Functions

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1 Functions

1.1 Definitions

Let $f \subseteq A \times B$ (so f is a relation). We say that f is a function, and write $f : A \to B$, if $\mathrm{dom} f = A$ and for all $x \in A$ and for all $y, z \in B$, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

If $f : A \to B$ is a function, then we often write $f(a) = b$ instead of $(a, b) \in f$ and use $f(a)$ to denote b.

If $f : A \to B$ is a function and $C \subseteq A$ then we use $f``C$ as shorthand for the set $\{y | \exists x (x \in C \wedge (x, y) \in f)\}$ and $f \restriction C$ for the set $\{(x, y) | (x, y) \in f \wedge x \in C\}$.

Let $f : A \to B$ be a function. We say that f is *injective* or *one-to-one* if for all $x, y \in A$ and for all $z \in B$, if $(x, z) \in f$ and $(y, z) \in f$, then $x = y$.

1.2 Composing Functions

Let f and g be functions with $\mathrm{ran} f \subseteq \mathrm{dom} g$. Then define $g \circ f$ to be the relation

$$g \circ f = \{(x, y) | \exists z ((x, z) \in f \wedge (z, y) \in g)\}$$

If f and g are functions with $\mathrm{ran} f \subseteq \mathrm{dom} g$ then $g \circ f$ is a function.

1.3 Inverse Functions

Let $f \subseteq A \times B$ be a function. The *inverse* of f , written f^{-1} is the relation

$$f^{-1} = \{(x, y) \in B \times A | (y, x) \in f\}$$

Let A be a set. The identity function on A is the function $id_A : A \to A$ defined by

$$id_A = \{(x, y) \in A \times A | x = y\}$$

Let $f : A \to B$ be a function. The relation f^{-1} is a function with $\mathrm{dom} f^{-1} = \mathrm{ran} f$ and $\mathrm{ran} f^{-1} = A$ if and only if f is injective. Moreover, f^{-1} is injective and $f \circ f^{-1} = f^{-1} \circ f = id_A$.

1.4 Surjective Functions

Let $f : A \to B$ be a function. We say that f is *surjective* or *onto* if for all $x \in B$, there exists $y \in A$, such that $(y, x) \in f$.

If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection.

1.5 Bijection

If a function $f : A \to B$ is both injective and surjective, then we say that f is a bijection.

2 Cardinality

2.1 Definitions

Let A and B be sets. We say that A and B have the same *cardinality*, and write $|A| = |B|$, if there exists a function $f : A \to B$ that is a bijection.

Let A and B be sets. We write $|A| \leqslant |B|$ if there exists a function $f : A \to B$ that is an injection.

Note that if $|A| \leqslant |B|$, then $|A| = |C|$, for some $C \subseteq B$ (namely $C = \text{ran} f$ where $f : A \to B$ is an injection).

2.2 Examples

1. $|\mathbb{N}| = |2\mathbb{N}|$ $(f : \mathbb{N} \to 2\mathbb{N}, f(n) = 2n)$
2. $|\mathbb{N}| = |\mathbb{N}/\{0\}|$ $(f : \mathbb{N} \to \mathbb{N}/\{0\}, f(n) = n + 1)$
3. $|\mathbb{Z}| = |\mathbb{N}/\{1\}|$

$$f : \mathbb{Z} \to \mathbb{N}/\{1\}, f((-1)^k n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n + k & \text{if } n \neq 0 \end{cases}$$

2.3 Theorem

$|\mathbb{Z}| = |\mathbb{N}|$.

$$f : \mathbb{Z} \to \mathbb{N}, f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n & \text{if } n \geqslant 1 \\ -2n - 1 & \text{if } n \leqslant -1 \end{cases}$$

2.4 Infinite Sets

We say that a set A is *infinite* if there exists $f : A \to A$ that is an injection but not a surjection. (Dedekind infinite)

