VE414 Lecture 10

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- In order to understand more sophisticated sampling schemes, we need more than just the concept of a random variable and its corresponding pdf.
- Recall a random variable X is a function

$$X : \Omega \to \mathcal{A} \qquad \text{where} \quad \mathcal{A} \subset \mathbb{R}$$

where Ω is the sample space.

Definition

A $\mathcal{E}$-valued discrete-time process is a function

$$\xi : \mathcal{I} \to \mathcal{E}$$

which maps values in the index set $\mathcal{I}$ to some other space $\mathcal{E}$.

- We will only concern discrete-time process, in which the index set $\mathcal{I}$ is $\mathbb{N}$.
- Of course, any index set isomorphic to $\mathbb{N}$ can be used by simply relabelling.
- Often $\mathcal{E} \subset \mathbb{R}$, so integer-valued, or real-valued process is very common.

- We are interested in discrete-time *stochastic* process, which can be viewed as a process in which, for each i in the index set $\mathbb{N}$, a random variable

$$X_i$$

taking values in $\mathcal{E}$ is assigned instead of a value in $\mathcal{E}$ is assigned at time i.

- Notice for any two finite collections of times that are non-overlapping

$$i_1, i_2, \cdots, i_t; \quad j_1, j_2, \cdots, j_s \qquad \text{where} \quad i_\ell \neq j_k \quad \text{for any} \quad k \leq t, l \leq s.$$

the joint cumulative distribution function over the first time set is defined as

$$\begin{aligned} F_{i_1,\ldots,i_t}(x_1,\ldots,x_t) &= \Pr(X_{i_1} \leq x_1, \ldots, X_{i_t} \leq x_t) \\ &= F_{i_1,\ldots,i_t,j_1,\ldots,j_s}(x_1,\ldots,x_t,\infty,\ldots,\infty) \end{aligned}$$

where $F_{i_1,\ldots,i_t,j_1,\ldots,j_s}$ denote joint cdf over the union of the two time sets.

- If $\mathcal{E}$ is discrete, for example, $\mathcal{E} = \mathbb{Z}$, then

$$F_{i_1,\ldots,i_t}(x_1,\ldots,x_t) = F_{i_1,\ldots,i_t,j_1,\ldots,j_s}(x_1,\ldots,x_t,\infty,\ldots,\infty)$$

$$= \sum_{\xi_1 \le x_1} \cdots \sum_{\xi_t \le x_t} \sum_{\xi_1^* \in \mathbb{Z}} \cdots \sum_{\xi_s^* \in \mathbb{Z}} f_{\mathbf{i},\mathbf{j}}(\xi_1,\ldots,\xi_t,\xi_1^*,\ldots,\xi_s^*)$$

  where $f_{\mathbf{i},\mathbf{j}}$ denotes joint pmf of $X_{i_1}, X_{i_2}, \ldots, X_{i_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_s}$.

- If $\mathcal{E}$ is continuous, for example, $\mathcal{E} = \mathbb{R}$, then

$$F_{i_1,\ldots,i_t,j_1,\ldots,j_s}(x_1,\ldots,x_t,\infty,\ldots,\infty)$$

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{i},\mathbf{j}}(\cdot)\, d\xi_1^* \ldots, d\xi_s^* d\xi_t \ldots, d\xi_1$$

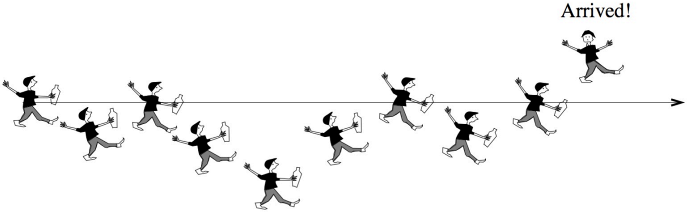
  where $f_{\mathbf{i},\mathbf{j}}$ denotes the joint pdf of $X_{i_1}, X_{i_2}, \ldots, X_{i_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_s}$.

- The above is known as a consistent requirement, which holds for common stochastic processes in practice, and we will exclusively consider those only.

