# Homework 3

**Answers:**

1. [5] For any $t_0$, we have

$$\int_{t_0}^{t_0+T_0} e^{j\omega_0 kt}dt = \frac{1}{j\omega_0 k}e^{j\omega_0 kt}|_{t_0}^{t_0+T_0} = \frac{1}{j\omega_0 k}(e^{j\omega_0 k(t_0+T_0)} - e^{j\omega_0 kt_0})$$

$$\Longrightarrow = \frac{1}{j\omega_0 k}e^{j\omega_0 kt_0}(e^{jk2\pi} - 1) = 0,\ k \neq 0$$

When $k = 0$, we have $e^{j\omega_0 k} = 1$ thus,

$$\int_{t_0}^{t_0+T_0} e^{j\omega_0 kt}dt = T_0$$

From the above calculation, we can conclude that,

$$< \phi_k, \phi_l >= \int_a^b \phi_k(t)\phi_l^*(t)dt = \int_{t_0}^{t_0+T_0} e^{j\omega_0 kt}e^{j\omega_0 lt}$$

$$\int_{t_0}^{t_0+T_0} e^{j\omega_0(k-l)t}dt = \begin{cases} T_0 & ,l = k \\ 0 & ,l \neq k \end{cases}$$

2. [5]

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k e^{jk\omega_0 t} + a_{-k}e^{-jk\omega_0 t})$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}[(a_k + a_{-k})\cos(k\omega_0 t) + (a_k - a_{-k})j\sin(k\omega_0 t)]$$

$$B[0] = a_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$B[k] = a_k + a_{-k} = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t} + \frac{1}{T}\int_0^T x(t)e^{jk\omega_0 t} = \frac{1}{T}\int_0^T x(t)(e^{-jk\omega_0 t} + e^{jk\omega_0 t})dt$$

$$B[k] = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt$$

For the same reason,

$$A[k] = (a_k - a_{-k})j = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt$$

3. [5] $T_1 = \frac{2}{3}$, $T_2 = \frac{1}{2}$, $T = lcm(T_1, T_2) = 2$, $\omega_0 = \pi$

$$x(t) = \sin(3\pi t) + \cos(4\pi t) = \frac{1}{2j}(e^{j(3)\pi t} - e^{j(-3)\pi t}) + \frac{1}{2}(e^{j(4)\pi t} + e^{j(-4)\pi t})$$

By inspection, we have

$$a_k = \begin{cases} \frac{1}{2} & ,k = \pm 4 \\ \frac{1}{2j} & ,k = 3 \\ \frac{-1}{2j} & ,k = -3 \\ 0 & ,otherwise \end{cases}$$

4. [6]

(a) It is an impulse train with a period of $T = 4$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t} = \frac{1}{4}$$

We can find that $a_k$ is a constant.

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{4}e^{jk\omega_0 t} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{2}\cos(k\frac{\pi}{2}t)$$

(b) Sketch $x(t)$ and we find it can be expressed as

$$x(t) = 1 + \sum_{n=-\infty}^{\infty} rect(t - 5n - \frac{1}{2})$$

$$x(t) = 1 + \sum_{k=-\infty}^{\infty} \frac{1}{5}sinc(\frac{k}{5})e^{jk(2\pi/5)(1/2)}e^{jk(2\pi/5)t}$$

$$x(t) = \frac{6}{5} + \sum_{k=1}^{\infty} \frac{1}{5}sinc(\frac{k}{5})\cos(\frac{2\pi(t-1/2)k}{5})$$

(c) For this signal $T = 2$, $\omega_0 = \pi$

$$a_k = \frac{1}{2}\int_0^1 e^{-t}e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{2}\int_0^1 e^{-(1+jk\pi)t}dt$$

$$a_k = \frac{1-e^{-(1+jk\pi)}}{2(1+jk\pi)}$$

$$x(t) = \frac{1-e^{-1}}{2} + \sum_{k=1}^{\infty}[\frac{1-e^{-1}\cos(k\pi)}{1+k^2\pi^2}\cos(k\pi t) + \frac{k\pi(1-e^{-1}\cos(k\pi))}{1+k^2\pi^2}\sin(k\pi t)]$$

5. [6]

(a) If $x(t)$ is real. Then $x(t) = x(t)^*$, which implies that $a_k = a^*_{-k}$. It is not true in this case, so $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. It is true in this case, so $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk\frac{2\pi}{T_0}a_k$$

Therefore,

$$b_k = \begin{cases} 0 & ,k = 0 \\ -k(\frac{1}{2})^{|k|}\frac{2\pi}{T_0}, & otherwise \end{cases}$$

Since $b_k$ is not even, $\frac{dx(t)}{xt}$ is not even.

