Logic

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1 Logic

1.1 Propositional Logic

A *proposition* is a declarative sentence. I.e. a statement that is either true (T) or false (F), but not both.

Connectives: $\neg$ (not), $\vee$ (disjunction), $\wedge$ (conjunction), $\Rightarrow$ (implication) and $\Leftrightarrow$ (biconditional).

A compound expression that is always true is called a *tautology*.

A compound expression that is always false is called a *contradiction*.

Two compound propositions A and B are called *logically equivalent* if $A \Leftrightarrow B$ is a tautology. We then write $A \equiv B$.

Here are some very important tautologies that you must memorize:

1. $A \Leftrightarrow \neg\neg A$
2. $A \Leftrightarrow A \vee A$, $A \Leftrightarrow A \wedge A$,
3. $A \vee B \Leftrightarrow B \vee A$, $A \wedge B \Leftrightarrow B \wedge A$
4. $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$, $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
5. $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$, $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
6. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$, $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$
7. $A \vee (A \wedge B) \Leftrightarrow A$, $A \wedge (A \vee B) \Leftrightarrow A$
8. $A \vee T \Leftrightarrow T$, $A \wedge F \Leftrightarrow F$
9. $A \vee F \Leftrightarrow A$, $A \wedge T \Leftrightarrow A$
10. $A \vee \neg A \Leftrightarrow T$
11. $A \wedge \neg A \Leftrightarrow F$
12. $A \Rightarrow B \Leftrightarrow \neg A \vee B$
13. $(A \Leftrightarrow B) \Leftrightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$

1.2 Exercises

1. Prove that $P \Rightarrow (Q \Rightarrow R) \equiv (P \wedge Q) \Rightarrow R$.
2. Simplify the compound expression $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$.
3. Prove that $((P \vee Q) \wedge \neg Q) \Rightarrow P$ is a tautology.
4. Prove that $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ is a tautology.

$$① \; P \Rightarrow (Q \Rightarrow R) \equiv P \Rightarrow (\neg Q \vee R)$$
$$\equiv \neg P \vee (\neg Q \vee R)$$
$$\equiv (\neg P \vee \neg Q) \vee R$$
$$\equiv \neg (P \wedge Q) \vee R \equiv (P \wedge Q) \Rightarrow R$$

$$② \; (\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$$
$$\equiv (\neg P \wedge (\neg Q \wedge R)) \vee (R \wedge (P \vee Q))$$
$$\equiv ((\neg P \wedge \neg Q) \wedge R) \vee ((P \vee Q) \wedge R)$$
$$\equiv ((\neg P \wedge \neg Q) \vee (P \vee Q)) \wedge R$$
$$\equiv (\neg (P \vee Q) \vee (P \vee Q)) \wedge R \equiv T \wedge R \equiv R$$

$$③ \; ((P \vee Q) \wedge \neg Q) \Rightarrow P$$
$$\equiv ((P \wedge \neg Q) \vee \underbrace{(Q \wedge \neg Q)}_{F}) \Rightarrow P$$
$$\equiv (P \wedge \neg Q) \Rightarrow P$$
$$\equiv \neg (P \wedge \neg Q) \vee P \equiv (\neg P \vee Q) \vee P \equiv (\neg P \vee P) \vee Q \equiv T$$

$$④ \; (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$$
$$\equiv (\neg A \vee (\neg B \vee C)) \Rightarrow (\neg B \vee (\neg A \vee C))$$
$$\equiv (\neg A \vee \neg B \vee C) \Rightarrow (\neg A \vee \neg B \vee C)$$
$$\equiv P \Rightarrow P \equiv T \quad (P = \neg A \vee \neg B \vee C)$$

1.3 Normal Form

A disjunctive normal form is a disjunction of clauses, where each clause is a conjunction of literals. (Like $\neg P \wedge (P \vee Q) \wedge (P \vee \neg Q \vee R)$)

A conjunctive normal form is a conjunction of clauses, where each clause is a disjunction of literals. (Like $P \vee (Q \wedge R) \vee (P \wedge \neg R)$)

Any compound proposition can be written as a disjunctive normal form and a conjunctive normal form.

