_2,\alpha)\right\}f = 0.\end{aligned} \tag{6.19}$$

Similarly, assume that $\phi_n(r;\alpha)$ satisfies the second order q-difference equation

$$A_n(r,\alpha)\delta_{q,r}^2 f + B_n(r,\alpha)\delta_{q,r} f + C_n(r,\alpha)f = 0, \tag{6.20}$$

where

$$\delta_{q,r} f = rD_{q,r}f. \tag{6.21}$$

Then $f_{m,n}(z_1,z_2)$, for $m \ge n$ and $\alpha = m-n+\beta$, satisfies the q-partial differential equations

$$A_n(z_1z_2,\alpha)\delta_{q,z_2}^2 f + B_n(z_1z_2,\alpha)\delta_{q,z_2} f + C_n(z_1z_2,\alpha)f = 0, \tag{6.22}$$

$$\begin{aligned}&A_n(z_1z_2,\alpha)\delta_{q,z_1}^2 f + \{q^\alpha B_n(z_1z_2,\alpha) - 2[\alpha]_q A_n(z_1z_2,\alpha)\}\delta_{q,z_1} f\\ &+ \left\{[\alpha]_q^2 A_n(z_1z_2,\alpha) - [\alpha]_q q^\alpha B_n(z_1z_2,\alpha) + q^{2\alpha}C_n(z_1z_2,\alpha)\right\}f = 0.\end{aligned} \tag{6.23}$$

Theorem 6.5. *Assume that*

$$w_{\alpha+\beta}(x) = x^\alpha w_\beta(x) \tag{6.24}$$

and

$$\phi_n(x;\alpha) = \frac{1}{w_\alpha(x)}\partial_x^n(w_\alpha(x)x^n), \quad n = 0,1,\dots. \tag{6.25}$$

Then for $m \ge n$ we have

$$f_{m,n}(z_1,z_2;\beta) = \left(\frac{z_1}{z_2}\right)^{m-n}\frac{\partial_{z_2}^n\left(w_\beta(z_1z_2)z_2^m\right)}{w_\beta(z_1z_2)}, \tag{6.26}$$

$$f_{m+1,n+1}(z_1,z_2;\beta) = \frac{\partial_{z_2}\left(\left(\frac{z_2}{z_1}\right)^{m+1-n}w_\beta(z_1z_2)f_{m+1,n}(z_1,z_2;\beta)\right)}{\left(\frac{z_2}{z_1}\right)^{m-n}w_\beta(z_1z_2)}, \tag{6.27}$$

$$f_{m,n}(z_1,z_2;\beta) = \frac{\partial_{z_1}^n\left(w_\beta(z_1z_2)z_1^m\right)}{w_\beta(z_1z_2)}, \tag{6.28}$$

$$f_{m+1,n+1}(z_1,z_2;\beta) = \frac{\partial_{z_1}\left(w_\beta(z_1z_2)f_{m+1,n}(z_1,z_2;\beta)\right)}{w_\beta(z_1z_2)} \tag{6.29}$$

and

$$\begin{aligned}&\partial_{z_2}\left(\left(\frac{z_2}{z_1}\right)^{m+1-n}w_\beta(z_1z_2)f_{m+1,n}(z_1,z_2;\beta)\right)\\ &= \partial_{z_1}\left(w_\beta(z_1z_2)f_{m+1,n}(z_1,z_2;\beta)\right)\end{aligned} \tag{6.30}$$

We next state a q-analogue of the previous result.

Theorem 6.6. *Assume that*

$$w_{\alpha+\beta}(x) = w_{\alpha+\beta}(x|q) = x^\alpha w_\beta(x|q) \tag{6.31}$$

and

$$\phi_n(x;\alpha) = \phi_n(x;\alpha|q) = \frac{1}{w_\alpha(x)}D_{q,x}^n(w_\alpha(x|q)x^n), \quad n = 0,1,\dots. \tag{6.32}$$

Let $f_{m,n}(z_1,z_2;\beta|q) = f_{m,n}(z_1,z_2;\beta)$, then for $m \ge n$ we have

$$f_{m,n}(z_1,z_2;\beta|q) = \frac{\left(\frac{z_1}{z_2}\right)^{m-n}\partial_{q,z_2}^n\left(w_\beta(z_1z_2|q)z_2^m\right)}{w_\beta(z_1z_2|q)}, \tag{6.33}$$

$$f_{m+1,n+1}(z_1,z_2;\beta|q) = \frac{\partial_{q,z_2}\left(\left(\frac{z_2}{z_1}\right)^{m+1-n}w_\beta(z_1z_2|q)f_{m+1,n}(z_1,z_2;\beta|q)\right)}{\left(\frac{z_2}{z_1}\right)^{m-n}w_\beta(z_1z_2|q)}, \tag{6.34}$$

$$f_{m,n}(z_1,z_2;\beta|q) = \frac{\partial_{q,z_1}^n\left(w_\beta(z_1z_2|q)z_1^m\right)}{w_\beta(z_1z_2|q)}, \tag{6.35}$$

$$f_{m+1,n+1}(z_1,z_2;\beta|q)=\frac{\partial_{q,z_1}\left(w_\beta(z_1z_2|q)f_{m+1,n}(z_1,z_2;\beta|q)\right)}{w_\beta(z_1z_2|q)} \tag{6.36}$$

and

$$\partial_{q,z_2}\left(\left(\frac{z_2}{z_1}\right)^{m+1-n}w_\beta(z_1z_2|q)f_{m+1,n}(z_1,z_2;\beta|q)\right) = \left(\frac{z_2}{z_1}\right)^{m-n}\partial_{q,z_1}\left(w_\beta(z_1z_2|q)f_{m+1,n}(z_1,z_2;\beta|q)\right). \tag{6.37}$$

7. 2D q-ultraspherical polynomials

The 2D-ultraspherical polynomials are

$$C^{\nu}_{m,n}(z_1,z_2)=\sum_{k=0}^{m\wedge n}\binom{m}{k}\binom{n}{k}k!(-1)^k(\nu)_{m+n-k}z_1^{m-k}z_2^{n-k},\quad \nu>-1. \tag{7.38}$$

They are also known as the disk polynomials or the Zernike polynomials, [11].

It is clear that

$$C^{\nu}_{m,n}(z_1,z_2)=(\nu)_{m+n}z_1^mz_2^n\,{}_2F_1\left(\begin{matrix}-m,-n\\ -m-n-\nu+1\end{matrix}\middle|\frac{1}{z_1z_2}\right) \tag{7.39}$$

which a constant multiple of the disk polynomials of Section 2.3 in [11].

They have the generating function

$$\sum_{m,n=0}^{\infty}C^{\nu}_{m,n}(z_1,z_2)\frac{u^m\,v^n}{(m!\,n!}=(1-uz_1-vz_2+uv)^{-\nu}, \tag{7.40}$$

whose proof is an exercise in the application of the binomial theorem. Next we solve the connection relation between $C^{\nu}_{m,n}(z_1,z_2)$ and $H_{m,\,n}(z_1,\,z_2)$. We claim that

$$C^{\nu}_{m,n}(z_1,z_2)=\sum_{p=0}^{m\wedge n}\frac{(\nu)_{m+n}\;m!\;n!}{p!\;(m-p)!\;(n-p)!}H_{m-p,n-p}(z_1z_2)\;{}_1 \times F_1(-p;-\nu-m-n+1;-1) \tag{7.41}$$

Write the right-hand side of (7.40) as

$$\begin{aligned}&\int_0^\infty\frac{t^{\nu-1}}{\Gamma(\nu)}e^{-t+tuz_1+tvz_2-tuv}dt\\ &=\int_0^\infty\frac{t^{\nu-1}}{\Gamma(\nu)}e^{-t+t(t-1)uv}\sum_{r,s=0}^{\infty}H_{r,s}(z_1z_2)\frac{u^r\,v^s}{r!\,s!}\;t^{r+s}dt\\ &=\sum_{r,s=0}^{\infty}H_{r,s}(z_1z_2)\frac{u^r\,v^s}{r!\,s!}\sum_{j,k=0}^{\infty}\frac{(-1)^j(uv)^{j+k}}{j!\,k!}\frac{\Gamma(\nu+r+s+j+2k)}{\Gamma(\nu)}.\end{aligned}$$

Equating coefficients of u^mv^n we see that $m=r+j+k, n=s+j+k$. Let $p=j+k$. Now (7.41) follows after some manipulations.

It is clear from the generating function (7.40) that the disk polynomials have the convolution property

$$C^{\nu+\mu}_{m,n}(z_1,z_2)=\sum_{j=0}^{m}\sum_{k=0}^{n}C^{\mu}_{j,k}(z_1,z_2)C^{\nu}_{m-j,n-k}(z_1,z_2) \tag{7.42}$$

Floris [16], see also [15] and [17], introduced the following q-analogue of the disk polynomials. For $\alpha,\beta>-1$ and $l,m\in\mathbb{Z}_+$, the Floris q-disk polynomials $R^{(\alpha)}_{l,m}\left(z,z*;q^2\right)$ are defined by

$$R^{(\alpha)}_{l,m}\left(z,z*;q^2\right)=\begin{cases}z^{l-m}P^{(\alpha,l-m)}_m\left(1-zz^*:q^2\right) & l\geq m\\ P^{(\alpha,m-l)}_m\left(1-zz^*:q^2\right)(z^*)^{m-l} & l\leq m,\end{cases}$$

where

$$z^*z=q^2zz^*+1-q^2,$$

and

$$P^{(\alpha,\beta)}_m(x:q)=p_m\left(x;q^\alpha,q^\beta;q\right)$$

is the little q-Jacobi polynomials. The q-disk polynomials $R^{(\alpha)}_{l,m}\left(z,z*;q^2\right)$ satisfy

$$R^{(\alpha)}_{l,m}\left(z,z*;q^2\right)^*=R^{(\alpha)}_{m,l}\left(z,z*;q^2\right)$$

and the orthogonality relation

$$\begin{aligned}&\frac{1}{2\pi}\int_0^1\int_0^{2\pi}R^{(\alpha)}_{l,m}\left(e^{i\theta}z,e^{-i\theta}z*;q^2\right)^*R^{(\alpha)}_{l',m'}\left(e^{i\theta}z,e^{-i\theta}z*;q^2\right)d\theta\\ &\times(1-zz^*)^\alpha d_{q^2}(1-zz^*)\\ &=\frac{\left(1-q^2\right)\left(q^2;q^2\right)_l\left(q^2;q^2\right)_mq^{2m(\alpha+1)}\delta_{ll'}\delta_{mm'}}{\left(1-q^{2(\alpha+l+m+1)}\right)\left(q^{2(\alpha+1)};q^2\right)_l\left(q^{2(\alpha+1)};q^2\right)_m},\end{aligned}$$

for $\alpha>-1,\ l,l',m,m'\in\mathbb{Z}_+$.

