Ve216 Introduction to Signals and Systems: Summary
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(Based on Lecture Notes by Prof. Jeffrey A. Fessler)

Signals and Systems: Summary 1

- Circuits: $v(t) = Ri(t)$, $v(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i(t)$, $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t)$
- Notation: $x(t) = \begin{cases} e^{-t}, & t > 2, \\ 0, & \text{otherwise} \end{cases} = e^{-t}u(t-2)$
- Time transformation:
  - $x\left(\frac{t-t_0}{w}\right)$. First scale according to w, then shift according to t_0.
  - $x(at-b)$. First time-delay by b, then time-scale by a
- Integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$
- Even symmetry: $x(-t) = x(t)$, Odd symmetry: $x(-t) = -x(t)$
- $\mathrm{Ev}\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$, $\mathrm{Od}\{x(t)\} = \frac{1}{2}(x(t) - x(-t))$, $x(t) = \mathrm{Ev}\{x(t)\} + \mathrm{Od}\{x(t)\}$.

- Average value: $A \stackrel{\triangle}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$
- Energy: $E \stackrel{\triangle}{=} \int_{-\infty}^{\infty} |x(t)|^2\,dt$
- Average power: $P \stackrel{\triangle}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$
- Energy signal: $E < \infty$, $P = 0$.
- Power signal: $E = \infty$, $0 < P < \infty$.
- Power of periodic signal: $P = \frac{1}{T_0}\int_{0}^{T_0} |x(t)|^2\,dt$

- Step function: $u(t) = 1$ for $t > 0$.
- Rect function: $\mathrm{rect}(t) = 1$ for $-1/2 < t < 1/2$, $\mathrm{rect}(t) = u(t+1/2) - u(t-1/2) = u(t+1/2)u(1/2-t)$
- Impulse functions
  - Sifting property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$ if $x(t)$ is continuous at t_0
  - Sampling property: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$ if $x(t)$ is continuous at t_0
  - unit area property: $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$ for any t_0
  - scaling property: $\delta(at+b) = \frac{1}{|a|}\delta(t+b/a)$ for $a \neq 0$.
  - symmetry property: $\delta(t) = \delta(-t)$
  - support property: $\delta(t-t_0) = 0$ for $t \neq t_0$
  - relationships with unit step function: $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$, $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

Continuous-time system properties

- Stability (BIBO): all bounded input signals produce bounded output signals
- Invertibility: each output signal is the response to only one input signal
- Causal: output signal value $y(t)$ at any time t depends only on present and past input signal values.
- Static (memoryless): output at any time only depends on input signal at the same time. (otherwise dynamic)
- Time invariant:

- $x(t) \xrightarrow{\mathcal{T}} y(t)$ implies that $x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0)$
- Recipe for showing time-invariance.
  - Determine output signal $y(t)$ due to a generic input signal $x(t)$.
  - Determine the delayed output signal $y(t-t_0)$, by replacing t with $t-t_0$ in $y(t)$ expression.
  - Determine output signal $y_d(t)$ due to a delayed input signal $x_d(t) = x(t-t_0)$.
  - If $y_d(t) = y(t-t_0)$, then system is time-invariant.
- Linear systems:
  - superposition property: $\mathcal{T}[\sum_k a_k x_k(t)] = \sum_k a_k \mathcal{T}[x_k(t)]$
  - additivity property: $\mathcal{T}[x_1(t) + x_2(t)] = \mathcal{T}[x_1(t)] + \mathcal{T}[x_2(t)]$
  - scaling property: $\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)]$

LTI systems

input-output relationship described by convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

Properties:
- Commutative property: $x(t) * h(t) = h(t) * x(t)$
- Associative property: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive property: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$
- The order of serial connection of LTI systems does not affect the overall impulse response.
- $x(t) * \delta(t) = x(t)$
- Delay property: $x(t) * \delta(t-t_0) = x(t-t_0)$
- $\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$
- Time-invariance: If $y(t) = x(t) * h(t)$, then $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$

LTI system properties
- causal: $h(t) = 0$ for all $t < 0$
- static: $h(t) = k\delta(t)$, otherwise dynamic
- stable: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$
- invertible: $h(t) * h_i(t) = \delta(t)$ for some $h_i(t)$
  If $h(t) * x(t) = 0$ for some nonzero signal $x(t)$, then not invertible
- step response: $h(t) = \frac{\mathrm{d}}{\mathrm{d}t} s(t)$, where $u(t) \xrightarrow{\text{LTI}} s(t)$

