Arguments

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1 Arguments

An *argument* is a finite sequence of propositions. All propositions except for the final statement are called *premises* while the final statement is called the *conclusion*. We say that an argument is *valid* if the truth of all premises implies the truth of the conclusion.

From the definition of an argument it is clear that an argument consisting of a sequence of premises $P_1, ..., P_n$ and a conclusion C is valid of and only if

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Rightarrow C$$

is a tautology.

1.1 Rules of Inference

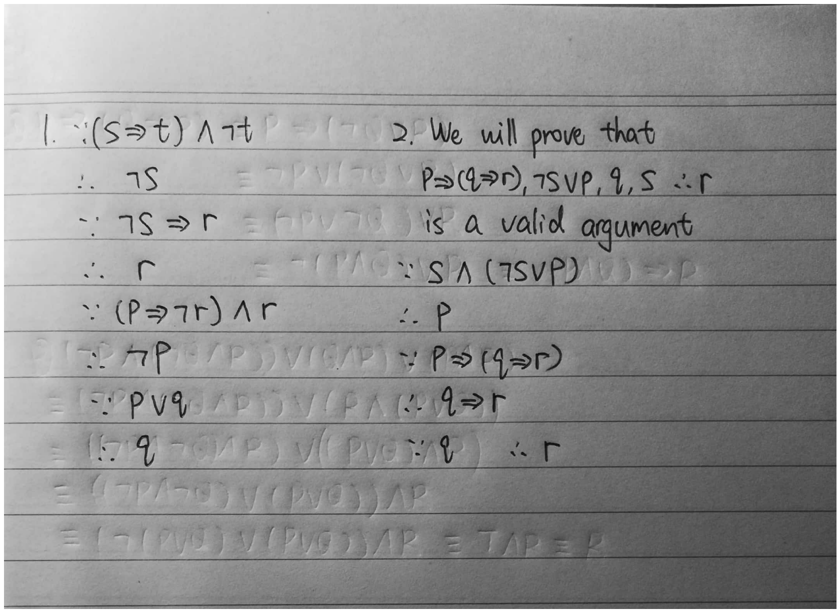
1. $A \quad \therefore A \vee B$
2. $A \wedge B \quad \therefore A$
3. $(A \Rightarrow B) \wedge A \quad \therefore B$
4. $(A \Rightarrow B) \wedge \neg B \quad \therefore \neg A$
5. $(A \vee B) \wedge \neg B \quad \therefore A$
6. $(A \Rightarrow B) \wedge (B \Rightarrow C) \quad \therefore A \Rightarrow C$
7. $(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \quad \therefore A \Leftrightarrow C$
8. $(A \Rightarrow B) \wedge (C \Rightarrow D) \wedge (A \vee B) \quad \therefore C \vee D$

Other useful rules:

1. Premises can be used anywhere in the proof.
2. Any conclusions that have been made can be used as premises in the follow-up steps of the proof.
3. If $(A_1, A_2, \cdots, A_n, A \quad \therefore B)$ is a valid argument, then $(A_1, A_2, \cdots, A_n \quad \therefore A \Rightarrow B)$ is also valid. (If the conclusion that needs to be proved is $A \Rightarrow B$, then A can be used as a premise.)

1.2 Exercise

1. Show that ($p \vee q, p \Rightarrow \neg r, s \Rightarrow t, \neg s \Rightarrow r, \neg t \quad \therefore q$) is a valid argument.
2. Show that ($p \Rightarrow (q \Rightarrow r), \neg s \vee p, q \quad \therefore s \Rightarrow r$) is a valid argument.