We now introduce our q-analogue of these polynomials. For $0<q<1$ and $b<q^{-1}$, let us define

$$p_{m,n}(z_1,z_2;b|q)=\sum_{k=0}^{\infty}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\binom{k}{2}}(q;q)_k(bq;q)_{m+n-k}}{(-1)^kz_1^{k-m}z_2^{k-n}}, \tag{7.43}$$

it is clear that

$$p_{m,n}(z_2,z_1;b|q)=p_{n,m}(z_1,z_2;b|q) \tag{7.44}$$

then for $m\geq n$ we have

$$p_{m,n}(z_1,z_2;b|q)=(-1)^nq^{\binom{n}{2}}(bq;q)_m\left(q^{m-n+1};q\right)_nz_1^{m-n}\times p_n\left(z_1z_2;q^{m-n},b|q\right), \tag{7.45}$$

where $p_n(x;\,a,\,b|q)$ is the little Jacobi polynomials.

Theorem 7.1. *For $0<q<1$ and $b<q^{-1}$, the polynomials $\{p_{m,n}(z,\bar{z};b|q)\}$ satisfy the following orthogonality*

$$\int_{\mathbb{C}}p_{m,n}(z,\bar{z};b|q)\overline{p_{s,t}(z,\bar{z};b|q)}d\mu(z,\bar{z}) = \frac{(bq;q)_\infty}{(q;q)_\infty}\frac{q^{mn}(q,bq;q)_m(q,bq;q)_n}{1-bq^{m+n+1}}\delta_{m,s}\delta_{n,t}, \tag{7.46}$$

where

$$d\mu(z,\bar{z})=\frac{d\theta}{2\pi}\otimes\sum_{k=0}^{\infty}\frac{(bq;q)_kq^k}{(q;q)_k}\delta\left(r-q^{k/2}\right), \tag{7.47}$$

$z=re^{i\theta}, r\in\mathbb{R}^+, \theta\in[0,2\pi]$ and $m,n,s,t\in\mathbb{N}_0$.

Theorem 7.2. *The polynomials $\{p_{m,\,n}(z_1,\,z_2;\,b|q)\}$ have the generating function*

$$\sum_{m,n=0}^{\infty}\frac{p_{m,n}(z_1,z_2;b|q)}{(q;q)_m(q;q)_n}u^mv^n = \frac{(bq,uv;q)_\infty}{(uz_1,z_2v;q)_\infty}{}_2\phi_1\left(\begin{matrix}uz_1,vz_2\\ uv\end{matrix}\middle|q;bq\right). \tag{7.48}$$

For $bq,cq<1$ and $b\neq0$, the connection relation between the $q-2D$ ultraspherical polynomials is given by

$$\frac{p_{m,n}(z_1,z_2;b|q)}{(bq;q)_\infty}=\sum_{j=0}^{\infty}\frac{\left(\frac{c}{b};q\right)_j}{(q;q)_j}\left(bq^{\frac{m+n}{2}+1}\right)^j\frac{p_{m,n}\left(z_1q^{\frac{j}{2}},z_2q^{\frac{j}{2}};c|q\right)}{(cq;q)_\infty}. \tag{7.49}$$

The connection relation between the $q-2D$ ultraspherical and $q-2D$ Hermite is given by

$$\frac{p_{m,n}(z_1,z_2;b|q)}{(bq;q)_\infty}=\sum_{j=0}^{\infty}\frac{\left(bq^{(m+n)/2+1}\right)^j}{(q;q)_j}H_{m,n}\left(z_1q^{j/2},z_2q^{j/2}|q\right). \tag{7.50}$$

Moreover we have the inverse relation

$$H_{m,n}(z_1,z_2|q)=\sum_{k=0}^{\infty}\frac{\left(-bq^{(m+n)/2+1}\right)^k}{(bq;q)_\infty(q;q)_k}q^{\binom{k}{2}}p_{m,n}\left(z_1q^{k/2},z_2q^{k/2};b|q\right). \tag{7.51}$$

Let us rewrite (7.48) in the form

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix}uz_1,vz_2\\uv\end{matrix}\,\middle|q;bq\right)\\&=\frac{(uz_1,z_2v;q)_\infty}{(bq,uv;q)_\infty}\sum_{m,n=0}^{\infty}\frac{p_{m,n}(z_1,z_2;b|q)}{(q;q)_m(q;q)_n}u^mv^n.\end{aligned} \tag{7.52}$$

Theorem 7.3. *The polynomials* $\{p_{m,n}(z_1,z_2;b|q)\}$ *satisfy the following properties,*

$$D_{q,z_1}p_{m,n}(z_1,z_2;b|q)=\frac{(1-bq)}{1-q}(1-q^m)p_{m-1,n}(z_1,z_2;bq|q), \tag{7.53}$$

$$D_{q,z_2}p_{m,n}(z_1,z_2;b|q)=\frac{(1-bq)}{1-q}(1-q^n)p_{m,n-1}(z_1,z_2;bq|q), \tag{7.54}$$

$$\begin{aligned}&D_{q^{-1},z_1}\left(\frac{(qz_1z_2;q)_\infty}{(bqz_1z_2;q)_\infty}p_{m,n}(z_1,z_2;b|q)\right)\\&=\frac{(qz_1z_2;q)_\infty p_{m,n+1}\left(z_1,z_2;bq^{-1}|q\right)}{q^{m-1}(q-1)(bz_1z_2;q)_\infty},\end{aligned} \tag{7.55}$$

$$\begin{aligned}&D_{q^{-1},z_2}\left(\frac{(qz_1z_2;q)_\infty}{(bqz_1z_2;q)_\infty}p_{m,n}(z_1,z_2;b|q)\right)\\&=\frac{(qz_1z_2;q)_\infty p_{m+1,n}\left(z_1,z_2;bq^{-1}|q\right)}{q^{n-1}(q-1)(bz_1z_2;q)_\infty},\end{aligned} \tag{7.56}$$

$$\begin{aligned}&p_{m,n}(z_1q,z_2;b|q)-bq^{n+m-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1q,z_2;b|q)\\&=p_{m,n}(z_1,z_2;b|q)q^m-q^{m-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2;b|q),\end{aligned} \tag{7.57}$$

$$\begin{aligned}&p_{m,n}(z_1q,z_2;b|q)-bq^{2m-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2q;b|q)\\&=p_{m,n}(z_1,z_2;b|q)-q^{m-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2;b|q),\end{aligned} \tag{7.58}$$

$$\begin{aligned}&p_{m,n}(z_1,z_2q;b|q)-bq^{2n-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1q,z_2;b|q)\\&=q^np_{m,n}(z_1,z_2;b|q)-q^{n-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2;b|q),\end{aligned} \tag{7.59}$$

$$\begin{aligned}&p_{m,n}(z_1,z_2q;b|q)-bq^{m+n}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2q;b|q)\\&=q^np_{m,n}(z_1,z_2;b|q)-q^{n-1}(1-q^m)(1-q^n)p_{m-1,n-1}(z_1,z_2;b|q).\end{aligned} \tag{7.60}$$

$$\begin{aligned}&p_{m,n}(z_1q,z_2;b|q)-bq^{n+1}z_1(1-q^m)p_{m-1,n}(z_1q,z_2;b|q)\\&=p_{m,n}(z_1,z_2;b|q)-z_1(1-q^m)p_{m-1,n}(z_1,z_2;b|q),\end{aligned} \tag{7.61}$$

$$\begin{aligned}&p_{m,n}(z_1q,z_2;b|q)-bq^mz_1(1-q^m)p_{m-1,n}(z_1,z_2q;b|q)\\&=p_{m,n}(z_1,z_2;b|q)-z_1(1-q^m)p_{m-1,n}(z_1,z_2;b|q),\end{aligned} \tag{7.62}$$

$$\begin{aligned}&\left(1+bq^{m+n}z_1\right)p_{m,n}(z_1,z_2;b|q)\\&\quad=z_1p_{m-1,n}(z_1,z_2;b|q)-q^{m-1}(1-q^n)(1-bq^n)p_{m-1,n-1}\\&\qquad\times(z_1,z_2;b|q),\end{aligned} \tag{7.63}$$

$$\begin{aligned}&\left(1-q^{m-n}\right)p_{m+1,n+1}(z_1,z_2;b|q)\\&\quad=z_1\left(1-bq^{m+1}\right)\left(1-q^{m+1}\right)p_{m,n+1}(z_1,z_2;b|q)\\&\qquad-z_2q^{m-n}\left(1-bq^{n+1}\right)\left(1-q^{n+1}\right)p_{m+1,n}(z_1,z_2;b|q),\end{aligned} \tag{7.64}$$

