Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

Review FS and FT

We have seen that Fourier methods are very useful in the study of many problems involving signals and LTI systems.

- We can represent a broad class of signals using linear combinations of complex exponential signals ($e^{j\omega t}$).

$$\text{FS} \quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\, dt, k = 0, \pm 1, \ldots$$

$$\text{FT} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega, \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

- Those complex exponential signals are eigenfunctions of LTI systems.
- Those complex exponential signals are solutions to linear constant coefficient differential equations too.

General complex exponential signal

- **Fourier series** and **Fourier transforms** use signals of the form e^{st} where $s = j\omega$, because signals of the form $e^{j\omega t}$ are eigenfunctions of LTI systems, which led to the concept of frequency response, filtering, etc.
- Many of the properties of e^{st} also apply when s is a general complex number $s = \sigma + j\omega$, rather than a pure imaginary number $s = j\omega$.

Laplace transform

In particular, the eigenfunction property holds, as shown previously:

$$e^{st} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow H(s)e^{st}, \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt.$$

Often there are advantages in reformulating some of the previously discussed ideas in the more general context of $s = \sigma + j\omega$.

The **Laplace transform** (LT) is the generalization of the **Fourier transform** to include general complex exponential signals of the form

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}.$$

Bilateral Laplace transform

Laplace transforms come in two flavors

- Bilateral Laplace transform (two-sided)
- Unilateral Laplace transform (on-sided)

- We will focus on the bilateral form of the LT, and will probably not have time to discuss the unilateral form.
- For causal systems and signals, the two forms are identical, so for many problems of interest there is no need to make a distinction. (Most properties are the same or very similar.)
- For brevity, I will speak simply of the “Laplace transform” and specify “bilateral” only occasionally.
- The treatment in these notes fairly closely follows that of Ch. 9 in textbook.

Overview

- bilateral
- ROC
- rational Laplace transforms
- pole-zero plots
- inverse LT
- ROC, causality, stability
- Magnitude response from pole-zero plot
- Properties
- application to LTI systems / filtering
- Feedback control

Outline

Bilateral Laplace transform

Definition

The **bilateral Laplace transform** is defined as

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}\, dt,$$

where $s = \sigma + j\omega$ is a complex variable with real part σ and imaginary part ω.

The following notation denotes LT pairs:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s).$$

Unilateral Laplace transform

Definition

The **unilateral Laplace transform** is defined as

$$X_+(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt,$$

where the 0^- is included to handle an impulse function at 0.

Laplace transform: example (1)

Example

Find LT of $x(t) = e^{-at}u(t)$.

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{1}{-(s+a)} e^{-(s+a)t} \Big|_{0}^{\infty} = \frac{1}{s+a}[1 - e^{-(s+a)\infty}] = ?$$

Question

When is this integral finite?

Region of convergence (ROC)

In general, the bilateral Laplace transform will exist for some values of real$\{s\}$ and not for others.

Definition

The set of values of s for which the bilateral Laplace transform is guaranteed to exist ($x(t)e^{-\text{real}\{s\}\,t}$ is absolutely integrable) is given by

$$\text{ROC} \triangleq \left\{ s : \int_{-\infty}^{\infty} |x(t)| e^{-\text{real}\{s\}\,t}\, dt < \infty \right\},$$

and is called the **region of convergence** or **ROC**.

The ROC depends on the signal $x(t)$.

ROC example

Question

What is the ROC in the preceding example $x(t) = e^{-at}u(t)$*?*

Laplace transform: example (2)

Example

Find LT and ROC of $x(t) = -e^{-at}u(-t)$.

Outline

Why an ROC?

The LT for any given s is the signed "area" within the product of $x(t)$ with e^{-st} and t-axis.

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

Example

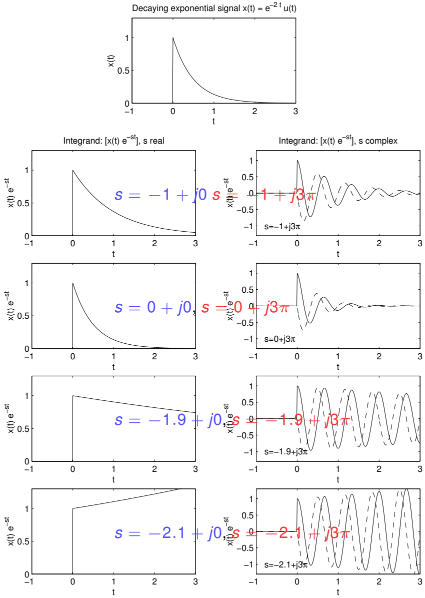
$$x(t) = e^{-2t}u(t)$$

- If $\mathrm{real}\{s\} > -2$, then the product of $x(t) = e^{-2t}u(t)$ with e^{-st} is a decaying exponential, so the area is finite.

$$e^{-2t}u(t)e^{-st} = e^{-(2+s)t}, \text{for } t \geq 0 \implies \mathrm{real}\{2+s\} > 0$$

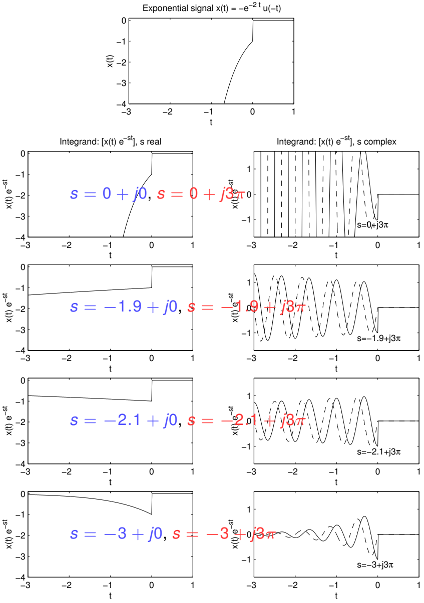
- If $\mathrm{real}\{s\} \leq -2$, then the product of the two functions has "infinite area" (more precisely the LT is undefined since things like $\infty - \infty$ would come into play).

