Ve203 Discrete Mathematics

Sample Exercises for the First Midterm Exam

The following exercises are sample exercises of a difficulty comparable to those found the actual first midterm exam. The exam will usually include of 7 to 8 such exercises to be completed in 100 minutes.

Exercise 1. Find $x, y \in \mathbb{Z}$ such that $24x + 138y = \gcd(24, 138)$.
(1 Mark)

Exercise 2. Find all solutions of $140x \equiv 133 \pmod{301}$.
(2 Marks)

Exercise 3. Calculate $3^{20} \bmod 99$.
(3 Marks)

Exercise 4. Find all solutions of $140x \equiv 133 \pmod{301}$.
(2 Marks)

Exercise 5. Let A, B, C be statements. Are the following tautologies:

$$\big((A \Rightarrow B) \Rightarrow C\big) \Leftrightarrow \big(A \Rightarrow (B \Rightarrow C)\big),$$
$$\big((A \Rightarrow B) \wedge (C \Rightarrow \neg B)\big) \Rightarrow (A \Rightarrow \neg C)?$$

Give proofs or counterexamples!
(2 + 2 Marks)

Exercise 6. Let A, B, C, D, E be statements. Prove that the argument

$$\begin{array}{rl} & A \Rightarrow C \\ & D \vee E \\ & \neg E \Rightarrow \neg B \\ & (\neg B \wedge D) \Rightarrow A \\ & \neg E \\ \hline \therefore & C \end{array}$$

is valid by succesively applying known rules of inference.
(3 Marks)

Exercise 7. Prove the following statement using induction in n:

$$\sum_{j=1}^{n} x^{n-j} y^{j-1} = \frac{x^n - y^n}{x - y}, \qquad\qquad x, y \in \mathbb{R},\ x \neq y,\ n \geq 1.$$

(4 Marks)

Exercise 8. We define the set $S \subset \mathbb{Z}^2$ by the following properties

- $(3,5) \in S$
- $(x,y) \in S \Rightarrow (x+2,y) \in S$
- $(x,y) \in S \Rightarrow (-x,y) \in S$
- $(x,y) \in S \Rightarrow (y,x) \in S$

Show that $S = T$, where

$$T = \{(x,y) \in \mathbb{Z}^2 \colon \underset{m,n\in\mathbb{Z}}{\exists} : (x,y) = (2m+1, 2n+1)\}.$$

Hint: show that $S \subset T$ and $T \subset S$.
(6 Marks)

Exercise 9.

i) Solve the system of congruences

$$x \equiv 2 \mod 3, \qquad x \equiv 5 \mod 7, \qquad x \equiv 6 \mod 8.$$

ii) Solve the congruence $x^2 \equiv 29 \mod 35$.

(4+4 Marks)

Exercise 10. Let M_q be an integer of the form $a^q - 1$, where a and q are natural numbers. M_q is called a *Mersenne number*. When M_q is prime and $a = 2$, M_q is called a *Mersenne prime*.

i) Prove that $(a-1) \mid (a^q - 1)$.

ii) Conclude that if M_q is prime then $a = 2$ or $q = 1$.

iii) Prove that if M_q is a Mersenne prime, then q is prime.

(2+2+3 Marks)