$$\begin{aligned}&p_{m+1,n+1}(z_1q,z_2;b|q)-p_{m+1,n+1}(z_1,z_2q;b|q)\\&=z_2\left(1-bq^{m+2}\right)\left(1-q^{n+1}\right)p_{m+1,n}(z_1q,z_2;b|q)\\&-z_1\left(1-bq^{m+1}\right)\left(1-q^{m+1}\right)p_{m,n+1}(z_1,z_2q;b|q),\end{aligned} \tag{7.65}$$

$$\begin{aligned}&z_2(1-q^n)p_{m,n-1}(z_1q,z_2;b|q)-z_1(1-q^m)p_{m-1,n}(z_1,z_2q;b|q)\\&=z_2(1-q^n)p_{m,n-1}(z_1,z_2;b|q)-z_1(1-q^m)p_{m-1,n}(z_1,z_2;b|q),\end{aligned} \tag{7.66}$$

$$\begin{aligned}&qz_1z_2(p_{m,n}(z_1q,z_2;b|q)-p_{m,n}(z_1,z_2q;b|q))\\&\quad+z_1z_2(1-q^m)(1-q^n)(p_{m-1,n-1}(z_1q,z_2;b|q)\\&\quad-p_{m-1,n-1}(z_1,z_2q;b|q))\\&\quad=qz_1z_2^2(1-q^n)(p_{m,n-1}(z_1q,z_2;b|q)-bp_{m,n-1}(z_1q,z_2q;b|q))\\&\quad-qz_1^2z_2(1-q^m)(p_{m-1,n}(z_1,z_2q;b|q)-bp_{m-1,n}(z_1q,z_2q;b|q))\\&\quad+z_1(1-q^m)(p_{m-1,n}(z_1q,z_2;b|q)-p_{m-1,n}(z_1q,z_2q;b|q))\\&\quad-z_2(1-q^n)(p_{m,n-1}(z_1,z_2q;b|q)-p_{m,n-1}(z_1q,z_2q;b|q)).\end{aligned} \tag{7.67}$$

The inversion transformation of quanta $q\to q^{-1}$ in (4.6) relates the properties of one family of polynomials for $q>1$ to the properties of another family of polynomials with $0<q<1$. The polynomials $p_{m,n}(z_1,z_2;b|q)$ are essentially invariant under the quanta inversion transformation,

$$\begin{aligned}&p_{m,n}\left(z_1,z_2;b|q^{-1}\right)\\&=(-b)^{m+n}\sum_{k=0}^{\infty}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-b)^{-k}(q;q)_k\left(\frac{q}{b};q\right)_{m+n-k}\\&\quad\times q^{k(k-m)+k(k-n)-\binom{k}{2}-\binom{k+1}{2}-\binom{m+n-k+1}{2}}z_1^{m-k}z_2^{n-k}\\&=(-b)^{m+n}q^{-\binom{m+n+1}{2}}\sum_{k=0}^{\infty}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}\left(-\frac{q}{b}\right)^k(q;q)_k\\&\quad\times\left(\frac{q}{b};q\right)_{m+n-k}z_1^{m-k}z_2^{n-k}\\&=(-1)^{m+n}(b/q)^{(1-\alpha)m+\alpha n}q^{-\binom{m+n}{2}}p_{m,n}\\&\quad\times\left((b/q)^\alpha z_1,\,(b/q)^{1-\alpha}z_2;\,\frac{1}{b}|q\right).\end{aligned}$$

Therefore, we have established the symmetry

$$\begin{aligned}&p_{m,n}\left(z_1,z_2;b|q^{-1}\right)\\&\quad=\frac{\left(bq^{-1}\right)^{(1-\alpha)m+\alpha n}}{(-1)^{m+n}q^{\binom{m+n}{2}}}p_{m,n}\left((b/q)^\alpha z_1,\,(b/q)^{1-\alpha}z_2;\,1/b|q\right),\end{aligned} \tag{7.68}$$

for $\alpha\in\mathbb{C}$.

We now come the asymptotics of $p_{m,n}(z_1,z_2;b|q)$.

Theorem 7.4. *Let* $z_1,z_2\in\mathbb{C}$, $bq<1$ *and* $z_1z_2\neq 0$, *then we have*

$$\lim_{m,n\to\infty}\frac{p_{m,n}(z_1,z_2;b|q)}{(bq;q)_\infty z_1^mz_2^n}=\left(\frac{1}{z_1z_2};q\right)_\infty, \tag{7.69}$$

uniformly on compact subsets of the z_1 *and* z_2 *planes.*

The theorem follows from the definition (7.43) and Tannery's theorem.

8. The polynomials $\{Z^{(\beta)}_{m,n}(z_1,z_2)\}$

Motivated by the class of general $1D$ systems in Section 2 we define polynomials $\{Z^{(\beta)}_{m,n}(z_1,z_2)\}$ by

$$Z^{(\beta)}_{m,n}(z_1,z_2) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(\beta+1)_m}{(\beta+1)_{m-k}}(-1)^{n-k}z_1^{m-k}z_2^{n-k}, \tag{8.1}$$

for $m \ge n$. When $m < n$ the polynomials are defined by

$$Z^{(\beta)}_{m,n}(z_1,z_2) = Z^{(\beta)}_{n,m}(z_2,z_1) \tag{8.2}$$

Here $\beta > -1$.

These polynomials arise through the choice $\phi_n(x;\alpha) = L_n^{(\alpha+\beta)}(x)$, where $L_n^{(a)}(x)$ is a Laguerre polynomial, [25,50],

$$L_n^{(a)}(x) = \frac{(a+1)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{(a+1)_k} = \frac{(a+1)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^{n-k}}{(a+1)_{n-k}}.$$

Indeed, for $m \ge n$

$$Z^{(\beta)}_{m,n}(z_1,z_2) = z_1^{m-n}L_n^{(\beta+m-n)}(z_1z_2) \tag{8.3}$$

and

$$z_1Z^{(\beta+1)}_{m,n}(z_1,z_2) = Z^{(\beta)}_{m+1,n}(z_1,z_2). \tag{8.4}$$

It is clear that when $\beta = 0$ we see that

$$H_{m,n}(z_1,z_2) = (-1)^n(m\wedge n)!Z^{(0)}_{m,n}(z_1,z_2). \tag{8.5}$$

Therefore in the notation of Section 2, we have

$$\zeta_n(\alpha) = \frac{\Gamma(\alpha+\beta+n+1)}{n!},$$
$$c_j(n,\alpha) = \frac{(-1)^{n-j}\,(\alpha+\beta+1)_n}{(n-j)!\;j!\;(\alpha+\beta+1)_{n-j}}. \tag{8.6}$$

Theorem 8.1. For $m,n,s,t = 0,1,\dots$ and $\beta > -1$, we have the orthogonality relation

$$\int_{\mathbb{R}^2} Z^{(\beta)}_{m,n}(z,\bar z)\,\overline{Z^{(\beta)}_{s,t}(z,\bar z)}\,(z\bar z)^{\beta}e^{-z\bar z}rdrd\theta$$
$$= \pi\,\frac{\Gamma(\beta+m\vee n+1)}{(m\wedge n)!}\,\delta_{m,s}\delta_{n,t}. \tag{8.7}$$

In view of (8.6) the recurrence relations (6.8), (6.7) and (6.13) become

$$z_1Z^{(\beta)}_{m,n}(z_1,z_2) = Z^{(\beta)}_{m+1,n}(z_1,z_2) - Z^{(\beta)}_{m,n-1}(z_1,z_2), \tag{8.8}$$

$$z_2Z^{(\beta)}_{m+1,n}(z_1,z_2) = -(n+1)Z^{(\beta)}_{m+1,n+1}(z_1,z_2) + (\beta+m+1)Z^{(\beta)}_{m,n}(z_1,z_2), \tag{8.9}$$

and

$$(\beta+m+n+1-z_1z_2)Z^{(\beta)}_{m,n}(z_1,z_2) = (n+1)Z^{(\beta)}_{m+1,n+1}(z_1,z_2) + (m+\beta)Z^{(\beta)}_{m-1,n-1}(z_1,z_2) \tag{8.10}$$

respectively, where $m \ge n$. We now discuss differential recurrence relations.