Decaying exponential



- $x(t) = e^{-2t}u(t)$
- Solid line: real
- Dashed line: imag
- ROC: $\text{real}\{s\} > \text{real}\{-2\}$

Exponential

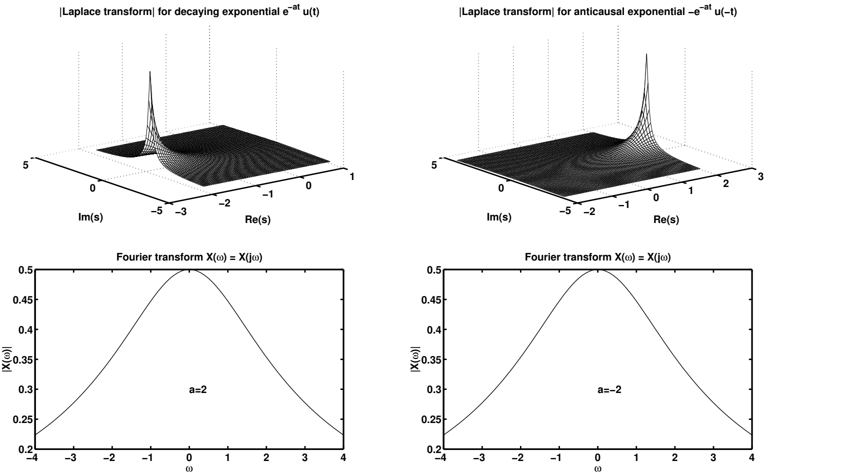


- $x(t) = -e^{-2t}u(-t)$
- Solid line: real
- Dashed line: imaginary
- ROC: $\text{real}\{s\} < \text{real}\{-2\}$

Plotting a Laplace transform

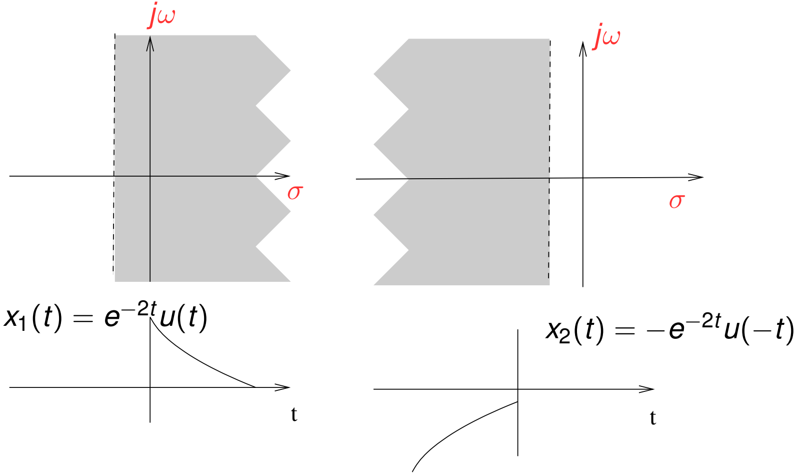
- Since $s = \sigma + j\omega$ varies with both its real part σ and its imaginary part ω, in a sense the Laplace transform is a 2D function.
- To "plot" a Laplace transform one would use a `mesh` or `surface` plot in MATLAB, showing the value of $X(s)$ as a function of (σ,ω) over the complex plane.
- This is rarely done, since we will see later that a **pole-zero plot** provides the same information more easily.

Plotting a Laplace transform



Display of the ROC (1)

Often we display the ROC of a signal using the complex s-plane as shown in the following figures.



Display of the ROC (2)

- The horizontal axis is usually called the σ **axis**, and the vertical axis is usually called the $j\omega$**axis**.
- The shaded region indicates the set of points in the *s*-plane where the bilateral Laplace transform exists, *i.e.*, the ROC.
- Dotted lines for boundaries if ROC does not include its edges.
- If the shaded region includes the $j\omega$axis, then the FT of the signal exists.

Display of the ROC: Example

Example

Find the LT of $x(t) = 3e^{-2t}u(t) + 4e^{t}u(-t) + \delta(t)$ and sketch its ROC.

Relation to Fourier transform

- When $\sigma = 0$, the Laplace transform integral is the same as the Fourier transform integral.
- Thus, the value of the bilateral Laplace transform along the $j\omega$axis is the FT of the signal. Mathematically

$$X(\omega) = X(s)|_{s=j\omega} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

if ROC includes $j\omega$axis.

- Note that $X(\omega)$ and $X(j\omega)$ are just different notations for the same thing (the integral above); this reuse of notation is common in most books and papers.
- The Fourier transform of a signal $x(t)$ exists if and only if the ROC of $X(s)$ includes the imaginary axis.