2.5 Examples

1. $\mathbb{N}$ is infinite. $(f : \mathbb{N} \to \mathbb{N}, f(n) = n + 1)$
2. $\mathbb{Z}$ is infinite. $(f : \mathbb{Z} \to \mathbb{Z}, f(x) = 2x)$
3. $\mathbb{Q}$ and $\mathbb{R}$ are infinite.

$$f : \mathbb{Q} \to \mathbb{Q}(\mathbb{R} \to \mathbb{R}), f(x) = \begin{cases} x+1 & \text{if } x \geqslant 0 \\ x & \text{if } x < 0 \end{cases}$$

2.6 Countable Sets

Let A be a set. We say that A is countable if $|A| \leqslant |N|$. We say that A is countably infinite if A is countable and A is infinite.

If B is a countable set and $A \subseteq B$, then A is countable.

2.7 Examples

1. Any finite set is countable.
2. $\mathbb{Z}$ is countably infinite.
3. If $A \subseteq N$, then A is countable. If A is also infinite, then A is countably infinite.

2.8 $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

1. Cantors Pairing Function is the function $: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$$

2. Cantors Pairing Function $: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.
3. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

2.9 Cantor's Theorem

If A is a set, then there is no injection $f : \mathcal{P}(A) \to A$.

proof by contradiction:

1. If $f : \mathcal{P}(A) \to A$ is an injection, then $f^{-1} : \text{ran} f \to \mathcal{P}(A)$ is an bijection.
2. Let $Z = \{x \in \text{ran} f | x \notin f^{-1}(x)\}$, and $Z \subseteq A$.
3. Let $z = f(Z)$. $z \in f^{-1}(z)$ or $z \notin f^{-1}(z)$?

Note that if A is a set, then $|A| < |\mathcal{P}(A)|$.

2.10 Uncountable Sets

We say that a set is uncountable if it is not countable.

2.11 Exercises

Show that if S is an infinite set, then there exists an element $a \in S$, such that a bijection $f : S \to S - \{a\}$ exits.

Since S is (Dedekind) infinite, then there exists a function $g : S \to S$, that is injective but not surjective. Therefore, ran $g \subset S$, and there exists an element $a \in S$ such that $a \notin$ran g.

We define the set $B = \{x \in S | \exists n \in \mathbb{N}, x = g^n(a)\}$,where $g^0(a) = a, g^k(a) = g(g^{k-1}(a))$. Note that $g^m(a) = g^n(a)$ if and only if $m = n$, since $g : S \to S$ is injective.

Let $f : S \to S - \{a\}$ be

$$f(x) = \begin{cases} x & \text{if } x \notin \text{B} \\ g(x) & \text{if } x \in \text{B} \end{cases}$$

Show that

1. any subset of a countable set is countable. (If $f : A \to \mathbb{N}$ is an injection, then $f : B \to \mathbb{N}$ is also an injection, where B is a subset of A.)
2. the union of two countable sets is countable. (If $f_1 : S_1 \to \mathbb{N}$ and $f_2 : S_2 \to \mathbb{N}$ are both injective, then $f : S_1 \cup S_2 \to \mathbb{N}$

   $$f(x) = \begin{cases} 2f_1(x) & \text{if } x \in S_1 \\ 2f_2(x) + 1 & \text{if } x \in S_2 \end{cases}$$

   is also an injection.)

Show that $|\mathbb{Q}^+| \leqslant |\mathbb{N} \times \mathbb{N}|$. (Obviously)
Show that $\mathbb{Q}$ is countable.($\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$)
Find a bijection $f : \mathbb{Z}^+ \to \mathbb{N} \times \mathbb{N}$. ($f : \mathbb{Z}^+ \to \mathbb{N} \times \mathbb{N}, 2^x(2y+1) \mapsto (x, y)$)
Find a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ other than Cantors Pairing Function. ($f : \mathbb{Z}^+ \to \mathbb{N} \times \mathbb{N}, 2^x(2y+1) - 1 \mapsto (x, y)$)
Show that $S = \{x \in \mathbb{R} | 1 < x < 2\}$ is infinite. ($f : S \to S, f(x) = (x+1)/2$)