VE414 Lecture 11

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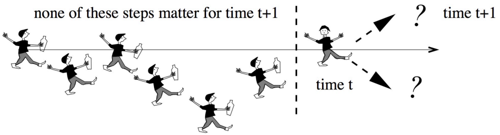
- The random walk is a type of stochastic process known as a Markov chain.



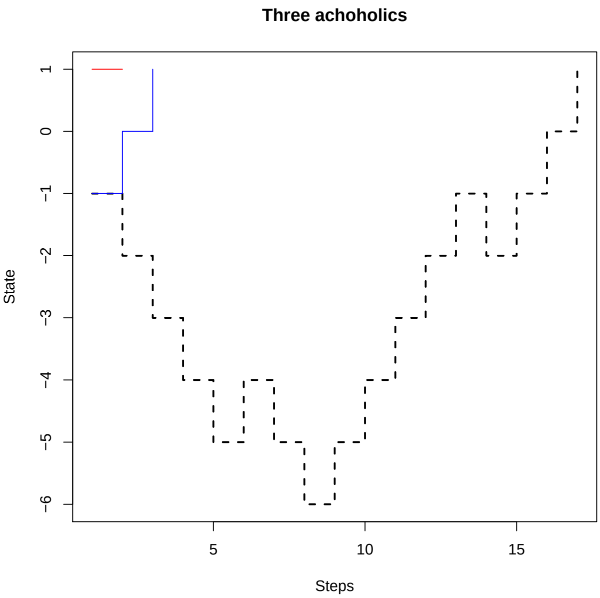
  in which the future depends only upon the present: not upon the past.

- The state of a Markov chain $\{Z_k\}$ at time t is the value of Z_t takes.
- The state space is the set of all values that Z_t can take, e.g. the state space of $\{Y_t\}$ is simply $\{-1, 1\}$. If we let the sum of Y_t be

$$Z_t = \sum_{j=1}^{t} Y_j$$

  then the state space of $\{Z_t\}$ is all integers.

- A trajectory of a Markov chain $\{Z_t\}$ is a particular sequence of values for Z_t.

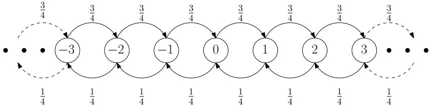

Three achoholics
State
Steps
1
0
−1
−2
−3
−4
−5
−6
5
10
15

- The defining property of a Markov chain is that only the most recent point in the trajectory effects what happens next, this is called the Markov Property:

$$\Pr\left(Z_{t+1} = j \mid Z_t, Z_{t-1}, \ldots, Z_1\right) = \Pr\left(Z_{t+1} = j \mid Z_t\right)$$

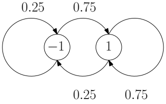
  that is, future depends only upon the present: not upon the past.

- A transition diagram is a way to describe a Markov chain with probabilities



- We could summarise the probabilities in a matrix if the state space is finite.

Transition diagram for Markov chain $\{Y_t\}$



$$\iff Y_k \left\{ \begin{array}{l} \text{Right} \\ \text{Left} \end{array} \right. \begin{array}{c} \overbrace{\text{Right Left}}^{Y_{k+1}} \\ \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix} \end{array} = \mathbf{P}$$

The corresponding Transition matrix

Transition matrix

$$X_t \left\{ \begin{array}{c} \text{All} \\ \text{possible} \\ \text{states} \end{array} \right. \Big\updownarrow \; \overbrace{\overset{\text{All possible states}}{\longleftrightarrow}}^{X_{t+1}} \hspace{-6em} \left[\begin{array}{c} \text{Transition} \\ \text{probabilities} \\ p_{ij} \end{array} \right] = \mathbf{P}$$

where the transition probabilities are given by

$$p_{ij} = \Pr\left(X_{t+1} = j \mid X_t = i\right)$$

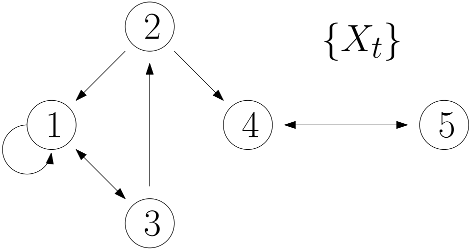
- It is clear the row sum of a transition matrix is always 1, since X_{t+1} must be one of the possible states, but there is no reason why the column sum is 1.

Q: Given $\mathbf{P}$, what is the probability of going from state i to j in two steps?

- States i and j are said to be in the same communicating class if there is a trajectory of getting from state i to j, *and* back from state j to i, denoted by

$$i \leftrightarrow j$$

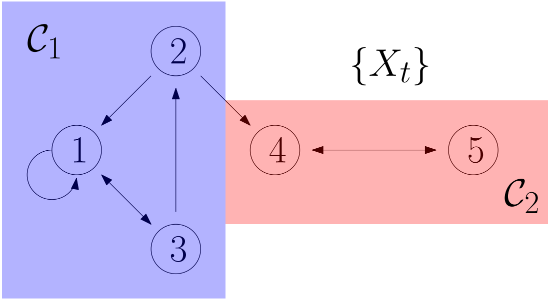
Q: What are the communicating classes of $\{X_t\}$ depicted by the following?



- In terms of transition matrix state i communicates with state j if there exist

$$t \quad \text{such that} \quad \left[\mathbf{P}^t\right]_{ij} > 0 \quad \text{and} \quad u \quad \text{such that} \quad \left[\mathbf{P}^u\right]_{ji} > 0$$

- A communicating class is closed if it is not possible to leave that class.

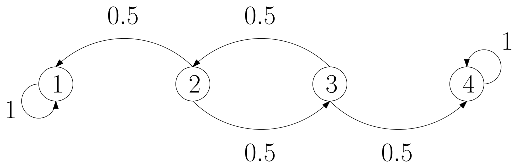


Q: What happens if there is no closed class in the Markov chain?

Definition

A Markov Chain or transition matrix $\mathbf{P}$ is said to be irreducible if $i \leftrightarrow j$ for all i, j in the state space $\mathcal{S}$, i.e. it is irreducible if $\mathcal{S}$ is a single communicating class.

- Suppose $\{X_t\}$ has the following transition diagram:



Q: What are the probabilities of reaching state 4 from other states?

Definition

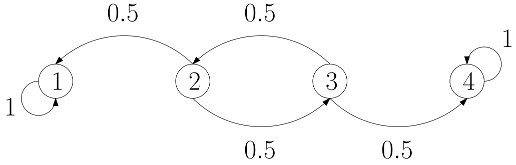
Let $\mathcal{A}$ be a subset of the state space $\mathcal{S}$. The reaching time of $\mathcal{A}$ is

$$T_{\mathcal{A}} = \min\{t \geq 0 \mid X_t \in \mathcal{A}\}$$

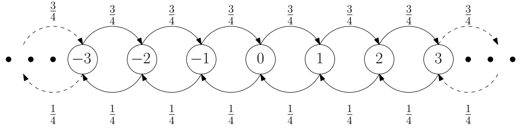
If the chain can never reach $\mathcal{A}$, then $T_{\mathcal{A}} = \infty$.

- $T_{\mathcal{A}}$ is the time or steps taken before reaching set $\mathcal{A}$ for the first time.

Q: What is the expected reaching time to $\mathcal{A} = \{1, 4\}$ starting from state 2?



- This is called the expected time to absorption since leaving $\mathcal{A}$ is impossible.
- Note we have covered all the terminologies to do with the opening problem



which involves using probability generating function since the state space is not finite, but it is essentially just a problem of expected reaching time.