Q1. Prove that a set A is (Dedekind) infinite if an only if there exists an injective function $f: \mathbb{N} \longrightarrow A$.

(3 marks)

Q2. Let (L, \preceq) be a lattice and let $a, b \in L$. Prove that the following are equivalent:

- (i) $a \leq b$
- (ii) $a \lor b = b$
- (iii) $a \wedge b = a$

(3 marks)

Q3. Let (L, \preceq) be a lattice. Prove that for all $a, b, c \in L$,

$$(a \lor b) \lor c = a \lor (b \lor c)$$

(2 marks)

Q4. Without using Cantor's Pairing Function, prove that $\mathbb{N} \times \mathbb{N}$ is countable. (2 marks)

Q5. For all $n \in \mathbb{N}$, use [n] to denote the set of predecessors of n in the natural numbers with the usual ordering, i.e. $[0] = \emptyset$ and for all $n \ge 1$, $[n] = \{0, \dots, n-1\}$. Define

$$S = \{ f \mid (\exists n \in \mathbb{N}) (f \text{ is a function with dom } f = [n] \text{ and ran } f \subseteq \mathbb{N}) \}$$

- (i) Use the fact that every natural number has a unique factorisation into primes to show that S is countable. Is $|S| = |\mathbb{N}|$?
- (ii) Define $\leq_1 \subseteq S \times S$ by: $f \leq_1 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$, and $A = [n] \cap [m]$ if and only if

$$(\exists i \in A)(f(i) < g(i) \land (\forall j < i)(f(j) = g(j))) \text{ or } (\forall i \in A)(f(i) = g(i)) \land ([n] \subseteq [m]),$$

where \leq and < are the usual orders on \mathbb{N} . Is (S, \leq_1) a partial order? Is (S, \leq_1) a linear order? Is (S, \leq_1) chain complete? Is (S, \leq_1) a lattice? Is (S, \leq_1) a well-order? Prove your answers.

(iii) Define $\leq_2 \subseteq S \times S$ by: $f \leq_2 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$ if and only if

$$[n] \subseteq [m]$$
 and $(\forall i \in [n])(f(i) \leq g(i))$

Is (S, \leq_2) a partial order? Is (S, \leq_2) chain complete? Is (S, \leq_2) a lattice? Is (S, \leq_2) a linear order? Prove your answers.

(7 marks)

Q6. Give an explicit formula that defines a bijection between \mathbb{N} and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. You do not have to prove that this formula works! (1 mark)

Q7. Every nonzero rational number has a unique representation in the form $\frac{(-1)^k a}{b}$ where $k \in \{0,1\}$ and $a,b \in \mathbb{N}$ with $a,b \neq 0$ and the greatest common divisor of a and b is 1. Use this to show that \mathbb{Q} is countable.

(3 marks)

Q8. Let $\pi(x,y)$ be Cantor's pairing function. Find $x,y \in \mathbb{N}$ such that $\pi(x,y) = 223$. You may do this question however you wish. (1 mark)

Q9*. Prove that $|\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})|$. (1 mark)

Q10. Prove that $|\mathbb{N}| < |\mathbb{R}|$. (Hint: Start with a bijection $f : \mathbb{N} \longrightarrow \mathbb{R}$ and try to construct a real that can not be in the range of f. There may also be a proof that uses continued fractions and Cantor's Theorem, but I have not thought about that too deeply.)

(3 marks)