1. $\because (s \Rightarrow t) \wedge \neg t$

$\therefore \neg s$

$\because \neg s \Rightarrow r$

$\therefore r$

$\because (p \Rightarrow \neg r) \wedge r$

$\therefore \neg p$

$\because p \vee q$

$\therefore q$

2. We will prove that

$p \Rightarrow (q \Rightarrow r), \neg s \vee p, q, s \therefore r$

is a valid argument

$\because s \wedge (\neg s \vee p)$

$\therefore p$

$\because p \Rightarrow (q \Rightarrow r)$

$\therefore q \Rightarrow r$

$\because q$

$\therefore r$

1.3 Predicate Logic

A *predicate* is a declaritive sentence involving variables. I.e. a statement involving variables such that when the variables a substituted with appropriate individuals we obtain a proposition. The *arity* of a predicate is the number of distinct variables appearing in the predicate.

Predicate logic consists of basic predicate variables $A, B, P, Q, \cdots$, variables $x, y, z, \cdots$, and constants $a, b, c, \cdots$. If A is, for example, a binary predicate, then we write A(x,y) to indicate that x and y are the variables appearing in A. The variables $x, y, z, \cdots$ take values from a universe called the *domain of discourse*. The constants $a, b, c, \cdots$ are specific individuals from the domian of discourse. Replacing all the variables by constants in a predicate or compound expression yields a declaritive statement (a proposition).

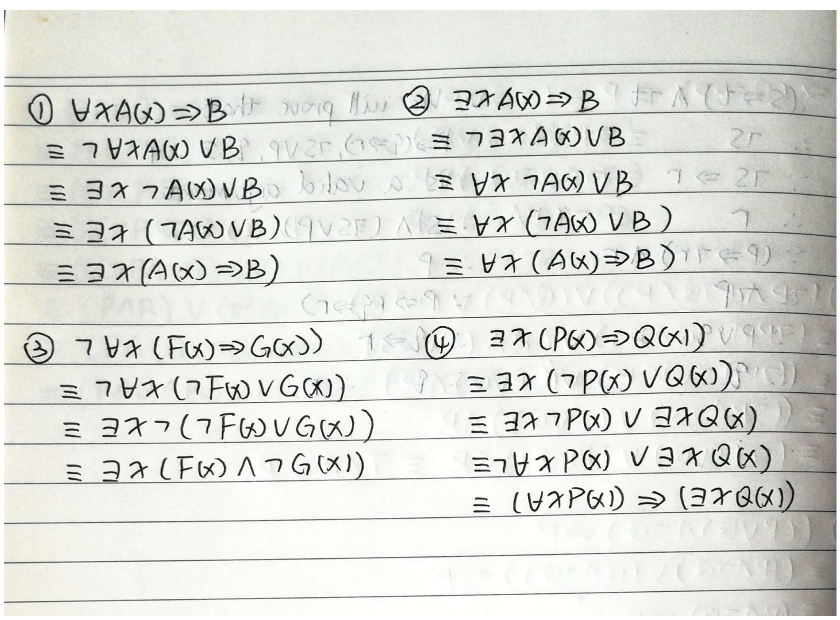
1.4 Logical Quantifier

Bounding variables in a compound expression using the *logical quantifier* $\forall$, that reads for all . . . , or the *logical quantifier* $\exists$, that reads there exists . . .

1. $\neg\forall x A(x) \equiv \exists x \neg A(x)$
2. $\neg\exists x A(x) \equiv \forall x \neg A(x)$
3. $\forall x A(x) \wedge B \equiv \forall x(A(x) \wedge B)$
4. $\forall x A(x) \vee B \equiv \forall x(A(x) \vee B)$
5. $\exists x A(x) \wedge B \equiv \exists x(A(x) \wedge B)$
6. $\exists x A(x) \vee B \equiv \exists x(A(x) \vee B)$
7. $\forall x(A(x) \wedge B(x)) \equiv \forall x A(x) \wedge \forall x B(x)$
8. $\exists x(A(x) \vee B(x)) \equiv \exists x A(x) \vee \exists x B(x)$

1.5 Exercise

1. $\forall x A(x) \Rightarrow B \equiv \exists x(A(x) \Rightarrow B)$
2. $\exists x A(x) \Rightarrow B \equiv \forall x(A(x) \Rightarrow B)$
3. $\neg\forall x(F(x) \Rightarrow G(x)) \equiv \exists x(F(x) \wedge \neg G(x))$
4. Show that $(\exists x(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology.



