Relations and Orders

September 24, 2019

1 Relations

1.1 Partitions

Let A be a non-empty set, and S be a set composed of subsets of A.

$$S = \{S_1, S_2, \cdots, S_m\}, S_i \subset A.$$

S is a *partition* of A if

1. $S_i \neq \emptyset$
2. $S_i \cap S_j = \emptyset, i \neq j$
3. $\bigcup_{i=1}^{m} S_i = A$

S_i (the elements of S) can be called *blocks*.

A *partition* of a set A is a collection of disjoint nonempty subsets of A that have A as their union.

1.2 Relations

A set R is called a *relation* if R only contains ordered pairs.

Let R be a relation. We define the *domain* of R to be the set

$$\text{dom } R = \{x | \exists y((x, y) \in R)\}$$

And we define the *range* of R to be the set

$$\text{ran } R = \{y | \exists x((x, y) \in R)\}$$

The *field* of R is the set Ran $R = \text{ran } R \cup \text{dom } R$.

Sometimes we write aRb instead of $(a, b) \in R$.

1.3 Definitions

We say that a relation R on a set M is

- *reflexive* if for all $a \in M, (a, a) \in R$
- *symmetric* if for all $a, b \in M$, if $(a, b) \in R$, then $(b, a) \in R$.
- *antisymmetric* if for all $a, b \in M$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.
- *asymmetric* if for all $a, b \in M$, if $(a, b) \in R$, then $(b, a) \notin R$.
- *transitive* if for all $a, b, c \in M$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

1.4 Equivalence Relation and Equivalence class

Let R be a relation on a set M. If R is *reflexive*, *symmetric* and *transitive*, then we say that R is an *equivalence relation* on M. If R is an equivalence relation on M and $a \in M$, then define the equivalence class of a to be

$$[a]_R = \{b \in M | (a, b) \in R\}$$

1.5 Quotient Set

Let R be an equivalence relation on a set A. S is called a *quotient set* of A induced by R if S is the set of all equivalence classes induced by R. S can be denoted as A/R. I.e.

$$A/R = \{[a]_R | a \in A\}.$$

where

$$[a]_R = \{b \in A | aRb\}.$$

1.6 Theorems

Here are some theorems about Equivalence Relations, Equivalence Classes, and Quotient Sets.

i)Let R be an equivalence relation on a non-empty set A. For all $a, b \in A$, $(a, b) \in R$ if and only if $[a]_R = [b]_R$. (You can prove it by yourself.)

ii)Let R be an equivalence relation on a non-empty set A. The quotient set A/R is a partition of A. (A partition can be induced from an equivalence relation.)

iii)Let $S = \{S_1, S_2, \cdots, S_m\}, S_i \subset S$ be a partition of a non-empty set A. Let $R = \{(a, b) | \exists i(a \in S_i \wedge b \in S_i)\}$. Then R is an equivalence relation on A. (An equivalence relation can be induced from a partition.)

iv)Let R_1 and R_2 be two equivalence relations on a non-empty set A. $R_1 = R_2$ if and only if $A/R_1 = A/R_2$.

1.7 Summary

- An equivalence relation corresponds to a partition. The equivalence classes are the blocks of a partition.
- A partition is a quotient set. A quotient set is a partition.
- Different partitions corresponds to different equivalence relations.

1.8 Exercise

How many different equivalence relations are there on a set $A = \{1, 2, 3, 4\}$. (15)

2 Orders

2.1 Definitions

Let R be a relation on a set M. If R is reflexive, antisymmetric and transitive, then R is called a *partial order*. We often write a partial order together with its domain, (M, R), and say that (M, R) is a *partially ordered set* or *poset*. (A partial order is often written as $\preceq$.)

Let R be a relation on a set M. If R is asymmetric and transitive, then R is called a *strict partial order*. We say that (M, R) is a *strict partially ordered set* or *strict poset*. (A strict partial order is often written as $\prec$.)

