

Q1. Let (L, \preceq) be a complete lattice and let $f : L \longrightarrow L$ be an order-preserving function.

- (i) Let $a, b \in L$ with $a \preceq b$. Define

$$[a, b] = \{x \in L \mid a \preceq x \preceq b\}$$

Show that $([a, b], \preceq)$ is a complete lattice.

- (ii) Consider $X = \{x \in L \mid f(x) = x\}$. The Tarski-Knaster Theorem shows that there exists $a \in X$ such that for all $x \in X$, $a \preceq x$. Let $S \subseteq X$. Since (L, \preceq) is a complete lattice, S has a g.l.b.

$$s = \bigwedge S$$

in L . Show that $a \preceq s$.

- (iii) By considering f restricted to $[a, s]$, for s and a defined in (ii), show that S has a g.l.b. in X . Note that it is not necessarily the case that the g.l.b. of S in X is the same as the g.l.b. of S in L .
- (iv) Show that (X, \preceq) is a complete lattice.
- (v) Define an order-preserving function $f : [0, 1] \longrightarrow [0, 1]$ such that the set of fixed points of f endowed with the usual order (\leq) on \mathbb{R} is a linear order with exactly 4 elements.

(8 marks)

Q2. Define $G : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ by: for all $(n, m) \in \mathbb{N} \times \mathbb{N}$,

$$G((n, m)) = (n + 1, 2^m)$$

Let $X = \{R \in \mathcal{P}(\mathbb{N} \times \mathbb{N}) \mid (0, 0) \in R\}$. Define $F : X \longrightarrow X$ by: for all $R \in X$, $F(R) = R \cup G^{\omega}R$.

- (i) Show that (X, \subseteq) is a complete lattice.
- (ii) Prove that F is an order preserving function on (X, \subseteq) .

By the Tarski-Knaster Theorem, we know that F must have a least fixed point in (X, \subseteq) . Let f be the least fixed point of F .

- (iii) Prove that f is a function with $\text{dom}(f) = \mathbb{N}$.
- (iv) Prove that for all $n \in \mathbb{N}$ with $n \geq 2$, $f(n)$ is even.
- (v) Prove that f is injective.

(7 marks)

Q3. Prove that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

(2 marks)

Q4. Prove that for all $n \geq 1$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$$

(1 mark)

Q5. Prove that for all $m \geq 0$ and for all $n \geq 1$,

$$\sum_{k=0}^n \binom{m+k}{m} = \binom{m+n+1}{m+1}$$

(2 marks)

Q6. What is coefficient of $x^{31}y^4$ in the expansion of $((x+y)^8 + y)^7$?

(2 mark)

Q7. Find the greatest term in the expansion of $(1+4x)^9$ when $x = \frac{1}{3}$.

(2 marks)

Q8. Find the number solutions to the equation $x_1+x_2+x_3+x_4 \leq 6$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.

(2 mark)

Q9*. Use the binomial theorem to evaluate $(1.2)^5$.

(1 marks)

Q10. This question refers to \mathbb{N}_{def} discussed in lectures.

(i) Prove that if $n \in \mathbb{N}_{\text{def}}$ with $n \neq \emptyset$, then there exists $m \in \mathbb{N}_{\text{def}}$ such that

$$n = S(m)$$

For (ii) you may assume the definition of $+$ on \mathbb{N}_{def} given in lectures ensures that \leq (defined in the first lecture) is a linear ordering of \mathbb{N}_{def} .

(ii) Prove that the usual \leq order on \mathbb{N}_{def} is a well order.

(6 marks)