① $\forall x A(x) \Rightarrow B$
$\equiv \neg\forall x A(x) \vee B$
$\equiv \exists x \neg A(x) \vee B$
$\equiv \exists x (\neg A(x) \vee B)$
$\equiv \exists x (A(x) \Rightarrow B)$

② $\exists x A(x) \Rightarrow B$
$\equiv \neg\exists x A(x) \vee B$
$\equiv \forall x \neg A(x) \vee B$
$\equiv \forall x (\neg A(x) \vee B)$
$\equiv \forall x (A(x) \Rightarrow B)$

③ $\neg\forall x (F(x) \Rightarrow G(x))$
$\equiv \neg\forall x (\neg F(x) \vee G(x))$
$\equiv \exists x \neg(\neg F(x) \vee G(x))$
$\equiv \exists x (F(x) \wedge \neg G(x))$

④ $\exists x (P(x) \Rightarrow Q(x))$
$\equiv \exists x (\neg P(x) \vee Q(x))$
$\equiv \exists x \neg P(x) \vee \exists x Q(x)$
$\equiv \neg\forall x P(x) \vee \exists x Q(x)$
$\equiv (\forall x P(x)) \Rightarrow (\exists x Q(x))$

1.6 Rules of inference for quantified expressions

Universal Instantiation, Universal Generalisation, Existential Instantiation, Existential Generalisation.

Show that $(\exists x(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology.

First, we will prove $(\exists x(P(x) \Rightarrow Q(x))) \Rightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology. We only need to show that $\because \exists x(P(x) \Rightarrow Q(x)) \therefore (\forall x P(x)) \Rightarrow (\exists x Q(x))$ is a valid argument.

1. $\because \exists x(P(x) \Rightarrow Q(x))$
2. $\therefore P(a) \Rightarrow Q(a)$, where a is some (unknown) element of the domain of discourse.
3. $\because \forall x P(x)$
4. $\therefore P(a)$ Universal Instantiation of 3
5. $\therefore Q(a)$ Modus Ponens from 2 and 4
6. $\therefore \exists x Q(x)$ Existential Generalisation of 5

Next, we will prove $((\forall x P(x)) \Rightarrow (\exists x Q(x))) \Rightarrow (\exists x(P(x) \Rightarrow Q(x)))$ is a tautology. We only need to show that $\because (\forall x P(x)) \Rightarrow (\exists x Q(x)) \therefore \exists x(P(x) \Rightarrow Q(x))$ is a valid argument.

1. $\because (\forall x P(x)) \Rightarrow (\exists x Q(x))$
2. $\therefore P(a) \Rightarrow Q(a)$, where a is some (unknown) element of the domain of discourse.
3. $\therefore \exists x(P(x) \Rightarrow Q(x))$ Existential Generalisation of 2

1.7 A Confusing Problem

What's wrong with the following inference? The premise is $\forall x \exists y F(x, y)$.

1. $\forall x \exists y F(x, y)$
2. $\exists y F(t, y)$ (UI)
3. $F(t, c)$ (EI)
4. $\forall x F(x, c)$ (UG)
5. $\exists y \forall x F(x, y)$ (EG)

Step 3 is wrong. Actually, step 3 should be $F(t, c(t))$. We cannot use EI here.

1.8 Vacuously true

If the domain of the universal quantifier $\forall$ is the empty set $M = \emptyset$, then the statement $(\forall x \in M)A(x)$ is defined to be true regardless of the predicate $A(x)$. It is then said that $A(x)$ is vacuously true.

1.9 Valid and sound

Valid: If the truth of all premises implies the truth of the conclusion.
Sound: If, in addition to being valid, an argument has only true premises, we say that the argument is sound. In that case, its conclusion is true.