

## Homework 2

### HW Notes:

- Box your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

### Problems:

1. [3!] Assume a system has the input-output relationship  $y(t) = f(t)x(t)$ , where  $x(t)$  is the input and  $y(t)$  is the output.  $f(t)$  is not constant, i.e., there exists  $t_0, t_1$  that  $f(t_0) \neq f(t_1)$ . Show that this system is time-variant. That is, a system with time-variant gain cannot be time-invariant. (*Hint: find a counterexample.*)
2. [9!] Here are input-output relationships for a few systems, all of which are linear. Some of them are time-invariant, some are not. Determine which are which. Find the impulse response of the time-invariant systems.

$$(a) \ y(t) = \int_{-\infty}^t \left[ \int_{-\infty}^s x(\tau - 5) d\tau \right] ds,$$

$$(b) \ y(t) = \int_{-1}^3 e^{-(t-\tau)^2} x(\tau) d\tau,$$

$$(c) \ y(t) = \int_{-3}^3 \tau^2 x(t - \tau) d\tau + \int_{-\infty}^{t+1} (t - \tau + 3)^{-2} x(\tau) d\tau.$$

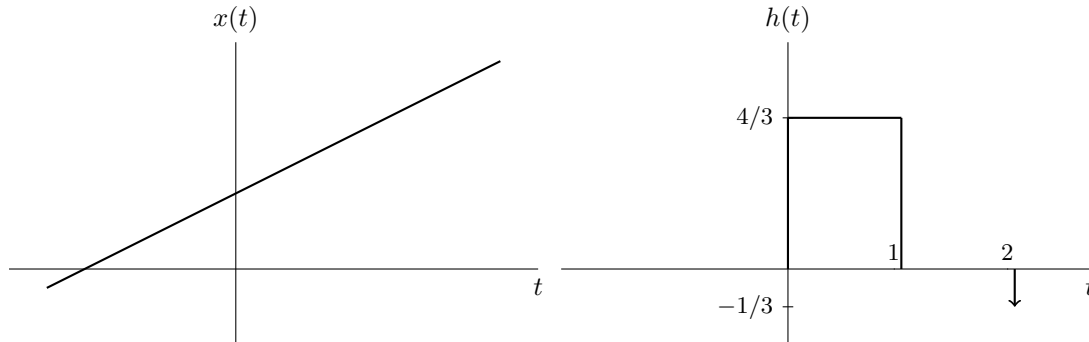
Hint: Note that if you can transform the above relationships into the exact form of convolution  $y(t) = g(t) * x(t)$ , then the system is immediately time-invariant with  $g(t)$  being the impulse response  $h(t)$ . That is because different from algebraic operators like multiplication, the convolution operator implies time-invariance itself.

3. [9!] One of following two statements is correct, and the other is incorrect. The symbol \* denotes *convolution*.
  - If  $y(t) = h(t) * x(t)$  then  $y(t - 3) = h(t - 3) * x(t - 3)$ ;
  - Or if  $y(t) = h(t) * x(t)$  then  $y(t - 3) = h(t) * x(t - 3)$ .
  - (a) Give a simple proof of the correct statement.
  - (b) Give a simple counterexample for the incorrect statement.
  - (c) Repeat (a) and (b) for the following two statements. The symbol · denotes *multiplication*.
    - If  $y(t) = h(t) \cdot x(t)$  then  $y(t - 3) = h(t - 3) \cdot x(t - 3)$ ;
    - Or if  $y(t) = h(t) \cdot x(t)$  then  $y(t - 3) = h(t) \cdot x(t - 3)$ .

Be careful with the notation  $h(t) * x(t)$ . More precise notation is  $(h * x)(t)$ , which makes it clear that convolution is an operation on two signals, not a point-wise operation like multiplication.

