

# Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

# Outline

- 1 1. Signals & Systems (Fundamentals)
  - Overview
  - Signal and System Definition
  - Classification of Signals
  - Signal Notation
  - Transformations of CT signals
  - Signal Characteristics
  - Exponential signals
  - Singularity functions (1.4)
  - Continuous-time systems
  - Summary

# Outline

## 1 1. Signals & Systems (Fundamentals)

### ● Overview

- Signal and System Definition
- Classification of Signals
- Signal Notation
- Transformations of CT signals
- Signal Characteristics
- Exponential signals
- Singularity functions (1.4)
- Continuous-time systems
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# Overview

## Signals

- definition
- classes
- notation
- transformations  
(operations)
- important signals  
(Skip: 1.3.2, 1.3.3, 1.4.1)

## Systems

- definition
- block diagrams
- system interconnection
- classes
- **linearity**, **time-invariance**

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Goal: eventually system design; must first learn to analyze!

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# Signal Definition

## Definition

A **signal** is any “physical” quantity that varies with time or space (or any other independent variable or variables).

Often when we discuss signals we refer to **mathematical representation** of the physical quantity.

## Example

An approaching **ambulance siren** produces a time-varying change in acoustic pressure that our ears perceive as sound.

$$s(t) = (1 + t) \sin\left(2\pi[1000t + 10t^2 + 300 \sin(2\pi t/2)]\right)$$

# Example: Ambulance Siren

$$s(t) = (1 + t) \sin\left(2\pi[1000t + 10t^2 + 300 \sin(2\pi t/2)]\right)$$

- The  $(1 + t)$  **amplitude term** represents increasing loudness as the ambulance approaches.
- The  $1000t$  term represents the 1kHz **siren oscillation**.
- The  $10t^2$  term represents **increasing pitch** due to the **Doppler effect** as the ambulance approaches.
- The  $300 \sin(2\pi t/2)$  term represents **the eeh-oo-eeh-oo periodic variation in pitch**.

# System Definition

## Definition

A **system** is a physical “device” that performs an operation on a signal.

## Example

The **human ear** converts **acoustic signals** into **electrical nerve synapses** (another signal) that are processed by the brain. The input and output signals are different physical quantities.



# Signal processing

One of the main roles of electrical engineers is to **design and analyze systems** that take some **input signal** and produce some related (but almost always different) **output signal**.

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We refer such operations as **signal processing**.

# Signals and Systems

## Example

For an **audio amplifier**, ideally the output signal is “simply” an amplified version of the input signal. (On paper it is easy:

$$s_{\text{out}}(t) = as_{\text{in}}(t).$$

But implementing this in analog hardware with **minimal distortion** is **nontrivial**.)

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This course will emphasize **continuous-time** or **analog** signals, and briefly introduce **discrete-time** or **digital** signals at the end.

(Portions of Chapters 1-10)

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# A signal is a function

Mathematically, a **signal** is a **function** of one or more independent variables.

## Question

*What is a function?*

# Classification of Signals: Dimensionality (1)

One way to classify signals is by the dimension of the **domain of the function**, *i.e.* how many arguments the function has.

## Definition

A **one-dimensional** signal is a function of a single variable, *e.g.* time.

# Classification of Signals: Dimensionality (2)

## Question

What is a ***M-dimensional*** signal?

# Classification of Signals: Dimensionality (3)

## Example

A sequence of BW TV pictures  $I(x, y, t)$  is a scalar valued function of two spatial coordinates  $x$  and  $y$  and time  $t$ , so it is a **3D signal**.

We will focus on **one-dimensional** signals in this course, generally considering the independent variable to be **time  $t$** .

# Classification of Signals: Dimensionality (3)

## Example

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# Classification of Signals: Dimensionality (4)

Another way to classify signals is by the **dimension of the range** of the function, *i.e.*, the space of values the function can take.

## Definition

A **scalar** or **single-channel** signal is a function of a real-valued scalar or complex-valued scalar.

## Question

What is a **multichannel** signal?

# Classification of Signals: Dimensionality (5)

## Example

A **color** TV picture can be described by a red, blue and green signal, whereas a **BW** TV picture is scalar valued.

We will focus on **scalar** signals in this course, both real and complex.

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Most of the design/analysis techniques **generalize to multichannel and multidimensional** signals.

# Continuous-time signals

## Definition

A **continuous-time signal** or **analog signal** is a function defined for all times  $t \in (-\infty, \infty)$ , or at least over some continuous interval  $(a, b)$ .

## Example

$$x(t) = e^{-t^2}, \quad -\infty < t < \infty. \text{ (Picture)}$$

# Discrete-time signals

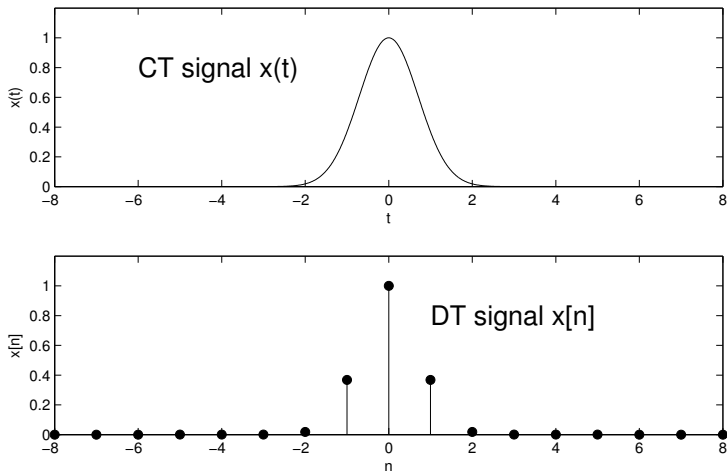
## Definition

A **Discrete-time signal** is a function defined only at certain specific values of time.

## Example

$$x[n] = x(t_n). \text{ (Picture)}$$

# Continuous-time signals vs. discrete-time signals



# Classification of Signals: time characteristics

Classify signals by **time characteristics**

- 1 Continuous-time signals or analog signals
- 2 Discrete-time signals

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Discrete-time signals arise from

- Sampling a continuous signal at discrete time instants
- Accumulating a quantity over a period of time

## Example

When counting number of heart attacks per month,  $n$  would index the month, and  $x[n]$  would be the number.

# Classification: Value characteristics

## Definition

A **continuous-valued signal** or **continuous-amplitude signal** can take any value in some continuous interval.

## Example

Voltage between 0 and 5 volts.

## Definition

A **discrete-valued signal** or **discrete-amplitude signal** only takes values from a discrete set of possible values.

## Example

In heart attack example,  $x[n]$  could be 0, 1, 2, ..., population of world.

# Deterministic vs Random signals

- 1 **Deterministic signals** can be described by an **explicit mathematical** representation.
- 2 **Random signals** evolve over time in an **unpredictable** manner.

## Example

“Hiss” or “noise” in an audio system.

We will focus on **deterministic signals**, although reducing noise (eliminating a random component) is often a goal in designing signal processing systems.



# Classification: Our focus

We will focus on

- single-channel, one-dimensional, continuous-valued, continuous-time signals.
- $x(t)$  is a scalar valued function of a real independent variable.
- Mathematically

$$x : \mathbb{R} \rightarrow \mathbb{R} \text{ or } x : \mathbb{R} \rightarrow \mathbb{C}$$

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# Signal notation (mathematical representation)(1)

① Graphically (**Picture**):

② Braces or piecewise notation:  $x(t) = \begin{cases} e^{-t}, & t > 0 \\ e^t, & t \leq 0. \end{cases}$

③ Formula:  $x(t) = e^{-|t|}$ .

④ In terms of other functions:  $x(t) = s(t) + s(-t)$  where

$$s(t) = \begin{cases} e^{-t}, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0. \end{cases}$$

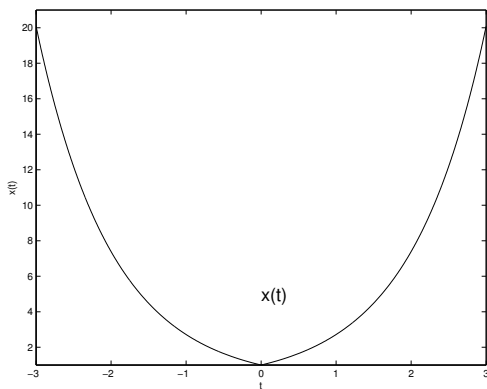
⑤ Fourier representation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 + \omega^2} e^{j\omega t} d\omega$$

**Skill:** *Convert between different signal representations.*

**Skill:** *Choose representation most appropriate for a given problem.*

# Signal notation (mathematical representation)(2)



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# Eventual goal

- **Eventual goal**: analyze **interesting signals** and to analyze and design **useful systems**.
- Such signals and systems consist of combinations of **simpler (less interesting?)** signals and systems.
- So we walk before we run...

