Sets

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1 Sets

1.1 Definitions

Let X be a set and let x be an object. We write $x \in X$ to indicate that x is a member of X.

If $P(x)$ is predicate then the set of all objects x that satisfy $P(x)$ is written:

$$X = \{x|P(x)\}$$

I.e. $x \in X$ if and only if $P(x)$.

Two sets X and Y are equal ($X = Y$) if for all x, $x \in X$ if and only if $x \in Y$.

If every object $x \in X$ is also an element of a set Y, we say that X is a *subset* of Y, writing $X \subseteq Y$; in other words,

$$X \subseteq Y \Leftrightarrow \forall x(x \in X \Rightarrow x \in Y)$$

. Note that $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

We say that X is a *proper subset* of Y if $X \subseteq Y$ but $X = Y$. In that case we write $X \subset Y$.

If X is a set, then the *powerset* of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X. I.e.

$$\mathcal{P}(X) = \{A|A \subseteq X\}$$

This means the expressions $A \in \mathcal{P}(X)$ and $A \subseteq X$ are equivalent.

1.2 Operations on Sets

Union, intersection, difference.

Logical equivalences immediately lead to several rules for set operations.

1. $A \cup A = A, A \cap A = A$
2. $A \cup B = B \cup A, A \cap B = B \cap A$
3. $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$
4. $A \cup \emptyset = A$, $A \cap E = A$
5. $A \cap \emptyset = \emptyset$, $A \cup E = E$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7. $A \cap (A \cup B) = A, A \cup (A \cap B) = A$

8. $A - (B \cup C) = (A - B) \cap (A - C), A - (B \cap C) = (A - B) \cup (A - C)$
9. $(A \cap B)^c = A^c \cup B^c, (A \cup B)^c = A^c \cap B^c$
10. $A - B = B^c \cap A, (A - B)^c = A^c \cup B$

other notations:

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup \cdots \cup A_n$$

$$\bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap \cdots \cap A_n$$

when $n \to \infty$,

$$x \in \bigcup_{k=0}^{\infty} A_k \Leftrightarrow (\exists k \in \mathbb{N})(x \in A_k)$$

$$x \in \bigcap_{k=0}^{\infty} A_k \Leftrightarrow (\forall k \in \mathbb{N})(x \in A_k)$$

1.3 Exercise

Here is the proof of $(A \cup B) \cup C = A \cup (B \cup C)$.

$$\begin{aligned} x \in (A \cup B) \cup C &\equiv x \in (A \cup B) \vee x \in C \\ &\equiv (x \in A \vee x \in B) \vee x \in C \\ &\equiv x \in A \vee (x \in B \vee x \in C) \\ &\equiv x \in A \cup (B \cup C) \end{aligned}$$

Show that

1. $(A \cap B) \cap C = A \cap (B \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $A - (B \cup C) = (A - B) \cap (A - C)$

1.4 Exercise

Show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

$$
\begin{aligned}
(A - B) \cup (B - A) &= (A \cap B^c) \cup (B \cap A^c) \\
&= ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c) \\
&= ((A \cup B) \cap (B^c \cup B)) \cap ((A \cup A^c) \cap (B^c \cup A^c)) \\
&= (A \cup B) \cap E \cap E \cap (B^c \cup A^c) \\
&= (A \cup B) \cap (B^c \cup A^c) \\
&= (A \cup B) \cap (B \cap A)^c \\
&= (A \cup B) - (A \cap B)
\end{aligned}
$$

1.5 Ordered Pairs

$(a, b) = \{\{a\}, \{a, b\}\}$

1.6 Cartesian Product of Sets

If A, B are sets and $a \in A, b \in B$, then we denote the set of all ordered pairs by

$$A \times B := \{(a, b) | a \in A \wedge b \in B\}.$$

$A \times B$ is called the cartesian product of A and B.

1.7 Russells Paradox

The set of all sets that are not members of themselves is not a set.