Tarski-Knaster Theorem, Schroder-Bernstein Theorem, and Mathematical Induction

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1 Tarski-Knaster Theorem and Schroder-Bernstein Theorem

1.1 Definitions

Let $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ be partial orders. We say that a function $f : P_1 \to P_2$ is *order-preserving* if for all $x, y \in P_1$,

$$\text{if } x \preceq_1 y, \text{ then } f(x) \preceq_2 f(y)$$

.
We might write $f : (P_1, \preceq_1)(P_2, \preceq_2)$ when we need to specify which orders we are talking about.

Let A be a set and let $f : A \to A$ be a function. We say that $x \in A$ is a *fixed point* of f if $f(x) = x$.

1.2 Tarski-Knaster Theorem

Let $(L, \preceq)$ be a complete lattice. If $f : (L, \preceq) \to (L, \preceq)$ is an order-preserving function, then f has a fixed point. Moreover, the set of fixed points of f on L is a complete lattice.

The proof is quite long. Please be patient.

1. $X = \{x \in L | f(x) \preceq x\}$ and $a = \bigwedge X$.
2. If $x \in X$, then $f(x) \in X$.
3. $f(a)$ is a lower bound of X.
4. $f(a) \preceq a$, which implies $a \in X$. Then, $f(a) \preceq a$.
5. $f(a) = a$. Therefore, a is a fixed point of f on L.

Let $S = \{x \in L | f(x) = x\}$. It's easy to a is the least element of S, since $S \subseteq X$. Similarly, we can find the greatest element of S. ($Y = \{y \in L | y \preceq f(y)\}$, and $b = \bigvee Y$.).

Let $(L, \preceq)$ be a complete lattice. If $f : (L, \preceq) \to (L, \preceq)$ is an order-preserving function, then f has a greatest fixed point (a) and a least fixed point (b). For all $s \in S = \{x \in L | f(x) = x\}$, $a \preceq s \preceq b$.

Let P be a subset of S, i.e. $P \subseteq S$, and let u be the g.l.b of P in L.(**The g.l.b of P in L is different from the g.l.b of P in S!**)

1. $[a,u] = \{x \in L | a \preceq x \preceq u\}$ is a complete lattice. (Why $a \preceq u$? Why is it a complete lattice?)

2. For any $x \in L$, if $x \in [a,u]$, then $f(x) \in [a,u]$. ($f(x)$ is a lower bound of S.)

3. Let $g : [a,u] \to [a,u]$ and $g(x) = f(x)$. (Just consider f restricted to $[a,u]$.) Since g is order-preserving and $[a,u]$ is a complete lattice, then we can find the greatest fixed point of g. Let m be the greatest fixed point of g. Then m is the g.l.b of P in S.

4. Similarly we can find the l.u.b of P is S.

1.3 Schroder-Bernstein Theorem

Let A and B be sets. If there exists $f : A \to B$ that is injective and $g : B \to A$ that is injective, then there exists a bijection $h : A \to B$.

I will provide two different methods to prove the Schroder-Bernstein Theorem. The first one is very similar to the proof covered in the lecture, but we do not explicitly apply Tarski-Knaster Theorem. The second one has nothing to do with Tarski-Knaster Theorem or fixed points.

Here, we use $f(A) = B$ to denote $B = \{y | \exists x, y = f(x)\}$, where $A \subseteq \mathrm{dom} f$. We also define $A^* = g(B)$,a nd $B^* = f(A)$.

Proof I:
Our goal is to find two sets $S \subseteq A$ and $T \subseteq B$, such that $f(S) = T, g(B - T) = A - S$. Then, we can find a bijection between A and B. (How?)
If $E \subseteq A$, $F = f(E)$, then $F \subseteq B^*, F \subseteq B$. Let $G = g(B - F)$, and $\widehat{E} = A - G = A - g(B - f(E))$.
We call a set $E \subseteq A$ a *magical* set, if $E \subseteq \widehat{E}$. Let $S \subseteq A$ be the union of all *magical* sets. For any two arbitrary sets E_1 and E_2, if $E_1 \subseteq E_2$, then $\widehat{E_1} \subseteq \widehat{E_2}$. (Why?)
Therefore, for any element $x \in S$, there exists a *magical* set E, such that $x \in E$. Since, $E \subseteq S$, then $\widehat{E} \subseteq \widehat{S}$. Also, E is a *magical* set, which implies that $E \subseteq \widehat{E}$. Then, $E \subseteq \widehat{S}$, and $x \in \widehat{S}$. Therefore, $S \subseteq \widehat{S}$.
The most interesting thing is that $S \subseteq \widehat{S}$ means $\widehat{S}$ is a *magical* set, and it implies that $\widehat{S} \subseteq S$. Therefore, $S = \widehat{S}$. Nice!
Now, $T = f(S)$, and $g(B - T) = g(B - f(S)) = A - \widehat{S} = A - S$.