Example

Example

$x(t) = -e^{-at}u(-t)$ again...
When does the FT of the preceding signal exist?

Why both LT and FT?

So the Laplace transform generalizes the FT in the sense that some signals have a LT that do not have a FT.

Question

Why both LT and FT are needed? Are there signals that have FT but no LT?

Example

Example

1. Find LT of the causal cosinusoidal signal, $x(t) = \cos(\omega_0 t)\, u(t)$.
2. Find LT of the anti-causal cosinusoidal signal, $x(t) = \cos(\omega_0 t)\, u(-t)$.
3. Find LT of the cosinusoidal signal, $x(t) = \cos(\omega_0 t)$.

Outline

Signals with rational Laplace transforms

Definition

If the Laplace transform of a signal $x(t)$ has the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where $N(s)$ and $D(s)$ are polynomials in s, then we say that the Laplace transform $X(s)$ is **rational**.

Poles and Zeros

Definition

Rational Laplace transform

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- The roots of the denominator are called **poles**, since if s_0 is a pole, then $X(s_0) = N(s_0)/0 = \infty$.
- The roots of the numerator are called **zeros**, since if s_0 is a zero, then $X(s_0) = 0/D(s_0) = 0$.
- The factor $G = b_m/a_n$ is called the **gain**.

Outline

Pole-zero plot

- A rational LT can be completely described by its **pole-zero plot**, along with a gain G.
  (Picture MIT Lecture 20.2-4)
- The corresponding signal $x(t)$ is completely specified provided we know 3 things: the pole-zero plot, the gain G, and the ROC.
- The ROC of a rational Laplace transform $X(s)$ is bounded by its poles (or by infinity).

Properties of any ROC (1)

Property

1. *The ROC consists of "strips" (which may be empty or include the entire s-plane) parallel to the $j\omega$axis in the s-plane.* ***(Picture)****(MIT Lecture 20.2-4)*
2. *The ROC of $X(s)$ does not contain any poles of $X(s)$, if $X(s)$ is rational.*

For some signals, such as $e^{|t|}$ or $\cos(\omega_0 t)$, the ROC is the empty set and we say that the Laplace transform of $x(t)$ does not exist.

Properties of ROC (2)

$$\mathrm{ROC} \triangleq \left\{ s : \int_{-\infty}^{\infty} |x(t)| e^{-\,\mathrm{real}\{s\}\,t}\, dt < \infty \right\},$$

absolutely integrable of $x(t)e^{-\,\mathrm{real}\{s\}\,t}$

For signals that have Laplace transforms that exist, the ROC always falls into one of the following categories, depending on the signal characteristics.

1. finite duration signal $\Longrightarrow$ the entire s-plane
2. right-sided signal $\Longrightarrow$ right half plane
3. left-sided signal $\Longrightarrow$ left half plane
4. two-sided signal $\Longrightarrow$ vertical strip

Finite duration signals

Property

If $x(t)$ is finite duration and absolutely integrable, then $\mathrm{ROC} = \mathbb{C}$ (the entire s-plane).

The intuition behind this result is suggested in

- $x(t)$multiplied by a decaying exponential (MIT Lecture 20.5)
- $x(t)$multiplied by a growing exponential (MIT Lecture 20.6)

Since the interval over which $x(t)$ is nonzero is finite, the exponential weighting is never unbounded, and consequently, it is reasonable that the integrability of $x(t)$ is not destroyed by this exponential weighting.
(formal verification, textbook, p.664)

Right-sided signals (1)

Property

If the signal $x(t)$ is a right-sided signal, i.e.,

$$x(t) = 0, \text{ for } t < T_1, \text{ where } T_1 \text{ is some constant,}$$

then the ROC of $x(t)$ will be a right half plane (RHP) of $\text{real}\{s\} > \sigma_0$*, for some σ_0.*

Suppose that the Laplace transform converges for some σ_0, then if $\sigma_1 > \sigma_0$, it must also be true that $x(t)e^{-\sigma_1 t}$ is absolutely integrable, since $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \to \infty$. ***(Picture)***(MIT Lecture 20.7)

Right-sided signals (2)

Property

If $X(s)$ has a rational form, then if $x(t)$ is right-sided, the RHP of ROC will be everything to the right of the rightmost pole.

Left-sided signals

Property

If the signal $x(t)$ is a left-sided signal, i.e.,

$$x(t) = 0, \text{ for } t > T_2, \text{ where } T_2 \text{ is some constant,}$$

then the ROC of $x(t)$ will be a left half plane (LHP) of $\text{real}\{s\} < \sigma_0$*, for some* σ_0*.*

Property

If $X(s)$ has a rational form, then if $x(t)$ is left-sided, the LHP of ROC *will be everything to the left of the leftmost pole.*

(Picture)(MIT Lecture 20.4)

Two-sided signals

Property

If the signal $x(t)$ is a two-sided signal, then the ROC will be a vertical strip.

(Explanation, textbook, p. 666)

Property

If $X(s)$ is rational, then the ROC will have the form $\sigma_1 < \text{real}\{s\} < \sigma_2$, for some $\sigma_1 < \sigma_2$, and in fact the ROC will be a strip between a pair of adjacent poles (not including any other poles).

(Picture)(MIT Lecture 20.3)

Example

Example

Where do causal signals fall into the above categories?