Theorem 8.2. *For $m \ge n$, the polynomials $\{Z^{(\beta)}_{m,n}(z_1,z_2)\}$ satisfy the differential recurrence relations*

$$\delta_{z_2}Z^{(\beta)}_{m,n}(z_1,z_2) = -z_2Z^{(\beta)}_{m,n-1}(z_1,z_2), \tag{8.11}$$

$$\delta_{z_2}Z^{(\beta)}_{m,n}(z_1,z_2) = nZ^{(\beta)}_{m,n}(z_1,z_2) - (\beta+m)Z^{(\beta)}_{m-1,n-1}(z_1,z_2), \tag{8.12}$$

$$\delta_{z_1}Z^{(\beta)}_{m,n}(z_1,z_2) = (m-n)Z^{(\beta)}_{m,n}(z_1,z_2) - z_2Z^{(\beta)}_{m,n-1}(z_1,z_2), \tag{8.13}$$

$$\delta_{z_1}Z^{(\beta)}_{m,n}(z_1,z_2) = mZ^{(\beta)}_{m,n}(z_1,z_2) - (m+\beta)Z^{(\beta)}_{m-1,n-1}(z_1,z_2), \tag{8.14}$$

$$(\delta_{z_1}-\delta_{z_2})Z^{(\beta)}_{m,n}(z_1,z_2) = (m-n)Z^{(\beta)}_{m,n}(z_1,z_2), \tag{8.15}$$

$$(n\delta_{z_1}-m\delta_{z_2})Z^{(\beta)}_{m,n}(z_1,z_2) = (m-n)(m+\beta)Z^{(\beta)}_{m-1,n-1}(z_1,z_2), \tag{8.16}$$

where $\delta_{z_1} = z_1\partial_{z_1}$, $\delta_{z_2} = z_2\partial_{z_2}$. Moreover they have the operational representation

$$Z^{(\beta)}_{m,n}(z_1,z_2) = \frac{(-1)^n}{n!}z_1^{-\beta}\exp(-\partial_{z_1}\partial_{z_2})z_1^{\beta+m}z_2^n \tag{8.17}$$

and the Rodrigues type representation

$$Z^{(\beta)}_{m,n}(z_1,z_2) = \frac{1}{n!}z_1^{-\beta}e^{z_1z_2}\partial_{z_1}^n\left(z_1^{m+\beta}e^{-z_1z_2}\right)$$
$$= \frac{(-1)^m}{n!}(z_1z_2)^{-\beta}e^{z_1z_2}\partial_{z_1}^n\left((z_1z_2)^{\beta}\partial_{z_2}^m\left(e^{-z_1z_2}\right)\right). \tag{8.18}$$

We shall use the notation

$$w_\beta(z_1z_2) = (z_1z_2)^{\beta}\exp(-z_1z_2). \tag{8.19}$$

It is clear that (8.18) implies the differentiation formulas

$$e^{z_1z_2}\partial_{z_2}\left(e^{-z_1z_2}Z^{(\beta)}_{m,n}(z_1,z_2)\right) = -z_1Z^{(\beta+1)}_{m,n}(z_1,z_2) = -Z^{(\beta)}_{m+1,n}(z_1,z_2) \tag{8.20}$$

and

$$\frac{1}{w_\beta(z_1z_2)}\partial_{z_1}\left(w_\beta(z_1z_2)Z^{(\beta)}_{m,n}(z_1,z_2)\right) = (n+1)Z^{(\beta)}_{m,n+1}(z_1,z_2) \tag{8.21}$$

for $m \ge n$. It also clear that

$$\delta_{z_j}w_\beta(z_1z_2) = (\beta - z_1z_2)w_\beta(z_1z_2), \quad j = 1,2 \tag{8.22}$$

and (8.8) which imply the relationships

$$\frac{\delta_{z_2}\left(w_\beta(z_1z_2)Z^{(\beta)}_{m,n}(z_1,z_2)\right)}{w_\beta(z_1z_2)} = \beta Z^{(\beta)}_{m,n}(z_1,z_2) - z_2Z^{(\beta)}_{m+1,n}(z_1,z_2), \tag{8.23}$$

and

$$\frac{\delta_{z_1}\left(w_\beta(z_1z_2)Z^{(\beta)}_{m,n}(z_1,z_2)\right)}{w_\beta(z_1z_2)}$$
$$= (\beta+m-n)Z^{(\beta)}_{m,n}(z_1,z_2) - z_2Z^{(\beta)}_{m+1,n}(z_1,z_2). \tag{8.24}$$

Theorem 8.3. The polynomials $\{Z^{(\beta)}_{m,n}(z_1,z_2)\}$ satisfy the second order partial differential equation

$$\partial_{\partial z_1}\partial_{\partial z_2}f + \left(\frac{\beta - z_1z_2}{z_1}\right)\partial_{\partial z_2}f = -nf, \tag{8.25}$$

for all $m \ge n$.

It is clear that the differential property (8.25) can be written in the form

$$\frac{\partial}{\partial z_1}\left[w_\beta(z_1z_2)\frac{\partial}{\partial z_2}Z^{(\beta)}_{m,n}(z_1,z_2)\right] = -nZ^{(\beta)}_{m,n}(z_1,z_2). \tag{8.26}$$

Note that (8.25) and (8.15) indicate that the polynomials $\{Z^{(\beta)}_{m,n}(z_1,z_2)\}$ are simultaneous eigenfunctions of the operators on their respective left-hand sides. The differential operators $\partial_{\partial z_1}\partial_{\partial z_2} + \left(\frac{\beta-z_1z_2}{z_1}\right)\partial_{\partial z_2}$ and $z_1\partial_{\partial z_1} - z_2\partial_{\partial z_2}$ indeed commute as can be directly verified.

Let $w(x,y) = (z\bar z)^{\beta}e^{-z\bar z}$ and define the inner product

$$< f,g >= \int_{\mathbb{R}^2} f(x,y)\overline{g(x,y)}w(x,y)dxdy. \tag{8.27}$$

Theorem 8.4. *We have the adjoint relations*

$$(\partial_z)^* = -\partial_{\bar z} - \frac{\beta}{\bar z} + z, \qquad (\partial_{\bar z})^* = -\partial_z - \frac{\beta}{z} + \bar z. \tag{8.28}$$

The proof is straightforward calculus exercise.

This allows us to write (8.25) or (8.26) in the selfadjoint form

$$((\partial_{\bar z})^* \partial_{\bar z}) f = nf. \tag{8.29}$$

Moreover we also have the following result.

Theorem 8.5. *The operator $A := (\partial_{\bar z})^* \partial_{\bar z}$ is positive in the sense that $< Af, f > \geq 0$ with $=$ if and only if $\partial_{\bar z} f = 0$, that is f depends only on z.*

This indicates that

$$\frac{1}{w(x,y)} \frac{\partial}{\partial z}\left[w(x,y) Z^{(\beta)}_{m,n}(z,\bar z)\right] = (n+1) Z^{(\beta)}_{m,n}(z,\bar z). \tag{8.30}$$

This is the adjoint of (8.11).

Ismail and Zeng [29] established the connection relations stated in the next theorem.

Theorem 8.6. *We have the connection relation*

$$Z^{(\beta)}_{m,n}(z_1,z_2) = \sum_{j=0}^{n} \frac{(\beta-\gamma)_j}{j!} (-1)^j Z^{(\gamma)}_{m-j,n-j}(z_1,z_2) \qquad (m \geq n). \tag{8.31}$$

For $m \geq n$, we have the special cases

$$n! Z^{(\beta)}_{m,n}(z_1,z_2) = \sum_{j=0}^{n} \binom{n}{j} (\beta)_j (-1)^j H_{m-j,n-j}(z_1,z_2), \tag{8.32}$$

$$H_{m,n}(z_1,z_2) = n! \sum_{j=0}^{n} \frac{(-\beta)_j}{j!} (-1)^j Z^{(\beta)}_{m-j,n-j}(z_1,z_2). \tag{8.33}$$

The Laguerre differential equation is [45,50],

$$xy'' + (1+\alpha-x)y' + ny = 0. \tag{8.34}$$

Therefore Theorem 6.4 shows that

$$z_2 \frac{\partial^2 Z^{(\beta)}_{m,n}}{\partial z_2^2} + (1+\beta+m-n-z_1z_2) \frac{\partial Z^{(\beta)}_{m,n}}{\partial z_2} + n z_1 Z^{(\beta)}_{m,n} = 0. \tag{8.35}$$

Ismail and Zeng [29] gave the generating function

$$\sum_{m\geq n\geq 0} \frac{u^m v^n}{(m-n)!} Z^{(\beta)}_{m,n}(z_1,z_2) = (1+uv)^{-\beta-1} \exp\left(\frac{uvz_1z_2 + z_1 u}{1+uv}\right). \tag{8.36}$$

It is clear that the generating function (8.36) implies the identity

$$Z^{(\beta+\gamma+1)}_{m,n}(z_1+z_3, z_2+z_4) = \sum_{m\geq j\geq k\geq 0} \frac{Z^{(\beta)}_{j,k}(z_1,z_2) Z^{(\gamma)}_{m-j,n-k}(z_3,z_4)}{(j-k)!(m-n-j+k)!}. \tag{8.37}$$

This is an analogue of the convolution identity

$$L^{(\alpha+\beta+1)}_n(x+y) = \sum_{k=0}^{n} L^{(\alpha)}_k(x) L^{(\beta)}_n(y). \tag{8.38}$$

Problem. Al-Salam and Chihara characterized all 1-*D* orthogonal polynomials satisfying formulas of convolution type in [2]. They discovered the Al-Salam–Chihara polynomials through this characterization, [25]. It will be interesting to solve the corresponding 2-*D* characterization problem.

