Discrete Mathematics Review2
PART I

Liu Ziwei

Fall Term 2019

refer to Qiu Tianyu,Yan xinyu

Contents

Induction

- 1. Principle of Induction (p120-p121)
- Let P(n) be a property, $n_0 \in N$ we can show P(n) holds for all $n_0 \in N(n \geq n_0, n \in N)$ using the following argument structure:

  1. Show that P(0)(or P(n0)) holds.

  2. Show that for arbitrary $n_0 \in N(n \geq n_0, n \in N)$,$P(n) \Rightarrow P(n+1)$

then, it follows for $n_0 \in N(n \geq n_0, n \in N)$,P(n) holds.

Theorem: For all $n \in N, n \geq 1$, $\sum_{k=1}^{n}(2k-1) = n^2$

proof: induction

Induction

- 2. Link between induction and the well-orderedness of (N;≤) (p124)
- Well-order: existence of least element
- Correctness of Principle of Induction (proof by contradiction): for all
- $n_0 \in N$ with $n \geq n_0$

1. Show that $P(n_0)$ holds
2. Suppose that $\{n \in \mathbb{N} \mid n \geq n_0 \wedge \neg P(n)\}$ is nonempty and let n' be the least element of this set
3. Let $m \geq n_0$ be such that $n' = m + 1$
4. Show that the fact that $P(m)$ holds implies that $P(n')$ holds, thus obtaining a contradiction

Induction

Theorem:
Let $(L, \preceq)$ be a lattice. If $X \subseteq L$ is finite with $|X| \geq 2$, then X has a least upper bound.
proof: induction, y $\vee$ xm

3. Strong Induction (p126)

An argument by strong induction that shows that a property A(n) holds for all $n \in N (n \geq n_0)$ proceeds as follows:

1. Show that $A(n_0)$ holds.
2. Show that for all $n \geq n_0$, if for all $n_0 \leq k \leq n$, A(k) holds, then A(n + 1) holds

Conclude that for all $n \in N$ with $n \geq n_0$, A(n) holds.

Recursive Definitions

Theorem:

For all $n \in \mathbb{N}$ with $n \geq 2$, n is either prime or the product of primes.

proof: strong induction

4. Recursive Definitions (p128-130)

  1. Initial value

  2. A rule that allows us to obtain the value of f (n + 1) from the values of f (n), f (n -1); ...(General: Construction Rules: C1,C2..Cn)

Recursive Definitions

5. Recursively Defined Sets and Structural Induction (p131-133)

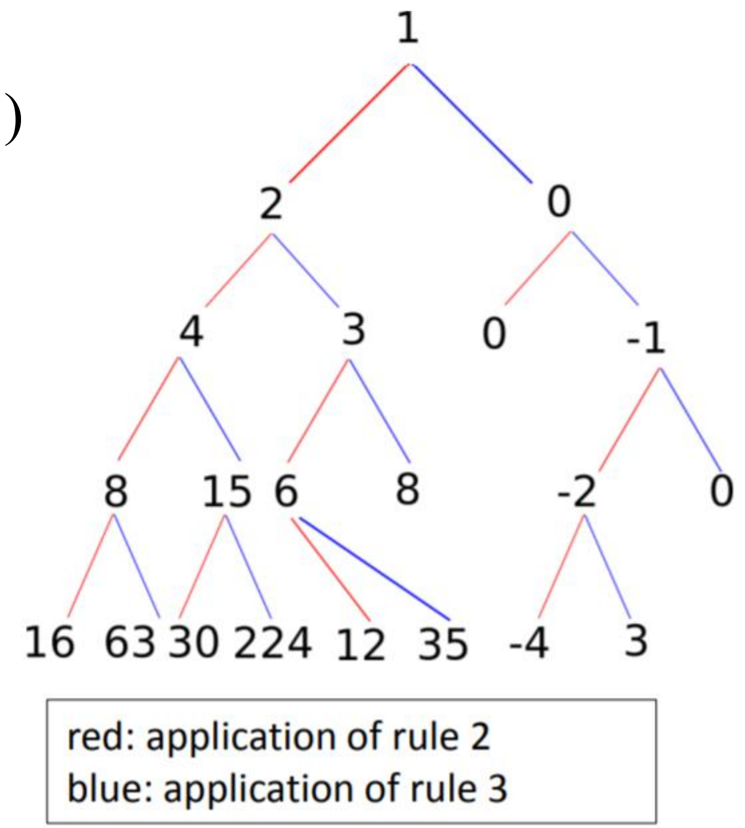
e.g.

