# Logic

statement(proposition) : anything we can regard as being either true or false, denoted by A(x)

argument : finite sequence of statements

valid : truth of premises $\Rightarrow$ truth of conclusion

tautology : $P(A_1) \wedge P(A_2) \wedge \ldots \wedge P(A_n) \Rightarrow C$

negation $\neg$

conjunction(and) $\wedge$

disjunction(or) $\vee$

implication(implies) $A \Rightarrow B$ $\quad \neg B \Rightarrow \neg A$ $\quad B \vee (\neg A)$

equivalence $A \Leftrightarrow B$ $\quad (\neg A) \Leftrightarrow (\neg B)$ $\quad (A \vee \neg B) \wedge (B \vee \neg A)$ $\quad (A \wedge B) \vee (\neg A \wedge \neg B)$

quantifiers: $\forall, \exists$ (understanding) $\quad \neg\forall \Leftrightarrow \exists$

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de Morgan rules $\neg(A \vee B) \equiv \neg A \wedge \neg B$ $\quad \neg(A \wedge B) \equiv \neg A \vee \neg B$

Commutativity: $A \wedge B \equiv B \wedge A$ $\quad A \vee B \equiv B \vee A$

Associativity: $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ $\quad (A \vee B) \vee C \equiv A \vee (B \vee C)$

Distributivity: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ $\quad A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

extension $(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (B \vee C) \wedge (A \vee D) \wedge (B \vee D)$

Absorption: $A \vee (A \wedge B) \equiv A$ $\quad A \wedge (A \vee B) \equiv A$

Rules of Inference:

$A \wedge B \Rightarrow A$ $\quad A \Rightarrow A \vee B$

$(A \Rightarrow B) \wedge (A \Rightarrow C) \equiv A \Rightarrow (B \wedge C)$ $\quad (A \Rightarrow B) \vee (A \Rightarrow C) \equiv A \Rightarrow (B \vee C)$

$(A \Rightarrow C) \wedge (B \Rightarrow C) \equiv (A \vee B) \Rightarrow C$ $\quad (A \Rightarrow C) \vee (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$

$\neg(A \Leftrightarrow B) \equiv A \Leftrightarrow (\neg B)$

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methods for proving Example : $(A \wedge (A \Rightarrow B)) \Rightarrow B$ and $F(a, b, c) = (ab)' + ac' + bc$

1. truth table less than 3 variables
2. calculate
3. logic $\wedge$ 连接
4. Normal form (tautology) $A \Leftrightarrow B$ A and B has the same expression Disjunctive normal form is 1

Disjunctive Normal form $A = A_1 \vee A_2 \vee \ldots \vee A_s$

Conjunctive Normal form $A = A_1 \wedge A_2 \wedge \ldots \wedge A_s$

Example : $(p \vee q) \wedge r = m_3 \vee m_5 \vee m_7 = M_0 \wedge M_1 \wedge M_2 \wedge M_4 \wedge M_6$

$$\begin{aligned}(p \vee q) \wedge r &= (p \wedge r) \vee (q \wedge r) = (q \wedge p \wedge r) \vee (\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r) \vee (q \wedge \neg p \wedge r) \\ &= (\neg q \wedge p \wedge r) \vee (q \wedge p \wedge r) \vee (q \wedge \neg p \wedge r) = m_{111} \vee m_{101} \vee m_{011}\end{aligned}$$

$$\begin{aligned}(p \vee q) \wedge r &= (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee r) \\ &= (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee r) \\ &= M_{000} \wedge M_{001} \wedge M_{100} \wedge M_{010} \wedge M_{110}\end{aligned}$$

$m_{1x1} = m_{101} \vee m_{111} \quad p \wedge r = p \wedge r \wedge T = p \wedge r \wedge (q \vee \neg q) = (p \wedge r \wedge q) \vee (p \wedge r \wedge \neg q)$

$M_{1x1} = M_{101} \wedge M_{111} \quad p \vee r = p \vee r \vee F = p \vee r \vee (q \wedge \neg q) = (p \vee r \vee q) \wedge (p \vee r \vee \neg q)$

| $p$ $q$ $r$ | $p \vee q$ | $(p \vee q) \wedge r$ |
|---|---|---|
| 0 0 0 | 0 | 0 |
| 0 0 1 | 0 | 0 |
| 0 1 0 | 1 | 0 |
| 0 1 1 | 1 | 1 |
| 1 0 0 | 1 | 0 |
| 1 0 1 | 1 | 1 |
| 1 1 0 | 1 | 0 |
| 1 1 1 | 1 | 1 |

Application:

**【例 1.21】** 某科研所有3名青年高级工程师 $A, B, C$. 所里要选派他们中的 1 到 2 人出国进修, 由于所里工作的需要选派时必须满足以下条件:

(1) 若 $A$ 去, 则 $C$ 也可以去;

(2) 若 $B$ 去, 则 $C$ 不能去;

(3) 若 $C$ 不去, 则 $A$ 或 $B$ 可以去.

