Something More about Q1 in Assignment 3

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1 Infinite Sets and Countable Infinite Sets

Q1. Prove that a set A is (Dedekind) infinite if and only if there exists an injective function $f : \mathbb{N} \to A$.

If A is (Dedekind) infinite, then there exists a function $g : A \to A$, that is injective but not surjective. Therefore, ran $g \subset A$, and there exists an element $a \in A$ such that $a \notin$ran g. We define the function $f : \mathbb{N} \to A$ as $f(0) = a$ and $f(n) = g^n(a)$, where $n \in \mathbb{N}$ and $g^k(a) = g(g^{(k-1)}(a))$ when $k > 1$. We will prove that f is an injective function.

If f is not injective and there exists $m, n \in \mathbb{N}$, $m > n$, such that $f(m) = f(n)$. It means $g^m(a) = g^n(a)$. Since g is injective, $g^m(a) = g^n(a)$ implies $g^{m-1}(a) = g^{n-1}(a)$. $g^{m-1}(a) = g^{n-1}(a)$ implies $g^{m-2}(a) = g^{n-2}(a)$. By induction, we can get $g^{m-n}(a) = a$. However, it contradicts with $a \notin$ran g. Therefore, f is an injective function.

If there exists an injective function $f : \mathbb{N} \to A$. ran$f$=$\{x \in A | \exists n \in \mathbb{N}, f(n) = x\}$ then we can get a function $g : A \to A$.

$$g(x) = \begin{cases} x & \text{if } a \notin \text{ ran } f \\ f(n+1) & \text{if there exists } n \in \mathbb{N},\ f(n) = a \end{cases}$$

It can be easily proved that g is injective but not surjective. Therefore, A is an infinite set.

2 Other Similar Problems

Show that if S is an infinite set, then there exists an element $a \in S$, such that a bijection $f : S \to S - \{a\}$ exits. (You can find the answer in my RC slides.)

Show that S is an infinite set if and only if $|S| = |S - \{a\}|$, where a is an arbitrary element in S.