Let (M, R) be a partially ordered set. If for all $x, y \in M$, $(x, y) \in R$ or $(y, x) \in R$, then R is called a *linear order* or *total order*, and we say that (M, R) is a *linearly ordered set* or *totally ordered set*. (Any two elements in a linearly ordered set can be linked with $\preceq$)

Let R be a linear order on a set M. We say that R is a well-order if for all $A \subseteq M$, if $A \neq \emptyset$, then there exists $x \in A$, such that for all $y \in A$, if $(y, x) \in R$ then $y = x$. We also say that (M, R) is a *well-ordered set*. (This says that every nonempty $A \subseteq M$ has a least element according to R.)

If R is a linear order on M and M is finite then R is a well-order.

2.2 Greatest and Least Element

a is the *greatest element* of the poset $(S, \preceq)$ if $b \preceq a$ for all $b \in S$.
a is the *least element* of the poset $(S, \preceq)$ if $a \preceq b$ for all $b \in S$.

2.3 Maximal and Minimal Elements

An element of a poset is called *maximal* if it is not less than any element of the poset. That is, a is *maximal* in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \preceq b \wedge a \neq b$. Similarly, an element of a poset is called *minimal* if it is not greater than any element of the poset. That is, a is *minimal* if there is no element $b \in S$ such that $b \preceq a \wedge a \neq b$.

2.4 Upper Bound and Lower Bound

Let $(L, \preceq)$ be a poset and let $S \subseteq L$. We say that $x \in L$ is an *upper bound* on S if for all $y \in S$, $y \preceq x$. We say that $x \in L$ is a *lower bound* on S if for all $y \in S, x \preceq y$.

Let $(L, \preceq)$ be a poset and let $S \subseteq L$. We say that $x \in L$ is a *least upper bound* **(l.u.b.)** on S if x is an upper bound on S and for all y, if y is an upper bound on S, then $x \preceq y$. We say that $x \in L$ is a *greatest lower bound* **(g.l.b.)** on S if x is a lower bound on S and for all y, if y is a lower bound on S then $y \preceq x$.

2.5 Theorems

1. The greatest (least) element is unique when it exists.

2. The greatest (least) element is always the maximal (minimal) element. However, the maximal (minimal) elements are **not** always the greatest (least) elements.

3. If $(S, \preceq)$ is a finite poset, then the maximal (minimal) elements exist but are not unique.

4. Let $(L, \preceq)$ be a poset and let $S \subseteq L$. If the least upper bound of S exists, then the least upper bound of S is unique. However, S can have several upper bounds.

2.6 Exercise

For the poset $S = (\{2, 4, 5, 10, 12, 20, 25\}, |)$,

1. Determine whether the poset S has a greatest element and a least element. (No.)
2. Which elements of the poset are maximal, and which are minimal? (Maximal: 12,20,25. Minimal: 2,5.)
3. Find the lower and upper bounds of the subset $\{4, 10, 20\}$. (Lower bounds: 2. Upper bounds: 20.)

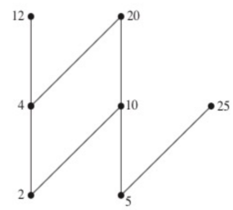

FIGURE 5 The Hasse Diagram of a Poset.

2.7 Lattices

Let $(L, \preceq)$ be a poset. We say that $(L, \preceq)$ is a *lattice* if for all $x, y \in L$, the set $\{x, y\}$ has both a l.u.b. and a g.l.b. If $(L, \preceq)$ is a lattice and $x, y \in L$, then we write $x \vee y$ for the l.u.b. of $\{x, y\}$ and $x \wedge y$ for the g.l.b. of $\{x, y\}$.

In fact, if $\preceq$ is a linear order on M, then $(M, \preceq)$ is a lattice.

Let $(L, \preceq)$ be a lattice. We say that $(L, \preceq)$ is complete if for every $X \subseteq L$, X has both a least upper bound and a greatest lower bound. If $(L, \preceq)$ is a complete lattice and $X \subseteq L$, then we use $\bigvee X$ to denote the least upper bound of X and $\bigwedge X$ to denote the greatest lower bound of X.