Linear, constant coefficient, differential equation systems
- LTI and causal if initially at rest
- dynamic unless $N = M = 0$
- homogenous solution, natural response:
  $y_h(t) = \sum_l C_l e^{s_l t}$, where s_l's are the N roots of the characteristic polynomial $\sum_{k=0}^{N} a_k s^k = 0$.
- particular solution, forced response: $y_p(t) = P_0 x(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \cdots$

Topics covered in Chap. 1-2

Chap. 1
- signal classes
- signal notation
- * time transformations
- amplitude transformations
- signal operations
- integrator system (running integral operation)
- * even/odd signals
- * energy and power signals
- periodicity
- * unit step / rect signals
- * unit impulse function
- * impulse function properties (sifting, sampling)
- CT systems
- block diagrams
- system classes
- * amplitude properties: linearity, stability, invertibility
- * time properties: causality, memory, time-invariance

Chap. 2
- impulse response $h(t)$
- convolution for CT LTI systems
- * graphical convolution
- * properties of convolution and LTI systems
- impulse response vs step response
- * properties of convolution and impulse response
- LTI system properties characterized by $h(t)$(causality, memory, stability, invertibility)
- diffeq systems
- solutions of diffeq

The items with a * are virtually guaranteed to be on the exam.

Signals and Systems: Summary 2

Fourier Series

- Analysis equation: $c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt,\ k = 0, \pm1, \pm2, \ldots.$
- DC value: $c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt.$
- Synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$
- Combined trigonometric form: $x(t) = c_0 + \sum_{k=1}^{\infty} 2\,|c_k|\cos(k\omega_0 t + \angle c_k)$, if $x(t)$ is real.
- Trigonometric form: $x(t) = c_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) - B_k \sin(k\omega_0 t)$, where $A_k = \mathrm{real}\{c_k\}$ and $B_k = \mathrm{Imag}(c_k)$

Convergence properties

- Error signal $e_N(t) = x(t) - x_N(t)$ where $x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$ when c_k's chosen according to above FS analysis equation
- Error signal energy $E_N = \int_{T_0} |e_N(t)|^2\ dt \to 0$ as $N \to \infty$, provided $x(t)$ square integrable: $\int_{T_0} |x(t)|^2\,dt < \infty$
- $e_N(t) \to 0$ as $N \to \infty$ provided Dirichlet conditions hold
- Near the discontinuity there will usually be overshoot and/or undershoot that persists even as N increases, which is called Gibbs phenomenon.

One-signal properties (Fourier series transformations)

- Amplitude transformation: $ax(t) + b \leftrightarrow \begin{cases} b + ac_0, & k = 0 \\ ac_k, & k \neq 0. \end{cases}$
- Time transformation: $x(at+b) \leftrightarrow \begin{cases} c_k e^{jk\omega_0 b}, & a > 0 \\ c_{-k} e^{jk\omega_0 b}, & a < 0. \end{cases} \quad \omega_1 = |a|\omega_0$
- Time shift: $x(t - t_0) \leftrightarrow c_k e^{-jk\omega_0 t_0}$
- Conjugation: $[x(t)]^* \leftrightarrow c_{-k}^*$
- Complex modulation (frequency shift): $x(t)e^{j\omega_0 tN} \leftrightarrow c_{k-N}$
- Differentiation: $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow jk\omega_0\, c_k$

Properties

- If $x(t)$ is real, then $c_{-k} = c_k^*$.
- Linearity (add coefficients if same period T_0)
- Multiplication $c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$. (discrete convolution)
- Filtering: see below
- Circular convolution: skip
- Total harmonic distortion: THD = $(1 - 2|c_1|^2/P)\cdot 100\%$
- Power of $ce^{jk\omega_0 t}$ is $|c|^2$
- Power of $A\cos(\omega t + \phi)$ is $A^2/2$
- Parseval's theorem: $P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\ dt = \sum_{k=-\infty}^{\infty} |c_k|^2$
- Power density spectrum: $|c_k|^2$
- Magnitude spectrum: $|c_k|$. Phase spectrum: $\angle c_k$

