Q1. Let $m \in \mathbb{N}$ with $m > 1$. Prove that if $ac \equiv bc \pmod{m}$, then

$$a \equiv b \left(\mod \frac{m}{\gcd(c, m)}\right)$$

(2 marks)

Q2. Find a solution to the linear congruence $36x \equiv 75 \pmod{1309}$.
(2 marks)

Q3. Find all solutions to the system:

$$\begin{aligned} 2x &\equiv 4 \pmod{5} \\ 3x &\equiv 5 \pmod{7} \\ 7x &\equiv 2 \pmod{13} \end{aligned}$$

(2 marks)

Q4. Find all, if any, solutions to the system:

$$\begin{aligned} x &\equiv 5 \pmod{6} \\ x &\equiv 3 \pmod{10} \\ x &\equiv 8 \pmod{15} \end{aligned}$$

(2 marks)

Q5. Find all, if any, solutions to the system:

$$\begin{aligned} x &\equiv 5 \pmod{5} \\ x &\equiv 3 \pmod{7} \\ x &\equiv 8 \pmod{11} \\ x &\equiv 2 \pmod{17} \end{aligned}$$

(2 marks)

Q6. Show that $n \log_2(n)$ is not $O(\log_2(n))$.
(1 mark)

Q7. Show that $\log_2(n)$, $\log_{10}(n)$ and $\ln(n)$ all have the same order.
(1 mark)

Q8. Show that $\lfloor x^3 - 4 \rfloor$ is order x^3.
(1 mark)

Q9. Let $k > 1$. Do n^n and n^{n-k} have the same order?
(2 marks)

Q10. For all $n \in \mathbb{N}\backslash\{0\}$ define

$$H(n) = \sum_{k=0}^{n-1} \frac{1}{n-k}$$

(i) Show that

$$\sum_{j=2}^{n} \frac{1}{j} < \int_{1}^{n} \frac{1}{x} dx$$

(ii) Prove that $H(n)$ is $O(\ln(n))$.

(4 marks)

Q11. The **Binary Insertion Sort Algorithm** is a variation of the Insertion Sort Algorithm that uses a binary search technique rather than a linear search technique to insert the i^{th} element in the correct place among the previously sorted elements.

(i) Express the Binary Insertion Sort Algorithm in pseudocode.

(ii) Compare the number of comparisons of elements used by the Insertion Sort Algorithm and the Binary Insertion Sort Algorithm when sorting the list $(7, 4, 3, 8, 1, 5, 4, 2)$.

(iii) Show that the Insertion Sort Algorithm uses $O(n^2)$ comparisons of elements.

(iv) Find the complexity of the Binary Insertion Sort Algorithm. Is it significantly faster than Insertion Sort?

(8 marks)