Outline

Table of Laplace transform pairs (1)

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

Table of Laplace transform pairs (2)

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\,u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{real}\{s\} > 0$
$\cos(\omega_0 t)\,u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\mathrm{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{real}\{s\} > \mathrm{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{real}\{s\} > \mathrm{real}\{-a\}$
$u_n(t) = \dfrac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\dfrac{1}{s^n}$	$\mathrm{real}\{s\} > 0$

Outline

Inverse Laplace transform (1)

Question

Given a LT $X(s)$, how do we invert it to find the signal $x(t)$?

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \Longrightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$\Longrightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\Longrightarrow x(t)e^{-\sigma t} \overset{\mathcal{F}}{\longleftrightarrow} X(\sigma + j\omega)$$

Inverse FT $\quad x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$

$$x(t) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

Inverse Laplace transform (2)

Combining $e^{\sigma t}$ and $e^{j\omega t}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}\, d\omega$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\, ds,$$

where

$$s = \sigma + j\omega, \quad ds = j\, d\omega$$

and σ is any fixed real number that lies in the ROC.

Laplace transform pair

Laplace transform pair:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt, \quad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

- In general, evaluation of the above inverse LT integral requires **contour integration**, a topic from complex analysis.
- In this course, we will only require inverse transforms of rational Laplace transforms, so the PFE/table-lookup method will be the method of choice.

Example

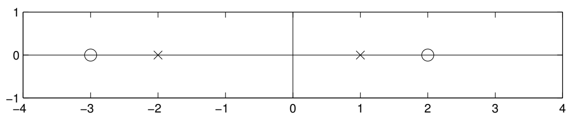
Example

Find $x(t)$ when $X(s) = \frac{1}{(s+1)^2+1}$, where $\mathrm{ROC} = \text{real}\{s\} > -1$.

Example of inverse LT given pole-zero plot

Example

Find “the” signal $x(t)$ having the following pole-zero plot, with a DC value of 12.



Outline

Stability

- Recall: An LTI system is BIBO stable if its impulse response is absolutely integrable, *i.e.*. $\int_{-\infty}^{\infty} |h(t)| < \infty$.
- If $h(t)$ is absolutely integrable, its FT $H(\omega)$ converges (exists).
- The value of the LT along the $j\omega$ axis is the FT of the signal.

> For a system with a rational system function $H(s)$, it is stable iff the ROC of $H(s)$ includes the $j\omega$ axis.

Causality

- Recall: An LTI system is causal iff its impulse response $h(t) = 0$ for all $t < 0$.
- $h(t)$ is a right-sided signal.

For a system with a rational system function $H(s)$, it is causal iff its ROC is a RHP.

Causality and Stability (1)

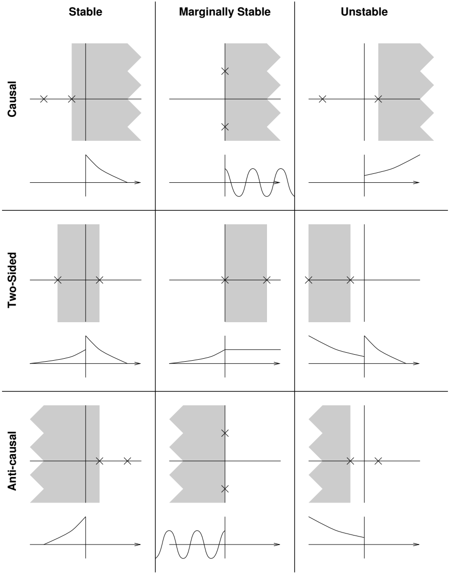
For a system with a rational system function $H(s)$:

- It is stable iff the ROC of $H(s)$ includes the $j\omega$ axis.
- It is causal iff its ROC is a RHP.

Question

How about stable and causal?

Causality and Stability (2)



Causality and Stability (3)

- The above figure summarizes the various cases.
- The zeros can be anywhere, without affecting causality or stability.
- The upper left plot is the case of greatest interest in practice: a causal system whose poles lie in the left half plane.

Example (1)

Example

Consider the system with transfer function $H(s) = s = \frac{s}{1}$. Is this system stable?

Differential equation systems: natural response

linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

(Recall Lecture 9, p.32)
The **natural response** part of the solution to a diffeq has the form (if no repeated roots)

$$y_h(t) = C_1 e^{s_1 t} + \cdots + C_N e^{s_N t},$$

where the s_k's are the roots the characteristic polynomial.
In general these s_k's can be real or complex numbers.

Differential equation systems: impulse response

(Proper) rational Laplace transform:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

$$= G\frac{(s - z_1)\cdots(s - z_m)}{(s - s_1)\cdots(s - s_n)} = \frac{r_1}{s - s_1} + \ldots + \frac{r_N}{s - s_N}$$

The impulse response for a diffeq system has the same form as the natural response, but with different coefficients.

$$\begin{aligned} h(t) &= r_1 e^{s_1 t} u(t) + \cdots + r_N e^{s_N t} u(t) \\ &\quad + \text{(more terms if repeated roots or if } M \geq N\text{)}. \end{aligned}$$

A term Ce^{st} is called a **system mode** or just **mode** of the system.

System mode

The behavior of the system mode depends on whether s is real or complex. We can always write

$$s = \sigma + j\omega.$$

We can then describe s by its location in the s-plane.

There are various cases to consider.