# Transformations of CT signals

- Time transformations
  - Folding/reflection/time-reversal
  - Time-scaling
  - Time-shifting/time-delay
  - General time transformation
- Amplitude transformations
  - Amplitude reversal
  - Amplitude scaling
  - Amplitude shifting
- More signal operations
  - Differentiator
  - Integrator
- Operations with two signals

# Change of variables

If  $x(t) = e^{-(t-2)}$  then  $y(t) = x\left(\frac{t-1}{3}\right)$  is another function;

$$y(t) = e^{-[(t-1)/3-2]} = e^{-\left(\frac{t-7}{3}\right)}.$$

In calculus, this type of transformation is called a **change of variables**.



# Time transformations

Here we give some new names to such transformations to reflect the **physical meaning** of the mathematics.

## Example

$$x(t) = \begin{cases} e^{-(t-2)}, & t \geq 2, \\ 0, & \text{otherwise.} \end{cases} \quad \textbf{(Picture)}.$$

(used throughout)

One can apply time transformations both **graphically** and **mathematically**. Both approaches are useful.

# Folding/reflection/time-reversal: $y(t) = x(-t)$ (1)

## Folding/reflection/time-reversal

$$y(t) = x(-t)$$

### Example

- Backwards play a movie/audio tape.
- A mirror is an optical system that does “space reversal”.

### Example

Find  $y(t) = x(-t)$  for  $x(t)$  above.

# Folding/reflection/time-reversal: $y(t) = x(-t)$ (2)

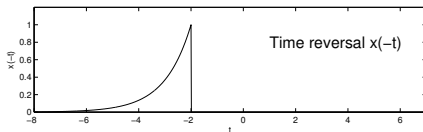
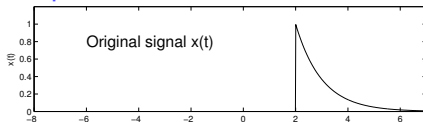
Mathematical method:

- Replace all  $t$ 's with  $-t$ ,
- Simplify where possible.

$$\begin{aligned}
 y(t) &= x(-t) \\
 &= \begin{cases} e^{-(-t-2)}, & -t \geq 2, \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} e^{t+2}, & t \leq -2, \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

Note that  $y(t)$  becomes a “mirror image” of  $x(t)$  around  $t = 0$ .

Graphical method:



# Time-scaling: $y(t) = x(at), a > 1$

## Time-scaling

$$y(t) = x(at)$$

$a > 1$  will shrink or compress the signal

### Example

Playing a recording at 3 times the normal speed.

### Example

Find  $y(t) = x(3t)$  for  $x(t)$  above.

# Time-scaling: $y(t) = x(at)$ , $a < 1$

## Time-scaling

$$y(t) = x(at)$$

$a < 1$  will stretch or expand the signal

### Example

slow-motion part of a movie

### Example

Find  $y(t) = x(t/2)$  for  $x(t)$  above.

# Time shifting: $y(t) = x(t - t_0)$

Time shifting:  $y(t) = x(t - t_0)$

- $t_0$  can be positive (delayed signal) or negative (advanced signal).
- Physical systems can only delay, not advance, in time.

## Example

“Park distance control” propagation delay

## Example

Find  $y(t) = x(t - 1)$  for  $x(t)$  above.

# General time transformation

General time transformation involves all three of the above time transformations (time reversal, time scaling, and time shifting).

Two distinct (but related) forms:

$$y(t) = x(at - b) = x\left(\frac{t - t_0}{w}\right)$$

where  $t_0 = b/a$  and  $w = 1/a$  or  
equivalently  $a = 1/w$  and  $b = t_0/w$ .

# Mathematical time transformation

mathematical recipe:

- 1 Replace **all** occurrences of  $t$  in the definition of  $x(t)$  with  $at - b$  or with  $\frac{t-t_0}{w}$ .
- 2 Manipulate algebraically to simplify.

## Example

Find  $y(t) = x(-t/2 + 5) = x\left(\frac{t-10}{-2}\right)$  for  $x(t)$  above.



# Graphical time transformations

## Question

*How to perform a general time transformation **graphically**?  
Should you “shift first” or “scale first?”*

# Form 1: $y(t) = x(at - b)$

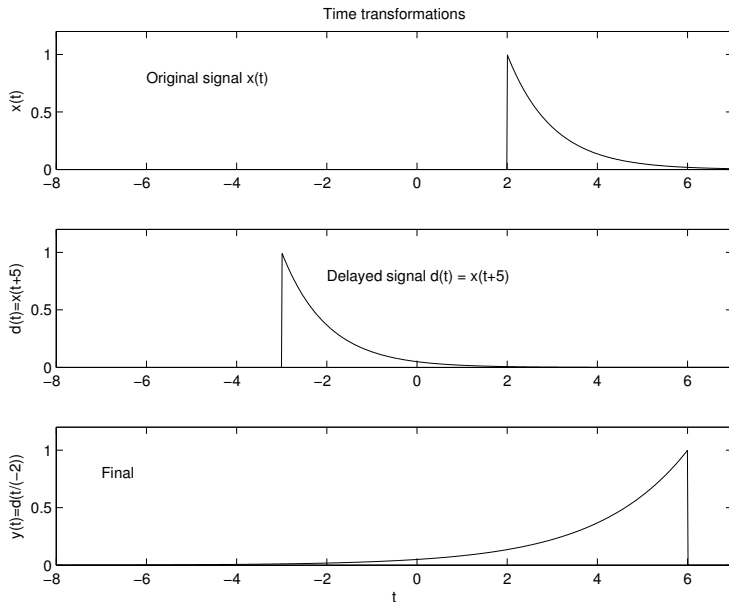
$$\text{Form 1 : } y(t) = x(at - b)$$

- Introduce an intermediate signal  $d(t) = x(t - b)$ .
- Clearly  $d(t)$  is just a **delayed** version of  $x(t)$  by amount  $b$ .
- But  $y(t) = d(at)$ , which is just a **scaled** version of the signal  $d(t)$ .

To find  $y(t) = x(at - b)$  graphically we must

- 1 **time-delay** the signal  $x(t)$  by  $b$ .
- 2 **time-scale** that delayed signal by  $a$ .

# Form 1: $y(t) = x(at - b)$ (Cont.)



## Form 2: $y(t) = x\left(\frac{t-t_0}{w}\right)$

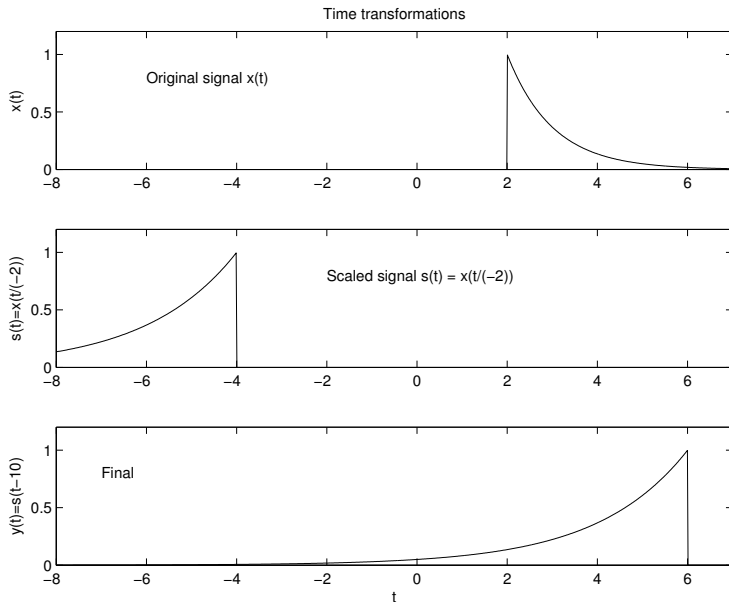
$$\text{Form 2 : } y(t) = x\left(\frac{t-t_0}{w}\right)$$

- Introduce an intermediate signal  $s(t) = x(t/w)$ .
- Clearly  $s(t)$  is a **time-scaled** version of  $x(t)$  by the factor  $1/w$ .
- But  $y(t) = s(t - t_0)$ , which is just time delay of the signal  $s(t)$ .