Foundations of Filtering

- $x(t) = e^{st} \xrightarrow{\text{LTI}} y(t) = H(s)e^{st}$
- Laplace transform of $h(t)$, aka system function: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$
- $x(t) = e^{j\omega t} \xrightarrow{\text{LTI}} y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}$
- Fourier transform of $h(t)$, aka frequency response: $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = H(s)|_{s=j\omega} = |H(j\omega)|e^{j\angle H(j\omega)}$
- $x(t) = \sum_k c_k e^{j\omega_k t} \xrightarrow{\text{LTI}} y(t) = \sum_k c_k H(j\omega_k)e^{j\omega_k t}$
- If $h(t)$ is real, then $H^*(s) = H(s^*)$ and $H(-j\omega) = H^*(j\omega)$. (Hermitian symmetry)
- If $h(t)$ is real, $x(t) = \cos(\omega t + \phi) \xrightarrow{\text{LTI}} y(t) = |H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega))$
- $x(t) = \sum_k A_k \cos(\omega_k t + \phi_k) \to \boxed{\text{LTI } h(t)} \to y(t) = \sum_k A_k |H(j\omega_k)|\cos(\omega_k t + \phi_k + \angle H(j\omega_k))$

Fourier Transform

- Fourier transform (analysis): $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt$.
- Inverse Fourier transform (synthesis): $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$
- The FT of a signal $f(t)$ exists (converges) if $f(t)$ is absolutely integrable, *i.e.*, $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. (more rigorously, $f(t)$ satisfies the Dirichlet conditions)
- For periodic signals: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$.
- $x(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 G(k\omega_0)\delta(\omega - k\omega_0)$
- Energy density spectrum: $|X(\omega)|^2$
- Energy over a spectral band: $E_B = \frac{1}{2\pi} \int_B |X(\omega)|^2\, d\omega$
- Symmetry properties $\begin{array}{ccccccccc} f(t) & = & f_R^e(t) & + & jf_I^e(t) & + & f_R^o(t) & + & jf_I^o(t) \\ & & \updownarrow & & \updownarrow & & & & \times \\ F(\omega) & = & F_R^e(\omega) & + & jF_I^e(\omega) & + & F_R^o(\omega) & + & jF_I^o(\omega) \end{array}$

Topics covered in Chap. 3-4

Ch. 3

- eigenfunctions of LTI systems (complex-exponential signals)
- * Fourier series (3.3)
- Convergence of Fourier series (Gibbs phenomenon)(3.4)
- Properties of Fourier series (3.5)
- trigonometric forms of FS
- system transfer function (Laplace)
- frequency response (Fourier)
- Parseval's Relation for CT Periodic Signals(3.5.7)
- Power density spectrum
- magnitude/phase spectrum
- * Fourier Series and LTI Systems (3.8)
- differentiation/modulation properties
- Filtering (3.9)
- * Filters described by diffeqs (3.10)
- Rational transfer functions for diffeq systems

Ch. 4

- Defined FT and inverse FT by limits of FS
- Existence of FT
- * FT of many important signals
- * FT properties
- FT of periodic signals
- Parseval's relation (Energy density spectrum)
- * convolution property and LTI systems
- * Application of FT to RLC and diffeq systems

The items with a * are virtually guaranteed to be on the exam.

Table of Fourier Series for Common Signals

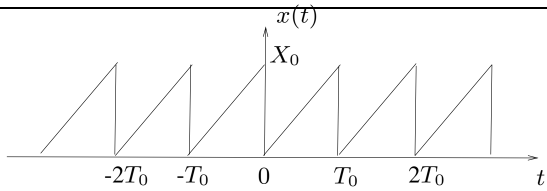
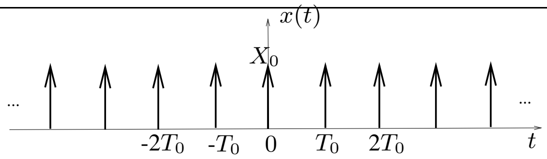
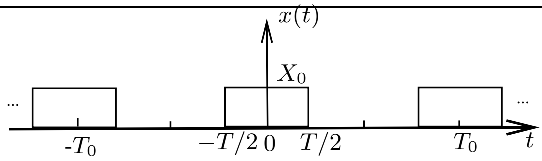
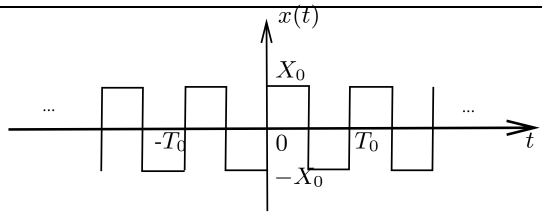
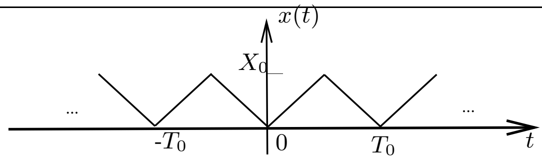
Name	Waveform	c_0	$c_k, k \neq 0$	Comments
Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	
Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\,\mathrm{sinc}\left(\frac{Tk\omega_0}{2\pi}\right)$	$\frac{Tk\omega_0}{2\pi} = \frac{Tk}{T_0}$
Square wave		0	$-j\frac{2X_0}{\pi k}$	$c_k = 0$, k even
Triangular wave sine		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$c_k = 0$, k even