6. [10]

(a) A sample code is shown below,

```
a0 = 0.25;
t= -15: 0.01: 15;
x = 0*t;
x = x + a0;
for k=1:5000
x = x+0.5.*cos(pi.*k.*t./2);
end
plot(t,x)
```

The result,

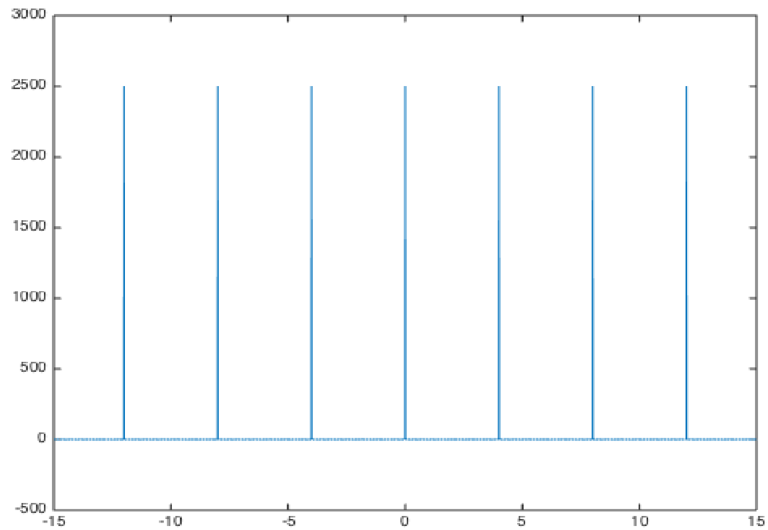



Figure 1: Results for (a)

(b) A sample code is shown below,

```
a0 = 1.2;
t= -15: 0.01: 15;
x = 0*t;
x = x + a0;
for k=1:50000
x = x+0.4.*sinc(k./5)*cos(2.*pi.*k.*(t-0.5)./5);
end
plot(t,x)
ylim([-1 3])
grid on
```

And the result is like this,

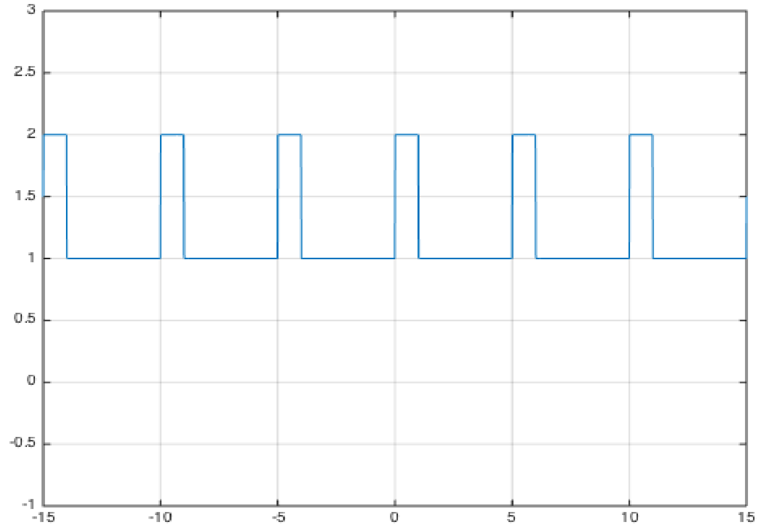



Figure 2: Results for (b)

(c) You may also have a look at the third one.

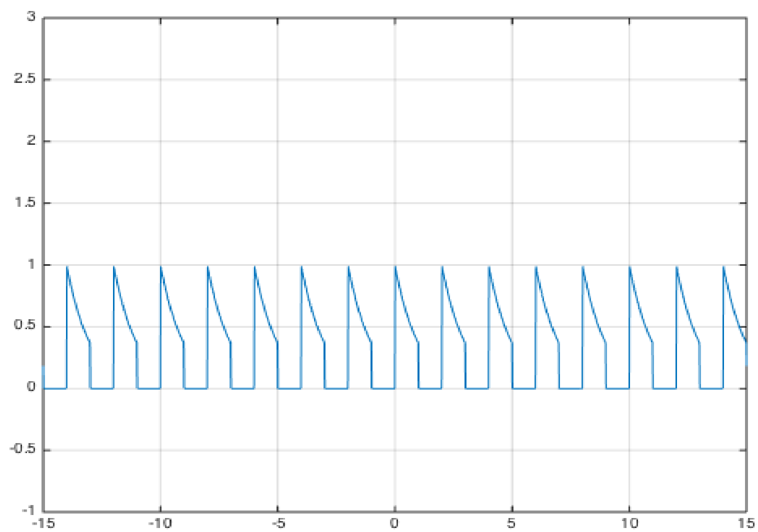



Figure 3: Results for (c)

7. [4]

(a) we have

$$x(t) = \sum_{odd\ k} a_k e^{jk(2\pi/T)t}$$

Therefore,

$$x(t + \frac{T}{2}) = \sum_{odd\ k} a_k e^{jk\pi} e^{jk(2\pi/T)t}$$

Since $e^{jk\pi} = -1$ for odd $k$.

$$x(t+\frac{T}{2}) = -x(t)$$

(b) The Fourier series coefficients of $x(t)$ are

$$a_k = \frac{1}{T}\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \frac{1}{T}\int_{T/2}^{T} x(t)e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{T}\int_0^{T/2} [x(t) + x(t+\frac{T}{2})e^{-jk\pi}]e^{-j\omega_0 t}dt$$

Note that $a_k$ evaluate to zero for even $k$ if we have $x(t) = -x(t+\frac{T}{2})$. So it is odd harmonious.

