# Basic Concepts in Set Theory

# Naive Set Theory: Sets via Predicates

We want to be able to talk about "collections of objects"; however, we will be unable to strictly define what an "object" or a "collection" is (except that we also want any collection to qualify as an "object"). The problem with "naive" set theory is that any attempt to make a formal definition will lead to a contradiction - we will see an example of this later. However, for our practical purposes we can live with this, as we won't generally encounter these contradictions.

We indicate that an object (called an ***element***) $x$ is part of a collection (called a ***set***) $X$ by writing $x \in X$. We characterize the elements of a set $X$ by some predicate $P$:

$$x \in X \qquad \Leftrightarrow \qquad P(x). \tag{1.1}$$

We write $X = \{x : P(x)\}$.

# Notation for Sets

We define the empty set $\varnothing := \{x : x \neq x\}$. The empty set has no elements, because the predicate $x \neq x$ is never true.

We may also use the notation $X = \{x_1, x_2, \dots, x_n\}$ to denote a set. In this case, $X$ is understood to be the set

$$X = \{x : (x = x_1) \vee (x = x_2) \vee \cdots \vee (x = x_n)\}.$$

We will frequently use the convention

$$\{x \in A : P(x)\} = \{x : x \in A \wedge P(x)\}$$

1.1. Example. The set of even positive integers is

$$\{n \in \mathbb{N} : \underset{k \in \mathbb{N}}{\exists} n = 2k\}$$

# Subsets and Equality of Sets

If every object $x \in X$ is also an element of a set $Y$, we say that $X$ is a subset of $Y$, writing $X \subset Y$; in other words,

$$X \subset Y \qquad \Leftrightarrow \qquad \forall x \in X : x \in Y.$$

We say that $X = Y$ if and only if $X \subset Y$ and $Y \subset X$.

We say that $X$ is a ***proper subset*** of $Y$ if $X \subset Y$ but $X \neq Y$. In that case we write $X \subsetneqq Y$.

Some authors write $\subseteq$ for $\subset$ and $\subset$ for $\subsetneqq$. Pay attention to the convention used when referring to literature.

# Examples of Sets and Subsets

## 1.2. Examples.

1. For any set $X$, $\varnothing \subset X$. Since $\varnothing$ does not contain any elements, the domain of the statement $\forall x \in X: x \in Y$ is empty. Therefore, it is vacuously true and hence $\varnothing \subset X$.
2. Consider the set $A = \{a, b, c\}$ where $a, b, c$ are arbitrary objects, for example, numbers. The set

   $$B = \{a, b, a, b, c, c\}$$

   is equal to $A$, because it satisfies $A \subset B$ and $B \subset A$ as follows:

   $$x \in A \quad \Leftrightarrow \quad (x = a) \vee (x = b) \vee (x = c) \quad \Leftrightarrow \quad x \in B.$$

   Therefore, neither order nor repetition of the elements affects the contents of a set.

   If $C = \{a, b\}$, then $C \subset A$ and in fact $C \subsetneqq A$. Setting $D = \{b, c\}$ we have $D \subsetneqq A$ but $C \not\subset D$ and $D \not\subset C$.

# Power Set and Cardinality

If a set $X$ has a finite number of elements, we define the ***cardinality*** of $X$ to be this number, denoted by $\#X$, $|X|$ or $\operatorname{card} X$.

We define the ***power set***

$$\mathscr{P}(X) := \{A \colon A \subset X\}.$$

Here the elements of the set $\mathscr{P}(X)$ are themselves sets; $\mathscr{P}(X)$ is the "set of all subsets of $X$." Therefore, the statements

$$A \subset X \qquad \text{and} \qquad A \in \mathscr{P}(X)$$

are equivalent.

1.3. Example. The power set of $\{a, b, c\}$ is

$$\mathscr{P}(\{a, b, c\}) = \big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\big\}.$$

The cardinality of $\{a, b, c\}$ is 3, the cardinality of the power set is $|\mathscr{P}(\{a, b, c\})| = 8$.

