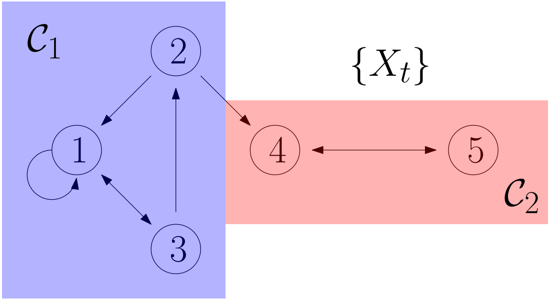
VE414 Lecture 13

Jing Liu

UM-SJTU Joint Institute

October 22, 2019

Q: Can you identify the recurrent states from a transition diagram?



Q: Can we have 1 recurrent and 1 transient state in a communicating class?

- A class is said to be recurrent if the states in that class are recurrent. If, on the other hand, the states are transient, the class is known as transient.

Q: For a finite Markov chain, is it possible to no recurrent class?

- A finite irreducible Markov chain is always recurrent.

- Given a recurrent state, it is natural to consider its mean return time:

Definition

Let i be a recurrent state, and

$$R_i = \min\{n \geq 1 \mid \Pr\left(X_n = i \mid X_0 = i\right) > 0\}$$

that is, R_i be the number of steps needed to return to state i

- If $m_i = \mathbb{E}\left[R_i\right] < \infty$, then i is said to positive recurrent.
- If $\mathbb{E}\left[R_i\right] = \infty$, then i is said to null recurrent.

Q: Can we have 1 positive and 1 null recurrent state in a communicating class?

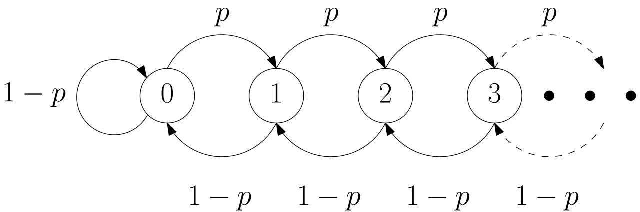
- A class is said to be positive recurrent if the states in that class are positive recurrent. If all states are null recurrent, then it is known as null recurrent.

Q: Can you think of a finite Markov chain which has a null recurrent state?

Q: How can we have a null recurrent state then?

Countably Infinite Markov Chain

- Consider the following Markov chain for $0 < p < 1$,



Theorem

Let $\{X_k\}$ be a countably infinite Markov chain, that is, the state space of $\{X_k\}$ is countably infinite. If the chain is irreducible, aperiodic and positive recurrent, then the limiting distribution exists.

Q: What is the limiting distribution of the Markov chain above for $0 < p < \frac{1}{2}$?

- If the limiting distribution exists,

$$\pi_j = \lim_{k \to \infty} \Pr\left(X_k = j \mid X_0 = i\right)$$

then it converges to the invariant distribution

$$\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}} \mathbf{P} \quad \text{where } \boldsymbol{\pi} \text{ is a column vector of } \pi_j.$$

- Let $\{X_t\}$ with a finite or countably infinite $\mathcal{S}$ be irreducible and aperiodic, then as you have seen previously, we simply solve the stationary equations,

$$\pi_j = \sum_{k \in \mathcal{S}} \pi_k P_{kj} \quad \text{for } j \in \mathcal{S}; \qquad \sum_{j \in \mathcal{S}} \pi_j = 1$$

if there is a unique solution, then we know the chain is positive recurrent and the invariant distribution is the limiting distribution of this chain.

- Given $\boldsymbol{\pi}$, we could also check whether it is the invariant distribution using

$$\pi_i \left[\mathbf{P}\right]_{ij} = \pi_j \left[\mathbf{P}\right]_{ji} \qquad \text{for all } i, j \in \mathcal{S}$$

which is known as the detailed balance equation or condition.