To find  $y(t) = x\left(\frac{t-t_0}{w}\right)$  graphically, we must

- 1 **time-scale** the signal  $x(t)$  by  $1/w$ .
- 2 **time-delay** that scaled signal by  $t_0$ .

# Form 2: $y(t) = x\left(\frac{t-t_0}{w}\right)$ (Cont.)



# Amplitude transformations

- ① **amplitude reversal**  $y(t) = -x(t)$
- ② **amplitude scaling**  $y(t) = ax(t)$
- ③ **amplitude shifting**  $y(t) = x(t) + b$

## Example

(Using all three.) Find  $y(t) = -3x(t) + 2$  for the  $x(t)$  above.

# Outline

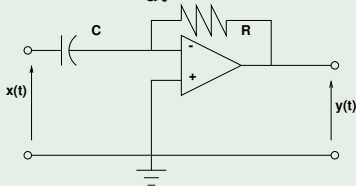
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# Differentiator $y(t) = \frac{d}{dt}x(t)$

$$\text{Differentiator } y(t) = \frac{d}{dt}x(t)$$

## Example

$$y(t) = -RC \frac{d}{dt}x(t)$$



## Example

Find the differentiated signal of  $x(t) = e^{-2|t|}$ .

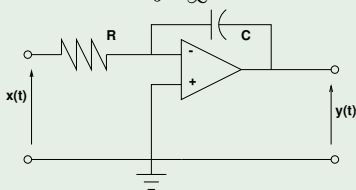


# Integrator: $y(t) = \int_{-\infty}^t x(\tau) d\tau$

Integrator:  $y(t) = \int_{-\infty}^t x(\tau) d\tau$

## Example

$$y(t) = -\frac{1}{RC} \int_{-\infty}^t x(\tau) d\tau$$



## Example

Find the integrated signal of  $x(t) = e^{-|t|}$ .

# Integrator: example

## Solution

- ① rewrite  $x(\cdot)$  *in terms of  $\tau$*

$$x(\tau) = \begin{cases} e^{-\tau}, & \tau > 0 \\ e^{\tau}, & \tau \leq 0. \end{cases}$$

- ② For  $t \leq 0$ :

$$y(t) = \int_{-\infty}^t e^{\tau} d\tau = e^t$$

- ③ For  $t \geq 0$ :

$$\begin{aligned} y(t) &= \int_{-\infty}^0 e^{\tau} d\tau + \int_0^t e^{-\tau} d\tau \\ &= e^0 + (-e^{-\tau}) \Big|_0^t = 1 + (1 - e^{-t}) \\ &= 2 - e^{-t}. \end{aligned}$$

# Integrator vs. Integration in calculus

## Question

*What is the **distinction** between **simple integration** of the kind learned in **calculus** (computing area under a curve) and the **integrator system** described here.*

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  - More signal operations
  - Operations with two signals
- Signal Characteristics
  - Periodic/aperiodic signals
  - Even and odd signals
  - Energy and power signals
- Exponential signals
- Singularity functions (1.4)
  - Unit step signal
  - Rect(angle) function
  - Unit impulse function  $\delta(t)$  (1.4.2, 2.5)
- Continuous-time systems

# Operations with two signals

## Operations with two signals

- 1 **sum** of two signals  $y(t) = x_1(t) + x_2(t)$
- 2 **product** of two signals  $y(t) = x_1(t) x_2(t)$ .

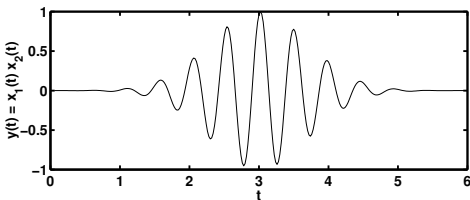
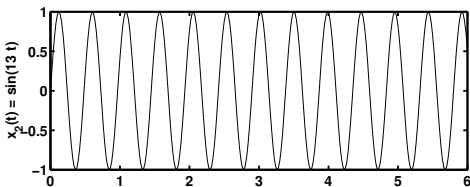
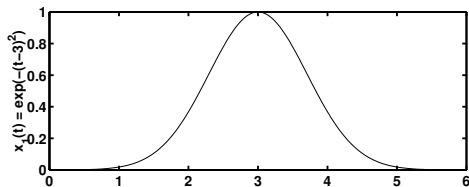
Add or multiply two signals **at every time point**.

## Example

If  $x_1(t) = e^{-(t-3)^2}$ ,  $x_2(t) = \sin(13t)$ , then  
 $y(t) = x_1(t)x_2(t) = e^{-(t-3)^2} \sin(13t)$ .

This is called **amplitude modulation**.

# amplitude modulation



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# Periodic signals

Why study **periodic signals**?

- important for analysis
- solution to ideal LC electrical circuits
- periodic physical phenomena: frictionless pendulums, earth rotation, heart rhythms, etc.

## Definition

$x(t)$  is **periodic** with a **period**  $T > 0$  iff

$$x(t + T) = x(t) \quad \forall t \quad (1)$$

## Definition

If no such  $T > 0$  exists,  $x(t)$  is called **aperiodic**



# Fundamental period

## Definition

The **fundamental period**  $T_0$  of a signal is the smallest value of  $T$  satisfying (1).

## Theorem

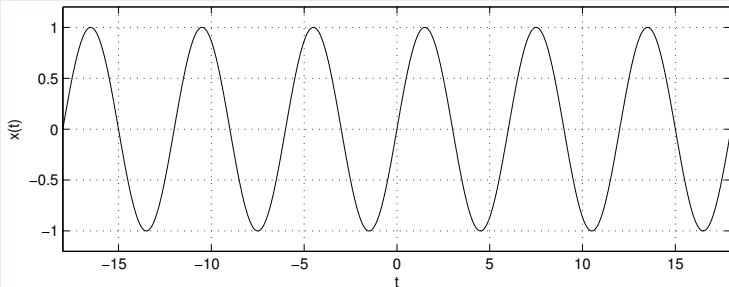
*A signal that is periodic with period  $T > 0$ , is also periodic with period  $nT$  for any integer  $n \neq 0$ , i.e.*

$$x(t + nT) = x(t).$$

# Periodic signals: example

## Example

$x(t) = \sin\left(\frac{\pi}{3}t\right) = \sin\left(\frac{2\pi}{T_0}t\right)$  What is fundamental period?



# Sums of two periodic signals

## Question

*Suppose  $x_1(t)$  is periodic with period  $T_1$  and  $x_2(t)$  is periodic with period  $T_2$  and  $x(t) = x_1(t) + x_2(t)$ .*

- Is  $x(t)$  periodic?*
- If so, what is a period  $T$  of  $x(t)$ ?*

# Sums of two periodic signals: solution (1)

## Solution

- ① *Easy case:* if  $T_1 = T_2$  then,  $x(t)$  is periodic, and  $T = T_1 = T_2$ .
- ② *General case.* We know  $x_1(t) = x_1(t + T_1)$  and  $x_2(t) = x_2(t + T_2)$  and  $x(t) = x_1(t) + x_2(t)$ .  
We want to determine if there is any value of  $T > 0$  such that  $x(t) = x(t + T)$ .

# Sums of two periodic signals: solution (2)

Suppose there is a value of  $T > 0$  that satisfies  $T = n_1 T_1$  and  $T = n_2 T_2$ , for some nonzero integers  $n_1$  and  $n_2$ . Then

$$\begin{aligned}x(t + T) &= x_1(t + T) + x_2(t + T) = x_1(t + n_1 T_1) + x_2(t + n_2 T_2) \\&= x_1(t) + x_2(t) = x(t),\end{aligned}$$

so  $x(t)$  is periodic with period  $T$ .

# Sums of two periodic signals: solution (3)

The conditions  $T = n_1 T_1$  and  $T = n_2 T_2$  are equivalent to requiring that

$$n_1 T_1 = n_2 T_2 \text{ so } T_1/T_2 = n_2/n_1,$$

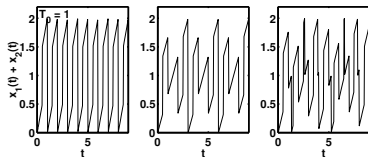
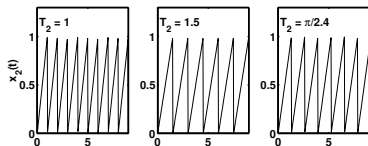
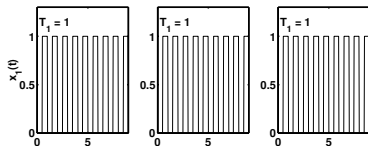
which means that  $T_1/T_2$  is **rational**, a ratio of two integers.