# Operations on Sets

If $A = \{x \colon P_1(x)\}$, $B = \{x \colon P_2(x)\}$ we define the ***union***, ***intersection*** and ***difference*** of $A$ and $B$ by

$$A \cup B := \{x \colon P_1(x) \vee P_2(x)\}, \qquad A \cap B := \{x \colon P_1(x) \wedge P_2(x)\},$$
$$A \setminus B := \{x \colon P_1(x) \wedge (\neg P_2(x))\}.$$

Let $A \subset M$. We then define the ***complement*** of $A$ by

$$A^{\mathrm{c}} := M \setminus A.$$

If $A \cap B = \emptyset$, we say that the sets $A$ and $B$ are ***disjoint***.

Occasionally, the notation $A - B$ is used for $A \setminus B$ and $A^{\mathrm{c}}$ is sometimes denoted by $\overline{A}$.

1.4. Example. Let $A = \{a, b, c\}$ and $B = \{c, d\}$. Then

$$A \cup B = \{a, b, c, d\}, \qquad A \cap B = \{c\}, \qquad A \setminus B = \{a, b\}.$$

## Operations on Sets

The laws for logical equivalencies immediately lead to several rules for set operations. For example, the distributive laws for $\wedge$ and $\vee$ imply

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Other such rules are, for example,

- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- $A \setminus B = B^{\mathrm{c}} \cap A$
- $(A \setminus B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B$

Some of these will be proved in the recitation class and the exercises.

# Operations on Sets

Occasionally we will need the following notation for the union and intersection of a finite number $n \in \mathbb{N}$ of sets:

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n,$$

$$\bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap A_2 \cap \cdots \cap A_n.$$

This notation even extends to $n = \infty$, but needs to be properly defined:

$$x \in \bigcup_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k\in\mathbb{N}}{\exists}\, x \in A_k,$$

$$x \in \bigcap_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k\in\mathbb{N}}{\forall}\, x \in A_k.$$

# Operations on Sets

In particular,

$$\bigcap_{k=0}^{\infty} A_k \subset \bigcup_{k=0}^{\infty} A_k.$$

1.5. Example. Let $A_k = \{0, 1, 2, \dots, k\}$ for $k \in \mathbb{N}$. Then

$$\bigcup_{k=0}^{\infty} A_k = \mathbb{N}, \qquad \bigcap_{k=0}^{\infty} A_k = \{0\}.$$

To see the first statement, note that $\mathbb{N} \subset \bigcup_{k=0}^{\infty} A_k$ since $x \in \mathbb{N}$ implies $x \in A_x$ implies $x \in \bigcup_{k=0}^{\infty} A_k$. Furthermore, $\bigcup_{k=0}^{\infty} A_k \subset \mathbb{N}$ since $x \in \bigcup_{k=0}^{\infty} A_k$ implies $x \in A_k$ for some $k \in \mathbb{N}$ implies $x \in \mathbb{N}$.

For the second statement, note that $\bigcap_{k=0}^{\infty} A_k \subset \mathbb{N}$. Now $0 \in A_k$ for all $k \in \mathbb{N}$. Thus $\{0\} \subset \bigcap_{k=0}^{\infty} A_k$. On the other hand, for any $x \in \mathbb{N} \setminus \{0\}$ we have $x \notin A_{x-1}$ whence $x \notin \bigcap_{k=0}^{\infty} A_k$.

## Ordered Pairs

A set does not contain any information about the order of its elements, e.g.,

$$\{a, b\} = \{b, a\}.$$

Thus, there is no such a thing as the "first element of a set". However, sometimes it is convenient or necessary to have such an ordering. This is achieved by defining an ***ordered pair***, denoted by

$$(a, b)$$

and having the property that

$$(a, b) = (c, d) \qquad \Leftrightarrow \qquad (a = c) \wedge (b = d). \tag{1.2}$$

We define

$$(a, b) := \{\{a\}, \{a, b\}\}.$$

It is not difficult to see that this definition guarantees that (1.2) holds.