Proof II:
This proof provides a direct method to find a bijection between A and A^*.
Let

$$\begin{array}{ll} A_1 = A & B_1 = B \\ A_2 = A^* & B_2 = B^* \\ A_3 = g(B_2) & B_3 = f(A_2) \\ \cdots & \cdots \\ A_{k+1} = g(B_k) & B_{k+1} = f(A_k) \end{array}$$

Therefore,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$$
$$B_1 \supseteq B_2 \supseteq B_3 \supseteq B_4 \supseteq \cdots$$

(Why?)
Let

$$D = \bigcap_{k=1}^{\infty} A_k$$

Then $A = D + (A_1 - A_2) + (A_2 - A_3) + (A_3 - A_4) + \cdots$, and $A^* = D + (A_2 - A_3) + (A_3 - A_4) + (A_4 - A_5) + \cdots$.
We can find a bijection $h_A : A \to A^*$, which is made up of three parts: h_D, h_{2k-1} and h_{2k}. All of them are bijections.
$h_D : D \to D$ is a bijection. (No problem.)
$h_{2k} : (A_{2k} - A_{2k+1}) \to (A_{2k} - A_{2k+1})$ is a bijection. (No problem.)
$h_{2k-1} : (A_{2k-1} - A_{2k}) \to (A_{2k+1} - A_{2k+2})$ is a bijection. (Hint: $g(f(A_{2k-1} - A_{2k})) = g(B_{2k} - B_{2k+1}) = A_{2k+1} - A_{2k+2}$)
Now, Let $h = g^{-1} \circ h_A$. $h : A \to B$ is a bijection.

The $\leqslant$ relation on cardinalities is antisymmetric. I.e. if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.

2 Mathematical Induction

2.1 A flawed definition of $\mathbb{N}$

I will not talk about it. The only thing you need to know is that Mathematical Induction is valid.

2.2 Induction Arguments

Let be $P(n)$ be a property. We can show that $P(n)$ holds for all $n \in \mathbb{N}$ using the following argument structure:
1. Show that $P(0)$ holds
2. Show that for arbitrary $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$, i.e. if $P(n)$ holds, then so does $P(n+1)$.

2.3 Strong Induction

An argument by strong induction that shows that a property $A(n)$ holds for all $n \in \mathbb{N}$ with $n \geqslant n_0$ proceeds as follows:
(i) Show that $A(n_0)$ holds
(ii) Show that for all $n \geqslant n_0$, if for all $n_0 \leqslant k \leqslant n$, $A(k)$ holds, then $A(n+1)$ holds
(iii) Conclude that for all $n \in \mathbb{N}$ with $n \geqslant n0$, $A(n)$ holds

2.4 Principle of Structural Induction

Let B be a set and let $C_1, ..., C_n$ be construction rules. Let A be recursively defined to be the $\subseteq$-least set such that $B \subseteq A$ and A is closed under the rules $C_1, ..., C_n$. Let $P(x)$ be a property. If
(i) for all $b \in B$, $P(b)$ holds
(ii) for all $a_1, ..., a_m$ and c and $1 \leqslant i \leqslant n$, if $P(a_1), ..., P(a_m)$ all hold and c is obtained from $a_1, ..., a_m$ by a single application of the rule C_i, then $P(c)$ holds
Then P(x) holds for every element of A.

I will just give you a rough idea of Structural Induction.
A is a set that is hard to write in an explicit form like $A = \{x|P(x)\}$. However, we know a set $B \subseteq A$, and all the elements in B are clear. We also know how to find other elements in A by applying some rules $C_1, C_2, \cdots, C_n$ to the elements in B. Meanwhile, we know that all the elements in A can be found by applying these rules to the elements in B. So, how to prove that $P(x)$ holds for all $x \in A$. We first have to prove that $P(x)$ holds for all $x \in B$. Next, we have to prove that $P(c)$ holds, where $c \notin B$. c can be obtained from $a_1, ..., a_m$ by a *single* application of the rule C_i. Therefore, we have to prove that if $P(a_1), ..., P(a_m)$ all hold, then $P(c)$ hold. How to prove $P(a_i)$? We just need to repeat what we have done before. We have to find a method to obtain a_i by a *single* application of the rule C_i to $b_1, b_2, \cdots b_k$, and $P(b_1), P(b_1), \cdots P(b_k)$ all hold. This process of repetition will finally stop, because all elements in A can be obtained from elements in B.

2.5 Exercise

Show that

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$$
$$B_1 \supseteq B_2 \supseteq B_3 \supseteq B_4 \supseteq \cdots$$

in the proof II of Tarski-Knaster Theorem.
Show that $(a+b)^n \geqslant a^n + na^{n-1}b$. $n \in \mathbb{Z}^+$, $a, b \in \mathbb{R}^+$
Show that

$$\left(\frac{a_1 + a_2 + \cdots a_n}{n}\right)^n \geqslant a_1 a_2 \cdots a_n$$

$n \in \mathbb{Z}^+$, $a_1, a_2, \cdots, a_n \in \mathbb{R}^+$
Hint. Let $s = a_1 + a_2 + \cdots a_k$.

$$\left(\frac{a_1 + \cdots a_k + a_{k+1}}{k+1}\right)^{k+1} = \left(\frac{s}{k} + \frac{ka_{k+1} - s}{k(k+1)}\right)^{k+1} \geqslant \left(\frac{s}{k}\right)^{k+1} + (k+1)\left(\frac{s}{k}\right)^k \left(\frac{ka_{k+1} - s}{k(k+1)}\right)$$

2.6 Prepare For Your Midterm Exam