4. [8!] Often we will be convolving two signals that are zero everywhere except over some finite range (called **finite support** signals). Suppose  $x_1(t)$  is non-zero over the range  $a \leq t \leq b$  and that  $x_2(t)$  is non-zero over the range  $c \leq t \leq d$ . Suppose  $y(t) = x_1(t) * x_2(t)$ .

- (a) Find the range of values of  $t$  for which  $y(t)$  is possibly non-zero.
- (b) Compute  $\text{rect}((t-2)/2) * \text{rect}((t+3)/4)$  (express answer with braces and carefully sketch). Check your result with part (a).
5. [6!] For each of the following pairs of waveforms, use convolution integral to find the response  $y(t)$  of the LTI system with impulse response  $h(t)$  and input  $x(t)$ . Sketch your results.
- (a)  $x(t) = e^{-\alpha t}u(t), h(t) = e^{-\beta t}u(t)$  (Do this both when  $\alpha = \beta$  and  $\alpha \neq \beta$ )
- (b)  $x(t)$  and  $h(t)$  as in the figure below, the slope for the straight line is  $a$  and the line intersects y-axis at  $(0, b)$ :



6. [6!]

- (a) Consider a linear system with input  $x(t)$  and output  $y(t)$  given by

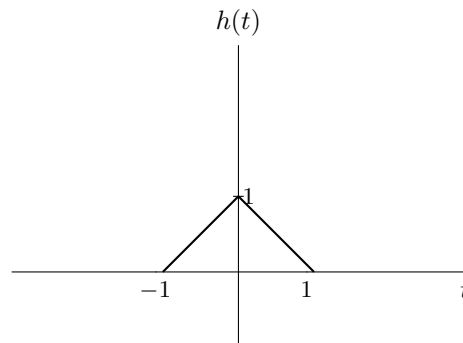
$$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT).$$

Is this system time-invariant?

- (b) Consider another LTI system. Let its impulse response  $h(t)$  be the triangular pulse shown below, and  $x(t)$  be the **impulse train**

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT),$$

SKETCH  $y(t) = x(t) * h(t)$  for  $T = 4, 2, 1.5$  and  $1$ . (No formulae are needed though you still want to label your graphs clearly.)



7. [6!] Let  $y(t) = (x * h)(t)$ . Show the following properties of convolution.

(a)  $\int_{-\infty}^{\infty} y(t) dt = \left[ \int_{-\infty}^{\infty} x(t) dt \right] \left[ \int_{-\infty}^{\infty} h(t) dt \right],$

(b)  $\frac{d}{dt}y(t) = \left[\frac{d}{dt}x(t)\right] * h(t) = x(t) * \left[\frac{d}{dt}h(t)\right],$

8. [6!] Compute the following convolution:

(a)  $u(t) * u(t),$

(b)  $u(t) * t^2 u(t).$

9. [3!] Suppose you have a "black box" LTI system and you discover that for a unit-step input signal the response of the system is the function  $(3 - t)\text{rect}(\frac{t-1}{2})$ . Determine the impulse response of the system. (Hint: See Problem 7.)

10. [6!] Determine whether the following systems are linear, stable, causal, time-invariant, and memoryless.