9. The polynomials $\{M^{(\beta,\gamma)}_n(z_1,z_2)\}$

For $\alpha > 0, \beta, \gamma > -1$ Ismail and Zeng [29] introduced the polynomials $\phi_n(r;\alpha) = P^{(\alpha+\gamma,\beta)}_n(1-2r)$, that is

$$\phi_n(r;\alpha) = (\alpha+\gamma+1)_n \sum_{k=0}^{n} \frac{(\alpha+\beta+\gamma+n+1)_{n-k}(-r)^{n-k}}{k!(n-k)!(\alpha+\gamma+1)_{n-k}}. \tag{9.1}$$

They satisfy the following orthogonality relation

$$\int_0^1 \phi_m(r;\alpha)\phi_n(r;\alpha) u^{\alpha+\gamma}(1-u)^\beta\, du = \zeta_n(\alpha+\gamma,\beta)\, \delta_{m,n}, \tag{9.2}$$

where

$$\zeta_n(\alpha+\gamma,\beta) = \frac{\Gamma(\alpha+\gamma+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+\gamma+n+1)(\alpha+\beta+\gamma+2n+1)}. \tag{9.3}$$

We define the two variable polynomial $M^{\beta,\gamma}_{m,n}(z_1,z_2)$ by

$$M^{(\beta,\gamma)}_{m,n}(z_1,z_2) = \sum_{k=0}^{n} \frac{(m+\beta+\gamma+1)_{n-k}(\gamma+1)_m z_1^{m-k}(-z_2)^{n-k}}{k!(n-k)!(\gamma+1)_{m-k}} \tag{9.4}$$

for $m \geq n$ and

$$M^{(\beta,\gamma)}_{m,n}(z_1,z_2) = M^{(\beta,\gamma)}_{n,m}(z_2,z_1), \quad m < n. \tag{9.5}$$

Then,

$$z_1 M^{(\beta,\gamma+1)}_{m,n}(z_1,z_2) = M^{(\beta,\gamma)}_{m+1,n}(z_1,z_2). \tag{9.6}$$

Clearly, $M^{(\beta,\gamma)}_{m,n}(z_1,z_2)$ satisfy the orthogonality relation

$$\begin{aligned} &\int_{|z|\leq 1} M^{(\beta,\gamma)}_{m,n}(z,\bar z)\overline{M^{(\beta,\gamma)}_{p,q}(z,\bar z)} r^{2\gamma}(1-r^2)^\beta r dr \frac{d\theta}{\pi} \\ &= \frac{\Gamma(\gamma+m+1)\,\Gamma(\beta+n+1)}{n!\Gamma(\beta+\gamma+m+1)(\beta+\gamma+m+n+1)}\, \delta_{m,p}\delta_{n,q}, \end{aligned} \tag{9.7}$$

for $m \geq n$ as it is stated in [29]. The disk polynomials defined in [11] can be expressed as

$$P^\beta_{m,n}(z) = \frac{(-1)^n n!}{(\beta+1)_n} M^{(\beta,0)}_{m,n}(z,\bar z), \quad m \geq n. \tag{9.8}$$

From (9.1) we get

$$c_k(n,\alpha) = \frac{(m+\beta+\gamma+1)_{n-k}(\gamma+1)_m(-1)^{n-k}}{k!(n-k)!(\gamma+1)_{m-k}}, \tag{9.9}$$

where $\alpha = m-n$ with $m \geq n$. Then

$$\begin{aligned} &(\beta+\gamma+m+n+2) z_2 M^{(\beta,\gamma)}_{m+1,n}(z_1,z_2) \\ &= (\gamma+m+1) M^{(\beta,\gamma)}_{m,n}(z_1,z_2) - (n+1) M^{(\beta,\gamma)}_{m+1,n+1}(z_1,z_2), \end{aligned} \tag{9.10}$$

$$\begin{aligned} &(\beta+\gamma+m+n+1) z_1 M^{(\beta,\gamma)}_{m,n}(z_1,z_2) \\ &= (\beta+\gamma+m+1) M^{(\beta,\gamma)}_{m+1,n}(z_1,z_2) - (\beta+n) M^{(\beta,\gamma)}_{m,n-1}(z_1,z_2) \end{aligned} \tag{9.11}$$

and

$$(c_n - z_1z_2) M^{(\beta,\gamma)}_{m,n}(z_1,z_2) = a_n M^{(\beta,\gamma)}_{m+1,n+1}(z_1,z_2) + b_n M^{(\beta,\gamma)}_{m-1,n-1}(z_1,z_2), \tag{9.12}$$

where

$$a_n = \frac{(n+1)(\beta+\gamma+m+1)}{(\beta+\gamma+m+n+1)(\beta+\gamma+m+n+2)}, \tag{9.13}$$

$$b_n = \frac{(\gamma+m)(\beta+n)}{(\beta+\gamma+m+n)(\beta+\gamma+m+n+1)}, \tag{9.14}$$

$$c_n = \frac{(n+1)(\gamma+m+1)}{\beta+\gamma+m+n+2} - \frac{n(\gamma+m)}{\beta+\gamma+m+n}. \tag{9.15}$$

It must be noted that (9.12) is essentially the three term recurrence relation for Jacobi polynomials [50].

It is clear that

$$A_n\delta_x^2\phi_n(x,\alpha) + B_n\delta_x\phi_n(x,\alpha) + C_n\phi_n(x,\alpha) = 0, \tag{9.16}$$

where

$$A_n = 1-x, \tag{9.17}$$

$$B_n = \alpha+\gamma-x(\alpha+\beta+\gamma+1), \tag{9.18}$$

$$C_n = xn(\alpha+\beta+\gamma+n+1). \tag{9.19}$$

Then for $m \geq n$, $M_{m,n}^{(\beta,\gamma)}(z_1,z_2)$ satisfy the following second order partial differential equation

$$\begin{aligned}(1-z_1z_2)\delta_{z_2}^2 f(z_1,z_2) + \{m-n+\gamma \\ - z_1z_2(\beta+\gamma+m-n+1)\}\delta_{z_2} f(z_1,z_2) \\ + z_1z_2 n(\beta+\gamma+m+1) f(z_1,z_2) = 0.\end{aligned} \tag{9.20}$$

An equivalent form is

$$\begin{aligned}&(1-z_1z_2)z_2\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2} \\ &+[1+m-n+\gamma-z_1z_2(2+\gamma+\beta+m-n)]\frac{\partial f(z_1,z_2)}{\partial z_2} \\ &+z_1 n(m+\beta+\gamma+1) f(z_1,z_2) = 0.\end{aligned} \tag{9.21}$$

Similarly we establish

$$\begin{aligned}&(1-z_1z_2)z_1\frac{\partial^2 f(z_1,z_1)}{\partial z_2^1} \\ &+[1-m+n+\gamma-z_1z_2(2+\gamma+\beta-m+n)]\frac{\partial f(z_1,z_2)}{\partial z_2} \\ &+z_2 m(n+\beta+\gamma+1) f(z_1,z_2) = 0.\end{aligned} \tag{9.22}$$

Theorem 9.1. For $m \geq n \geq 0, \beta > -1, \gamma > -1$ we have

$$\frac{\partial}{\partial z_2}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) = -(\beta+\gamma+m+1)M_{m,n}^{(\beta+1,\gamma)}(z_1,z_2)$$

$$\begin{aligned}\delta_{z_2}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) &= nM_{m,n}^{(\beta,\gamma)}(z_1,z_2) - (\gamma+m)M_{m-1,n-1}^{(\beta+1,\gamma)}(z_1,z_2) \\ &= (\beta+\gamma+m+1)\left(M_{m,n}^{(\beta+1,\gamma)}(z_1,z_2) - M_{m,n}^{(\beta,\gamma)}(z_1,z_2)\right),\end{aligned}$$

$$\begin{aligned}\delta_{z_1}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) &= mM_{m,n}^{(\beta,\gamma)}(z_1,z_2) - (\gamma+m)M_{m-1,n-1}^{(\beta+1,\gamma)}(z_1,z_2) \\ &= (\beta+\gamma+m+1)M_{m,n}^{(\beta+1,\gamma)}(z_1,z_2) \\ &\quad -(\beta+\gamma+n+1)M_{m,n}^{(\beta,\gamma)}(z_1,z_2),\end{aligned}$$

$$\begin{aligned}&(\beta+\gamma+m+1)\delta_{z_1}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) \\ &\quad -(\beta+\gamma+n+1)\delta_{z_2}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) \\ &= (m-n)(\beta+\gamma+m+1)M_{m,n}^{(\beta+1,\gamma)}(z_1,z_2),\end{aligned}$$

$$(\delta_{z_1}-\delta_{z_2})M_{m,n}^{(\beta,\gamma)}(z_1,z_2) = (m-n)M_{m,n}^{(\beta,\gamma)}(z_1,z_2),$$

$$\begin{aligned}&(\beta+\gamma+m+1)\delta_{z_1}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) \\ &\quad -(\beta+\gamma+n+1)\delta_{z_2}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) \\ &= (m-n)(\beta+\gamma+m+1)M_{m,n}^{(\beta+1,\gamma)}(z_1,z_2).\end{aligned}$$

and

$$\begin{aligned}&\delta_{z_2}M_{m,n}^{(\beta,\gamma)}(z_1,z_2) \\ &= \sum_{k=0}^{n}\frac{(\beta+\gamma+m+1)_{n-k}(\gamma+1)_m z_1^{m-k}(-z_2)^{n-k}(n-k)}{k!(n-k)!(\gamma+1)_{m-k}} \\ &= n\sum_{k=0}^{n}\frac{(\beta+\gamma+m+1)_{n-k}(\gamma+1)_m z_1^{m-k}(-z_2)^{n-k}}{k!(n-k)!(\gamma+1)_{m-k}} \\ &\quad -\sum_{k=1}^{n}\frac{(\beta+\gamma+m+1)_{n-k}(\gamma+1)_m z_1^{m-k}(-z_2)^{n-k}}{(k-1)!(n-k)!(\gamma+1)_{m-k}} \\ &= nM_{m,n}^{(\beta,\gamma)}(z_1,z_2) - (\gamma+m)M_{m-1,n-1}^{(\beta+1,\gamma)}(z_1,z_2).\end{aligned}$$

Similarly,

Ismail and Zeng [29] established the following generating functions:

$$\begin{aligned}&\sum_{m,n=0}^{\infty}M_{m,n}^{(\beta,\gamma)}(z_1,z_2)u^m v^n \\ &= \frac{2^{\beta+\gamma}}{\rho}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\gamma} \\ &\quad\times\left[\frac{1}{1-2z_1u/(1+\rho-uv)} + \frac{1}{1-2vz_2/(1+\rho-uv)} - 1\right],\end{aligned} \tag{9.23}$$

and

$$\begin{aligned}&\sum_{m\geq n\geq 0}M_{m,n}^{(\beta,\gamma)}(z_1,z_2)\frac{u^m v^n}{(m-n)!} \\ &= \frac{2^{\beta+\gamma}}{\rho}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{-\gamma} \\ &\quad\times\exp\left(\frac{2z_1u}{1-uv+\rho}\right),\end{aligned} \tag{9.24}$$

where

$$\rho = (1-2uv(1-2z_1z_2)+u^2v^2)^{1/2} \tag{9.25}$$

They also verified the limiting relation

$$\lim_{\beta\to\infty}M_{m,n}^{(\beta,\gamma)}(z_1,z_2/\beta) = z_1^{m-n}L_n^{(\gamma+m-n)}(z_1z_2) = (-1)^n Z_{m,n}^{(\gamma)}(z_1,z_2),$$

and found the generating relation

$$\sum_{m\geq n\geq 0}u^m v^n M_{m,n}^{(\beta,\gamma)}(z_1,z_2) = \frac{2^{\beta+\gamma}(1+uv+\rho)^{-\beta}(1-uv+\rho)^{1-\gamma}}{\rho(1-uv-2uz_1+\rho)}, \tag{9.26}$$

with ρ as in (9.25).

