Hierarchical Linear Models

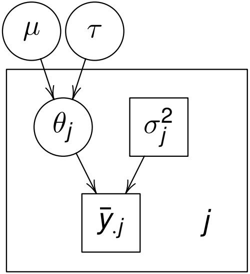
- What are Hierarchical Linear Models?
  - Also known as mixed models, multilevel models, and random effects models
  - Common in social sciences
  - Observations are nested in some manner (Workers nested within firms nested within sectors/Stratified sampling/ Cluster sampling)

Simple varying-coefficients model

Hierarchical normal model for group means

$\theta_j|\mu, \tau \sim \mathrm{N}(\mu, \tau)$

$\bar{y}_{.j}|\theta_j \sim \mathrm{N}(\theta_j, \sigma_j^2)$



Mixed-effect models

Standard notation for mixed-effect models:

$$y = X\beta + Zu + e$$

where

- y is an $n \times 1$ response vector
- X is an $n \times p$ design matrix for fixed effects
- β is a $p \times 1$ unknown fixed effect parameter vector
- Z is an $n \times q$ design matrix for random effects
- u is a $q \times 1$ unknown random effect parameter vector
- e is an $n \times 1$ unknown error vector

Coefficients exchangeable in batches

Assumptions

$$y = X\beta + Zu + e$$

Typically assume

- $E[u] = E[e] = 0$
- $Var[u] = \Omega$ and $Var[e] = \Lambda$
- $Cov[u, e] = 0$

These assumptions imply

- $E[y|\beta, \Omega, \Lambda] = X\beta$
- $Var[y|\beta, \Omega, \Lambda] = Z\Omega Z^T + \Lambda = \Sigma_y$

Common addition assumptions

- $Var[e] = \Lambda = \sigma_e^2 I$,
- $Var[u] = \Omega = \text{diag}\{\sigma_{u,\cdot}^2\}$, (or $Var[u] = \Omega = \sigma_u^2 I$ for single source), and
- u and e are normally distributed.

Rewrite as a standard linear regression model

We can rewrite

$$y = X\beta + Zu + e$$

as

$$y = \tilde{X}\tilde{\beta} + e$$

where $\tilde{X}$ is $n \times (p+q)$ with

$$\tilde{X} = [X \; Z]$$

and $\tilde{\beta}$ is a $(p+q) \times 1$ vector with

$$\tilde{\beta} = \left[\begin{array}{c} \beta \\ u \end{array} \right].$$

The fixed and random effects have been concatenated into the same vector.

Hierarchical linear model

Assume $y \sim N(\tilde{X}\tilde{\beta}, \Lambda)$. A Bayesian analysis proceeds by assigning prior distributions to $\tilde{\beta}$ and Λ. In constructing the prior for $\tilde{\beta}$, consider the components β and u separately. Assume

$$\beta \sim N(b, B) \qquad \text{and} \qquad u \sim N(0, \Omega)$$

independently.

For the

- **fixed** effects β, we select b and B while for the
- **random** effects u, we assign a prior for Ω.

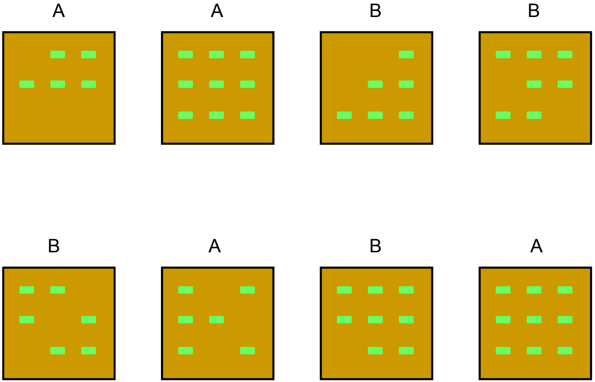
Therefore we have created a hierarchical model for the random effects and thus refer to this as a hierarchical linear model.

Seedling weight example

Example from biostat

> *Researchers were interested in comparing the dry weight of maize seedlings from two different genotypes (A and B). For each genotype, nine seeds were planted in each of four trays. The eight trays in total were randomly positioned in a growth chamber. Three weeks after the emergence of the first seedling, emerged seedlings were harvested from each tray and, after drying, weighed.*

A picture



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A mixed effect model for seedling weight

Let y_{gts} be the seedling weight of the

- g^{th} genotype with $g = 1, 2$,
- t^{th} tray $t = 1, 2, 3, 4$ of the g^{th} genotype, and
- s^{th} seedling with $s = 1, \ldots, n_{gt}$.

Then, we assume

$$y_{gts} = \gamma_g + \tau_{gt} + e_{gts}$$

where

- $\tau_{gt} \overset{ind}{\sim} N(0, \sigma^2_\tau)$ and, independently,
- $e_{gts} \overset{ind}{\sim} N(0, \sigma^2_e)$.

The main quantity of interest is the difference in mean seedling weight: $\gamma_2 - \gamma_1$.

As a general mixed effects model

Let X have the following 2 columns

- col1: all ones (intercept) [γ_1]
- col2: ones if genotype B and zeros otherwise [$\gamma_2 - \gamma_1$]

Let Z have the following 8 columns

- col1: ones if genotype 1, tray 1 and zeros otherwise [τ_{11}]
- col2: ones if genotype 1, tray 2 and zeros otherwise [τ_{12}]
- $\vdots$
- col8: ones if genotype 2, tray 4 and zeros otherwise [τ_{24}]

Then

$$y = X\beta + Zu + e$$

with $u \sim N(0, \sigma_\tau^2 I)$ and, independently, $e \sim N(0, \sigma_e^2 I)$.