## Theorem

*A sum of two periodic signals is periodic iff the ratio of their periods is rational.*

# Sums of two periodic signals: example

Are the signals in the third row periodic?



# Sums of two periodic signals: fundamental period

## Question

*The least common multiple of the fundamental periods of the two signals is a period of the sum. Is it the fundamental period?*



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# Even and odd symmetry

## Definition

$x(t)$  has **even symmetry** iff  $x(-t) = x(t) \forall t$

## Definition

$x(t)$  has **odd symmetry** iff  $x(-t) = -x(t) \forall t$

Note that if  $x(t)$  has **odd** symmetry, then  $x(0) = -x(0)$  so  $x(0) = 0$ .

## Question

*If  $x(0) = 0$ , does  $x(t)$  have odd symmetry?*

# Even and odd components

We can decompose any signal into even and odd components:

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) \triangleq \frac{1}{2} [x(t) + x(-t)], \quad x_o(t) \triangleq \frac{1}{2} [x(t) - x(-t)]$$

## Question

*Is the following  $x(t)$  even or odd?*

$$x(t) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{otherwise,} \end{cases}$$

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- 1 1. Signals & Systems (Fundamentals)
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  - Transformations of CT signals
  - **Signal Characteristics**
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  - Singularity functions (1.4)
  - Continuous-time systems
  - Summary

# Average value and energy

## Definition

The **average value** of a signal is defined as

$$A \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt.$$

## Example

The average value of an odd signal is zero.

## Definition

The **energy** of a signal  $x(t)$  is defined as

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

(Not necessarily physical energy. If  $x(t)$  is a voltage across a  $1\Omega$  resistor, then  $E$  is energy.)

# Average power and energy signal

## Definition

The **average power** of a signal is defined as

$$P \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt.$$

## Definition

If  $E$  is finite ( $E < \infty$ ) then  $x(t)$  is called an **energy signal** and  $P = 0$ .

# Energy and power signals

## Definition

If  $E$  is infinite, then  $P$  can be either finite or infinite. If  $P$  is finite and nonzero, then  $x(t)$  is called a **power signal**.

Some signals are neither energy signals nor power signals, such as  $x(t) = t^2$ , for which  $E = \infty$  and  $P = \infty$ . Such signals are generally of little practical engineering importance.

# Energy and power signals: example

## Example

consider  $x(t) = 5 + a \cos t$  where  $0 < a < \infty$ .

- Find the average value of  $x(t)$  ?
- Is  $x(t)$  a power signal, an energy signal, or neither?



# Outline

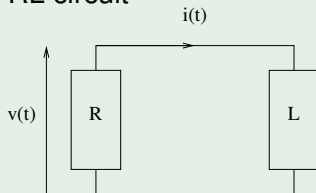
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# Exponential signals (1)

Sinusoidal signals, exponential signals, and complex exponentials signals are particularly important because they arise from the solutions of linear constant-coefficient differential equations.

## Example

an RL circuit



$$\frac{d}{dt}v(t) = \left(-\frac{R}{L}\right)v(t) = av(t)$$

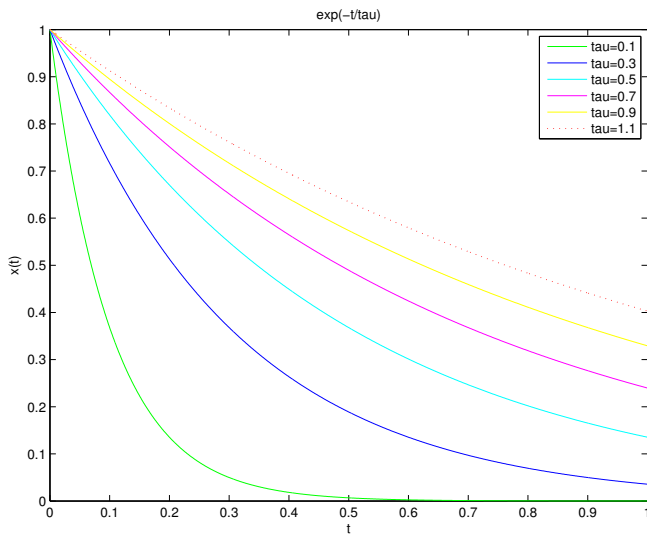
where  $a = -R/L$ . Solution for  $t > 0$  is

$$v(t) = v(0)e^{at} = v(0)e^{-t/\tau}$$

$$v(t) = L \frac{d}{dt}i(t) = L \frac{d}{dt} \left( -\frac{v(t)}{R} \right)$$

where  $\tau = -1/a = L/R$  is called the **time constant** of the circuit. **(Picture)**

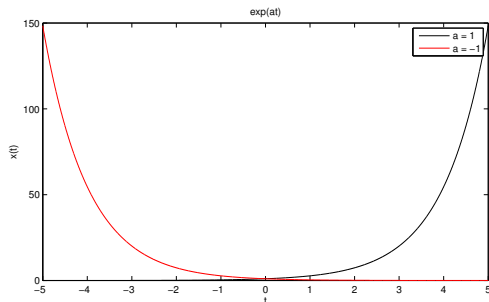
# Exponential signals (2)



# Exponential signals (3)

Signals of the form  $x(t) = ce^{at}$  are very important, for both real and complex  $c$  and  $a$ .

- $a > 0$  real *e.g.*, population growth
- $a < 0$  real *e.g.*, radioactive decay,



# Exponential signals (4)

- If  $a$  is purely imaginary, we get  $x(t) = ce^{j\omega_0 t}$ , a **complex exponential** signal.
- If  $c = Ae^{j\phi}$ , then  $x(t) = Ae^{j\phi} e^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$ , a **sinusoid** signal.
- If  $x(t) = e^{st}$  where  $s = a + j\omega_0$  and  $a < 0$ , then  $x(t) = e^{at}(\cos \omega_0 t + j\sin \omega_0 t)$  which is called a **damped sinusoid** signal. (See textbook p21.)

Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

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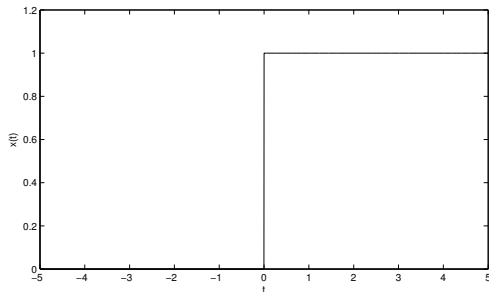
# Unit step function/Signal

## Definition

A **unit step function(signal)** is defined as

$$u(t) \triangleq \begin{cases} 1, & t > 0 \text{ or } t \geq 0 \\ 0, & t < 0 \end{cases}$$

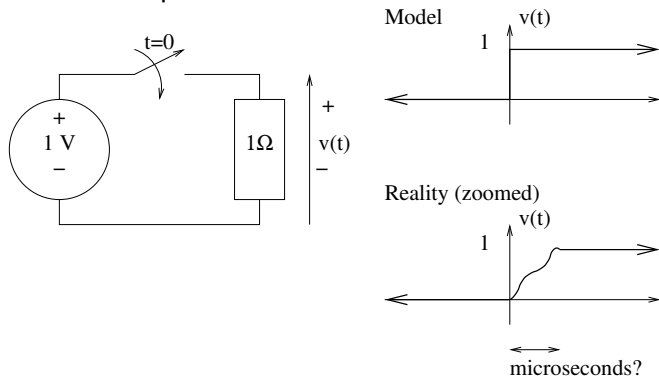
The value at  $t = 0$  is arbitrary and unimportant! Reasonable choices are 0, 1, and  $\frac{1}{2}$ ; any will do.





# Modeling a switch

The unit step is a useful model for a **switch**.



## Question

- For a real switch in above is the voltage exactly a step function?
- Does the final voltage exactly equal 1 Volt?

# Simplifying notation

The step function is useful for simplifying notation.