10. q-Laguerre type 2 – D polynomials

For $\alpha > -1$ the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q) = (q^{\alpha+1};q)_n\sum_{k=0}^{n}\frac{q^{(\alpha+n-k)(n-k)}(-x)^{n-k}}{(q;q)_k(q,q^{\alpha+1};q)_{n-k}}, \tag{10.1}$$

are orthogonal to infinitely many measures. Some orthogonality measures are

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)x^\alpha d\mu(x|q) = \zeta_n(\alpha)\delta_{m,n}, \tag{10.2}$$

are given by

$$d\mu(x|q) = \frac{x^\alpha dx}{(-x;q)_\infty},$$

$$\zeta_n(\alpha) = \Gamma(-\alpha)\Gamma(\alpha+1)\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{q^n(q;q)_n}, \tag{10.3}$$

$$d\mu(x|q) = \sum_{k=-\infty}^{\infty}\frac{x\delta(x-cq^k)}{(-x;q)_\infty},$$

$$\zeta_n(\alpha) = \frac{(q,-cq^{\alpha+1},-\frac{q^{-\alpha}}{c};q)_\infty}{(q^{\alpha+1},-c,-\frac{q}{c};q)_\infty c^{-\alpha-1}}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \tag{10.4}$$

and

$$d\mu(x|q) = \frac{x^{-c}\left(-\lambda x, -\frac{q}{\lambda x}; q\right)_\infty dx}{\left(-x, -\lambda q^c x, -\frac{q}{\lambda q^c x}; q\right)_\infty},$$

$$\zeta_n(\alpha) = \int_0^\infty \frac{x^{\alpha-c}\left(-\lambda x, -\frac{q}{\lambda x}; q\right)_\infty dx}{\left(-x, -\lambda q^c x, -\frac{q}{\lambda q^c x}; q\right)_\infty}, \tag{10.5}$$

where $c, \lambda > 0$. Following the procedure outline in Sections 6 we let

$$z_{m,n}^{(\beta)}(z_1, z_2|q) = \begin{cases} z_1^{m-n} L_n^{(\beta+m-n)}(z_1 z_2; q), & m \ge n, \\ z_{n,m}^{(\beta)}(z_2, z_1|q), & m < n, \end{cases} \tag{10.6}$$

and use the notation

$$z_{m,n}^{(\beta)}(z_1, z_2|q) = \sum_{k=0}^{n} \frac{q^{(\beta+m-k)(n-k)}(q^{\beta+1}; q)_m z_1^{m-k} z_2^{n-k}}{(-1)^{n-k}(q; q)_k (q; q)_{n-k}(q^{\beta+1}; q)_{m-k}}. \tag{10.7}$$

Therefore

$$z_1 z_{m,n}^{(\beta+1)}(z_1, z_2|\beta) = z_{m+1,n}^{(\beta)}(z_1, z_2|\beta),$$
$$z_{m,n}^{(0)}(z_1, z_2|q) = \frac{(-1)^n}{(q; q)_n} h_{m,n}(z_1, z_2|q), \quad m \ge n. \tag{10.8}$$

Theorem 10.1. *Let $\beta > -1$. Then the following orthogonality relations*

$$\int_{\mathbb{R}^2} z_{m,n}^{(\beta)}(z, \bar{z}|q) \overline{z_{s,t}^{(\beta)}(z, \bar{z}|q)} r^{2\beta} d\mu(r^2|q) d\theta$$
$$= \zeta_{m\wedge n}(|m-n| + \beta)\delta_{m,s}\delta_{n,t}, \tag{10.9}$$

for $m, n, s, t = 0, 1, \ldots$ and $d\mu(x|q), \zeta_n(\alpha)$ may be any pair from (10.3), (10.4) or (10.5).

In the case

$$c_k(n, \alpha) = \frac{(q^{\alpha+1}; q)_n q^{(\alpha+n-k)(n-k)}(-1)^{n-k}}{(q; q)_k (q, q^{\alpha+1}; q)_{n-k}} \tag{10.10}$$

Applying formulas (6.7) and (6.8) lead to the following theorem.

Theorem 10.2. *For $\beta > -1$ and $m \ge n$ we have*

$$q^{(m+1+\beta)} z_2 z_{m+1,n}^{(\beta)}(z_1, z_2) = (q^{n+1} - 1) z_{m+1,n+1}^{(\beta)}(z_1, z_2)$$
$$+ (1 - q^{m+\beta+1}) z_{m,n}^{(\beta)}(z_1, z_2), \tag{10.11}$$

$$q^n z_1 z_{m,n+1}^{(\beta)}(z_1, z_2) = z_{m+1,n}^{(\beta)}(z_1, z_2) - z_{m,n-1}^{(\beta)}(z_1, z_2) \tag{10.12}$$

and

$$\left(1 + q(1 - q^n - q^{m+\beta}) - z_1 z_2 q^{\beta+m+n+1}\right) z_{m,n}^{(\beta)}(z_1, z_2)$$
$$= (1 - q^{1+n}) z_{m+1,n+1}^{(\beta)}(z_1, z_2) + q(1 - q^{\beta+m}) z_{m-1,n-1}^{(\beta)}(z_1, z_2). \tag{10.13}$$

The raising and lowering relations of the q-Laguerre polynomials lead to the following q-difference relations,

Theorem 10.3. *For $\beta > -1$ let $w_\beta(x; q) = x^\beta/(-x; q)_\infty$. Then we have*

$$\frac{D_{q,z_1}\left(z_1^{n-m} z_{m,n}^{(\beta)}(z_1, z_2)\right)}{z_1^{n-m}} = \frac{q^\beta}{q-1} \frac{z_2}{z_1} z_{m,n-1}^{(\beta)}(qz_1, z_2), \tag{10.14}$$

$$D_{q,z_2}\left(z_{m,n}^{(\beta)}(z_1, z_2)\right) = \frac{q^\beta}{q-1} z_{m,n-1}^{(\beta)}(qz_1, z_2), \tag{10.15}$$

$$z_{m,n}^{(\beta)}(z_1, z_2) = \frac{(1-q)^{m+n} D_{q,z_1}^n \left\{ z_2^\beta D_{q,z_2}^m \left\{ z_2^{-\beta} w_\beta(z_1 z_2) \right\} \right\}}{(-1)^m (q; q)_n w_\beta(z_1 z_2)}, \tag{10.16}$$

$$\frac{z_2 D_{q,z_2}\left\{ w_\beta(z_1 z_2) z_{m,n}^{(\beta)}(z_1, z_2) \right\}}{w_\beta(z_1 z_2)}$$
$$= \frac{1 - q^\beta}{1 - q} z_{m,n}^{(\beta)}(z_1, z_2) - \frac{q^\beta z_2}{1 - q} z_{m+1,n}^{(\beta)}(z_1, z_2), \tag{10.17}$$

where $m \ge n$ and

$$\frac{D_{q,z_1}\left\{ w_\beta(z_1 z_2; q) z_{m,n}^{(\beta)}(z_1, z_2) \right\}}{w_\beta(z_1 z_2; q)} = \frac{1 - q^{n+1}}{1 - q} z_{m,n+1}^{(\beta)}(z_1, z_2), \tag{10.18}$$

$$\frac{D_{q,z_2}\left\{ z_2^{m-n} w_\beta(z_1 z_2; q) z_{m,n}^{(\beta)}(z_1, z_2) \right\}}{z_2^{m-n} w_\beta(z_1 z_2; q)} = \frac{1 - q^{n+1}}{1 - q} \frac{z_1}{z_2} z_{m,n+1}^{(\beta)}(z_1, z_2), \tag{10.19}$$

where $m > n$.