1. $1 \in S$
2. if $x \in S$, then $2x \in S$
3. if $x \in S$, then $(x^2 - 1) \in S$

Recursively defined set is a set S that is defined as follows:

1. All members of a given set G are in S.
2. For some given functions f_k, whenever $s \in S$, then $f_k(s) \in S$ for all k.

*In a recursively defined set, there is no natural linear order- no "first", "second", etc.



Recursive Definitions

5. Recursively Defined Sets and Structural Induction (p131-133)

Structural Induction is a way of proving that all elements of a recursively defined set have a certain property.

Principle of Structural Induction: Let B be a set and let $C_1, \ldots, C_n$ be construction rules. Let A be recursively defined to be the $\subseteq$-least set such that $B \subseteq A$ and A is closed under the rules $C_1, \ldots, C_n$. Let $P(x)$ be a property. If

(i) for all $b \in B$, $P(b)$ holds

(ii) for all $a_1, \ldots a_m$ and c and $1 \leq i \leq n$, if $P(a_1), \ldots, P(a_m)$ all hold and c is obtained from $a_1, \ldots, a_m$ by a single application of the rule C_i, then $P(c)$ holds

Then $P(x)$ holds for every element of A.

Counting and Cardinality

1. Subsets of size k (p136-137)

A is a finite set, 0≤k≤|A|, $k \notin N, n \in N \setminus \{0\}$

1. $\mathscr{P}_k(A) = \{x \in \mathscr{P}(A) \mid |x| = k\}$

The collection of subsets whose cardinality is k of A.

2. [n] ={0,1,....,n-1} [0]=∅

3. cardinality of the $\mathscr{P}_k([n])$: $\binom{n}{k}$

Counting and Cardinality

2. Pascal's Triangle (p138-139)

Lemma

For all $n \in \mathbb{N}$ *and for all* $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$

proof: F(x) = [n]\x bijection

Theorem

For all $n \in \mathbb{N}$ *with* $n \geq 1$ *and for all* $0 < k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

proof: $A = \{x \cup \{n\} \mid x \in \mathscr{P}_{k-1}([n])\}$ and $B = \mathscr{P}_k([n])$

Counting and Cardinality

2. Binomial Theorem (p140-p142)

Theorem

(Binomial Theorem) For all $n \in \mathbb{N}$ with $n \geq 1$ and for all numbers x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

proof: induction   n=1; n->n+1

Corollary $(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$

Corollary $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Counting and Cardinality

3. Other finite sets (p143-p144)

Theorem

$$|\mathscr{P}_n([2n])| = \sum_{k=0}^{n} \binom{n}{k}^2$$

proof: $(1+x)^n(1+x)^n = (1+x)^{2n}$

$$\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) = \sum_{k=0}^{2n}\binom{2n}{k}x^k \quad ; \quad \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

Theorem

$$|\mathcal{P}([n])| = 2^n$$

proof: $|\mathscr{P}([n])| = \sum_{k=0}^{n} |\mathscr{P}_k([n])| = \sum_{k=0}^{n}\binom{n}{k} = 2^n$ /

For every number in [n], we can either put it into A or not.

Counting and Cardinality

3. Counting (p146-p147)

Theorem

Let $n, r \in \mathbb{N}$. The number of solutions to the equation $x_1 + \cdots + x_n = r$ with $x_1, \ldots, x_n \in \mathbb{N}$ is

$$\binom{n+r-1}{r}$$

proof: allocate balls (n person,n+r balls, at least one per person) /

$$A = \{(x_1, \cdots, x_n) \in \mathbb{N}^n | x_1 + \cdots + x_n = r\}$$

injective

$$F : A \longrightarrow \mathcal{P}_{n-1}([n+r-1])$$

surjective

$$F(x_1, \ldots, x_n) = \left\{x_1, x_1 + x_2 + 1, \ldots, n-2+\sum_{i=1}^{n-1} x_i\right\}$$

$\{y_1 < \cdots < y_{n-1}\} \in \mathscr{P}_{n-1}([n+r-1])$, then by letting $x_1 = y_1$, $x_2 = y_2 - (x_1 + 1), \ldots, x_{n-1} = y_{n-1} - (x_1 + \cdots + x_{n-2} + n - 2)$ and $x_n = r - (x_1 + \cdots + x_{n-1})$ we get an element