问所里应如何选派他们?

$$(A \Rightarrow C) \wedge (B \Rightarrow \neg C) \wedge (\neg C \Rightarrow (A \vee B)) = (\neg A \vee C) \wedge (\neg B \vee \neg C) \wedge (C \vee A \vee B)$$

$$= M_{000} \wedge M_{011} \wedge M_{100} \wedge M_{110} \wedge M_{111} = m_{001} \vee m_{010} \vee m_{101}$$

Tedious Calculation:

Let M be a set and let $X, Y, Z \subseteq M$. We define the symmetric difference:

$$X \triangle Y := (X \cup Y) \setminus (X \cap Y)$$

(i) Prove that $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$

(ii) Prove that $(M \setminus X) \triangle (M \setminus Y) = X \triangle Y$

(iii) Show that the symmetric difference is associative, $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$

(iv) Prove that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$

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$\forall x(A(x) \vee B) \Leftrightarrow \forall x A(x) \vee B$ $\quad$ $\forall x(A(x) \wedge B) \Leftrightarrow \forall x A(x) \wedge B$

$\forall x(A(x) \rightarrow B) \Leftrightarrow \exists x A(x) \rightarrow B$ $\quad$ $\forall x(B \rightarrow A(x)) \Leftrightarrow \forall x A(x)$

$\exists x(A(x) \vee B) \Leftrightarrow \exists x A(x) \vee B$ $\quad$ $\exists x(A(x) \wedge B) \Leftrightarrow \exists x A(x) \wedge B$

$\exists x(A(x) \rightarrow B) \Leftrightarrow \forall x A(x) \rightarrow B$ $\quad$ $\exists x(B \rightarrow A(x)) \Leftrightarrow B \rightarrow \exists x A(x)$

$\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall x A(x) \wedge \forall x B(x)$ $\quad$ $\exists x(A(x) \vee B(x)) \Leftrightarrow \exists x A(x) \vee \exists x B(x)$

**important**: $\quad \forall x(A(x) \vee B(x)) \not\Leftrightarrow \forall x A(x) \vee \forall x B(x) \quad \exists x(A(x) \wedge B(x)) \not\Leftrightarrow \exists x A(x) \wedge \exists x B(x)$

Exercise:

1. Show that $(\exists x(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology
2. $\exists x(P(x) \wedge \neg Q(x))$

   $\forall x(P(x) \Rightarrow R(x))$

   $\therefore \exists x(R(x) \wedge \neg Q(x))$ $\qquad$ is a valid argument

3. Show that the following sentences are not tautologies.

   (i) $(\exists x P(x) \wedge \exists y Q(y)) \Rightarrow \exists z(P(z) \wedge Q(z))$

   (ii) $(\forall z(Q(z) \vee P(z))) \Rightarrow ((\forall x Q(x)) \vee (\forall y P(y)))$

   (iii) $((\forall x P(x)) \Rightarrow (\exists y Q(y))) \Rightarrow \forall z(P(z) \Rightarrow \exists w Q(w))$

Fun Exercise:

A and B find themselves trapped in a dark and cold dungeon. After a quick search they find three doors; the first one red, the second one blue, and the third one green. Behind one of the doors is a path to freedom. Behind the other two doors, however, is almost certain death. On each door there is an inscription: the red door says “freedom is behind this door”, the blue door says “freedom is not behind this door” and the green door says “freedom is not behind the blue door”. Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead A and B to safety?

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## Set theory

We indicate that an object (called an element) $x$ is part of a collection (called a set) $X$ by writing $x \in X$. We characterize the elements of a set $X$ by some predicate $P$: $\quad x \in X \Leftrightarrow P(x)$
We write $X = \{x : P(x)\}$

$\{x \in A : P(x)\} = \{x : x \in A \wedge P(x)\}$

$X = Y$ if and only if $X \subset Y$ and $Y \subset X$

$X \subset Y \Leftrightarrow \forall x \in X : x \in Y$

$X$ is a proper subset of $Y$ if $X \subset Y$ but $X \neq Y$

power set $P(X) := \{A : A \subset X\}$

$\cup_{k=0}^{n} A_k := A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n$ $\qquad$ $\cap_{k=0}^{n} A_k := A_0 \cap A_1 \cap A_2 \cap \cdots \cap A_n$

Rules:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$

$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$A \setminus B = B^c \cap A$ $(A \setminus B)^c = A^c \cup B$

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ordered pair $(a, b) := \{\{a\}, \{a, b\}\}$.

cartesian product $A \times B := \{(a, b) : a \in A, b \in B\}$

Russel Antinomy. The predicate $P(x) : x \notin x$ does not define a set $A = \{x : P(x)\}$