1. s pure real ($s = \sigma$) and $\omega = 0$.
   Mode is $e^{\sigma t}$
2. s pure complex ($s = j\omega$) and $\sigma = 0$.
   Mode is $e^{j\omega t}$
3. s general complex ($s = \sigma + j\omega$).
   Mode is $e^{\sigma t}e^{j\omega t}$

Complex-conjugate pairs (1)

- When a_k's and b_k's are real, any complex roots will appear in **complex-conjugate pairs**.
- So if $\sigma + j\omega$ is a root, then $\sigma - j\omega$ is also a root of the characteristic polynomial.
- Furthermore, the coefficients in PFE also occurred in complex-conjugate pairs. This is useful for simplifying results.

Complex-conjugate pairs (2)

Question

Show that PFE coefficients occurs in complex-conjugate pairs for complex-conjugate roots (in the distinct-root case with real coefficients).

Complex-conjugate pairs (3)

The coefficients in the natural response corresponding to a complex-conjugate pair of roots will also be complex-conjugates, so the natural response will include terms of the form $Ce^{st} + C^* e^{s^* t}$.

This is useful for simplifying results.

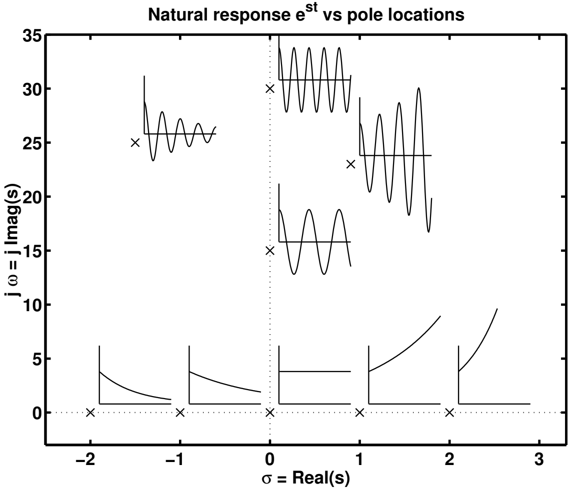
$$Ce^{st} + C^* e^{s^* t} = |C|e^{j\theta} e^{(\sigma + j\omega)t} + |C|e^{-j\theta} e^{(\sigma - j\omega)t}$$

$$= |C|e^{\sigma t}\left[e^{j(\theta+\omega t)} + e^{-j(\theta+\omega t)}\right] = |C|e^{\sigma t} 2\cos(\omega t + \theta)$$

where $C = |C|e^{j\theta}$ in polar form.

Natural response: single pole

This figure shows singe poles in various *s*-plane locations with the corresponding term of the natural response.



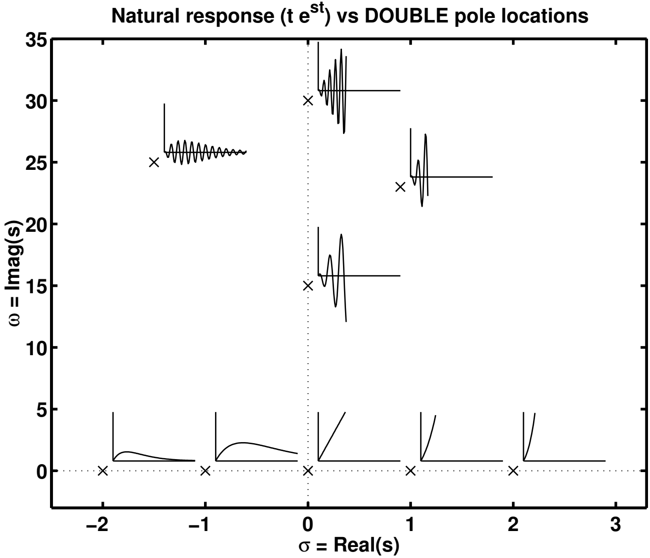
Real coefficients diffeq systems

For a double root, the modes are of the form te^{st}.

$$t^n e^{-at} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{n!}{(s+a)^{n+1}}, \quad \text{real}\{s\} > \text{real}\{-a\}$$

Natural response: double pole

For a double pole (repeated root), the mode has the form te^{st}



Stability of diffeq systems

A diffeq system (initially at rest) is stable iff all roots of its characteristic polynomial are in the left half plane ($\sigma < 0$).

Example

Is the following system stable?

$$6y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

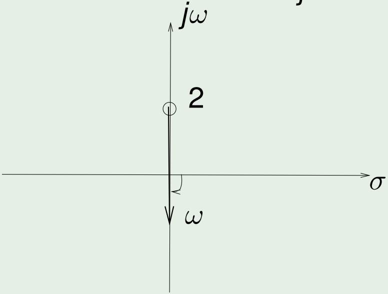
Outline

Geometric properties of FT from pole-zero plot

Given the pole-zero plot corresponding the the transfer function $H(s)$ of an LTI system, one can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$ of the system! This is very useful for understanding general system properties.

Example

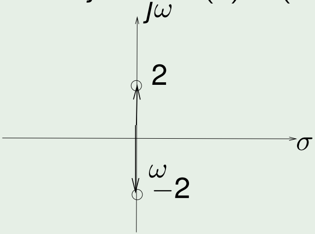
$H(s) = s - j2$ which has a zero at $s = j2$.



Example (2)

Example

Consider two zeros at $\pm 2j$. then $H(s) = (s - j2)(s + j2)$.