8. [4]

(a) Sample code for $N = 19$ is as follows:

```
clc
close all
a0 = 0.5 * (1 - exp(-1));
t = linspace(-0.5, 4.5, 1000);
x = zeros(1, 1000);
x = x + a0;
x0 = a0;
x1 = a0;
for k = 1:19
b = (1 - exp(-1)*(-1)^k)/(1 + k^2*pi^2);
c = b * k * pi;
x = x + b * cos(pi * k .* t) + c * sin(pi * k .* t);
x0 = x0 + b * cos(pi * k .* 0) + c * sin(pi * k .* 0);
x1 = x1 + b * cos(pi * k .* 1) + c * sin(pi * k .* 1);
end
fprintf('When k = %d, x(0) = %f, x(1) = %f\n', k, x0, x1);
plot(t,x);
xlabel('t'), ylabel('x(t)')
```

And the plot for $N = 19$ is given below (other values for N are omitted):

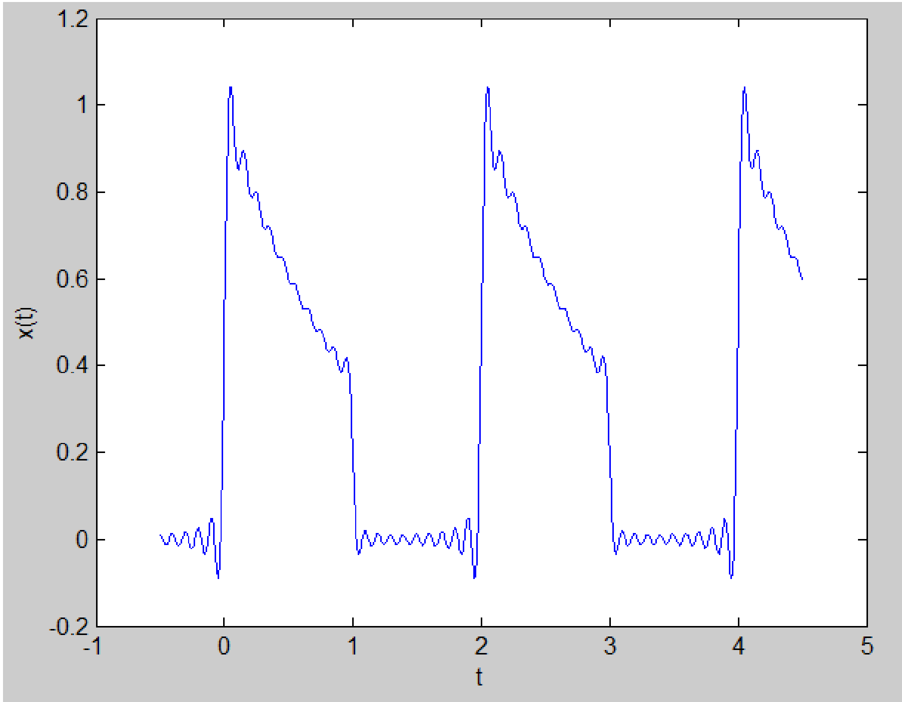



Figure 4: HW3-8

One should be able to see that as $N$ becomes larger, $S_N(t)$ becomes more like $x(t)$. (One could also talk about the Gibbs phenomena.)

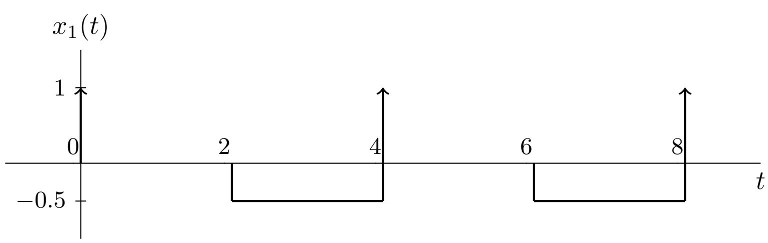
(b) Using the code above, Matlab gives the following results:

```
When k = 5, x(0) = 0.482248, x(1) = 0.189060
When k = 10, x(0) = 0.490193, x(1) = 0.187942
When k = 15, x(0) = 0.493544, x(1) = 0.186133
When k = 19, x(0) = 0.494855, x(1) = 0.185718
```

In fact when $N \to \infty$, $S_N(0) = \boxed{1/2}$ and $S_N(1) = \boxed{e^{-1}/2}$.

9. [6]

(a) Take the derivative of $x(t)$, we get $x_1(t)$:




Same as $x(t)$, for $x_1(t)$ $T_0 = 4$ and $\omega_0 = \pi/2$. After some calculation we get the FS coefficients of $x_1(t)$:

$$a_0 = 0, \quad \text{and}$$

$$a_k = \frac{1}{4}\left(1 + \frac{1-(-1)^k}{jk\pi}\right) \quad (k \neq 0).$$

Using the differentiating property, for all $k \neq 0$, we have

$$b_k = \frac{a_k}{jk\omega_0} = \frac{2a_k}{jk\pi}$$
$$= \frac{1}{2jk\pi} + \begin{cases} 0 & (\text{k even} \quad k \neq 0) \\ \frac{-1}{k^2\pi^2} & (\text{k odd}), \end{cases}$$

where $b_k$ denotes the FS coefficients for $x(t)$. And for $b_0$, we have $b_0 = \frac{1}{4}\int x(t)\mathrm{d}t = \frac{3}{4}$.