The following second order q-difference equation

$$(1 + qx)\theta_{q,x}^2 y(x) + \frac{1 - q^\alpha - 2q^{\alpha+1}x + q^{\alpha+n+1}x}{q^\alpha(1-q)} \theta_{q,x} y(x)$$
$$+ \frac{qx(1 - q^n)}{(1-q)^2} y(x) = 0. \tag{10.20}$$

where $\theta_{q,z} = zD_{q,z}$ has a solution $y(x) = L_n^{(\alpha)}(x; q)$. This leads to:

Theorem 10.4. *For $m \ge n$ the function $f = z_{m,n}^{(\beta)}(z_1, z_2)$ satisfies the second order q-difference equations*

$$(1 + qz_1 z_2)\theta_{q,z_2}^2 f + \frac{q^{n-m-\beta} - 1 - (2 - q^n) q z_1 z_2}{1 - q} \theta_{q,z_2} f$$
$$+ \frac{q z_1 z_2 (1 - q^n)}{(1-q)^2} f = 0 \tag{10.21}$$

and

$$(1 + qz_1 z_2)\theta_{q,z_1}^2 f - \frac{1 - q^{\beta+m-n} + (2 - q^{\beta+m+1}) q z_1 z_2}{1 - q} \theta_{q,z_1} f$$
$$+ \frac{q(1 - q^{\beta+m}) z_1 z_2}{(1-q)^2} f = 0. \tag{10.22}$$

Starting with the little q-Laguerre [38] or Wall polynomials [9]

$$p_n(x; q^\alpha|q) = \sum_{k=0}^{n} \frac{(q; q)_n q^{\frac{(k-n)(n+k-1)}{2}} (-x)^{n-k}}{(q; q)_k (q, q^{\alpha+1}; q)_{n-k}} \tag{10.23}$$

we define the 2D polynomials $w_{m,n}^{(\beta)}(z_1, z_2|q)$ through

$$w_{m,n}^{(\beta)}(z_1, z_2|q) = \begin{cases} z_1^{m-n} p_n(z_1 z_2; q^{\beta+m-n}|q), & m \ge n, \\ w_{n,m}^{(\beta)}(z_2, z_1|q), & m < n, \end{cases} \tag{10.24}$$

for $\beta > -1$ and $m, n = 0, 1, \ldots$. Then for $m \ge n$ find that

$$z_1 w_{m,n}^{(\beta+1)}(z_1, z_2|\beta) = w_{m+1,n}^{(\beta)}(z_1, z_2|\beta), \tag{10.25}$$

$$w_{m,n}^{(\beta)}(z_1, z_2|q) = \sum_{k=0}^{n} \frac{(q; q)_n (q^{\beta+1}; q)_{m-n} z_1^{m-k} z_2^{n-k} q^{-\binom{k}{2} - \binom{n}{2}}}{(-1)^{n-k}(q; q)_k (q; q)_{n-k}(q^{\beta+1}; q)_{m-k}}. \tag{10.26}$$

The orthogonality relation for the little q-Laguerre polynomials is

$$\sum_{k=0}^{\infty} q^{(\alpha+1)k}(q^{k+1}; q)_\infty p_m(q^k; q^\alpha|q) p_n(q^k; q^\alpha|q)$$
$$= \frac{(q; q)_\infty q^{(\alpha+1)n}(q; q)_n \delta_{m,n}}{(q^{\alpha+1}; q)_\infty (q^{\alpha+1}; q)_n}, \quad m, n = 0, 1, \ldots. \tag{10.27}$$

This leads to the orthogonality relation below.

Theorem 10.5. *For $m, n, s, t = 0, 1, \dots$ and $\beta > -1$ we have the following orthogonality relations*

$$\int_{\mathbb{R}^2} w^{(\beta)}_{m,n}(z, \bar{z}|q)\overline{w^{(\beta)}_{s,t}(z, \bar{z}|q)}r^{2\beta}d\mu(r^2|q)d\theta = \zeta_{m\wedge n}(|m-n| + \beta)\delta_{m,s}\delta_{n,t}, \tag{10.28}$$

where $z = re^{i\theta}$,

$$d\mu(r|q) = \sum_{k=0}^{\infty} x(xq; q)_\infty \delta(x - q^k),$$

$$\zeta_n(\alpha) = \frac{(q; q)_\infty q^{(\alpha+1)n}(q; q)_n}{(q^{\alpha+1}; q)_\infty (q^{\alpha+1}; q)_n}. \tag{10.29}$$

The function $y(x) = p_n(x; q^\alpha|q)$ satisfies the q-difference equation

$$q^{\alpha+n-1}\theta^2_{q,z}y(z) + \frac{q^{n-1} + q^{\alpha+n-1} - z}{1-q}\theta_{q,z}y(z) + \frac{z(1-q^n)}{(1-q)^2}y(z) = 0 \tag{10.30}$$

where $\theta_{q,z} = zD_{q,z}$. This gives the following theorem.

Theorem 10.6. *For $\beta > -1$, $m \ge n$ the polynomial $w^{(\beta)}_{m,n}(z_1, z_2|q)$, satisfies the q-partial difference equations*

$$q^{\beta+m-1}\theta^2_{q,z_2}f + \frac{q^{n-1} - q^{\beta+m-1} - z_1z_2}{1-q}\theta_{q,z_2}f + \frac{(1-q^n)z_1z_2}{(1-q)^2}f = 0, \tag{10.31}$$

$$q^n\theta^2_{q,z_1}f - \frac{q^n - q^{\beta+m} + qz_1z_2}{1-q}\theta_{q,z_1}f + \frac{q(1-q^{\beta+m})z_1z_2}{(1-q)^2}f = 0. \tag{10.32}$$

Let

$$w(x; \alpha) = (qx; q)_\infty x^\alpha. \tag{10.33}$$

The following theorem follows from the raising and lowering relations of the little q-Laguerre polynomials.

Theorem 10.7. *For $\beta > -1$ and $m \ge n$ we have*

$$\frac{D_{q,z_1}\left\{z_1^{n-m}w^{(\beta)}_{m,n}(z_1, z_2|q)\right\}}{z_1^{n-m}} = -\frac{z_2}{z_1}\frac{q^{1-n}(1-q^n)w^{(\beta)}_{m,n-1}(z_1, z_2|q)}{(1-q)(1-q^{\beta+m-n+1})}, \tag{10.34}$$

$$D_{q,z_2}\left\{w^{(\beta)}_{m,n}(z_1, z_2|q)\right\} = -\frac{q^{1-n}(1-q^n)w^{(\beta)}_{m,n-1}(z_1, z_2|q)}{(1-q)(1-q^{\beta+m-n+1})}, \tag{10.35}$$

and

$$w(z_1z_2; \beta|q)w^{(\beta)}_{m,n}(z_1, z_2|q) = \frac{q^{m(n-1)+n(\beta-1)-\binom{n}{2}}(1-q)^{m+n}}{(-1)^m(q^{\beta+m-n+1}; q)_n} \times D^n_{q^{-1},z_1}\left\{(z_1z_2)^\beta D^m_{q^{-1},z_2}\left\{(z_1z_2)^{-\beta}w(z_1z_2; \beta|q)\right\}\right\}. \tag{10.36}$$

For $m > n$ we have

$$\frac{D_{q^{-1},z_1}\left\{w(z_1z_2; \beta|q)w^{(\beta)}_{m,n}(z_1, z_2|q)\right\}}{w(z_1z_2; \beta|q)} = \frac{(1-q^{\beta+m-n})w^{(\beta)}_{m,n+1}(z_1, z_2|q)}{q^{\beta+m-n-1}(1-q)}, \tag{10.37}$$

$$\frac{D_{q^{-1},z_2}\left\{z_2^{m-n}w(z_1z_2; \beta|q)w^{(\beta)}_{m,n}(z_1, z_2|q)\right\}}{z_2^{m-n}w(z_1z_2; \beta|q)} = \frac{z_1}{z_2}\frac{(1-q^{\beta+m-n})w^{(\beta)}_{m,n+1}(z_1, z_2|q)}{q^{\beta+m-n-1}(1-q)} \tag{10.38}$$

and

$$\frac{z_2D_{q^{-1},z_2}\left\{w(z_1z_2; \beta|q)w^{(\beta)}_{m,n}(z_1, z_2|q)\right\}}{w^{(\beta)}_{m,n}(z_1, z_2|q)} = \frac{1-q^\beta}{1-q}w^{(\beta)}_{m,n}(z_1, z_2|q) - \frac{q^{\beta+1-n}}{1-q}\frac{1-q^{\beta+m+1}}{1-q^{\beta+m-n+1}}z_2w^{(\beta)}_{m+1,n}(z_1, z_2|q). \tag{10.39}$$

Applying the procedure of Section 6 to

$$c_k(n, \alpha) = \frac{(q; q)_n q^{\frac{(k-n)(n+k-1)}{2}}(-1)^{n-k}}{(q; q)_k(q, q^{\alpha+1}; q)_{n-k}} \tag{10.40}$$

we can derive the recurrences in the following theorem.

Theorem 10.8. *For $\beta > -1$, $m \ge n$ we have*

$$\frac{z_2w^{(\beta)}_{m+1,n}(z_1, z_2|q)}{q^n(1-q^{m-n+\beta+1})} = w^{(\beta)}_{m,n}(z_1, z_2|q) - w^{(\beta)}_{m+1,n+1}(z_1, z_2|q), \tag{10.41}$$

$$z_1(1-q^{m-n+\beta+1})w^{(\beta)}_{m,n}(z_1, z_2|q) = (1-q^{m+\beta+1})w^{(\beta)}_{m+1,n}(z_1, z_2|q) - q^{m-n+\beta+1}(1-q^n)w^{(\beta)}_{m,n-1}(z_1, z_2|q), \tag{10.42}$$

and

$$\left(q^n + q^{m+\beta}(1-q^n-q^{n+1}) - z_1z_2\right)w^{(\beta)}_{m,n}(z_1, z_2|q) = q^n(1-q^{m+\beta+1})w^{(\beta)}_{m+1,n+1}(z_1, z_2|q) + q^{m+\beta}(1-q^n)w^{(\beta)}_{m-1,n-1}(z_1, z_2|q). \tag{10.43}$$

11. The polynomials $\{M^{(\alpha,\beta)}_n(z_1, z_2|q)\}$

For $\alpha, \gamma > -1$, the little q-Jacobi polynomials

$$p_n(x; q^\alpha, q^\gamma|q) = \frac{(q; q)_n}{q^{\binom{n}{2}}}\sum_{k=0}^n \frac{(q^{\alpha+\gamma+n+1}; q)_{n-k}q^{\binom{k}{2}}(-x)^{n-k}}{(q; q)_k(q, q^{\alpha+1}; q)_{n-k}} \tag{11.1}$$

satisfy the orthogonality relation (6.1) with

$$d\mu(x; \gamma|q) = \sum_{k=0}^\infty \frac{(qx; q)_\infty x}{(q^{\gamma+1}x; q)_\infty}\delta(x - q^k),$$

$$\zeta_n(\alpha, \gamma|q) = \frac{(q, q^{\alpha+\gamma+n+1}; q)_\infty (q; q)_n q^{n(\alpha+1)}}{(q^{\alpha+1}, q^{\gamma+n+1}; q)_\infty (q^{\alpha+1}; q)_n(1-q^{\alpha+\gamma+2n+1})}, \tag{11.2}$$