General cases

If the LT of a signal $h(t)$ is rational, then it can be expressed

$$H(s) = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)},$$

where the z_k's and p_k's are the zeros and poles of the system, and G is a constant scale factor.

Mathematically, the frequency response of the system is given by

$$H(\omega) = H(s)|_{s=j\omega} = G\frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega - p_1)\cdots(j\omega - p_n)}.$$

Magnitude response (1)

The **magnitude response** is given by

$$|H(\omega)| = |G| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}.$$

- The magnitude response at that frequency ω is proportional to the product of the geometric distances in the complex plane from the point $(0, j\omega)$ to each of the zeros, divided by the product of the distances to each of the poles.
- As one "slides along the $j\omega$ axis" to make a plot of $|H(\omega)|$ vs ω, as the point $(0, j\omega)$ gets closer to a zero, the contribution of that term decreases, and as we get closer to a pole, the contribution of that term increases.

Magnitude response (2)

The **magnitude response** is given by

$$|H(\omega)| = |G| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}.$$

If there are any zeros along the $j\omega$ axis, then the frequency response will be exactly zero at that point.

Example

This property was used to design a 60Hz notch filter much earlier in the course.

Phase response

The **phase response** is given by

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \cdots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \cdots - \angle(j\omega - p_n).$$

One can add and subtract the angles formed by the line segment between the point $(0, j\omega)$ and each zero or pole location in the s-plane to determine the overall phase response for each frequency ω.

Second order system

Example

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_n \frac{\mathrm{d}}{\mathrm{d}t}y(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 X(s)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(Picture MIT Lecture 21.1-4)

Video [MIT Lecture 21, 15:08-29:00min]

Outline

Important properties

Most important: linearity, differentiation, convolution. With these 3 we can solve most of the LTI systems problems of greatest interest to us.

Linearity

Property

Linearity

If $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$ *with* ROC_1 *and* $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$ *with* ROC_2 *then*

$$x(t) = a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\mathcal{L}} X(s) = a_1 X_1(s) + a_2 X_2(s),$$

and the ROC *of* $X(s)$ *is at least as large as the intersection of the* ROC_1 *and* ROC_2.

Question

Can the ROC *of* $X(s)$ *be larger?*

Differentiation property

Property

The ***differentiation property*** *(in time)*

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$$

the ROC of $sX(s)$ will be at least as large (due to possible cancellation with a pole at $s = 0$) as before.

Differentiation property: example

Example

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} 1/s, \quad \text{real}\{s\} > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \delta(t) \stackrel{\mathcal{L}}{\longleftrightarrow} 1, \forall s$$

Convolution property

Property

*The **convolution property***

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = H(s)X(s)$$

The ROC is at least as large as the intersection of the ROC's of $X(s)$ and $H(s)$.

Convolution property: example

Example

$$X_1(s) = \frac{s+1}{s+2},, \quad \text{real}\{s\} > -2$$

$$X_2(s) = \frac{s+2}{s+1},, \quad \text{real}\{s\} > -1$$

then

$$X_1(s)X_2(s) = 1, \quad \forall s$$

Time shift property

Property

time shift property

$$x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-t_0 s} X(s)$$

Same ROC as $X(s)$.

Caution: after time shifting, a formerly rational LT becomes irrational due to $e^{-t_0 s}$.

modulation property

Property

modulation property *(shifting in s-domain)*

$$e^{s_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0),\ \mathrm{ROC}_{\mathrm{new}} = \mathrm{ROC}_{\mathrm{old}} + \mathrm{real}\{s_0\}$$

The ROC *associated with* $X(s-s_0)$ *is that of* $X(s)$ *shifted by* real$\{s_0\}$ ***(Picture textbook, Figure 9.23)***

$$e^{j\omega_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-j\omega_0),\ \mathrm{ROC}\ \text{unchanged}$$

Time scaling property

Property

time scaling $a \neq 0$

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{s}{a}), \ \mathrm{ROC}_{\mathrm{new}} = a\mathrm{ROC}_{\mathrm{old}}$$

Differentiation in s-domain

Property

differentiation in s-domain

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{\mathrm{d}}{\mathrm{d}s}X(s), \ \mathrm{ROC}\ \textit{unchanged}$$

Running integration in time

Property

running integration in time

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s)$$

$\mathrm{ROC}_{\mathrm{new}}$ *must contain* $\mathrm{ROC}_{\mathrm{old}} \cap \{\mathrm{real}\{s\} > 0\}$

Initial and final value theorem

skip

initial value theorem

$$\lim_{t \to 0} x(t) = x(0+) = \lim_{s \to \infty} sX(s)$$

if $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, *i.e.* $M < N$ for rational $X(s)$ (no poles at infinity).

---

final value theorem

$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$ if $x(t)$ has a final value and $x(t) = 0$ for $t < 0$

Example

Example

Find the step response of the causal LTI system described by the following differential equation:

$$y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

Solution (3)

$$y(t) = -\frac{1}{2}e^{-0.5t}u(t) + u(t).$$

This answer consists of two parts.

- The $e^{-0.5t}u(t)$ part is called the **natural response**, and note that the decay rate is associated with the pole location $s = -1/2$. Since the natural response decays to zero, we also call it the **transient response**.
- The $u(t)$ part is called the **forced response** and is associated with the input signal. Since the forced response persists indefinitely, we also call it the **steady-state response**.