A simple stochastic process

Simple Random Walk

A very drunk person staggers to left and right as he walks alone. With each step he takes, he staggers one pace to the left with probability 0.75, and one pace to the right with probability 0.25.



What is the expected number of paces he must take before ends up one pace to the left of his starting point?

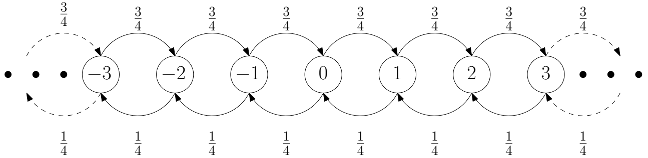
Q: What are the random variables involved in this stochastic process?

- The obvious random variables are the directions of staggering:

  Let Y_k be the step taken at time k;

$$Y_k = \begin{cases} 1 & \text{if the step is one pace to the left with probability } 0.75, \\ -1 & \text{if the step is one pace to the right with probability } 0.25. \end{cases}$$

- Suppose Y_1, Y_2, $\ldots$ are independent, then the *sum* of Y_k can be graphically represented as the following, which is known as a transition diagram,



  where each state represents one of possible sums of Y_k.

- Notice there are infinitely many paths to state 1 from the initial state of 0, each path has a different number of steps, and likelihood of happening.