[38]. Following our general construction in Section 6, we define the polynomials

$$M^{(\beta,\gamma)}_{m,n}(z_1, z_2|q) = \begin{cases} z_1^{m-n}p_n(z_1z_2, q^{\beta+m-n}, q^\gamma|q), & m \ge n, \\ M^{(\beta,\gamma)}_{n,m}(z_2, z_1|q), & m < n. \end{cases} \tag{11.3}$$

In other words

$$M^{(\beta,\gamma)}_{m,n}(z_1, z_2|q) = \frac{(q; q)_n(q^{\beta+1}; q)_{m-n}}{(q^{\beta+\gamma+1}; q)_m q^{\binom{n}{2}}}\sum_{k=0}^n \frac{(q^{\beta+\gamma+1}; q)_{m+n-k}q^{\binom{k}{2}}z_1^{m-k}(-z_2)^{n-k}}{(q; q)_k(q; q)_{n-k}(q^{\beta+1}; q)_{m-k}}, \tag{11.4}$$

$$z_1M^{(\beta+1,\gamma)}_{m,n}(z_1, z_2|q) = M^{(\beta,\gamma)}_{m+1,n}(z_1, z_2|q) \tag{11.5}$$

for $m \ge n$. Now Theorem 6.1 leads to:

Theorem 11.1. *For $m, n, s, t = 0, 1, \ldots$ and $\beta, \gamma > -1$ we have the following orthogonality relation*

$$\int_{\mathbb{R}^2} M_{m,n}^{(\beta,\gamma)}(z,\bar{z}|q)\overline{M_{s,t}^{(\beta,\gamma)}(z,\bar{z}|q)}r^{2\beta}d\mu(r^2;\gamma|q)d\theta = \zeta_{m\wedge n}(|m-n|+\beta,\gamma|q)\delta_{m,s}\delta_{n,t}, \tag{11.6}$$

where $d\mu(x;\gamma|q), \zeta_n(\alpha,\gamma|q)$ are given in (11.2).

Following the construction of §6 we derive the recurrence below via the choice

$$c_k(n,\alpha) = \frac{(q;q)_n(q^{\alpha+\gamma+n+1};q)_{n-k}(-1)^{n-k}}{(q;q)_k(q,q^{\alpha+1};q)_{n-k}q^{\binom{n}{2}-\binom{k}{2}}}. \tag{11.7}$$

$$\frac{(1-q^{\beta+\gamma+m+n+2})}{q^n(1-q^{\beta+m-n+1})}z_2M_{m+1,n}^{(\beta,\gamma)}(z_1,z_2|q) = -M_{m+1,n+1}^{(\beta,\gamma)}(z_1,z_2|q)+M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q), \tag{11.8}$$

$$z_1M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q) = \frac{(1-q^{\beta+m+1})(1-q^{\beta+\gamma+m+1})}{(1-q^{\beta+m-n+1})(1-q^{\beta+\gamma+m+n+1})}M_{m+1,n}^{(\beta,\gamma)}(z_1,z_2|q) + \frac{q^{m-n+\beta+1}(1-q^n)(1-q^{\gamma+n})}{(1-q^{\beta+m-n+1})(1-q^{\beta+\gamma+m+n+1})}M_{m,n-1}^{(\beta,\gamma)}(z_1,z_2|q). \tag{11.9}$$

The above formulas hold for $\beta,\gamma > -1$ and $m \geq n$. Next we note that the polynomials $\{p_n(x;q^\alpha,q^\gamma|q)\}$ satisfy the following second order difference equation

$$(1-q^{\gamma+2}x)\theta_{q,x}^2y(x) + \frac{q^{-\alpha}-1-q^{1-n-\alpha}x-q^{\gamma+2}(2-q^n)x}{1-q}\theta_{q,x}y(x) - \frac{1+q^{1-\alpha}x-(q^{1-n-\alpha}+q^{\gamma+n+2})x}{(1-q)^2}y(x) = 0. \tag{11.10}$$

where $y(x) = p_n(x;q^\alpha,q^\gamma|q)$, $\theta_{q,z} = zD_{q,z}$.

Theorem 11.2. *For $m \geq n$ and $\beta,\gamma > -1$, the function $f = M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q)$ satisfies*

$$\theta_{q,z_2}^2 f + \left\{\frac{q^{n-m-\beta}-1-q^{1-m-\beta}z_1z_2-q^{\gamma+2}(2-q^n)z_1z_2}{(1-q)(1-q^{\gamma+2}z_1z_2)}\right\}\theta_{q,z_2}f - \frac{1+(q^{n+1-m-\beta}-q^{1-m-\beta}-q^{\gamma+n+2})z_1z_2}{(1-q)^2(1-q^{\gamma+2}z_1z_2)}f = 0 \tag{11.11}$$

and

$$\theta_{q,z_1}^2 f + \frac{z_1z_2\left(2q^{\gamma+2}-q^{1-n}-q^{m+\beta+\gamma+2}\right)+q^{m-n+\beta}-1}{(1-q)(1-q^{\gamma+2}z_1z_2)}\theta_{q,z_1}f + \frac{\left(q^{1-n}-q^{1+m-n+\beta}+q^{\gamma+2}(q^{m+\beta}+q^{2(m-n+\beta)}-1)\right)z_1z_2-q^{2(m-n+\beta)}}{(1-q)^2(1-q^{\gamma+2}z_1z_2)}f = 0. \tag{11.12}$$

When $\beta,\gamma > -1$ the raising and powering operators for the polynomials $\{p_n(x;q^\alpha,q^\gamma|q)\}$ with

$$w(x;q^\alpha,q^\gamma|q) = \frac{(qx;q)_\infty x^\alpha}{(q^{\gamma+1}x;q)_\infty}. \tag{11.13}$$

lead to the following ladder operators

$$M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q) - M_{m,n}^{(\beta,\gamma)}(z_1,qz_2|q) = -\frac{q^{1-n}(1-q^n)(1-q^{m+\beta+\gamma-1})}{1-q^{m-n+\beta}}z_2M_{m,n-1}^{(\beta,\gamma+1)}(z_1,z_2|q), \tag{11.14}$$

where $m \geq n$, and

$$D_{q^{-1},z_1}\left(w(z_1z_2;q^\beta,q^\gamma|q)M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q)\right) = \frac{(1-q^{m-n+\beta})w(z_1z_2;q^\beta,q^{\gamma-1}|q)}{q^{m-n+\beta}(1-q)}M_{m,n+1}^{(\beta,\gamma-1)}(z_1,z_2|q), \tag{11.15}$$

$$z_2^{n-m}D_{q^{-1},z_2}\left(z_2^{m-n}w(z_1z_2;q^\beta,q^\gamma|q)M_{m,n}^{(\beta,\gamma)}(z_1,z_2|q)\right) = \frac{z_1(1-q^{m-n+\beta})w(z_1z_2;q^\beta,q^{\gamma-1}|q)}{z_2q^{m-n+\beta-1}(1-q)}M_{m,n+1}^{(\beta,\gamma-1)}(z_1,z_2|q), \tag{11.16}$$

where $m \geq n+1$.

12. Polynomial solutions to differential equations

In this section we study polynomial solutions to partial differential equations. We are looking for polynomial solutions to the second order partial differential equation (8.25). Let $f = \sum_{m,n=0}^\infty z_1^jz_2^k$ and substitute in (8.25).

Theorem 12.1. *The partial differential equation (8.25), namely,*

$$\partial_{\partial z_1}\partial_{\partial z_2}f + \left(\frac{\beta - z_1z_2}{z_1}\right)\partial_{\partial z_2}f = -nf, \tag{12.1}$$

has a power series solution

$$f(z_1,z_2) = \sum_{j,k=0}^\infty a_{j,k}z_1^jz_2^k \tag{12.2}$$

if and only if

$$f(z_1,z_2) = \sum_{j=1}^\infty a_{j,o}\frac{n!\,z_1^j}{(\beta+j+1)_n}L_n^{(\beta+j)}(z_1z_2) + \sum_{k=0}^n a_{0,k}z_2^k{}_2F_1\left(\left.\begin{matrix}-n+k,1\\k+1,\beta+1\end{matrix}\right|z_1z_2\right). \tag{12.3}$$

Proof. Substitute the power series (12.2) for f in (8.25) and equate coefficients of like powers of z_1 and z_2 to find that

$$a_{j,k} = \frac{k-1-n}{k(\beta+j)}a_{j-1,k-1}, \quad j,k > 0.$$

We iterate this and find that

$$a_{j,k} = \begin{cases}\dfrac{(-n)_ka_{j-k,0}}{k!(\beta+j-k+1)_k}, & j \geq k\\[2ex] \dfrac{(k-j-n)_j(k-j)!}{k!(\beta+1)j}a_{0,k-j}, & j \leq k,\end{cases}$$

This proves the theorem. □

Theorem 12.2. *In order for the equation*

$$\partial_{z_1}\partial_{z_2}f + \left(\frac{\beta}{z_1} - z_2\right)\partial_{z_2}f = \lambda f \tag{12.4}$$

to have a polynomials solution in z_2, it is necessary and sufficient that $\lambda = -n, n = 0,1,2,\ldots$, in which case the function f will be given as in Theorem 12.1.

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