(b) $z(t) = \frac{1}{2}(z_1(t)+z_2(t))$, where $z_1(t) = x(t)e^{2j\frac{\pi}{2}t}$, and $z_2(t) = x(t)e^{-2j\frac{\pi}{2}t}$. Let $z_1(t)$ have FS coefficients $b_k$ and $z_2(t)$ have FS coefficients $c_k$. Then

$$b_k = a_{k-2} = \begin{cases} \frac{3}{4} & k = 2 \\ \frac{1}{2j\pi(k-2)} + \frac{(-1)^k-1}{2(k-2)^2\pi^2} & k \neq 2, \end{cases}$$
$$c_k = a_{k+2} = \begin{cases} \frac{3}{4} & k = -2 \\ \frac{1}{2j\pi(k+2)} + \frac{(-1)^k-1}{2(k+2)^2\pi^2} & k \neq -2, \end{cases}$$

and from linearity of FS, $d_k = \frac{1}{2}(b_k + c_k) = (omitted)$.

And $d_2 = \boxed{\frac{3}{8} + \frac{1}{j16\pi}}$, $d_{-3} = \boxed{-\frac{3}{j10\pi} - \frac{13}{25\pi^2}}$.

(c) From (a), we easily obtain

$$y(t) = 4x(t/3-2)+5 = \boxed{8 + \sum_{k=1}^{\infty} \frac{4}{\pi k}\sin\left(k\frac{\pi}{2}(t/3-2)\right) - \sum_{k=1,\ \text{odd}}^{\infty} \frac{8}{k^2\pi^2}\cos\left(k\frac{\pi}{2}(t/3-2)\right)}.$$

If you want to fully simplify this, it is equivalent as

$$\boxed{y(t) = 8 + \sum_{k=1,\ \text{odd}}^{\infty} \frac{8}{k^2\pi^2}\cos(k\frac{\pi}{6}t) + \sum_{k=1}^{\infty} \frac{4(-1)^k}{\pi k}\sin(k\frac{\pi}{6}t).}$$

10. [18]

(a) $x(t) = \mathrm{LC}\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \mathrm{RC}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t)$, so the diffeq is given by $\boxed{\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t)}$.

(b) The output of $e^{j\omega t}$ should be in the form $H(j\omega)e^{j\omega t}$, where $H(j\omega)$ have nothing to do with time $t$. Hence $\frac{d^2}{dt^2}(H(j\omega)e^{j\omega t})+\frac{d}{dt}(H(j\omega)e^{j\omega t})+H(j\omega)e^{j\omega t} = e^{j\omega t}$. From which we have $H(j\omega) = \boxed{\frac{1}{-\omega^2+j\omega+1}}$.

(c) $H(s) = \boxed{\frac{1}{s^2+s+1}}$.

(d) $|H(j\omega)| = \left(\frac{1}{(1-\omega^2)^2+\omega^2}\right)^{1/2}$. Use Matlab `ezplot('1/((1-x^ 2)^ 2 + x^ 2) ^  (1/2)', [0 6])` to plot this function of $\omega$:

(e) Sample code:

```
num = [1];
den = [1 1 1];
x = linspace(0, 6);
H = freqs(num, den, x);
plot(x, abs(H));
xlabel('Angluar Freqs w (rad/s)'),
ylabel('Magnitude Response |H(jw)|'),
title('Hw6-3e')
```

And the output figure:

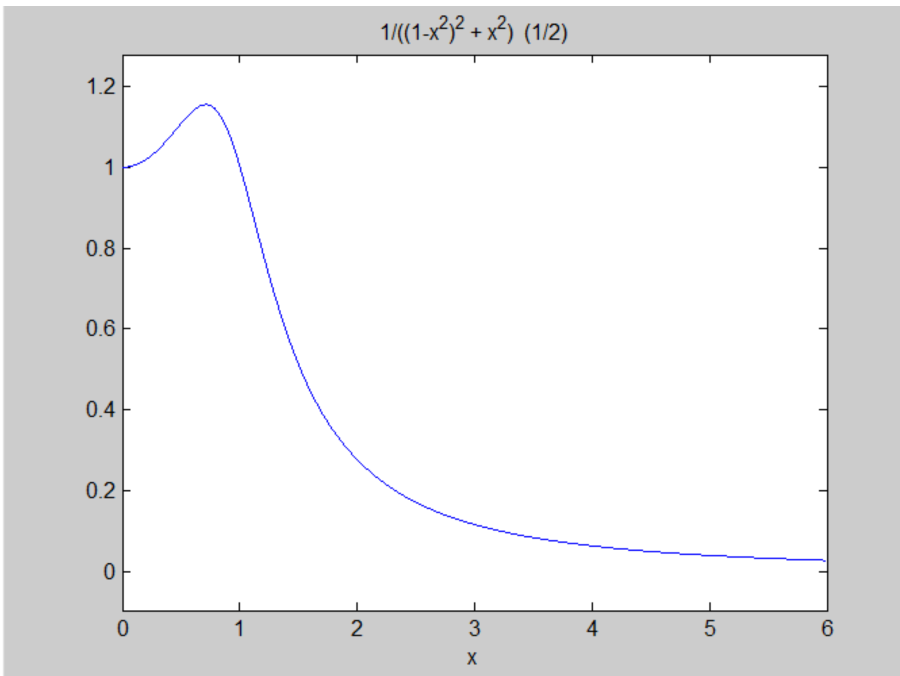



Figure 5: HW3-10d

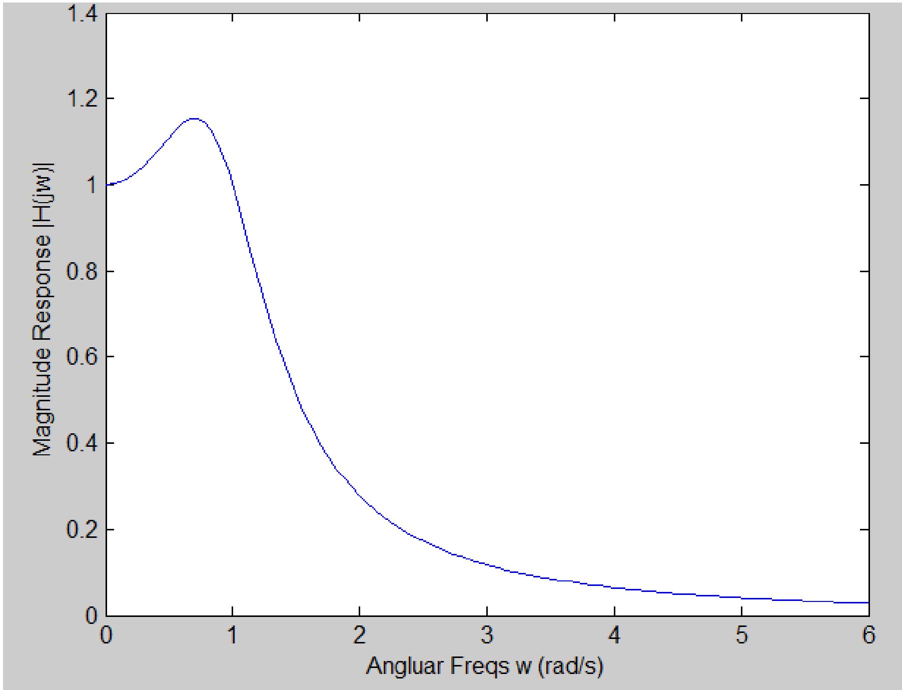



Figure 6: HW3-10e

We can see that the figure is identical to part (d), as expected.

(f) $T_0 = 2\pi$ and $\omega_0 = 1$. And $x(t) = \boxed{1 + \frac{1}{2j}(e^{jt} - e^{-jt}) + \frac{1}{2j}(e^{4jt} - e^{-4jt})}$. Note that this already implies $c_1 = c_4 = 1/(2j)$, $c_{-1} = c_{-4} = -1/(2j)$ and $c_0 = 1$. After calculating the $|c_k|^2$'s, we use the following code to generate a PDS:

```
x = [-4 -1 0 1 4];
y = [0.25 0.25 1 0.25 0.25];
stem(x,y,'filled');
axis([-5 5 0 1.5]);
xlabel('Angular frequncy \omega (k\omega_0)'),
ylabel('Component power |c_k|^2'),
title('Power Density Spectrum (Hw6-3f)')
```

(g) By using computer, we get $\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^2 = 2$. And we have $\sum |c_k|^2 = 1 + 0.25 + 0.25 + 0.25 + 0.25 = 2$. Hence the Parseval's relation is verified.

(h) $y(t) = 1 * H(0) + \frac{1}{2j}(H(1)e^{jt} - H(-1)e^{-jt} + H(4)e^{j4t} - H(-4)e^{-j4t}) = 1 + \frac{1}{2j}(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt} + \frac{1}{-15+j4}e^{j4t} - \frac{1}{-15-j4}e^{-j4t}) = \boxed{1 - \cos(t) - \frac{4}{241}\cos(4t) - \frac{15}{241}\sin(4t)}$. The code for PDS is very similar to part (f), and the figure is shown below: (*Note that the stem at* $\pm$ *4 should have height 1/964.*)

(i) The high frequency component (4 rad/s sine wave) is attenuated. The circuit serves as a $\boxed{\text{lowpass}}$ filter.