## Example

The following step function “switches on” the signal from a guitar string plucked at time  $t = 2$ .

$$x(t) = \begin{cases} e^{-t} \sin(5t), & t > 2 \\ 0, & \text{otherwise} \end{cases} = e^{-t} \sin(5t) u(t - 2).$$

The first notation is messy, the second way is neat, and hides the braces within the definition of  $u(t - 2)$ .

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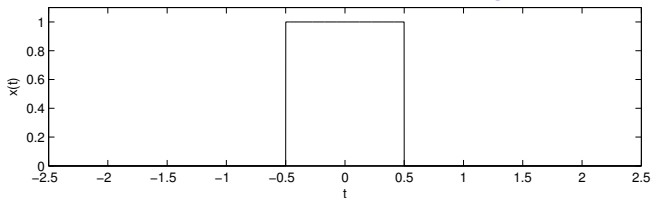
# Rect(angle) function

## Definition

A **rect(angle) function** is defined as

$$\text{rect}(t) \triangleq \begin{cases} 1, & -1/2 < t < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

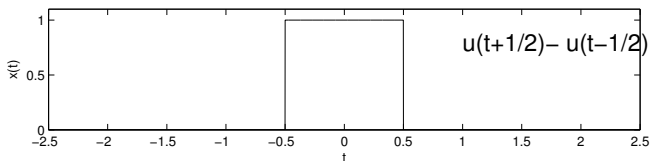
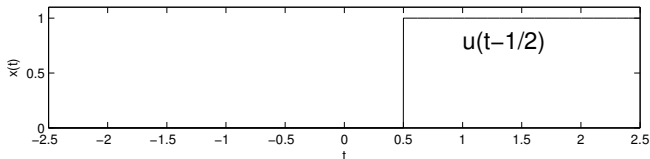
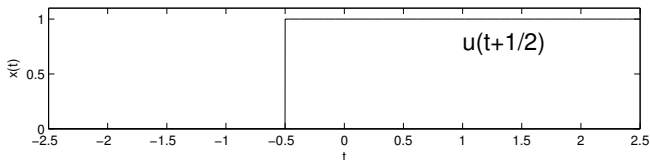
Centered at zero with **unit width and unit height**.



# Rect function and step functions (1)

Can be represented using step functions:

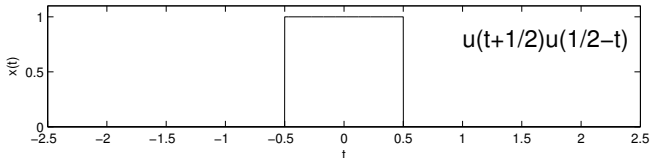
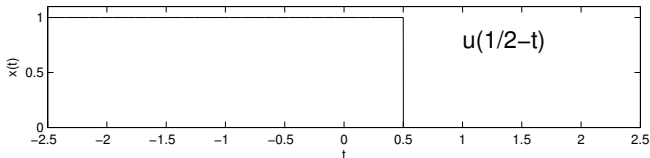
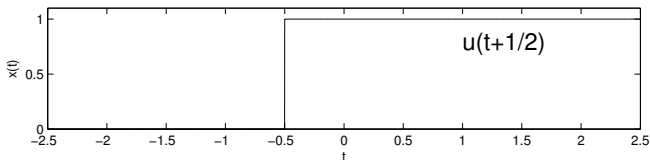
$$\text{rect}(t) = u(t + 1/2) - u(t - 1/2)$$



# Rect function and step functions (2)

Can be represented using step functions:

$$\text{rect}(t) = u(t + 1/2)u(1/2 - t)$$

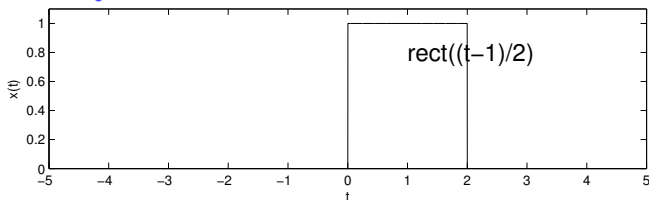


# Transformed rect functions

Time-scaled and time-shifted rect function

$$\begin{aligned}\text{rect}\left(\frac{t - t_0}{T}\right) &= \begin{cases} 1, & -1/2 < \frac{t - t_0}{T} < 1/2 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & t_0 - T/2 < t < t_0 + T/2 \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Centered at  $t_0$  with width  $T$

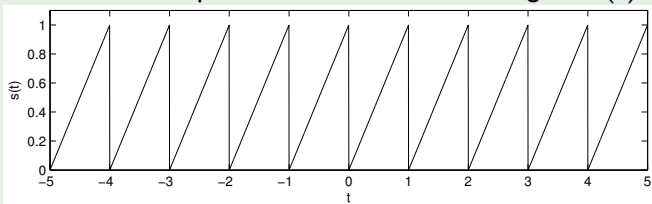


# Rect function: example

Useful for “switching on and off” other functions, or for “extracting” on part of a signal, such as one period of a periodic signal.

## Example

Find mathematical expression for a sawtooth signal  $s(t)$ .





# Outline

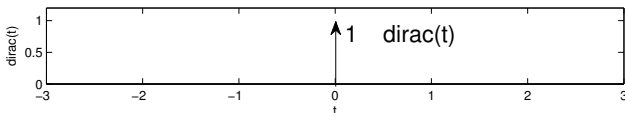
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# Unit impulse function $\delta(t)$

- **Unit impulse function**  $\delta(t)$ , aka **Dirac delta function** or just **delta function**.
- It is another **mathematical idealization** that cannot occur in nature (like the unit step function), but is nevertheless useful for modeling certain phenomena, just as the step function is a useful idealization of a switch.
- More importantly, **it will greatly simplify our analysis of LTI systems later.**

# Unit impulse function $\delta(t)$ (2)

$\delta(t)$  is like a pulse of zero width but infinite height and unit area.



Graphical representation using upward arrow, labeled with area (called **weight**). (see text p.34 for scaled impulse.)

Such a thing is clearly not quite a “function” in the usual sense defined in calculus.

We can “define” an impulse function through its **properties**.

# Minor Properties

## Property

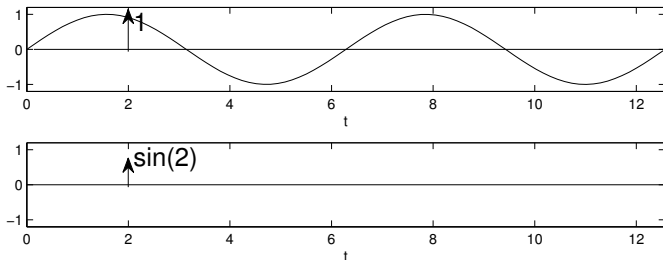
- ① *unit area property*  $\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1$  for any  $t_0$
- ② *scaling property*  $\delta(at + b) = \frac{1}{|a|} \delta(t + b/a)$  for  $a \neq 0$ .
- ③ *symmetry property*  $\delta(t) = \delta(-t)$
- ④ *support property*  $\delta(t - t_0) = 0$  for  $t \neq t_0$
- ⑤ *relationships with unit step function*:  $\delta(t) = \frac{d}{dt} u(t)$ ,  
 $u(t) = \int_{-\infty}^t \delta(\tau) d\tau$

# Major properties: Sampling property

## Property

**Sampling property** holds when  $x(t)$  is continuous at  $t_0$ :

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

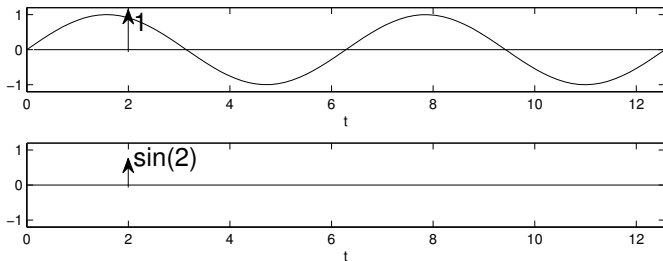


# Major properties: Sifting property

## Property

**Sifting property** holds when  $x(t)$  is continuous at  $t_0$ :

$$\int_{-\infty}^{\infty} x(t) \delta(t - t_0) dt = x(t_0).$$



# Algebraic property

Property

*Algebraic property*

$$t\delta(t) = 0$$

# Scaling property

## Example

Show that  $\delta(at) = \frac{1}{|a|}\delta(t)$  for  $a \neq 0$ .