Table of Fourier transform pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega) = \delta\left(\frac{\omega}{2\pi}\right)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
$\mathrm{e}^{\jmath\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$
$\sin\omega_0 t$	$\frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)$
e^{-bt^2}	$\sqrt{\pi/b}\,\mathrm{e}^{-\omega^2/(4b)}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\sum_{k=-\infty}^{\infty}\omega_0\,\delta(\omega-k\omega_0)$

$f(t)$	$F(\omega)$
$\frac{1}{b^2+t^2}$	$\frac{\pi}{b}\,\mathrm{e}^{-b\lvert\omega\rvert}$
$\mathrm{e}^{-b\lvert t\rvert}$	$\frac{2b}{b^2+\omega^2}$
$\operatorname{rect}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}\left(T\frac{\omega}{2\pi}\right)$
$\operatorname{tri}(t)$	$\operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\frac{\omega_0}{2\pi}\operatorname{sinc}\left(\frac{\omega_0}{2\pi}t\right)$	$\operatorname{rect}\left(\frac{\omega}{\omega_0}\right)$
$\operatorname{sinc}^2(t)$	$\operatorname{tri}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{e}^{-at}\,u(t)$	$\frac{1}{j\omega+a}$
$\frac{t^{n-1}}{(n-1)!}\,\mathrm{e}^{-at}\,u(t)$	$\frac{1}{(j\omega+a)^n}$
$\frac{j}{\pi t}$	$\operatorname{sgn}(\omega)$

b is a real positive number throughout. a is a real or complex number throughout, with positive real part.

Properties of the Continuous-Time Fourier Transform

	Time	Fourier
Synthesis, Analysis	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$
Eigenfunction	$h(t) * e^{j\omega_0 t} = H(\omega_0)e^{j\omega_0 t}$	$H(\omega)2\pi\delta(\omega-\omega_0)$ $= H(\omega_0)2\pi\delta(\omega-\omega_0)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Time transformation	$f(at+b),\ a \neq 0$	$\frac{1}{\lvert a\rvert}e^{j\omega b/a}F(\omega/a)$
Time shift	$f(t-\tau)$	$F(\omega)e^{-j\omega\tau}$
Time reversal	$f(-t)$	$F(-\omega)$
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Convolution	$f_1(t) * f_2(t)$	$F_1(\omega)\cdot F_2(\omega)$
Time-domain Multiplication	$f_1(t)\cdot f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Modulation (cosine)	$f(t)\cos(\omega_0 t)$	$\frac{F(\omega-\omega_0)+F(\omega+\omega_0)}{2}$
Time. Differentiation	$\frac{d^n}{dt^n}f(t)$	$(j\omega)^n F(\omega)$
Freq. Differentiation	$(-jt)^n f(t)$	$\frac{d^n}{d\omega^n}F(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$
Conjugation	$f^*(t)$	$F^*(-\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Relation to Laplace	$F(\omega) = F(s)\rvert_{s=j\omega}$, if ROC includes $j\omega$ axis	
Parseval's Theorem	$\int_{-\infty}^{\infty} f_1(t)f_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2^*(\omega)\, d\omega$	
Parseval/Rayleigh Theorem	$E = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2\, d\omega$	
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$	

A function that satisfies $f(t) = f^*(-t)$ is said to have **Hermitian symmetry**.

Tips for Exam Preparation

- **Study lecture slides.** Read through lecture slides and the summary notes carefully. Make sure that you fully understand all the lecture materials.
- **Study homework solutions.** Review your HW sets and the posted solutions on Canvas.
- **Attend exam recitation classes.** TAs will posted times on Canvas.