## Cartesian Product of Sets

If $A, B$ are sets and $a \in A$, $b \in B$, then we denote the set of all ordered pairs by

$$A \times B := \{(a, b) \colon a \in A,\ b \in B\}.$$

$A \times B$ is called the ***cartesian product*** of $A$ and $B$.

In this manner we can define an ***ordered triple*** $(a, b, c)$ or, more generally, an ordered ***$n$-tuple*** $(a_1, \dots, a_n)$ and the $n$-fold cartesian product $A_1 \times \cdots \times A_n$ of sets $A_k$, $k = 1, \dots, n$.

If we take the cartesian product of a set with itself, we may abbreviate it using exponents, e.g.,

$$\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}.$$

# Problems in Naive Set Theory

If one simply views sets as arbitrary "collections" and allows a set to contain arbitrary objects, including other sets, then fundamental problems arise. We first illustrate this by an analogy:

Suppose a library contains not only books but also catalogs of books, i.e., books listing other books. For example, there might be a catalog listing all mathematics books in the library, a catalog listing all history books, etc. Suppose that there are so many catalogs, that you are asked to create catalogs of catalogs, i.e., catalogs listing other catalogs. In particular, you are asked to create the following:

(i) A catalog of all catalogs in the library. This catalog lists all catalogs contained in the library, so it must of course also list itself.

(ii) A catalog of all catalogs that list themselves. Does this catalog also list itself?

(iii) A catalog of all catalogs that do not list themselves. Does this catalog also list itself?

# The Russel Antinomy

In the previous analogy, we can view “catalogs” as “sets” and being “listed in a catalog” as “being an element of a set”. Then we have

(i) The set of all sets must have itself as an element.

(ii) The set of all sets that have themselves as elements may or may not contain itself. (This may be decidable by adding some rule to set theory.)

(iii) It is not decidable whether the “set of all sets that do not have themselves as elements” has itself as an element.

# The Russel Antinomy

Formally, this paradox is known as the ***Russel antinomy:***

1.6. Russel Antinomy. The predicate $P(x)\colon x \notin x$ does not define a set $A = \{x \colon P(x)\}$.

Proof.
If $A = \{x \colon x \notin x\}$ were a set, then we should be able to decide for any set $y$ whether $y \in A$ or $y \notin A$. We show that for $y = A$ this is not possible because either assumption leads to a contradiction:

(i) Assume $A \in A$. Then $P(A)$ by (1.1), i.e., $A \notin A$. $\lightning$

(ii) Assume $A \notin A$. Then $\neg P(A)$ by (1.1), therefore $A \in A$. $\lightning$

Since we cannot decide whether $A \in A$ or $A \notin A$, $A$ can not be a set. $\square$

## The Russel Antinomy

There are several examples in classical literature and philosophy of the Russel antimony:

(i) Epimenides the Cretan says, “All Cretans are liars.”

(ii) In a mountain village, there is a barber. Some villagers shave themselves (always) while the others never shave themselves. The barber shaves those and only those villagers that never shave themselves. Who shaves the barber?

# Russel Antinomy

We will simply ignore the existence of such contradictions and build on naive set theory. There are further paradoxes (antinomies) in naive set theory, such as ***Cantor's paradox*** and the ***Burali-Forti paradox***. All of these are resolved if naive set theory is replaced by a ***modern axiomatic set theory*** such as ***Zermelo-Fraenkel set theory.***

## Further Information:

- ***Set Theory***, Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/set-theory/
- P.R. Halmos, ***Naive Set Theory***,Aavilable here: http://link.springer.com/book/10.1007/978-1-4757-1645-0
- T. Jech, ***Set Theory: The Third Millennium Edition, Revised and Expanded***, Available here: http://link.springer.com/book/10.1007/3-540-44761-X