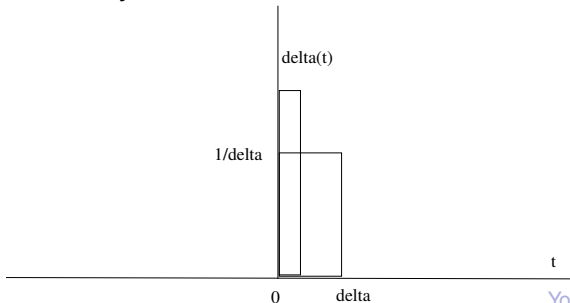
# Practical impulse function

## Definition

**Practical impulse function**, defined for any  $\Delta > 0$ :

$$\delta_{\Delta}(t) \triangleq \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Note that area is **unity**, width approaches zero as  $\Delta \rightarrow 0$ ; height approaches infinity as  $\Delta \rightarrow 0$ .



# Practical impulse function: example

## Example

drumstick striking a drum (applied force vs time)

## Example

metal hammer tapping a pendulum (applied force vs time)

# Practical impulse function: example

It is tempting to try to write

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$

but the limit is **not well defined mathematically**. Nevertheless, this is the intuition.

---

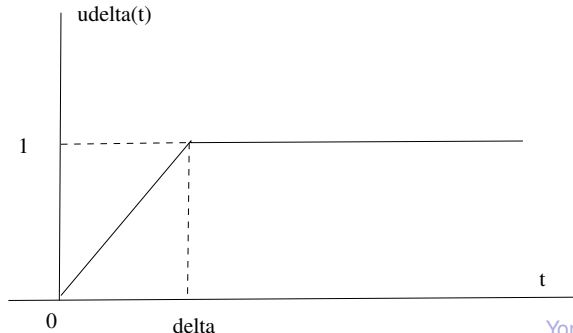
Instead we “define”  $\delta(t)$  in terms of its **properties**, making sure that the properties are consistent with the above “limit”. Such objects are called **generalized functions** in mathematics.

# Relationship to unit step function (1)

Explanation of  $\delta(t) = \frac{d}{dt}u(t)$  using limiting step function.

Define a practical almost-step function as

$$u_{\Delta}(t) \triangleq \begin{cases} 0, & t \leq 0 \\ t/\Delta, & 0 < t < \Delta \\ 1, & t \geq \Delta \end{cases}$$



## Relationship to unit step function (2)

$$\frac{d}{dt}u_{\Delta}(t) = \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases} = \delta_{\Delta}(t)$$

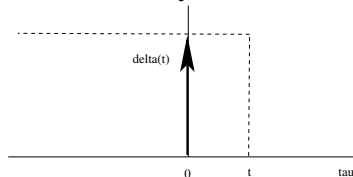
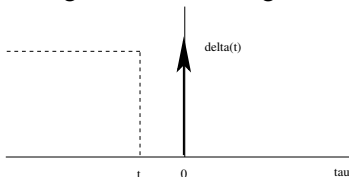
Then since  $u(t) = \lim_{\Delta \rightarrow 0} u_{\Delta}(t)$ , by taking the limit of both sides we have that

$$\frac{d}{dt}u(t) = \lim_{\Delta \rightarrow 0} \frac{d}{dt}u_{\Delta}(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t) = \delta(t).$$

## Relationship to unit step function (3)

Show graphically that  $\int_{-\infty}^t \delta(\tau) d\tau = u(t)$ .

- For any  $t < 0$ , the range of integration over  $(-\infty, t)$  will **not cover zero**, so the integral is simply **zero**.
- For **any**  $t > 0$ , integration over the range  $(-\infty, t)$  **covers zero**, so the entire unit area of the impulse is included in the integral, so the integral evaluates to **1** for any  $t > 0$ .



# Example (1)

## Example

- By Newton's laws, velocity is the time-integral of acceleration.
- When a hammer taps a stationary pendulum, the pendulum (almost) instantaneously changes from being stationary to moving with some velocity that is related to how “hard” the hammer taps the pendulum.
- The acceleration is like a Dirac delta function, and the velocity is like a step function.

## Example (2)

### Example

Let  $x(t) = 2 \operatorname{rect}(t/2 - 3)$ . Find  $y(t) = \frac{d}{dt}x(t)$ .



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# Continuous-time systems

## Definition

A **continuous-time system** is a device or process that, according to some well-defined rule, transforms one CT signal called the **input signal** or **excitation** into another CT signal called the **output signal** or **response**.

The input signal  $x(t)$  is **transformed** by the system into a signal  $y(t)$ , which we express mathematically as

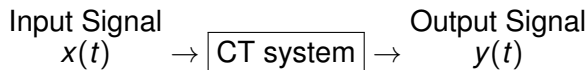
$$y(\cdot) = \mathcal{T}[x(\cdot)] \quad \text{or} \quad y(t) = \mathcal{T}[x(\cdot)](t) \quad \text{or} \quad x(\cdot) \xrightarrow{\mathcal{T}} y(\cdot).$$

# Notation

## Question

*The notation  $y(t) = \mathcal{T}[x(t)]$  is mathematically vague. Why?*

# Diagram



The arrows in this diagram are not necessarily wires! They represent whatever medium transports the signal from one part of the system to another part.

---

At the systems level, we are less interested in the details of the implementation than in the **mathematical relationships** and **system properties**.

# Example (1)

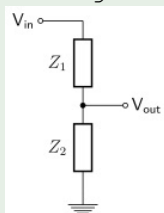
## Example

voice (acoustic pressure)  $\rightarrow$  microphone  $\rightarrow$  electrical current

## Example

voltage divider

([https://en.wikipedia.org/wiki/Voltage\\_divider](https://en.wikipedia.org/wiki/Voltage_divider))



For identical resistors, the output is  $y(t) = \frac{1}{2}x(t)$ . Called a **static** system.

# Example (2)

## Example

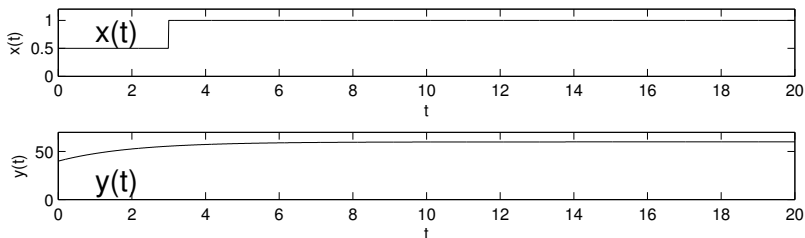
accelerator pedal position  $\rightarrow$  engine/car  $\rightarrow$  car velocity

- Input signal induces a response of the system.
- $x(t) = \frac{1}{2} + \frac{1}{2}u(t-3)$  (one pushes the gas pedal to the floor)
- $y(t) = 40 + 20(1 - e^{-t/\tau})$  (rise time or transient response)
- $y(t)$  is not solely a function of  $x(t)$  at time  $t$ , but also a function of previous input signal values and the present and past state of the system.
- Called a **dynamic** system.

# Example (3)

$$x(t) = \frac{1}{2} + \frac{1}{2}u(t-3)$$

$$y(t) = 40 + 20(1 - e^{-t/\tau}), \quad \tau = 2$$



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# Input-output description of systems (1)

Pictures/diagrams are a starting point, but for **quantitative analysis** every system must have an **input-output relationship**.

## Definition

**Input-output relationship** is a mathematical expression that precisely defines how the output signal is related to the input signal.