Outline

Parallel interconnection (1)

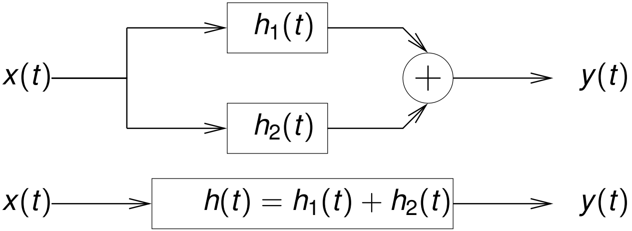
We have seen that when two LTI systems are connected in parallel, *i.e.*

$$y(t) = [h_1(t) * x(t)] + [h_2(t) * x(t)],$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where} \quad h(t) = h_1(t) + h_2(t).$$

Distributive



Parallel interconnection (2)

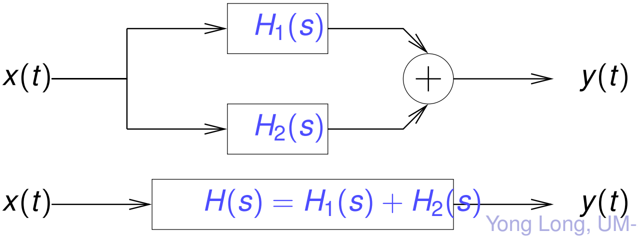
Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$

Distributive



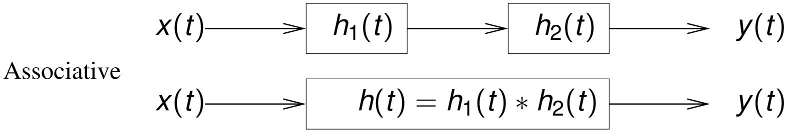
Series combination (1)

When two LTI systems are connected in series, *i.e.*

$$y(t) = h_2(t) * [h_1(t) * x(t)], ,$$

the output signal is

$$y(t) = h(t) * x(t), \quad \text{where} \quad h(t) = h_1(t) * h_2(t).$$



Series combination (2)

Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s)H_2(s).$$

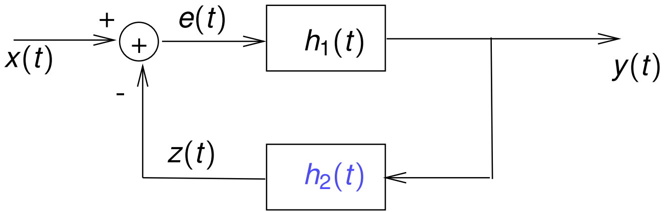
$$x(t) \longrightarrow \boxed{H_1(s)} \longrightarrow \boxed{H_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{H(s) = H_1(s)H_2(s)} \longrightarrow y(t)$$

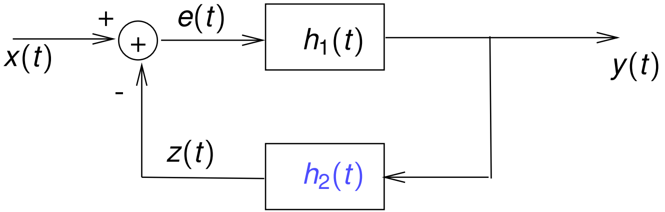
Feedback interconnection (1)

A **feedback system** uses the output of a system to control or modify the input.



Feedback interconnection (1)

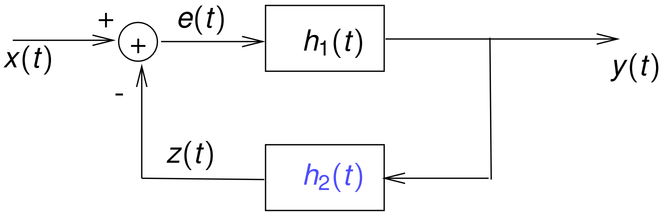
A **feedback system** uses the output of a system to control or modify the input.



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \Longrightarrow y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

Feedback interconnection (1)

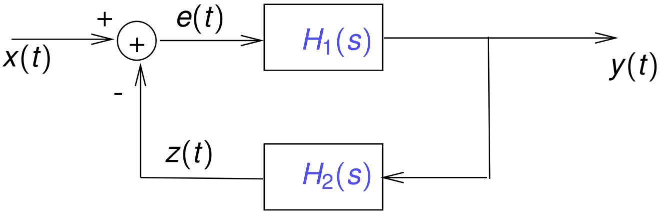
A **feedback system** uses the output of a system to control or modify the input.



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \implies y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

Feedback interconnection (1)



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \Longrightarrow y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Outline

Example # 1

Example

Draw the block diagram of the causal LTI system with system function

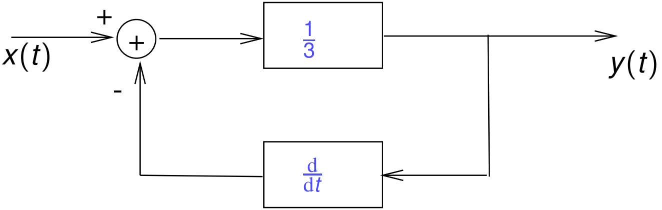
$$H(s) = \frac{1}{s+3}$$

Solution (1)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3} \Longrightarrow (s+3)Y(s) = X(s)$$