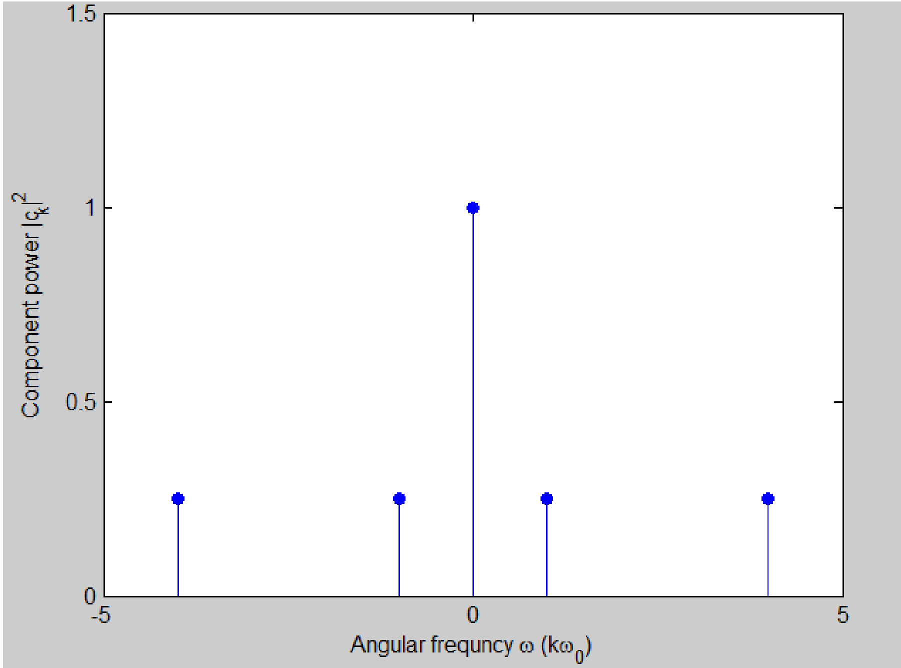



Figure 7: HW3-10f

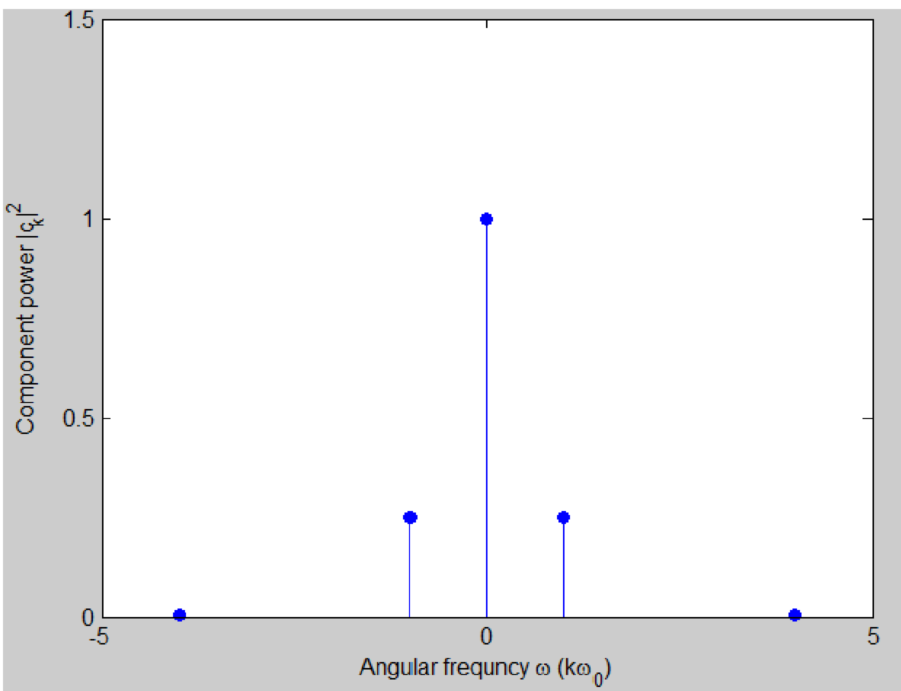



Figure 8: HW3-10h

11. [5] We know that

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t},$$

where $\omega_0 = 12$ from the question. This implies $H(j\omega)$ must be zero for $|\omega| > 100$, so $|k|\omega_0 > 100$. That is, for $\boxed{|k| \geq 9}$, $a_k$ is guaranteed to be zero.

12. [6] The sample code and the three plots are given below:

```
close all;
clear all;

[num, den] = besself(5,1e4);
f = linspace(0, 2e4, 1e5);
H = freqs(num, den, f);
figure, plot(f, abs(H));

[u, t] = gensig('square',0.002, 1, 1e-4);
figure, plot(t,u);
axis([0 0.01 -1 2]);

sys = tf(num, den);
y = lsim(sys, u, t);
figure, plot(t, y);
axis([0 0.01 -1 2]);
```