## Example

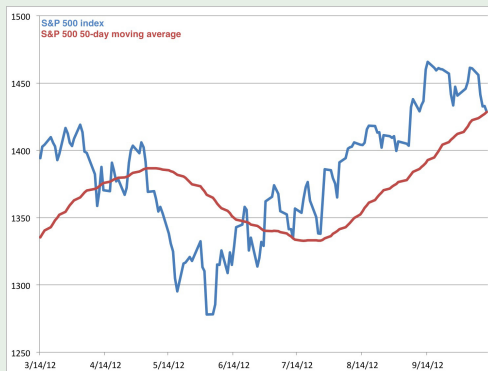
**integrator** system  $y(t) = \int_{-\infty}^t x(\tau) d\tau$

# Moving average

## Example

**moving average** filter

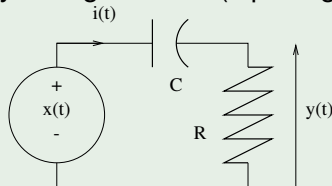
$$y(t) = \frac{1}{T} \int_{t-T}^t x(\tau) d\tau.$$



# Input-output description of systems (2)

## Example

RC circuit driven by voltage source (input signal).



$$\frac{1}{CR}y(t) + \frac{d}{dt}y(t) = \frac{d}{dt}x(t).$$

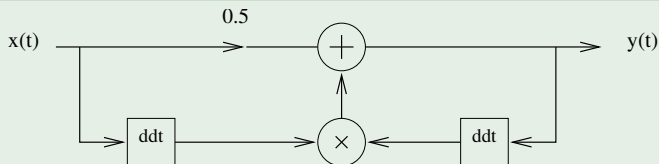
- This input-output relation is not of the form  $y(t) = \text{some\_function}[x(t)]$ .
- When combined with an appropriate initial condition (such as 0 charge on the capacitor at time  $t = 0$ ) one can solve the diffeq to determine  $y(t)$  for any  $x(t)$  (**later lectures.**)

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# Block diagram representation of CT systems

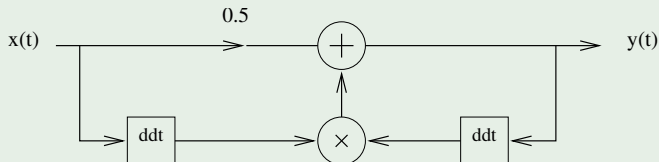
## Example



The lower-right part is called a **feedback connection**.

# Block diagram representation of CT systems

## Example



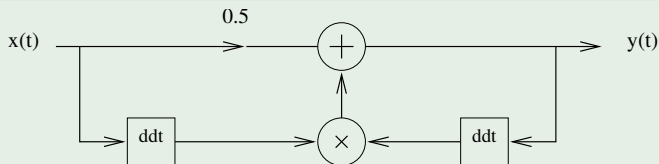
The lower-right part is called a **feedback connection**.

Basic elements:

- **adder** (see text Fig. 2.29 on P. 126)
- **constant multiplier (amplifier)** (see text Fig. 2.29 on P. 126)
- **signal multiplier**
- **differentiator** (see text Fig. 2.29 on P. 126)
- **integrator** (see text Fig. 2.31 & 2.32 on P. 127)

# Block diagram representation of CT systems

## Example



The lower-right part is called a **feedback connection**.

Input-output relationship defined by the diagram:

$$y(t) = 0.5x(t) + \left( \frac{d}{dt}x(t) \right) \cdot \left( \frac{d}{dt}y(t) \right).$$

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# Interconnection of systems

## 1 Series connection

$$x(t) \rightarrow \boxed{\mathcal{T}_1} \rightarrow \boxed{\mathcal{T}_2} \rightarrow y(t)$$

Mathematically:  $y(t) = \mathcal{T}_2[\mathcal{T}_1[x(t)]]$ .

## 2 Parallel connection

(See text Fig. 1.42(b) on P. 42 )

Mathematically:  $y(t) = \mathcal{T}_1[x(t)] + \mathcal{T}_2[x(t)]$ .

# Example

## Example

$x(t) \rightarrow$  amplifier, gain=5  $\rightarrow$  differentiator  $\rightarrow y(t)$

$$y(t) = 5 \frac{d}{dt} x(t)$$

In this example the order of interconnection is irrelevant. We will learn soon that this is because both subsystems are linear.

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  - Signal Characteristics
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  - Singularity functions (1.4)
  - **Continuous-time systems**
  - Summary

# Classification of CT systems

Two general aspects to categorize:

- **Amplitude** properties
  - A-1 linearity (1.6.6)
  - A-2 stability (1.6.4)
  - A-3 invertibility (1.6.2)
- **Time** properties
  - T-1 causality (1.6.3)
  - T-2 memory (1.6.1)
  - T-3 time-invariance (1.6.5)

**Skill: *Determining classifications of a given CT system***

# A-1 Linearity (1)

## Definition

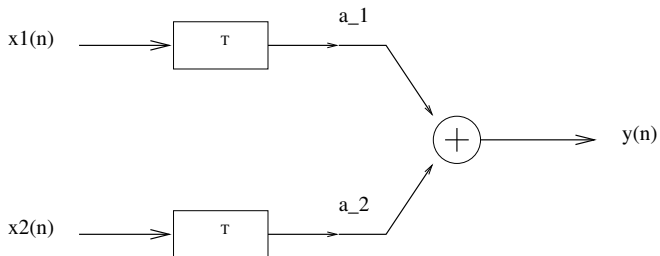
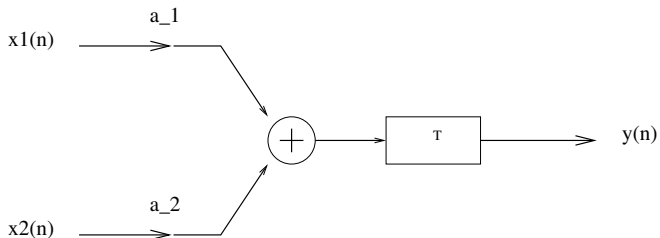
A system  $\mathcal{T}$  is **linear** iff

$$\mathcal{T}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{T}[x_1(t)] + a_2 \mathcal{T}[x_2(t)]$$

for **any signals**  $x_1(t)$ ,  $x_2(t)$  and **any (even complex) constants**  $a_1$  and  $a_2$ . Otherwise the system is called **nonlinear**.

Response to a weighted sum of input signals is the weighted sum of the individual responses. **(Picture)**

# A-1 Linearity (2)

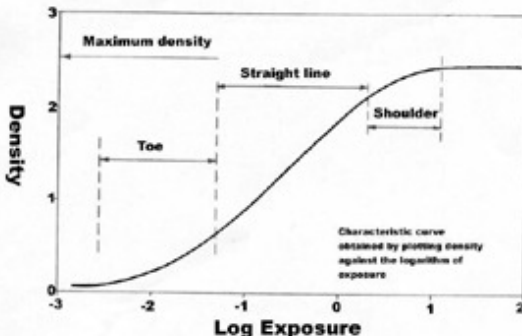


# A-1 Linearity (3)

## Question

*We will focus on linear systems.* Why?

# A-1 Linearity (4)



Real systems are **never perfectly linear**, but often they are **approximately linear over an appropriate operating range**.



# Two important special cases of linearity property (1)

## Property

*scaling property or homogeneity property:*

$$\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)]$$

Note that from  $a = 0$  we see that **zero input signal implies zero output signal for a linear system.**

## Two important special cases of linearity property (2)

### Property

*additivity property:*

$$\mathcal{T}[x_1(t) + x_2(t)] = \mathcal{T}[x_1(t)] + \mathcal{T}[x_2(t)]$$

Using proof-by-induction, one can easily extend this property to the general superposition property

### Property

*general superposition property*

$$\mathcal{T}\left[\sum_{k=1}^K x_k(t)\right] = \sum_{k=1}^K \mathcal{T}[x_k(t)].$$

In words: the response of a linear system to the sum of several signals is the sum of the response to each of the signals.

# General superposition property

- In general superposition need not hold for **infinite sums**; additional continuity assumptions are required.
- We assume the superposition summation holds even for infinite sums without further comment in this course.
- In fact, we even assume that superposition holds for **integrals**:

$$\mathcal{T} \left[ \int x(t; \nu) d\nu \right] = \int \mathcal{T} [x(t; \nu)] d\nu$$

# Determining a system is linear or nonlinear

## **Skill: *Determining a system is linear or nonlinear.***

- ① Find output signal  $y_1(t)$  for a **general** input signal  $x_1(t)$ .
- ② “Repeat” for input  $x_2(t)$  and  $y_2(t)$ .
- ③ Find output signal  $y(t)$  when input signal is  $x(t) = a_1 x_1(t) + a_2 x_2(t)$ .
- ④ If  $y(t) = a_1 y_1(t) + a_2 y_2(t) \forall t$ , then the system is linear.
- ⑤ If it does not appear that  $y(t) = a_1 y_1(t) + a_2 y_2(t) \forall t$ , then find a **specific counter-example**.

# Example (1)

## Example

Prove that the integrator is a linear system, where

$$y(t) = \int_{-\infty}^t x(\tau) d\tau.$$

## Example (2)

### Example

Determine whether linearity holds for  $y(t) = \int_{-\infty}^t x^3(\tau) d\tau$ .

## Example (3)

### Example

Are the following systems linear?