- We could use simulations to obtain a roughly understanding of the problem.

```
julia> using Distributions      # for the variable d below
julia> using Random             # for the variable u below
julia>
julia> function simple_rand_walk(n)
       ## Compute max, mean and var no. of steps
               # array to store no. of steps
               k_vec = Array{Int64, 1}(undef, n);

               # Looping over simulations
               for i = 1:n
                       global ysum = 0;          # state
                       global k = 0;             # no. of steps
                       while ysum < 1
                               d = Uniform(0, 1);
                               u = rand(d, 1);
                               if u[1] < 0.75
                                       global ysum += 1;
                               else
                                       global ysum -= 1;
                               end
                               global k += 1;
                       end
                       k_vec[i] = k;
               end
               return maximum(k_vec), mean(k_vec), var(k_vec)
       end
julia> simple_rand_walk(7654321)
```

```
(71, 2.000600575805483, 6.000702118431554)
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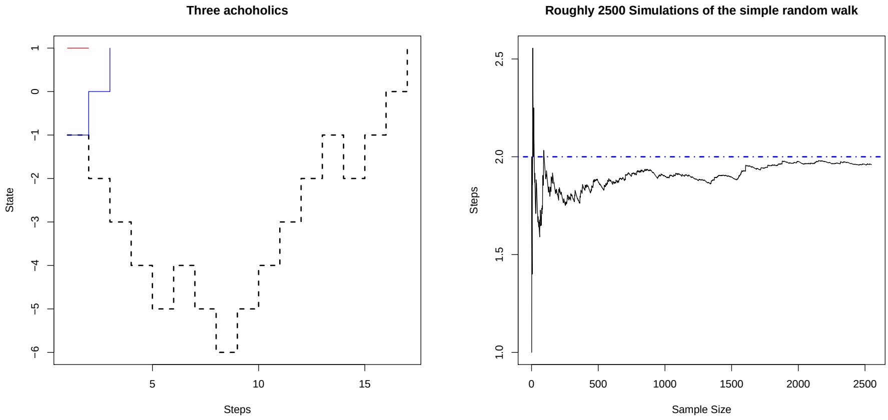


- According to the law of large numbers, we know the expected number of steps before ends up one pace to the left of his starting point is close to 2.
- To obtain the exact value, we need some results in calculus!

- Let $T_{i,j}$ be the number of steps to get from state i to state j for any, i, j.
- Using law of total expectation, we can partition the sample space over the Y_1

$$
\begin{aligned}
\mathbb{E}\left[T_{0,1}\right] &= \mathbb{E}\Big[\mathbb{E}\left[T_{0,1} \mid Y_1\right]\Big] \\
&= \left(1 + \mathbb{E}\left[T_{1,1} \mid Y_1 = 1\right]\right) \Pr\left(Y_1 = 1\right) \\
&\qquad + \left(1 + \mathbb{E}\left[T_{-1,1} \mid Y_1 = -1\right]\right) \Pr\left(Y_1 = -1\right) \\
&= (1 + 0) \cdot \frac{3}{4} + \left(1 + \mathbb{E}\left[T_{-1,1} \mid Y_1 = -1\right]\right) \cdot \frac{1}{4} \\
&= 1 + \frac{1}{4}\mathbb{E}\left[T_{-1,1}\right] \\
&= 1 + \frac{1}{4}\left(\mathbb{E}\Big[\mathbb{E}\left[T_{-1,1} \mid Y_2\right]\Big]\right)
\end{aligned}
$$

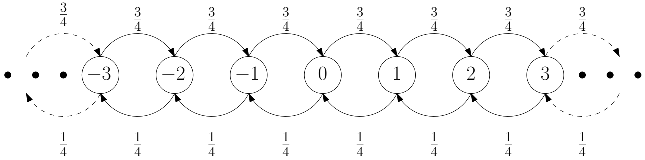
- We could keep going and obtain a series and see what it converges to.
- However, probability generating function provides a systematic approach.

- Recall we have defined the following two sets of random variables:

  Y_k denotes the step taken at time k

$$Y_k = \begin{cases} 1 & \text{if the step is one pace to the left with probability } 0.75, \\ -1 & \text{if the step is one pace to the right with probability } 0.25. \end{cases}$$

  $T_{i,j}$ denotes the number of steps to get from state i to state j for any, i, j.



Q: Which probability generating function do we need?

- Since we are after the expected number of steps before ends up one pace to the left of his starting point, i.e. $\mathbb{E}[T_{0,1}]$, we need to find PGF of $T_{0,1}$.

- We do not know PMF of $T_{0,1}$, so the trick is to treat it as an expectation

$$G_T(s) = \mathbb{E}\left[s^{T_{0,1}}\right]$$

and partition it over the first step Y_1 like we have done before to $\mathbb{E}\left[T_{0,1}\right]$,

$$\begin{aligned}
G_T(s) &= \mathbb{E}\Bigg[\mathbb{E}\left[s^{T_{0,1}} \mid Y_1\right]\Bigg] \\
&= \mathbb{E}\left[s^{T_{0,1}} \mid Y_1 = 1\right] \Pr\left(Y_1 = 1\right) + \mathbb{E}\left[s^{T_{0,1}} \mid Y_1 = -1\right] \Pr\left(Y_1 = -1\right) \\
&= \frac{3}{4} \cdot \mathbb{E}\left[s^{T_{0,1}} \mid Y_1 = 1\right] + \frac{1}{4} \cdot \mathbb{E}\left[s^{T_{0,1}} \mid Y_1 = -1\right] \\
&= \frac{3}{4} \cdot \mathbb{E}\left[s^{1+T_{1,1}} \mid Y_1 = 1\right] + \frac{1}{4} \cdot \mathbb{E}\left[s^{1+T_{-1,1}} \mid Y_1 = -1\right] \\
&= \frac{3}{4} \cdot s + \frac{1}{4} \cdot \mathbb{E}\left[s^{1+T_{-1,0}+T_{0,1}} \mid Y_1 = -1\right] = \frac{3}{4} \cdot s + \frac{1}{4} \cdot s \left(G_T(s)\right)^2
\end{aligned}$$

Q: Do you understand the reason behind every step?

- Thus we obtain the following quadratic equation by considering the first-step

$$s\big[G_T(s)\big]^2 - 4G_T(s) + 3s = 0$$

$$\Longrightarrow\ G_T(s) = \frac{4 \pm \sqrt{16 - 12s^2}}{2s} = \frac{2 \pm \sqrt{4 - 3s^2}}{s}$$

Q: Why only the negative root gives a valid PGF of $T_{0,1}$?

$$G_T(s) = \frac{2 - \sqrt{4 - 3s^2}}{s} \qquad \text{for} \quad |s| < 1.$$

- Therefore, the expected number of steps that we are looking for is given by

$$G_T'(1^-) = \frac{3s^2(4 - 3s^2)^{-1/2} - 2 + (4 - 3s^2)^{-1/2}}{s^2}\bigg|_{s=1} = 2$$

- As suggested by our simulation result, the theoretic result is exactly 2!