1.4 Example

Find the (principal) disjunctive normal form and the (principal) conjunctive normal form for $A \Leftrightarrow B$.
The principal conjunctive normal form:

$$A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A) \equiv (\neg A \vee B) \wedge (\neg B \vee A) \equiv (\neg A \vee B) \wedge (A \vee \neg B)$$

and the principal disjunctive normal form

$$\begin{aligned} A \Leftrightarrow B &\equiv (\neg A \vee B) \wedge (A \vee \neg B) \\ &\equiv ((\neg A \vee B) \wedge A) \vee ((\neg A \vee B) \wedge \neg B) \\ &\equiv ((\neg A \wedge A) \vee (B \wedge A)) \vee ((\neg A \wedge \neg B) \vee (B \wedge \neg B)) \\ &\equiv (A \wedge B) \vee (\neg A \wedge \neg B) \end{aligned}$$

It is easy to verify that $A \Leftrightarrow B \equiv (\neg A \vee B) \wedge (A \vee \neg B)$ and $A \Leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$ by the truth table.

1.5 Exercise

Find the (principal) disjunctive normal form and the (principal) conjunctive normal form for $((P \vee Q) \Rightarrow R) \Leftrightarrow P$.

We can find that $(P \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$ is a disjunctive normal form for $((P \vee Q) \Rightarrow R) \Leftrightarrow P$, but it is not the principal disjunctive normal form.

The principal disjunctive normal form for $((P \vee Q) \Rightarrow R) \Leftrightarrow P$ is $(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$. It can also be written as $m_{111} \vee m_{101} \vee m_{010}$ or $\sum(2,5,7)$.

The principal conjunctive normal form for $((P \vee Q) \Rightarrow R) \Leftrightarrow P$ can be easily got from its principal disjunctive normal form $\sum(2,5,7)$. The principal conjunctive normal form is $\prod(0,1,3,4,6)$ or $M_{000} \wedge M_{001} \wedge M_{011} \wedge M_{100} \wedge M_{110}$.
Or the principal conjunctive normal form can be written as

$$(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$$

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$$
\begin{aligned}
&((P \vee Q) \Rightarrow R) \Leftrightarrow P \\
&\equiv (((P \vee Q) \Rightarrow R) \wedge P) \vee (\neg((P \vee Q) \Rightarrow R) \wedge \neg P) \\
&\equiv ((\neg(P \vee Q) \vee R) \wedge P) \vee (\neg(\neg(P \vee Q) \vee R) \wedge \neg P) \\
&\equiv (((\neg P \wedge \neg Q) \vee R) \wedge P) \vee (((P \vee Q) \wedge \neg R) \wedge \neg P) \\
&\equiv (((\neg P \wedge \neg Q) \wedge P) \vee (R \wedge P)) \vee ((P \wedge \neg R \wedge \neg P) \vee (Q \wedge \neg R \wedge \neg P)) \\
&\equiv (P \wedge R) \vee (P \wedge Q \wedge \neg R) \quad (*) \\
&\equiv (P \wedge (Q \vee \neg Q) \wedge R) \vee (P \wedge Q \wedge \neg R) \\
&\equiv (P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R)
\end{aligned}
$$

1.6 Truth Table and Normal Form

The principal disjunctive normal form and the principal conjunctive normal form for a compound expression can be easily got from the truth table.

Let $G = ((P \vee Q) \Rightarrow R) \Leftrightarrow P$

P	Q	R	G	m (minimum)	M (maximum)
T	T	T	T	111	-
T	T	F	F	-	110
T	F	T	T	101	-
T	F	F	F	-	100
F	T	T	F	-	011
F	T	F	T	010	-
F	F	T	F	-	001
F	F	F	F	-	000

Note that:

1. $G = T$, minimum (m) for disjunctive normal form, and m_{001} means $\neg P \wedge \neg Q \wedge R$.

2. $G = T$, maximum (M) for conjunctive normal form, and M_{001} means $P \vee Q \vee \neg R$.

1.7 Exercise

Find the principal disjunctive normal form of $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$, and prove that it is logically equivalent to R.

Answer: $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \equiv m_{001} \vee m_{011} \vee m_{101} \vee m_{111}$