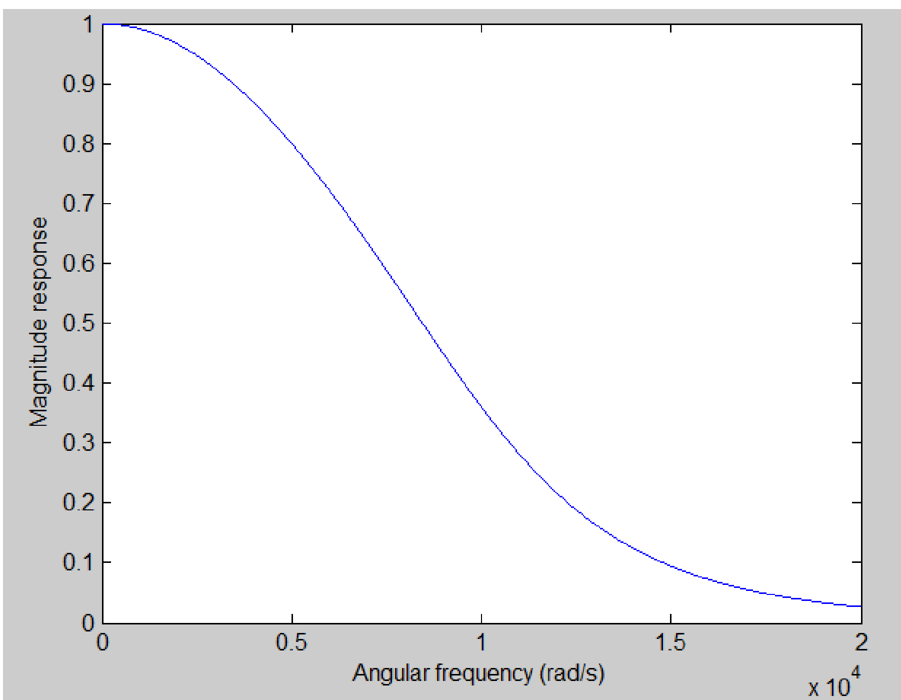



Figure 9: HW3-12a

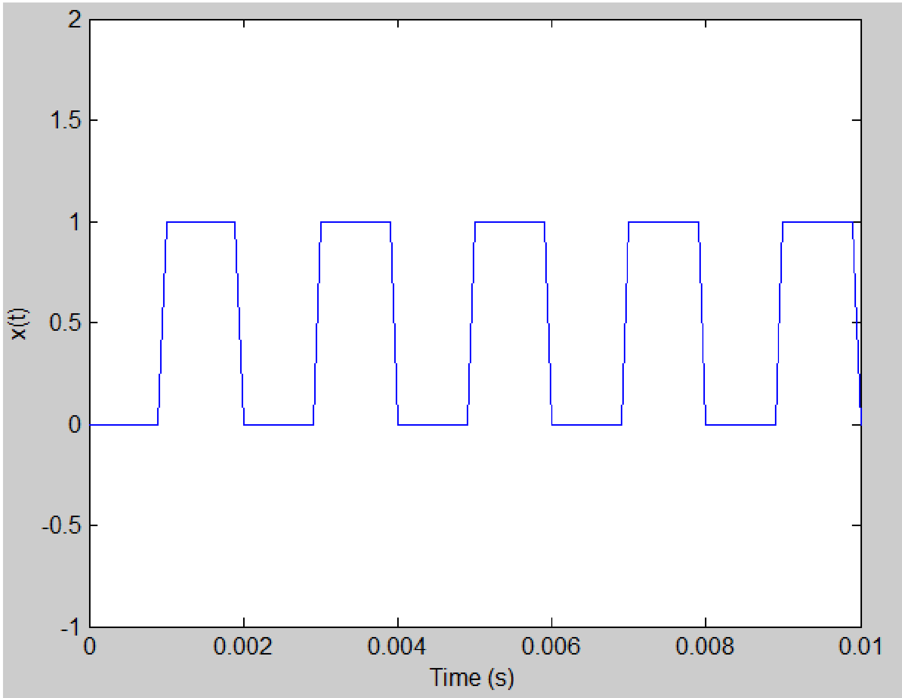



Figure 10: HW3-12b

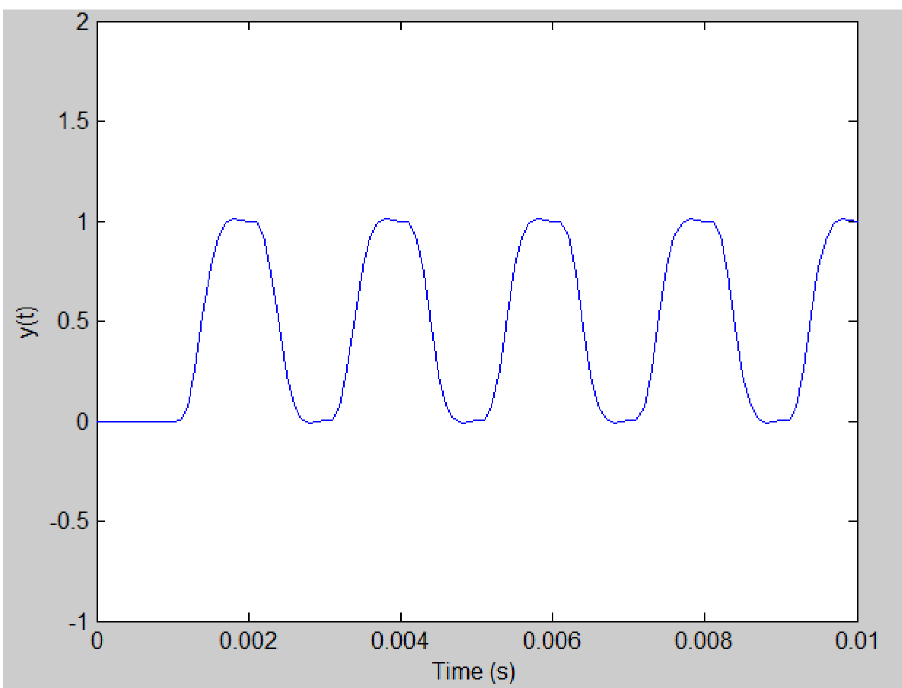



Figure 11: HW3-12c

13. [5] If the FS coefficients of $x(t)$ are periodic with period $N$, then $a_k = a_{k-N}$ for all $k$. This implies that $x(t) = x(t)e^{j(2\pi/T)Nt}$ for all $t$. This is possible only if $x(t)$ is zero for all $t$ other than when $(2\pi/T)Nt = 2\pi k$, where $k \in \mathbb{Z}$. Therefore, $x(t)$ must be in the form $x(t) = \sum_{k=-\infty}^{\infty} g[k]\delta(t - kT/N)$. On the other hand, because $x(t + NT/N) = x(t + T) = x(t)$ from the periodicity of $x(t)$, we easily obtain $g[k + N] = g[k]$ for all $k$.

14. [5]

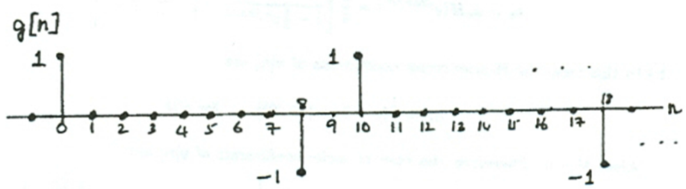
$\sin^5 x = (1/(2j))^5(e^{jx} - e^{-jx})^5 = \frac{1}{32j}(e^{j5x} - 5e^{j3x} + 10e^{jx} - 10e^{-jx} + 5e^{-j3x} - e^{-j5x}) = \frac{10\sin x - 5\sin(3x) + \sin(5x)}{16}$.
Thus $y(t) = 7\left[\sin 3t + \frac{10b}{16}\sin 3t - \frac{5b}{16}\sin 9t + \frac{b}{16}\sin 15t\right]$. So $c_1 = 7(1 + 10b/16)$.
Thus THD $= 1 - \frac{(1+10b/16)^2}{(1+10b/16)^2 + (5b/16)^2 + (b/16)^2} 100\% = \boxed{0.0899\%}$ for $b = 0.10$.

15. [6] The solution is as follows:

(a) For $0 \le n \le 9$, We have

$$g[n] = x[n] - x[n-1] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 7 \\ -1, & n = 8 \\ 0, & n = 9 \end{cases}$$