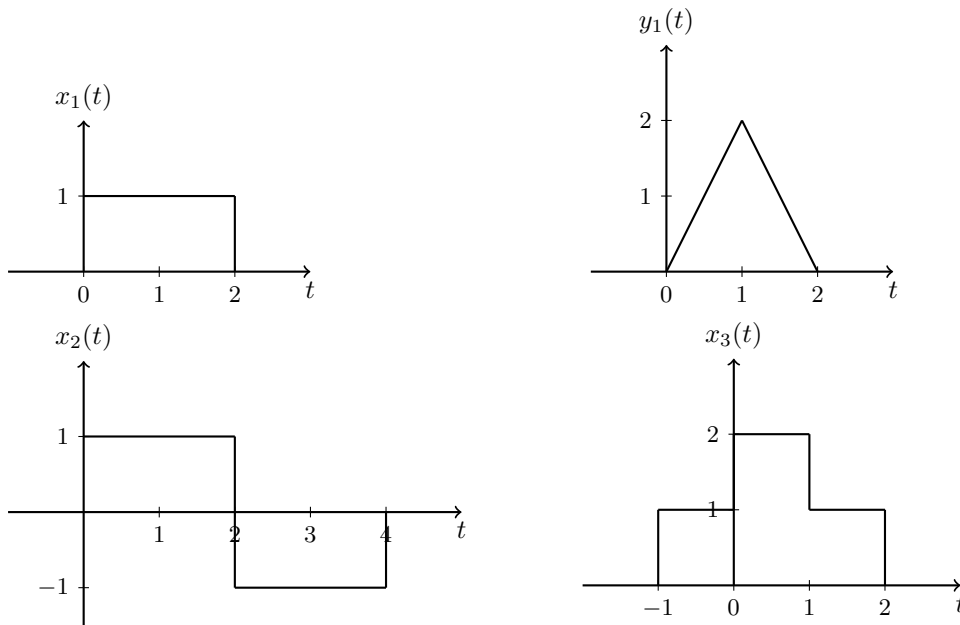
(a)  $y(t) = x(\sin(t))$

(b)  $y(t) = \frac{d}{dt}\{e^{-t}x(t)\}$

11. [6!] Consider an LTI system whose response to the signal  $x_1(t)$  is the signal  $y_1(t)$  which are illustrated below.

(a) Determine and sketch carefully the response of the system to the input  $x_2(t)$  depicted below.

(b) Determine and sketch carefully the response of the system to the input  $x_3(t)$  depicted below.



12. [6!] The triangular pulse is defined as  $\text{tri}(t) = (1 - |t|)\text{rect}(t/2)$ .

Compute  $x(t) = \text{tri}(t/2) * \text{rect}(\frac{t-1}{2})$ . Express your answer using braces, and carefully sketch.

13. [6!] Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.

(a)  $y(t) = \int_{-\infty}^t (t - \tau)e^{-(t-\tau)}x(\tau)d\tau$

(b)  $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}x(\tau)d\tau$

14. [3!] Consider an LTI system  $S$  and a signal  $x(t) = 2e^{-3t}u(t-1)$ . If

$$x(t) \rightarrow y(t)$$

and

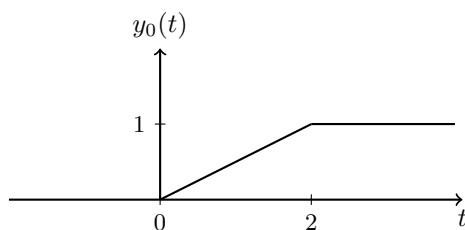
$$\frac{dx(t)}{dt} \rightarrow -3y(t) + e^{-2t}u(t)$$

determine the impulse response  $h(t)$  of  $S$ .

15. [12!] We are given a certain LTI system with impulse response  $h_0(t)$ . We are told that when the input is  $x_0(t)$  the output is  $y_0(t)$ , which is sketched below. We are then given the following set of inputs to LTI systems with the indicated impulse responses.

	Input $x(t)$	Impulse response $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t - 2)$	$h(t) = h_0(t + 1)$
(c)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(e)	$x(t) = x'_0(t)$	$h(t) = h_0(t)$
(f)	$x(t) = x'_0(t)$	$h(t) = h'_0(t)$

In each of these cases, determine whether or not we have enough information to determine the output  $y(t)$  when the input is  $x(t)$  and the system has impulse response  $h(t)$ . If so, provide an accurate sketch of it with numerical values clearly indicated on the graph.



16. [5!] Find the expression of response of the CT system described by the linear constant-coefficient differential equation.

$$\frac{d}{dt}y(t) + 10y(t) = 2x(t), \quad y(0) = 1, \quad x(t) = u(t)$$