- $y(t) = x^3(t)$
- $y(t) = 2x(t) + 3$

## Example (4)

### Example

Is  $y(t) = \text{Real}[x(t)]$  linear?



# A-2 Stability (1)

## Definition

A system is **bounded-input bounded-output (BIBO) stable** iff every bounded input produces a bounded output.

If  $\exists M_x$  s.t.  $|x(t)| \leq M_x < \infty \forall t$ , then there must exist an  $M_y$  s.t.  
 $|y(t)| \leq M_y < \infty \forall t$ .

Usually  $M_y$  will depend on  $M_x$ .

Otherwise the system is called **unstable**, and it is possible that a small input signal will make the output “blow up.”

# A-2 Stability: example (1)

## Example

Is the integrator system  $y(t) = \int_{-\infty}^t x(\tau) d\tau$  BIBO stable?

# Triangle inequality

The **triangle inequality** is sometimes useful for proving that a system is BIBO stable.

- $|a + b| \leq |a| + |b|$  (Easily proved by considering 4 cases where  $a$  and  $b$  are positive or negative.)
- $|\sum_n a_n| \leq \sum_n |a_n|$
- $|\int f(t) dt| \leq \int |f(t)| dt$

## A-2 Stability: example (2)

### Example

Is the moving average  $y(t) = \frac{1}{T} \int_{t-T}^t x(\tau) d\tau$  for  $T > 0$  system BIBO stable?

## A-2 Stability: example (3)

### Example

Is  $y(t) = x^5(t)$  BIBO stable?

# A-3 Invertibility

## Definition

A system  $\mathcal{T}$  is called **invertible** iff each (possible) output signal is the response to only one input signal. Otherwise  $\mathcal{T}$  is not **invertible**.

## Property

*If a system  $\mathcal{T}$  is invertible, then there exists a system  $\mathcal{T}^{-1}$  such that*

$$x(t) \rightarrow \boxed{\mathcal{T}} \rightarrow y(t) \rightarrow \boxed{\mathcal{T}^{-1}} \rightarrow z(t) = x(t).$$

*Mathematically:*

$$\mathcal{T}^{-1}[\mathcal{T}[x(t)]] = x(t)$$

Design of  $\mathcal{T}^{-1}$  is important in many signal processing applications.

# A-3 Invertibility: example (1)

## Example

encryption/decryption for secure communication. Needs to be invertible for no loss of information.

## Example

digital speedometer

velocity  $\rightarrow$  speed sensor  $\rightarrow$  voltage

$\rightarrow$  mathematical inverse of sensor law  $\rightarrow$  velocity display.

We display the velocity, not the voltage, so there should be a one-to-one relationship between the two.

## A-3 Invertibility: example (2)

### Example

Is the full-wave rectifier:  $y(t) = |x(t)|$  invertible?



## A-3 Invertibility: example (3)

### Example

Is the exponential-law device:  $y(t) = e^{x(t)}$  invertible?

## A-3 Invertibility: example (4)

### Example

Is the ideal amplifier:  $y(t) = 2x(t)$  invertible?

# T-1 Causal systems

## Definition

For a **causal** system, the output  $y(t)$  at any time  $t$  depends **only** on the “present” and (possibly) “past” inputs *i.e.* on  $x(t)$  and on various  $x(t_0)$  for  $t_0 \leq t$  only, but **not** on future inputs. Otherwise **noncausal** system.

Causality is necessary for **real-time implementation**. Noncausal systems arise primarily when  $t$  is some other variable than time, such as space.

# T-1 Causal systems: example (1)

## Example

Is the integrator:  $y(t) = \int_{-\infty}^t x(\tau) d\tau$  causal?

# T-1 Causal systems: example (2)

## Example

Is the symmetric moving average:  $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\tau) d\tau$  causal? (Useful for image processing.)

# T-2 Memory

## Definition

For a **static system** or **memoryless** system, the output  $y(t)$  depends only on the current input  $x(t)$ , not on previous or future values of the input signal.

Otherwise it is a **dynamic system** and must have memory.

## Example

- $y(t) = e^{x(t)} / \sqrt{|t+3|}$ .
- moving average  $y(t) = \frac{1}{T} \int_{t-T}^t x(\tau) d\tau, T > 0$

**Dynamic systems** are the interesting ones and will be **our focus**. (This time we take the more complicated choice!)

# Memory vs. causality

## Question

- *Is a memoryless system necessarily causal?*
- *Is a dynamic system necessarily noncausal?*

# T-3 Time-invariance (1)

Systems whose input-output behavior does not change with time are called **time-invariant** will be **our focus**.

- “**Easier**” to analyze.
- Time-invariance is a **desired property** of many systems.

We will focus primarily, but not exclusively, on time-invariant systems.



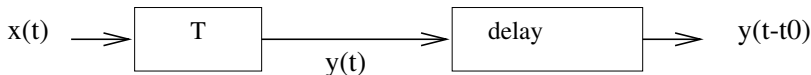
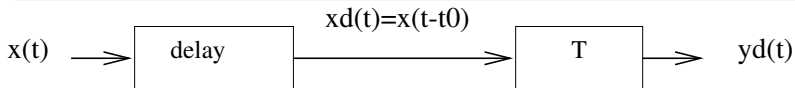
# T-3 Time-invariance (2)

## Definition

A system  $\mathcal{T}$  is called **time invariant** or **shift invariant** iff

$$x(t) \xrightarrow{\mathcal{T}} y(t) \text{ implies that } x(t - t_0) \xrightarrow{\mathcal{T}} y(t - t_0)$$

for **every** input signal  $x(t)$  and time shift  $t_0$ . Otherwise the system is called **time variant** or **shift variant**.



# Recipe for showing time-invariance

## Recipe for showing time-invariance

- 1 Determine output signal  $y(t)$  due to a generic input signal  $x(t)$ .
- 2 Determine the **delayed output** signal  $y(t - t_0)$ , by **replacing  $t$  with  $t - t_0$  in  $y(t)$  expression**.
- 3 Determine output signal  $y_d(t)$  due to a **delayed input** signal  $x_d(t) = x(t - t_0)$ .
- 4 If  $y_d(t) = y(t - t_0)$ , then system is time-invariant.

# Time-invariance: example (1)

## Example

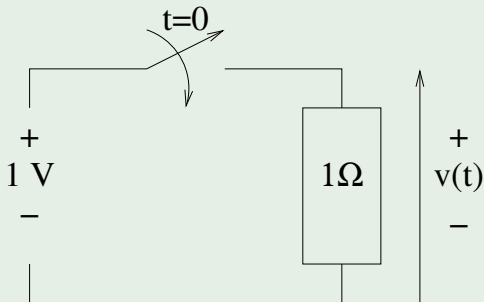
Is the symmetric moving average filter

$y(t) = \frac{1}{3}[x(t-1) + x(t) + x(t+1)]$  time-invariant?

# Time-invariance: example (2)

## Example

A switch that closes at  $t = 1$ .



- 1 How to represent input-output relationship mathematically?
- 2 Is it Time invariant? If no, find a counter-example.

# Time-invariance: example (3)

## Example

Is the modulator  $y(t) = \cos(\pi t) x(t)$  time-invariant?

# Time-invariance: example (4)

## Example

Is the amplified time reversal  $y(t) = 3x(-t)$  time-invariant?

# Time-invariance: example (5)

## Example

Is the integrator  $y(t) = \int_{-\infty}^t x(\tau) d\tau$  time-invariant?

# Outline

- 1 1. Signals & Systems (Fundamentals)
  - Overview
  - Signal and System Definition
  - Classification of Signals
  - Signal Notation
  - Transformations of CT signals
  - Signal Characteristics
  - Exponential signals
  - Singularity functions (1.4)
  - Continuous-time systems
  - **Summary**



# Summary (1)

- signal notation
- signal transformations
  - time transformations
  - amplitude transformations
  - differentiator / integrator systems
  - two-signal operations
- signal classes
  - even/odd signals
  - energy/power signals
  - periodic/apperiodic signals
  - exponential signals

# Summary (2)

- singularity functions
  - unit step / rect signals
  - unit impulse function
  - impulse function properties (sifting, sampling, scaling)
- CT systems
- block diagrams
- system classes
  - amplitude properties: linearity, stability, invertibility
  - time properties: causality, memory, time-invariance

# Key concepts/skills to study

## Key concepts/skills to study

- time transformations
- braces/plots to rects/steps
- running integral operation
- properties of  $\delta(t)$
- identifying signal properties
- identifying (all six) system properties