Counting

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1 A Problem in Midterm 1

Show that $|\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.
My answer: $f_1(x) = 2x$, $f_2(x) = 2x + 1$.
$f : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}), f(A, B) = f_1(A) \cup f_2(B)$.

2 Counting

2.1 Definition

Let A be a finite set and let $0 \leqslant k \leqslant |A|$. Define

$$\mathcal{P}_k(A) = \{x \in \mathcal{P}(A) || x| = k\}$$

For all $n \in \mathbb{N}\backslash\{0\}$, we will use $[n]$ to denote the set

$$\{0, ..., n - 1\}$$

and $[0] = \emptyset$. $[n]$ will be the canonical set of size n.

Let $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$. Define $\left(\begin{array}{c} n \\ k \end{array} \right)$ to be the cardinality of the set $\mathcal{P}_k([n])$.
$\left(\begin{array}{c} n \\ 0 \end{array} \right)$=$\left(\begin{array}{c} n \\ n \end{array} \right)$=1.

2.2 Lemma

For all $n \in \mathbb{N}$ and for all $0 \leqslant k \leqslant n$, $\left(\begin{array}{c} n \\ k \end{array} \right)$=$\left(\begin{array}{c} n \\ n-k \end{array} \right)$.

2.3 Theorem

For all $n \in \mathbb{N}$ with $n \geqslant 1$ and for all $0 < k \leqslant n$,

$$\left(\begin{array}{c} n+1 \\ k \end{array} \right) = \left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right)$$

.

Proof. How to choose k different numbers from $[n + 1]$? One way is to choose k different numbers from $[n + 1]$ directly. Another way is to separate it into two cases. Choose k different numbers from $[n]$, or choose the number n and choose the other $k - 1$ different numbers from $[n - 1]$.

2.4 Binomial Theorem

For all $n \in N$ with $n \geqslant 1$ and for all numbers x and y ,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Prove it by yourself.

2.5 Corollary

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

2.6 Theorem

$$|\mathcal{P}_n([2n])| = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof. $(1+x)^n(1+x)^n = (1+x)^{2n}$. Calculate the coefficients of x^n on each side through the Binomial Theorem.

$$|\mathcal{P}([n])| = 2^n$$

Proof. How to get a subset A of $[n]$. For every number in $[n]$, we can either put it into A or not. Therefore, there are 2^n different subsets of $[n]$.

2.7 Theorem

Let $n, r \in \mathbb{N}$. The number of solutions to the equation $x_1 + \cdots + x_n = r$ with $x_1, ..., x_n \in \mathbb{N}$ is

$$\binom{n+r-1}{r}$$

Proof. Let $y_i = x_i + 1$. We will count the solutions to the equation $y_1 + \cdots + y_n = n + r$. Imagine $n + r$ identical balls are placed in a straight line. Now, try to separate these balls into n groups. We can put $n-1$ plates between the balls to make them into n groups. The ith group contains y_i balls. One separation corresponds to one solution of $y_1 + \cdots + y_n = n + r$. There are $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$ different separations.

Another better proof. We still want to count the solutions to the equation $y_1 + \cdots + y_n = n + r$. Let $S_i = y_1 + \cdots + y_i$. Then, $S_n = n + r$. We can find a bijection between $\mathcal{P}_{n-1}([n+r-1])$ and $\{S_1, ..., S_{n-1}\}$, where $1 \leqslant S_1 < S_2 < \cdots < S_{n-1} \leqslant n + r$.(How?)

2.8 Theorem

The number of ways of selecting r objects from n objects when the order does not matter and repetitions are allowed is

$$\binom{n+r-1}{r}$$

Proof. It is equivalent to $x_1 + \cdots + x_n = r$ where $x_1, ..., x_n \in \mathbb{N}$.

2.9 Theorem

The number of bijections from $[n]$ to $[n]$ is $n!$. I.e.

$$|\{f | f : [n] \to [n] \text{ is a bijection}\}| = n!$$

Proof. A bijections from $[n]$ to $[n]$ can be mapped to a permutation of $[n]$.

2.10 Theorem

Let $n \in \mathbb{N}$ and let $0 \leqslant k \leqslant n$. The number of of ordered k-tuples of distinct elements of [n] is

$$\binom{n}{k} k!$$

I.e. $|\{(x1, ..., xk) \in [n]^k | \text{for all } 0 \leqslant i < j \leqslant k, x_i \neq x_j\}| = \binom{n}{k} k!$

Prove. Choose k different numbers from $[n]$, consider all the permutations.

2.11 Toolbox

1. $\binom{n}{r} = \binom{n}{n-r}$
2. $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$
3. $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
4. $\binom{n}{r}\binom{r}{m} = \binom{n}{m}\binom{n-m}{r-m}$
5. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

6. $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$

7. $\binom{n}{r} + \binom{n+1}{r} + \cdots + \binom{n+k}{r} = \binom{n+k-1}{r+1} - \binom{n}{r+1}$

2.12 Warm-up Exercise

Explain the following equations with minimal calculation.

1. $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$

2. $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$

3. $\binom{n}{r}\binom{r}{m} = \binom{n}{m}\binom{n-m}{r-m}$

1. See 2.3.

2. How to choose r different numbers from $[n]$? Can we just choose $r-1$ different numbers from $[n]\backslash\{x\}$ plus the the number x for all $x \in [n]$?

3. How to choose r different numbers from $[n]$? We first separate $[n]$ into two groups with m numbers in group I and $n-m$ numbers in group II. Can we just choose all the numbers in group I plus $r-m$ numbers from group II for all the different separations?

Show that

$$\binom{n}{r} + \binom{n+1}{r} + \cdots + \binom{n+k}{r} = \binom{n+k-1}{r+1} - \binom{n}{r+1}$$

.

Apply the equation $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$ again and again.

2.13 Difficult Exercise

1.Calculate

$$\binom{4n-1}{0} + \binom{4n-1}{3} + \binom{4n-1}{4} + \binom{4n-1}{7} + \cdots + \binom{4n-1}{4n-4} + \binom{4n-1}{4n-1}$$

Hint. Apply Binomial Theorem to $\left(1+i^k\right)^{4n-1}$, $k=0,1,2,3$.
It is equal to

$$\frac{1-i}{4}(1+i)^{4n-1}+\frac{1}{2}(1+i^2)^{4n-1}+\frac{1+i}{4}(1+i^3)^{4n-1}$$

2.Calculate

$$\binom{n}{1}-3\binom{n}{3}+5\binom{n}{5}-7\binom{n}{7}+\cdots$$

Hint. $\binom{n}{r}=\frac{n}{r}\binom{n-1}{r-1}$. Calculate the $Re[(1+i)^{n-1}]$.

3.Let $S_n(k)$ be the set of all the bijection in $[n]$ with exact k fixed points. Show that

$$\sum_{k=0}^{n} k|S_n(k)|=n!$$

Hint. $|S_n(k)|=\binom{n}{k}|S_{n-k}(0)|$. $\sum_{k=0}^{n-1}|S_{n-1}(k)|=(n-1)!$

4. Show that

$$S_{m,n}=\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}=(-1)^m\delta_{m,n}\quad(m\leqslant n),$$

where

$$\delta_{m,n}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$$

5. Let $\{a_n\}$ be a sequence of real numbers. Show that if

$$b_n=(-1)^k\sum_{k=0}^{n}\binom{n}{k}a_k, n=0,1,2,\cdots,$$

then

$$a_n=(-1)^k\sum_{k=0}^{n}\binom{n}{k}b_k, n=0,1,2,\cdots.$$

Hint. Use the conclusion of question 4.