Exercise For Midterm 1

October 15, 2019

1 Logic

1.Write the principal disjunctive normal form of the following table.

P	Q	R	G	m (minimum)	M (maximum)
T	T	T	T	111	-
T	T	F	F	-	110
T	F	T	F	-	101
T	F	F	F	-	100
F	T	T	T	011	-
F	T	F	F	-	010
F	F	T	T	001	-
F	F	F	F	-	000

Answer: $G = m_{111} \vee m_{011} \vee m_{001} = (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$ or $G = M_{110} \wedge M_{101} \wedge M_{100} \wedge M_{010} \wedge M_{000} = (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \wedge R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \wedge R)$
Note that:

1. $G = T$, minimum (m) for disjunctive normal form, and m_{001} means $\neg P \wedge \neg Q \wedge R$.
2. $G = F$, maximum (M) for conjunctive normal form, and M_{001} means $P \vee Q \vee \neg R$.

2. Show that $A \Rightarrow B, \neg(B \Rightarrow C) \Rightarrow \neg A \mid \therefore A \Rightarrow C$
Answer:

1. $\because A$
2. $\because \neg(B \Rightarrow C) \Rightarrow \neg A$
3. $\therefore \neg(\neg(B \Rightarrow C)) \equiv B \Rightarrow C$
4. $\because A \Rightarrow B$
5. $\therefore C$

Note: If $(A_1, A_2, \cdots, A_n, A \quad \therefore B)$ is a valid argument, then $(A_1, A_2, \cdots, A_n \quad \therefore A \Rightarrow B)$ is also valid. (If the conclusion that needs to be proved is $A \Rightarrow B$, then A can be used as a premise.)

3. Show that $(\exists x(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology. Note

1. $\forall x(A(x) \wedge B(x)) \equiv \forall x A(x) \wedge \forall x B(x)$
2. $\exists x(A(x) \vee B(x)) \equiv \exists x A(x) \vee \exists x B(x)$

Answer: see my RC slides.

2 Set

No exercise for set.

3 Relation and Orders

Just memorize all the definitions.

4 Lattice

4.One of the most important properties of lattice is the property of order-preserving. Let $(L, \preceq)$ be a lattice, and $a, b, c \in L$. Show that

1. $(a \vee b) \vee c = a \vee (b \vee c)$.
2. $a \vee (a \wedge b) = a$.
3. $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$.

Answer: see my RC slides.
Note that: Let $(L, \preceq)$ be a lattice, and $a, b, c \in L$.

1. (a) $a \preceq a \vee b, a \wedge b \preceq a$.
   (b) If $a \preceq b, c \preceq d$, then $a \wedge c \preceq b \wedge d, a \vee c \preceq b \vee d$.
   proof: $a \wedge c \preceq a \preceq b$, and $a \wedge b \preceq b \preceq d$ ($\preceq$ is transitive.)
   $a \wedge c$ is a lower bound of $\{b, d\}$, and thus, $a \wedge c \preceq b \wedge d$
   (c) If $b \preceq c$, then $a \wedge b \preceq a \wedge c, a \vee b \preceq a \vee c$.
2. $a \vee b = b \vee a, a \wedge b = b \wedge a$.
3. $a \vee a = a, a \wedge a = a$.

5 Function

5. Find an injection from the power set of $\mathbb{N}$ to $\mathbb{R}[0,1] = \{x \in \mathbb{R} | 0 \leqslant x \leqslant 1\}$
Answer:

$$f(M) = \sum_{k \in M} 10^{-(k+1)},\ M \subseteq \mathbb{N}$$

5.1 Cantor's Theorem (Important)

If A is a set, then there is no injection $f : \mathcal{P}(A) \to A$.
6. Show that $\mathbb{R}[0,1]$ is not a countable set by showing that $|\mathcal{P}(\mathbb{N})| \leqslant |\mathbb{R}[0,1)|$ Answer: the conclusion of 5.

6 Tarski-Knaster Theorem and Schroder-Bernstein Theorem

7.Prove the Tarski-Knaster Theorem. (See the lecture slide).

8. Show that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}[0,1)|$ By Schroder-Bernstein Theorem.
Answer: we have already found a an injection from the power set of $\mathbb{N}$ to $\mathbb{R}[0,1) = \{x \in \mathbb{R} | 0 \leqslant x \leqslant 1\}$. An injection from $\mathbb{R}[0,1)$ to the power set of $\mathbb{N}$ is $g : \mathbb{R}[0,1) \to \mathcal{P}(\mathbb{N})$

$$\sum_{k \in M} 2^{-k} \to M$$

7 Countable Sets and Infinite Sets

9.Show that a set S is infinite if and only if $|S - \{a\}| = |S|$, where a is an arbitrary element of S.
Answer: Let P be a countable infinite subset of S, and $a \in P$. If $a \notin P$, then $P = P \cup \{a\}$, and P is still a countable infinite subset. It is easy to find a bijection $f : P \to P - \{a\}$.
10.$S = \{X \in \mathcal{P}(\mathbb{Z}) | \text{X is finite}\}$. Is S a countable set?
Answer: yes.