Counting and Cardinality

3. Counting (p148-p149)

Theorem

The number of ways of selecting r objects from n objects when the order does not matter and repetitions are allowed is

$$\binom{n+r-1}{r}$$

proof: equivalent to $x_1 + \cdots x_n = r$, where $x_1 \cdots x_n \in N$

Theorem

The number of bijections from [n] to [n] is n!. I.e.

$$|\{f \mid f : [n] \longrightarrow [n] \text{ is a bijection}\}| = n!$$

proof: bijection from [n] to [n] can be mapped to a permutation of [n].

Counting and Cardinality

3. Counting (p148-p149)

Theorem

Let $n \in \mathbb{N}$ and let $0 \leq k \leq n$. The number of ordered k-tuples of distinct elements of $[n]$ is

$$\binom{n}{k} k!$$

I.e. $|\{(x_1, \ldots, x_k) \in [n]^k \mid \textit{for all } 0 \leq i < j \leq k, x_i \neq x_j\}| = \binom{n}{k} k!$

proof： 1.Choose k different numbers from [n]

2. For chosen k numbers consider its permutations.

Permutation & Combination

1. Permutation & Combination with/without repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutation & Combination

1. distinguishable Objects + distinguishable Boxs

The number of ways to distribute n distinguishable objects into k distinguishable boxes,

where there are

n_1 distinguishable objects in box 1 , n_2 distinguishable objects in box 2... , and n_k distinguishable objects in box k

$$C_n^{n_1} \cdot C_{n-n_1}^{n_2} \cdot \ldots \cdot C_{n_k}^{n_k}$$
$$= \frac{n!}{n_1! n_2! \cdots n_k!}$$
$$= \frac{P_n^n}{P_{n_1}^{n_1} \cdot P_{n_2}^{n_2} \cdot \ldots \cdot P_{n_k}^{n_k}}$$

Permutation & Combination

2. indistinguishable Objects + distinguishable Boxs

The number of ways to distribute n indistinguishable objects into k distinguishable boxes

$$\binom{n+k-1}{k-1}$$

3. distinguishable Objects + indistinguishable Boxs

$\sum_{j=1}^{k} S(n,j)$ Stirling Numbers of the second kind （no need to know)

$$\text{S(k,n)} = \frac{1}{n!}\sum_{i=0}^{n-1}(-1)^i n_{C_i}(n-i)^k$$

$$= \frac{1}{n!}[n_{C_0}(n-0)^k - n_{C_1}(n-1)^k + n_{C_2}(n-2)^k + \cdots + (-1)^{n-1} n_{C_{n-1}}(1)^k]$$

4. indistinguishable Objects + indistinguishable Boxs

Enumeration

Permutation & Combination

Distribution of		How many balls boxes can contain			
k Balls	into n Boxes	No Restrictions	≤ 1 (At most one)	≥ 1 (At least one)	= 1 (Exactly one)
Distinct	Distinct	n^k	${}^nP_k$	$S(k,n) \times n!$	${}^nP_n = n!$ if $k = n$ 0 if $k \neq n$
Identical	Distinct	${}^{(k+n-1)}C_{(n-1)}$	${}^nC_k$	${}^{(k-1)}C_{(n-1)}$	1 if $k = n$ 0 if $k \neq n$
Distinct	Identical	$\sum_{i=1}^{n} S(k,i)$	1 if $k \leq n$ 0 if $k > n$	$S(k,n)$	1 if $k = n$ 0 if $k \neq n$
Identical	Identical	$\sum_{i=1}^{n} P(k, i)$	1 if $k \leq n$ 0 if $k > n$	$P(k, n)$	1 if $k = n$ 0 if $k \neq n$

Permutation & Combination

1. Tool box

1. $\binom{n}{r} = \binom{n}{n-r}$

2. $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$

3. $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$

4. $\binom{n}{r}\binom{r}{m} = \binom{n}{m}\binom{n-m}{r-m}$

5. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

6. $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

7. $\binom{n}{r} + \binom{n+1}{r} + \cdots + \binom{n+k}{r} = \binom{n+k-1}{r+1} - \binom{n}{r+1}$

8. $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$