This period begin to show again in the following 10 points, it's clearly to draw the conclusion that $g[n]$ is periodic with period of 10. The graph of signal $g[n]$ is shown in Figure 3.




(b) It is known that $T = 10$. So that the FS coefficients of $g[n]$ is $b_k$, which is

$$\begin{aligned} b_k &= \frac{1}{N}\sum_{N} g[n]e^{-jk(2\pi/N)n} \\ &= \frac{1}{10}\sum_{n=0}^{9} g[n]e^{-jk(\pi/5)n} \\ &= \frac{1}{10}(1 - e^{-j8k(\pi/5)}) \end{aligned}$$

(c) Since $g[n] = x[n] - x[n-1]$, the FS coefficients $a_k$ and $b_k$ must be related as

$$b_k = a_k - e^{-jk(\pi/5)}a_k.$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-jk(\pi/5)}} = \frac{1}{10}\frac{1 - e^{-j8k(\pi/5)}}{1 - e^{-jk(\pi/5)}}$$

16. [4] The solution is as follows:

(a)

$$x(t) = cos(4\pi t) = \frac{1}{2}e^{4\pi t} + \frac{1}{2}e^{-4\pi t}$$

So that the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$.

(b)

$$y(t) = sin(4\pi t) = \frac{1}{2j}e^{4\pi t} - \frac{1}{2j}e^{-4\pi t}$$

So that the nonzero FS coefficients of $x(t)$ are $b_1 = 1/2j$, $b_{-1} = -1/2j$.

(c) Using the *multiplication* property, we know that

$$z(t) = x(t)y(t) \overset{FS}{\leftrightarrow} c_k = \sum_{t=-\infty}^{\infty} a_l b_{k-l}$$

Therefore,

$$c_k = a_k * b_k = \sum_{t=-\infty}^{\infty} a_l b_{k-l} = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$.

(d) We have

$$z(t) = sin(4t)cos(4t) = \frac{1}{2}sin(8t) = \frac{1}{4j}e^{8\pi t} - \frac{1}{4j}e^{-8\pi t}$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$, the same with part (c).