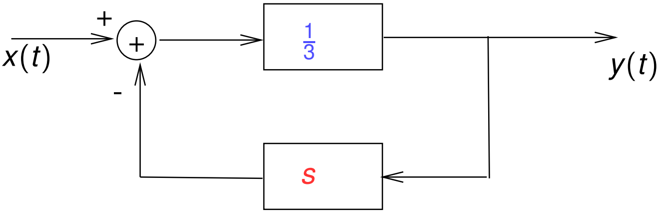
$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t)$$

$$\Longrightarrow y(t) = \frac{1}{3}\left[x(t) - \frac{\mathrm{d}}{\mathrm{d}t}y(t)\right]$$



Solution (2)

$$y(t) = \frac{1}{3}\left[x(t) - \frac{\mathrm{d}}{\mathrm{d}t}y(t)\right] \Longrightarrow Y(s) = \frac{1}{3}\left[X(s) - sY(s)\right]$$

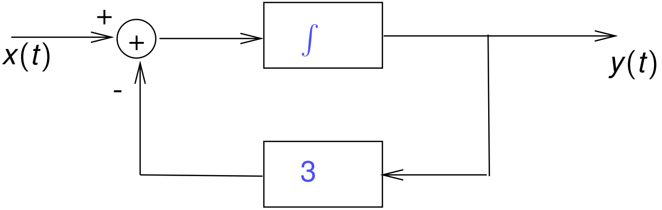


Solution (3)

- The previous diagram is a valid representation.
- But the differentiator is both difficult to implement and extremely sensitive to noise.

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t)$$

$$\Longrightarrow y(t) = \int_{-\infty}^{t} [x(\tau) - 3y(\tau)]\ d\tau$$



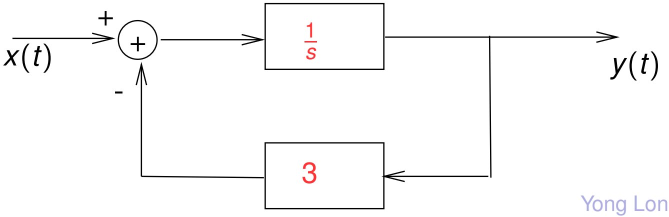
Solution (4)

$$y(t) = \int_{-\infty}^{t} [x(\tau) - 3y(\tau)] \, d\tau$$

$$\Longrightarrow y(t) = \int_{-\infty}^{\infty} [x(\tau) - 3y(\tau)] \, u(t - \tau) \, d\tau$$

$$= [x(\tau) - 3y(\tau)] * u(t)$$

$$\Longrightarrow Y(s) = [X(s) - 3Y(s)] \frac{1}{s}$$



Example # 2

Example

Draw the block diagram of the causal LTI system with transfer function

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

Example # 2

(Picture Block diagram from spring semester)
Block diagram representation in direct form, cascade form and parallel form. (Textbook, Example 9.30)

Outline

Cruise control system

Example

The cruise control system in a car.

- The "input" to a car is the applied forces
  - external: wind, gravity (hills), road friction, etc.
  - internal: engine (controlled by gas pedal)
- The output is the car's velocity, which a cruise control system should hold approximately constant (by adjusting gas pedal) even as road conditions/hills vary.

Transfer function

System diagram:
force $f(t) \to \boxed{\text{Car}} \to v(t)$ velocity.

Newton's laws say

$$f(t) = ma(t)$$

where $a(t)$ is the acceleration and m is the mass of the car. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = a(t) = \frac{f(t)}{m}$$

is the input-output relationship for this system, so in the Laplace domain

$$sV(s) = F(s)/m$$

so the transfer function of this system is:

$$H(s) = V(s)/F(s) = \frac{1}{sm}.$$

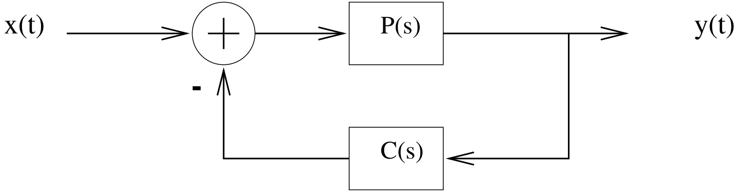
Stability

Question

Is this system stable?

Negative feedback

A **negative feedback** control system looks like the following:

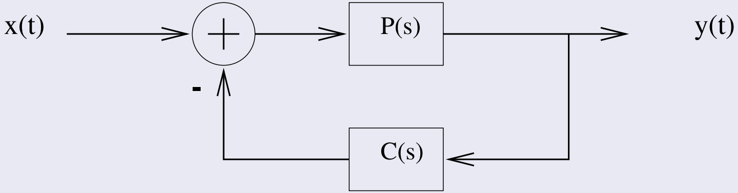


- $P(s)$ is the transfer function of the "plant" to be controlled (in this case the car).
- $C(s)$ is the transfer function of the controller system.
  - The simplest form of feedback is proportional control, where $C(s) = c$, simply some constant.
  - Intuition: if the car velocity is too high, then decrease the force (acceleration) a little to compensate. (And vice versa).

Overall transfer function

Question

For the car system, $P(s) = 1/(ms)$. Find overall transfer function of system with controller in place, when $C(s) = c$. Is this system stable?



Outline

Summary

- Laplace transform definition / computation by integration
- ROC of Laplace transform / properties
- relation to Fourier transform
- rational Laplace transforms / pole-zero plot
- inverse Laplace transform by PFE
- FT magnitude from pole-zero plot
- properties of LT
- application of LT to LTI systems