If $(L, \preceq)$ is a complete lattice, then $(L, \preceq)$ has a maximal element given by $\bigvee L$. This maximal element is sometimes denoted $\mathbf{1}$.

If $(L, \preceq)$ is a complete lattice, then $(L, \preceq)$ has a minimal element given by $\bigwedge L$. This minimal element is sometimes denoted $\mathbf{0}$.

2.8 Properties of Lattices (Order-Preserving)

Let $(L, \preceq)$ be a lattice, and $a, b, c \in L$.

1. (a) $a \preceq a \vee b, a \wedge b \preceq a$.

   (b) If $a \preceq b, c \preceq d$, then $a \wedge c \preceq b \wedge d, a \vee c \preceq b \vee d$.
   $a \wedge c \preceq a \preceq b$, and $a \wedge b \preceq b \preceq d$ ($\preceq$ is transitive.)
   $a \wedge c$ is a lower bound of $\{b, d\}$, and thus, $a \wedge c \preceq b \wedge d$

   (c) If $b \preceq c$, then $a \wedge b \preceq a \wedge c, a \vee b \preceq a \vee c$.

2. $a \vee b = b \vee a, a \wedge b = b \wedge a$.

3. $a \vee a = a, a \wedge a = a$.

4. $(a \vee b) \vee c = a \vee (b \vee c), (a \wedge b) \wedge c = a \wedge (b \wedge c)$.

5. $a \vee (a \wedge b) = a, a \wedge (a \vee b) = a$.

6. $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c), (a \wedge b) \vee (a \wedge c) \preceq a \wedge (b \vee c)$.

7. $a \preceq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$

2.9 Exercises

Let $(L, \preceq)$ be a lattice, and $a, b, c \in L$. Show that

1. $(a \vee b) \vee c = a \vee (b \vee c)$.

2. $a \vee (a \wedge b) = a$.

3. $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$.

① Association: $(a \vee b) \vee c = a \vee (b \vee c)$

$$\left.\begin{array}{l} \left.\begin{array}{l} a \leq a \vee (b \vee c) \\ b \leq b \vee c \leq a \vee (b \vee c) \end{array}\right\} \Rightarrow a \vee b \leq a \vee (b \vee c) \\ c \leq b \vee c \leq a \vee (b \vee c) \end{array}\right\} \Rightarrow (a \vee b) \vee c \leq a \vee (b \vee c)$$

Similarly, $a \vee (b \vee c) \leq (a \vee b) \vee c$

Therefore, $(a \vee b) \vee c = a \vee (b \vee c)$

② Absorption: $a \vee (a \wedge b) = a$

$$\left.\begin{array}{l} \left.\begin{array}{l} a \leq a \\ a \wedge b \leq a \end{array}\right\} \Rightarrow a \vee (a \wedge b) \leq a \\ a \leq a \vee (a \wedge b) \end{array}\right\} \Rightarrow a \vee (a \wedge b) = a$$

③ Distribution: $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$$\left.\begin{array}{l} \left.\begin{array}{l} a \leq a \vee b \\ a \leq a \vee c \end{array}\right\} \Rightarrow a \leq (a \vee b) \wedge (a \vee c) \\ \left.\begin{array}{l} b \leq a \vee b \\ c \leq a \vee c \end{array}\right\} \Rightarrow b \wedge c \leq (a \vee b) \wedge (a \vee c) \end{array}\right\} \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

2.10 Chain Complete Posets

Let $(P, \preceq)$ be a partial order. We say that $X \subseteq P$ is a chain if $(X, \preceq)$ is a linear order.

Let $(P, \preceq)$ be a partial order. We say that $(P, \preceq)$ is chain complete if for all $X \subseteq P$, if X is a chain then X has a least upper bound.

Note that this definition ensures the existence of a unique least element. Since $\emptyset \subseteq P$ and $\emptyset$ is a chain, and all $a \in P$ is the upper bound of $\emptyset$, therefore, P must has a unique least element, which is also the l.u.b of $\emptyset$.

Every complete lattice is a chain complete poset.